



# Vertex-disjoint rainbow triangles in edge-colored graphs

Jie Hu, Hao Li, Donglei Yang

## ► To cite this version:

Jie Hu, Hao Li, Donglei Yang. Vertex-disjoint rainbow triangles in edge-colored graphs. *Discrete Mathematics*, 2020, 343 (12), pp.112117. 10.1016/j.disc.2020.112117 . hal-03327197

**HAL Id: hal-03327197**

**<https://hal.science/hal-03327197>**

Submitted on 9 Sep 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

# Vertex-disjoint rainbow triangles in edge-colored graphs

Jie Hu<sup>a,\*</sup>, Hao Li<sup>a,†</sup>, Donglei Yang<sup>b,‡</sup>

<sup>a</sup> Laboratoire de Recherche en Informatique,  
 CNRS–Université Paris-Saclay, Orsay Cedex 91405, France

<sup>b</sup> Data Science Institute,  
 Shandong University, Jinan 250100, China

## Abstract

Let  $G$  be an edge-colored graph of order  $n$ . The minimum color degree of  $G$ , denoted by  $\delta^c(G)$ , is the largest integer  $k$  such that for every vertex  $v$ , there are at least  $k$  distinct colors on edges incident to  $v$ . We say that an edge-colored graph is rainbow if all its edges have different colors. In this paper, we consider vertex-disjoint rainbow triangles in edge-colored graphs. Li [Rainbow  $C_3$ 's and  $C_4$ 's in edge-colored graphs, Discrete Math., 313 (2013), 1893-1896] showed that if  $\delta^c(G) \geq (n+1)/2$ , then  $G$  contains a rainbow triangle and the lower bound is tight. Motivated by this result, we prove that if  $n \geq 20$  and  $\delta^c(G) \geq (n+2)/2$ , then  $G$  contains two vertex-disjoint rainbow triangles. In particular, we conjecture that if  $\delta^c(G) \geq (n+k)/2$ , then  $G$  contains  $k$  vertex-disjoint rainbow triangles. For any integer  $k \geq 2$ , we show that if  $n \geq 16k - 12$  and  $\delta^c(G) \geq n/2 + k - 1$ , then  $G$  contains  $k$  vertex-disjoint rainbow triangles. Moreover, we provide sufficient conditions for the existence of  $k$  edge-disjoint rainbow triangles.

**Keywords:** Edge-colored graph; Rainbow triangles; Color degree.

## 1 Introduction

All graphs considered in this paper are simple and finite. All terminology and notation used but not defined here, please follow [4]. An *edge-colored graph*  $G$  is a graph with an edge coloring  $c$ . For an edge  $e \in E(G)$ , denote by  $c(e)$  the color of  $e$ . We say that  $G$  is *properly colored* if any two adjacent edges of  $G$  have different colors, and  $G$  is *rainbow* if all edges of  $G$  have pairwise different colors. Given a vertex  $v \in V(G)$ , the *color degree*  $d^c(v)$  is the number of distinct colors appearing on incident edges of  $v$ . The *minimum color degree*  $\delta^c(G)$  of an edge-colored graph  $G$  is the minimum  $d^c(v)$  over all vertices  $v$  in  $G$ . Let  $\Delta^{mon}(v)$  be the maximum number of incident edges of  $v$  with the same color. Then the *maximum monochromatic degree*  $\Delta^{mon}(G)$  of  $G$  is the maximum  $\Delta^{mon}(v)$  over all vertices  $v$  in  $G$ .

---

\*Supported by the China Scholarship Council (No. 201806220051). *E-mail address:* jie.hu@lri.fr.

†Corresponding author. *E-mail address:* li@lri.fr.

‡*E-mail address:* dlyang120@163.com.

It has been a long and rich history to study the existence of cycles in graphs, digraphs and edge-colored graphs. For graphs, there is a trivial fact that every graph with minimum degree at least two contains a cycle. In 1963, Corrádi and Hajnal [6] generalized this fact as follows.

**Theorem 1.** (Theorem 1 of [6]) *For any positive integer  $k$ , every graph of order at least  $3k$  and minimum degree at least  $2k$  contains  $k$  vertex-disjoint cycles.*

Compared to graphs without edge-colorings, establishing color degree conditions forcing properly colored or rainbow cycles in edge-colored graphs seems essentially more complicated. Indeed, Wang and Li (Proposition 2.2 of [18]) proved that for any positive integer  $l$ , there exists an edge-colored graph  $G$  with  $\delta^c(G) \geq l$ , but  $G$  contains no properly colored cycle. Recently, Fujita, Li and Zhang [8] obtained a tight minimum color degree condition as follows.

**Theorem 2.** (Theorem 2 of [8]) *For all positive integers  $n$  and  $d$  with  $d! \sum_{i=0}^d 1/i! \geq n + 1$ , every edge-colored graph  $G$  of order  $n$  with  $\delta^c(G) \geq d$  contains a properly colored cycle.*

As an analogy of Theorem 1, it seems more challenging to determine sufficient conditions for the existence of vertex-disjoint properly colored cycles. Compared to Theorem 2, we first give an alternative tight color degree condition for a properly colored cycle, which will be of independent interest. To see this, we say that a star  $K_{1,s}$  is *nontrivial* when  $s > 1$ . Given an edge-colored graph  $G$  and a vertex  $v \in V(G)$ , let  $m(v)$  be the number of maximal monochromatic nontrivial stars centred at  $v$ . For any positive integer  $k$ , let  $f(k)$  be the smallest integer such that every digraph of minimum outdegree at least  $f(k)$  contains  $k$  vertex-disjoint directed cycles. There has been plenty of work around  $f(k)$ . Alon (Theorem 1.1 of [1]) proved that  $f(k) \leq 64k$ . Recently, Bucić (Theorem 4.1 of [5]) improved on this bound to show  $f(k) \leq 18k$ .

**Theorem 3.** *For every positive integer  $k$  and every edge-colored graph  $G$ , we have the following.*

- (i) *If  $d^c(v) \geq 2 + m(v)$  for all vertices  $v \in V(G)$ , with at most one exception  $v_0$  satisfying  $d^c(v_0) = 1 + m(v_0)$ , then  $G$  contains a properly colored cycle.*
- (ii) *If  $d^c(v) \geq 2f(k) + m(v)$  for each vertex  $v \in V(G)$ , then  $G$  contains  $k$  vertex-disjoint properly colored cycles.*

It is worth noting that the color degree condition given in Theorem 3 (i) is tight by the following construction.

**Example 1.** *Let  $G_1$  be an edge-colored  $K_2$  with color  $c_0$ . For all  $i \geq 1$ , let  $G_{i+1}$  be obtained from  $2i$  vertex-disjoint copies  $\{H_1, H_2, \dots, H_{2i}\}$  of  $G_i$  and a new edge  $x_i y_i$  with color  $c_i$  as follows. First, join  $x_i$  completely to each  $H_j$  with  $1 \leq j \leq i$ , and  $y_i$  completely to each  $H_j$  with  $i + 1 \leq j \leq 2i$ . Then color all the edges between  $x_i$  (or  $y_i$ ) and  $H_j$  with one color  $c_i^j$ . Note that all the colors  $c_i$  and  $c_i^j$  are different.*

By the above construction, we can easily observe that in each  $G_{i+1}$ ,  $d^c(x_i) = d^c(y_i) = i + 1$ ,  $m(x_i) = m(y_i) = i$ ,  $d^c(v) \geq 2 + m(v)$  for each  $v \in V(G_{i+1}) \setminus \{x_i, y_i\}$  and  $G_{i+1}$  contains no properly colored cycle.

Let  $K_n^c$  be an edge-colored complete graph of order  $n$ . Li, Broersma and Zhang [14] initiated the study of vertex-disjoint properly colored short cycles in  $K_n^c$  and proposed the following conjecture.

**Conjecture 1.** (Conjecture 5 of [14]) *If  $\Delta^{mon}(K_n^c) \leq n - 3k + 1$ , then  $K_n^c$  contains  $k$  vertex-disjoint properly colored cycles of length at most 4.*

In the same paper, they confirmed this conjecture for  $k = 1, 2$  (Observation 3 and Theorem 6 of [14]). Moreover, they constructed some examples to show the tightness of the bound in Conjecture 1.

Closely related questions concern rainbow cycles. For instance, Li and Wang (Theorem 2.1 of [13]) showed that every edge-colored graph  $G$  of order  $n$  with  $\delta^c(G) \geq (n + 1)/2$  contains a rainbow cycle. Furthermore, they conjectured that if  $\delta^c(G) \geq (n + 1)/2$ , then  $G$  contains a rainbow triangle. Later on, Li (Theorem 6 of [12]) proved this conjecture. In particular, Li, Ning, Xu and Zhang [11] generalized this result as follows.

**Theorem 4.** (Theorem 2 of [11]) *Every edge-colored graph  $G$  of order  $n$  with  $\sum_{v \in V(G)} d^c(v) \geq n(n + 1)/2$  contains a rainbow triangle.*

To the end, it is worth mentioning that Czygrinow, Molla, Nagle and Oursler (Theorem 1.2 of [7]) recently confirmed that the minimum color degree condition  $\delta^c(G) \geq (n + 1)/2$  for a graph  $G$  also guarantees a rainbow cycle  $C_\ell$  for any given odd integer  $\ell$ , where the order of  $G$  is essentially large with respect to  $\ell$ , and the bound is also tight.

Inspired by Conjecture 1 and the result of Li [12], we study the minimum color degree condition which guarantees  $k$  vertex-disjoint rainbow triangles, where  $k \geq 2$ . Therefore the first open case is  $k = 2$ , and we almost resolve the case as follows.

**Theorem 5.** *For all integers  $n \geq 20$ , every graph  $G$  of order  $n$  with  $\delta^c(G) \geq (n + 2)/2$  contains two vertex-disjoint rainbow triangles.*

It has been observed from a result of Hajnal and Szemerédi [10] that for all positive integers  $n$  and  $k$  with  $n \geq 3k$ , every graph of order  $n$  with minimum degree at least  $(n + k)/2$  contains  $k$  vertex-disjoint triangles. As an edge-colored version, we propose the following conjecture.

**Conjecture 2.** *For all positive integers  $n$  and  $k$  with  $n \geq 3k$ , every edge-colored graph  $G$  of order  $n$  with  $\delta^c(G) \geq (n + k)/2$  contains  $k$  vertex-disjoint rainbow triangles.*

If Conjecture 2 holds, then the lower bound is tight by the following example.

**Example 2.** Let  $H$  be a properly colored graph where  $V(H) = X_1 \cup X_2 \cup Y$  with  $|X_1| = \lceil (n - k + 1)/2 \rceil$ ,  $|X_2| = \lfloor (n - k + 1)/2 \rfloor$  and  $|Y| = k - 1$ .  $H[X_1 \cup X_2]$  is a balanced complete bipartite graph and  $H[Y]$  is empty. In addition, each  $y \in Y$  is complete to  $X_1 \cup X_2$ .

Hence  $\delta^c(H) = \lfloor (n + k - 1)/2 \rfloor$ . Since every rainbow triangle in  $H$  must contain at least one vertex from  $Y$  and  $|Y| = k - 1$ ,  $H$  contains at most  $k - 1$  vertex-disjoint rainbow triangles.

We say that two rainbow triangles  $uvw$  and  $xyz$  have *totally different colors* if  $\{c(uv), c(vw), c(wu)\} \cap \{c(xy), c(yz), c(zx)\} = \emptyset$ . Note that when  $n$  and  $k$  have the same parity,  $\delta^c(G) \geq (n + k)/2$  cannot guarantee  $k$  vertex-disjoint rainbow triangles such that any two of them have totally different colors. To see this, we obtain the following construction  $H^*$  by modifying  $H$ .

**Example 3.** Let  $V(H^*) = X_1 \cup X_2 \cup Y$  and  $|X_1| = |X_2| = (n - k)/2$ ,  $|Y| = k$ , where  $n$  and  $k$  have the same parity. Moreover,  $H^*[X_1 \cup X_2]$  is a balanced complete bipartite graph and  $H^*[Y]$  is a clique. Also, each  $y \in Y$  is complete to  $X_1 \cup X_2$ . Arbitrarily take two vertices  $y_1, y_2$  from  $Y$  and color all the edges in  $H^*[X_1 \cup \{y_1\}]$  and  $H^*[X_2 \cup \{y_2\}]$  with one color  $c_0$ . Next we properly color the remaining edges of  $H^*$  with other colors.

Hence  $\delta^c(H^*) = (n + k)/2$  and among any collection of  $k$  vertex-disjoint rainbow triangles in  $H^*$ , the two rainbow triangles containing  $y_1$  and  $y_2$ , respectively, always have a common color  $c_0$ .

By induction on  $k$ , one can easily observe the following fact.

**Fact 1.** If  $\delta^c(G) \geq (n + 3k)/2$ , then  $G$  contains  $k$  vertex-disjoint rainbow triangles.

To provide more evidence for Conjecture 2, we obtain the following extension of Theorem 5.

**Theorem 6.** For all integers  $n, k$  with  $k \geq 2$  and  $n \geq 16k - 12$ , every edge-colored graph  $G$  of order  $n$  with  $\delta^c(G) \geq n/2 + k - 1$  contains  $k$  vertex-disjoint rainbow triangles.

By similar arguments in the proof of Theorem 6, we obtain the following result.

**Proposition 1.** For all positive integers  $n, k$  with  $n \geq k + 4$ , every edge-colored graph  $G$  of order  $n$  with  $\delta^c(G) \geq (n + k)/2$  contains  $k$  edge-disjoint rainbow triangles.

## 2 Preliminaries

Recall that  $G$  is an edge-colored graph of order  $n$ . Let  $\delta = \delta^c(G)$  and  $\Delta = \Delta^{mon}(G)$ . By deleting edges in  $G$ , we may assume that  $G$  is *edge-minimal*, i.e., any additional edge deletion would lead to a decrease in  $d^c(v)$  for some vertex  $v$  in  $G$ . Therefore, every monochromatic subgraph in  $G$  is a disjoint union of stars. For every vertex  $v \in V(G)$ , let  $N_G(v)$  be the set of neighbors of  $v$  in  $G$ . We say  $xy \in E(G)$  is a *good edge* for  $v$  if  $vxy$  is a rainbow triangle, and denote by  $g(v)$  the number of good edges for  $v$ .

**Lemma 1.** *Let  $v$  be a vertex of  $G$  with  $\Delta^{mon}(v) = \Delta$ . Then  $g(v) \geq (2\delta - n)(\delta + \Delta - 1)/2$ .*

*Proof.* Let  $A$  be the maximal subset of  $N_G(v)$  such that  $c(vx) = c(vy)$  for any two vertices  $x, y \in A$ . Hence  $|A| = \Delta$ . Let  $B$  be a maximal subset of  $N_G(v) \setminus A$  such that  $c(vx) \neq c(vy)$  for any two vertices  $x, y \in B$ . So we have  $|B| \geq \delta - 1$  and  $|V(G) \setminus (A \cup B \cup \{v\})| \leq n - \delta - \Delta$ .

Now we define a directed graph  $D$  on  $A \cup B$  such that  $\overrightarrow{xy}$  is an arc in  $D$  if and only if  $xy \in E(G)$  and  $c(xy) \neq c(vx)$ . Hence for any two vertices  $x, y \in A$ , there is either a 2-cycle or no arcs between  $x$  and  $y$ . Denote by  $e_a$  the number of 2-cycles in  $D[A]$ . Moreover, if  $\overrightarrow{xy}$  is a 2-cycle with at least one end-vertex in  $B$ , then  $xy$  is a good edge for  $v$ . Let  $e_0$  be the number of such 2-cycles in  $D$ . Obviously,  $g(v) \geq e_0$ . In the remaining proof, we obtain a lower bound of  $e_0$ .

For every  $x \in A \cup B$ , write  $d_D^+(x)$  and  $d_D^-(x)$  for its outdegree and indegree in  $D$ , respectively. We have

$$d_D^+(x) \geq d^c(x) - 1 - |V(G) \setminus (A \cup B \cup \{v\})| \geq 2\delta + \Delta - n - 1.$$

Therefore,

$$\sum_{x \in A \cup B} d_D^+(x) \geq (\Delta + |B|)(2\delta + \Delta - n - 1).$$

Next we consider  $\sum_{x \in A \cup B} d_D^-(x)$ . If an arc  $\overrightarrow{xy}$  is not contained in a 2-cycle, then we have  $c(xy) = c(vy)$ . Hence for every vertex  $x \in A \cup B$ , there are at most  $\Delta^{mon}(x) - 1 \leq \Delta - 1$  arcs  $\overrightarrow{zx}$  which are not contained in 2-cycles. Moreover, the arcs in  $D[A]$  and the arcs from  $B$  to  $A$  are all contained in 2-cycles. Thus, by double counting, we have

$$\sum_{x \in A \cup B} d_D^-(x) \leq |B|(\Delta - 1) + 2(e_a + e_0).$$

Since  $\sum_{x \in A \cup B} d_D^-(x) = \sum_{x \in A \cup B} d_D^+(x)$  and  $2e_a \leq \Delta(\Delta - 1)$ , we conclude that

$$e_0 \geq (2\delta - n)(|B| + \Delta)/2 \geq (2\delta - n)(\delta + \Delta - 1)/2.$$

□

The following lemma is due to Li, Ning, Xu and Zhang [11].

**Lemma 2.** (Theorem 3 of [11]) *Let  $G$  be an edge-colored graph of order  $n$ . If  $d^c(v) \geq n/2$  for every vertex  $v \in V(G)$  and  $G$  contains no rainbow triangles, then  $n$  is even and  $G$  is the complete bipartite graph  $K_{n/2, n/2}$ , unless  $G = K_4 - e$  or  $K_4$  when  $n = 4$ .*

The rest of the paper is organized as follows. In Section 3, we first prove Theorem 3. Next we proceed to prove Theorem 6 while the proof of Theorem 5 is included in that of Theorem 6 as an initial step. The proof of Proposition 1 is presented at the end of this section. Section 4 will propose some related problems.

### 3 Main proofs

**Proof of Theorem 3.** Let  $G$  be given in the statement of (i) (or (ii)) and suppose  $G$  contains no properly colored cycles (or  $k$  vertex-disjoint properly colored cycles). We still assume that  $G$  is edge-minimal. Therefore, every monochromatic subgraph of  $G$  is a union of vertex-disjoint stars. Now we define a directed graph  $D$  on  $G$  in the following way: for every edge  $xy$  in  $G$ , we have an arc from  $x$  to  $y$  in  $D$  if there is no  $z \in V(G) \setminus \{x, y\}$  with  $c(xy) = c(xz)$ . Note that since there is no monochromatic path of length 3 in  $G$ , for every edge  $xy$ , there is an arc or a directed 2-cycle corresponding to it in  $D$ . We can easily observe that every directed cycle of length at least 3 in  $D$  is actually a properly colored cycle in  $G$ . Moreover, for each vertex  $v \in V(D)$ , we have  $d^+(v) = d^c(v) - m(v)$ . By assumption, we can greedily find a directed cycle of length at least 3 if each vertex  $v$  has  $d^+(v) \geq 2$ , even though there exists one exception  $v_0$  with  $d^+(v) = d^c(v) - m(v) = 1$ .

We proceed the proof of (ii) by defining an orientation  $D$  of  $G$  in the same way as above. It follows that  $d^+(v) = d^c(v) - m(v) \geq 2f(k)$  for all vertices  $v \in V(D)$ . By the observation that every directed cycle of length at least 3 in  $D$  is indeed a properly colored cycle in  $G$ , it remains to prove that  $D$  contains  $k$  vertex-disjoint directed cycles of length at least 3.

**Claim 3.1.** *There exists a subdigraph  $D'$  of  $D$  such that  $d_{D'}^+(v) \geq f(k)$  for each  $v \in V(D)$  and  $D'$  contains no 2-cycle.*

*Proof.* Let  $H$  be a spanning subgraph of  $G$  with  $E(H) = \{xy \in E(G) \mid \overrightarrow{xy} \in D\}$ . Assume that  $H$  is connected, otherwise we can consider each component separately. We pair up the vertices of odd degree in  $H$  and join each pair with an extra edge to form a multigraph  $H'$ . Hence,  $H'$  has an Eulerian circuit  $C$ , that is, a walk that begins and ends at the same vertex, and contains each edge in  $H'$  exactly once. We can see  $C$  as a directed cycle, and split it into a family of directed walks  $\{W_1, W_2, \dots, W_l\}$  by deleting the extra edges added to  $H$  to form  $H'$ . Hence each walk  $W_i$  can be treated as a subdigraph of  $D$ . Now we obtain  $D'$  from  $D$  by deleting all the arcs in  $W_i$  for each  $i \in [l]$ . It follows that  $D'$  contains no 2-cycle and  $d_{D'}^+(v) \geq d_D^+(v) - \lceil d_H(v)/2 \rceil \geq f(k)$  for each  $v \in V(D)$ .  $\square$

By the definition of  $f(k)$  and Claim 3.1, one can easily obtain  $k$  vertex-disjoint properly colored cycles of length at least 3. This completes the proof of Theorem 3.  $\square$

**Proof of Theorem 6.** As an initial step, we first prove Theorem 5. Let  $v$  be a vertex of  $G$  with  $\Delta^{mon}(v) = \Delta$ . We assume that  $\Delta \geq 2$ , otherwise  $G$  is a properly colored graph with  $\delta(G) = \delta^c(G) \geq (n+2)/2$ . It follows that one can find two vertex-disjoint triangles by a result of Hajnal and Szemerédi [10]. Since  $\delta^c(G-v) \geq (|G-v|+1)/2$ , there is a rainbow triangle, say  $xyz$ , in  $G-v$ . For all good edges  $uw$  for  $v$ , we assume that  $\{u, w\} \cap \{x, y, z\} \neq \emptyset$ , otherwise we

can find two vertex-disjoint rainbow triangles  $xyz$  and  $uvw$  in  $G$ . So there are at least  $g(v)/3$  good edges for  $v$  which are incident with one vertex, say  $x$ , from  $\{x, y, z\}$ . Let  $N^*(v, x)$  be the set of vertices  $u$  such that  $uvx$  is a rainbow triangle. By Lemma 1, we have  $|N^*(v, x)| \geq g(v)/3 \geq 4$ . Let  $G' = G - \{v, x\}$ , then  $\delta^c(G') \geq |G'|/2$ . If there exists a rainbow triangle, say  $x'y'z'$  in  $G'$ , then we can find a vertex  $u \in N^*(v, x) \setminus \{x', y', z'\}$  such that  $uvx$  and  $x'y'z'$  are two vertex-disjoint rainbow triangles in  $G$ . Therefore we assume that there is no rainbow triangle in  $G'$ . By Lemma 2, we know that  $G'$  is a properly colored balanced complete bipartite graph. Since  $\delta^c(G) \geq (n+2)/2$ , we have  $vu, xu \in E(G)$  and  $d_G^c(u) = (n+2)/2$  for every vertex  $u \in V(G')$ . Thus, we can easily find two vertex-disjoint rainbow triangles containing  $x$  and  $v$ , respectively. This completes the proof of Theorem 5.

Now we proceed the proof of Theorem 6 by induction on  $k$ . The basic case  $k = 2$  is easily derived from Theorem 5. Let  $k \geq 3$  and assume that Theorem 6 is true for  $k - 1$ . Similar to the proof of Theorem 5, let  $v$  be a vertex of  $G$  with maximum monochromatic degree. Since  $\delta^c(G - v) \geq n/2 + k - 2 > (n-1)/2 + (k-1) - 1$  and  $n-1 \geq 16(k-1) - 12$ , there are  $k-1$  vertex-disjoint rainbow triangles in  $G - v$  by induction hypothesis. Let  $S$  be the set of vertices of such  $k-1$  rainbow triangles. For any good edge  $uw$  for  $v$ , we may assume that  $\{u, w\} \cap S \neq \emptyset$ , otherwise one can find  $k$  vertex-disjoint rainbow triangles in  $G$ . By averaging arguments, there is a vertex  $x \in S$  which is incident with at least  $g(v)/(3k-3)$  good edges for  $v$ . We also use  $N^*(v, x)$  to denote the set of vertices  $u$  such that  $uvx$  is a rainbow triangle. Since  $\delta^c(G - \{v, x\}) \geq (n-2)/2 + (k-1) - 1$  and  $n-2 \geq 16(k-1) - 12$ , there are  $k-1$  vertex-disjoint rainbow triangles in  $G - \{v, x\}$  by induction hypothesis. Since  $|N^*(v, x)| \geq g(v)/(3k-3)$  and by Lemma 1, we have  $|N^*(v, x)| \geq 3k-2$ . It follows that there is a vertex  $u \in N^*(v, x) \setminus S$  such that  $uvx$  together with the  $k-1$  vertex-disjoint rainbow triangles in  $G - \{v, x\}$  forms  $k$  vertex-disjoint rainbow triangles in  $G$ .  $\square$

**Proof of Proposition 1.** We prove Proposition 1 by induction on  $k$ . The case  $k = 1$  follows from the result of Li [12]. Let  $k \geq 2$  and suppose Proposition 1 holds for  $k - 1$ . Let  $v$  be a vertex of  $G$  with maximum monochromatic degree. Since  $\delta^c(G - v) \geq (n+k)/2 - 1 = [(n-1) + (k-1)]/2$  and  $n-1 \geq (k-1) + 4$ ,  $G - v$  contains  $k-1$  edge-disjoint rainbow triangles by induction hypothesis. Since  $g(v) \geq k(k+2)/2 \geq 3(k-1) + 1$ , there is a good edge  $uw$  for  $v$  such that  $uvw$  and the  $k-1$  edge-disjoint rainbow triangles in  $G - v$  form  $k$  edge-disjoint rainbow triangles in  $G$ .  $\square$

## 4 Concluding Remarks

In this paper, we mainly consider vertex-disjoint rainbow triangles in edge-colored graphs. Theorem 3 concerns vertex-disjoint properly colored cycles. Based on Theorems 2 and 3, it will be interesting to determine the minimum color degree condition for  $k$  vertex-disjoint properly colored cycles. For digraphs, Henning and Yeo (Conjecture 1 of [9]) conjectured that every digraph of minimum



out-degree at least 4, contains two vertex-disjoint directed cycles of different lengths. Later on, Lichiardopol confirmed it (Theorem 1.1 of [15]) and proposed the following conjecture.

**Conjecture 3.** (Conjecture 3.1 of [15]) *For every integer  $k \geq 2$ , there exists an integer  $g(k)$  such that any digraph of minimum out-degree at least  $g(k)$  contains  $k$  vertex-disjoint cycles of different lengths.*

Recently, Bensmail, Harutyunyan, Le, Li and Lichiardopol [2] proved Conjecture 3 for tournaments, regular digraphs and digraphs of small order. Moreover, for undirected graphs, they proved that every graph with minimum degree at least  $(k^2 + 5k - 2)/2$  contains  $k$  vertex-disjoint cycles of different lengths (Theorem 12 of [2]), where the degree bound is best possible. In particular, using the same arguments in the proof of Theorem 3, we can easily obtain the following result.

**Proposition 2.** *If Conjecture 3 holds for each positive integer  $k$ , then every edge-colored graph  $G$  with  $d^c(v) \geq 2g(k) + m(v)$  for all vertices  $v \in V(G)$  contains  $k$  vertex-disjoint properly colored cycles of different lengths.*

It will also be interesting to establish the minimum color degree condition for  $k$  vertex-disjoint properly colored cycles of different lengths, and we will return to this topic in the near future.

## References

- [1] N. Alon, Disjoint directed cycles, J. Combin. Theory Ser. B, 68 (1996), 167-178.
- [2] J. Bensmail, A. Harutyunyan, N. K. Le, B. Li and N. Lichiardopol, Disjoint cycles of different lengths in graphs and digraphs, Electron. J. Combin., 24 (2017), P4.37.
- [3] N. C. Bermond and C. Thomassen, Cycles in digraphs – A survey, J. Graph Theory, 5 (1981), 1-43.
- [4] J. A. Bondy and U. S. R. Murty, Graph theory with applications, Elsevier Science, North-Holland, New York, 1976.
- [5] M. Bucić, An improved bound for disjoint directed cycles, Discrete Math., 341 (2018), 2231-2236.
- [6] K. Corrádi and A. Hajnal, On the maximal number of independent circuits of a graph, Acta Math. Acad. Sci. Hungar., 14 (1963), 423-443.
- [7] A. Czygrinow, T. Molla, B. Nagle and R. Oursler, On odd rainbow cycles in edge-colored graphs, arXiv:1910.03745.

- [8] S. Fujita, R. Li and S. Zhang, Color degree and monochromatic degree conditions for short properly colored cycles, *J. Graph Theory*, 87 (2018), 362-373.
- [9] M. A. Henning and A. Yeo, Vertex disjoint cycles of different length in digraphs, *SIAM J. Discrete Math.*, 26 (2012), 687-694.
- [10] A. Hajnal and E. Szemerédi, Proof of a conjecture of P. Erdős, *Combinatorial Theory and Its Applications*, 2 (1970), 601-623.
- [11] B. Li, B. Ning, C. Xu and S. Zhang, Rainbow triangles in edge-colored graphs, *European J. Combin.*, 36 (2014), 453-459.
- [12] H. Li, Rainbow  $C_3$ 's and  $C_4$ 's in edge-colored graphs, *Discrete Math.*, 313 (2013), 1893-1896.
- [13] H. Li and G. Wang, Color degree and heterochromatic cycles in edge-colored graphs, *European J. Combin.*, 33 (2012), 1958-1964.
- [14] R. Li, H. Broersma and S. Zhang, Vertex-disjoint properly edge-colored cycles in edge-colored complete graphs, *J. Graph Theory*, 94 (2020), 476-493.
- [15] N. Lichiardopol, Proof of an conjecture of Henning and Yeo on vertex-disjoint directed cycles, *SIAM J. Discrete Math.*, 28 (2014), 1618-1627.
- [16] N. Lichiardopol, A. Por and J. Sereni, A step toward the Bermond-Thomassen conjecture about disjoint cycles in digraphs, *SIAM J. Discrete Math.*, 23(2) (2009), 979-992.
- [17] C. Thomassen, Disjoint cycles in digraphs, *Combinatorica*, 3 (1983), 393-396.
- [18] G. Wang and H. Li, Color degree and alternating cycles in edge-colored graphs, *Discrete Math.*, 309 (2009), 4349-4354.