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► **To cite this version:**

François Bachoc, Gilles Blanchard, Pierre Neuvial. On the Post Selection Inference constant under Restricted Isometry Properties. 2018. hal-03327160v1

HAL Id: hal-03327160

<https://hal.science/hal-03327160v1>

Preprint submitted on 20 Apr 2018 (v1), last revised 26 Aug 2021 (v3)

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On the Post Selection Inference constant under Restricted Isometry Properties

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Abstract: Uniformly valid confidence intervals post model selection in regression can be constructed based on Post-Selection Inference (PoSI) constants. PoSI constants are minimal for orthogonal design matrices, and can be upper bounded in function of the sparsity of the set of models under consideration, for generic design matrices.

In order to improve on these generic sparse upper bounds, we consider design matrices satisfying a Restricted Isometry Property (RIP) condition. We provide a new upper bound on the PoSI constant in this setting. This upper bound is an explicit function of the RIP constant of the design matrix, thereby giving an interpolation between the orthogonal setting and the generic sparse setting. We show that this upper bound is asymptotically optimal in many settings by constructing a matching lower bound.

1. Introduction

Fitting a statistical model to data is often preceded by a model selection step. The construction of valid statistical procedures in such post model selection situations is quite challenging (cf. [19, 20, 21], [15] and [23], and the references given in that literature), and has recently attracted a considerable amount of attention. Among various recent references in this context, we can mention those addressing sparse high dimensional settings with a focus on lasso-type model selection procedures [4, 5, 26, 27], those aiming for conditional coverage properties for polyhedral-type model selection procedures [25, 18, 12, 17, 24] and those achieving valid post selection inference universally over the model selection procedure [6, 1, 2].

In this paper, we shall focus on the latter type of approach and adopt the setting of [6]. In that work, a linear Gaussian regression model is considered, based on an $n \times p$ design matrix X . A model $M \subset \{1, \dots, p\}$ is defined as a subset of indices of the p covariates. For a family $\mathcal{M} \subset \{M, M \subset \{1, \dots, p\}\}$ of admissible models, it is shown in [6] that a universal coverage property is achievable (see Section 2) by using a family of confidence intervals whose sizes are proportional to a constant $K(X, \mathcal{M}) > 0$. This constant $K(X, \mathcal{M})$ is called a PoSI (Post-Selection Inference) constant in [6]. This setting was later extended to prediction problems in [1] and to misspecified non-linear settings in [2].

The focus of this paper is on the order of magnitude of the PoSI constant $K(X, \mathcal{M})$ for large p . We shall consider $n \geq p$ for simplicity of exposition in the rest of this section (and asymptotics $n, p \rightarrow \infty$). It is shown in [6] that $K(X, \mathcal{M}) = \Omega(\sqrt{\log(p)})$; this rate is reached in particular when X has orthogonal columns. On the other hand, in full generality $K(X, \mathcal{M}) = O(\sqrt{p})$ for all X . It

can also be shown, as discussed in an intermediary version of [28], that when \mathcal{M} is composed of s -sparse submodels, the sharper upper bound $K(X, \mathcal{M}) = O(\sqrt{s \log(p/s)})$ holds. Hence, intuitively, design matrices that are close to orthogonal and sparse models yield smaller PoSI constants.

In this paper, we obtain additional quantitative insights for this intuition, by considering design matrices X satisfying restricted isometry property (RIP) conditions. RIP conditions have become central in high dimensional statistics and compressed sensing [13, 8, 9]. In the s -sparse setting and for design matrices X that satisfy a RIP property of order s with RIP constant $\delta \rightarrow 0$, we show that $K(X, \mathcal{M}) = O(\sqrt{\log(p)} + \delta\sqrt{s \log(p/s)})$. This corresponds to the intuition that for such matrices, any subset of s columns of X is ‘‘approximately orthogonal’’. Thus, under the RIP condition we improve the upper bound of [28] for the s -sparse case, by up to a factor $\delta \rightarrow 0$. We show that our upper bound is complementary to the bounds recently proposed in [16]. In addition, we obtain lower bounds on $K(X, \mathcal{M})$ for a class of design matrices that extends the equi-correlated design matrix in [6]. From these lower bounds, we show that the new upper bound we provide is optimal, in a large range of situations.

The rest of the paper is organized as follows. In Section 2 we introduce in more details the setting and the PoSI constant $K(X, \mathcal{M})$. In Section 3 we introduce the RIP condition and provide the upper bound on $K(X, \mathcal{M})$. In Section 4 we provide the lower bound and the optimality result for the upper bound. All the proofs are given in the appendix.

2. Settings and notation

2.1. PoSI confidence intervals

Consider a Gaussian vector of observation

$$Y = \mu + \epsilon, \tag{1}$$

where the $n \times 1$ mean vector μ is fixed and unknown, and ϵ follows the $\mathcal{N}(0, \sigma^2 I_n)$ distribution where $\sigma^2 > 0$ is unknown. We consider a $n \times p$ fixed design matrix X , whose columns correspond to explanatory variables for μ . Importantly, we do not assume that μ belongs to the image of X or that $n \geq p$.

A model M corresponds to a subset of selected variables in $\{1, \dots, p\}$. We consider a set of models $\mathcal{M} \subset \mathcal{M}_{all} = \{M, M \subset \{1, \dots, p\}\}$. Following [6], for any $M \in \mathcal{M}$, the projection based vector of regression coefficients β_M is a target of inference, with

$$\beta_M := \underset{\beta \in \mathbb{R}^{|M|}}{\text{Arg Min}} \|\mu - X_M \beta\|^2 = (X_M^t X_M)^{-1} X_M^t \mu, \tag{2}$$

where X_M is the submatrix of X formed of the columns of X with indices in M , and where we assume that for each $M \in \mathcal{M}$, X_M has full rank and M is non-empty. We refer to [6] for an interpretation of the vector β_M and a justification for considering it as a target of inference. In [6], a family of confidence intervals $(\text{CI}_{i,M}; i \in M \in \mathcal{M})$ for β_M is introduced, taking the form

$$\text{CI}_{i,M} := (\hat{\beta}_M)_{i,M} \pm \hat{\sigma} \|v_{M,i}\| K(X, \mathcal{M}, \alpha, r); \tag{3}$$

the different quantities involved, which we now define, are standard ingredients for univariate confidence intervals for regression coefficients in the Gaussian model, except for the last factor (the ‘‘PoSI constant’’) which will account for multiplicity of covariates and models. The confidence interval is centered at $\hat{\beta}_M := (X_M^t X_M)^{-1} X_M^t Y$, the ordinary least squares estimator of β_M ; also, if $M = \{j_1, \dots, j_{|M|}\}$ with $j_1 < \dots < j_{|M|}$, for $i \in M$ we denote by $i.M$ the number $k \in \mathbb{N}$ for which $j_k = i$, that is, the rank of the i -th element in the subset M . The quantity $\hat{\sigma}$ is an unbiased estimator of σ , more specifically it is assumed that it is an observable random variable, such that $\hat{\sigma}^2$ is independent of $P_X Y$ and is distributed as σ^2/r times a chi-square distributed random variable with r degrees of freedom (P_X denoting the orthogonal projection onto the column space of X). We allow for $r = \infty$ corresponding to $\hat{\sigma} = \sigma$, i.e., the case of known variance (also called

Gaussian limiting case). In [6], it is assumed that $\hat{\sigma}$ exists and it is shown that this indeed holds in some specific situations. A further analysis of the existence of $\hat{\sigma}$ is provided in [1, 2].

The next quantity to define is

$$v_{M,i}^t := (e_{i.M}^{|M|})^t G_M^{-1} X_M^t \in \mathbb{R}^n, \quad (4)$$

where e_a^b is the a -th base column vector of \mathbb{R}^b ; and $G_M := X_M^t X_M$ is the $|M| \times |M|$ Gram matrix formed from the columns of X_M . Observe that $v_{M,i}$ is nothing more than the row corresponding to covariate i in the estimation matrix $G_M^{-1} X_M^t$, in other words $(\hat{\beta}_M)_{i.M} = v_{M,i}^t Y$.

Finally, $K(X, \mathcal{M}, \alpha, r)$ is called a PoSI constant and we turn to its definition. We shall occasionally write for simplicity $K(X, \mathcal{M}, \alpha, r) = K(X, \mathcal{M})$. Furthermore, if the value of r is not specified in $K(X, \mathcal{M})$, it is implicit that $r = \infty$.

Definition 2.1. Let $\mathcal{M} \subset \mathcal{M}_{all}$ for which each $M \in \mathcal{M}$ is non-empty, and so that X_M has full rank. Let also

$$w_{M,i} = \begin{cases} v_{M,i} / \|v_{M,i}\|, & \text{if } \|v_{M,i}\| \neq 0; \\ 0 \in \mathbb{R}^n & \text{else.} \end{cases}$$

Let ξ be a Gaussian vector with zero mean vector and identity covariance matrix on \mathbb{R}^n . Let N be a random variable, independent of ξ , and so that rN^2 follows a chi-square distribution with r degrees of freedom. If $r = \infty$, then we let $N = 1$. For $\alpha \in (0, 1)$, $K(X, \mathcal{M}, \alpha, r)$ is defined as the $1 - \alpha$ quantile of

$$\gamma_{\mathcal{M},r} := \frac{1}{N} \max_{M \in \mathcal{M}, i \in M} |w_{M,i}^t \xi|.$$

We remark that $K(X, \mathcal{M}, \alpha, r)$ is the same as in [6]. For $j = 1, \dots, p$, let X_j be the column j of X . We also remark, from [6], that the vector $v_{M,i} / \|v_{M,i}\|^2$ in (4) is the residual of the regression of X_i with respect to the variables $\{j, j \in M \setminus \{i\}\}$; in other words, it is the component of the vector X_i orthogonal to $\text{Span}\{X_j, j \in M \setminus \{i\}\}$. It is shown in [6] that we have, with probability larger than $1 - \alpha$,

$$\forall M \in \mathcal{M}, \quad \forall i \in M, \quad (\beta_M)_{i.M} \in \text{CI}_{i,M}. \quad (5)$$

Hence, the PoSI confidence intervals guarantee a simultaneous coverage of all the projection-based regression coefficients, over all models M in the set \mathcal{M} .

For a square symmetric non-negative matrix A , we let $\text{corr}(A) = (\text{diag}(A)^\dagger)^{1/2} A (\text{diag}(A)^\dagger)^{1/2}$, where $\text{diag}(A)$ is obtained by setting all the non-diagonal elements of A to zero and where B^\dagger is the Moore-Penrose pseudo-inverse of B . Then we show in the following lemma that $K(X, \mathcal{M})$ depends on X only through $\text{corr}(X^t X)$.

Lemma 2.2. *Let X and Z be two $n \times p$ and $m \times p$ matrices satisfying $\text{corr}(X^t X) = \text{corr}(Z^t Z)$. Then $K(X, \mathcal{M}, \alpha, r) = K(Z, \mathcal{M}, \alpha, r)$.*

2.2. Order of magnitude of the PoSI constant

The confidence intervals in (3) are similar in form to the standard confidence intervals that one would use for a single fixed model M and a fixed $i \in M$. For a standard interval, $K(X, \mathcal{M})$ would be replaced by a standard Gaussian or Student quantile. Of course, the standard intervals do not account for multiplicity and do not have uniform coverage over $i \in M \in \mathcal{M}$ (see [1, 2]). Hence $K(X, \mathcal{M})$ is the inflation factor or correction over standard intervals to get uniform coverage; it must go to infinity as $p \rightarrow \infty$ [6]. Studying the asymptotic order of magnitude of $K(X, \mathcal{M})$ is thus an important problem, as this order of magnitude corresponds to the price one has to pay in order to obtain universally valid post model selection inference.

We now present the existing results on the asymptotic order of magnitude of $K(X, \mathcal{M})$. Let us define

$$\gamma_{\mathcal{M},\infty} := \max_{M \in \mathcal{M}, i \in M} |w_{M,i}^t \xi|, \quad (6)$$

so that $\gamma_{\mathcal{M},r} = \gamma_{\mathcal{M},\infty}/N$, where we recall that rN^2 follows a chi-square distribution with r degrees of freedom.

We can relate the quantiles of $\gamma_{\mathcal{M},r}$ (which coincide with the PoSI constants $K(X, \mathcal{M})$) to the expectation $\mathbb{E}[\gamma_{\mathcal{M},\infty}]$ by the following argument based on Gaussian concentration (see Appendix A):

Proposition 2.3. *Let $T(\mu, r, \alpha)$ denote the α -quantile of a noncentral T distribution with r degrees of freedom and noncentrality parameter μ . Then*

$$K(X, \mathcal{M}, \alpha, r) \leq T(\mathbb{E}[\gamma_{\mathcal{M},\infty}], r, 1 - \alpha/2).$$

To be more concrete, we observe that we can get a rough estimate of the latter quantile via

$$T(\mathbb{E}[\gamma_{\mathcal{M},\infty}], r, 1 - \alpha/2) \leq \frac{\mathbb{E}[\gamma_{\mathcal{M},\infty}] + \sqrt{2 \log(4/\alpha)}}{(1 - 2\sqrt{2 \log(4/\alpha)/r})_+};$$

furthermore, as $r \rightarrow +\infty$, this quantile reduces to the $(1 - \alpha/2)$ quantile of a Gaussian distribution with mean $\mathbb{E}[\gamma_{\mathcal{M},\infty}]$ and unit variance.

The point of the above estimate is that the dependence in the set of models \mathcal{M} is only present through $\mathbb{E}[\gamma_{\mathcal{M},\infty}]$. Therefore, we will focus in this paper on the problem of bounding $\mathbb{E}[\gamma_{\mathcal{M},\infty}]$, which is nothing more than the Gaussian width [13, chapter 9] of the set $\Gamma_{\mathcal{M}} = \{\pm w_{M,i} | M \in \mathcal{M}, i \in M\}$.

When $n \geq p$, it is shown in [6] that $\mathbb{E}[\gamma_{\mathcal{M},\infty}]$ is no smaller than $\sqrt{2 \log(2p)}$ and asymptotically no larger than \sqrt{p} . These two lower and upper bound are reached by respectively orthogonal design matrices and equi-correlated design matrices (see [6]).

We now concentrate on s -sparse models and let $\mathcal{M}_s = \{M, M \subset \{1, \dots, p\}, |M| \leq s\}$ for some $s \leq p$. In this case using a direct cardinality-based bound one gets the following generic upper bound (proved in Appendix B).

Lemma 2.4. *For any $s, n, p \in \mathbb{N}$, with $s \leq n$, we have*

$$\mathbb{E}[\gamma_{\mathcal{M}_s, \infty}] \leq \sqrt{2s \log(6p/s)}. \tag{7}$$

We remark that an asymptotic version of the bound in Lemma 2.4 (as p and s go to infinity) appears in an intermediary version of [28].

3. Upper bound under RIP conditions

3.1. Main result

The RIP constant $\kappa(X, s)$ associated to a design matrix X and a sparsity condition s [13, Chap.6] is defined as

$$\kappa(X, s) = \sup_{|M| \leq s} \|X_M^t X_M - I_{|M|}\|_{op}. \tag{8}$$

We remark that, with $\kappa = \kappa(X, s)$, we have [13, Chap.6],

$$\forall M \subset \{1, \dots, p\} \text{ s.t. } |M| \leq s: \quad \forall \beta \in \mathbb{R}^{|M|} \quad (1 - \kappa)_+ \|\beta\|^2 \leq \|X_M \beta\|^2 \leq (1 + \kappa) \|\beta\|^2. \tag{9}$$

Remark 3.1. The RIP condition may also be stated between norms instead of squared norms in (9). Following [13, Chap.6] we will consider the formulation in terms of squared norms, which is more convenient here.

Since the PoSI constant $K(X, \mathcal{M})$ only depends on $\text{corr}(X^t X)$ (see Lemma 2.2), we shall rather consider the RIP constant associated to $\text{corr}(X^t X)$. We let

$$\delta(X, s) = \sup_{|M| \leq s} \|\text{corr}(X_M^t X_M) - I_{|M|}\|_{op}. \tag{10}$$

Any upper bound for $\kappa(X, s)$ yields an upper bound for $\delta(X, s)$ as shown in the following lemma.

Lemma 3.2. *Let $\kappa = \kappa(X, s)$. If $\kappa \in [0, 1)$, then*

$$\delta(X, s) \leq \frac{2\kappa}{1 - \kappa}.$$

The next theorem is the main result of the paper. It provides a new upper bound on the PoSI constant, under RIP conditions and with sparse submodels. We remark that in this theorem, we do not necessarily assume that $n \geq p$.

Theorem 3.3. *Let X be a $n \times p$ matrix with $n, p \in \mathbb{N}$. Let $\delta = \delta(X, s)$. We have*

$$\mathbb{E}[\gamma_{\mathcal{M}_s, \infty}] \leq \sqrt{2 \log(2p)} + 2\delta \left(\frac{\sqrt{1 + \delta}}{1 - \delta} \right) \sqrt{2s \log(6p/s)}.$$

This upper bound is of the form

$$U_{\text{RIP}}(p, s, \delta) = U_{\text{orth}}(p) + 2\delta c(\delta) U_{\text{sparse}}(p, s),$$

where $U_{\text{orth}}(p) = \sqrt{2 \log(2p)}$ is the upper bound in the orthogonal case, $U_{\text{sparse}}(p, s)$ is the right-hand side of (7) corresponding to the cardinality-based upper bound in the sparse case, and $c(\delta) = \sqrt{1 + \delta}/(1 - \delta)$ satisfies: $c(\delta) \geq 0$, $c(\delta) \rightarrow 1$ as $\delta \rightarrow 0$, and c is increasing. We observe that if $\delta \rightarrow 0$, our bound U_{RIP} is $o(U_{\text{sparse}})$, and if $\delta \sqrt{s} \sqrt{1 - \log s / \log p + 1 / \log p} \rightarrow 0$, then U_{RIP} is even asymptotically equivalent to U_{orth} . In particular, this is the case if $\delta \sqrt{s} \rightarrow 0$.

We now consider the specific case where X is a subgaussian random matrix, that is, X has independent subgaussian entries [13, Definition 9.1]. We discuss in which situations $\delta = \delta(X, s) \rightarrow 0$. The estimate of κ in [13, Theorem 9.2] combined with Lemma 3.2 yields

$$\delta = O_P \left(\sqrt{s \log(ep/s)/n} \right), \tag{11}$$

so that $\delta \rightarrow 0$ as soon as $n/(s \log(ep/s)) \rightarrow +\infty$. In practice it is typically too costly to compute $\delta(X, s)$ when $n < p$. Nevertheless, if one knows that X is a subgaussian random matrix, they can compute $U_{\text{RIP}}(p, s, \tilde{\delta})$ where $\tilde{\delta}$ is an upper bound of δ with high probability given in [13]. We remark that using the values of $\tilde{\delta}$ currently available in the literature, one would need n to be very large for U_{RIP} to improve U_{sparse} .

When $n > p$, we have $\delta(X, s) \leq \delta(X, p)$ and $\delta(X, p)$ can be computed in practice for a given X . Specifically, $\delta(X, p)$ is the largest eigenvalue of $\text{corr}(X^t X) - I_p$ in absolute value. When X is a subgaussian random matrix, $\delta(X, p) \sim \sqrt{p/n}$ [3, 22].

Remark 3.4. For $q, r \in \mathbb{N}$ and $\ell \in (0, 1)$, let $B_\ell(q, r)$ be defined as the smallest $t > 0$ so that

$$\mathbb{E}_G \left(\min \left(1, r \left[1 - F_{\text{Beta}, 1/2, (q-1)/2} (t^2/G^2) \right] \right) \right) \leq \ell,$$

where $F_{\text{Beta}, 1/2, (q-1)/2}$ is the cumulative distribution function of the Beta(1/2, (q-1)/2) distribution, and where G^2 follows a chi-squared distribution with q degrees of freedom. B_ℓ is also defined and further described in [2, Section 2.5.2].

It can be seen from the proof of Theorem 3.3 (see specifically (21) which also holds without the expectation operators), and from the arguments in [2], that we have

$$K(X, \mathcal{M}_s, \alpha) \leq B_{t\alpha}(n \wedge p, p) + 2\delta c(\delta) B_{(1-t)\alpha}(n \wedge p, |\mathcal{M}_s|)$$

for any $t \in (0, 1)$. This upper bound can be minimized with respect to t and is generally sharper than that of Theorem 3.3 combined with Proposition 2.3, though less explicit (but it can be easily computed numerically).

3.2. Comparison with upper bounds based on Euclidean norms

We now compare our upper bound in Theorem 3.3 to upper bounds recently and independently obtained in [16]. Recall the notation Y , μ , β_M and $\hat{\beta}_M$ from Section 2 and let $r = \infty$ for simplicity of exposition. The authors in [16] address the case where X is random (random design) and consider deviations of $\hat{\beta}_M$ to $\bar{\beta}_M = \mathbb{E}[X_M^t X_M]^{-1} \mathbb{E}[X_M^t Y]$, the population version of the regression coefficients β_M , assuming that the lines of X are independent random vectors in dimension p . They derive uniform bounds over $M \in \mathcal{M}_s$ for $\|\bar{\beta}_M - \hat{\beta}_M\|_2$. They also consider briefly (Remark 4.3 in [16]) the fixed design case with $\beta_M = (X_M^t X_M)^{-1} X_M^t \mu$ as in the present paper. This target β_M can be interpreted as the random design model conditional to X . They assume that the individual coordinates of X and Y have exponential moments bounded by a constant independently from n, p (thus their setting is more general than the Gaussian regression setting, but for the purpose of this discussion we assume Gaussian noise).

Let us additionally assume that the RIP property $\kappa(X/\sqrt{n}, s) \leq \kappa$ is satisfied (on an event of probability tending to 1) and for κ restricted to a compact of $[0, 1)$ independently of n, p ; note that we used the rescaling of X by \sqrt{n} , which is natural in the random design case. Then some simple estimates obtained as a consequence of Theorems¹ 3.1 and 4.1 in [16] lead to

$$\sup_{M \in \mathcal{M}_s} \|\beta_M - \hat{\beta}_M\|_2 = O_P \left(\sigma \sqrt{\frac{s \log(ep/s)}{n}} \right), \quad (12)$$

as $p, n \rightarrow \infty$ and assuming $s \log^2 p = o(n)$. On our side, under the same assumptions we have that

$$\sup_{M \in \mathcal{M}_s, i \in M} \left(\left(\frac{X_M^t X_M}{n} \right)^{-1} \right)_{i.Mi.M}$$

is bounded on an event of probability tending to 1, leading to $\|v_{i.M}\| = O_P(1/\sqrt{n})$ uniformly for all $M \in \mathcal{M}_s, i \in M$. Hence, from Theorem 3.3, (3), (5), we obtain

$$\sup_{M \in \mathcal{M}_s} \|\beta_M - \hat{\beta}_M\|_\infty = O_P \left(\sigma \left(\sqrt{\frac{\log(p)}{n}} + \delta \sqrt{\frac{s \log(ep/s)}{n}} \right) \right). \quad (13)$$

Thus, if $\delta = \Omega(1)$, since the Euclidean norm upper bounds the supremum norm, the results of [16] imply ours (at least in the sense of these asymptotic considerations). On the other hand, in the case where $\delta \rightarrow 0$, which is the case we are specifically interested in, we obtain a sharper bound (in the weaker supremum norm).

In particular, if X is a subgaussian random matrix (as discussed in the previous section), due to (11) we obtain

$$\sup_{M \in \mathcal{M}_s} \|\beta_M - \hat{\beta}_M\|_\infty = O_P \left(\sigma \left(\sqrt{\frac{\log(p)}{n}} + \frac{s \log(ep/s)}{n} \right) \right). \quad (14)$$

This improves over the estimate deduced from (12) as soon as $s \log(ep/s) = o(n)$, which corresponds to the case where (12) tends to 0. Conversely, in this situation our bound (14) yields for the Euclidean norm (using $\|w\|_2 \leq \|w\|_0 \|w\|_\infty$):

$$\sup_{M \in \mathcal{M}_s} \|\beta_M - \hat{\beta}_M\|_2 = O_P \left(\sigma \left(\sqrt{\frac{s \log(p)}{n}} + \frac{s^{3/2} \log(ep/s)}{n} \right) \right). \quad (15)$$

Assuming $s = O(p^\lambda)$ for some $\lambda < 1$ for ease of interpretation, we see that (15) is of the same order as (12) when $s^2 \log(p) = O(n)$, and is of a strictly larger order otherwise. In this sense, it seems that (13) and (12) are complementary to each other since we are using a weaker norm, but obtain a sharper bound in the case $\delta \rightarrow 0$.

¹The technical conditions assumed by [16] imply a slightly weaker version of the RIP property $\kappa(X/\sqrt{n}, s) \leq \kappa < 1$.

4. Lower bound

4.1. Equi-correlated design matrices

The goal of this section is to find a matching lower bound for Theorem 3.3. To investigate this, we consider the design matrix $X^{(c,k)} = (e_1^p, e_2^p, \dots, e_{p-1}^p, x_k(c))$, where

$$x_k(c) = \underbrace{(c, c, \dots, c)}_k, \underbrace{(0, 0, \dots, 0)}_{p-1-k}, \underbrace{(\sqrt{1-kc^2})}_1^t,$$

where we assume $k < p$, and the constant c satisfies $c^2 < 1/k$, so that $X^{(c,k)}$ has full rank. By definition, the correlation between any of the first k columns of $X^{(c,k)}$ and the last one is c , and $X^{(c,k)}$ restricted to its first $p-1$ columns is the identity matrix I_{p-1} . The case where $k = p-1$ is studied in [6, Example 6.2]: Theorem 6.2 in [6] implies that the PoSI constant $K(X^{(c,k)}, \mathcal{M})$ is of the order of \sqrt{p} when $k = p-1$ and $\mathcal{M} = \mathcal{M}_{all}$. The Gram matrix of $X^{(c,k)}$ is the 3×3 block matrix with sizes $(k, p-k-1, 1) \times (k, p-k-1, 1)$ defined by

$$(X^{(c,k)})^t X^{(c,k)} = \begin{bmatrix} I_k & [0] & [c] \\ [0] & I_{p-k-1} & [0] \\ [c] & [0] & 1 \end{bmatrix}, \quad (16)$$

where $[a]$ means that all the entries of the corresponding block are identical to a . We begin by studying the RIP coefficient $\delta(X, s)$ for matrices X yielding the Gram matrix (16).

Lemma 4.1. *Let X be a $n \times p$ matrix for which $X^t X$ is given by (16). Then for $s \leq k \leq p-1$, we have $\kappa(X, s) = \delta(X, s) \leq c\sqrt{s-1}$.*

4.2. A matching lower bound

We now provide a lower bound of $K(X^{(c,k)}, \mathcal{M}_s)$ in the following proposition.

Proposition 4.2. *For any $s \leq k < p$, $c^2 < 1/k$ and $\alpha \leq \frac{1}{2}$ we have*

$$K(X^{(c,k)}, \mathcal{M}_s, \alpha, \infty) \geq A \frac{c(s-1)}{\sqrt{1-(s-1)c^2}} \sqrt{\log[k/s]} - \sqrt{2 \log 2},$$

where $A > 0$ is a universal constant.

From the previous lemma, we now show that the upper bound of Theorem 3.3 is optimal (up to a multiplicative constant) for a large range of behavior of s and δ relatively to p . As discussed after Theorem 3.3, in the case where $\delta\sqrt{s}\sqrt{1-\log s/\log p + 1/\log p} = O(1)$, the upper bound we obtain is optimal, since it can be written as $O(\sqrt{\log p})$. In the next Corollary, we show that the upper bound of Theorem 3.3 is also optimal when $\delta\sqrt{s}\sqrt{1-\log s/\log p + 1/\log p} \rightarrow \infty$, and when $\delta = O(p^{-\lambda})$ for some $\lambda > 0$.

Corollary 4.3 (Optimality of the RIP-PoSI bound). *Let $(s_p, \delta_p)_{p \geq 0}$ be sequences of values such that $s_p < p$, $\delta_p > 0$, $\delta_p \rightarrow 0$ and $\delta_p\sqrt{s_p}\sqrt{1-\log s_p/\log p + 1/\log p} \rightarrow \infty$ as $p \rightarrow \infty$. Then Theorem 3.3 implies*

$$\sup_{\substack{n \in \mathbb{N} \\ s \leq s_p, X \in \mathbb{R}^{n \times p} \\ \text{s.t. } \delta(X, s) \leq \delta_p}} K(X, \mathcal{M}_{s_p}) \leq B \delta_p \sqrt{s_p} \sqrt{\log(6p/s_p)}, \quad (17)$$

where B is a constant. Moreover, there exists a sequence of design matrices X_p such that $\delta(X_p, s_p) \leq \delta_p$ and

$$K(X_p, \mathcal{M}_{s_p}) \geq A \delta_p \sqrt{s_p} \sqrt{\log(\min(1/\delta_p^2, \lfloor (p-1)/s_p \rfloor))}, \quad (18)$$

where A is a constant.

In particular, if $\delta_p = O(p^{-\lambda})$ for some $\lambda > 0$ and if $\lfloor (p-1)/s_p \rfloor \geq 2$, then the above upper and lower bounds have the same rate.

Therefore, the upper bound in Theorem 3.3 is optimal in most configurations of s_p and δ_p , except if δ_p goes to 0 slower than any inverse power of p .

5. Concluding remarks

In this paper, we have proposed an upper bound on PoSI constants in s -sparse situations where the $n \times p$ design matrix X satisfies a RIP condition. As the value of the RIP constant δ increases from 0, this upper bound provides an interpolation between the case of an orthogonal X and an existing upper bound only based on sparsity and cardinality. We have shown that this upper bound is asymptotically optimal for many configurations of (s, δ, p) by giving a matching lower bound. In the case of random design matrices with independent entries, since δ decreases with n , our upper bound compares increasingly more favorably to the cardinality-based upper bound as n gets larger. It is also complementary to the bounds recently proposed in [16].

The PoSI constant corresponds to confidence intervals on β_M in (2). In section 3.2 we also mention another target of interest in the case of random X , $\bar{\beta}_M = \mathbb{E}[X_M^t X_M]^{-1} \mathbb{E}[X_M^t Y]$. This quantity depends on the distribution of X rather than on its realization, which is a desirable property as discussed in [1, 16] where the same target has also been considered. In [1], it is shown that valid confidence intervals for β_M are also asymptotically valid for $\bar{\beta}_M$, provided that p is fixed. These results require that μ belongs to the column space of X and hold for models M such that μ is close to the column space of X_M . It would be interesting to study whether assuming RIP conditions on X enables to alleviate these assumptions.

Acknowledgements

This work has been supported by ANR-16-CE40-0019 (SansSouci). The second author acknowledges the support from the german DFG, under the Research Unit FOR-1735 ‘‘Structural Inference in Statistics - Adaptation and Efficiency’’, and under the Collaborative Research Center SFB-1294 ‘‘Data Assimilation’’.

Appendix

Appendix A: Gaussian concentration

To relate the expectation of a supremum of Gaussian variables to its quantiles, we use the following classical Gaussian concentration inequality [11] (see e.g. [14], Section B.2.2. for a short exposition):

Theorem A.1 (Cirel’son, Ibragimov, Sudakov). *Assume that $F : \mathbb{R}^d \rightarrow \mathbb{R}$ is a 1-Lipschitz function (w.r.t. the Euclidean norm of its input) and Z follows the $\mathcal{N}(0, \sigma^2 I_d)$ distribution. Then, there exists two one-dimensional standard Gaussian variables ζ, ζ' such that*

$$\mathbb{E}[F(Z)] - \sigma|\zeta'| \leq F(Z) \leq \mathbb{E}[F(Z)] + \sigma|\zeta|. \quad (19)$$

It is known that in certain situations one can expect an even tighter concentration, through the phenomenon known as superconcentration [10]. While such situations are likely to be relevant for the setting considered in this paper, we leave such improvements as an open issue for future work.

We use the previous property in our setting as follows:

Proposition A.2. *Let \mathcal{C} be finite a family of unit vectors of \mathbb{R}^n , ξ a standard Gaussian vector in \mathbb{R}^n and N an independent nonnegative random variable so that rN^2 follows a chi-squared distribution with r degrees of freedom. Define the random variable*

$$\gamma_{\mathcal{C}, r} := \frac{1}{N} \max_{v \in \mathcal{C}} |v^t \xi|.$$

Then the $(1 - \alpha)$ quantile of $\gamma_{\mathcal{C},r}$ is upper bounded by the $(1 - \alpha/2)$ quantile of a noncentral T distribution with r degrees of freedom and noncentrality parameter $\mathbb{E}[\max_{v \in \mathcal{C}} |v^t \xi|]$.

Proof. Observe that $\xi \mapsto \max_{v \in \mathcal{C}} |v^t \xi|$ is 1-Lipschitz since the vectors of \mathcal{C} are unit vectors. Therefore we conclude by Theorem A.1 that there exists a standard normal variable ζ (which is independent of N since N is independent of ξ) so that the following holds:

$$\gamma_{\mathcal{C}} \leq \frac{1}{N} \left(\mathbb{E} \left[\max_{v \in \mathcal{C}} |v^t \xi| \right] + |\zeta| \right).$$

We can represent the above right-hand side as $\max(T_+, T_-)$ where

$$T_{\pm} = \frac{1}{N} \left(\mathbb{E} \left[\max_{v \in \mathcal{C}} |v^t \xi| \right] \pm \zeta \right),$$

i.e. T_+, T_- are two (dependent) noncentral t distributions with r degrees of freedom and noncentrality parameter $\mathbb{E}[\max_{v \in \mathcal{C}} |v^t \xi|]$. Finally since

$$\mathbb{P}[\max(T_+, T_-) > t] \leq \mathbb{P}[T_+ > t] + \mathbb{P}[T_- > t] = 2\mathbb{P}[T_+ > t],$$

we obtain the claim. \square

Since a noncentral distribution is (stochastically) increasing in its noncentrality parameter, any bound obtained for $\mathbb{E}[\max_{v \in \mathcal{C}} |v^t \xi|]$ will result in a corresponding bound on the quantiles of the corresponding noncentral T distribution and therefore of those of $\gamma_{\mathcal{C}}$. In the limit $r \rightarrow \infty$, the quantiles of the noncentral T distribution reduce to those of a shifted Gaussian distribution with unit variance.

Here is a naive bound on (some) quantiles of a noncentral T :

Lemma A.3. *The $1 - \alpha$ quantile of a noncentral T distribution with r degrees of freedom and noncentrality parameter $\mu \geq 0$ is upper bounded by:*

$$(\mu + \sqrt{2 \log(2/\alpha)} / (1 - 2\sqrt{2 \log(2/\alpha)/r}))_+.$$

Proof. Let

$$T = \frac{\mu + \zeta}{\sqrt{V/r}},$$

where $\zeta \sim \mathcal{N}(0, 1)$ and $V \sim \chi^2(r)$. We have (as a consequence of e.g. [7], Lemma 8.1), for any $\eta \in (0, 1]$:

$$\mathbb{P} \left[\sqrt{V} \leq \sqrt{r} - 2\sqrt{2 \log \eta^{-1}} \right] \leq \eta,$$

as well as the classical bound

$$\mathbb{P} \left[\zeta \geq \sqrt{2 \log \eta^{-1}} \right] \leq \eta.$$

It follows that

$$\mathbb{P} \left[T \geq (\mu + \sqrt{2 \log \eta^{-1}}) / (1 - 2\sqrt{2 \log(\eta^{-1})/r})_+ \right] \leq 2\eta.$$

The claimed estimate follows. \square

Appendix B: Proofs

Proof of Lemma 2.2. With the notation of Definition 2.1, $K(X, \mathcal{M}, \alpha, r)$ is the $1 - \alpha$ quantile of $(1/N) \|z\|_{\infty}$ where $z = (z_{M,i}, M \in \mathcal{M}, i \in \mathcal{M})$ is a Gaussian vector, independent of N , with mean vector zero and covariance matrix $\text{corr}(\Sigma)$, where Σ is defined by, for $i \in M \in \mathcal{M}$ and $i' \in M' \in \mathcal{M}$,

$$\begin{aligned} \Sigma_{(M,i),(M',i')} &= v_{M,i}^t v_{M',i'} \\ &= (e_{i,M}^{|M|})^t (X_M^t X_M)^{-1} X_M^t X_{M'} (X_{M'}^t X_{M'})^{-1} e_{i',M'}^{|M'|}. \end{aligned}$$

Hence, Σ depends on X only through $X^t X$. Also, if X is replaced by XD , where D is a diagonal matrix with positive components, Σ becomes the matrix Λ with for $i \in M \in \mathcal{M}$ and $i' \in M' \in \mathcal{M}$,

$$\begin{aligned}\Lambda_{(M,i),(M',i')} &= (e_{i.M}^{|M|})^t D_{M,M}^{-1} (X_M^t X_M)^{-1} X_M^t X_{M'} (X_{M'}^t X_{M'})^{-1} D_{M',M'}^{-1} e_{i'.M'}^{|M'|} \\ &= D_{i,i}^{-1} D_{i',i'}^{-1} \Sigma_{(M,i),(M',i')}.\end{aligned}$$

Hence, $\text{corr}(\Sigma) = \text{corr}(\Lambda)$. This shows that Σ depends on X only through $\text{corr}(X^t X)$ (we remark that because $\cup_{\mathcal{M}} M = \{1, \dots, p\}$ and each $X_M^t X_M$ is invertible we have that $\|X_i\| > 0$ for $i = 1, \dots, p$). Hence $K(X, \mathcal{M}, \alpha, r)$ depends on X only through $\text{corr}(X^t X)$. \square

Proof of Lemma 2.4. Using a direct cardinality-based bound we have the well-known inequality $\mathbb{E}[\gamma_{\mathcal{M}_s, \infty}] \leq \sqrt{2 \log(2|\mathcal{M}_s|)}$, hence

$$\mathbb{E}[\gamma_{\mathcal{M}_s, \infty}] \leq \sqrt{2 \log \left(2 \sum_{i=1}^s i \binom{p}{i} \right)},$$

moreover

$$\sum_{i=1}^s i \binom{p}{i} \leq s \sum_{i=0}^s \binom{p}{i} \leq s \left(\frac{pe}{s} \right)^s,$$

the last inequality being classical and due to

$$\left(\frac{s}{p} \right)^s \sum_{i=0}^s \binom{p}{i} \leq \sum_{i=0}^s \left(\frac{s}{p} \right)^i \binom{p}{i} \leq \left(1 + \frac{s}{p} \right)^p \leq e^s.$$

Since $\log s \leq s/e$, and using $e^{1+2/e} \leq 6$, we obtain

$$\log \left(2 \sum_{i=1}^s i \binom{p}{i} \right) \leq \log 2s + s \log \left(\frac{pe}{s} \right) \leq s \log \left(\frac{pe}{s} e^{1+2/e} \right) \leq s \log \left(\frac{6p}{s} \right),$$

implying (7). \square

Proof of Lemma 3.2. Put $\kappa = \kappa(X, s) < 1$. Then, $\|X_i\| \geq (1 - \kappa)^{1/2}$ for $i = 1, \dots, p$ so that for $i \in M \in \mathcal{M}_s$, $\text{corr}(X_M^t X_M) = D_M X_M^t X_M D_M$ where D_M is a $|M| \times |M|$ matrix defined by $[D_M]_{i.M, i.M} = 1/\|X_i\|$. Hence $\|D_M\|_{op} \leq 1/\sqrt{1 - \kappa}$. We have, by applications of the triangle inequality and since $\|\cdot\|_{op}$ is a matrix norm,

$$\begin{aligned}\|\text{corr}(X_M^t X_M) - I_{|M|}\|_{op} &= \|(D_M - I_{|M|}) X_M^t X_M D_M + X_M^t X_M (D_M - I_{|M|}) + X_M^t X_M - I_{|M|}\|_{op} \\ &\leq \|D_M - I_{|M|}\|_{op} \|X_M^t X_M\|_{op} \|D_M\|_{op} + \|D_M - I_{|M|}\|_{op} \|X_M^t X_M\|_{op} \\ &\quad + \|X_M^t X_M - I_{|M|}\|_{op} \\ &= \|D_M - I_{|M|}\|_{op} \|X_M^t X_M\|_{op} (\|D_M\|_{op} + 1) \\ &\quad + \|X_M^t X_M - I_{|M|}\|_{op}.\end{aligned}\tag{20}$$

From (8)-(9), we have for all $M \in \mathcal{M}_s$: $\|X_M^t X_M\|_{op} \leq 1 + \kappa$, as well as

$$\|D_M - I_{|M|}\|_{op} \leq \max_{i=1, \dots, p} \left| \frac{1}{\|X_i\|} - 1 \right| \leq \max \left(1 - \frac{1}{\sqrt{1 + \kappa}}, \frac{1}{\sqrt{1 - \kappa}} - 1 \right) = \frac{1}{\sqrt{1 - \kappa}} - 1.$$

Plugging this into (20), we obtain

$$\begin{aligned}\delta(X, s) &\leq \left(\frac{1}{\sqrt{1 - \kappa}} - 1 \right) (1 + \kappa) \left(\frac{1}{\sqrt{1 - \kappa}} + 1 \right) + \kappa \\ &= \frac{2\kappa}{1 - \kappa}.\end{aligned}$$

\square

Proof of Theorem 3.3. From Lemma 2.2, it is sufficient to treat the case where, for any M , $G_M = X_M^t X_M$ has ones on the diagonal; in that case $\delta(X, s) = \kappa(X, s)$. We have

$$\begin{aligned} v_{M,i}^t &= (e_{i,M}^{|M|})^t G_M^{-1} X_M^t \\ &= (e_{i,M}^{|M|})^t I_{|M|} X_M^t + (e_{i,M}^{|M|})^t (G_M^{-1} - I_{|M|}) X_M^t \\ &= X_i^t + r_{M,i}^t, \end{aligned}$$

say. We have

$$\begin{aligned} r_{M,i}^t r_{M,i} &= (e_{i,M}^{|M|})^t (G_M^{-1} - I_{|M|}) G_M (G_M^{-1} - I_{|M|}) e_{i,M}^{|M|} \\ &\leq \|e_{i,M}^{|M|}\|^2 \|G_M^{-1} - I_{|M|}\|_{op}^2 \|G_M\|_{op}. \end{aligned}$$

From (9), the eigenvalues of G_M are all between $(1 - \delta)$ and $(1 + \delta)$, hence we have

$$r_{M,i}^t r_{M,i} \leq \left(\frac{\delta}{1 - \delta} \right)^2 (1 + \delta),$$

so that

$$\|r_{M,i}\| \leq \delta \left(\frac{\sqrt{1 + \delta}}{1 - \delta} \right),$$

and

$$\begin{aligned} \|w_{M,i} - X_i\| &= \left\| \frac{v_{M,i}}{\|v_{M,i}\|} - X_i \right\| = \left\| \frac{v_{M,i}}{\|v_{M,i}\|} (1 - \|v_{M,i}\|) + v_{M,i} - X_i \right\| \\ &\leq 2\|r_{M,i}\|, \end{aligned}$$

from two applications of the triangle inequality, and using that $\|X_i\| = 1$ since we assumed that G_M has ones on its diagonal for all M . Hence, we have

$$\begin{aligned} \mathbb{E}[\gamma_{\mathcal{M}_s, \infty}] &= \mathbb{E} \left[\sup_{M \in \mathcal{M}_s; i \in M} |w_{M,i}^t \xi| \right] \\ &\leq \mathbb{E} \left[\sup_{M \in \mathcal{M}_s; i \in M} |X_i^t \xi| \right] + \mathbb{E} \left[\sup_{M \in \mathcal{M}_s; i \in M} |(w_{M,i} - X_i)^t \xi| \right] \\ &\leq \mathbb{E} \left[\sup_{i=1, \dots, p} |X_i^t \xi| \right] + 2\delta \left(\frac{\sqrt{1 + \delta}}{1 - \delta} \right) \mathbb{E} \left[\sup_{M \in \mathcal{M}_s; i \in M} \left| \left(\frac{w_{M,i} - X_i}{\|w_{M,i} - X_i\|} \right)^t \xi \right| \right] \quad (21) \\ &\leq \sqrt{2 \log(2p)} + 2\delta \left(\frac{\sqrt{1 + \delta}}{1 - \delta} \right) \sqrt{2s \log(6p/s)}, \end{aligned}$$

where in the last step we have used Lemma 2.4. \square

Proof of Lemma 4.1. Since $\|X_i\| = 1$ for $i = 1, \dots, p$ we have $\text{corr}(X^t X) = X^t X$ and so $\kappa(X, s) = \delta(X, s)$. The Gram matrix in (16) can be written as $I_p + cU_{p,k}$, where $U_{p,k}$ is the 3×3 block matrix with sizes $(k, p - k - 1, 1) \times (k, p - k - 1, 1)$ defined by

$$U_{p,k} = \begin{bmatrix} [0] & [0] & [1] \\ [0] & [0] & [0] \\ [1] & [0] & [0] \end{bmatrix}.$$

Consider a model M with $|M| = s \leq k \leq p - 1$, and denote by G_M its Gram matrix. If $p \notin M$, then $G_M = I_s$ and $\|G_M - I_s\|_{op} = 0$. If $p \in M$, then $G_M = I_s + cU_{s,m}$, where $m = m(M) = |(M \setminus \{p\}) \cap \{1, \dots, k\}| \leq s - 1$. The operator norm of $G_M - I_s$ is the square root of the largest eigenvalue of $(cU_{s,m})^2$, where $U_{s,m}^2$ is a 3×3 block matrix with sizes $(m, s - m - 1, 1) \times (m, s - m - 1, 1)$ defined by

$$U_{s,m}^2 = \begin{bmatrix} [1] & [0] & [0] \\ [0] & [0] & [0] \\ [0] & [0] & m \end{bmatrix}.$$

The first block is a $m \times m$ matrix with all entries equal to 1, hence its only non-null eigenvalue is m . This is also the (only) eigenvalue of the last block (an 1×1 matrix). Thus, the largest eigenvalue of $U_{s,m}^2$ is m . Therefore, as $m \leq s - 1$, we have $\|G_M - I_s\|_{op} = c\sqrt{s - 1}$ for all M such that $|M| = s \leq k \leq p - 1$, which concludes the proof. \square

Proof of Proposition 4.2. The proof is an extension of the proof of [6, Theorem 6.2]. For $m \geq 0$, consider a model M such that $M \ni p$, $M \cap \{k + 1, \dots, p - 1\} = \emptyset$, and $|M| = m + 1$; in other words, $M = \{i_1, \dots, i_m, p\}$ such that i_1, \dots, i_m are elements of $\{1, \dots, k\}$. Denote as $\mathcal{M}_{m:k}^{+p}$ the set of all such models. Let $u_{M,p} = X_p - P_{M \setminus \{p\}}(X_p)$, where X_p is the last column of $X^{(c,k)}$, and where $P_{M \setminus \{p\}}(X_p)$ is the orthogonal projection of X_p onto the span of the columns with indices $M \setminus \{p\}$. Observe that the column i_j of $X^{(c,k)}$ is the i_j -th base column vector of \mathbb{R}^p that we write e_{i_j} , therefore

$$P_{M \setminus \{p\}}(X_p) = \sum_{j=1}^m (e_{i_j}^t X_p) e_{i_j} = c(e_{i_1} + \dots + e_{i_m}).$$

Hence, we have, for $M \in \mathcal{M}_{m:k}^{+p}$,

$$[u_{M,p}]_j = \begin{cases} 0 & \text{for } j = k + 1, \dots, p - 1, \\ 0 & \text{for } j = 1, \dots, k; j \in M, \\ c & \text{for } j = 1, \dots, k; j \notin M, \\ \sqrt{1 - kc^2} & \text{for } j = p. \end{cases}$$

Recall that we have $w_{M,p} = u_{M,p}/\|u_{M,p}\|$. Hence, for $M \in \mathcal{M}_{m:k}^{+p}$,

$$[w_{M,p}]_j = \begin{cases} 0 & \text{for } j = k + 1, \dots, p - 1, \\ 0 & \text{for } j = 1, \dots, k; j \in M, \\ c/\sqrt{1 - mc^2} & \text{for } j = 1, \dots, k; j \notin M, \\ \sqrt{1 - kc^2}/\sqrt{1 - mc^2} & \text{for } j = p. \end{cases}$$

Hence, we have

$$\begin{aligned} \mathbb{E}[\gamma_{\mathcal{M}_s, \infty}] &= \mathbb{E} \left[\max_{|M| \leq s, i \in M} |w_{M,i}^t \xi| \right] \\ &\geq \mathbb{E} \left[\max_{M \in \mathcal{M}_{(s-1):k}^{+p}} w_{M,p}^t \xi \right] \\ &= \mathbb{E} \left[\frac{\sqrt{1 - kc^2}}{\sqrt{1 - (s-1)c^2}} \xi_p + \frac{c}{\sqrt{1 - (s-1)c^2}} \sum_{j=1}^{k-s+1} \xi_{k-j:k} \right], \end{aligned}$$

where $\xi_{1:k} \leq \dots \leq \xi_{k:k}$ are the order statistics of ξ_1, \dots, ξ_k . Hence, since $s - 1 < k$, we obtain

$$\begin{aligned} \mathbb{E}[\gamma_{\mathcal{M}_s, \infty}] &\geq 0 + \frac{c}{\sqrt{1 - (s-1)c^2}} \mathbb{E} \left[\sum_{j=1}^k \xi_j - \sum_{j=1}^{s-1} \xi_{j:k} \right] \\ &= \frac{c}{\sqrt{1 - (s-1)c^2}} \mathbb{E} \left[\sum_{j=1}^{s-1} \xi_{k-j:k} \right] \\ &\geq \frac{c}{\sqrt{1 - (s-1)c^2}} \mathbb{E} \left[\sum_{j=1}^{s-1} \max_{l=1, \dots, \lfloor k/s \rfloor} \xi_{(j-1)\lfloor k/s \rfloor + l} \right]. \end{aligned}$$

In the above display, each of the maxima has mean value larger than $A\sqrt{\log[k/s]}$ with $A > 0$ a universal constant (see e.g. Lemma A.3 in [10]). Hence, we have

$$\mathbb{E}[\gamma_{\mathcal{M}_s, \infty}] \geq A \frac{c(s-1)}{\sqrt{1-(s-1)c^2}} \sqrt{\log[k/s]}.$$

Finally, a consequence of Gaussian concentration (Theorem A.1) is that mean and median of $\gamma_{\mathcal{M}_s, \infty}$ are within $\sqrt{2\log 2}$ of each other. Since we assumed $\alpha \leq \frac{1}{2}$, $K(X^{(c,k)}, \mathcal{M}_s, \alpha, \infty) \geq \mathbb{E}[\gamma_{\mathcal{M}_s, \infty}] - \sqrt{2\log 2}$, which concludes the proof. \square

Proof of Corollary 4.3. When $\delta_p \sqrt{s_p} \sqrt{1 - \log s_p / \log p + 1 / \log p} \rightarrow \infty$, one can see that in Theorem 3.3, the first term is negligible compared to the second one. Since $\delta_p \rightarrow 0$, the first result (17) follows from Theorem 3.3.

We now apply Proposition 4.2 with $c_p = \delta_p / \sqrt{s_p - 1}$ and $k_p = \min(p-1, \lfloor 1/c_p^2 - 1 \rfloor)$. From Lemma 4.1, $\delta(X^{(c_p, k_p)}, s_p) \leq c_p \sqrt{s_p - 1} = \delta_p$. We then have, with two positive constants A' and A ,

$$\begin{aligned} K(X^{(c_p, k_p)}, \mathcal{M}_s, \alpha, \infty) &\geq A' \delta_p \sqrt{s_p} \sqrt{\log \left(\left\lfloor \frac{\min(p-1, \lfloor 1/c_p^2 - 1 \rfloor)}{s_p} \right\rfloor \right)} \\ &\geq A \delta_p \sqrt{s_p} \sqrt{\log(\min(\lfloor (p-1)/s_p \rfloor, 1/\delta_p^2))}. \end{aligned}$$

This concludes the proof of (18). \square

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