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CRYSTALLINITY OF THE HOMOGENIZED ENERGY DENSITY OF PERIODIC LATTICE SYSTEMS

ANTONIN CHAMBOLLE AND LEONARD KREUTZ

ABSTRACT. We study the homogenized energy densities of periodic ferromagnetic Ising systems. We prove that, for finite range interactions, the homogenized energy density, identifying the effective limit, is crystalline, i.e. its Wulff crystal is a polytope, for which we can (exponentially) bound the number of vertices. This is achieved by deriving a dual representation of the energy density through a finite cell formula. This formula also permits easy numerical computations: we show a few experiments where we compute periodic patterns which minimize the anisotropy of the surface tension.

1. INTRODUCTION

The study of discrete interfacial energies has attracted widespread attention in the mathematical community over last decades, with applications in various contexts such as computer vision [8], crystallization problems [9], fracture mechanics [7, 19, 36], or statistical physics [41, 42]. To give examples, in computer vision the understanding of these energies allows us to investigate functional correctness of segmentation algorithms [23]. Whereas for crystallization problems it gives fluctuation estimates on the macroscopic shape of the crystal cluster of ground state configurations [28, 34, 35, 40].

In this work, we consider energies defined on discrete periodic sets $\mathcal{L} \subset \mathbb{R}^d$ and corresponding Ising systems. We refer to [1, 13, 21, 22, 29, 31, 37, 38, 39] for an abundant literature on the derivation of continuum limits of such systems and their effective behavior. More precisely, we consider \mathcal{L} satisfying the following two conditions (see Figure 1)

- (i) (Discreteness) There exists $c > 0$ such that $\text{dist}(x, \mathcal{L} \setminus \{x\}) \geq c$ for all $x \in \mathcal{L}$;
- (ii) (Periodicity) There exists $T \in \mathbb{N}$ such that for all $z \in \mathbb{Z}^d$, it holds that $\mathcal{L} + Tz = \mathcal{L}$;

To each function $u : \mathcal{L} \rightarrow \{0, 1\}$ and each $A \subset \mathbb{R}^d$ we associate an energy

$$E(u, A) = \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j))^+, \quad (1)$$

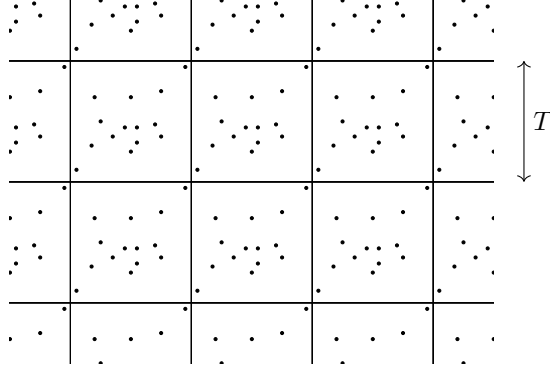
where $(z)^+$ denotes the positive part of $z \in \mathbb{R}$, $c_{i,j} : \mathcal{L} \times \mathcal{L} \rightarrow [0, +\infty)$ are T -periodic, that is $c_{i+Tz, j+Tz} = c_{i,j}$ for all $i, j \in \mathcal{L}$ and $z \in \mathbb{Z}^d$ and satisfy the following decay assumption

- (iii) (Decay of interactions) For all $i \in \mathcal{L}$ there holds

$$\sum_{j \in \mathcal{L}} c_{i,j} |i - j| < +\infty.$$

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FIGURE 1. An example of the set \mathcal{L}

Assuming conditions (i)-(iii) (and some additional coercivity assumption) ensures that the asymptotic behavior of (1) is well described (in a variational sense) by a continuum perimeter energy. More precisely, let us introduce a scaling parameter $\varepsilon > 0$. We consider the scaled energies

$$E_\varepsilon(u) = \sum_{i,j \in \varepsilon\mathcal{L}} \varepsilon^{d-1} c_{i,j}^\varepsilon (u(i) - u(j))^+,$$

where $c_{i,j}^\varepsilon = c_{i/\varepsilon, j/\varepsilon}$ and $u: \varepsilon\mathcal{L} \rightarrow \{0, 1\}$. By identifying u with its piecewise constant interpolation taking the value $u(i)$ on the Voronoi cell centered at $i \in \varepsilon\mathcal{L}$ we may regard the energies as defined on $L_{\text{loc}}^1(\mathbb{R}^d, \{0, 1\})$. Integral representation results [2, 4, 18] then guarantee that the energies E_ε Γ -converge (see [12, 30] for an introduction to that subject) with respect to the $L_{\text{loc}}^1(\mathbb{R}^d)$ -topology to a continuum energy of the form

$$E_0(u) = \int_{\partial^* \{u=1\}} \varphi(\nu_u(x)) \, d\mathcal{H}^{d-1} \quad u \in BV_{\text{loc}}(\mathbb{R}^d; \{0, 1\}).$$

Here, $BV_{\text{loc}}(\mathbb{R}^d; \{0, 1\})$ denotes the space of functions with (locally) bounded variation and values in $\{0, 1\}$, $\partial^* \{u = 1\}$ denotes the reduced boundary of the level set $\{u = 1\}$, $\nu_u(x)$ its measure theoretic normal at the point $x \in \partial^* \{u = 1\}$, and \mathcal{H}^{d-1} denotes the $(d-1)$ -dimensional Hausdorff measure, see [5] for the precise definitions of these notions. The energy density $\varphi: \mathbb{R}^d \rightarrow [0, +\infty)$ can be recovered via the asymptotic cell formula (for the precise definition see Definition 2.1)

$$\varphi(\nu) := \lim_{\delta \rightarrow 0} \lim_{S \rightarrow +\infty} \frac{1}{S^{d-1}} \inf \left\{ E(u, Q_S^\nu) : u: \mathcal{L} \rightarrow \{0, 1\}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\}, \quad (2)$$

where

$$u_\nu(x) = \begin{cases} 1 & \text{if } \langle x, \nu \rangle \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here, Q_S^ν is a suitable rotation of the coordinate cube with side-length S such that two faces are parallel to $\{x: \langle \nu, x \rangle = 0\}$. In the case $\mathcal{L} = \mathbb{Z}^2$, $c_{i,j} = 1$ if $|i - j| = 1$ and $c_{i,j} = 0$ otherwise, we have that $\varphi(\nu) = 2\|\nu\|_1$, see Figure 2.

The goal of this article is to investigate the energy density φ . In particular, we show that for finite interaction range $c_{i,j}$, that is there exists $R > 0$ such that $c_{i,j} = 0$ if $|i - j| > R$, the

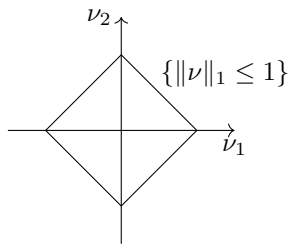


FIGURE 2. The energy density in the case of nearest neighbor interactions on \mathbb{Z}^2

function φ is *crystalline*, see [32, Definition 3.2]. This means that the solution to

$$\min \left\{ \int_{\partial^* A} \varphi(\nu_A(x)) d\mathcal{H}^{d-1} : |A| = 1 \right\}$$

is a convex polytope. The finite range of interaction is crucial. Indeed, Example 2.8 shows that for infinite range interactions this is in general not true. In [16, 17] it is shown that, as the periodicity T of the interactions tends to $+\infty$, it is possible to approximate any norm as surface energy density satisfying suitable growth conditions. We refer to [2] for a random setting where it is shown that an isotropic energy density (and thus non-crystalline) can be obtained in the limit.

The proof of the crystallinity in the case of finite range interactions relies on the following alternative representation formula of the density, proven in Proposition 2.6. Namely, we prove that

$$\varphi(\nu) = \frac{1}{T^d} \inf \{ E(u, Q_T) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \text{ is } T\text{-periodic} \}. \quad (3)$$

This representation formula is reminiscent of the representation formula of the energy density of integral functionals obtained via homogenization of T -periodic integral functionals in $W^{1,p}$ [15]. To motivate this, consider the positively 1-homogeneous extension of E_ε defined by

$$F_\varepsilon(u) = \sum_{i,j \in \varepsilon\mathcal{L}} \varepsilon^{d-1} c_{i,j}^\varepsilon (u(i) - u(j))^+,$$

for $u : \varepsilon\mathcal{L} \rightarrow \mathbb{R}$. The Γ -limit F_0 of the above sequence is clearly positively 1-homogeneous and convex as the sequence of functionals satisfies these properties. Thus, F_0 admits an integral representation of the form

$$F_0(u) = \int f_0(\nabla u) dx + \int f_0 \left(\frac{dD_s u}{|dD_s u|} \right) d|D_s u|,$$

where $f_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is convex and positively 1-homogeneous, see [14]. (We like to stress however, that this integral representation for the spin energies considered above is not proven in the literature.) Here, the important point is that the density of the singular part and the density of absolutely continuous parts agree. In [24], it has been shown that for continuous and convex densities that satisfy a *generalized coarea formula* [45], the Γ -convergence over sets of finite perimeter or in the space of BV -functions is equivalent. This is used there to derive Γ -limits for a particular type of energies on the lattice \mathbb{Z}^d . This is used as well in the continuous setting in [27], to show that in this case, the densities agree. The density of the absolutely continuous part can be calculated via (3). This property eventually allows us to express φ via (3) since the density of the absolutely continuous part can be calculated via (3) and the density of the singular part agrees with the energy density in (2), see Proposition 2.6. Using convex duality

(see [43]) and using (3) we show in Theorem 2.7 that φ is crystalline, and estimate an upper bound on the number of extreme points of the corresponding Wulff shape. We would like to stress that (3) is not only a useful tool in our proof but it can be used also for computational purposes as it is a finite and not an asymptotic cell formula.

The paper is organized as follows. In Section 2 we describe the mathematical setting and state the main theorems of our paper. In Section 3 we prove Proposition 2.6, the alternate representation formula for φ . In Section 4 we show that, in the case of finite range interactions, the density φ is always crystalline. In Section 5 we discuss some differentiability properties of φ . We present some numerical simulations of our findings in the last chapter.

2. SETTING OF THE PROBLEM AND STATEMENT OF THE MAIN RESULT

2.1. Notation. We denote by $\mathcal{B}(\mathbb{R}^d)$ the collection of all Borel sets in \mathbb{R}^d . For every $A \subset \mathbb{R}^d$ we denote by $|A|$ its d -dimensional Lebesgue measure. Given $r > 0$, we denote by $(A)_r := \{x \in \mathbb{R}^d : \text{dist}(x, A) < r\}$ the r -neighbourhood of A . Given $\tau \in \mathbb{R}^d$, we set $A + \tau := \{x + \tau : x \in A\}$ and for $A, B \subset \mathbb{R}^d$ we define $A + B := \{a + b : a \in A, b \in B\}$ their Minkowski sum. The set $\mathbb{S}^{d-1} := \{\nu \in \mathbb{R}^d : |\nu| = 1\}$ is the set of all d -dimensional unit vectors. For $v, w \in \mathbb{R}^d$ we denote by $\langle v, w \rangle$ their scalar product. We denote by $\{e_1, \dots, e_d\} \subset \mathbb{R}^d$ the standard orthonormal basis of \mathbb{R}^d . Given $C \subset \mathbb{R}^d$ convex, we denote by $\text{extreme}(C)$ its extreme points. Given $\rho > 0$, we denote by $Q_\rho := [-\rho/2, \rho/2]^d$ the half open cube centred in 0 with side-length ρ . For $\nu \in \mathbb{S}^{d-1}$, we set $Q_\rho^\nu := R^\nu Q_\rho$, where R^ν is an arbitrary rotation such that $R^\nu e_d = \nu$. Furthermore, given $x \in \mathbb{R}^d$ we set $Q_\rho^\nu(x) := x + Q_\rho^\nu$ (resp. $Q_\rho(x) = x + Q_\rho$). Given $x \in \mathbb{R}^d$ and $r > 0$ we denote by $B_r(x)$ the open ball with radius $r > 0$ and center x . For $A \subset \mathbb{R}^d$ we denote by $\chi : \mathbb{R}^d \rightarrow \{0, 1\}$ the characteristic function of the set A given by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

We denote by ω_d the volume of the unit ball in \mathbb{R}^d . Given $\nu \in \mathbb{S}^{d-1}$ we define

$$u_\nu(x) := \begin{cases} 1 & \text{if } \langle \nu, x \rangle \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

For $z \in \mathbb{R}$ we denote by $(z)^+ := \max\{z, 0\}$ the positive part of z and by $\lfloor z \rfloor$ its integer part.

2.2. Discrete energies and homogenized surface energy density. In this paragraph we define the discrete energies we want to consider and the homogenized surface energy density.

Let $\mathcal{L} \subset \mathbb{R}^d$ satisfy the following two conditions:

(L1) (Discreteness) There exists $c > 0$ such that for all $x \in \mathcal{L}$ there holds

$$\text{dist}(x, \mathcal{L} \setminus \{x\}) \geq c.$$

(L2) (Periodicity) There exists $T \in \mathbb{N}$ such that for all $z \in \mathbb{Z}^d$ there holds

$$\mathcal{L} + Tz = \mathcal{L}.$$

Note that the two assumptions (L1) and (L2) include multi-lattices, such as the hexagonal closed packing lattice in three dimensions, and Bravais lattices, such as \mathbb{Z}^d , or the face-centered cubic lattice in three dimensions.

We consider interaction coefficients $c_{i,j} : \mathcal{L} \times \mathcal{L} \rightarrow [0, +\infty)$ and the corresponding (localized) ferromagnetic spin energies of the form

$$E(u, A) := \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j))^+, \quad (6)$$

where $u : \mathcal{L} \rightarrow \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R}^d)$. If $A = \mathbb{R}^d$ we omit the dependence on the set and write $E(u) := E(u, \mathbb{R}^d)$. We want to remark that we are considering interactions on the directed graph instead of the undirected graph.

We introduce the following three hypotheses on the interaction coefficients $c_{i,j} : \mathcal{L} \times \mathcal{L} \rightarrow [0, +\infty)$:

(H1) (Periodicity) There holds

$$c_{i+Tz, j+Tz} = c_{i,j}$$

for all $i, j \in \mathcal{L}$, $z \in \mathbb{Z}^d$.

(H2) (Decay of Interactions) For all $i \in \mathcal{L}$ there holds

$$\sum_{j \in \mathcal{L}} c_{i,j} |i - j| < +\infty.$$

(H3) (Finite Range Interactions) There exists $R > 1$ such that

$$c_{i,j} = 0$$

for all $i, j \in \mathcal{L}$ such that $|i - j| \geq R$.

It is obvious, that hypothesis (H3) implies hypothesis (H2). Note that, if (H1) and (H2) are satisfied then

$$\max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| = \max_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| < +\infty.$$

Additionally, due to (H1), by changing the order of summation, we have

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| &= \sum_{i \in \mathcal{L} \cap Q_T} \sum_{z \in \mathbb{Z}^d} \sum_{j \in \mathcal{L} \cap Q_T} c_{i, j+Tz} |i - j - Tz| \\ &= \sum_{j \in \mathcal{L} \cap Q_T} \sum_{z \in \mathbb{Z}^d} \sum_{i \in \mathcal{L} \cap Q_T} c_{i-Tz, j} |i - Tz - j| = \sum_{j \in \mathcal{L} \cap Q_T} \sum_{i \in \mathcal{L}} c_{i,j} |i - j|. \end{aligned}$$

Therefore, employing (H2), there holds

$$\max_{j \in \mathcal{L}} \sum_{i \in \mathcal{L}} c_{i,j} |i - j| < +\infty. \quad (7)$$

As a consequence, letting for $R > 0$:

$$C_R := \max \left\{ \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}, |j-i| \geq R} c_{i,j} |i - j|, \max_{j \in \mathcal{L}} \sum_{i \in \mathcal{L}, |i-j| \geq R} c_{i,j} |i - j| \right\} \quad (8)$$

one has $\lim_{R \rightarrow \infty} C_R = 0$: this property will be used in several instances throughout the proofs.

Definition 2.1. Let $c_{i,j}$ satisfy (H1) and (H2). We then define the *homogenized surface energy density* $\varphi : \mathbb{R}^d \rightarrow [0, +\infty)$ as the positively homogeneous function of degree one such that for all $\nu \in \mathbb{S}^{d-1}$ we have

$$\varphi(\nu) := \lim_{\delta \rightarrow 0} \liminf_{S \rightarrow +\infty} \frac{1}{S^{d-1}} \inf \left\{ E(u, Q_S^\nu) : u : \mathcal{L} \rightarrow \{0, 1\}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\}, \quad (9)$$

with u_ν defined in (5).

In (9), the rotation R^ν such that $Q_S^\nu = R^\nu Q_S$ is arbitrary and can vary with S . In fact, as we will show in Lemma 3.17, for any family of rotations $(R_S^\nu)_{S \geq 1}$ with $R_S^\nu e_d = \nu$, letting for each S , $Q_S^\nu = R_S^\nu Q_S$, it holds:

$$\begin{aligned} \varphi(\nu) &= \lim_{\delta \rightarrow 0} \liminf_{S \rightarrow +\infty} \frac{1}{S^{d-1}} \inf \left\{ E(u, Q_S^\nu) : u : \mathcal{L} \rightarrow \{0, 1\}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\} \\ &= \lim_{\delta \rightarrow 0} \limsup_{S \rightarrow +\infty} \frac{1}{S^{d-1}} \inf \left\{ E(u, Q_S^\nu) : u : \mathcal{L} \rightarrow \{0, 1\}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\}. \end{aligned}$$

Remark 2.2. The definition above can be interpreted as a passage from discrete to continuum description as follows. Given $\varepsilon > 0$, we consider the scaled energies

$$E_\varepsilon(u) := \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} \varepsilon^{d-1} c_{i,j} (u(\varepsilon i) - u(\varepsilon j))^+,$$

where $u : \varepsilon \mathcal{L} \rightarrow \{0, 1\}$. Upon identifying u with its piecewise-constant interpolation, we can regard these energies to be defined on $L_{\text{loc}}^1(\mathbb{R}^d)$. We know that their Γ -limit is infinite outside the space $BV_{\text{loc}}(\mathbb{R}^d; \{0, 1\})$, where it has the form

$$E_0(u) := \int_{\partial^* \{u=1\}} \varphi(\nu) d\mathcal{H}^{d-1}$$

with φ given by (9), see for example [2].¹ Here, $\partial^* \{u = 1\}$ denotes the reduced boundary of the set $\{u = 1\}$ and \mathcal{H}^{d-1} denotes the $(d-1)$ -dimensional Hausdorff measure in \mathbb{R}^d (cf. [5], Chapters 2.8 and 3.5).

Remark 2.3. Testing with u_ν in (9), the forthcoming Lemma 3.7 shows that $\varphi(\nu) \leq C$ for all $\nu \in \mathbb{S}^{d-1}$. Therefore, thanks to the convexity (deduced from Lemma 3.9, Lemma 3.17, and Lemma 3.12) and the fact that it is a positively one homogeneous function of degree one, φ is Lipschitz continuous. The latter fact can be proved from the definition, see Remark 3.16.

2.3. Statement of the main result. In this section we state the main result.

Definition 2.4. Given $\varphi : \mathbb{R}^d \rightarrow [0, +\infty)$ convex, positively homogeneous of degree one, we define the Wulff set of φ by

$$W_\varphi := \{ \zeta \in \mathbb{R}^d : \langle \zeta, \nu \rangle \leq \varphi(\nu) \text{ for all } \nu \in \mathbb{S}^{d-1} \}.$$

We say that φ is crystalline, if W_φ is a polytope.

Remark 2.5. From the definition of the Wulff set, it is clear that

$$\varphi(\nu) = \sup_{\zeta \in W_\varphi} \langle \nu, \zeta \rangle.$$

Furthermore, one can check, that if φ is crystalline, then the set $\{\varphi \leq 1\}$ is a polytope.

¹Actually, the integral representation for the Γ -limit has only been shown for undirected graphs. However, a slight modification of the proof shows that it is still true for directed graphs.

The next proposition shows that, we obtain a finite cell formula in order to calculate φ instead of the asymptotic one, given in (9). We think that this result in itself is interesting, since it allows for calculations on finite size systems in order to compute φ for general Ising systems. This result is in spirit very close to [10, 14], where convex and positively 1-homogenous continuum energies are considered. In this case, the surface energy density and the energy with respect to the absolutely continuous part coincide.

For $k \in \mathbb{N}$ let $\Lambda := Tk$ and denote by

$$\mathcal{A}_{\text{per}}(Q_\Lambda; \mathbb{R}) := \{u: \mathcal{L} \rightarrow \mathbb{R}: u(x + \Lambda z) = u(x) \text{ for all } z \in \mathbb{Z}^d\} \quad (10)$$

be the space of Λ -periodic functions.

Proposition 2.6. *Let $c_{i,j}: \mathcal{L} \times \mathcal{L} \rightarrow [0, +\infty)$ be interaction coefficients such that (H1) and (H2) hold true. Then*

$$\varphi(\nu) = \frac{1}{T^d} \inf \{E(u, Q_T): u: \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})\}. \quad (11)$$

Theorem 2.7. *Let $c_{i,j}: \mathcal{L} \times \mathcal{L} \rightarrow [0, +\infty)$ be interaction coefficients such that (H1) and (H3) hold true. Then, the homogenized surface energy density φ is crystalline. Denote by*

$$N := \#\{(i, j) \in \mathcal{L} \cap Q_T \times \mathcal{L}: c_{i,j} \neq 0\}.$$

Then,

$$\#\text{extreme}(W_\varphi) \leq 3^N.$$

The next example shows that without assumption (H3) Theorem 2.7 fails to hold true.

Example 2.8. To construct the example we first observe that if $f: \mathbb{R}^d \rightarrow [0, +\infty)$ is crystalline, then D^2f is a Radon-measure with support contained in finitely many hyper-planes. To see this, note that if $f: \mathbb{R}^d \rightarrow [0, +\infty)$ is crystalline, then there exist $\{\xi_k\}_{k=1}^N \subset \mathbb{R}^d$ such that

$$f(\nu) = \max_{1 \leq k \leq N} \langle \xi_k, \nu \rangle.$$

Here, we assume that $\{\xi_k\}_{k=1}^N$ is chosen minimal, i.e. if we set $V_k = \{\nu \in \mathbb{R}^d: f(\nu) = \langle \xi_k, \nu \rangle\}$, then $|V_k| > 0$ for all $k = 1, \dots, N$. This assumption ensures that all the vectors ξ_k play an active role in the definition of f . Now, $Df \in BV_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ is given by

$$Df(\nu) = \sum_{k=1}^N \chi_{V_k}(\nu) \xi_k,$$

with V_k defined above. Then

$$D^2f(\nu) = \sum_{1 \leq k < j \leq N} (\xi_k - \xi_j) \otimes \nu_{kj} \mathcal{H}^{d-1} \llcorner_{\partial V_k \cap \partial V_j},$$

where $\partial V_k \cap \partial V_j = \{\nu \in \mathbb{R}^d: f(\nu) = \langle \xi_k, \nu \rangle = \langle \xi_j, \nu \rangle\}$ and $\nu_{kj} \in \mathbb{S}^{d-1}$ denotes the normal pointing towards the set V_k .

Let now $\mathcal{L} = \mathbb{Z}^d$ and $c_{i,j} = c_{j-i} = c_{i-j}$ (in the following denoted by $\{c_\xi\}_{\xi \in \mathbb{Z}^d}$) be such that $c_\xi > 0$ for all $\xi \in \mathbb{Z}^d$ and

$$\sum_{\xi \in \mathbb{Z}^d} c_\xi |\xi| < +\infty.$$

It is then obvious that $c_{i,j}$ is 1-periodic, (H1) and (H2) hold true, but (H3) is violated. Therefore, due to Proposition 2.6, we have

$$\varphi(\nu) = \sum_{\xi \in \mathbb{Z}^d} c_\xi |\langle \xi, \nu \rangle|.$$

This is true, since the only admissible functions in the minimum problem given by Proposition 2.6 are $u_\nu(i) = \langle \nu, i \rangle + c$ for some $c \in \mathbb{R}$. We claim that

$$D\varphi(\nu) = \sum_{\xi \in \mathbb{Z}^d} \text{sign}(\langle \xi, \nu \rangle) c_\xi \xi,$$

where $\text{sign}: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\text{sign}(t) = \begin{cases} 1 & t \geq 0; \\ -1 & t < 0. \end{cases}$$

Therefore

$$D^2\varphi = 2 \sum_{\xi \in \mathbb{Z}^d} c_\xi \xi \otimes \frac{\xi}{|\xi|} \mathcal{H}^{d-1} \llcorner_{\{\nu: \langle \xi, \nu \rangle = 0\}}. \quad (12)$$

This can be seen by approximation. Consider $\varphi_R: \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$\varphi_R(\nu) = \sum_{\substack{\xi \in \mathbb{Z}^d \\ |\xi| \leq R}} c_\xi |\langle \xi, \nu \rangle|, \quad D\varphi_R(\nu) = \sum_{\substack{\xi \in \mathbb{Z}^d \\ |\xi| \leq R}} \text{sign}(\langle \xi, \nu \rangle) c_\xi \xi,$$

Then

$$D^2\varphi_R = 2 \sum_{\substack{\xi \in \mathbb{Z}^d \\ |\xi| \leq R}} c_\xi \xi \otimes \frac{\xi}{|\xi|} \mathcal{H}^{d-1} \llcorner_{\{\nu: \langle \xi, \nu \rangle = 0\}}.$$

Now

$$|D^2\varphi_R|(B_r) \leq Cr^{d-1} \sum_{\xi \in \mathbb{Z}^d} c_\xi |\xi|,$$

so the total variation of $D^2\varphi_R$ is (locally) uniformly bounded with limiting measure $D^2\varphi$ and $D\varphi_R \rightarrow D\varphi$ in $L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$, actually weakly in BV . Hence, (12) is shown. Now, since $c_\xi > 0$ for all $\xi \in \mathbb{Z}^d$ it is obvious that $D^2\varphi$ is not supported on finitely many hyper-planes. Thus φ cannot be crystalline.

Note that φ is differentiable in totally irrational directions.² A similar property is known to hold, in the continuous setting [6, 25], for homogenized surface tensions. We can state a result showing that this is still the case in the discrete setting, under assumptions (H1) and (H2).

Proposition 2.9. *Assume (H1), (H2). Let $\nu \in \mathbb{S}^{d-1}$ be a totally irrational direction such that a minimizer to (11) exists. Then, φ is differentiable in ν .*

See Example 3.11 for a discussion on the possible non-existence of minimizers and a condition for existence. It is expected that φ should be, “in general”, not differentiable in the other directions, at least whenever the minimizers u in (11) are constant on an infinite set, however the proofs in [6, 25] rely on ellipticity properties of the problem and are less easy to transfer to the discrete case. The proof of Proposition 2.9, which mimicks the proof in [25], is postponed to Section 5, and relies on the dual representation (105) introduced later on.

² p is totally irrational if there is no $q \in \mathbb{Z}^d \setminus \{0\}$ such that $\langle q, p \rangle = 0$.

3. PROOF OF PROPOSITION 2.6

This section is devoted to the proof of Proposition 2.6. We split this section into several subsections. In Subsection 3.1 we establish some basic properties of the energy E . In Subsection 3.2 we show that the functions ϕ and ψ , to be defined in (13) and (15), respectively, are continuous functions that satisfy linear growth conditions. In Subsection 3.3 we analyze the asymptotic periodic cell formula and show that it can be calculated on a finite cell and coincides with the asymptotic cell formula with affine boundary conditions. Finally, in Subsection 3.4 we show that the surface energy density φ coincides with the cell formulas introduced below establishing thereby also Proposition 2.6. We assume throughout this section that assumptions (L1), (L2) and (H1), (H2) are satisfied. The proof consists in showing that φ can be characterized by several (equivalent) cell formulas and therefore passing from (9) to (11).

We introduce the following (shown to be equivalent) cell formulas:

We define $\phi: \mathbb{R}^d \rightarrow [0, +\infty]$ by

$$\phi(\nu) := \lim_{\delta \rightarrow 0} \liminf_{S \rightarrow +\infty} \frac{1}{S^d} \inf \{E(u, Q_S): u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}\} \quad (13)$$

and $\phi_{\text{per}}: \mathbb{R}^d \rightarrow [0, +\infty]$ is defined by

$$\phi_{\text{per}}(\nu) := \liminf_{k \rightarrow +\infty} \frac{1}{(kT)^d} \inf \{E(u, Q_{kT}): u: \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kT}; \mathbb{R})\}. \quad (14)$$

Finally, we define $\psi: \mathbb{R}^d \rightarrow [0, +\infty]$ as the positively homogeneous function of degree one that for $\nu \in \mathbb{S}^{d-1}$ is given by

$$\psi(\nu) = \lim_{\delta \rightarrow 0} \liminf_{S \rightarrow +\infty} \frac{1}{S^d} \inf \left\{ E(u, Q_S^\nu): u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\}. \quad (15)$$

The function ψ differs from the function ϕ in the domain where one calculates the energy. For the function ϕ we take the coordinate cube Q_T whereas for ν we take a cube Q_T^ν .

Remark 3.1. By standard subadditivity arguments, see e.g. [3, Proposition 4.2], it can be shown that one can take any sequence $S \rightarrow +\infty$ in the definition of (15). In fact

$$\begin{aligned} \psi(\nu) &= \lim_{\delta \rightarrow 0} \liminf_{S \rightarrow +\infty} \frac{1}{S^d} \inf \left\{ E(u, Q_S^\nu): u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\} \\ &= \lim_{\delta \rightarrow 0} \limsup_{S \rightarrow +\infty} \frac{1}{S^d} \inf \left\{ E(u, Q_S^\nu): u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\}. \end{aligned}$$

In addition, with the same techniques as in the proof of Lemma 3.14, one can also show that these limits do not depend on the particular choice of the rotations R^ν defining Q_S^ν .

We show that $\phi_{\text{per}} = \phi$ in Lemma 3.12, $\psi = \varphi$ in Lemma 3.17, and $\psi = \phi$ in Lemma 3.18.

3.1. Elementary properties. In this section, we will state and prove some elementary properties of E that will be used throughout the subsequent sections.

Lemma 3.2. *Let $A \in \mathcal{B}(\mathbb{R}^d)$ and let $c > 0$ be as in (L1).*

- (i) *There exists a universal constant $C > 0$ (depending only on $c_{i,j}$ and c in (L1)) such that for all $\nu \in \mathbb{R}^d$ we have*

$$E(\langle \nu, \cdot \rangle, A) \leq C|\nu| |(A)_c|.$$

(ii) Let $u: \mathcal{L} \rightarrow \mathbb{R}$. For all $t \in \mathbb{R}, \lambda > 0$ there holds

$$E(\lambda u + t, A) = \lambda E(u, A)$$

and $u \mapsto E(u, A)$ is convex. In particular,

$$E(u + v, A) \leq E(u, A) + E(v, A)$$

for all $u, v: \mathcal{L} \rightarrow \mathbb{R}$.

(iii) Let $u: \mathcal{L} \rightarrow \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R}^d)$ be such that $A \subset B$. Then

$$E(u, A) \leq E(u, B).$$

(iv) Let $u: \mathcal{L} \rightarrow \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R}^d)$ be such that $A \cap B = \emptyset$. Then

$$E(u, A \cup B) = E(u, A) + E(u, B).$$

(v) We have

$$\#\{i \in \mathcal{L} \cap A\} \leq \frac{1}{c^d \omega_d} |(A)_c|.$$

(vi) Let $u: \mathcal{L} \rightarrow \mathbb{R}$. Then, the function $\tau \mapsto E(u(\cdot - \tau), A + \tau)$ is T -periodic.

Proof. We start by proving (ii)-(iv) in Step 1, then (v) and (vi) in Step 2 and Step 3 respectively, and finally (i) in Step 4.

Step 1.(Proof of (ii) - (iv)) All the three statements are a direct consequence of (6) and the fact that $c_{i,j} \geq 0$.

Step 2. (Proof of (v)) Note that

$$\bigcup_{i \in \mathcal{L} \cap A} B_c(i) \subset (A)_c$$

and therefore, due to (L1),

$$c^d \omega_d \#\{i \in \mathcal{L} \cap A\} = \left| \bigcup_{i \in \mathcal{L} \cap A} B_c(i) \right| \leq |(A)_c|.$$

This is the claim.

Step 3.(Proof of (vi)) Let $u: \mathcal{L} \rightarrow \mathbb{R}$ and $z \in \mathbb{Z}^d$. Then, using (H1) and (L2),

$$\begin{aligned} E(u(\cdot - Tz), A + Tz) &= \sum_{i \in \mathcal{L} \cap (A + Tz)} \sum_{j \in \mathcal{L}} c_{i,j} (u(i - Tz) - u(j - Tz))^+ \\ &= \sum_{i \in \mathcal{L} \cap A} \sum_{j \in (\mathcal{L} + Tz)} c_{i+Tz, j+Tz} (u(i) - u(j))^+ \\ &= \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j))^+ = E(u, A). \end{aligned}$$

Step 4.(Proof of (i)) Let $\nu \in \mathbb{R}^d$, then, due to (v), (L1), (L2), (H1), and (H2), we have

$$E(\langle \nu, \cdot \rangle, A) = \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} |\langle \nu, i - j \rangle| \leq |\nu| \#\{i \in \mathcal{L} \cap A\} \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| \leq C |\nu| |(A)_c|.$$

□

The next Lemma shows that our energy satisfies a generalized coarea formula [24, 45].

Lemma 3.3. *Let $u: \mathcal{L} \rightarrow \mathbb{R}$ and $A \subset \mathbb{R}^d$. Then*

$$E(u, A) = \int_{-\infty}^{+\infty} E(\chi_{\{u>t\}}, A) dt. \quad (16)$$

Proof. For $a, b \in \mathbb{R}$ we have

$$(a - b)^+ = \int_{-\infty}^{+\infty} (\chi_{\{a>t\}} - \chi_{\{b>t\}})^+ dt.$$

Therefore,

$$E(u, A) = \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j))^+ = \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} \int_{-\infty}^{+\infty} (\chi_{\{u(i)>t\}} - \chi_{\{u(j)>t\}})^+ dt.$$

Now, by Fubini's Theorem (note that $c_{i,j} \geq 0$), we obtain

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} \int_{-\infty}^{+\infty} (\chi_{\{u(i)>t\}} - \chi_{\{u(j)>t\}})^+ dt &= \int_{-\infty}^{+\infty} \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} (\chi_{\{u(i)>t\}} - \chi_{\{u(j)>t\}})^+ dt \\ &= \int_{-\infty}^{+\infty} E(\chi_{\{u>t\}}, A) dt \end{aligned}$$

and thus the claim. \square

Lemma 3.4. *Let $S > 0$, $\delta > 0$ and $\nu \in \mathbb{S}^{d-1}$. Then*

$$\begin{aligned} &\inf \left\{ E(u, Q_S^\nu) : u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\} \\ &= \inf \left\{ E(u, Q_S^\nu) : u: \mathcal{L} \rightarrow \{0, 1\}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\}. \end{aligned}$$

Proof. The infimum on the left hand side is taken over a larger class of admissible function, since here the image of the competitor u is a subset of \mathbb{R} and not just of $\{0, 1\}$. Hence, one inequality is trivial. The other inequality follows from a coarea formula satisfied by our energies.

Step 1. (Proof of ' \leq ') This inequality is clear, since the infimum on the left hand side is taken over a larger class of functions.

Step 2. (Proof of ' \geq ') Let us take $u: \mathcal{L} \rightarrow \mathbb{R}$ such that $u = u_\nu$ on $\mathcal{L} \setminus Q_{(1-\delta)S}^\nu$ and denote by $u^s = \chi_{\{u>s\}}$. Then, using Lemma 3.3, there exists $t \in (0, 1)$ such that

$$E(u^t, Q_S^\nu) \leq \int_0^1 E(u^s, Q_S^\nu) ds \leq \int_{-\infty}^{+\infty} E(u^s, Q_S^\nu) ds = E(u, Q_S^\nu).$$

Noting that $u^t(i) \in \{0, 1\}$ for all $i \in \mathcal{L}$ and $u^t = u_\nu$ on $\mathcal{L} \setminus Q_S^\nu$, this concludes Step 2. \square

To conclude this subsection, we show Lemma 3.5, which will be used many times, such as in the proof of Lemma 3.17. Let $\nu \in \mathbb{S}^{d-1}$ be fixed. In the following, given $A \subset x + \nu^\perp$ a $(d-1)$ -dimensional subset, with $x \in \mathbb{R}^d$ an arbitrary translation, we denote $A_\nu = \{y \in \mathbb{R}^d : y - \langle y - x, \nu \rangle \nu \in A\}$ the cylinder over A , and $[A]_{\nu, s} := \{y + z : y \in A, z \in \nu^\perp, \|z\|_2 \leq s\}$. Furthermore, let

$$\Pi_{x, \nu}^+ := \{i \in \mathcal{L} : \langle i - x, \nu \rangle > 0\}, \quad A_\nu^- := \{i \in \mathcal{L} \cap A_\nu : \langle i - x, \nu \rangle \leq 0\}.$$

Then the following holds:

Lemma 3.5. *There exists $C \geq 1$ (depending on \mathcal{L} and the dimension) such that for any $\nu \in \mathbb{S}^{d-1}$, $x \in \mathbb{R}^d$, and $A \subset x + \nu^\perp$ as above, there holds:*

$$\sum_{i \in A_\nu^-} \sum_{j \in \Pi_{x,\nu}^+} c_{i,j} \leq CT \mathcal{H}^{d-1}([A]_{\nu, \frac{3\sqrt{d}}{2}T}) \left(\max_{i \in Q_T} \sum_{j \in \mathcal{L}} c_{i,j} |i-j| \right). \quad (17)$$

Remark 3.6. Thanks to (7), formula (17) holds true with the weights $c_{i,j}$ replaced with $c_{j,i}$, yielding a bound on $\sum_{i \in \Pi_{x,\nu}^+} \sum_{j \in A_\nu^-} c_{i,j}$. Furthermore, we also observe that for any $R > 0$:

$$\sum_{i \in A_\nu^-} \sum_{\substack{j \in \Pi_{x,\nu}^+ \\ |i-j| \geq R}} c_{i,j} \leq CT \mathcal{H}^{d-1}([A]_{\nu, \frac{3\sqrt{d}}{2}T}) \left(\max_{i \in Q_T} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq R}} c_{i,j} |i-j| \right). \quad (18)$$

Indeed, the sum on the left-hand side does not involve any of the coefficients $c_{i,j}$ with $|i-j| < R$, and (18) follows by invoking Lemma 3.5 after replacing the weights $c_{i,j}$ with $c'_{i,j}$ defined as:

$$c'_{i,j} = \begin{cases} 0 & \text{if } |i-j| < R, \\ c_{i,j} & \text{else.} \end{cases}$$

Corollary 3.7. *Let Q_S a cube of size S . Then, $\lim_{S \rightarrow \infty} \frac{1}{S^{d-1}} \sum_{i \in Q_{(1-\delta)S} \cap \mathcal{L}} \sum_{j \in \mathcal{L} \setminus Q_S} c_{i,j} = 0$.*

Proof. The corollary follows by applying Remark 3.6 to each side of the cube and noting that if $i \in \mathcal{L} \cap Q_{(1-\delta)S}$ and $j \in \mathcal{L} \setminus Q_S$, then $|i-j| \geq \delta S/2$. We will show a refined version of this result in Lemma 3.15. \square

Proof of Lemma 3.5. As will be clear from the proof, we may assume without loss of generality that $x = 0$ and $A \subset \nu^\perp$. For a non-negative integer n let $D_n = \{x \in A_\nu^- : -(n+1)T < \langle \nu, x \rangle \leq -nT\}$, $P_n = \{x \in \Pi_{0,\nu}^+ : nT < \langle \nu, x \rangle \leq (n+1)T\}$. One has

$$\sum_{i \in A_\nu^-} \sum_{j \in \Pi_{0,\nu}^+} c_{i,j} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i \in D_n} \sum_{j \in P_m} c_{i,j} = \sum_{l=0}^{\infty} \sum_{n=0}^l \sum_{i \in D_n} \sum_{j \in P_{l-n}} c_{i,j}$$

Denote $C_{\mathcal{L}} := \#(\mathcal{L} \cap B_{\sqrt{d}T}(0))T^{-d}$. Due to Lemma 3.2(v), $\mathcal{L} \cap D_0 = \mathcal{L} \cap (A + [-T, 0]\nu)$ may be covered by disjoint (up to their boundaries) translates of Q_T^ν (a cube of size T with one facet orthogonal to ν), each containing at most $C_{\mathcal{L}}T^d$ points and with global volume less than $T\mathcal{H}^{d-1}([A]_{\nu, \sqrt{d}T})$. It follows that $\#(\mathcal{L} \cap D_0) \leq C_{\mathcal{L}}T\mathcal{H}^{d-1}([A]_{\nu, \sqrt{d}T})$ so that the first term in the sum may be bounded as follows:

$$\sum_{i \in D_0} \sum_{j \in P_0} c_{i,j} \leq \#(D_0 \cap \mathcal{L}) \max_i \sum_j c_{i,j} \leq C_{\mathcal{L}}T\mathcal{H}^{d-1}([A]_{\nu, \sqrt{d}T}) \max_i \sum_j c_{i,j}. \quad (19)$$

Then, for each $n \geq 1$, we approximate $nT\nu$ by an integer valued translation, letting $k_i(n) = \lfloor n\nu_i \rfloor$ for $i = 1, \dots, d$. One has, by definition, $k_i(n) \leq n\nu_i < k_i(n) + 1$ so that for any set E ,

$E + k(n)T \subset E + nT\nu + [-T, 0]^d$. By periodicity of $c_{i,j}$, we have that for any $l \geq 1$, $0 \leq n \leq l$,

$$\begin{aligned} \sum_{i \in D_n} \sum_{j \in P_{l-n}} c_{i,j} &\leq \sum_{i \in D_n} \sum_{j \in P_{l-n}} \frac{c_{i,j}|i-j|}{lT} \\ &= \sum_{i \in D_n + k(n)T} \sum_{j \in P_{l-n} + k(n)T} \frac{c_{i,j}|i-j|}{lT} \\ &\leq \frac{1}{lT} \sum_{i \in D_0 + [-T, 0]^d} \sum_{j \in P_l + [-T, 0]^d} c_{i,j}|i-j|. \end{aligned}$$

Hence,

$$\sum_{n=0}^l \sum_{i \in D_n} \sum_{j \in P_{l-n}} c_{i,j} \leq \frac{2}{T} \sum_{i \in D_0 + [-T, 0]^d} \sum_{j \in P_l + [-T, 0]^d} c_{i,j}|i-j|.$$

Eventually, each point in \mathcal{L} belonging only to a uniformly bounded number of sets $P_l + [-T, 0]^d$, there exists $C \geq 1$ with:

$$\begin{aligned} \sum_{l \geq 1} \sum_{n=0}^l \sum_{i \in D_n} \sum_{j \in P_{l-n}} c_{i,j} &\leq \frac{C}{T} \sum_{i \in D_0 + [-T, 0]^d} \sum_j c_{i,j}|i-j| \\ &\leq \frac{C}{T} \#((D_0 + [-T, 0]^d) \cap \mathcal{L}) \max_i \sum_j c_{i,j}|i-j|. \end{aligned}$$

The cardinality of $(D_0 + [-T, 0]^d) \cap \mathcal{L}$ is evaluated as before: as $[-T/2, T/2]^d \subset \overline{B}_{\sqrt{dT}/2}(0)$ the set is contained in a translate of $D_0 + \overline{B}_{\sqrt{dT}/2}(0)$, hence in a translate of $[A]_{\nu, \frac{\sqrt{dT}}{2}} + [-(1 + \sqrt{d})T, 0]\nu$. Covering this set with disjoint translates of Q_T^ν (up to the boundaries), we deduce $\#((D_0 + [-T, 0]^d) \cap \mathcal{L}) \leq C_{\mathcal{L}}([1 + \sqrt{d}] + 1)T\mathcal{H}^{d-1}([A]_{\nu, \frac{3\sqrt{dT}}{2}})$. It yields:

$$\sum_{l \geq 1} \sum_{n=0}^l \sum_{i \in D_n} \sum_{j \in P_{l-n}} c_{i,j} \leq CT\mathcal{H}^{d-1}([A]_{\nu, \frac{3\sqrt{dT}}{2}}) \max_i \sum_j c_{i,j}|i-j|$$

for some $C > 0$. Together with (19), we deduce (17). \square

3.2. Continuity properties of ϕ and ψ . In this section we show some continuity properties of the functions ϕ , ψ defined in (13) and (15), respectively.

Lemma 3.8. $\psi: \mathbb{R}^d \rightarrow [0, +\infty]$ satisfies the following properties:

- (i) There exists $C > 0$ such that $\psi(\nu) \leq C|\nu|$ for all $\nu \in \mathbb{R}^d$,
- (ii) ψ is a continuous function.

Proof. We divide the proof into two steps. We first prove (i) and then (ii). Throughout the proofs let $1 \ll S$.

Step 1.(Proof of (i)) Let $\nu \in \mathbb{S}^{d-1}$; it suffices to prove

$$\psi(\nu) \leq C.$$

The general case then follows by one-homogeneity. In order to prove (i) we insert $u(i) = \langle \nu, i \rangle$ for all $i \in \mathcal{L}$ as a competitor in the cell formula. Using Lemma 3.2(i), we then have

$$E(u, Q_S^\nu) = E(\langle \nu, \cdot \rangle, Q_S^\nu) \leq C|\nu| |(Q_S)_c| \leq CS^d.$$

Dividing by S^d and letting $S \rightarrow +\infty$ yields the claim.

Step 2. (Proof of (ii)) Due to the one-homogeneity, it suffices to consider the case where $\nu_1, \nu_2 \in \mathbb{S}^{d-1}$. Let $\eta > 0$ and $\nu_1, \nu_2 \in \mathbb{S}^{d-1}$ be such that $|\nu_1 - \nu_2| \leq \eta$. Our goal is to prove that there exists $C > 0$ independent of ν_1 and ν_2 such that

$$|\psi(\nu_1) - \psi(\nu_2)| \leq C\eta. \quad (20)$$

We only prove

$$\psi(\nu_1) - \psi(\nu_2) \leq C\eta, \quad (21)$$

since then (20) follows by exchanging ν_1 and ν_2 in (21). To this end let $\delta > 0$ small enough, $S > 0$ large enough, $u_1: \mathcal{L} \rightarrow \mathbb{R}$ be such that $u_1(i) = \langle \nu_1, i \rangle$ on $\mathcal{L} \setminus Q_{(1-\delta)S}^{\nu_1}$ and

$$\frac{1}{S^d} E(u_1, Q_S^{\nu_1}) \leq \phi(\nu_1) + \eta. \quad (22)$$

We assume that

$$\|u_1\|_{L^\infty(Q_S^{\nu_1})} \leq S. \quad (23)$$

If this were not the case, we consider

$$\tilde{u}_1(i) = \begin{cases} (u_1(i) \wedge S) \vee (-S) & i \in Q_{2S}^{\nu_1}, \\ u_1(i) & \text{otherwise.} \end{cases}$$

Note that for $i, j \in Q_{2S}^{\nu_1}$, due to truncation, $(\tilde{u}_1(i) - \tilde{u}_1(j))^+ \leq (u_1(i) - u_1(j))^+$, whereas in general there holds $|\tilde{u}_1(i) - \tilde{u}_1(j)| \leq CS + |i - j|$. From this, using Lemma 3.2(v) and (H2), we deduce

$$\begin{aligned} E(\tilde{u}_1, Q_S^{\nu_1}) &= \sum_{i \in \mathcal{L} \cap Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \cap Q_{2S}^{\nu_1}} c_{i,j} (\tilde{u}_1(i) - \tilde{u}_1(j))^+ + \sum_{i \in \mathcal{L} \cap Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \setminus Q_{2S}^{\nu_1}} c_{i,j} (\tilde{u}_1(i) - \tilde{u}_1(j))^+ \\ &\leq \sum_{i \in \mathcal{L} \cap Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \cap Q_{2S}^{\nu_1}} c_{i,j} (u_1(i) - u_1(j))^+ + C \sum_{i \in \mathcal{L} \cap Q_S^{\nu_1}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq S/2}} c_{i,j} |i - j| \\ &\leq E(u_1, Q_S^{\nu_1}) + C \#(\mathcal{L} \cap Q_S^{\nu_1}) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq S/2}} c_{i,j} |i - j| \leq E(u_1, Q_S^{\nu_1}) + C_S S^d, \end{aligned}$$

where $C_S \rightarrow 0$ as $S \rightarrow \infty$ (cf. (8)). In particular $C_S \leq \eta$ for S large enough. Hence, we can assume (23). There exists $C > 0$ such that for $\tilde{S} = (1 + C\eta)S$ there holds $Q_{(1-\delta)\tilde{S}}^{\nu_2} \supset Q_{(1+\delta)S}^{\nu_1}$. We now define $u_2: \mathcal{L} \rightarrow \mathbb{R}$ by

$$u_2(i) = \langle \nu_2 - \nu_1, i \rangle + u_1(i).$$

First, note that $u_2(i) = \langle \nu_2, i \rangle$ for all $i \in \mathcal{L} \setminus Q_{(1-\delta)\tilde{S}}^{\nu_2}$ and therefore

$$\inf \left\{ E(u, Q_{\tilde{S}}^{\nu_2}) : u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu_2, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)\tilde{S}}^{\nu_2} \right\} \leq E(u_2, Q_{\tilde{S}}^{\nu_2}). \quad (24)$$

We claim that

$$E(u_2, Q_{\tilde{S}}^{\nu_2}) \leq E(u_1, Q_S^{\nu_1}) + \frac{C_S^\delta}{\delta} S^d + C\eta S^d + C\delta S^d, \quad (25)$$

where $C_S^\delta \rightarrow 0$ as $S \rightarrow +\infty$. We postpone the proof of (25) and show first how it implies (21). Dividing (25) by \tilde{S}^d , letting \tilde{S} (therefore also S) tend to $+\infty$, $\delta \rightarrow 0$, and using (24) as well as (22), we get

$$\phi(\nu_2) \leq \phi(\nu_1) + C\eta \leq \phi(\nu_1) + C\eta.$$

This is (21). We now prove (25). Due to Lemma 3.2(ii), there holds

$$E(u_2, Q_S^{\nu_2}) \leq E(u_1, Q_S^{\nu_2}) + E(\langle \nu_2 - \nu_1, \cdot \rangle, Q_S^{\nu_1}). \quad (26)$$

Now, due to Lemma 3.2(i) and the fact that $\tilde{S} \leq 2S$, there holds

$$E(\langle \nu_2 - \nu_1, \cdot \rangle, Q_S^{\nu_1}) \leq C|\nu_2 - \nu_1|S^d \leq C\eta S^d. \quad (27)$$

Next, we prove

$$E(u_1, Q_S^{\nu_2}) \leq E(u_1, Q_S^{\nu_1}) + C\delta S^d + \frac{C_S^\delta}{\delta} S^d, \quad (28)$$

where $C_S^\delta \rightarrow 0$ as $S \rightarrow +\infty$. We use Lemma 3.2(iv), to obtain

$$E(u_1, Q_S^{\nu_2}) = E(u_1, Q_S^{\nu_1}) + E(u_1, Q_S^{\nu_2} \setminus Q_S^{\nu_1}).$$

In order to prove (28) it suffices to prove

$$E(u_1, Q_S^{\nu_2} \setminus Q_S^{\nu_1}) \leq C\eta S^d + \frac{C_S^\delta}{\delta} S^d, \quad (29)$$

where $C_S^\delta \rightarrow 0$ as $S \rightarrow +\infty$. To see this we write

$$\begin{aligned} E(u_1, Q_S^{\nu_2} \setminus Q_S^{\nu_1}) &= \sum_{i \in \mathcal{L} \cap Q_S^{\nu_2} \setminus Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \cap Q_{(1-\delta)S}^{\nu_1}} c_{i,j} (u_1(i) - u_1(j))^+ \\ &+ \sum_{i \in \mathcal{L} \cap Q_S^{\nu_2} \setminus Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \setminus Q_{(1-\delta)S}^{\nu_1}} c_{i,j} (u_1(i) - u_1(j))^+. \end{aligned} \quad (30)$$

To estimate the first term, note that due to (23), we have $|u_1(i) - u_1(j)| \leq CS + |i - j|$, and therefore, up to changing C , using (H2), and Lemma 3.2(iv), we get

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap Q_S^{\nu_2} \setminus Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \cap Q_{(1-\delta)S}^{\nu_1}} c_{i,j} (u_1(i) - u_1(j))^+ &\leq \frac{C}{\delta} \sum_{i \in \mathcal{L} \cap Q_S^{\nu_2} \setminus Q_S^{\nu_1}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S/2}} c_{i,j} |i - j| \\ &\leq \frac{C}{\delta} \#(\mathcal{L} \cap Q_S^{\nu_2}) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S/2}} c_{i,j} |i - j| \leq \frac{C_S^\delta}{\delta} S^d, \end{aligned} \quad (31)$$

where $C_S^\delta \rightarrow 0$ as $S \rightarrow +\infty$ (cf. (8)). To estimate the second term in the right-hand side of (30), we use the fact that $u_1(i) = \langle \nu_1, i \rangle$ on $\mathcal{L} \setminus Q_{(1-\delta)S}^{\nu_1}$, and Lemma 3.2 (i), to obtain

$$\sum_{i \in \mathcal{L} \cap Q_S^{\nu_2} \setminus Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \setminus Q_{(1-\delta)S}^{\nu_1}} c_{i,j} (u_1(i) - u_1(j))^+ \leq E(\langle \nu_1, \cdot \rangle, Q_S^{\nu_2} \setminus Q_S^{\nu_1}) \leq C|\nu_1| |(Q_S^{\nu_2} \setminus Q_S^{\nu_1})_c| \leq C\eta S^d.$$

This together with (30) and (31) implies (29) which in turn, together with (26) and (27) implies (25) and therefore the conclusion of Step 2. \square

Lemma 3.9. $\phi: \mathbb{R}^d \rightarrow [0, +\infty]$ satisfies the following properties:

- (i) There exists $C > 0$ such that $\phi(\nu) \leq C|\nu|$ for all $\nu \in \mathbb{R}^d$,
- (ii) ϕ is a positively homogeneous function of degree one,
- (iii) ϕ is a convex function. In particular, ϕ is Lipschitz continuous.

Proof. We divide the proof into two steps. Throughout the proofs let $1 \ll S$. Furthermore, due to Lemma 3.12 we can take $S \rightarrow +\infty$ to be a fixed sequence such that

$$\phi(\nu) = \lim_{\delta \rightarrow 0} \lim_{S \rightarrow +\infty} \frac{1}{S^d} \inf \{E(u, Q_S) : u : \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}\}.$$

Step 1.(Proof of (i) and (ii)) In order to prove (i) we insert $u(i) = \langle \nu, i \rangle$ for all $i \in \mathcal{L}$ as a competitor in the cell formula. Using Lemma 3.2(i), we then have

$$E(u, Q_S^\nu) = E(\langle \nu, \cdot \rangle, Q_S^\nu) \leq C|\nu| |(Q_S)_c| \leq CS^d.$$

Dividing by S^d and letting $S \rightarrow +\infty$ yields the claim. (ii) follows by using Lemma (3.2)(ii) to obtain $E(\lambda u, Q_S) = \lambda E(u, Q_S)$ for all $\lambda > 0$ and by noting that, given $\nu \in \mathbb{R}^d$, if $u : \mathcal{L} \rightarrow \mathbb{R}$ satisfies $u(i) = \langle \nu, i \rangle$ on $\mathcal{L} \setminus Q_{(1-\delta)S}$, then $\lambda u(i) = \langle \lambda \nu, i \rangle$ on $\mathcal{L} \setminus Q_{(1-\delta)S}$. Employing this in (13) it is easy to see that ϕ is a positively homogeneous function of degree one.

Step 2.(Proof of (iii)) We show that for every $S > 0$ and $\delta > 0$, $S\delta \gg 1$ the function $\phi_S^\delta : \mathbb{R}^d \rightarrow [0, +\infty]$ given by

$$\phi_S^\delta(\nu) := \frac{1}{S^d} \inf \{E(u, Q_S) : u : \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}\} \quad (32)$$

is a convex function. Note that

$$\phi(\nu) = \lim_{\delta \rightarrow 0} \lim_{S \rightarrow +\infty} \phi_S^\delta(\nu) \text{ for all } \nu \in \mathbb{R}^d.$$

Thus, the convexity for ϕ_S^δ also implies the convexity of ϕ . This together with (i) and (ii) implies that ϕ is also Lipschitz continuous. Now we prove that ϕ_S^δ is a convex function. Given $\lambda \in [0, 1]$, $\nu_1, \nu_2 \in \mathbb{R}^d$, let $\varepsilon > 0$ $u_1 : \mathcal{L} \rightarrow \mathbb{R}$, $u_2 : \mathcal{L} \rightarrow \mathbb{R}$ be such that $u_k(i) = \langle \nu_k, i \rangle$ on $\mathcal{L} \setminus Q_{(1-\delta)S}$ and

$$E(u_k, Q_S) \leq \phi_S^\delta(\nu_k) + \frac{1}{2}\varepsilon \text{ for } k = 1, 2.$$

We have that $u(i) := \lambda u_1(i) + (1-\lambda)u_2$ is admissible for $\phi_S^\delta(\lambda \nu_1 + (1-\lambda)\nu_2)$ and by Lemma 3.2(ii) we obtain

$$\begin{aligned} \phi_S^\delta(\lambda \nu_1 + (1-\lambda)\nu_2) &\leq E(u, Q_S) = E(\lambda u_1 + (1-\lambda)u_2, Q_S) \leq \lambda E(u_1, Q_S) + (1-\lambda)E(u_2, Q_S) \\ &\leq \lambda \phi_S^\delta(\nu_1) + (1-\lambda)\phi_S^\delta(\nu_2) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this yields the claim. \square

3.3. The periodic energy density. In this subsection we show that the asymptotic periodic cell formula given in (14) can be calculated via a finite cell formula and it coincides with the asymptotic cell formula with affine boundary conditions defining ϕ , see (13).

The next lemma shows that ϕ_{per} can be calculated via a finite cell formula. Additionally, it shows that the liminf in the definition of (14) is actually a limit.

Lemma 3.10. *Let $\nu \in \mathbb{S}^{d-1}$. For all $k \in \mathbb{N}$ there holds*

$$\begin{aligned} &\frac{1}{(kT)^d} \inf \{E(u, Q_{kT}) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kT}; \mathbb{R})\} \\ &= \frac{1}{T^d} \inf \{E(u, Q_T) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})\}. \end{aligned} \quad (33)$$

In particular,

$$\phi_{\text{per}}(\nu) = \frac{1}{T^d} \inf \{E(u, Q_T) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})\} \quad (34)$$

and the lim inf in (14) is actually a limit since the sequence, in fact, does not depend on k .

The following example shows that, without any further assumption on $c_{i,j}$, the minimum in

$$\inf \{E(u, Q_T) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})\} \quad (35)$$

is not achieved by any $u : \mathcal{L} \rightarrow \mathbb{R}$.

Example 3.11. Let $\mathcal{L} = \mathbb{Z}$ and let $c_{i,j}$ be 2-periodic. We set

$$c_{i,j} = \begin{cases} 2^{-k} & \text{if } j - i = 2k + 1, k \in \mathbb{N}, i \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $s \in \mathbb{R}$, $u(0) = 0$ and $u(1) = s$ be 2-periodic. Then, for $\nu = -1$ we have

$$E(u, Q_2) = \sum_{k=0}^{\infty} c_{0,2k+1} (u(0) - u(2k+1) + 2k+1)^+ = \sum_{k=0}^{\infty} 2^{-k} (2k+1-s)^+.$$

We have that

$$\inf_{u \in \mathcal{A}_{\text{per}}(Q_2; \mathbb{R})} E(u, Q_2) = 0.$$

However, clearly, for all $u \in \mathcal{A}_{\text{per}}(Q_2; \mathbb{R})$, $E(u, Q_2) > 0$. In order to ensure the existence in (35) and the other minimum problems a coercivity condition might be: For any $i, j \in \mathcal{L}$ there exists a path $\gamma = (i_0, \dots, i_N)$ such that $i_0 = i$, $i_N = j$ and such that $c_{i_k, i_{k+1}} > 0$. By considering the directed graph $G = (\mathcal{L}, \mathcal{E})$, where $\mathcal{E} := \{(i, j) \in \mathcal{L} \times \mathcal{L} : \text{there exists } z \in \mathbb{Z}^d \text{ such that } c_{i, j+Tz} > 0\}$ this condition ensures that for two vertices $i, j \in G$ there always exists a path of edges (in the infinite graph) with positive weights connecting them.

Proof of Lemma 3.10. We split the proof into two steps by first observing the (obvious) inequality that the right hand side in (33) is less than or equal to the left hand side. Then, we prove the converse inequality by using a superposition argument.

Step 1. (Proof of ' \leq ') given u with $u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$, then obviously $u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kT}; \mathbb{R})$ and

$$E(u, Q_{kT}) = \frac{1}{k^d} E(u, Q_T)$$

so this inequality is obvious.

Step 2. (Proof of ' \geq ') This is a standard convexity argument: given u now with $u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kT}; \mathbb{R})$, then for $i \in Q_T$ we let:

$$u_T(i) = \langle \nu, i \rangle + \frac{1}{k^d} \sum_{z \in \{0, \dots, k-1\}^d} (u(i + Tz) - \langle \nu, i + Tz \rangle).$$

Then clearly by construction, $u_T(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$ and by convexity,

$$E(u_T, Q_T) = \frac{1}{k^d} E(u_T, Q_{kT}) \leq \frac{1}{k^d} E(u, Q_{kT}),$$

which shows the lemma. \square

The following lemma uses a standard cutoff-argument. However, due to the infinite range of interactions, the arguments for the case of finite range interactions need to be adapted.

Lemma 3.12. *Let $\nu \in \mathbb{R}^d$. Then: $\phi_{\text{per}}(\nu) = \phi(\nu)$ and ϕ is independent on the sequence $S_n \rightarrow +\infty$. More precisely,*

$$\phi(\nu) = \lim_{\delta \rightarrow 0} \liminf_{S \rightarrow +\infty} \frac{1}{S^d} \inf \{E(u, Q_S) : u : \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}\} \quad (36)$$

$$= \lim_{\delta \rightarrow 0} \limsup_{S \rightarrow +\infty} \frac{1}{S^d} \inf \{E(u, Q_S) : u : \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}\}. \quad (37)$$

Proof. We first show $\phi_{\text{per}}(\nu) \leq \phi(\nu)$ in Step 1, and then the reverse inequality. In order to do so, we modify competitors of the respective cell formulas in order to obtain a competitor for the other formula. Due to the one homogeneity of both functions, we may assume that $\nu \in \mathbb{S}^{d-1}$.

Step 1.(Proof of ' \leq ') In this Step we show that

$$\phi_{\text{per}}(\nu) \leq \lim_{\delta \rightarrow 0} \liminf_{S \rightarrow +\infty} \frac{1}{S^d} \inf \{E(u, Q_S) : u : \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}\}. \quad (38)$$

Due to Lemma 3.10, the limit in the definition of $\phi_{\text{per}}(\nu)$ exists. Let $S \rightarrow +\infty$ be such that

$$\phi(\nu) = \lim_{\delta \rightarrow 0} \lim_{S \rightarrow +\infty} \frac{1}{S^d} \inf \{E(u, Q_S) : u : \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}\}.$$

Let $\delta > 0$, $\varepsilon > 0$, and let $u_S^\delta : \mathcal{L} \rightarrow \mathbb{R}$ be such that $u_S^\delta(i) = \langle \nu, i \rangle$ on $\mathcal{L} \setminus Q_{(1-\delta)S}$ and

$$E(u_S^\delta, Q_S) \leq \inf \{E(u, Q_S) : u : \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}\} + \varepsilon. \quad (39)$$

We assume that

$$\|u_S^\delta\|_{L^\infty(Q_{(1+\delta)S})} \leq 2S. \quad (40)$$

If that were not true we perform the following construction with $\tilde{u}_S^\delta(i) = (u_S^\delta(i) \vee (-2S)) \wedge (2S)$. In the following we show that we can assume without loss of generality that $S = kT$ for some diverging sequence $k \in \mathbb{N}$. To this end, let $kT = (\lfloor S/T \rfloor + 1)T$. By Lemma 3.2(iv) we have

$$E(u_S^\delta, Q_{kT}) = E(u_S^\delta, Q_S) + E(u_S^\delta, Q_{kT} \setminus Q_S)$$

We show that

$$E(u_S^\delta, Q_{kT} \setminus Q_S) \leq CTS^{d-1}(1 + C/\delta) \quad (41)$$

and therefore we can assume without loss of generality that $S = kT$ for some $k \in \mathbb{N}$. By (40) and the fact that $u(i) = \langle \nu, i \rangle$ on $\mathcal{L} \setminus Q_{(1-\delta)S}$ we have that

$$\begin{aligned} E(u_S^\delta, Q_{kT} \setminus Q_S) &\leq \sum_{i \in \mathcal{L} \cap Q_{kT} \setminus Q_S} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \leq \delta S}} c_{i,j} |i-j| + \frac{C}{\delta} \sum_{i \in \mathcal{L} \cap Q_{kT} \setminus Q_S} \sum_{\substack{j \in \mathcal{L} \\ |i-j| > \delta S}} c_{i,j} |i-j| \\ &\leq \#(\mathcal{L} \cap Q_{kT} \setminus Q_S)(1 + C/\delta) \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j} |i-j| \leq CTS^{d-1}(1 + C/\delta). \end{aligned}$$

This shows (41). By the definition of the sequence S , dividing with $(kT)^d$ and letting $k \rightarrow +\infty$, we observe that it is not restrictive to assume $S = kT$ for a diverging subsequence of $k \in \mathbb{N}$.

Since $S = kT$ we denote by $u_k^\delta = u_S^\delta$ and define $v_k^\delta : \mathcal{L} \rightarrow \mathbb{R}$ by setting

$$v_k^\delta(i) - \langle \nu, i \rangle = u_k^\delta(i_0) - \langle \nu, i_0 \rangle \text{ if } i = i_0 + kTz, i_0 \in Q_{kT}, z \in \mathbb{Z}^d \quad (42)$$

so that $v_k^\delta(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kT}; \mathbb{R})$. Then clearly, writing $i = i_0 + kTz$ and $j = j_0 + kTz'$ as above:

$$|v_k^\delta(i) - v_k^\delta(j)| \leq |u_k^\delta(i_0) - u_k^\delta(j_0)| + |i_0 - j_0| + |i - j| \leq CkT + |i - j|, \quad (43)$$

and

$$v_k^\delta(i) = u_k^\delta(i) \text{ for } i \in Q_{(1+\delta)kT}, \quad (44)$$

since $u_k^\delta(i) - \langle \nu, i \rangle = 0$ for $i \notin Q_{(1-\delta)kT}$. Additionally,

$$\inf \{E(u, Q_{kT}) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kT}; \mathbb{R})\} \leq E(v_k^\delta, Q_{kT}). \quad (45)$$

We are finished with Step 1 if we prove

$$E(v_k^\delta, Q_{kT}) \leq E(u_k^\delta, Q_{kT}) + \frac{C_k^\delta}{\delta} (kT)^d, \quad (46)$$

where $C_k^\delta \rightarrow 0$ as $k \rightarrow +\infty$. In fact, using (39), (45), (46), dividing by $(kT)^d$, letting $k \rightarrow +\infty$, and then $\delta \rightarrow 0$, we obtain (38) by noting that $\varepsilon > 0$ is chosen arbitrarily. Let us prove (46). We have, using (44),

$$\begin{aligned} E(v_k^\delta, Q_{kT}) &= \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{j \in \mathcal{L}} c_{i,j} (v_k^\delta(i) - v_k^\delta(j))^+ \\ &= \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{j \in \mathcal{L} \cap Q_{(1+\delta)kT}} c_{i,j} (v_k^\delta(i) - v_k^\delta(j))^+ + \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{j \in \mathcal{L} \setminus Q_{(1+\delta)kT}} c_{i,j} (v_k^\delta(i) - v_k^\delta(j))^+ \\ &\leq E(u_k^\delta, Q_{kT}) + \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j} |v_k^\delta(i) - v_k^\delta(j)|. \end{aligned}$$

Hence, in order to show (46), it remains to prove

$$\sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j} |v_k^\delta(i) - v_k^\delta(j)| \leq \frac{C_k^\delta}{\delta} (kT)^d, \quad (47)$$

where $C_k^\delta \rightarrow 0$ as $k \rightarrow +\infty$. Using (43), (H2), and Lemma 3.2(v), we have

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j} |v_k^\delta(i) - v_k^\delta(j)| &\leq \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j} (CkT + |i - j|) \\ &\leq \left(\frac{C}{\delta} + 1 \right) \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j} |i - j| \\ &\leq \frac{C}{\delta} \#(\mathcal{L} \cap Q_{kT}) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j} |i - j| \leq \frac{C_k^\delta}{\delta} (kT)^d, \end{aligned}$$

where $C_k^\delta \rightarrow 0$ as $k \rightarrow +\infty$ (cf. (8)). This yields (47) and therefore the claim of Step 1.

Step 2. (Proof of ' \geq ') In this Step we show that

$$\lim_{\delta \rightarrow 0} \limsup_{S \rightarrow +\infty} \frac{1}{S^d} \inf \{E(u, Q_S) : u : \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}\} \leq \phi_{\text{per}}(\nu). \quad (48)$$

Once this is shown, using also Step 1, it follows that $\phi(\nu) = \phi_{\text{per}}(\nu)$ for all $\nu \in \mathbb{R}^d$. Let $\varepsilon > 0$ and $u: \mathcal{L} \rightarrow \mathbb{R}$ be such that $u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$ and

$$E(u, Q_T) \leq \inf \{E(u, Q_T): u: \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})\} + \varepsilon.$$

Fix $\delta > 0$ and $S > 0$ such that $\delta S \gg 1$. Since $u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$, we have

$$E(u, Q_T(x_0)) = E(u, Q_T) \text{ for all } x_0 \in T\mathbb{Z}^d$$

and therefore

$$E(u, Q_S) \leq \frac{(S+T)^d}{T^d} E(u, Q_T). \quad (49)$$

There exists a constant $C > 0$ (we omit the dependence on T) such that, due to the fact that $u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$, there holds

$$\max_{i \in \mathcal{L}} |u(i) - \langle \nu, i \rangle| = \max_{i \in \mathcal{L} \cap Q_T} |u(i) - \langle \nu, i \rangle| \leq C_\varepsilon. \quad (50)$$

Let $\zeta_S \in C_c^\infty(\mathbb{R}^d; [0, 1])$ be a cut-off function such that

$$\zeta_S(x) = 1 \text{ for } x \in Q_{(1-3\delta)S}, \quad \text{supp } \zeta_S(x) \subset Q_{(1-2\delta)S}, \quad \text{and } \|\nabla \zeta_S\|_\infty \leq \frac{C}{\delta S}.$$

Define $u_S: \mathcal{L} \rightarrow \mathbb{R}$ by

$$u_S(i) = \zeta_S(i)u(i) + (1 - \zeta_S(i))\langle \nu, i \rangle.$$

Then, $u_S(i) = \langle \nu, i \rangle$ for $i \in \mathcal{L} \setminus Q_{(1-\delta)S}$ and therefore

$$\inf \{E(u, Q_S): u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}\} \leq E(u_S, Q_S). \quad (51)$$

For all $i, j \in \mathcal{L}$ there holds

$$u_S(i) - u_S(j) = \zeta_S(i)(u(i) - u(j)) + (1 - \zeta_S(i))\langle \nu, i - j \rangle + (\zeta_S(i) - \zeta_S(j))(u(j) - \langle \nu, j \rangle),$$

which, together with (50), implies for all $i, j \in \mathcal{L}$

$$\begin{aligned} (u_S(i) - u_S(j))^+ &\leq (u(i) - u(j))^+ + |i - j| + \frac{C}{\delta S} |u(j) - \langle \nu, j \rangle| |i - j| \\ &\leq (u(i) - u(j))^+ + C|i - j|, \end{aligned} \quad (52)$$

where we assume that $S\delta \geq C_\varepsilon$ (we will first send $S \rightarrow +\infty$, then $\delta \rightarrow 0$, and finally $\varepsilon \rightarrow 0$). For all $i, j \in Q_{(1-3\delta)S}$ we have

$$(u_S(i) - u_S(j))^+ = (u(i) - u(j))^+. \quad (53)$$

Using (49), (52), and (53), we obtain

$$\begin{aligned} E(u_S, Q_S) &\leq \sum_{i \in \mathcal{L} \cap Q_S} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j))^+ + C \sum_{i \in \mathcal{L} \cap Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L} \setminus Q_{(1-3\delta)S}} c_{i,j} |i - j| \\ &\quad + C \sum_{i \in \mathcal{L} \cap Q_S \setminus Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| \\ &= \frac{S^d}{(kT)^d} E(u_k, Q_{kT}) + C \sum_{i \in \mathcal{L} \cap Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L} \setminus Q_{(1-3\delta)S}} c_{i,j} |i - j| \\ &\quad + C \sum_{i \in \mathcal{L} \cap Q_S \setminus Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L}} c_{i,j} |i - j|. \end{aligned} \quad (54)$$

We show that

$$\sum_{i \in \mathcal{L} \cap Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L} \setminus Q_{(1-3\delta)S}} c_{i,j} |i-j| \leq C_S^\delta S^d, \quad (55)$$

and

$$\sum_{i \in \mathcal{L} \cap Q_S \setminus Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L}} c_{i,j} |i-j| \leq C\delta S^d, \quad (56)$$

where $C_S^\delta \rightarrow 0$ as $S \rightarrow +\infty$ and $C > 0$ is a universal constant. Note that, due to Lemma 3.10, since $\varepsilon > 0$ is chosen arbitrary, from (55) and (56) we obtain (48) - the claim of Step 2 - by using (51), (54), dividing by S^d , taking the limsup as $S \rightarrow +\infty$ and then $\delta \rightarrow 0$.

We first prove (55). Note that, for S large enough, due to (H2) and Lemma 3.2(v), we have

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L} \setminus Q_{(1-3\delta)S}} c_{i,j} |i-j| &\leq \sum_{i \in \mathcal{L} \cap Q_{(1-6\delta)S}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S}} c_{i,j} |i-j| \\ &\leq \#(\mathcal{L} \cap Q_S) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S}} c_{i,j} |i-j| \\ &\leq C_S^\delta S^d, \end{aligned}$$

where $C_S^\delta \rightarrow 0$ as $S \rightarrow +\infty$ (cf. (8)). Next, we show (56). Using (H2), and Lemma 3.2(v), we obtain

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap Q_S \setminus Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L}} c_{i,j} |i-j| &\leq \sum_{i \in \mathcal{L} \cap Q_S \setminus Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L}} c_{i,j} |i-j| \\ &\leq \#(\mathcal{L} \cap Q_S \setminus Q_{(1-6\delta)S}) \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j} |i-j| \\ &\leq C\delta S^d. \end{aligned}$$

This is (56) and hence the claim of Step 2. Now (36) follows from (38) and (48). \square

3.4. Comparison between surface and affine cell formula. In this subsection we show that the asymptotic cell formula describing the surface energy density is equal to the asymptotic cell formula with affine boundary conditions. In Lemma 3.17 and Lemma 3.18 we use the following remark.

Remark 3.13. We point out that $\mathbb{S}^{d-1} \cap \mathbb{Q}^d$ is dense in \mathbb{S}^{d-1} . This follows from the fact that \mathbb{Q}^{d-1} is dense in \mathbb{R}^{d-1} and that the inverse of the stereographic projection $P_d: \mathbb{R}^{d-1} \rightarrow \mathbb{S}^{d-1} \setminus \{e_d\}$ is a rational and continuous function.

First, we show the following:

Lemma 3.14. *Given $\nu \in \mathbb{Q}^d \cap \mathbb{S}^{d-1}$ and $t \in (-1/2, 1/2)$, one has*

$$\begin{aligned} \varphi(\nu) &= \lim_{\delta \rightarrow 0} \liminf_{S \rightarrow +\infty} \frac{1}{S^{d-1}} \inf \{E(u, Q_S^\nu): u: \mathcal{L} \rightarrow \{0, 1\}, u(i) = \chi_{\{\langle \nu, i \rangle > tS\}} \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu\} \\ &= \lim_{\delta \rightarrow 0} \limsup_{S \rightarrow +\infty} \frac{1}{S^{d-1}} \inf \{E(u, Q_S^\nu): u: \mathcal{L} \rightarrow \{0, 1\}, u(i) = \chi_{\{\langle \nu, i \rangle > tS\}} \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu\}. \end{aligned} \quad (57)$$

Moreover, $\varphi(\nu)$ is independent of the particular choices of the rotations R^ν defining Q^ν .

Before proceeding with the proof of Lemma 3.14, we show an elementary Lemma that will soon be useful. For a polyhedron P we denote by $\mathcal{F}(P)$ the set of its $(d-1)$ -dimensional facets and for each $F \in \mathcal{F}(P)$ we denote by $\nu_F \in \mathbb{S}^{d-1}$ its outer unit normal.

Lemma 3.15. *Let P be a polyhedron and let $R > 0$. Then*

$$\sum_{i \in P \cap \mathcal{L}} \sum_{j \in P^c \cap \mathcal{L}} c_{i,j} \leq CT \sum_{F \in \mathcal{F}(P)} \mathcal{H}^{d-1}([F]_{\nu_F, R + \frac{3}{2}\sqrt{dT}}) \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j} |i-j| + \frac{C_R}{R} |P|, \quad (58)$$

where $C_R \rightarrow 0$ as $R \rightarrow +\infty$.

Proof. We start by observing that

$$\sum_{i \in P \cap \mathcal{L}} \sum_{j \in P^c \cap \mathcal{L}} c_{i,j} = \sum_{i \in P \cap \mathcal{L}} \sum_{\substack{j \in P^c \cap \mathcal{L} \\ |i-j| \leq R}} c_{i,j} + \sum_{i \in P \cap \mathcal{L}} \sum_{\substack{j \in P^c \cap \mathcal{L} \\ |i-j| > R}} c_{i,j}, \quad (59)$$

and we estimate each term on the right hand side separately. The second term is:

$$\sum_{i \in P \cap \mathcal{L}} \sum_{\substack{j \in P^c \cap \mathcal{L} \\ |i-j| > R}} c_{i,j} \leq R^{-1} \#(P \cap \mathcal{L}) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| > R}} c_{i,j} |i-j| \leq \frac{C_R}{R} |P|, \quad (60)$$

where we used (H1), Lemma 3.2(v), and $C_R \rightarrow 0$ as $R \rightarrow +\infty$ (cf. (8)). The other term is estimated by noting that if $i \in P$, $j \in P^c$, then the segment $[i, j]$ connecting i and j intersects a facet $F \in \mathcal{F}(P)$ with unit normal $\nu_F \in \mathbb{S}^{d-1}$. Denote $x + \nu_F^\perp$ the plane that contains F . We estimate the contribution of the interactions of points $i \in P, j \in P^c$ such that $|i-j| \leq R$ and $[i, j] \cap F \neq \emptyset$. Note that in this case, since the projection onto the plane ν_F^\perp is 1-Lipschitz, $i \in ([F]_{\nu_F, R})_{\nu_F}^-$. Hence, thanks to Lemma 3.5, by summing over all facets $F \in \mathcal{F}(P)$ we obtain:

$$\sum_{i \in P \cap \mathcal{L}} \sum_{\substack{j \in P^c \cap \mathcal{L} \\ |i-j| \leq R}} c_{i,j} \leq CT \sum_{F \in \mathcal{F}(P)} \mathcal{H}^{d-1}([F]_{\nu_F, R + \frac{3}{2}\sqrt{dT}}) \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j} |i-j|.$$

This together with (59) and (60) yields (58). \square

Proof of Lemma 3.14. We show that the lim inf and lim sup in (57) (which are non-decreasing with respect to δ) differ at most by a quantity which goes to 0 as $\delta \rightarrow 0$. Given $\delta > 0$, $t \in (-1/2, 1/2)$, R^ν a rotation which sends e_d to ν , and $S > 0$, set:

$$\varphi(t, S, R^\nu, \delta) := \frac{1}{S^{d-1}} \inf \{ E(u, Q_S^\nu) : Q_S^\nu = R^\nu Q_S, u: \mathcal{L} \rightarrow \{0, 1\}, u(i) = \chi_{\{(\nu, i) > tS\}} \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \}. \quad (61)$$

Let $\delta > 0$. Given $t, t' \in (-1/2, 1/2)$, let $(S_n, R_n^\nu), (S'_n, R_n^{\nu'})$ be subsequences such that:

$$\liminf_{S \rightarrow +\infty} \varphi(t, S, R_S^\nu, \delta) = \lim_{n \rightarrow \infty} \varphi(t, S_n, R_n^\nu, \delta), \quad \limsup_{S \rightarrow +\infty} \varphi(t', S', R_{S'}^{\nu'}, \delta) = \lim_{n \rightarrow \infty} \varphi(t', S'_n, R_n^{\nu'}, \delta).$$

Up to extracting a further subsequence, we may assume that there exist limiting $R^\nu, R^{\nu'}$ such that $R_n^\nu \rightarrow R^\nu$, $R_n^{\nu'} \rightarrow R^{\nu'}$ and $S'_n/S_n \rightarrow +\infty$. For each $n \in \mathbb{N}$ we also consider u_n , which almost reach the infimums in $\varphi(t, S_n, R_n^\nu, \delta)$ up to errors which go to zero as $n \rightarrow \infty$. We recall that $0 \leq u_n \leq 1$. We denote $\nu_l = R^\nu e_l$ and the cubes $Q_{S_n}^\nu$ are given by $\{\sum_{l=1}^d t_l R_n^{\nu'} e_l : -S'_n/2 < t_l \leq S'_n/2\}$. We fix $0 < \varepsilon < \min\{\frac{1}{6\sqrt{d}}(1/2 - |t'|), \frac{1}{2\sqrt{d}}(1/2 - |t|)\}$ and find an orthonormal basis $(\nu_l^\varepsilon)_{l=1}^d$ with $\nu_l^\varepsilon \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ and $|\nu_l^\varepsilon - \nu_l| \leq \varepsilon$ for $l = 1, \dots, d$. Then, there exists $\lambda \in \mathbb{N}$

(possibly very large), such that $\lambda\nu_l^\varepsilon \in T\mathbb{Z}^d$ for all $l \in \{1, \dots, d\}$, where T is as in (L2) and (H1) respectively.

We assume n is large enough so that $|R_n^\nu e_l - \nu_l^\varepsilon| \leq 2\varepsilon$ for each l . In the following we set $\nu^\varepsilon = \nu_d^\varepsilon$ that is the rational approximation of ν . Then, we let $Q_{\tilde{S}_n}^{\nu^\varepsilon} := \{\sum_{l=1}^d t_l \nu_l^\varepsilon : -\tilde{S}_n/2 < t_l \leq \tilde{S}_n/2\}$ be a cube, centered in 0 and of size $\tilde{S}_n \in \lambda\mathbb{N}$, which contains $Q_{S_n}^\nu$. This may be built as follows. For $\hat{S}_n = S_n(1 + 2\sqrt{d}\varepsilon)$ we have for $Q_{\hat{S}_n}^{\nu^\varepsilon} \subset Q_{\tilde{S}_n}^{\nu^\varepsilon}$. This is true since for $x \in Q_{\hat{S}_n}^{\nu^\varepsilon}$ there holds

$$\langle x, \nu_l^\varepsilon \rangle \leq \langle x, R_n^\nu e_l \rangle + |\langle x, R_n^\nu e_l - \nu_l^\varepsilon \rangle| < S_n/2 + \sqrt{d}S_n\varepsilon = S_n/2(1 + 2\sqrt{d}\varepsilon)$$

(The condition that $\langle x, \nu_l^\varepsilon \rangle \geq -\hat{S}_n/2$ is verified in the same fashion). Now let $\tilde{S}_n = (1 + \lfloor S_n(1 + 2\sqrt{d}\varepsilon)/\lambda \rfloor)\lambda$ be the smallest integer multiple of λ such that $\hat{S}_n \leq \tilde{S}_n$. One has $S_n \leq \tilde{S}_n \leq S_n(1 + 2\sqrt{d}\varepsilon) + \lambda$. To simplify in what follows we drop the subscript n .

We estimate $E(u, Q_{\tilde{S}}^{\nu^\varepsilon} \setminus Q_S^\nu)$ from above as follows:

$$E(u, Q_{\tilde{S}}^{\nu^\varepsilon}) - E(u, Q_S^\nu) = E(u, Q_{\tilde{S}}^{\nu^\varepsilon} \setminus Q_S^\nu) \leq C\tilde{S}^{d-2}(\tilde{S} - S) + \sum_{i \in Q_{\tilde{S}}^{\nu^\varepsilon} \setminus Q_S^\nu} \sum_{|j-i| \geq \delta S} c_{i,j}$$

for some constant $C > 0$, where the first error term accounts for the interfacial energy at $\{x : \langle x, \nu \rangle = 0\}$, estimated using Lemma 3.5, and the second for the other facets of the domain. Since

$$\sum_{i \in Q_{\tilde{S}}^{\nu^\varepsilon} \setminus Q_S^\nu} \sum_{|j-i| \geq \delta S} c_{i,j} \leq \sum_{i \in Q_{\tilde{S}}^{\nu^\varepsilon} \setminus Q_S^\nu} \sum_{|j-i| \geq \delta S} c_{i,j} \frac{|i-j|}{\delta S}$$

it follows this second term is less than $C\tilde{S}^{d-1}(\tilde{S} - S)C_S^\delta/(\delta S)$ for some C_S^δ with $\lim_{S \rightarrow \infty} C_S^\delta = 0$. Hence (using that \tilde{S} and S are of same order),

$$E(u, Q_{\tilde{S}}^{\nu^\varepsilon}) \leq E(u, Q_S^\nu) + C\delta S^{d-1} \quad (62)$$

for S large enough. Let now $v = \chi_{(Q_{(1-\delta)S}^\nu)^c}(\chi_{\{\langle \nu^\varepsilon, i \rangle > \tilde{t}S\}} - \chi_{\{\langle \nu, i \rangle > tS\}})$ where $\tilde{t} = t - \varepsilon\sqrt{d}\tilde{S}/S$ is chosen such that $v \geq 0$ in $Q_{\tilde{S}}^{\nu^\varepsilon}$. Then, the set where $v = 1$ in $Q_{\tilde{S}}^{\nu^\varepsilon}$ is given by $\mathcal{L} \cap \{x : \langle \nu^\varepsilon, x \rangle > \tilde{t}S, \langle \nu, x \rangle \leq tS\} \cap Q_{\tilde{S}}^{\nu^\varepsilon} \setminus Q_{(1-\delta)S}^\nu$. Denote P the latter set. Its boundary is made out of $4d - 2$ facets (8 in two dimensions), two of which (4 in two dimensions) are given by $\{\langle \nu, x \rangle = tS\} \cap Q_{\tilde{S}}^{\nu^\varepsilon} \setminus Q_{(1-\delta)S}^\nu$ and $\{\langle \nu^\varepsilon, x \rangle = \tilde{t}S\} \cap Q_{\tilde{S}}^{\nu^\varepsilon} \setminus Q_{(1-\delta)S}^\nu$ while the others are given by $\partial Q_{(1-\delta)S}^\nu \cap \{x : \langle \nu^\varepsilon, x \rangle > \tilde{t}S, \langle \nu, x \rangle \leq tS\}$ and $\partial Q_{\tilde{S}}^{\nu^\varepsilon} \cap \{x : \langle \nu^\varepsilon, x \rangle > \tilde{t}S, \langle \nu, x \rangle \leq tS\}$. We apply Lemma 3.15, with $R = (\delta + \varepsilon)S$. We note that $P \subset Q_{\tilde{S}}^{\nu^\varepsilon}$ so that $|P| \leq \tilde{S}^d$, and that for all $F \in \mathcal{F}(P)$ we have $\mathcal{H}^{d-1}([F]_{\nu_F, \frac{3}{2}\sqrt{d}T+(\delta+\varepsilon)S}) \leq C(T + (\delta + \varepsilon)S)S^{d-2}$. Hence we obtain for S large enough:

$$E(v, Q_{\tilde{S}}^{\nu^\varepsilon}) \leq CS^{d-1}(\delta + \varepsilon) + \frac{C_S^{\delta+\varepsilon}}{\delta + \varepsilon} S^{d-1} \leq CS^{d-1}(\delta + \varepsilon), \quad (63)$$

where we used that $C_S^{\delta+\varepsilon} \rightarrow 0$ as $S \rightarrow +\infty$. Now, we let $\tilde{u} = u + v$, that is:

$$\tilde{u}(i) = \begin{cases} u(i) & \text{if } i \in Q_{(1-\delta)S}^\nu, \\ \chi_{\{\langle \nu^\varepsilon, i \rangle > \tilde{t}S\}} & \text{else.} \end{cases} \quad (64)$$

Thanks to Lemma 3.2(ii) and (64), and using also (62) and (63):

$$E(\tilde{u}, Q_{\tilde{S}}^{\nu^\varepsilon}) \leq E(u, Q_{\tilde{S}}^{\nu^\varepsilon}) + E(v, Q_{\tilde{S}}^{\nu^\varepsilon}) \leq E(u, Q_S^\nu) + CS^{d-1}(\delta + \varepsilon). \quad (65)$$

Now, choose $\tilde{k}_d = (1 + \lfloor (t' + \sqrt{d}\varepsilon)S'/\tilde{S} \rfloor)$ and we let

$$I := \left\{ \vec{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d : k_d = \tilde{k}_d, Q_{\tilde{S}}^{\nu^\varepsilon} + \sum_{l=1}^d k_l \tilde{S} \nu_l^\varepsilon \subset Q_{(1-\delta)S'}^{\nu^\varepsilon} \right\}.$$

The choice of \tilde{k}_d implies that for any $\vec{k} \in I$ and any $x \in Q_{\tilde{S}}^{\nu^\varepsilon}$ with $\langle \nu^\varepsilon, x \rangle > \tilde{t}S$, one has $\langle \nu, x + \sum_{l=1}^d k_l \tilde{S} \nu_l^\varepsilon \rangle > t'S' + \tilde{t}S$. Note that, for $x \in Q_{\tilde{S}}^{\nu^\varepsilon}$

$$t'S' - \sqrt{d}/2\tilde{S} \leq \langle \nu, x + \sum_{l=1}^d k_l \tilde{S} \nu_l^\varepsilon \rangle \leq t'S' + 2\sqrt{d}\tilde{S} + 2\sqrt{d}\varepsilon S'$$

and thus for the strip of cubes

$$\mathcal{Q} := \bigcup_{\vec{k} \in I} \left(Q_{\tilde{S}}^{\nu^\varepsilon} + \sum_{l=1}^d k_l \tilde{S} \nu_l^\varepsilon \right)$$

we have for S, S' large enough

$$\mathcal{Q} \subset Q_{S'}^{\nu} \cap \{t''S' \leq \langle x, \nu \rangle \leq t'''S'\} \text{ with } |t''' - t''| \leq 3\sqrt{d}(\varepsilon + \tilde{S}/S') \leq 6\sqrt{d}\varepsilon \quad (66)$$

for values $t'' \leq t' \leq t'''$.

We now let, for $i' \in \mathcal{L}$

$$u'(i') = \begin{cases} \tilde{u}(i) & \text{if } \vec{k} \in I, i \in Q_{\tilde{S}}^{\nu^\varepsilon}, i' = i + \sum_{l=1}^d k_l \tilde{S} \nu_l^\varepsilon, \\ \chi_{\{\langle \nu^\varepsilon, i \rangle > \tilde{t}S + \tilde{k}_d \tilde{S}\}} & \text{else.} \end{cases}$$

The function $\chi_{\{\langle \nu^\varepsilon, i \rangle > \tilde{t}S + \tilde{k}_d \tilde{S}\}}$ is chosen in order to match the boundary conditions of the function $\tilde{u}(i)$ for $i \in \mathcal{Q}$ and $\text{dist}(i, \partial Q_{(1-\delta)S'}^{\nu}) \leq \tilde{S}$. Note that given a cube $Q_{\vec{k}} = Q_{\tilde{S}}^{\nu^\varepsilon} + \sum_{l=1}^d k_l \tilde{S} \nu_l^\varepsilon$, $\vec{k} \in I$, one has $u'(i') = \tilde{u}(i)$ for $i' = i + \sum_{l=1}^d k_l \tilde{S} \nu_l^\varepsilon$, $i \in Q_{\tilde{S}}^{\nu^\varepsilon}$ and thus, using Lemma 3.2(vi) and (65),

$$\begin{aligned} E(u', Q_{\vec{k}}) &\leq E(\tilde{u}, Q_{\tilde{S}}^{\nu^\varepsilon}) + \sum_{i \in Q_{\vec{k}}} \sum_{|j-i| \geq \delta S} c_{i,j} \leq E(\tilde{u}, Q_{\tilde{S}}^{\nu^\varepsilon}) + C\tilde{S}^d \max_{i \in Q_T} \sum_{|j-i| \geq \delta S} c_{i,j} \frac{|i-j|}{\delta S} \\ &\leq E(u, Q_S^{\nu}) + CS^{d-1}(\delta + \varepsilon + C_\delta^S) \end{aligned}$$

where $\lim_{S \rightarrow \infty} C_\delta^S = 0$, and where we have used (65). We now remark that $\#I$ is of order $((1-\delta)S')^{d-1}/(\langle \nu, \nu^\varepsilon \rangle)/\tilde{S}^{d-1}$ or even a bit less, due to the different rotations $R^\nu, R^{\nu'}$ of the cubes, and it can be bounded by $(S'/\tilde{S})^{d-1}/(1-2\varepsilon^2)$. Hence (assuming that S is large enough, so that $\lambda/S + C_\delta^S \leq \varepsilon$):

$$E(u', \mathcal{Q}) \leq \frac{S'^{d-1}}{(1-2\varepsilon^2)\tilde{S}^{d-1}} E(u, Q_S^{\nu}) + CS'^{d-1}(\delta + \varepsilon). \quad (67)$$

Now, we bound $E(u', Q_{S'}^{\nu} \setminus \mathcal{Q})$. For this, we will invoke again Lemma 3.5, Remark 3.6, and Lemma 3.15 to estimate the contribution of the interface where u' jumps and the contribution of the boundary $\partial \mathcal{Q}$. Here, we distinguish between the points close to $\partial Q_{S'}^{\nu}$ and the bulk points whose interactions cross $\partial \mathcal{Q}$ where it is orthogonal to ν^ε , see Figure 3.

First of all, we estimate the contributions of the interactions crossing the plane $A := \{\langle \nu^\varepsilon, x \rangle = (\tilde{k}_d + \frac{1}{2})\tilde{S}\}$. Note that here, given i below A and j above A , thanks to (64) and the definition of u' , we have $u'(i) = u'(j) = 1$ as long as $|i-j| \leq \delta S/2 \wedge |\tilde{t}S - \frac{1}{2}\tilde{S}|$ if $j \in \mathcal{Q}$ and $|i-j| \leq |\tilde{t}S - \frac{1}{2}\tilde{S}|$ if

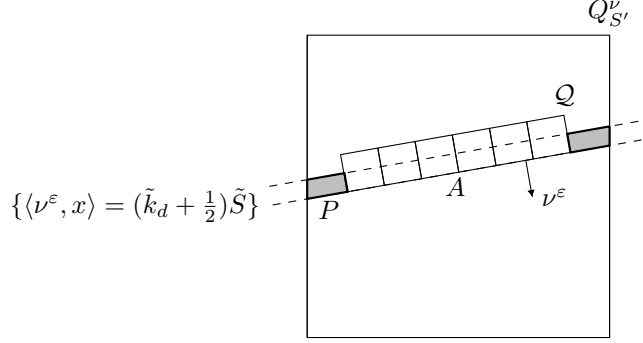


FIGURE 3. The two sets we distinguish in order to estimate u' outside \mathcal{Q} . We distinguish between the gray set P and the interactions crossing the hyperplane A .

$j \notin \mathcal{Q}$. Therefore, for $\delta > 0$ small enough, $u'(i) - u'(j) \neq 0$ only if $|i - j| \geq \delta S/2 \wedge |\tilde{t}S - \frac{1}{2}\tilde{S}| =: \eta S$. (Since $t \in (-1/2, 1/2)$, then, due to the choice of $\varepsilon > 0$ for all S large enough $\tilde{t} \in (-1/2, 1/2)$). Thus by Remark 3.6 (applied with $-\nu_\varepsilon$ and using its notation), there holds, for $S' > \sqrt{\delta}T$ large enough,

$$\begin{aligned} \sum_{i \in \Pi_{x, \nu^\varepsilon}^+ \cap Q_{S'}^{\nu^\varepsilon}} \sum_{j \in A^-} c_{i,j} (u(i) - u(j))^+ &\leq \sum_{i \in (A \cap Q_{2S'}^{\nu^\varepsilon})_{-\nu^\varepsilon}^-} \sum_{\substack{j \in \Pi_{x, -\nu^\varepsilon}^+ \\ |i-j| \geq \eta S}} c_{i,j} \\ &\leq C T \mathcal{H}^{d-1}([A \cap Q_{2S'}^{\nu^\varepsilon}]_{-\nu^\varepsilon, \frac{3\sqrt{\delta}}{2}T}) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \eta S}} c_{i,j} |i - j| \leq C (S')^{d-1} C_S^\eta, \end{aligned} \quad (68)$$

where $C_S^\eta \rightarrow 0$ as $S \rightarrow +\infty$ (cf. (8)). Next, we bound the contribution of the interactions in $\{x: \tilde{t}S + \tilde{k}_d\tilde{S} \leq \langle \nu^\varepsilon, x \rangle \leq (\tilde{k}_d + 1/2)\tilde{S}\} \cap (Q_{S'}^{\nu^\varepsilon} \setminus \mathcal{Q})$. This is a polyhedral set P which has two “large” facets $\{F_{\text{large}}^i\}_{i=1,2}$ with outer normal $\nu_1^i \in \mathbb{S}^{d-1}$ (4 in two dimensions) given by $\{\langle \nu^\varepsilon, x \rangle = (\tilde{k}_d + 1/2)\tilde{S}\} \cap (Q_{S'}^{\nu^\varepsilon} \setminus \mathcal{Q})$ and $\{\langle \nu^\varepsilon, x \rangle = \tilde{t}S + \tilde{k}_d\tilde{S}\} \cap (Q_{S'}^{\nu^\varepsilon} \setminus \mathcal{Q})$ where $\mathcal{H}^{d-1}([F_{\text{large}}^i]_{\nu_1^i, \delta\tilde{S}}) \leq C\delta(S' + \tilde{S})(S')^{d-2}$, and smaller facets: an outer boundary made of $2d$ facets $\{F_{\text{outer}}^i\}_{i=1, \dots, 2d}$ with outer normals $\{\nu_o^i\}_{i=1, \dots, 2d} \subset \mathbb{S}^{d-1}$ contained in $\partial Q_{S'}^{\nu^\varepsilon}$, where $\mathcal{H}^{d-1}([F_{\text{outer}}^i]_{\nu_o^i, \delta\tilde{S}}) \leq C\tilde{S}(S')^{d-2}$, and an “internal” boundary given as the union of $\{F_{\text{int}}^i\}_{i \in \mathcal{I}_{\text{int}}}$ with outer normals $\{\nu_i^i\}_{i \in \mathcal{I}_{\text{int}}} \subset \mathbb{S}^{d-1}$, where $\#\mathcal{I}_{\text{int}} \leq C(S'/\tilde{S})^{d-2}$ and $\mathcal{H}^{d-1}([F_{\text{int}}^i]_{\nu_i^i, \delta\tilde{S}}) \leq C\tilde{S}^{d-1}$. These facets are facets of cubes corresponding to I that lie in $(\partial Q_{(1-\delta)S'})_{\sqrt{\delta}\tilde{S}}$ (in order to estimate the cardinality note that k_1 is fixed in the definition of I). The volume $|P|$ is of order $\delta\tilde{S}S'^{d-1}$ and

$$\begin{aligned} \sum_{F \in \mathcal{F}(P)} \mathcal{H}^{d-1}([F]_{\nu_F, \delta\tilde{S}}) &= \sum_{i=1}^2 \mathcal{H}^{d-1}([F_{\text{large}}^i]_{\nu_1^i, \delta\tilde{S}}) + \sum_{i=1}^{2d} \mathcal{H}^{d-1}([F_{\text{outer}}^i]_{\nu_o^i, \delta\tilde{S}}) + \sum_{i \in \mathcal{I}_{\text{int}}} \mathcal{H}^{d-1}([F_{\text{int}}^i]_{\nu_i^i, \delta\tilde{S}}) \\ &\leq C \left(\delta(S' + \tilde{S})S'^{d-2} + \tilde{S}S'^{d-2} + \left(\frac{S'}{\tilde{S}}\right)^{d-2} \tilde{S}^{d-1} \right) \\ &\leq C(\delta S'^{d-1} + \tilde{S}S'^{d-2}). \end{aligned}$$

We use Lemma 3.15 with $R = \delta\tilde{S}$. Then, the volume term $C_R|P|/R$ in (58) is of order $C_{S'}S'^{d-1}$ with $C_{S'} \rightarrow 0$ as $S' \rightarrow \infty$ and therefore

$$\sum_{i \in \mathcal{L} \cap P} \sum_{j \in \mathcal{L} \cap P^c} c_{i,j} \leq C_{S'}S'^{d-1} + C(\delta S'^{d-1} + \tilde{S}S'^{d-2}) \leq C\delta S'^{d-1}$$

if S, S' are large enough (recalling that $\tilde{S}/S' \rightarrow 0$).

Combining this with (68), we obtain for S, S' large enough:

$$E(u', Q_{S'}^\nu \setminus \mathcal{Q}) \leq C\delta(S')^{d-1} + C(S')^{d-1}C_S^\eta \leq C\delta(S')^{d-1},$$

where we used that $C_S^\eta \rightarrow 0$ as $S \rightarrow +\infty$. We deduce, using (67), that for S, S' large:

$$\frac{1}{S'^{d-1}}E(u', Q_{S'}^\nu) \leq \frac{1}{1-2\varepsilon^2} \frac{1}{\tilde{S}^{d-1}}E(u, Q_S^\nu) + C(\delta + \varepsilon). \quad (69)$$

To conclude, we need to correct the boundary condition for u' , in order to make it admissible for the problem defining $\varphi(t', S', R^{\nu'}, \delta)$. For this, we let $u'' = u' + v'$ where for $i \in \mathcal{L}$,

$$v'(i) = \left(\chi_{\{\langle \nu, i \rangle > t'S'\}} - \chi_{\{\langle \nu^\varepsilon, i \rangle > \tilde{t}S + \tilde{k}_d\tilde{S}\}} \right) \chi_{(Q_{(1-\delta)S'}^\nu)^c}(i).$$

Then u'' is admissible. To estimate the energy, we use Lemma 3.2(ii) and the definition of u'' :

$$E(u'', Q_{S'}^{\nu'}) \leq E(u', Q_{S'}^\nu) + E(v', Q_{S'}^{\nu'})$$

and it remains to bound $E(v', Q_{S'}^{\nu'})$. Here we use that by construction (by the choice of \tilde{k}_d), we have $v'(i) \in \{0, 1\}$ in $Q_{S'}^\nu$, in addition the set where $v'(i) = 1$ is $\mathcal{L} \cap P$ where P is the polyhedron $(Q_{S'}^\nu \setminus Q_{(1-\delta)S'}^\nu) \cap \{x : t'S' < \langle x, \nu \rangle, \langle x, \nu^\varepsilon \rangle \leq \tilde{t}S + \tilde{k}_d\tilde{S}\}$. Its boundary has $4d - 2$ facets (8 in two dimensions), two of which are given by $\{\langle \nu, x \rangle = t'S'\} \cap Q_{S'}^\nu \setminus Q_{(1-\delta)S'}^\nu$ and $\{\langle \nu^\varepsilon, x \rangle = \tilde{t}S + \tilde{k}_d\tilde{S}\} \cap Q_{S'}^\nu \setminus Q_{(1-\delta)S'}^\nu$, while the others are given by $\partial Q_{(1-\delta)S'}^\nu \cap \{x : \langle \nu^\varepsilon, x \rangle \leq \tilde{t}S + \tilde{k}_d\tilde{S}, \langle \nu, x \rangle \geq t'S'\}$ and $\partial Q_{S'}^\nu \cap \{x : \langle \nu^\varepsilon, x \rangle \leq \tilde{t}S + \tilde{k}_d\tilde{S}, \langle \nu, x \rangle \geq t'S'\}$. Recalling (66), we note that for each facet $F \in \mathcal{F}(P)$ (whose outer normal is denoted by $\nu_F \in \mathbb{S}^{d-1}$) we have $\mathcal{H}^{d-1}([F]_{\nu_F, (\delta+\varepsilon)S'}) \leq C(\varepsilon + \delta)(S')^{d-1}$ and the total volume $|P|$ is of order $\varepsilon\delta S'^d$. Therefore, applying Lemma 3.15, with $R = (\delta + \varepsilon)S'$, we obtain, for S, S' large enough, that $E(v', Q_{S'}^{\nu'}) \leq C(\varepsilon + \delta)(S')^{d-1} + C_{(\delta+\varepsilon)S'}(S')^{d-1}$, with $C_{(\delta+\varepsilon)S'} \rightarrow 0$ as $S' \rightarrow \infty$, yielding a bound $C(\varepsilon + \delta)(S')^{d-1}$ for S' sufficiently large. In the end we obtain, using also (69):

$$\varphi(t', S', R^{\nu'}, \delta) \leq \frac{1}{1-2\varepsilon^2} \left(\frac{S}{\tilde{S}} \right)^{d-1} (\varphi(t, S, R^\nu, \delta) + o(1)) + C(\varepsilon + \delta)$$

and as $n \rightarrow \infty$ and then $\delta \rightarrow 0$ we obtain, thanks to the choice of the sequences (S_n, R_n^ν) and $(S'_n, R_n^{\nu'})$ and recalling $S_n \leq \tilde{S}_n$, that:

$$\lim_{\delta \rightarrow 0} \limsup_{S' \rightarrow +\infty} \varphi(t', S', R_n^{\nu'}, \delta) \leq \frac{1}{1-2\varepsilon^2} \lim_{\delta \rightarrow 0} \liminf_{S \rightarrow +\infty} \varphi(t, S, R_S^\nu, \delta) + C\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we deduce that Lemma 3.14 is true. \square

Remark 3.16. (Continuity of φ .) One can adapt the previous proof to the case $t = 0$ and $\nu \neq \nu'$ (close enough) to show that there exists $C > 0$ such that $\varphi(\nu) \leq \varphi(\nu') + C|\nu - \nu'|$, that is, φ is Lipschitz-continuous.

Lemma 3.17. *Let $\nu \in \mathbb{R}^d$. Then: $\psi(\nu) = \varphi(\nu)$.*

Proof. Due the fact that both ψ and φ are positively homogeneous functions of degree one, it suffices to consider the case where $\nu \in \mathbb{S}^{d-1}$. Furthermore, since both functions are continuous, see Lemma 3.8(ii) and Remark 2.3, it suffices to prove the claim for $\nu \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$. For each such vector we can find $\nu_1, \dots, \nu_{d-1} \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ such that the set $\{\nu_1, \dots, \nu_{d-1}, \nu\}$ forms an orthonormal basis of \mathbb{R}^d . Then, there exists $\lambda \in \mathbb{N}$ such that

$$\lambda \nu_n = z_n \text{ for some } z_n \in \mathbb{Z}^d \text{ for all } n \in \{1, \dots, d\}. \quad (70)$$

Step 1.($\varphi \geq \psi$) We show

$$\varphi(\nu) \geq \psi(\nu). \quad (71)$$

To this end let $\{\nu_1, \dots, \nu_{d-1}, \nu_d = \nu\} \subset \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ be an orthonormal basis as previously described and let $1 \ll S_1 \ll S_2$. We assume that $S_1 = \lambda T$, where λ satisfies (70) and T is given by (H1). Note that if λ satisfies (70), also $k\lambda$ satisfies (70) and therefore we can find a sequence $S_k = k\lambda T$ such that $S_k \rightarrow +\infty$ of the desired form. Lemma 3.14 and Remark 3.1 permit us to assume that S is of the specific form. Let $u_1: \mathcal{L} \rightarrow \{0, 1\}$ be such that $u_1(i) = u_\nu(i)$ on $\mathcal{L} \setminus Q_{(1-\delta)S_1}^\nu$ and

$$E(u_1, Q_{S_1}^\nu) \leq \inf \{E(u, Q_{S_1}^\nu) : u: \mathcal{L} \rightarrow \{0, 1\}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S_1}^\nu\} + 1. \quad (72)$$

Due to the assumption on S_1 and Lemma 3.2(vi), we have

$$E(u_1(\cdot - z), Q_{S_1}^\nu(z)) = E(u_1, Q_{S_1}^\nu) \text{ for all } z = \lambda T \sum_{n=1}^d k_n \nu_n, k \in \mathbb{Z}^d. \quad (73)$$

Set (omitting the dependence on S_1 and S_2)

$$\mathcal{Z} := \left\{ z = S_1 \sum_{n=1}^d k_n \nu_n : k \in \mathbb{Z}^d, Q_{S_1}^\nu(z) \subset Q_{(1-\delta)S_2}^\nu \right\}.$$

We define $u_2: \mathcal{L} \rightarrow \mathbb{R}$ by

$$u_2(i) = \begin{cases} S_1 (u_1(i - z) - \frac{1}{2}) + \langle \nu, z \rangle & \text{if } z \in \mathcal{Z}, i \in Q_{S_1}^\nu(z), \\ \langle \nu, i \rangle & \text{otherwise.} \end{cases} \quad (74)$$

We claim that

$$|u_2(i) - u_2(j)| \leq C(S_1 + |i - j|). \quad (75)$$

We postpone the proof of (75) to the end of Step 1. By the definition of u_2 , it is clear that

$$\inf \left\{ E(u, Q_{S_2}^\nu) : u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S_2}^\nu \right\} \leq E(u_2, Q_{S_2}^\nu). \quad (76)$$

It remains to show that

$$E(u_2, Q_{S_2}^\nu) \leq \frac{S_2^d}{S_1^{d-1}} E(u_1, Q_{S_1}^\nu) + C\delta S_2^d + CS_1^2 S_2^{d-1} + \frac{S_2^d}{\delta} C_{S_1}^\delta, \quad (77)$$

where $C_{S_1}^\delta \rightarrow 0$ as $S_1 \rightarrow +\infty$. In fact, once we have shown (77), Step 1 follows from (76) and (72) by dividing by S_2^d and letting first $S_2 \rightarrow +\infty$, then $S_1 \rightarrow +\infty$, and finally $\delta \rightarrow 0$. We are left to prove (77). In order to prove (77) we introduce

$$\begin{aligned} r_1 &= r_1(S_1, S_2, \delta) = (1 - \delta)S_2 - 3\sqrt{d}S_1, \\ r_2 &= r_2(S_1, S_2, \delta) = (1 - \delta)S_2 + 3\sqrt{d}S_1. \end{aligned} \quad (78)$$

We use Lemma 3.2(iv) to obtain

$$E(u_2, Q_{S_2}^\nu) = E(u_2, Q_{r_1}^\nu) + E(u_2, Q_{r_2}^\nu \setminus Q_{r_1}^\nu) + E(u_2, Q_{S_2}^\nu \setminus Q_{r_2}^\nu) \quad (79)$$

and we estimate the three terms on the right hand side separately. We claim that

$$E(u_2, Q_{r_1}^\nu) \leq \frac{S_2^d}{S_1^{d-1}} E(u_1, Q_{S_1}^\nu) + \frac{S_2^d}{\delta} C_{S_1}^\delta, \quad (80)$$

where $C_{S_1}^\delta \rightarrow 0$ as $S_1 \rightarrow +\infty$. Indeed, if $i \in Q_{S_1}^\nu(z)$ with $z = S_1 \sum_{n=1}^d k_n \nu_n \in \mathcal{Z}$ and $Q_{S_1}^\nu(z) \cap Q_{r_1}^\nu \neq \emptyset$ then

$$u_2(i) = S_1 \left(u_1(i - z) - \frac{1}{2} \right) + \langle \nu, z \rangle \text{ for all } i \in Q_{(1+\delta)S_1}^\nu(z). \quad (81)$$

Due to (74), this is clearly true for $i \in Q_{S_1}^\nu(z)$, while if $i \in Q_{(1+\delta)S_1}^\nu(z) \setminus Q_{S_1}^\nu(z)$ there exists $z' = S_1 \sum_{n=1}^d k_n \nu_n \in \mathcal{Z}$, $k \in \mathbb{Z}^d$ such that $\|k - k'\|_\infty = 1$, $Q_{S_1}^\nu(z') \subset Q_{r_1}^\nu$, and $i \in Q_{S_1}^\nu(z') \setminus Q_{(1-\delta)S_1}^\nu(z')$. Then, due to the boundary conditions of u_1 , we have

$$\begin{aligned} u_2(i) &= S_1 \left(u_1(i - z') - \frac{1}{2} \right) + \langle \nu, z' \rangle = S_1 \left(u_\nu(i - z') - \frac{1}{2} \right) + \langle \nu, z \rangle + \langle \nu, z' - z \rangle \\ &= S_1 \left(u_\nu(i - z) - \frac{1}{2} \right) + \langle \nu, z \rangle = S_1 \left(u_1(i - z) - \frac{1}{2} \right) + \langle \nu, z \rangle. \end{aligned}$$

Here, the third equality follows, from the fact that $\|k - k'\|_\infty = 1$ and therefore $\langle \nu, z' - z \rangle \in \{-S_1, 0, S_1\}$. To obtain the previous equality, we distinguish the following two cases:

$$\begin{aligned} \langle \nu, z' - z \rangle = \pm S_1 &\implies u_\nu(i - z') - u_\nu(i - z) = \mp 1 \text{ and} \\ \langle \nu, z' - z \rangle = 0 &\implies u_\nu(i - z') = u_\nu(i - z). \end{aligned}$$

Now (81) together with (73) implies for $z \in \mathcal{Z}$ such that $Q_{S_1}^\nu(z) \cap Q_{r_1}^\nu \neq \emptyset$

$$\begin{aligned} E(u_2, Q_{S_1}^\nu(z)) &= \sum_{i \in \mathcal{L} \cap Q_{S_1}^\nu(z)} \sum_{j \in \mathcal{L} \cap Q_{(1+\delta)S_1}^\nu(z)} c_{i,j} (u_2(i) - u_2(j))^+ \\ &\quad + \sum_{i \in \mathcal{L} \cap Q_{S_1}^\nu(z)} \sum_{j \in \mathcal{L} \setminus Q_{(1+\delta)S_1}^\nu(z)} c_{i,j} (u_2(i) - u_2(j))^+ \quad (82) \\ &\leq S_1 E(u_1, Q_{S_1}^\nu(z)) + \sum_{i \in \mathcal{L} \cap Q_{S_1}^\nu(z)} \sum_{j \in \mathcal{L} \setminus Q_{(1+\delta)S_1}^\nu(z)} c_{i,j} |u_2(i) - u_2(j)| \\ &= S_1 E(u_1, Q_{S_1}^\nu(z)) + \sum_{i \in \mathcal{L} \cap Q_{S_1}^\nu(z)} \sum_{j \in \mathcal{L} \setminus Q_{(1+\delta)S_1}^\nu(z)} c_{i,j} |u_2(i) - u_2(j)|. \end{aligned}$$

We estimate the second term on the right hand side of (82) to obtain (80). Note for $i \in Q_{S_1}^\nu(z)$, $j \in \mathcal{L} \setminus Q_{(1+\delta)S_1}^\nu$ we have $|i - j| \geq \delta S_1 / 2$ and thus, due to (75), we obtain

$$|u_2(i) - u_2(j)| \leq \frac{C}{\delta} |i - j| \text{ for all } i \in Q_{S_1}^\nu(z), j \in \mathcal{L} \setminus Q_{(1+\delta)S_1}^\nu(z) \quad (83)$$

for some $C > 0$ independent of S_1 , S_2 and δ . Now, we get

$$\begin{aligned} \sum_{i \in Q_{S_1}^\nu(z)} \sum_{j \in \mathcal{L} \setminus Q_{(1+\delta)S_1}^\nu(z)} c_{i,j} |u_2(i) - u_2(j)| &\leq \frac{C}{\delta} \sum_{i \in \mathcal{L} \cap Q_{S_1}^\nu(z)} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S_1/2}} c_{i,j} |i-j| \\ &\leq \#(\mathcal{L} \cap Q_{S_1}^\nu(z)) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S_1/2}} c_{i,j} |i-j| \leq \frac{C_{S_1}^\delta}{\delta} S_1^d, \end{aligned}$$

where $C_{S_1}^\delta \rightarrow 0$ as $S_1 \rightarrow +\infty$ (cf. (8)). Hence, noting that for $z, z' \in \mathcal{Z}$ such that $z \neq z'$, we have $Q_{S_1}^\nu(z) \cap Q_{S_1}^\nu(z') = \emptyset$ and therefore $\#\mathcal{Z} \leq S_2^d/S_1^d$, we get

$$\begin{aligned} E(u_2, Q_{r_1}^\nu) &\leq \sum_{\substack{z \in \mathcal{Z} \\ Q_{S_1}^\nu(z) \cap Q_{r_1}^\nu \neq \emptyset}} E(u_2, Q_{S_1}^\nu(z)) \leq \#\mathcal{Z} (S_1 E(u_1, Q_{S_1}^\nu) + \frac{C_{S_1}^\delta}{\delta} S_1^d) \\ &\leq \frac{S_2^d}{S_1^{d-1}} E(u_1, Q_{S_1}^\nu) + \frac{C_{S_1}^\delta}{\delta} S_2^d, \end{aligned}$$

where $C_{S_1}^\delta \rightarrow 0$ as $S_1 \rightarrow +\infty$ (recall (8)). This is (80). Next, we prove

$$E(u_2, Q_{r_2}^\nu \setminus Q_{r_1}^\nu) \leq C S_1^2 S_2^{d-1}. \quad (84)$$

We use (75) to obtain

$$E(u_2, Q_{r_2}^\nu \setminus Q_{r_1}^\nu) = \sum_{i \in Q_{r_2}^\nu \setminus Q_{r_1}^\nu} \sum_{j \in \mathcal{L}} c_{i,j} (u_2(i) - u_2(j))^+ \leq C S_1 \#(\mathcal{L} \cap Q_{r_2}^\nu \setminus Q_{r_1}^\nu) \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j} |i-j|,$$

where we used that, owing to (L1), we have $|i-j| \geq c$ if $i \neq j$. Using Lemma 3.2(v), (H2), and (78) we have that $\#(\mathcal{L} \cap Q_{r_2}^\nu \setminus Q_{r_1}^\nu) \leq C S_1 S_2^{d-1}$ and thus we obtain (84). As for the third term on the right hand side we prove

$$E(u_2, Q_{S_2}^\nu \setminus Q_{r_2}^\nu) \leq C \delta S_2^d + \frac{C_{S_1}^\delta}{\delta} S_2^d, \quad (85)$$

where $C_{S_1}^\delta \rightarrow 0$ as $S_1 \rightarrow +\infty$. To this end we split the summation over j to obtain

$$\begin{aligned} E(u_2, Q_{S_2}^\nu \setminus Q_{r_2}^\nu) &= \sum_{i \in \mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \leq S_1 \delta/2}} c_{i,j} (u_2(i) - u_2(j))^+ \\ &\quad + \sum_{i \in \mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu} \sum_{\substack{j \in \mathcal{L} \\ |i-j| > S_1 \delta/2}} c_{i,j} (u_2(i) - u_2(j))^+. \end{aligned} \quad (86)$$

Let us note first that $Q_{S_2}^\nu \setminus Q_{r_2}^\nu \subset Q_{S_2}^\nu \setminus Q_{(1-2\delta)S_2}^\nu$ and therefore, due to 3.2(v), we have

$$\#(\mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu) \leq C \delta S_2^d. \quad (87)$$

Now, for the first term on the right hand side of (86), employing (74), we note that $u_2(i) = \langle \nu, i \rangle$ and $u_2(j) = \langle \nu, j \rangle$. Hence,

$$\sum_{i \in \mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \leq S_1 \delta/2}} c_{i,j} |u_2(i) - u_2(j)| \leq \#(\mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu) \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j} |i-j| \leq C \delta S_2^d, \quad (88)$$

where we used (H2) and (87). For the second term on the right hand side of (86), we use (75) and (87) to obtain

$$\begin{aligned}
\sum_{i \in \mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu} \sum_{\substack{j \in \mathcal{L} \\ |i-j| > S_1 \delta / 2}} c_{i,j} |u_2(i) - u_2(j)| &\leq \frac{C}{\delta} \sum_{i \in \mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu} \sum_{\substack{j \in \mathcal{L} \\ |i-j| > S_1 \delta / 2}} c_{i,j} |i - j| \\
&\leq \frac{C}{\delta} \#(\mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| > S_1 \delta / 2}} c_{i,j} |i - j| \quad (89) \\
&\leq \frac{C_{S_1}^\delta}{\delta} S_2^d,
\end{aligned}$$

where $C_{S_1}^\delta \rightarrow 0$ as $S_1 \rightarrow +\infty$ (cf. (8)). Inequality (85) follows from (86), (88), and (89). Now (79), (80), (84), (85) give (77). To conclude Step 1, it remains to prove (75). There are four cases to consider:

- (a) $i = i_0 + z, j = j_0 + z, i_0 \in Q_{S_1}^\nu, j_0 \in Q_{S_1}^\nu, z \in \mathcal{Z}$;
- (b) $i = i_0 + z, j = j_0 + z', i_0 \in Q_{S_1}^\nu, j_0 \in Q_{S_1}^\nu, z, z' \in \mathcal{Z}$;
- (c) $i = i_0 + z, i_0 \in Q_{S_1}^\nu, z \in \mathcal{Z}$ and $j \notin Q_{S_1}^\nu(z')$ for any $z' \in \mathcal{Z}$;
- (d) $i \notin Q_{S_1}^\nu(z)$ for any $z \in \mathcal{Z}$ and $j \notin Q_{S_1}^\nu(z')$ for any $z' \in \mathcal{Z}$.

Case (a): This case follows since $\|u_1\|_{L^\infty(Q_{S_1}^\nu)} \leq 1$.

Case (b): Note that in the case where $i = i_0 + z, j = j_0 + z'$ for some $i_0, j_0 \in Q_{S_1}^\nu$ and for some $z, z' \in \mathcal{Z}$, we have

$$|u_2(i) - u_2(j)| \leq |\langle \nu, z - z' \rangle| + CS_1 \leq |\langle \nu, z + i_0 - z' - j_0 \rangle| + |i_0 - j_0| + CS_1 \leq |i - j| + CS_1$$

and therefore (75) holds true.

Case (c): Note that in the case where $i = i_0 + z, i_0 \in Q_{S_1}^\nu, z \in \mathcal{Z}$ and $j \notin Q_{S_1}^\nu(z)$ for any $z \in \mathcal{Z}$, we have

$$|u_2(i) - u_2(j)| \leq CS_1 + |\langle \nu, z - j \rangle| \leq CS_1 + |\langle \nu, i - j \rangle| + |i_0| \leq CS_1 + |i - j|.$$

Also here (75) holds true.

Case (d): In this case $u_2(i) = \langle \nu, i \rangle$ and $u_2(j) = \langle \nu, j \rangle$ and therefore (75) holds true. This shows (75) in general.

Step 2. ($\varphi \leq \psi$) Given $\delta > 0, t \in (-1/2, 1/2), S \gg 1$, and $\varepsilon > 0$ we find $u_1: \mathcal{L} \rightarrow \mathbb{R}$ such that $u_1(i) = \langle \nu, i \rangle$ for $i \in \mathcal{L} \setminus Q_{(1-\delta)S}^\nu$ and

$$E(u_1, Q_S^\nu) \leq S^d(\psi(\nu) + \varepsilon). \quad (90)$$

Due to Lemma 3.3, there holds

$$\begin{aligned}
E(u_1, Q_S^\nu) &= \int_{-\infty}^{+\infty} E(\chi_{\{u_1 > t\}}, Q_S^\nu) dt \geq \int_{-S(1-\delta)/2}^{S(1-\delta)/2} E(\chi_{\{u_1 > t\}}, Q_S^\nu) dt \\
&= S \int_{-(1-\delta)/2}^{(1-\delta)/2} E(\chi_{\{u_1 > tS\}}, Q_S^\nu) dt.
\end{aligned} \quad (91)$$

Note that for all $t \in (-(1-\delta)/2, (1-\delta)/2)$, due to $u_1(i) = \langle \nu, i \rangle$ we have that $\chi_{\{u_1 > tS\}}(i) = \chi_{\{\langle \nu, i \rangle > tS\}}$ for $i \in \mathcal{L} \setminus Q_{(1-\delta)S}^\nu$ and thus, recalling (61),

$$E(\chi_{\{u_1 > tS\}}, Q_S^\nu) \geq S^{d-1} \varphi(t, S, R^\nu, \delta). \quad (92)$$

Therefore,

$$\varepsilon + \psi(\nu) \geq \int_{-(1-\delta)/2}^{(1-\delta)/2} \varphi(t, S, R^\nu, \delta) dt$$

and then thanks to Fatou's lemma, taking first the liminf as $S \rightarrow +\infty$, then $\delta \rightarrow 0$, and using Lemma 3.14, we deduce:

$$\varepsilon + \psi(\nu) \geq \int_{-1}^1 \varphi(\nu) dt = \varphi(\nu).$$

Since ε is arbitrary, this concludes Step 2 and also the proof. \square

In the next Lemma we show that, assuming affine boundary conditions, the calculation of the asymptotic cell formula with respect to the coordinate cube and the calculation of the asymptotic cell formula with respect to the rotated cube are equivalent.

Lemma 3.18. *Let $\nu \in \mathbb{R}^d$. Then: $\psi(\nu) = \phi(\nu)$.*

Proof. After reducing to rational directions, we define a sequence of cell problems defined on cubes $Q_{\lambda T}^\nu$ and exploit Lemma 3.12 to show that it suffices to compare the two sequences of cell problems with periodic boundary conditions.

Since both ψ and ϕ are positively homogeneous functions of degree one (cf. (15) and Lemma 3.9(ii)) it suffices to consider the case where $\nu \in \mathbb{S}^{d-1}$. Thanks to Lemma 3.8(ii) and Lemma 3.9(iii) both functions are continuous. Thus it suffices to prove the claim for $\nu \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$. For each such vector we can find $\{\nu_1, \dots, \nu_{d-1}\} \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ such that the set $\{\nu_1, \dots, \nu_{d-1}, \nu_d = \nu\}$ forms an orthonormal basis of \mathbb{R}^d . For such an orthonormal basis, it is clear that there exists $\lambda \in \mathbb{N}$ such that

$$\lambda \nu_n = z_n \text{ for some } z_n \in \mathbb{Z}^d \text{ for all } n \in \{1, \dots, d\}. \quad (93)$$

Let $\{\nu_1, \dots, \nu_{d-1}, \nu_d = \nu\} \subset \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ be the orthonormal basis described previously. For $\nu \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ fixed, we set

$$\psi_{\text{per}}(\nu) := \lim_{k \rightarrow \infty} \frac{1}{(\lambda k T)^d} \inf\{E(u, Q_{\lambda k T}^\nu) : u : \mathcal{L} \rightarrow \mathbb{R}, \quad (94)$$

$$u(\cdot + \lambda k T \nu_n) - \langle \nu, \cdot + \lambda k T \nu_n \rangle = u(\cdot) - \langle \nu, \cdot \rangle \text{ for all } n = 1, \dots, d\}$$

Step 1. ($\psi_{\text{per}} = \psi$) We claim that

$$\psi_{\text{per}}(\nu) = \psi(\nu). \quad (95)$$

This follows exactly as the proof of Lemma 3.12 by replacing the coordinate cubes with the cubes $Q_{\lambda k T}^\nu$.

Step 2. ($\psi_{\text{per}} = \phi_{\text{per}}$) Due to Step 1 and Lemma 3.12 it suffices to show that

$$\psi_{\text{per}}(\nu) = \phi_{\text{per}}(\nu). \quad (96)$$

Step 2.1. (Proof of ' \geq ') To this end, let $\varepsilon > 0$, $k \in \mathbb{N}$ be large enough, and $u_k : \mathcal{L} \rightarrow \mathbb{R}$ be such that $u_k(\cdot + \lambda k T \nu_n) - \langle \nu, \cdot + \lambda k T \nu_n \rangle = u_k(\cdot) - \langle \nu, \cdot \rangle$ for all $n = 1, \dots, d$ and

$$E(u_k, Q_{\lambda k T}^\nu) \leq (\lambda k T)^d (\psi_{\text{per}}(\nu) + \varepsilon). \quad (97)$$

Thanks to (93) and the fact that $\{\nu_1, \dots, \nu_d\}$ is a basis, we have that $u_k - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kmT}; \mathbb{R})$ for some $m \in \mathbb{N}$ depending only on $\{\nu_1, \dots, \nu_d\}$ and λ . Fix now $M \in \mathbb{N}$ and integer multiple of m such that $M \gg m$ and let

$$\mathcal{Z} := \left\{ z = \lambda kT \sum_{n=1}^d \mu_n \nu_n : \mu \in \mathbb{Z}^d, Q_{\lambda kT}^\nu(z) \cap Q_{MkT} \neq \emptyset \right\}.$$

Since, $u_k - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kmT}; \mathbb{R})$ we have that $u_k - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kMT}; \mathbb{R})$ and thus

$$\inf \{E(u, Q_{kMT}) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kMT}; \mathbb{R})\} \leq E(u_k, Q_{kMT}). \quad (98)$$

Due to the periodicity of u_k , the assumption on λ , and Lemma 3.2(vi), we have that

$$E(u_k, Q_{\lambda kT}^\nu(z)) = E(u_k(\cdot - z), Q_{\lambda kT}^\nu(z)) = E(u_k, Q_{\lambda kT}^\nu) \text{ for all } z = \lambda kT \sum_{n=1}^d \mu_n \nu_n, \mu \in \mathbb{Z}^d. \quad (99)$$

Note that for $z \in \mathcal{Z}$ we have $Q_{\lambda kT}^\nu(z) \subset Q_{(M+\sqrt{d}\lambda)kT}$ and thus $\#\mathcal{Z} \leq (M^d + CM^{d-1}\lambda kT)/\lambda^d$. Therefore,

$$E(u_k, Q_{kMT}) \leq \sum_{z \in \mathcal{Z}} E(u_k, Q_{\lambda kT}^\nu(z)) \leq \#\mathcal{Z} E(u_k, Q_{\lambda kT}^\nu) \leq \frac{M^d + CM^{d-1}\lambda}{\lambda^d} E(u_k, Q_{\lambda kT}^\nu). \quad (100)$$

Dividing by $(kMT)^d$, letting first M tend to $+\infty$, then k to $+\infty$, and lastly $\varepsilon \rightarrow 0$, and noting (97) as well as (98), we obtain the conclusion of Step 2.1.

Step 2.2. (Proof of ' \leq ') The reverse inequality follows as in Step 2.1 by noting that if $u \in \mathcal{A}_{\text{per}}(Q_{kT}; \mathbb{R})$ for some $k \in \mathbb{N}$, then, due to (93), we have that $u(\cdot - \lambda mT \nu_n) - \langle \nu, \cdot - \lambda mT \nu_n \rangle = u(\cdot) - \langle \nu, \cdot \rangle$ for some $m \in \mathbb{N}$ and all $n = 1, \dots, d$. This allows us to perform the same construction as in Step 2.1. This concludes the proof. \square

Proof of Proposition 2.6. Our goal is to prove

$$\varphi(\nu) = \frac{1}{T^d} \inf \{E(u, Q_T) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})\} \quad (101)$$

for all $\nu \in \mathbb{R}^d$. Due to Lemma 3.18, Lemma 3.17, and Lemma 3.12, we have

$$\varphi(\nu) = \psi(\nu) = \phi(\nu) = \phi_{\text{per}}(\nu). \quad (102)$$

Additionally, Lemma 3.10 ensures that

$$\phi_{\text{per}}(\nu) = \frac{1}{T^d} \inf \{E(u, Q_T) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})\}.$$

This shows (101) and concludes the proof. \square

4. CRYSTALLINITY OF THE HOMOGENIZED SURFACE ENERGY DENSITY

This section is devoted to the proof of Theorem 2.7. We assume throughout this section that assumptions (L1), (L2), (H1) and (H2) are satisfied. In Step 2 and Step 3 we additionally assume (H3).

We define the set of edges \mathcal{E} by

$$\mathcal{E} = \{(i, j) \in (\mathcal{L} \cap Q_T) \times \mathcal{L} : c_{i,j} \neq 0\} \text{ and } N = \#\mathcal{E}. \quad (103)$$

Proof of Theorem 2.7. We divide the proof into three steps. First, we derive a dual representation of φ . Then, using this representation, we show that φ is crystalline.

Step 1.(Dual representation) For this step we assume (H1) and (H2). We define

$$\mathcal{C} = \left\{ \alpha_{i,j} \in [0, c_{i,j}] : \alpha_{i+Tz, j+Tz} = \alpha_{i,j} \text{ for all } z \in \mathcal{L}, \right. \\ \left. \sum_{j \in \mathcal{L}} (\alpha_{j,i} - \alpha_{i,j}) = 0 \text{ for all } i \in Q_T \cap \mathcal{L} \right\}. \quad (104)$$

Our goal is to prove the following identity for all $\nu \in \mathbb{R}^d$:

$$\varphi(\nu) = \frac{1}{T^d} \sup_{(\alpha_{i,j})_{i,j} \in \mathcal{C}} \left\langle \nu, \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (i - j) \right\rangle. \quad (105)$$

Let $\nu \in \mathbb{R}^d$. Due to Proposition 2.6 there holds

$$\begin{aligned} \varphi(\nu) &= \phi_{\text{per}}(\nu) = \frac{1}{T^d} \inf \{ E(u, Q_T) : u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R}) \} \\ &= \frac{1}{T^d} \inf \{ E(u + \langle \nu, \cdot \rangle, Q_T) : u \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R}) \} \\ &= \frac{1}{T^d} \inf_{u \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})} \sum_{i \in Q_T} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j) + \langle \nu, i - j \rangle)^+. \end{aligned}$$

Note that, we can write

$$\phi_{\text{per}}(\nu) = \frac{1}{T^d} \inf_{u \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})} \sup_{\substack{0 \leq \alpha_{i,j} \leq c_{i,j} \\ \alpha_{i+Tz, j+Tz} = \alpha_{i,j}}} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (u(i) - u(j) + \langle \nu, i - j \rangle). \quad (106)$$

To see this, we observe that for all $u \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$ and all $0 \leq \alpha_{i,j} \leq c_{i,j}$ such that $\alpha_{i+Tz, j+Tz} = \alpha_{i,j}$ for all $z \in \mathbb{Z}^d$ we have

$$\alpha_{i,j} (u(i) - u(j) + \langle \nu, i - j \rangle) \leq c_{i,j} (u(i) - u(j) + \langle \nu, i - j \rangle)^+$$

with equality for

$$\alpha_{i,j} = \begin{cases} c_{i,j} & \text{if } (u(i) - u(j) + \langle \nu, i - j \rangle) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Given $0 \leq \alpha_{i,j} \leq c_{i,j}$ such that $\alpha_{i+Tz, j+Tz} = \alpha_{i,j}$ for all $z \in \mathbb{Z}^d$, and $u: \mathcal{L} \rightarrow \mathbb{R}$ T -periodic, we have

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (u(i) - u(j)) &= \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} u(i) - \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} u(j) \\ &= \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} u(i) - \sum_{j \in \mathcal{L} \cap Q_T} \sum_{z \in \mathbb{Z}^d} \sum_{i \in \mathcal{L} \cap Q_T} \alpha_{i, j+Tz} u(j+Tz) \\ &= \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} u(i) - \sum_{j \in \mathcal{L} \cap Q_T} \sum_{z \in \mathbb{Z}^d} \sum_{i \in \mathcal{L} \cap Q_T} \alpha_{i-Tz, j} u(j) \\ &= \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} u(i) - \sum_{j \in \mathcal{L} \cap Q_T} \sum_{i \in \mathcal{L}} \alpha_{i,j} u(j) \\ &= \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} (\alpha_{i,j} - \alpha_{j,i}) u(i). \end{aligned}$$

Note that, since in all steps, due to (H2) and $|\alpha_{i,j} u(j)| \leq c_{i,j} \max_{i \in \mathcal{L} \cap Q_T} |u(i)|$, the double series $\sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} u(j)$ (resp. changing $u(i)$ with $u(j)$) is absolutely convergent, the order of

summation can be changed without changing the value of the various sums. This implies that, given $0 \leq \alpha_{i,j} \leq c_{i,j}$ such that $\alpha_{i+Tz,j+Tz} = \alpha_{i,j}$ for all $z \in \mathcal{L}$, we have

$$\inf_{u \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (u(i) - u(j)) = \begin{cases} 0 & \text{if } \sum_{j \in \mathcal{L}} (\alpha_{i,j} - \alpha_{j,i}) = 0 \text{ for all } i \in Q_T \cap \mathcal{L}, \\ -\infty & \text{otherwise.} \end{cases} \quad (107)$$

Hence, using (102), (104), (106), and (107), we obtain

$$\varphi(\nu) \geq \frac{1}{T^d} \sup_{(\alpha_{i,j})_{i,j \in \mathcal{C}}} \left\langle \nu, \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (i - j) \right\rangle. \quad (108)$$

As for the other inequality in finite dimension, that is when $c_{i,j}^R$ is such that $c_{i,j}^R = c_{i,j}$ if $|i - j| < R$ and $c_{i,j}^R = 0$ if $|i - j| \geq R$, the equality is true due to [43, Corollary 31.2.1]. More precisely, we obtain:

$$\begin{aligned} \varphi_R(\nu) &= \frac{1}{T^d} \inf_{u \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})} \sum_{i \in Q_T} \sum_{j \in \mathcal{L}} c_{i,j}^R (u(i) - u(j) + \langle \nu, i - j \rangle)^+ \\ &= \frac{1}{T^d} \sup_{(\alpha_{i,j})_{i,j \in \mathcal{C}_R}} \left\langle \nu, \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (i - j) \right\rangle, \end{aligned} \quad (109)$$

where $\mathcal{C}_R = \mathcal{C} \cap \prod_{(i,j)} [0, c_{i,j}^R]$. Note that $\mathcal{C}_R \subset \mathcal{C}$ and thus by (108) and (109), we have

$$\varphi_R(\nu) \leq \frac{1}{T^d} \sup_{(\alpha_{i,j})_{i,j \in \mathcal{C}}} \left\langle \nu, \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (i - j) \right\rangle \leq \varphi(\nu).$$

To conclude, we show $\lim_{R \rightarrow \infty} \varphi_R(\nu) = \varphi(\nu)$, from which (105) will follow. For $R > 0$ let $u_R \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$ be such that

$$\sum_{i \in Q_T} \sum_{j \in \mathcal{L}} c_{i,j}^R (u_R(i) - u_R(j) + \langle \nu, i - j \rangle)^+ \leq T^d \varphi_R(\nu) + \frac{1}{R} \leq T^d \varphi(\nu) + \frac{1}{R}. \quad (110)$$

One has

$$T^d \varphi(\nu) \leq \sum_{i \in Q_T} \sum_{j \in \mathcal{L}} c_{i,j} (u_R(i) - u_R(j) + \langle \nu, i - j \rangle)^+$$

hence we need to show that

$$\lim_{R \rightarrow \infty} \sum_{i \in Q_T} \sum_{j \in \mathcal{L}, |i-j| \geq R} c_{i,j} (u_R(i) - u_R(j) + \langle \nu, i - j \rangle)^+ = 0. \quad (111)$$

This will follow from (7) if we show that $(u_R(i) - u_R(j))^+$ is bounded (and use $(u_R(i) - u_R(j) + \langle \nu, i - j \rangle)^+ \leq (u_R(i) - u_R(j))^+ + |i - j|$ which follows from the subadditivity of $x \mapsto x^+$).

From (110), we have:

$$\sum_{i \in Q_T} \sum_{j \in \mathcal{L}} c_{i,j}^R (u_R(i) - u_R(j))^+ \leq T^d \varphi(\nu) + \frac{1}{R} + \sum_{i \in Q_T} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| =: \ell < +\infty.$$

On the other hand,

$$\sum_{i \in Q_T} \sum_{j \in \mathcal{L}} c_{i,j}^R (u_R(i) - u_R(j))^+ = \sum_{i \in Q_T} \sum_{j \in Q_T} \left(\sum_{z \in \mathbb{Z}^d} c_{i,j+zT}^R \right) (u_R(i) - u_R(j))^+.$$

If R is large enough, $\sum_{z \in \mathbb{Z}^d} c_{i,j+zT}^R \geq \frac{1}{2} \sum_{z \in \mathbb{Z}^d} c_{i,j+zT}$ for all $(i, j) \in (Q_T \cap \mathcal{L})^2$. Whenever $\sum_{z \in \mathbb{Z}^d} c_{i,j+zT} > 0$, we deduce $(u_R(i) - u_R(j))^+ \leq 2\ell / (\sum_{z \in \mathbb{Z}^d} c_{i,j+zT}) < +\infty$. As a consequence, we find a uniform bound for $(u_R(i) - u_R(j))^+$ for all the pairs (i, j) which appear in the sum (111). This achieves the proof of the dual representation.

For the remainder of this section we assume (H1) and (H3).

Step 2.(Crystallinity) By Remark 2.5, we have

$$\varphi(\nu) = \sup_{\zeta \in W_\varphi} \langle \nu, \zeta \rangle.$$

We shall prove that (105) implies

$$W_\varphi = \left\{ \frac{1}{T^d} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (i - j) : (\alpha_{i,j})_{i,j} \in \mathcal{C} \right\}, \quad (112)$$

with \mathcal{C} given in (104). Recall N and \mathcal{E} defined in (103). Define $L: \mathbb{R}^N \rightarrow \mathbb{R}^d$ by

$$L(\alpha_{i,j})_{(i,j) \in \mathcal{E}} = \frac{1}{T^d} \sum_{(i,j) \in \mathcal{E}} \alpha_{i,j} (i - j). \quad (113)$$

Hence, we observe that

$$W := \left\{ \frac{1}{T^d} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (i - j) : (\alpha_{i,j})_{i,j} \in \mathcal{C} \right\} = L(\mathcal{C}),$$

where \mathcal{C} is a convex closed set as the intersection of two convex and closed sets. Thus, as the image of the convex and closed set \mathcal{C} through the linear map L , W is a closed and convex set. Then (112) follows by Fenchel duality, since

$$\varphi(\nu) = \mathbf{I}_W^*(\nu) = \mathbf{I}_{W_\varphi}^*(\nu),$$

where $\mathbf{I}_A: \mathbb{R}^d \rightarrow [0, +\infty]$ denotes the support function of the set A given by

$$\mathbf{I}_A(\zeta) = \begin{cases} 0 & \text{if } \zeta \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

Now, since both W_φ and W are closed and convex, we have

$$\mathbf{I}_{W_\varphi}(\zeta) = \mathbf{I}_{W_\varphi}^{**}(\zeta) = \varphi^*(\zeta) = \mathbf{I}_W^{**}(\zeta) = \mathbf{I}_W(\zeta).$$

This shows (112). Furthermore, we find

$$W_\varphi = L \left(V \cap \prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}] \right), \quad (114)$$

where $V \subset \mathbb{R}^N$ is a linear subspace of co-dimension $k := \#(Q_T \cap \mathcal{L}) - 1$ given by

$$V = \left\{ \alpha_{i,j} \in \mathbb{R}^N : \sum_{j \in \mathcal{L}} (\alpha_{i,j} - \alpha_{j,i}) = 0 \text{ for all } i \in Q_T \cap \mathcal{L} \right\}. \quad (115)$$

Hence, due to (114), W_φ is the image of the linear map L , given in (113), of a N -dimensional polytope $\prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}]$ intersected with the linear subspace V , given in (115). The intersection of a cube with a linear subspace is a polytope, and thus also its image through a linear map.

This proves that φ is crystalline.

Step 3.(Estimate on the number of vertices) Our goal is to prove that

$$\#\text{extreme}(W_\varphi) \leq 3^N, \quad (116)$$

where we recall N defined in (103). Let us note that, due to the Krein-Milman Theorem (cf. [20], Theorem 1.13) and (114), it is easy to see that there holds

$$\#\text{extreme}(W_\varphi) = \#\text{extreme} \left(L \left(V \cap \prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}] \right) \right) \leq \#\text{extreme} \left(V \cap \prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}] \right).$$

In order to show (116), it remains to show

$$\#\text{extreme} \left(V \cap \prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}] \right) \leq 3^N. \quad (117)$$

In order to obtain this estimate we note that the extreme points of $V \cap \prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}]$ lie on the k -dimensional (here k is the co-dimension of the linear subspace V) facets of $\prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}]$. Furthermore, we can find an injective relation between extreme points and these facets. In fact, if the matrix determining the intersection of the facet with V is full rank, then the point of intersection is unique. If that is not the case, then the solution set is itself a subspace and one can add one additional condition to obtain the extreme point. This implies that here the extreme point is shared by more k -dimensional facets on a lower dimensional facet. Note that there are at most $\binom{N}{k} 2^{N-k}$ such facets and by the binomial formula we have that

$$\binom{N}{k} 2^{N-k} \leq \sum_{j=0}^N \binom{N}{j} 2^{N-j} = 3^N.$$

This concludes Step 3. \square

5. DIFFERENTIABILITY OF THE EFFECTIVE SURFACE TENSION

In this Section, we prove Proposition 2.9 which states that φ is differentiable in totally irrational directions whenever there exists a solution in the primal problem. It is a corollary of the two lemmas which we state and prove below.

Lemma 5.1. *Let $\nu \in \mathbb{S}^{d-1}$, let u be a minimizer in (11) and assume that for any $s \in \mathbb{R}$, the set $\{u = s\}$ is finite. Then φ is differentiable in ν .*

Proof. The expression (105) shows that φ is a convex, one-homogeneous function with subgradient at ν given by

$$\partial\varphi(\nu) = \left\{ \frac{1}{T^d} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (i - j) : \alpha = (\alpha_{i,j})_{i,j} \in \mathcal{C} \text{ maximizer in (105)} \right\}$$

It is differentiable at ν if and only if the above set has exactly one element.

Let $\alpha, \alpha' \in \mathcal{C}$ be two maximizers in (105). Classical optimality conditions guarantee that for any i, j , if $u(i) \neq u(j)$, then:

$$\alpha_{i,j} = \alpha'_{i,j} = \begin{cases} c_{i,j} & \text{if } u(i) - u(j) > 0 \\ 0 & \text{if } u(i) - u(j) < 0. \end{cases} \quad (118)$$

Let us denote by $p, p' \in \partial\varphi(\nu)$ the subgradients given by the dual variables, respectively, α and α' , we claim that $p = p'$. One has:

$$p - p' = \frac{1}{T^d} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j: u(j)=u(i)} (\alpha_{i,j} - \alpha'_{i,j})(i - j). \quad (119)$$

Let $s \in \mathbb{R}$, $i_0 \in \mathcal{L} \cap Q_T$ with $u(i_0) = s$ and such that the finite set $J_s := \{j : u(j) = s\}$ has more than one element. For any i, j , let $\beta_{i,j} := \alpha_{i,j} - \alpha'_{i,j}$. Then

$$\begin{aligned} \sum_{i \in J_s} \sum_{j \in J_s} \beta_{i,j}(i - j) &= \sum_{z \in \mathbb{Z}^d} \sum_{i \in J_s \cap (Tz + Q_T)} \sum_{j \in J_s} \beta_{i,j}(i - j) \\ &= \sum_{z \in \mathbb{Z}^d} \sum_{i \in (J_s - Tz) \cap Q_T} \sum_{j \in J_s - Tz} \beta_{i,j}(i - j) \end{aligned}$$

where for the last line we have substituted (i, j) with $(i - Tz, j - Tz)$ and used that β is Q_T -periodic. In addition, we have that $u(i) = u(j)$ if and only if $u(i - Tz) = u(j - Tz)$ so that this can be rewritten:

$$\sum_{i \in J_s} \sum_{j \in J_s} \beta_{i,j}(i - j) = \sum_{z \in \mathbb{Z}^d} \sum_{i \in (J_s - Tz) \cap Q_T} \sum_{j: u(j)=u(i)} \beta_{i,j}(i - j)$$

By assumption, the sets $(J_s - Tz) \cap Q_T$, $z \in \mathbb{Z}^d$ are all disjoint. Otherwise, there would be i, z with $s = u(i - Tz) = u(i) + T\langle \nu, z \rangle = s$, yielding in particular that $\langle \nu, z \rangle = 0$, and one would deduce that $i - kTz \in J_s$ for all $k \in \mathbb{Z}$, a contradiction since we assumed J_s was finite. As a consequence, showing that (119) vanishes is equivalent to showing that

$$\sum_{i \in J_s} \sum_{j \in J_s} \beta_{i,j}(i - j) = 0 \quad (120)$$

for any $s \in \mathbb{R}$ (such that J_s is not empty and contains more than one point). Obviously, the expression in (120) is also

$$\sum_{i \in J_s} \sum_{j \in J_s} (\beta_{i,j} - \beta_{j,i})i$$

Thanks to the definition (104) of \mathcal{C} , one has for any i that $\sum_j \beta_{i,j} - \beta_{j,i} = 0$, so that:

$$\sum_{i \in J_s} \sum_{j \in J_s} (\beta_{i,j} - \beta_{j,i})i = \sum_{i \in J_s} \sum_{j \notin J_s} (\beta_{j,i} - \beta_{i,j})i = 0$$

thanks to (118). Hence, (120) holds and we deduce $p = p'$, which shows the lemma. \square

Lemma 5.2. *Let $\nu \in \mathbb{S}^{d-1}$ be totally irrational and let u be a minimizer in (11). Then for any $s \in \mathbb{R}$, the set $\{u = s\}$ is finite.*

Proof. Recalling the notation in the previous proof, let $s \in \mathbb{R}$ and consider the set $J_s := \{u = s\}$. For $z \in \mathbb{Z}^d$, let $J_s^z = J_s \cap (Q_T + Tz) - Tz \subset Q_T$. For $i \in J_s^z$, $u(i) = s + T\langle z, \nu \rangle$. Since ν is totally irrational, we deduce that $J_s^z \cap J_s^{z'} = \emptyset$ for any $z \neq z'$, showing that all sets J_s^z but a finite number must be empty. Hence J_s is finite. \square

6. NUMERICAL ILLUSTRATION

6.1. A simplified framework. In this section, we address, as an illustrative experiment, the following issue. We consider a basic 2D cartesian graph $\{(i, j) : 0 \leq i \leq M-1, 0 \leq j \leq N-1\}$, representing for instance the pixels of an image, and we want to approximate on this discrete grid the two-dimensional total variation $\int_{\Omega} |Du|$, $u \in BV(\Omega)$. Here it is assumed that $\Omega \subset \mathbb{R}^2$ is a rectangle and that $\{0, \dots, M-1\} \times \{0, \dots, N-1\}$ is a discretization of Ω at a length scale $\sim 1/N \sim 1/M$.

There are of course many ways to do this, but we propose here to consider a family of discrete “graph” total variations, defined for a family $(u_{i,j})_{i,j} \in \mathbb{R}^{M \times N}$ by:

$$J(u) = \sum_{i,j} c_{i+\frac{1}{2},j}^+ (u_{i+1,j} - u_{i,j})^+ + c_{i+\frac{1}{2},j}^- (u_{i,j} - u_{i+1,j})^+ \\ + c_{i,j+\frac{1}{2}}^+ (u_{i,j+1} - u_{i,j})^+ + c_{i,j+\frac{1}{2}}^- (u_{i,j} - u_{i,j+1})^+ \quad (121)$$

and which involves only nearest-neighbour interactions in horizontal and vertical directions.

We assume in addition that the weight c^{\pm} are T -periodic for some $T \in \mathbb{N}$, $T > 0$, that is, $c_{a+kT,b+lT}^{\pm} = c_{a,b}^{\pm}$ for any $(k, l) \in \mathbb{Z}^2$, $(a, b) = (i + \frac{1}{2}, j)$ or $(i, j + \frac{1}{2})$, as long as the points fall inside the grid.

For $T = 1$, $c_{a,b}^{\pm} \equiv 1$, it is standard that (121) approximates, in the continuum limit, the anisotropic total variation $\int_{\Omega} |\partial_1 u| + |\partial_2 u|$, which, if used for instance as a regularizer for image denoising or reconstruction, may produce undesired artefacts (although hardly visible on standard applications, see Figure 9).

A standard way to mitigate this issue (besides, of course, resorting to numerical analysis based on finite differences or elements in order to define more refined discretizations), is to add to (121) diagonal interactions, with appropriate weights, in order to improve the isotropy of the limit (see for instance [11]), with the drawback of complexifying the graph and the optimization. We show here that a similar effect can be attained by homogenization. To illustrate this, let us first consider the simplest situation, for $T = 2$.

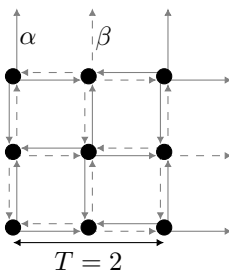
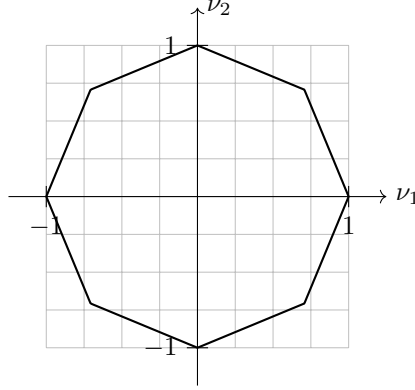


FIGURE 4. The alternating 2-periodic coefficients yielding the smallest anisotropy

In that case, one can explicitly build coefficients $c_{a,b}^{\pm}$, taking two values α, β (see Figure 4), which will yield the homogenized surface tension

$$\varphi(\nu) = (\sqrt{2} - 1) \left(|\nu_1| + |\nu_2| + \frac{|\nu_1 + \nu_2|}{\sqrt{2}} + \frac{|\nu_1 - \nu_2|}{\sqrt{2}} \right) \quad (122)$$

whose 1-level set (or *Frank diagram*) is shown in Figure 5. Observe that this is the same

FIGURE 5. The Frank diagram $\{\nu : \varphi(\nu) \leq 1\}$ given by (122)

anisotropy which would be obtained by using constant coefficients and adding interactions along the edges $((i, j), (i + 1, j + 1))$ and $((i, j), (i + 1, j - 1))$.

In order to obtain (122), one needs to tune α, β so that a vertical edge and a diagonal edge, in the most favorable position, have the same length (with a $\sqrt{2}$ factor for the diagonal, whose intersection with the periodicity cell is of course longer). This is ensured if $\alpha + \beta = 4\alpha/\sqrt{2}$, that is, $\beta = (2\sqrt{2} - 1)\alpha$. We find that choosing

$$\begin{cases} \alpha = \frac{1}{4\sqrt{2}} \approx 0.1768 \\ \beta = (2\sqrt{2} - 1)\alpha \approx 0.3232 \end{cases} \quad (123)$$

yields (122), as an effective homogenized anisotropy.

For larger periodicity cells, it seems difficult to do a similar analysis, first of all, because one should not expect the optimal minimizers, in most directions (if not all), to be given by straight lines, but rather by periodic perturbations of straight lines. We propose an optimization process in order to compute the optimal weights $c_{a,b}^{\pm}$.

6.2. The optimization method. The effective surface tension is obtained by solving the cell problem:

$$\begin{aligned} \phi(\nu) = \min_u \left\{ \sum_{(i,j) \in Y} c_{i+\frac{1}{2},j}^+(u_{i+1,j} - u_{i,j})^+ + c_{i+\frac{1}{2},j}^-(u_{i,j} - u_{i+1,j})^+ \right. \\ \left. + c_{i,j+\frac{1}{2}}^+(u_{i,j+1} - u_{i,j})^+ + c_{i,j+\frac{1}{2}}^-(u_{i,j} - u_{i,j+1})^+ : \right. \\ \left. u_{i,j} - \nu \cdot \begin{pmatrix} i \\ j \end{pmatrix} Y\text{-periodic} \right\} \end{aligned} \quad (124)$$

where $Y = \mathbb{Z}^2 \cap ([0, T) \times [0, T))$ is the periodicity cell. This is easily solved, for instance by a saddle-point algorithm [26] which aims at finding a solution to:

$$\begin{aligned} \phi(\nu) = \min_{v \text{ } Y\text{-periodic}} \max_{0 \leq w_{\bullet}^{\pm} \leq 1} \sum_{(i,j) \in Y} (w_{i+\frac{1}{2},j}^+ c_{i+\frac{1}{2},j}^+ - w_{i+\frac{1}{2},j}^- c_{i+\frac{1}{2},j}^-)(v_{i+1,j} - v_{i,j} + \nu_1) \\ + (w_{i,j+\frac{1}{2}}^- c_{i,j+\frac{1}{2}}^- - w_{i,j+\frac{1}{2}}^+ c_{i,j+\frac{1}{2}}^+)(v_{i,j+1} - v_{i,j} + \nu_2), \end{aligned}$$

where we have replaced the variable u with the periodic vector $v_{i,j} = u_{i,j} - \nu \cdot (i, j)^T$. For technical reasons, we need to “regularize” slightly this problem in order to make it differentiable with respect to the coefficients $\mathbf{c} = (c_{\bullet}^{\pm})$. This is done by introducing $\varepsilon > 0$ a (very) small parameter and adding to the previous objective the penalization

$$-\frac{\varepsilon}{2} \sum_{(i,j) \in Y} (w_{i+\frac{1}{2},j}^+)^2 + (w_{i+\frac{1}{2},j}^-)^2 + (w_{i,j+\frac{1}{2}}^-)^2 + (w_{i,j+\frac{1}{2}}^+)^2 + \frac{\varepsilon}{2} \sum_{(i,j) \in Y} v_{i,j}^2$$

which makes the problem strongly convex/concave and the solutions w, v unique. We call $\phi_{\varepsilon}(\nu)[\mathbf{c}]$ the corresponding value. The advantage of this regularization is that one can easily show that $\mathbf{c} \mapsto \phi_{\varepsilon}(\nu)[\mathbf{c}]$ is locally $C^{1,1}$, with a gradient given by:

$$\lim_{t \rightarrow 0} \frac{\phi_{\varepsilon}(\nu)[\mathbf{c} + t\mathbf{d}] - \phi_{\varepsilon}(\nu)[\mathbf{c}]}{t} = \sum_{(i,j) \in Y} (w_{i+\frac{1}{2},j}^+ d_{i+\frac{1}{2},j}^+ - w_{i+\frac{1}{2},j}^- d_{i+\frac{1}{2},j}^-)(v_{i+1,j} - v_{i,j} + \nu_1) + (w_{i,j+\frac{1}{2}}^- d_{i,j+\frac{1}{2}}^- - w_{i,j+\frac{1}{2}}^+ d_{i,j+\frac{1}{2}}^+)(v_{i,j+1} - v_{i,j} + \nu_2)$$

where (w, v) solves the saddle-point problem which defines $\phi_{\varepsilon}(\nu)[\mathbf{c}]$.

Then, to find coefficients which ensure that ϕ is as “isotropic” as possible, one fixes a finite set of directions (ν_1, \dots, ν_k) (typically, $(\cos(2\ell\pi/k), \sin(2\ell\pi/k))$ for $\ell = 1, \dots, k$), and uses a first order gradient descent algorithm to optimize:

$$\mathcal{L}(\mathbf{c}) = \sum_{\ell=1}^k (\phi_{\varepsilon}(\nu_{\ell})[\mathbf{c}] - 1)^2$$

The problem is easily solved for $Y = \{0, 1\} \times \{0, 1\}$, $k = 8$ and $(\nu_{\ell})_{\ell=1}^8$ given as above. For larger periodicity cells and more directions, it easily gets trapped in local minima and we use a random initialization in order to be able to find satisfactory solutions. We then test the result by computing the un-regularized surface tension ϕ with the resulting coefficients \mathbf{c} . We show some results in the next section. Of course, taking a large value of ε will make the problem easier to solve, but the learned coefficients will not allow to reconstruct a satisfactory surface tension: we need to choose ε small, an order of magnitude below the error which we expect on the anisotropy of ϕ .

6.3. Numerical results. We show the outcome of the optimization, in the periodicity cell $Y = \{0, \dots, T-1\} \times \{0, \dots, T-1\}$ for $T = 2, 4, 6, 8$. We plot first the set $\{\varphi \leq 1\}$ or Frank diagram for the effective surface tensions. Figure 6 shows the diagram obtained, for $T = 2, 4, 8$.

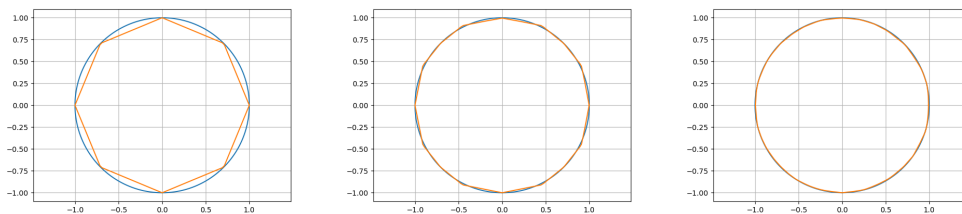


FIGURE 6. Frank diagrams of the effective anisotropies for $T = 2, 4, 8$.

For $T = 2$, the optimization yields the same anisotropy as our construction in (123) which gives the anisotropy (122) and which we conjecture to be optimal for $\mathcal{L}(\mathbf{c})$ taking $k = 8$ (compare

with Fig. 5). However, except when initialized with the values in (123), the algorithm usually outputs different values with the same effective anisotropy, see Fig. 7 (the values in (123) are in some sense better, as for instance a vertical edge will always have the same effective energy with these values, while with the computed values displayed in Fig. 7, it will need to pass through the edges in the second column of the cell in order to get the minimal energy).

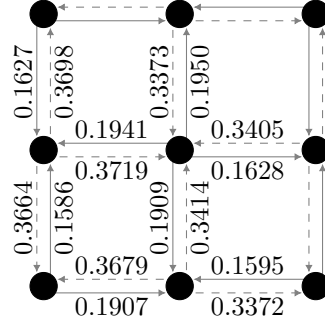


FIGURE 7. An example of optimized 2-periodic coefficients yielding the same anisotropy as the choice (123)



FIGURE 8. A minimal half-plane in the orientation $(\cos 3\pi/8, \sin 3\pi/8)$. Top left, boundary datum, the region where the perimeter is minimized is in gray. Top, middle: $\varphi(\nu) = |\nu_1| + |\nu_2|$. Top, right: optimal effective φ for $T = 2$. Bottom: for $T = 4, 6, 8$.

For $T = 4$, one sees that the behavior is almost isotropic, while for $T = 8$, the relative error with the perfect unit disk is about 1%. Here, we estimated this error as $(\max_{\ell} \phi(\nu_{\ell}) - \min_{\ell} \phi(\nu_{\ell})) / \min_{\ell} \phi(\nu_{\ell})$, where $\ell \in \{1, \dots, k\}$ and $k = 180$. We illustrate this on an “inpainting” example, which consists in finding the minimal line in a given direction. We consider as an example the direction $(\cos 3\pi/8, \sin 3\pi/8)$, which is irrational, so that there cannot be a fully

periodic solution. The figure 8 displays several minimal half-planes in this orientation. Observe that for this orientation, the results for $T = 4$ or 6 look nicer than the result obtained for $T = 8$.

We also show a denoising example based on the “ROF” method (which consists simply in minimizing the total variation (defined by the surface tension φ) of an image with a quadratic penalization of the distance to a noisy data, in order to produce a denoised version, see [44]) with the anisotropic tension $\varphi(\nu) = |\nu_1| + |\nu_2|$ (“ $T = 1$ ”) and the optimized homogenized surface tension for $T = 4$. The original image is degraded with a Gaussian noise with 10% standard deviation (with respect to the range of the values). Here, the difference between the two regularizers is hardly perceptible (since the data term strongly influences the position of the discontinuities), yet a close-up (bottom row) allows to see a slight difference, for instance on the cheek where the $T = 1$ anisotropy produces block structures.



FIGURE 9. “ROF” denoising example. Left: noisy image. Middle, denoised with $\varphi(\nu) = |\nu_1| + |\nu_2|$. Right: with the effective tension computed for $T = 4$.

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