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# Controllability results for cascade systems of $m$ coupled $N$ -dimensional Stokes and Navier-Stokes systems by $N - 1$ scalar controls

Takéo Takahashi <sup>\*1</sup>, Luz de Teresa <sup>†2</sup>, and Ying Wu-Zhang<sup>‡2</sup>

<sup>1</sup>Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France

<sup>2</sup>Instituto de Matematicas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., C. P. 04510 Ciudad de México, México

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## Abstract

In this paper we deal with the controllability properties of a system of  $m$  coupled Stokes systems or  $m$  coupled Navier-Stokes systems. We show the null-controllability of such systems in the case where the coupling is in a cascade form and when the control acts only on one of the systems. Moreover, we impose that this control has a vanishing component so that we control a  $m \times N$  state (corresponding to the velocities of the fluids) by  $N - 1$  distributed scalar controls. The proof of the controllability of the coupled Stokes system is based on a Carleman estimate for the adjoint system. The local null-controllability of the coupled Navier-Stokes systems is then obtained by means of the source term method and a Banach fixed point.

**Keywords:** Null controllability, Navier-Stokes systems, Carleman estimates

**2010 Mathematics Subject Classification.** 76D05, 35Q30, 93B05, 93B07, 93C10

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\*takeo.takahashi@inria.fr

†ldeteresa@im.unam.mx. Partially supported by Conacyt, project A1-S-17475

‡yngwu@gmail.com

# 1 Introduction

Controllability issues related to a single parabolic equation or to a single Stokes or Navier-Stokes system have been intensively studied in the last fifty years giving rise to interesting techniques, new challenges and open problems. See some seminal results [12, 16, 25] for the heat equation and [8, 14, 21] for the Navier-Stokes system. The literature is vast and it is difficult to mention all the intensive studies about this subject. However, it is only in the last fifteen years that the challenging issue of controlling coupled parabolic systems has attracted the interest of the control community. This kind of systems appears mathematically in optimal control theory as a characterization of the optimal control (with one equation coupled to its adjoint) but also appears, for example, in the study of chemical reactions (see e.g. [11], [7]), and in a wide variety of mathematical biology and physical situations (see e.g. [20]). In the case of scalar (heat) coupled equations an important number of challenging problems has been solved (see [1] for a survey of results until 2011) and sometimes the results have been surprising [2–4]. In the case of coupled Stokes or Navier-Stokes systems, to our knowledge, only some cases of *two* coupled systems have been treated [5, 6, 18, 29]. Here our aim is to generalize results for a  $m$  scalar cascade system [17] to a  $m$   $N$ -dimensional Stokes or Navier-Stokes cascade system but including an extra deal: to eliminate one component on the  $N$ -dimensional control.

Let us be more specific: we consider a bounded domain  $\Omega$  of  $\mathbb{R}^N$  ( $N = 2, 3$ ) whose boundary  $\partial\Omega$  is regular enough. Let  $T > 0$  and let  $\omega \subset \Omega$  be a (arbitrary small) nonempty open subset which will usually be referred as the *control domain*. We will use the notation  $Q = \Omega \times (0, T)$  and  $\Sigma = \partial\Omega \times (0, T)$ .

In this article, we are interested in the null controllability of a coupled system of  $m$  Stokes or Navier-Stokes systems, with  $m \geq 2$ :

$$\begin{cases} \partial_t y^{(i)} - \Delta y^{(i)} + \nabla p^{(i)} + \varepsilon (y^{(i)} \cdot \nabla) y^{(i)} = \sum_{j=1}^m A_{i,j} y^{(j)} + D_i v 1_\omega & \text{in } Q, \quad (1 \leq i \leq m) \\ \nabla \cdot y^{(i)} = 0 & \text{in } Q, \quad (1 \leq i \leq m) \\ y^{(i)} = 0 & \text{on } \Sigma, \quad (1 \leq i \leq m) \\ y^{(i)}(\cdot, 0) = y_0^{(i)} & \text{in } \Omega, \quad (1 \leq i \leq m) \end{cases}$$

with  $\varepsilon = 0$  for Stokes systems and  $\varepsilon = 1$  for Navier-Stokes systems, where  $A_{i,j} \in \mathcal{M}_N(\mathbb{R})$  and where  $D_i \in \mathcal{M}_{N,r}(\mathbb{R})$  for some  $r \in \mathbb{N}^*$ . We have denoted by  $1_\omega$  the characteristic function of  $\omega$ . Let us notice that we take in this work the viscosities of the fluids constant and equal to 1 to simplify.

We can write the above systems in a more compact way as

$$\begin{cases} \partial_t y - \Delta y + \nabla p = Ay + Dv 1_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y|_{t=0} = y_0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

or

$$\begin{cases} \partial_t y - \Delta y + (y \cdot \nabla)y + \nabla p = Ay + Dv 1_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y|_{t=0} = y_0 & \text{in } \Omega, \end{cases} \quad (1.2)$$

where we set

$$\begin{aligned} y &= (y^{(1)}, \dots, y^{(m)}), \quad p = (p^{(1)}, \dots, p^{(m)}), \quad y_0 = (y_0^{(1)}, \dots, y_0^{(m)}), \\ \Delta y &= (\Delta y^{(1)}, \dots, \Delta y^{(m)}), \quad \nabla p = (\nabla p^{(1)}, \dots, \nabla p^{(m)}), \\ (y \cdot \nabla)y &= ((y^{(1)} \cdot \nabla)y^{(1)}, \dots, (y^{(m)} \cdot \nabla)y^{(m)}), \quad \nabla \cdot y = (\nabla \cdot y^{(1)}, \dots, \nabla \cdot y^{(m)}) \end{aligned}$$

and

$$Ay = \left( \sum_{j=1}^m A_{1,j} y^{(j)}, \dots, \sum_{j=1}^m A_{m,j} y^{(j)} \right) \quad Dv = (D_1 v, \dots, D_m v).$$

In this work, we will focus in the particular case where the partitioned matrix  $A$  has the form

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \dots & A_{1,m} \\ A_{2,1} & A_{2,2} & A_{2,3} & \dots & A_{2,m} \\ 0 & A_{3,2} & A_{3,3} & \dots & A_{3,m} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & A_{m,m-1} & A_{m,m} \end{pmatrix} \quad (1.3)$$

with all the blocks under the diagonal non zero and such that all its blocks are scalar matrices. More precisely, our hypotheses on  $A$  are

$$A_{i,j} = a_{i,j}I_N, \quad a_{i,i-1} \neq 0 \quad (2 \leq i \leq m), \quad a_{i,j} = 0 \quad \text{if } i \geq j + 2. \quad (1.4)$$

We also control (1.1) or (1.2) by acting only on one of the Stokes or of the Navier-Stokes systems, for instance the first one, and with  $N - 1$  scalar controls on this system. Thus, without loss of generality, we assume

$$r = N - 1, \quad D_j = 0 \quad (j \geq 2), \quad D_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\text{if } N = 2) \quad \text{or} \quad D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (\text{if } N = 3). \quad (1.5)$$

The above choice of the matrix  $A$  corresponds to a particular coupling considered in the context of the null-controllability of systems of  $m$  linear heat equations, see [17] and our aim is to extend this result in the case of coupled of Stokes or Navier-Stokes systems.

In order to state our main results, we recall some standard functional spaces associated with the Stokes system:

$$H = \{y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega, y \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \quad (1.6)$$

and

$$V = \{y \in H_0^1(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega\}. \quad (1.7)$$

Our main result is the following theorem.

**Theorem 1.1.** *Assume (1.4) and (1.5). Then, for any  $T > 0$  and for any  $y_0 \in H^m$ , there exists a control  $v \in L^2(0, T; L^2(\omega)^{N-1})$  such that the corresponding solution  $y = (y^{(1)}, \dots, y^{(m)})$  to (1.1) satisfies*

$$y(\cdot, T) = 0 \quad \text{in } \Omega.$$

**Remark 1.2.** As a consequence, we deduce that we can control the system (1.1) of  $N \times m$  scalar equations with  $N - 1$  scalar controls.

From Theorem 1.1 and a general method to deal with the controllability of nonlinear parabolic systems, we deduce the local null controllability of the system (1.2):

**Theorem 1.3.** *Assume (1.4) and (1.5). Then, for any  $T > 0$ , there exists  $\delta > 0$  such that, for any  $y_0 \in V^m$  satisfying*

$$\|y_0\|_{V^m} \leq \delta,$$

*there exists  $v \in L^2(0, T; L^2(\omega)^{N-1})$  such that the corresponding solution  $y$  to (1.2) satisfies*

$$y(\cdot, T) = 0 \quad \text{in } \Omega.$$

In order to prove Theorem 1.1, we introduce the adjoint system:

$$\begin{cases} -\partial_t \varphi^{(i)} - \Delta \varphi^{(i)} + \nabla \pi^{(i)} = \sum_{j=1}^{i+1} A_{j,i} \varphi^{(j)} & \text{in } Q, \quad (1 \leq i \leq m-1) \\ -\partial_t \varphi^{(m)} - \Delta \varphi^{(m)} + \nabla \pi^{(m)} = \sum_{j=1}^m A_{j,m} \varphi^{(j)} & \text{in } Q, \\ \nabla \cdot \varphi^{(i)} = 0 & \text{in } Q, \quad (1 \leq i \leq m) \\ \varphi^{(i)} = 0 & \text{on } \Sigma, \quad (1 \leq i \leq m) \\ \varphi^{(i)}(\cdot, T) = \varphi_T^{(i)} & \text{in } \Omega, \quad (1 \leq i \leq m) \end{cases} \quad (1.8)$$

with  $\varphi_T^{(i)} \in H$  ( $1 \leq i \leq m$ ) and we denote by  $\varphi_j^{(i)}$ ,  $j = 1, \dots, N$  the coordinates of  $\varphi^{(i)}$ . Note that by setting

$$\varphi^{(m+1)} \equiv 0, \quad a_{m+1,i} = 0 \quad (1 \leq i \leq m),$$

we can write the first above equations as

$$-\partial_t \varphi^{(i)} - \Delta \varphi^{(i)} + \nabla \pi^{(i)} = \sum_{j=1}^{i+1} a_{j,i} \varphi^{(j)} \quad \text{in } Q, \quad (1 \leq i \leq m). \quad (1.9)$$

Following a standard duality argument (see, for instance, [34, Theorem 11.2.1, p.357]), Theorem 1.1 will be obtained as a consequence of the following observability inequality:

$$\int_{\Omega} \sum_{i=1}^m \left| \varphi^{(i)}(x, 0) \right|^2 dx \leq C(T) \int \int_{\omega \times (0, T)} \sum_{j=1}^{N-1} \left| \varphi_j^{(1)} \right|^2 dx dt. \quad (1.10)$$

for some  $C$  depending on  $T$ ,  $\Omega$  and  $\omega$ .

The proof of (1.10) is based on (*global*) Carleman inequalities introduced in [16] for the controllability of parabolic equations. Such inequalities have been used by many authors to deal with Stokes or Navier-Stokes systems (for instance, [22] or [14]). The case of controls with some vanishing components was considered in [15], [9] and [6]. We follow here the method introduced in [9]. As a first step, one can get rid of the pressure by applying a differential operator on (1.9) (or on components of (1.9)) such as curl or  $\Delta$ . This leads to a system of coupled heat equations but without prescribed boundary conditions. Using results such as [13] or [24], one can obtain a Carleman estimate with boundary terms that can be absorbed by some standard arguments. Let us note that to recover the observability on the components that are not observed, one has to use the divergence-free condition on  $\varphi^{(i)}$  and the Dirichlet boundary conditions. Unhappily in this process, one loses part of the weights on these components. The last part of the proof of (1.10) consists in estimating the local terms associated with  $\varphi_j^{(i)}$ ,  $i > 1$  and this is done by using (1.4). An important work related to this subject is [10] where the authors obtain the local null controllability of the Navier-Stokes system in dimension 3 with a control having two vanishing components. In that case, the method is based on a different linearization and on a different approach based on results of Gromov.

This paper is organized as follows. In Section 2, we introduce the weights for our Carleman estimates and we recall some results, in particular some Carleman estimates for other systems. Section 3 corresponds to the statement and to the proof of the Carleman estimate for (1.8). Finally, in Section 4, we use this estimate to prove Theorem 1.1 and Theorem 1.3.

## 2 Preliminaries

In the whole article, we use the notation  $C$  for a generic positive constant that depends on  $\Omega, \omega$ . We can assume to simplify that  $T \in (0, 1)$  and that will allow us to avoid some dependence on  $T$  for some constants. Finally we only write the proof in the case  $N = 2$ , the case  $N = 3$  can be done similarly.

To write our Carleman inequalities, we introduce standard weights and functions. First, we consider a nonempty domain  $\omega_0$  such that  $\overline{\omega_0} \subset \omega$ . Then, using [16] (see also [34, Theorem 9.4.3, p.299]), there exists  $\eta^0 \in C^2(\overline{\Omega})$  satisfying

$$\eta^0 > 0 \text{ in } \Omega, \quad \eta^0 = 0 \text{ on } \partial\Omega, \quad \max_{\Omega} \eta^0 = 1, \quad \nabla \eta^0 \neq 0 \text{ in } \overline{\Omega} \setminus \overline{\omega_0}. \quad (2.1)$$

Then, we define the following functions:

$$\alpha(x, t) = \frac{\exp\{\lambda(2\ell + 2)\} - \exp\{\lambda(2\ell + \eta^0(x))\}}{t^\ell(T - t)^\ell}, \quad \xi(x, t) = \frac{\exp\{\lambda(2\ell + \eta^0(x))\}}{t^\ell(T - t)^\ell}, \quad (2.2)$$

$$\alpha^*(t) = \max_{x \in \Omega} \alpha(x, t) = \frac{\exp\{\lambda(2\ell + 2)\} - \exp\{2\lambda\ell\}}{t^\ell(T - t)^\ell}, \quad \xi^*(t) = \min_{x \in \Omega} \xi(x, t) = \frac{\exp\{2\lambda\ell\}}{t^\ell(T - t)^\ell}, \quad (2.3)$$

where  $\ell \geq 14$ ,  $\lambda > 1$ . In literature  $\ell > 3$  is enough. However, to prove our result we need to use  $\ell \geq 14$  (see Proof of Proposition 3.2).

Note that we have the following useful relations: there exists  $C > 0$  depending on  $\Omega$  such that

$$|\partial_t \alpha| + |\partial_t \xi| \leq CT \xi^{1+1/\ell}, \quad (2.4)$$

$$|(\alpha^*)'| + |(\xi^*)'| \leq CT (\xi^*)^{1+1/\ell}, \quad |(\alpha^*)''| + |(\xi^*)''| \leq CT^2 (\xi^*)^{1+2/\ell}, \quad (2.5)$$

$$\xi^* \geq \frac{C}{T^{2\ell}}, \quad (2.6)$$

$$|\nabla \alpha| = |\nabla \xi| \leq C\lambda \xi, \quad |\Delta \alpha| = |\Delta \xi| \leq C\lambda^2 \xi. \quad (2.7)$$

Weights of the kind (2.2) were first considered in [16]. In its present form, these weights have already been used in [19] in order to obtain Carleman estimates for the controllability of strongly coupled parabolic equations and later in [18] for the existence of insensitizing controls for Stokes systems.

Now, we recall some standard results. The first one is a Carleman estimate for the gradient operator, it is stated and proved in [9]:

**Lemma 2.1.** *Let  $r \in \mathbb{R}$ . There exists  $C > 0$  depending only on  $r, \Omega$  and  $\omega_0$  such that, for every  $T > 0$ ,  $s \geq C$  and every  $u \in L^2(0, T; H^1(\Omega))$ ,*

$$\iint_Q e^{-2s\alpha} s^{r+2} \xi^{r+2} |u|^2 dxdt \leq C \left( \iint_Q e^{-2s\alpha} s^r \xi^r |\nabla u|^2 dxdt + \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} s^{r+2} \xi^{r+2} |u|^2 dxdt \right).$$

The second result is a Carleman estimate for the heat equation with non-homogeneous Dirichlet boundary conditions. It is proved in [24].

**Lemma 2.2.** *There exists a constant  $C > 0$  such that for any  $\lambda \geq C$ ,  $s \geq C$ ,  $f_0, f_1, \dots, f_N \in L^2(Q)$  and  $u \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$*

*satisfying*

$$\partial_t u - \Delta u = f_0 + \sum_{j=1}^N \partial_j f_j \quad \text{in } Q,$$

*we have*

$$\begin{aligned} \iint_Q e^{-2s\alpha} (s^{-1} \xi^{-1} |\nabla u|^2 + s \xi |u|^2) dxdt &\leq C \left( \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} s \xi |u|^2 dxdt \right. \\ &+ \left\| e^{-s\alpha^*} s^{-1/4} (\xi^*)^{-1/4+1/\ell} u \right\|_{L^2(\Sigma)}^2 + \left\| e^{-s\alpha^*} s^{-1/4} (\xi^*)^{-1/4} u \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)}^2 \\ &\left. + \iint_Q e^{-2s\alpha} s^{-2} \xi^{-2} |f_0|^2 dxdt + \sum_{j=1}^N \iint_Q e^{-2s\alpha} |f_j|^2 dxdt \right). \quad (2.8) \end{aligned}$$

Recall that

$$\|u\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)} = \left( \|u\|_{H^{1/4}(0, T; L^2(\partial\Omega))}^2 + \|u\|_{L^2(0, T; H^{1/2}(\partial\Omega))}^2 \right)^{1/2}.$$

The next result concerns the regularity of the solutions of the Stokes system

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f & \text{in } Q, \\ \nabla \cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \quad (2.9)$$

This result can be found in [33].

**Lemma 2.3.** *Assume  $T > 0$ . For any  $f \in L^2(Q)^N$ , there exists a unique solution*

$$u \in L^2(0, T; H^2(\Omega)^N) \cap H^1(0, T; H), \quad p \in L^2(0, T; H^1(\Omega)/\mathbb{R})$$

to the Stokes system (2.9) and there exists a constant  $C > 0$  depending only on  $\Omega$  such that

$$\|u\|_{L^2(0, T; H^2(\Omega)^N)}^2 + \|u\|_{H^1(0, T; L^2(\Omega)^N)}^2 \leq C \|f\|_{L^2(0, T; L^2(\Omega)^N)}^2. \quad (2.10)$$

Assume moreover that

$$f \in L^2(0, T; V)$$

then

$$u \in L^2(0, T; H^3(\Omega)^N) \cap H^1(0, T; V)$$

and there exists a constant  $C > 0$  depending only on  $\Omega$  such that

$$\|u\|_{L^2(0, T; H^3(\Omega)^N)}^2 + \|u\|_{H^1(0, T; V)}^2 \leq C \|f\|_{L^2(0, T; V)}^2. \quad (2.11)$$

Assume moreover that

$$f \in L^2(0, T; H^3(\Omega)^N) \cap H^1(0, T; V), \quad f(\cdot, 0) = 0 \text{ in } \Omega.$$

Then,  $u \in L^2(0, T; H^5(\Omega)^N) \cap H^1(0, T; H^3(\Omega)^N) \cap H^2(0, T; V)$  and there exists a constant  $C > 0$  depending only on  $\Omega$  such that

$$\|u\|_{L^2(0, T; H^5(\Omega)^N)}^2 + \|u\|_{H^1(0, T; H^3(\Omega)^N)}^2 + \|u\|_{H^2(0, T; V)}^2 \leq C \left( \|f\|_{L^2(0, T; H^3(\Omega)^N)}^2 + \|\partial_t f\|_{L^2(0, T; V)}^2 \right).$$

### 3 The Carleman estimate for the adjoint system

As before, we denote by  $C$  various positive constants which depends only on  $\Omega$  and  $\omega$  (they depend also in general on the choice of  $\eta^0$  and  $\omega_0$  but one can consider that  $\eta^0$  as well as  $\omega_0$  depend  $\Omega$  and  $\omega$ ). Without any lack of generality, we treat the case of dimension  $N = 2$  and  $j = 2$ . The same proof can be performed in the general case.

Our aim is to estimate the following quantity associated with the solutions of the system (1.8):

$$\begin{aligned} I_1(s, \varphi^{(i)}) := & \iint_Q e^{-2s\alpha} \left( s^{-1}\xi^{-1} \left| \nabla^3 \Delta \varphi_1^{(i)} \right|^2 + s\xi \left| \nabla^2 \Delta \varphi_1^{(i)} \right|^2 + s^3 \xi^3 \left| \nabla \Delta \varphi_1^{(i)} \right|^2 + s^5 \xi^5 \left| \Delta \varphi_1^{(i)} \right|^2 \right) dxdt \\ & + \iint_Q e^{-2s\alpha^*} (s\xi^*)^5 \left| \varphi^{(i)} \right|^2 dxdt. \end{aligned} \quad (3.1)$$

Thus, our main result states as follows:

**Theorem 3.1.** *There exists  $C > 0$  depending on the geometry such that for any  $s \geq C$  and for any  $\varphi_T \in H^m$ , the solution  $(\varphi^{(1)}, \dots, \varphi^{(m)})$  of (1.8) satisfies*

$$\sum_{i=1}^m I_1(s, \varphi^{(i)}) \leq C \iint_{\omega \times (0, T)} e^{-2s\alpha} s^{2m+3-6} \xi^{2m+3-6} \left| \varphi_1^{(1)} \right|^2 dxdt. \quad (3.2)$$

In order to prove the above proposition, we first start by estimating each  $I_1(s, \varphi^{(i)})$  ( $i = 1, \dots, m$ ) independently.

**Proposition 3.2.** *Let  $\widehat{\omega} \subset \Omega$  be a nonempty open set such that  $\omega_0 \in \widehat{\omega}$ . Then, there exists a constant  $C$  such that for any  $s \geq C$  and for any  $\varphi_T \in H^m$ , the solution  $\varphi$  of (1.8) satisfies*

$$I_1(s, \varphi^{(i)}) \leq C \left( \iint_{\widehat{\omega} \times (0, T)} e^{-2s\alpha} s^5 \xi^5 \left| \Delta \varphi_1^{(i)} \right|^2 dxdt + \sum_{j=1}^m \iint_{\widehat{Q}} \left( e^{-2s\alpha^*} s^{9/2} (\xi^*)^5 \left| \varphi^{(j)} \right|^2 + e^{-2s\alpha} s^{-2} \xi^{-2} \left| \nabla^2 \Delta \varphi_1^{(j)} \right|^2 \right) dxdt \right) \quad (1 \leq i \leq m). \quad (3.3)$$

*Proof of Proposition 3.2.* First taking the divergence of (1.9), we remark that

$$\Delta \pi^{(i)} = 0 \quad \text{in } Q \quad (1 \leq i \leq m).$$

Thus, using a standard method for the Stokes system (see, for instance [9]), we apply the operator  $\nabla^2 \Delta$  to the first components of (1.9), and we deduce

$$-\partial_t \nabla^2 \Delta \varphi_1^{(i)} - \Delta \nabla^2 \Delta \varphi_1^{(i)} = \sum_{j=1}^{i+1} a_{j,i} \nabla^2 \Delta \varphi_1^{(j)} \quad \text{in } Q, \quad (1 \leq i \leq m). \quad (3.4)$$

Applying Lemma 2.2 to the above equations, we deduce that for  $\lambda \geq \widehat{\lambda}_1$  and for  $s \geq \widehat{s}_1$ ,

$$\begin{aligned} & \iint_Q e^{-2s\alpha} \left( s^{-1} \xi^{-1} \left| \nabla^3 \Delta \varphi_1^{(i)} \right|^2 + s \xi \left| \nabla^2 \Delta \varphi_1^{(i)} \right|^2 \right) dxdt \\ & \leq C \left( \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} s \xi \left| \nabla^2 \Delta \varphi_1^{(i)} \right|^2 dxdt + \left\| e^{-s\alpha^*} s^{-1/4} (\xi^*)^{-1/4+1/\ell} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{L^2(\Sigma)^4}^2 \right. \\ & \quad \left. + \left\| e^{-s\alpha^*} (s \xi^*)^{-1/4} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)^4}^2 + \sum_{j=1}^{i+1} \iint_Q e^{-2s\alpha} s^{-2} \xi^{-2} \left| \nabla^2 \Delta \varphi_1^{(j)} \right|^2 dxdt \right). \quad (3.5) \end{aligned}$$

The rest of the proof is divided into several steps:

- In Step 1, we complete the left-hand side of (3.5) with weighted integrals of  $\varphi^{(i)}$  in  $Q$ , and adding some local terms in the right-hand side.
- In Step 2, we obtain an upper bound of the boundary terms.
- Finally, in Step 3, we estimate the local terms that do not appear in (3.3).

**Step 1.** We apply Lemma 2.1 with  $u = \nabla \Delta \varphi_1^{(i)}$  and  $r = 1$ : for any  $s \geq C$  and  $\lambda \geq C$ ,

$$\iint_Q e^{-2s\alpha} s^3 \xi^3 \left| \nabla \Delta \varphi_1^{(i)} \right|^2 dxdt \leq C \left( \iint_Q e^{-2s\alpha} s \xi \left| \nabla^2 \Delta \varphi_1^{(i)} \right|^2 dxdt + \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} s^3 \xi^3 \left| \nabla \Delta \varphi_1^{(i)} \right|^2 dxdt \right). \quad (3.6)$$

Then, we apply Lemma 2.1 with  $u = \Delta \varphi_1^{(i)}$  and  $r = 3$ : for any  $s \geq C$  and  $\lambda \geq C$ ,

$$\iint_Q e^{-2s\alpha} s^5 \xi^5 \left| \Delta \varphi_1^{(i)} \right|^2 dxdt \leq C \left( \iint_Q e^{-2s\alpha} s^3 \xi^3 \left| \nabla \Delta \varphi_1^{(i)} \right|^2 dxdt + \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} s^5 \xi^5 \left| \Delta \varphi_1^{(i)} \right|^2 dxdt \right). \quad (3.7)$$

Now, using the divergence condition of  $\varphi^{(i)}$ , we have

$$\left| \partial_2 \varphi_2^{(i)} \right| = \left| \partial_1 \varphi_1^{(i)} \right| \leq \left| \nabla \varphi_1^{(i)} \right|.$$

Then using the Poincaré inequality and the ellipticity of the Laplace operator with Dirichlet boundary conditions, we deduce the existence of a constant  $C$  depending on  $\Omega$  such that

$$\int_{\Omega} \left| \varphi^{(i)} \right|^2 dx \leq C \int_{\Omega} \left| \Delta \varphi_1^{(i)} \right|^2 dx.$$

Combining the above relation with (3.5), (3.6) and (3.7), we deduce that  $I_1(s, \varphi^{(i)})$  defined by (3.1) satisfies

$$\begin{aligned} I_1(s, \varphi^{(i)}) \leq & C \left( \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \left( s\xi \left| \nabla^2 \Delta \varphi_1^{(i)} \right|^2 + s^3 \xi^3 \left| \nabla \Delta \varphi_1^{(i)} \right|^2 + s^5 \xi^5 \left| \Delta \varphi_1^{(i)} \right|^2 \right) dxdt \right. \\ & + \left\| e^{-s\alpha^*} s^{-1/4} (\xi^*)^{-1/4+1/\ell} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{L^2(\Sigma)^4}^2 + \left\| e^{-s\alpha^*} (s\xi^*)^{-1/4} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)^4}^2 \\ & \left. + \sum_{j=1}^{i+1} \iint_Q e^{-2s\alpha} s^{-2} \xi^{-2} \left| \nabla^2 \Delta \varphi_1^{(j)} \right|^2 dxdt \right). \quad (3.8) \end{aligned}$$

**Step 2.** In this step, we get rid of the boundary terms in the right-hand side of (3.8). To estimate the first term, we notice that since  $\ell \geq 4$ ,  $(\xi^*)^{-1/4+1/\ell}$  is bounded in  $(0, T)$  (see (2.6)). Thus

$$\begin{aligned} & \left\| e^{-s\alpha^*} s^{-1/4} (\xi^*)^{-1/4+1/\ell} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{L^2(\Sigma)^4}^2 \leq C s^{-1/2} \left\| e^{-s\alpha^*} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{L^2(\Sigma)^4}^2 \\ & \leq C s^{-1/2} \left( \left\| e^{-s\alpha^*} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{L^2(Q)^4}^2 + \left\| s^{1/2} e^{-s\alpha^*} (\xi^*)^{1/2} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{L^2(Q)^4} \left\| s^{-1/2} e^{-s\alpha^*} (\xi^*)^{-1/2} \nabla^3 \Delta \varphi_1^{(i)} \right\|_{L^2(Q)^s} \right) \\ & \leq C s^{-1/2} \iint_Q e^{-2s\alpha^*} \left( s\xi \left| \nabla^2 \Delta \varphi_1^{(i)} \right|^2 + s^{-1} \xi^{-1} \left| \nabla^3 \Delta \varphi_1^{(i)} \right|^2 \right) dxdt \leq C s^{-1/2} I_1(s, \varphi^{(i)}). \quad (3.9) \end{aligned}$$

For the second boundary term, we will use a trace inequality (see, for instance, [27, Theorem 2.1, p.9]) and by an interpolation argument (see, for instance, [26, Corollary 9.2, p.46]), we have

$$\begin{aligned} & \left\| (\xi^*)^{-1/4} e^{-s\alpha^*} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)^4}^2 \leq C \left\| (\xi^*)^{-1/4} e^{-s\alpha^*} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{L^2(0, T; H^1(\Omega)^4) \cap H^{1/2}(0, T; L^2(\Omega)^4)}^2 \\ & \leq C \left\| (\xi^*)^{-1/4} e^{-s\alpha^*} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{L^2(0, T; H^1(\Omega)^4) \cap H^1(0, T; H^{-1}(\Omega)^4)}^2. \quad (3.10) \end{aligned}$$

Our goal is to estimate the last term. First we consider the function

$$\theta_1(t) := s^{3/2} (\xi^*)^{3/2-1/\ell} e^{-s\alpha^*}.$$

From (2.5)-(2.6), for  $s \geq C$ ,

$$|\theta_1'| \leq C s^{5/2} (\xi^*)^{5/2} e^{-s\alpha^*}. \quad (3.11)$$

Then, from (1.8),  $(\theta_1 \varphi^{(i)}, \theta_1 \pi^{(i)})$  is the solution of

$$\begin{cases} -\partial_t (\theta_1 \varphi^{(i)}) - \Delta (\theta_1 \varphi^{(i)}) + \nabla (\theta_1 \pi^{(i)}) = \sum_{j=1}^{i+1} a_{j,i} \theta_1 \varphi^{(j)} - \theta_1' \varphi^{(i)} & \text{in } Q, \\ \nabla \cdot (\theta_1 \varphi^{(i)}) = 0 & \text{in } Q, \\ (\theta_1 \varphi^{(i)}) = 0 & \text{on } \Sigma, \\ (\theta_1 \varphi^{(i)})|_{t=T} = 0 & \text{in } \Omega. \end{cases} \quad (3.12)$$

Applying Lemma 2.3 to the above system and using (3.11), we obtain

$$\begin{aligned} & \left\| \theta_1 \varphi^{(i)} \right\|_{L^2(0,T;H^2(\Omega)^2) \cap H^1(0,T;L^2(\Omega)^2)}^2 \\ & \leq C \left( \sum_{j=1}^{i+1} \left\| s^{3/2} (\xi^*)^{3/2} e^{-s\alpha^*} \varphi^{(j)} \right\|_{L^2(Q)^2}^2 + \left\| s^{5/2} (\xi^*)^{5/2} e^{-s\alpha^*} \varphi^{(i)} \right\|_{L^2(Q)^2}^2 \right). \end{aligned}$$

Now, combining the above estimate with an interpolation inequality, we deduce

$$\begin{aligned} & \left\| s^2 (\xi^*)^{2-1/(2\ell)} e^{-s\alpha^*} \varphi^{(i)} \right\|_{L^2(0,T;H^1(\Omega)^2)}^2 \leq C \left\| \theta_1 \varphi^{(i)} \right\|_{L^2(0,T;H^2(\Omega)^2)} \left\| s^{5/2} (\xi^*)^{5/2} e^{-s\alpha^*} \varphi^{(i)} \right\|_{L^2(Q)^2} \\ & \leq C \left( \sum_{j=1}^{i+1} \left\| s^{3/2} (\xi^*)^{3/2} e^{-s\alpha^*} \varphi^{(j)} \right\|_{L^2(Q)^2}^2 + \left\| s^{5/2} (\xi^*)^{5/2} e^{-s\alpha^*} \varphi^{(i)} \right\|_{L^2(Q)^2}^2 \right). \end{aligned} \quad (3.13)$$

Second, we introduce

$$\theta_2 = s (\xi^*)^{1-3/(2\ell)} e^{-s\alpha^*}.$$

From (2.5)-(2.6), for  $s \geq C$ ,

$$|\theta_2'| \leq C s^2 (\xi^*)^{2-1/(2\ell)} e^{-s\alpha^*}. \quad (3.14)$$

Then, from (1.8),  $(\theta_2 \varphi^{(i)}, \theta_2 \pi^{(i)})$  is the solution of

$$\begin{cases} -\partial_t (\theta_2 \varphi^{(i)}) - \Delta (\theta_2 \varphi^{(i)}) + \nabla (\theta_2 \pi^{(i)}) = \sum_{j=1}^{i+1} a_{j,i} \theta_2 \varphi^{(j)} - \theta_2' \varphi^{(i)} & \text{in } Q, \\ \nabla \cdot (\theta_2 \varphi^{(i)}) = 0 & \text{in } Q, \\ (\theta_2 \varphi^{(i)}) = 0 & \text{on } \Sigma, \\ (\theta_2 \varphi^{(i)})|_{t=T} = 0 & \text{in } \Omega. \end{cases} \quad (3.15)$$

Using that the right-hand side of the first equation is divergence free and with null trace on  $\partial\Omega$ , we can apply Lemma 2.3 to the above system and we deduce

$$\left\| \theta_2 \varphi^{(i)} \right\|_{L^2(0,T;H^3(\Omega)^2) \cap H^1(0,T;V)}^2 \leq C \left( \sum_{j=1}^{i+1} \left\| \theta_2 \varphi^{(j)} \right\|_{L^2(0,T;V)}^2 + \left\| \theta_2' \varphi^{(i)} \right\|_{L^2(0,T;V)}^2 \right).$$

Using  $\theta_2 \leq C s^2 (\xi^*)^{2-1/(2\ell)} e^{-s\alpha^*}$  and combining the above result with (3.14) and (3.13)

$$\left\| \theta_2 \varphi^{(i)} \right\|_{L^2(0,T;H^3(\Omega)^2) \cap H^1(0,T;V)}^2 \leq C \sum_{j=1}^m \left\| s^{5/2} (\xi^*)^{5/2} e^{-s\alpha^*} \varphi^{(j)} \right\|_{L^2(Q)^2}^2. \quad (3.16)$$

Finally, we introduce

$$\theta_3 = (\xi^*)^{-5/(2\ell)} e^{-s\alpha^*}.$$

From (2.5)-(2.6), for  $s \geq C$ ,

$$|\theta_3'| \leq C \theta_2, \quad |\theta_3''| \leq C s^2 (\xi^*)^{2-1/(2\ell)} e^{-s\alpha^*}. \quad (3.17)$$

Then, from (1.8),  $(\theta_3 \varphi^{(i)}, \theta_3 \pi^{(i)})$  is the solution of

$$\begin{cases} -\partial_t (\theta_3 \varphi^{(i)}) - \Delta (\theta_3 \varphi^{(i)}) + \nabla (\theta_3 \pi^{(i)}) = \sum_{j=1}^{i+1} A_{j,i} \theta_3 \varphi^{(j)} - \theta_3' \varphi^{(i)} & \text{in } Q, \\ \nabla \cdot (\theta_3 \varphi^{(i)}) = 0 & \text{in } Q, \\ (\theta_3 \varphi^{(i)}) = 0 & \text{on } \Sigma, \\ (\theta_3 \varphi^{(i)})|_{t=T} = 0 & \text{in } \Omega. \end{cases} \quad (3.18)$$

We can again apply Lemma 2.3 to the above estimate to deduce

$$\begin{aligned} & \left\| \theta_3 \varphi^{(i)} \right\|_{L^2(0,T;H^5(\Omega)^2) \cap H^1(0,T;H^3(\Omega)^2) \cap H^2(0,T;V)}^2 \\ & \leq C \left( \sum_{j=1}^{i+1} \left\| \theta_3 \varphi^{(j)} \right\|_{L^2(0,T;H^3(\Omega)^2)}^2 + \left\| \partial_t \left( \theta_3 \varphi^{(j)} \right) \right\|_{L^2(0,T;H^1(\Omega)^2)}^2 \right. \\ & \quad \left. + \left\| \theta'_3 \varphi^{(i)} \right\|_{L^2(0,T;H^3(\Omega)^2)}^2 + \left\| \partial_t \left( \theta'_3 \varphi^{(i)} \right) \right\|_{L^2(0,T;H^1(\Omega)^2)}^2 \right). \end{aligned} \quad (3.19)$$

Using  $\theta_3 \leq C\theta_2$  and applying (3.17), (3.16), and (3.13), we deduce from (3.19)

$$\left\| \theta_3 \varphi^{(i)} \right\|_{L^2(0,T;H^5(\Omega)^2) \cap H^1(0,T;H^3(\Omega)^2) \cap H^2(0,T;V)}^2 \leq C \left( \sum_{j=1}^m \left\| s^{5/2} (\xi^*)^{5/2} e^{-s\alpha^*} \varphi^{(j)} \right\|_{L^2(Q)^2}^2 \right). \quad (3.20)$$

Using the trace inequality (3.10) and that  $\ell \geq 14$ , we have

$$\begin{aligned} & \left\| (\xi^*)^{-1/4} e^{-s\alpha^*} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)^4}^2 \leq C \left\| (\xi^*)^{-1/4} e^{-s\alpha^*} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{L^2(0,T;H^1(\Omega)^4) \cap H^1(0,T;H^{-1}(\Omega)^4)}^2 \\ & \leq C \left\| (\xi^*)^{-1/4} e^{-s\alpha^*} \varphi_1^{(i)} \right\|_{L^2(0,T;H^5(\Omega)) \cap H^1(0,T;H^3(\Omega))}^2 \leq C \left\| \theta_3 \varphi^{(i)} \right\|_{L^2(0,T;H^5(\Omega)^2) \cap H^1(0,T;H^3(\Omega)^2)}^2. \end{aligned} \quad (3.21)$$

Combining (3.20) and (3.21), we deduce that

$$\left\| e^{-s\alpha^*} (s\xi^*)^{-1/4} \nabla^2 \Delta \varphi_1^{(i)} \right\|_{H^{\frac{1}{4}, \frac{1}{2}}(\Sigma)^4}^2 \leq C s^{-1/2} \sum_{j=1}^m \left\| s^{5/2} (\xi^*)^{5/2} e^{-s\alpha^*} \varphi^{(j)} \right\|_{L^2(Q)^2}^2.$$

Putting together (3.8), the above relation and (3.9), we deduce at this step the following inequality:

$$\begin{aligned} I_1(s, \varphi^{(i)}) & \leq C \left( \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} \left( s\xi \left| \nabla^2 \Delta \varphi_1^{(i)} \right|^2 + s^3 \xi^3 \left| \nabla \Delta \varphi_1^{(i)} \right|^2 + s^5 \xi^5 \left| \Delta \varphi_1^{(i)} \right|^2 \right) dxdt \right. \\ & \quad \left. + \sum_{j=1}^m \iint_Q \left( e^{-2s\alpha} s^{9/2} (\xi^*)^5 \left| \varphi^{(j)} \right|^2 + e^{-2s\alpha} s^{-2} \xi^{-2} \left| \nabla^2 \Delta \varphi_1^{(j)} \right|^2 \right) dxdt \right). \end{aligned} \quad (3.22)$$

**Step 3.** To estimate the local terms, we proceed in a standard way: we consider  $\omega_1$  an open subset satisfying  $\omega_0 \Subset \omega_1 \Subset \widehat{\omega}$  and

$$\eta_1 \in C_c^2(\omega_1), \quad \eta_1 \equiv 1 \text{ in } \omega_0, \quad \eta_1 \geq 0.$$

Then, an integration by parts gives

$$\begin{aligned} & \iint_{\omega_0 \times (0,T)} e^{-2s\alpha} s\xi \left( \frac{\partial^2}{\partial x_k \partial x_q} \Delta \varphi_1^{(i)} \right)^2 dxdt \leq \iint_{\omega_1 \times (0,T)} \eta_1 e^{-2s\alpha} s\xi \left( \frac{\partial^2}{\partial x_k \partial x_q} \Delta \varphi_1^{(i)} \right)^2 dxdt \\ & = - \iint_{\omega_1 \times (0,T)} \frac{\partial}{\partial x_k} \left( \eta_1 e^{-2s\alpha} s\xi \right) \frac{\partial^2}{\partial x_k \partial x_q} \Delta \varphi_1^{(i)} \frac{\partial}{\partial x_q} \Delta \varphi_1^{(i)} dxdt \\ & \quad - \iint_{\omega_1 \times (0,T)} \eta_1 e^{-2s\alpha} s\xi \frac{\partial^3}{\partial x_k^2 \partial x_q} \Delta \varphi_1^{(i)} \frac{\partial}{\partial x_q} \Delta \varphi_1^{(i)} dxdt. \end{aligned}$$

Using (2.7) and Young's inequality, we deduce from the above relation that there exists  $C > 0$  such that for all  $\varepsilon > 0$ ,

$$\begin{aligned} \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} s\xi \left| \nabla^2 \Delta \varphi_1^{(i)} \right|^2 dxdt &\leq \varepsilon \iint_Q e^{-2s\alpha} \left( s^{-1} \xi^{-1} \left| \nabla^3 \Delta \varphi_1^{(i)} \right|^2 + s\xi \left| \nabla^2 \Delta \varphi_1^{(i)} \right|^2 \right) dxdt \\ &\quad + \frac{C}{\varepsilon} \iint_{\omega_1 \times (0, T)} e^{-2s\alpha} s^3 \xi^3 \left| \nabla \Delta \varphi_1^{(i)} \right|^2 dxdt. \end{aligned} \quad (3.23)$$

Now we estimate, in an analogous way, the local term associated with  $\nabla \Delta \varphi_1^{(i)}$ : we consider

$$\eta_2 \in C_c^2(\widehat{\omega}), \quad \eta_2 \equiv 1 \text{ in } \omega_1, \quad \eta_2 \geq 0.$$

Then, integrating by parts, we obtain

$$\begin{aligned} \iint_{\omega_1 \times (0, T)} e^{-2s\alpha} s^3 \xi^3 \left( \frac{\partial}{\partial x_k} \Delta \varphi_1^{(i)} \right)^2 dxdt &\leq \iint_{\widehat{\omega} \times (0, T)} \eta_2 e^{-2s\alpha} s^3 \xi^3 \left( \frac{\partial}{\partial x_k} \Delta \varphi_1^{(i)} \right)^2 dxdt \\ &= - \iint_{\widehat{\omega} \times (0, T)} \frac{\partial}{\partial x_k} (\eta_2 e^{-2s\alpha} s^3 \xi^3) \frac{\partial}{\partial x_k} \Delta \varphi_1^{(i)} \Delta \varphi_1^{(i)} dxdt \\ &\quad - \iint_{\widehat{\omega} \times (0, T)} \eta_2 e^{-2s\alpha} s^3 \xi^3 \frac{\partial^2}{\partial x_k^2} \Delta \varphi_1^{(i)} \Delta \varphi_1^{(i)} dxdt. \end{aligned}$$

Using (2.7) and the Young's inequality, we deduce from the above relation that there exists  $C > 0$  such that for all  $\varepsilon > 0$ ,

$$\begin{aligned} \iint_{\omega_1 \times (0, T)} e^{-2s\alpha} s^3 \xi^3 \left| \nabla \Delta \varphi_1^{(i)} \right|^2 dxdt &\leq \varepsilon \iint_Q e^{-2s\alpha} \left( s^3 \xi^3 \left| \nabla \Delta \varphi_1^{(i)} \right|^2 + s\xi \left| \nabla^2 \Delta \varphi_1^{(i)} \right|^2 \right) dxdt \\ &\quad + \frac{C}{\varepsilon} \iint_{\widehat{\omega} \times (0, T)} e^{-2s\alpha} s^5 \xi^5 \left| \Delta \varphi_1^{(i)} \right|^2 dxdt. \end{aligned} \quad (3.24)$$

The above estimate, together with (3.22) and (3.23) implies (3.3). This concludes the proof of Proposition 3.2.  $\square$

We are now in a position to prove Theorem 3.1.

*Proof of Theorem 3.1.* Summing (3.3) for  $i = 1, \dots, m$ , and taking  $s \geq C$  for a constant  $C$  large enough, we deduce that

$$\sum_{i=1}^m I_1(s, \varphi^{(i)}) \leq C \sum_{i=1}^m \iint_{\widehat{\omega} \times (0, T)} e^{-2s\alpha} s^5 \xi^5 \left| \Delta \varphi_1^{(i)} \right|^2 dxdt. \quad (3.25)$$

In order to get rid of the local terms in the right-hand side (except the term corresponding to  $i = 1$ ), we introduce a sequence of open sets  $\mathcal{O}_i$ , ( $0 \leq i \leq m$ ) such that

$$\widehat{\omega} =: \mathcal{O}_0 \Subset \mathcal{O}_1 \Subset \dots \Subset \mathcal{O}_i \Subset \dots \Subset \mathcal{O}_m \Subset \omega$$

and functions

$$\zeta_i \in C_c^2(\mathcal{O}_i) \quad \text{such that} \quad \zeta_i \equiv 1 \text{ in } \mathcal{O}_{i-1}, \quad \zeta_i \geq 0 \quad (1 \leq i \leq m).$$

Then, we consider the equation  $m-1$  of (1.9), we apply the Laplace operator on the first component of this equation and we multiply it by  $\zeta_1$ :

$$\zeta_1 a_{m, m-1} \Delta \varphi_1^{(m)} = -\zeta_1 \partial_t \Delta \varphi_1^{(m-1)} - \zeta_1 \Delta^2 \varphi_1^{(m-1)} - \zeta_1 \sum_{j=1}^{m-1} a_{j, m-1} \Delta \varphi_1^{(j)}. \quad (3.26)$$

Then, using the above equation combined with (1.4) and integrating by parts, we deduce

$$\begin{aligned} \iint_{\widehat{\omega} \times (0, T)} e^{-2s\alpha} s^5 \xi^5 \left| \Delta \varphi_1^{(m)} \right|^2 dxdt &\leq \iint_{\mathcal{O}_1 \times (0, T)} \zeta_1 e^{-2s\alpha} s^5 \xi^5 \left| \Delta \varphi_1^{(m)} \right|^2 dxdt \\ &= \frac{-1}{a_{m, m-1}} \iint_{\mathcal{O}_1 \times (0, T)} \zeta_1 e^{-2s\alpha} s^5 \xi^5 \left( \Delta \varphi_1^{(m)} \right) \left( \partial_t \Delta \varphi_1^{(m-1)} + \Delta^2 \varphi_1^{(m-1)} + \sum_{j=1}^{m-1} a_{j, m-1} \Delta \varphi_1^{(j)} \right) dxdt. \end{aligned} \quad (3.27)$$

Integrating by parts and applying  $\Delta$  to the first component of the equation (1.9) with  $i = m$ , we obtain

$$\begin{aligned} \iint_{\mathcal{O}_1 \times (0, T)} \zeta_1 e^{-2s\alpha} s^5 \xi^5 \Delta \varphi_1^{(m)} \partial_t \Delta \varphi_1^{(m-1)} dxdt &= - \iint_{\mathcal{O}_1 \times (0, T)} \zeta_1 \partial_t (e^{-2s\alpha} s^5 \xi^5) \Delta \varphi_1^{(m)} \Delta \varphi_1^{(m-1)} dxdt \\ &\quad + \iint_{\mathcal{O}_1 \times (0, T)} \zeta_1 e^{-2s\alpha} s^5 \xi^5 \Delta^2 \varphi_1^{(m)} \Delta \varphi_1^{(m-1)} dxdt \\ &\quad + \sum_{j=1}^m \iint_{\mathcal{O}_1 \times (0, T)} \zeta_1 e^{-2s\alpha} s^5 \xi^5 a_{j, m} \Delta \varphi_1^{(j)} \Delta \varphi_1^{(m-1)} dxdt. \end{aligned} \quad (3.28)$$

By integrating by parts, we also find

$$\begin{aligned} \iint_{\mathcal{O}_1 \times (0, T)} \zeta_1 e^{-2s\alpha} s^5 \xi^5 \left( \Delta \varphi_1^{(m)} \right) \left( \Delta^2 \varphi_1^{(m-1)} \right) dxdt &= \iint_{\mathcal{O}_1 \times (0, T)} \Delta (\zeta_1 e^{-2s\alpha} s^5 \xi^5) \Delta \varphi_1^{(m)} \Delta \varphi_1^{(m-1)} dxdt \\ &\quad + \iint_{\mathcal{O}_1 \times (0, T)} \zeta_1 e^{-2s\alpha} s^5 \xi^5 \Delta^2 \varphi_1^{(m)} \Delta \varphi_1^{(m-1)} dxdt \\ &\quad + \iint_{\mathcal{O}_1 \times (0, T)} 2\nabla (\zeta_1 e^{-2s\alpha} s^5 \xi^5) \cdot \nabla \Delta \varphi_1^{(m)} \Delta \varphi_1^{(m-1)} dxdt. \end{aligned} \quad (3.29)$$

Let us now estimate the right-hand side of (3.28) and (3.29) and the last term in (3.27).

Using (2.4), we have

$$\left| \partial_t (e^{-2s\alpha} s^5 \xi^5) \right| \leq C e^{-2s\alpha} s^6 \xi^{6+1/\ell}.$$

Combining this inequality and Young's inequality, we deduce the existence of  $C$  such that for any  $s \geq C$  and for any  $\varepsilon > 0$ ,

$$\begin{aligned} \left| \iint_{\mathcal{O}_1 \times (0, T)} \zeta_1 \partial_t (e^{-2s\alpha} s^5 \xi^5) \Delta \varphi_1^{(m)} \Delta \varphi_1^{(m-1)} dxdt \right| \\ \leq \varepsilon \iint_Q e^{-2s\alpha} s^5 \xi^5 \left| \Delta \varphi_1^{(m)} \right|^2 dxdt + \frac{C}{\varepsilon} \iint_{\mathcal{O}_1 \times (0, T)} e^{-2s\alpha} s^7 \xi^{7+2/\ell} \left| \Delta \varphi_1^{(m-1)} \right|^2 dxdt. \end{aligned} \quad (3.30)$$

Using (2.7), we have

$$\left| \Delta (\zeta_1 e^{-2s\alpha} s^5 \xi^5) \right| \leq C e^{-2s\alpha} s^7 \xi^7.$$

Combining this inequality and Young's inequality, we deduce the existence of  $C$  such that for any  $s \geq C$  and for any  $\varepsilon > 0$ ,

$$\begin{aligned} \left| \iint_{\mathcal{O}_1 \times (0, T)} \Delta (\zeta_1 e^{-2s\alpha} s^5 \xi^5) \Delta \varphi_1^{(m)} \Delta \varphi_1^{(m-1)} dxdt \right| \\ \leq \varepsilon \iint_Q e^{-2s\alpha} s^5 \xi^5 \left| \Delta \varphi_1^{(m)} \right|^2 dxdt + \frac{C}{\varepsilon} \iint_{\mathcal{O}_1 \times (0, T)} e^{-2s\alpha} s^9 \xi^9 \left| \Delta \varphi_1^{(m-1)} \right|^2 dxdt. \end{aligned} \quad (3.31)$$

Using (2.7), we have

$$|\nabla(\zeta_1 e^{-2s\alpha} s^5 \xi^5)| \leq C e^{-2s\alpha} s^6 \xi^6.$$

Combining this inequality and Young's inequality, we deduce the existence of  $C$  such that for any  $s \geq C$  and for any  $\varepsilon > 0$ ,

$$\begin{aligned} & \left| \iint_{\mathcal{O}_1 \times (0, T)} 2\nabla(\zeta_1 e^{-2s\alpha} s^5 \xi^5) \cdot \nabla \Delta \varphi_1^{(m)} \Delta \varphi_1^{(m-1)} dxdt \right| \\ & \leq \varepsilon \iint_Q e^{-2s\alpha} s^3 \xi^3 |\nabla \Delta \varphi_1^{(m)}|^2 dxdt + \frac{C}{\varepsilon} \iint_{\mathcal{O}_1 \times (0, T)} e^{-2s\alpha} s^9 \xi^9 |\Delta \varphi_1^{(m-1)}|^2 dxdt. \end{aligned} \quad (3.32)$$

Finally, the other terms can be estimated by using Young's inequality:

$$\begin{aligned} & \left| \iint_{\mathcal{O}_1 \times (0, T)} \zeta_1 e^{-2s\alpha} s^5 \xi^5 \Delta^2 \varphi_1^{(m)} \Delta \varphi_1^{(m-1)} dxdt \right| \\ & \leq \varepsilon \iint_Q e^{-2s\alpha} s \xi |\Delta^2 \varphi_1^{(m)}|^2 dxdt + \frac{C}{\varepsilon} \iint_{\mathcal{O}_1 \times (0, T)} e^{-2s\alpha} s^9 \xi^9 |\Delta \varphi_1^{(m-1)}|^2 dxdt, \end{aligned} \quad (3.33)$$

$$\begin{aligned} & \left| \sum_{j=1}^m \iint_{\mathcal{O}_1 \times (0, T)} \zeta_1 e^{-2s\alpha} s^5 \xi^5 a_{j,m} \Delta \varphi_1^{(j)} \Delta \varphi_1^{(m-1)} dxdt \right| \\ & \leq \varepsilon \sum_{j=1}^m \iint_Q e^{-2s\alpha} s^5 \xi^5 |\Delta \varphi_1^{(j)}|^2 dxdt + \frac{C}{\varepsilon} \iint_{\mathcal{O}_1 \times (0, T)} e^{-2s\alpha} s^5 \xi^5 |\Delta \varphi_1^{(m-1)}|^2 dxdt, \end{aligned} \quad (3.34)$$

$$\begin{aligned} & \left| \frac{-1}{a_{m,m-1}} \iint_{\mathcal{O}_1 \times (0, T)} \zeta_1 e^{-2s\alpha} s^5 \xi^5 \Delta \varphi_1^{(m)} \sum_{j=1}^{m-1} a_{j,m-1} \Delta \varphi_1^{(j)} dxdt \right| \\ & \leq \varepsilon \iint_Q e^{-2s\alpha} s^5 \xi^5 |\Delta \varphi_1^{(m)}|^2 dxdt + \frac{C}{\varepsilon} \sum_{j=1}^{m-1} \iint_{\mathcal{O}_1 \times (0, T)} e^{-2s\alpha} s^5 \xi^5 |\Delta \varphi_1^{(j)}|^2 dxdt. \end{aligned} \quad (3.35)$$

The combination of (3.27) with (3.30), (3.31), (3.32), (3.33), (3.34), and (3.35) yields the existence of a constant  $C$  such that for any  $s \geq C$  and for any  $\varepsilon > 0$ ,

$$\begin{aligned} \iint_{\widehat{\omega} \times (0, T)} e^{-2s\alpha} s^5 \xi^5 |\Delta \varphi_1^{(m)}|^2 dxdt & \leq \varepsilon \sum_{j=1}^m I_1(s, \varphi^{(j)}) + \frac{C}{\varepsilon} \iint_{\mathcal{O}_1 \times (0, T)} e^{-2s\alpha} s^9 \xi^9 |\Delta \varphi_1^{(m-1)}|^2 dxdt \\ & \quad + \frac{C}{\varepsilon} \sum_{j=1}^{m-2} \iint_{\mathcal{O}_1 \times (0, T)} e^{-2s\alpha} s^5 \xi^5 |\Delta \varphi_1^{(j)}|^2 dxdt. \end{aligned} \quad (3.36)$$

We can repeat the same analysis for  $\Delta \varphi_1^{(m-1)}$  and we obtain the existence of a constant  $C$  such that for any  $s \geq C$

and for any  $\varepsilon > 0$ ,

$$\begin{aligned} \iint_{\mathcal{O}_1 \times (0, T)} e^{-2s\alpha} s^9 \xi^9 \left| \Delta \varphi_1^{(m-1)} \right|^2 dxdt &\leq \varepsilon \sum_{j=1}^m I_1(s, \varphi^{(j)}) + \frac{C}{\varepsilon} \iint_{\mathcal{O}_2 \times (0, T)} e^{-2s\alpha} s^{17} \xi^{17} \left| \Delta \varphi_1^{(m-2)} \right|^2 dxdt. \\ &+ \frac{C}{\varepsilon} \sum_{j=1}^{m-3} \iint_{\mathcal{O}_2 \times (0, T)} e^{-2s\alpha} s^{13} \xi^{13} \left| \Delta \varphi_1^{(j)} \right|^2 dxdt. \end{aligned} \quad (3.37)$$

Proceeding by induction, we can estimate all the local terms and we deduce from (3.25) that

$$\sum_{i=1}^m I_1(s, \varphi^{(i)}) \leq C \iint_{\mathcal{O}_m \times (0, T)} e^{-2s\alpha} s^{2(m+1)+1} \xi^{2(m+1)+1} \left| \Delta \varphi_1^{(1)} \right|^2 dxdt. \quad (3.38)$$

Finally, we estimate the above local term in terms of  $\varphi_1^{(1)}$ . In order to do this, we consider  $\tilde{\omega}$  an open subset satisfying  $\mathcal{O}_m \Subset \tilde{\omega} \Subset \omega$  and

$$\tilde{\zeta} \in C_c^2(\tilde{\omega}) \quad \text{such that} \quad \tilde{\zeta} \equiv 1 \text{ in } \mathcal{O}_m, \quad \tilde{\zeta} \geq 0.$$

Then by integrating by parts, we obtain

$$\begin{aligned} \iint_{\mathcal{O}_m \times (0, T)} e^{-2s\alpha} s^{2(m+1)+1} \xi^{2(m+1)+1} \left| \Delta \varphi_1^{(1)} \right|^2 dxdt &\leq \iint_{\tilde{\omega} \times (0, T)} \tilde{\zeta} e^{-2s\alpha} s^{2(m+1)+1} \xi^{2(m+1)+1} \left| \Delta \varphi_1^{(1)} \right|^2 dxdt \\ &= - \iint_{\tilde{\omega} \times (0, T)} \nabla \left( \tilde{\zeta} e^{-2s\alpha} s^{2(m+1)+1} \xi^{2(m+1)+1} \right) \Delta \varphi_1^{(1)} \cdot \nabla \varphi_1^{(1)} dxdt \\ &\quad - \iint_{\tilde{\omega} \times (0, T)} \tilde{\zeta} e^{-2s\alpha} s^{2(m+1)+1} \xi^{2(m+1)+1} \Delta \nabla \varphi_1^{(1)} \cdot \nabla \varphi_1^{(1)} dxdt. \end{aligned} \quad (3.39)$$

Using (2.7), we have

$$\left| \nabla \left( \tilde{\zeta} e^{-2s\alpha} s^{2(m+1)+1} \xi^{2(m+1)+1} \right) \right| \leq C e^{-2s\alpha} s^{2(m+1)+2} \xi^{2(m+1)+2}.$$

Combining this inequality and Young's inequality, we deduce the existence of  $C$  such that for any  $s \geq C$  and for any  $\varepsilon > 0$ ,

$$\begin{aligned} \iint_{\mathcal{O}_m \times (0, T)} e^{-2s\alpha} s^{2(m+1)+1} \xi^{2(m+1)+1} \left| \Delta \varphi_1^{(1)} \right|^2 dxdt &\leq \varepsilon \iint_Q e^{-2s\alpha} \left( s^3 \xi^3 \left| \nabla \Delta \varphi_1^{(1)} \right|^2 + s^5 \xi^5 \left| \Delta \varphi_1^{(1)} \right|^2 \right) dxdt \\ &+ \frac{C}{\varepsilon} \iint_{\tilde{\omega} \times (0, T)} e^{-2s\alpha} s^{2(m+2)-1} \xi^{2(m+2)-1} \left| \nabla \varphi_1^{(1)} \right|^2 dxdt. \end{aligned} \quad (3.40)$$

Then, we consider

$$\zeta \in C_c^2(\omega) \quad \text{such that} \quad \zeta \equiv 1 \text{ in } \tilde{\omega}, \quad \zeta \geq 0$$

and we integrate by parts:

$$\begin{aligned} \iint_{\tilde{\omega} \times (0, T)} e^{-2s\alpha} s^{2(m+2)-1} \xi^{2(m+2)-1} \left| \nabla \varphi_1^{(1)} \right|^2 dxdt &\leq \iint_{\omega \times (0, T)} \zeta e^{-2s\alpha} s^{2(m+2)-1} \xi^{2(m+2)-1} \left| \nabla \varphi_1^{(1)} \right|^2 dxdt \\ &= - \iint_{\omega \times (0, T)} \nabla \left( \zeta e^{-2s\alpha} s^{2(m+2)-1} \xi^{2(m+2)-1} \right) \cdot \nabla \varphi_1^{(1)} \varphi_1^{(1)} dxdt \\ &\quad - \iint_{\omega \times (0, T)} \zeta e^{-2s\alpha} s^{2(m+2)-1} \xi^{2(m+2)-1} \Delta \varphi_1^{(1)} \varphi_1^{(1)} dxdt. \end{aligned}$$

Using (2.7), we have

$$\left| \nabla (\zeta e^{-2s\alpha} s^{2(m+2)-1} \xi^{2(m+2)-1}) \right| \leq C e^{-2s\alpha} s^{2(m+2)} \xi^{2(m+2)}.$$

Combining this inequality and Young's inequality, we deduce the existence of  $C$  such that for any  $s \geq C$  and for any  $\varepsilon > 0$ ,

$$\begin{aligned} & \iint_{\bar{\omega} \times (0, T)} e^{-2s\alpha} s^{2(m+2)-1} \xi^{2(m+2)-1} \left| \nabla \varphi_1^{(1)} \right|^2 dx dt \\ & \leq \varepsilon \left( \iint_Q e^{-2s\alpha} s^6 \xi^6 \left| \nabla \varphi_1^{(1)} \right|^2 dx dt + \iint_Q e^{-2s\alpha} s^5 \xi^5 \left| \Delta \varphi_1^{(1)} \right|^2 dx dt \right) \\ & \quad + \frac{C}{\varepsilon} \iint_{\omega \times (0, T)} e^{-2s\alpha} s^{2(m+3)-6} \xi^{2(m+3)-6} \left| \varphi_1^{(1)} \right|^2 dx dt. \end{aligned} \quad (3.41)$$

Using Lemma 4 in [9] (see [23] for the proof) and the definition (3.1) of  $I_1$ , we deduce that

$$\begin{aligned} I_1(s, \varphi^{(i)}) + \iint_{\omega \times (0, T)} e^{-2s\alpha} s^8 \xi^8 \left| \varphi_1^{(i)} \right|^2 dx dt & \geq \iint_Q e^{-2s\alpha} s^5 \xi^5 \left| \Delta \varphi_1^{(i)} \right|^2 dx dt + \iint_{\omega \times (0, T)} e^{-2s\alpha} s^8 \xi^8 \left| \varphi_1^{(i)} \right|^2 dx dt \\ & \geq C \left( \iint_Q e^{-2s\alpha} s^8 \xi^8 \left| \varphi_1^{(i)} \right|^2 dx dt + \iint_Q e^{-2s\alpha} s^6 \xi^6 \left| \nabla \varphi_1^{(i)} \right|^2 dx dt \right). \end{aligned}$$

This estimate, combined with (3.38), (3.40) and (3.41) yields the conclusion of Theorem 3.1.  $\square$

## 4 Proof of the main results

### 4.1 Final state observability

In this section, we use Theorem 3.1 in order to prove the final state observability of the adjoint system (1.8).

**Lemma 4.1.** *Assume  $T \in (0, 1)$  and  $\omega$  is non empty open set of  $\Omega$ . Then, there exists  $C > 0$  and  $\ell \geq 14$  such that for any  $\varphi_0 \in H^m$ , the solution  $\varphi$  of (1.8) satisfies*

$$\int_{\Omega} \sum_{i=1}^m \left| \varphi^{(i)}(x, 0) \right|^2 dx \leq C e^{\frac{C}{T^{2\ell}}} \iint_{\omega \times (0, T)} \left| \varphi_1^{(1)} \right|^2 dx dt. \quad (4.1)$$

*Proof.* First, we consider an energy estimate of the adjoint system (1.8). Multiplying each equation (1.9) by  $\varphi^{(i)}$  and integrating by parts, we deduce

$$-\frac{1}{2} \frac{d}{dt} \int_{\Omega} \sum_{i=1}^m \left| \varphi^{(i)} \right|^2 dx + \int_{\Omega} \sum_{i=1}^m \left| \nabla \varphi^{(i)} \right|^2 dx = \int_{\Omega} \sum_{i,j=1}^m A_{j,i} \varphi^{(i)} \cdot \varphi^{(j)} dx. \quad (4.2)$$

Thus, using the Grönwall lemma, there exists  $C > 0$  such that

$$t \mapsto e^{Ct} \int_{\Omega} \sum_{i=1}^m \left| \varphi^{(i)}(x, t) \right|^2 dx$$

is nondecreasing. In particular, for some constant  $C > 0$ ,

$$\int_{\Omega} \sum_{i=1}^m \left| \varphi^{(i)}(x, 0) \right|^2 dx \leq \frac{2}{T} e^{CT} \int_{T/4}^{3T/4} \int_{\Omega} \sum_{i=1}^m \left| \varphi^{(i)}(x, t) \right|^2 dx dt. \quad (4.3)$$

On the other hand, from (3.2) and (3.1), we deduce that

$$\iint_{\tilde{Q}} e^{-2s\alpha^*} (s\xi^*)^5 \sum_{i=1}^m \left| \varphi^{(i)} \right|^2 dx dt \leq C \iint_{\omega \times (0, T)} e^{-2s\alpha} (s\xi)^{2^{m+3}-6} \left| \varphi_1^{(1)} \right|^2 dx dt. \quad (4.4)$$

Using that for  $t \in [T/4, 3T/4]$ ,

$$\frac{3T^2}{16} \leq t(T-t) \leq \frac{T^2}{4},$$

we deduce, from (2.3), the existence of two constants  $C_1, C_2 > 0$  such that for  $t \in [T/4, 3T/4]$ ,

$$\alpha^*(t) \leq \frac{C_1}{T^{2\ell}}, \quad \xi^*(t) \geq \frac{C_2}{T^{2\ell}}$$

and consequently, for some constant  $C_3 > 0$ ,

$$e^{-2s\alpha^*} (s\xi^*)^5 \geq e^{-\frac{C_3}{T^{2\ell}}}. \quad (4.5)$$

Similarly, from (2.2), there exist two constants  $c_1, c_2 > 0$  such that for  $(x, t) \in \Omega \times [0, T]$ ,

$$\alpha(x, t) \geq \frac{c_1}{t^\ell(T-t)^\ell}, \quad \xi(x, t) \leq \frac{c_2}{t^\ell(T-t)^\ell}$$

and consequently, for some constant  $c_3 > 0$ ,

$$e^{-2s\alpha} (s\xi)^{2^{m+3}-6} \leq e^{-2s\frac{c_1}{t^\ell(T-t)^\ell}} \left( s \frac{c_2}{t^\ell(T-t)^\ell} \right)^{2^{m+3}-6} \leq c_3. \quad (4.6)$$

Combining (4.3), (4.4), (4.5) and (4.6), we deduce that for some constant  $C$ ,

$$\int_{\Omega} \sum_{i=1}^m \left| \varphi^{(i)}(x, 0) \right|^2 dx \leq \frac{C}{T} e^{C(T+\frac{1}{T^{2\ell}})} \iint_{\omega \times (0, T)} \left| \varphi_1^{(1)} \right|^2 dx dt.$$

This implies that (4.1). □

## 4.2 Proof of Theorem 1.1

We recall that  $H$  and  $V$  are defined by (1.6) and (1.7). We also set

$$\mathcal{H} := H^m, \quad \mathcal{V} := V^m. \quad (4.7)$$

We define  $P_0 : L^2(\Omega)^N \rightarrow H$  the Leray projector and the projection  $\mathcal{P}$  defined as:

$$\mathcal{P} : [L^2(\Omega)^N]^m \rightarrow \mathcal{H}, \quad y = (y^{(1)}, \dots, y^{(m)}) \mapsto (P_0 y^{(1)}, \dots, P_0 y^{(m)}).$$

We also consider the unbounded operator in  $\mathcal{H}$  defined by

$$\mathcal{D}(\mathcal{A}) := [H^2(\Omega)^N \cap V]^m, \quad \mathcal{A}y := \mathcal{P}\Delta y + \mathcal{A}y. \quad (4.8)$$

Finally, we define the control operator  $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{H})$  by

$$\mathcal{U} := L^2(\omega)^{N-1}, \quad \mathcal{B}v = \mathcal{B}(v_1, \dots, v_{N-1}) := (P_0((v_1, \dots, v_{N-1}, 0) \mathbf{1}_\omega), 0, \dots, 0). \quad (4.9)$$

With the above definition, we can write (1.1) as

$$\begin{cases} y' = \mathcal{A}y + \mathcal{B}v, \\ y(0) = y_0. \end{cases} \quad (4.10)$$

As it is well-known (see, for instance, [34, p.357]), system (4.10) is null-controllable in time  $T > 0$  if and only if there exists  $K(T) > 0$  such that

$$\left\| e^{T\mathcal{A}^*} \varphi_0 \right\|_{\mathcal{H}}^2 \leq K(T)^2 \int_0^T \left\| \mathcal{B}^* e^{t\mathcal{A}^*} \varphi_0 \right\|_{\mathcal{H}}^2 dt \quad (\varphi_0 \in \mathcal{H}). \quad (4.11)$$

One can check that

$$\mathcal{D}(\mathcal{A}^*) = [H^2(\Omega)^N \cap V]^m, \quad \mathcal{A}^*y = \mathcal{P}\Delta y + A^*y,$$

where  $A^*$  is the transpose of the matrix  $A$  and that

$$\mathcal{B}^*\varphi = \left( \left( \varphi_1^{(1)} \right)_{|\omega}, \dots, \left( \varphi_{N-1}^{(1)} \right)_{|\omega} \right).$$

Thus, we deduce Theorem 1.1 from Lemma 4.1.

### 4.3 Proof of Theorem 1.3

In order to prove Theorem 1.3, we recall a method introduced in [28, 32] to deal with the controllability of nonlinear parabolic systems. We consider  $\mathcal{H}$  and  $\mathcal{U}$  two Hilbert spaces,  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{H}$  the infinitesimal generator of an analytic semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$  and  $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$  a control operator. We also assume the final state observability for all  $T > 0$

$$\left\| e^{T\mathcal{A}^*} \varphi_0 \right\|_{\mathcal{H}}^2 \leq K(T)^2 \int_0^T \left\| \mathcal{B}^* e^{t\mathcal{A}^*} \varphi_0 \right\|_{\mathcal{H}}^2 dt \quad (\varphi_0 \in \mathcal{H}), \quad (4.12)$$

with  $K : (0, \infty) \rightarrow [0, \infty)$  continuous and non increasing. Let us consider  $T > 0$  and suppose there exist  $\rho_0, \rho_1, \rho \in C^0([0, T], \mathbb{R}^+)$ , non increasing, positive in  $[0, T)$  such that  $\rho_0(T) = \rho_1(T) = \rho(T) = 0$  and such that, for some constant  $q > 1$ ,

$$\rho_0(t) := \rho_1(q^2(t-T) + T)K((q-1)(T-t)) \quad \left( t \in \left[ T \left( 1 - \frac{1}{q^2} \right), T \right] \right), \quad (4.13)$$

$$\rho_0 \leq C\rho, \quad \rho_1 \leq C\rho, \quad |\rho'| \rho_0 \leq C\rho^2 \quad (t \in [0, T]). \quad (4.14)$$

for some constant  $C > 0$ . We denote by  $L_{\rho_1}^2(0, T; \mathcal{H})$  the space

$$L_{\rho_1}^2(0, T; \mathcal{H}) := \left\{ f \in L^2(0, T; \mathcal{H}) ; \frac{f}{\rho_1} \in L^2(0, T; \mathcal{H}) \right\}$$

and we define similarly  $L_{\rho_0}^2(0, T; \mathcal{U})$ .

Then we can consider the control problem

$$\begin{cases} y' = \mathcal{A}y + \mathcal{B}v + f, \\ y(0) = y_0. \end{cases} \quad (4.15)$$

We have the following result (see [28]):

**Theorem 4.2.** *With the above assumptions, there exists a bounded operator*

$$\mathcal{E}_T \in \mathcal{L} \left( \mathcal{D} \left( (-\mathcal{A})^{1/2} \right) \times L_{\rho_1}^2(0, T; \mathcal{H}), L_{\rho_0}^2(0, T; \mathcal{U}) \right)$$

such that for any  $y_0 \in \mathcal{D} \left( (-\mathcal{A})^{1/2} \right)$  and for any  $f \in L_{\rho_1}^2(0, T; \mathcal{H})$ , the solution  $y$  of (4.15) with  $u = \mathcal{E}_T(y_0, f)$  satisfies

$$\frac{y}{\rho} \in L^2(0, T; \mathcal{D}(\mathcal{A})) \cap C^0 \left( [0, T]; \mathcal{D} \left( (-\mathcal{A})^{1/2} \right) \right) \cap H^1(0, T; \mathcal{H}).$$

Moreover there exists a constant  $C$  such that

$$\left\| \frac{y}{\rho} \right\|_{L^2(0, T; \mathcal{D}(\mathcal{A})) \cap C^0([0, T]; \mathcal{D}((- \mathcal{A})^{1/2})) \cap H^1(0, T; \mathcal{H})} \leq C \left( \|y_0\|_{\mathcal{D}((- \mathcal{A})^{1/2})} + \|f\|_{L_{\rho_1}^2(0, T; \mathcal{H})} \right).$$

**Remark 4.3.** Note that in [28],  $\mathcal{A}$  is assumed to be self-adjoint negative but the result can be extended to the case where  $\mathcal{A}$  is the generator of an analytic semigroup. Indeed, the hypothesis used in the proof is the maximal regularity of (4.15) for  $v = 0$ .

**Remark 4.4.** Since  $\rho(T) = 0$ , the above result implies in particular that  $y(T) = 0$ , that is the null-controllability of (4.15).

In the previous section, we have defined for our problem the spaces  $\mathcal{H}$ ,  $\mathcal{U}$ ,  $\mathcal{A}$  and  $\mathcal{B}$ , see (4.7), (4.8) and (4.9). To show that  $\mathcal{A}$  is the generator of an analytic semigroup, we first note that  $\mathcal{P}\Delta$  is self-adjoint negative (see, for instance, [31, Theorem 2.1.1, p.128]) and thus is the generator of an analytic semigroup. Then using a perturbation argument (see, for instance, [30, Theorem 2.1, p.80]), we deduce that  $\mathcal{A}$  is the generator of an analytic semigroup in  $\mathcal{H}$ .

Finally, applying Lemma 4.1, we deduce that (4.12) holds with

$$K(T) = C_K e^{\frac{C_K}{T^{2\ell}}},$$

for some constant  $C_K > 0$ .

Let us consider

$$q \in \left( 1, 2^{\frac{1}{4\ell}} \right).$$

and let us set

$$\rho_0(t) := C_K e^{-\frac{C_0}{(T-t)^{2\ell}}}, \quad \rho_1(t) := e^{-\frac{C_1}{(T-t)^{2\ell}}}, \quad \rho(t) := e^{-\frac{C_\star}{(T-t)^{2\ell}}}$$

with  $C_0, C_1, C_\star$  some positive constants such that

$$C_0 := \frac{C_1}{q^{4\ell}} - \frac{C_K}{(q-1)^{2\ell}} > \frac{C_1}{2}, \quad \frac{C_1}{2} < C_\star < C_0 < C_1.$$

Then we can check that (4.13) and (4.14) hold and we have moreover that

$$\rho^2 \leq \rho_1. \tag{4.16}$$

Consequently, we deduce from Theorem 4.2 a controllability result on the system

$$\begin{cases} \partial_t y^{(1)} - \Delta y^{(1)} + \nabla p^{(1)} = \sum_{j=1}^m A_{1,j} y^{(j)} + v e_1 1_\omega + f^{(1)} & \text{in } Q, \\ \partial_t y^{(i)} - \Delta y^{(i)} + \nabla p^{(i)} = \sum_{j=i-1}^m A_{i,j} y^{(j)} + f^{(i)} & \text{in } Q, \quad (2 \leq i \leq m) \\ \nabla \cdot y^{(i)} = 0 & \text{in } Q, \quad (1 \leq i \leq m) \\ y^{(i)} = 0 & \text{on } \Sigma, \quad (1 \leq i \leq m) \\ y^{(i)}(\cdot, 0) = y_0^{(i)} & \text{in } \Omega. \quad (1 \leq i \leq m) \end{cases} \tag{4.17}$$

More precisely, there exists

$$\mathcal{E}_T \in \mathcal{L} \left( \mathcal{V} \times L_{\rho_1}^2(0, T; [L^2(\Omega)^N]^m), L_{\rho_0}^2(0, T; L^2(\omega)) \right)$$

such that for any  $y_0 \in \mathcal{V}$  and for any  $f = (f^{(1)}, \dots, f^{(m)}) \in L_{\rho_1}^2(0, T; [L^2(\Omega)^N]^m)$ , the solution  $y$  of (4.17) with the control  $v = \mathcal{E}_T(y_0, f)$  satisfies

$$\frac{y}{\rho} \in L^2(0, T; [H^2(\Omega)^N]^m) \cap C^0([0, T]; [H^1(\Omega)^N]^m) \cap H^1(0, T; \mathcal{H}). \quad (4.18)$$

Moreover we have the following estimate

$$\left\| \frac{y}{\rho} \right\|_{L^2(0, T; [H^2(\Omega)^N]^m) \cap C^0([0, T]; [H^1(\Omega)^N]^m) \cap H^1(0, T; \mathcal{H})} \leq C \left( \|y_0\|_{\mathcal{V}} + \|f\|_{L_{\rho_1}^2(0, T; [L^2(\Omega)^N]^m)} \right). \quad (4.19)$$

We are now in a position to prove Theorem 1.3:

*Proof of Theorem 1.3.* First we notice that  $y$  is solution of (1.2) if it is a solution of (4.17) with

$$f = \left( -(y^{(1)} \cdot \nabla)y^{(1)}, \dots, -(y^{(m)} \cdot \nabla)y^{(m)} \right).$$

Thus, we consider the mapping

$$\mathcal{N}_T : f \in B_R \mapsto \left( -(y^{(1)} \cdot \nabla)y^{(1)}, \dots, -(y^{(m)} \cdot \nabla)y^{(m)} \right),$$

where

$$B_R := \left\{ f \in L_{\rho_1}^2(0, T; [L^2(\Omega)^N]^m) ; \left\| \frac{f}{\rho_1} \right\|_{L^2(0, T; [L^2(\Omega)^N]^m)} \leq R \right\}$$

where  $R > 0$  is such that

$$\|y_0\|_{\mathcal{V}} \leq R.$$

We are going to show that for  $R$  small enough (and thus  $\|y_0\|_{\mathcal{V}}$  small enough),  $\mathcal{N}_T(B_R) \subset B_R$  and that  $(\mathcal{N}_T)|_{B_R}$  is a strict contraction. Using the Banach fixed point theorem we deduce the existence of a fixed point of  $\mathcal{N}_T$ . The corresponding solution  $y$  of (4.17) is a solution of (1.2) and from (4.18), we deduce that  $y(\cdot, T) = 0$ .

It thus remains to prove that for  $R$  small enough,  $\mathcal{N}_T(B_R) \subset B_R$  and that  $(\mathcal{N}_T)|_{B_R}$  is a strict contraction. In order to do this, we first note that, using (4.16), Sobolev's embeddings and Hölder's inequalities, we have

$$\begin{aligned} \int_0^T \int_{\Omega} \left| \frac{v \cdot \nabla w}{\rho_1} \right|^2 dxdt &\leq \int_0^T \int_{\Omega} \left| \left( \frac{v}{\rho} \right) \cdot \nabla \left( \frac{w}{\rho} \right) \right|^2 dxdt \leq C \left\| \frac{v}{\rho} \right\|_{L^\infty(0, T; L^6(\Omega)^N)}^2 \left\| \frac{\nabla w}{\rho} \right\|_{L^2(0, T; L^6(\Omega)^N)}^2 \\ &\leq C \left\| \frac{v}{\rho} \right\|_{L^\infty(0, T; H^1(\Omega)^N)}^2 \left\| \frac{w}{\rho} \right\|_{L^2(0, T; H^2(\Omega)^N)}^2. \end{aligned} \quad (4.20)$$

Using this relation and (4.19), we deduce that

$$\left\| \frac{\mathcal{N}_T(f)}{\rho_1} \right\|_{L^2(0, T; [L^2(\Omega)^N]^m)} \leq C \left( \|y_0\|_{\mathcal{V}} + \|f\|_{L_{\rho_1}^2(0, T; [L^2(\Omega)^N]^m)} \right)^2 \leq 4CR^2 \leq R,$$

for  $R$  small enough. For such  $R$ , we have  $\mathcal{N}_T(B_R) \subset B_R$ .

Now, let us consider  $\tilde{f}, \hat{f} \in B_R$  and let us write  $f = \tilde{f} - \hat{f}$ . We consider the solution  $\tilde{y}$  (resp.  $\hat{y}$ ) the solution of (4.17) associated with the control  $\tilde{v} = \mathcal{E}_T(y_0, \tilde{f})$  (resp.  $\hat{v} = \mathcal{E}_T(y_0, \hat{f})$ ). Then,  $y := \tilde{y} - \hat{y}$  is the solution of (4.17) associated with the control  $v := \mathcal{E}_T(0, f)$  and thus

$$\left\| \frac{y}{\rho} \right\|_{L^2(0, T; [H^2(\Omega)^N]^m) \cap C^0([0, T]; [H^1(\Omega)^N]^m) \cap H^1(0, T; \mathcal{H})} \leq C \|f\|_{L_{\rho_1}^2(0, T; [L^2(\Omega)^N]^m)}.$$

Using this and (4.20), we obtain

$$\begin{aligned} \left\| \frac{\mathcal{N}_T(\tilde{f})}{\rho_1} - \frac{\mathcal{N}_T(\hat{f})}{\rho_1} \right\|_{L^2(0,T;[L^2(\Omega)^N]^m)} &\leq \left\| \left( \frac{\tilde{y}}{\rho} \right) \cdot \nabla \left( \frac{y}{\rho} \right) \right\|_{L^2(0,T;[L^2(\Omega)^N]^m)} + \left\| \left( \frac{y}{\rho} \right) \cdot \nabla \left( \frac{\hat{y}}{\rho} \right) \right\|_{L^2(0,T;[L^2(\Omega)^N]^m)} \\ &\leq CR \|f\|_{L^2_{\rho_1}(0,T;[L^2(\Omega)^N]^m)}. \end{aligned}$$

Thus for  $R$  small enough,  $(\mathcal{N}_T)_{|_{B_R}}$  is a strict contraction and this ends the proof of Theorem 1.3  $\square$

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