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THE WAVE FRONT SET CORRESPONDENCE FOR DUAL PAIRS WITH ONE MEMBER COMPACT

M. MCKEE, A. PASQUALE, AND T. PRZEBINDA

ABSTRACT. Let W be a real symplectic space and $(\mathrm{G}, \mathrm{G}')$ an irreducible dual pair in $\mathrm{Sp}(\mathrm{W})$, in the sense of Howe, with G compact. Let $\widetilde{\mathrm{G}}$ be the preimage of G in the metaplectic group $\widetilde{\mathrm{Sp}}(\mathrm{W})$. Given an irreducible unitary representation Π of $\widetilde{\mathrm{G}}$ that occurs in the restriction of the Weil representation to $\widetilde{\mathrm{G}}$, let Θ_Π denote its character. We prove that, for a suitable embedding T of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ in the space of tempered distributions on W, the distribution $T(\check{\Theta}_\Pi)$ admits an asymptotic limit, and the limit is a nilpotent orbital integral. As an application, we compute the wave front set of Π', the representation of $\widetilde{\mathrm{G}'}$ dual to Π, by elementary means.

CONTENTS

2010 *Mathematics Subject Classification.* Primary: 22E45; secondary: 22E46, 22E30.

Key words and phrases. Reductive dual pairs, Howe duality, Weil representation, wave front set, orbital integrals.

The second author is grateful to the University of Oklahoma for hospitality and financial support. The third author gratefully acknowledges hospitality and financial support from the Université de Lorraine and partial support from the NSA grant H98230-13-1-0205 and the NSF grant DMS-2225892.

1. **Introduction.**

Let $(\mathrm{G}, \mathrm{G}')$ be an irreducible reductive dual pair with G compact. Thus there is a division algebra $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ with an involution $\mathbb{D} \ni a \to \overline{a} \in \mathbb{D}$ over $\mathbb{R}$, a finite dimensional right $\mathbb{D}$-vector space V, with a positive definite hermitian form $(\cdot, \cdot)$, a finite dimensional right $\mathbb{D}$-vector space V' with a skew-hermitian form $(\cdot, \cdot)'$ so that G is the isometry group of $(\cdot, \cdot)$ and G' is the isometry group of $(\cdot, \cdot)'$. [1] Explicitly, $(\mathrm{G}, \mathrm{G}')$ is one of the pairs [2]

$$(\mathrm{O}_d, \mathrm{Sp}_{2n}(\mathbb{R}))\,, \qquad (\mathrm{U}_d, \mathrm{U}_{p,q})\,, \qquad (\mathrm{Sp}_d, \mathrm{O}^*_{2n}). \tag{1}$$

These groups act on $\mathrm{W} = \mathrm{Hom}_{\mathbb{D}}(\mathrm{V}, \mathrm{V}')$ via post-multiplication and pre-multiplication by the inverse. We set $d = \dim_{\mathbb{D}} \mathrm{V}$ and $d' = \dim_{\mathbb{D}} \mathrm{V}'$.

There is a map

$$\mathrm{Hom}_{\mathbb{D}}(\mathrm{V}, \mathrm{V}') \ni w \to w^* \in \mathrm{Hom}_{\mathbb{D}}(\mathrm{V}', \mathrm{V})$$

defined by

$$(wv, v')' = (v, w^* v') \qquad (v \in \mathrm{V}\,,\ v' \in \mathrm{V}')\,,$$

a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$ on the real vector space W

$$\langle w', w \rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(w^* w') \qquad (w, w' \in \mathrm{W})\,,$$

preserved by the actions of G and G'. Here $\mathrm{tr}_{\mathbb{D}/\mathbb{R}}$ denotes the trace of an endomorphism considered over $\mathbb{R}$. Moreover, we have the unnormalized moment maps

$$\tau : \mathrm{W} \ni w \to w^* w \in \mathfrak{g}\,, \quad \tau' : \mathrm{W} \ni w \to w w^* \in \mathfrak{g}'\,, \tag{2}$$

where $\mathfrak{g}$ and $\mathfrak{g}'$ are the Lie algebras of G and G', respectively. These maps are GG'-equivariant in the sense that

$$\tau(g g'(w)) = g \tau(w) g^{-1}\,, \quad \tau'(g g'(w)) = g' \tau'(w) g'^{-1} \qquad (g \in \mathrm{G}\,,\ g' \in \mathrm{G}'\,,\ w \in \mathrm{W})\,.$$

In particular the fiber $\tau^{-1}(0) \subseteq \mathrm{W}$ is a union of GG'-orbits, which are well known and easy to describe. We collect the relevant facts in the two lemmas below. Since we could not find a reference, their proofs are provided in Appendices A.1 and A.2.

Lemma 1. *Let m be the minimum of d and the Witt index of the form $(\cdot, \cdot)'$. In particular, $d = m$ means that the pair $(\mathrm{G}, \mathrm{G}')$ is in the stable range with* G *the smaller member. Then*

$$\tau^{-1}(0) = \mathcal{O}_m \cup \mathcal{O}_{m-1} \cup \cdots \cup \mathcal{O}_0\,, \tag{3}$$

[1] We use the notation G' for the second member of a dual pair because it is the centralizer of G in Sp(W). We also use the notation $\cdot'$ for all the objects associated with G', such as $\mathfrak{g}'$, Π', ... Unfortunately, this collides with the usual notation for the dual of a linear topological space in functional analysis, also used in this paper, such as $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$, ... We hope the reader will guess from the context the correct meaning of the notation.

[2] The notation for Lie groups is as in Howe [How89]. In particular, we denote the quaternion unitary group $\mathrm{U}_d(\mathbb{H})$ by Sp_d.

where:

- $\mathcal{O}_k \subseteq \mathrm{Hom}(\mathrm{V}, \mathrm{V}')$ *is the subset of elements with isotropic range and rank* k *,*
- $\mathcal{O}_k \cup \mathcal{O}_{k-1} \cup \cdots \cup \mathcal{O}_0$ *is the closure of* $\mathcal{O}_k$ *for* $0 \leq k \leq m$*,*
- $\dim \mathcal{O}_k = \dim_{\mathbb{R}}(\mathbb{D}) \cdot ((d'-k)k + (d-k)d) + \dim_{\mathbb{R}} \mathcal{H}_k(\mathbb{D})$ (4)

and

$$\dim_{\mathbb{R}} \mathcal{H}_k(\mathbb{D}) = \dim_{\mathbb{R}}(\mathbb{D}) \cdot \frac{k(k-1)}{2} + k$$

is the dimension, over $\mathbb{R}$*, of the space* $\mathcal{H}_k(\mathbb{D})$ *of hermitian matrices of size* k *with entries in* $\mathbb{D}$*.*

Set $\mathcal{O}'_k = \tau'(\mathcal{O}_k)$*. Then*

$$\tau'\tau^{-1}(0) = \mathcal{O}'_m \cup \mathcal{O}'_{m-1} \cup \cdots \cup \mathcal{O}'_0$$

where:

- $\mathcal{O}'_k \cup \mathcal{O}'_{k-1} \cup \cdots \cup \mathcal{O}'_0$ *is the closure of* $\mathcal{O}'_k$ *for* $0 \leq k \leq m$ *,*
- $\dim \mathcal{O}'_k = d'k \, \dim_{\mathbb{R}}(\mathbb{D}) - 2 \dim_{\mathbb{R}} \mathcal{SH}_k(\mathbb{D})$ *,* (5)

and

$$2 \dim_{\mathbb{R}} \mathcal{SH}_k(\mathbb{D}) = \begin{cases} k(k-1) & \text{if } \mathbb{D} = \mathbb{R}, \\ 2k^2 & \text{if } \mathbb{D} = \mathbb{C}, \\ 2k(2k+1) & \text{if } \mathbb{D} = \mathbb{H} \end{cases} \tag{6}$$

is twice the dimension, over $\mathbb{R}$*, of the space* $\mathcal{SH}_k(\mathbb{D})$ *of skew-hermitian matrices of size* k *with entries in* $\mathbb{D}$*.*

For an open set U in a finite dimensional real vector space and $t > 0$ such that $tU \subseteq U$, let $M_t^* : \mathcal{D}'(U) \to \mathcal{D}'(U)$ denote the pullback of distributions defined by the submersion $M_t : U \ni v \to tv \in U$, [Hör83, Example 6.1.4]. In particular a distribution $u \in \mathcal{D}'(U)$ is homogeneous of degree $a \in \mathbb{C}$ if $M_t^* u = t^a u$ for every $t > 0$.

Lemma 2. *For each* $k = 0, 1, 2, \ldots, m$*, the orbital integral* $\mu_{\mathcal{O}_k}$ *is a* GG'*-invariant distribution on* W*, homogeneous of degree* $\deg \mu_{\mathcal{O}_k} = \dim \mathcal{O}'_k - \dim \mathrm{W}$*.*

Recall the embedding of the metaplectic group $\widetilde{\mathrm{Sp}}(\mathrm{W})$ into the space of the tempered distributions $\mathcal{S}'(\mathrm{W})$,

$$T : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathcal{S}'(\mathrm{W}) \, , \tag{7}$$

[AP14, Definition 4.23] and the corresponding Weil representation [AP14, Theorem 4.27]. Let $\widetilde{\mathrm{G}}$ be the preimage of G in $\widetilde{\mathrm{Sp}}(\mathrm{W})$.

The main goal of this article is to prove the following theorem and its corollary.

Theorem 3. *Let* Θ_Π *be the character be an irreducible representation* Π *of* $\widetilde{\mathrm{G}}$ *that occurs in the restriction of the Weil representation to* $\widetilde{\mathrm{G}}$*. Then, in the topology of* $\mathcal{S}'(\mathrm{W})$*,*

$$t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* T(\check{\Theta}_\Pi) \underset{t \to 0+}{\longrightarrow} C \mu_{\mathcal{O}_m},$$

where $C \neq 0$,

$$T(\check{\Theta}_\Pi) = \int_{\mathrm{G}} \Theta_\Pi(\tilde{g}^{-1}) T(\tilde{g})\, dg\,,$$

dg is a Haar measure on the group G *and the product* $\Theta_\Pi(\tilde{g}^{-1})T(\tilde{g})$ *does not depend on the element* $\tilde{g}$ *in the preimage of* g *in* G.

Remark 1. *If* Π *is an irreducible admissible representation of a real reductive group* G *with Gelfand-Kirillov dimension* κ, *then* [BV80] *shows that there is a function* u_κ, *homogeneous of degree* $-\kappa$ *and defined on the set* $\mathfrak{g}^{rs}$ *of regular semisimple elements of the Lie algebra* $\mathfrak{g}$ *of* G, *such that*

$$\lim_{t\to 0^+} t^\kappa \Theta_\Pi(\exp(tx)) = u_\kappa(x) \qquad (x \in \mathfrak{g}^{rs})\,. \tag{8}$$

The function u_κ *extends to a tempered distribution on* $\mathfrak{g}$. *Its Fourier transform is a sum of nilpotent orbital integrals over nilpotent orbits of the same dimension* 2κ. *However, the Fourier transform of the left-hand side of* (8) *might even not be well defined. On the other hand, Theorem 3 shows that for* G *compact,* $T(\check{\Theta}_\Pi)$ *admits an asymptotic limit, and the limit is a nilpotent orbital integral on* W.

The limit in Theorem 3 was previously computed in [Prz93, Theorem 6.12], even for dual pairs with a noncompact G, but only on an open dense subset of W. The explicit formula for the intertwining distribution from [MPP21] – see also section 5 – allows us to compute the limit on the entire space W.

Let Π' be the irreducible representation of $\widetilde{\mathrm{G}}'$ corresponding to Π in the Howe's correspondence. As a corollary of Theorem 3, we obtain an elementary computation of $WF(\Pi')$, the wave front of the character $\Theta_{\Pi'}$ at the identity.

Corollary 4. *For any representation* $\Pi \otimes \Pi'$ *that occurs in the restriction of the Weil representation to the dual pair* $(\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}')$,

$$WF(\Pi') = \tau'(\tau^{-1}(0)) = \overline{\mathcal{O}_m}\,.$$

In [Prz93], the wave front set was determined using a computation of the Gelfand-Kirillov dimension and Vogan's results in [Vog78]. For completeness, one should also recall that this dimension was independently computed in [Prz93], [NOT$^+$01] and [EW04]. In this paper, we do not use the notion of Gelfand-Kirillov dimension.

The proofs of Theorem 3 and Corollary 4 are given in sections 5 and 6, respectively.

List of symbols

Function spaces

U = open subset of a smooth manifold
$C_c^\infty(U) = \mathcal{D}(U)$ = the space of compactly supported smooth functions on U
$\mathcal{D}'(U)$ = the space of distributions on U
$\mathcal{S}(U)$ = the Schwartz space on an open subset $U \subseteq \mathbb{R}^n$
$\mathcal{S}'(U)$ = the space of tempered distributions on an open subset $U \subseteq \mathbb{R}^n$

Dual pairs

page, (equation), or theorem

$\mathbb{D}$ $= \mathbb{R}$ (the reals), $\mathbb{C}$ (the complex numbers), or $\mathbb{H}$ (the quaternions) 2

V right $\mathbb{D}$-vector space with positive definite hermitian form $(\cdot,\cdot)$ 2

d $= \dim_{\mathbb{D}} \mathrm{V}$ 2

$\mathsf{V}_{\overline{0}}$ $= \mathbb{D}^d$ as a right $\mathbb{D}$-vector-space, which coincides with the above V in the supergroup realization of dual pairs 6

V' right $\mathbb{D}$-vector space with skew-hermitian form $(\cdot,\cdot)'$ 2

d' $= \dim_{\mathbb{D}} \mathrm{V}'$ 2

$\mathsf{V}_{\overline{1}}$ $= \mathbb{D}^{d'}$ as a right $\mathbb{D}$-vector-space, which coincides with the above V' in the supergroup realization of dual pairs 6

W $= \mathrm{Hom}_{\mathbb{D}}(\mathrm{V}, \mathrm{V}')$, $\mathbb{R}$-vector space with non-degenerate symplectic form $\langle w', w\rangle$ 2

$= \mathrm{Hom}_{\mathbb{D}}(\mathsf{V}_{\overline{0}}, \mathsf{V}_{\overline{1}}) = M_{d',d}(\mathbb{D})$ in the supergroup realization of dual pairs 6

G the isometry group of $(\cdot,\cdot)$ 2

$= \{g \in M_d(\mathbb{D});\ \overline{g}^t g = I_d\}$ in the supergroup realization of dual pairs 6

G^0 the connected component of the identity in G 18

F a fixed element in $\mathrm{GL}_{d'}(\mathbb{D})$ satisfying $F = -\overline{F}^t$ 6, (27)

G' the isometry group of $(\cdot,\cdot)'$ 2

$= \{g \in M_{d'}(\mathbb{D});\ \overline{g}^t F g = F\}$ in the supergroup realization of dual pairs 6

$\mathfrak{g}, \mathfrak{g}'$ the Lie algebras of G and G', respectively 2

l, l' the ranks of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively 15

$\mathfrak{s}_{\overline{0}}$ $= \mathfrak{g} \times \mathfrak{g}'$ diagonally embedded in $M_{d+d'}(\mathbb{D})$ ..... (10)

$\mathfrak{s}_{\overline{1}}$ $= \left\{ x_w = \begin{pmatrix} 0 & w^* \\ w & 0 \end{pmatrix} \in M_{d+d'}(\mathbb{D}); w \in \mathrm{W} \right\}$ ..... (10), (13)

$\mathfrak{s}$ $= \mathfrak{s}_{\overline{0}} \oplus \mathfrak{s}_{\overline{1}}$ ..... (10)

S $= \mathrm{G} \times \mathrm{G}'$ diagonally embedded as a subgroup of $\mathrm{GL}_{d+d'}(\mathbb{D})$ ..... (10)

x_w, w_x the idenfication of W and $\mathfrak{s}_{\overline{1}}$ via $w \to x_w$ and its inverse $x \to w_x$ ..... (13), (14)

$\mathcal{S}'(\mathrm{W})$, $\mathcal{S}'(\mathrm{W})^{\mathrm{S}}$, $C_c^\infty(\mathrm{W})$ are $\mathcal{S}'(\mathfrak{s}_{\overline{1}})$, $\mathcal{S}'(\mathfrak{s}_{\overline{1}})^{\mathrm{S}}$, $C_c^\infty(\mathfrak{s}_{\overline{1}})$, when we identify W and $\mathfrak{s}_{\overline{1}}$ 15

$s.x$ $= \mathrm{Ad}(s)x = sxs^{-1} \quad (s \in \mathrm{S}, x \in \mathfrak{s})$ ..... (11)

$x(w)$ $= \mathrm{ad}(x)(w) = xw - wx \quad (x \in \mathfrak{s}_{\overline{0}},\ w \in \mathfrak{s}_{\overline{1}})$ ..... (11)

τ, τ' the unnormalized moment maps ..... (2)

as maps $\tau : \mathfrak{s}_{\overline{1}} \to \mathfrak{g}$ and $\tau' : \mathfrak{s}_{\overline{1}} \to \mathfrak{g}'$ in the supergroup realization ..... (32)

θ automorphism of $\mathfrak{s}$ restricting to a Cartan involution on $\mathfrak{s}_{\overline{0}}$ and to a negative-definite compatible complex structure on $\mathfrak{s}_{\overline{1}}$ 8

B positive-definite symmetric bilinear form on W, or on $\mathfrak{s}_{\overline{1}}$ using $w \longleftrightarrow x_w$, ... (19)

$\perp_B$ orthogonality with respect to B 8

$\mathfrak{h}$ the symbol used for both the linear span of $\tau(\mathfrak{h}_{\overline{1}}) \subset \mathfrak{g}$ and the linear span of $\tau'(\mathfrak{h}_{\overline{1}}) \subset \mathfrak{g}'$, which are isomorphic. It is a Cartan subalgebra of $\mathfrak{g}$ if $l \le l'$ and a Cartan subalgebra of $\mathfrak{g}'$ if $l \ge l'$ 15

$\mathfrak{z}$ the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ 15

$\mathfrak{z}'$ the centralizer of $\mathfrak{h}$ in $\mathfrak{g}'$ 15

$\pi_{\mathfrak{g}/\mathfrak{h}}$ if $\mathfrak{h}$ is a Cartan subalgebra of a Lie algebra $\mathfrak{g}$, this is the product of a choice of positive roots of $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ 15

$\pi_{\mathfrak{g}/\mathfrak{z}}$ if $\mathfrak{z}$ is the centralizer in $\mathfrak{g}$ of a Cartan subalgebra $\mathfrak{h}$, this is the product of the positive roots of $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ such that the corresponding root spaces do not occur in $\mathfrak{z}_{\mathbb{C}}$ 15

$\mathfrak{h}_{\overline{1}}$ a Cartan subspace in $\mathfrak{s}_{\overline{1}}$ 15

$\mathfrak{h}_{\overline{1}}^{reg}$ the set of regular elements in $\mathfrak{h}_{\overline{1}}$ 15

$\widetilde{\mathrm{Sp}}(\mathrm{W})$ the metaplectic group 3

Θ the distribution character of the Weil representation ..... (68)

T the embedding of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ into the space of the tempered distributions $\mathcal{S}'(\mathrm{W})$ ..... (7), (68)

Orbital integrals and their limits

An integral over the slice through a nilpotent element

Other symbols

2. **A slice through a nilpotent element in the symplectic space.**

We will need the realization of the dual pair $(\mathrm{G}, \mathrm{G}')$ as a supergroup $(\mathrm{S}, \mathfrak{s})$, [Prz06]. We present it in terms of matrices.

Consider $\mathsf{V}_{\overline{0}} = \mathbb{D}^d$ as a right vector space over $\mathbb{D}$ via

$$av := va \qquad (v \in \mathsf{V}_{\overline{0}},\ a \in \mathbb{D})\,.$$

The space $\mathrm{End}_{\mathbb{D}}(\mathsf{V}_{\overline{0}})$ may be identified with the space of square matrices $M_d(\mathbb{D})$ acting on $\mathbb{D}^d$ via left multiplication. Let

$$(v, v') = \overline{v}^t v' \qquad (v, v' \in \mathbb{D}^d\,).$$

This is a positive definite hermitian form on $\mathbb{D}^d$. The isometry group of this form is

$$\mathrm{G} = \{g \in M_d(\mathbb{D});\ \overline{g}^t g = I_d\}\,.$$

Similarly, $\mathsf{V}_{\overline{1}} = \mathbb{D}^{d'}$ is a left vector space over $\mathbb{D}$ and

$$\mathrm{G}' = \{g \in M_{d'}(\mathbb{D});\ \overline{g}^t F g = F\}\,,$$

for a suitable $F = -\overline{F}^t \in \mathrm{GL}_{d'}(\mathbb{D})$. This is the isometry group of the form

$$(v, v')' = \overline{v}^t F v' \qquad (v, v' \in \mathbb{D}^{d'})\,.$$

Set

$$\mathrm{W} = \mathrm{Hom}_{\mathbb{D}}(\mathsf{V}_{\overline{0}}, \mathsf{V}_{\overline{1}}) = M_{d',d}(\mathbb{D})\,,$$

with symplectic form

$$\langle w', w\rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(w^* w') \qquad (w, w' \in M_{d',d}(\mathbb{D}))\,, \tag{9}$$

where $w^* = \overline{w}^t F$. Let

$$\begin{aligned}
&\mathfrak{s}_{\overline{0}} = \mathfrak{g} \times \mathfrak{g}' \quad \text{diagonally embedded in } M_{d+d'}(\mathbb{D})\,,\\
&\mathfrak{s}_{\overline{1}} = \left\{ \begin{pmatrix} 0 & w^* \\ w & 0 \end{pmatrix} \in M_{d+d'}(\mathbb{D}); w \in \mathrm{W} \right\},\\
&\mathrm{S} = \mathrm{G} \times \mathrm{G}' \quad \text{diagonally embedded as a subgroup of } \mathrm{GL}_{d+d'}(\mathbb{D})\,.
\end{aligned} \tag{10}$$

Then $(\mathrm{S}, \mathfrak{s})$ is a real Lie supergroup, i.e. a real Lie group S together with a real Lie superalgebra $\mathfrak{s} = \mathfrak{s}_{\overline{0}} \oplus \mathfrak{s}_{\overline{1}}$, whose even component $\mathfrak{s}_{\overline{0}}$ is the Lie algebra of S. We denote by $[\cdot,\cdot]$ the Lie superbracket on $\mathfrak{s}$. It agrees with the Lie bracket on $\mathfrak{s}_{\overline{0}}$ and with the anticommutator $\{x, y\} = xy + yx$ on $\mathfrak{s}_{\overline{1}}$.

The group S acts on $\mathfrak{s}$ by conjugation. We shall employ the notation

$$s.x = \mathrm{Ad}(s)x = sxs^{-1} \qquad (s \in \mathrm{S}\,,\ x \in \mathfrak{s})\,, \tag{11}$$

$$x(w) = \mathrm{ad}(x)(w) = xw - wx \qquad (x \in \mathfrak{s}_{\overline{0}}\,,\ w \in \mathfrak{s}_{\overline{1}})\,. \tag{12}$$

We shall also write

$$\mathrm{W} = M_{d',d}(\mathbb{D}) \in w \longrightarrow x_w = \begin{pmatrix} 0 & w^* \\ w & 0 \end{pmatrix} \in \mathfrak{s}_{\overline{1}} \tag{13}$$

for the natural vector space isomorphism between W and $\mathfrak{s}_{\overline{1}}$, and

$$\mathrm{W} = M_{d',d}(\mathbb{D}) \in w_x \longleftarrow x \in \mathfrak{s}_{\overline{1}} \tag{14}$$

for its inverse. Under this isomorphisms, the adjoint action of $g \in \mathrm{G} \subseteq \mathrm{S}$ on $\mathfrak{s}_{\overline{1}}$ becomes the action on W by right multiplication by g^{-1}. Similarly, the adjoint action of $g' \in \mathrm{G}' \subseteq \mathrm{S}$ on $\mathfrak{s}_{\overline{1}}$ becomes the action on W by left multiplication by g'. Explicitly,

$$g.x_w = x_{wg^{-1}} \qquad (g \in \mathrm{G}, w \in \mathrm{W})\,, \tag{15}$$

$$g'.x_w = x_{g'w} \qquad (g' \in \mathrm{G}', w \in \mathrm{W})\,. \tag{16}$$

For an endomorphism $h \in \mathrm{End}(\mathrm{W})$, we denote by the same symbol the corresponding endomorphism of $\mathfrak{s}_{\overline{1}}$, given by

$$h(x_w) = x_{h(w)} \qquad (w \in \mathrm{W})\,. \tag{17}$$

Notice that two elements $w, w' \in M_{d',d}(\mathbb{D})$, viewed as members of $\mathfrak{s}_{\overline{1}}$, anticommute if and only if

$$ww'^* + w'w^* = 0 \ \text{ and } \ w^*w' + w'^*w = 0\,. \tag{18}$$

Remark 2. *The unified realization of the dual pair and the symplectic space in the Lie supergroup* $(\mathrm{S}, \mathfrak{s}_{\overline{1}})$ *is convenient in many computations. Distinguishing between the symplectic space* W *and its isomorphic space* $\mathfrak{s}_{\overline{1}}$ *makes the matrix algebra more transparent. Still, most of the representation-theoretic applications of Howe duality prefer focusing on the symplectic space* W *rather than on* $\mathfrak{s}_{\overline{1}}$*. So, later in the paper, when working on orbital integrals in section 3, we will choose to come back to the symplectic picture, which in practice corresponds to identifying* W *and* $\mathfrak{s}_{\overline{1}}$ *under the isomorphism* (13)*. With this identification, we will for instance write* $g.w$*,* $g'.w$ *or* $s.w$ *instead of* $g.x_w$*,* $g'.x_w$ *or* $s.x_w$*, as we did in* (15) *and* (16)*. Correspondingly, the* S*-orbit* $\mathrm{S}.x_w$ *of* $x_w \in \mathfrak{s}_{\overline{1}}$ *will be written* $\mathrm{S}.w$*, called the* S*-orbit of* $w \in \mathrm{W}$ *and denoted* $\mathcal{O}(w)$*. This identification will allow us to refer to the existing literature on the subjet without any serious change of notation.*

We denote by θ the automorphism of $\mathfrak{s}$ defined in [Prz06, sec. 2.1]. See also [DKP05, §5.3]. The construction of θ is done case-by-case and we shall not need these details. It can also be found in [BS17, Proposition 1.1 and §2]. Its restriction to $\mathfrak{s}_{\overline{0}}$ is a Cartan involution and the restriction of $-\theta$ to $\mathfrak{s}_{\overline{1}}$ is a positive definite compatible complex structure. Using (13), we can think of θ and $\langle\cdot,\cdot\rangle$ as maps either on $\mathfrak{s}_{\overline{1}}$ or W. The bilinear form $B(\cdot,\cdot) = -\langle\theta\cdot,\cdot\rangle$ is symmetric and positive definite. Moreover, $-\theta(w) = F^{-1}w$ for $w \in \mathrm{W}$. Hence

$$B(w', w) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\overline{w}^t w') \qquad (w, w' \in \mathrm{W})\,. \tag{19}$$

We can now get into the topic of this section. Fix an element $N \in \mathfrak{s}_{\overline{1}}$. Then $N + [\mathfrak{s}_{\overline{0}}, N] \subseteq \mathfrak{s}_{\overline{1}}$ may be thought of as the tangent space at N to the S-orbit in $\mathfrak{s}_{\overline{1}}$ through N. Denote by $[\mathfrak{s}_{\overline{0}}, N]^{\perp_B} \subseteq \mathfrak{s}_{\overline{1}}$ the B-orthogonal complement of $[\mathfrak{s}_{\overline{0}}, N]$. Since the form B is positive definite, we have a direct sum orthogonal decomposition

$$\mathfrak{s}_{\overline{1}} = [\mathfrak{s}_{\overline{0}}, N] \oplus [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}\,. \tag{20}$$

Consider the map

$$\sigma : \mathrm{S} \times \left(N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}\right) \ni (s, u) \to s.u \in \mathfrak{s}_{\overline{1}}\,. \tag{21}$$

The derivative of σ at (s, u) coincides with the following linear map:

$$\mathfrak{s}_{\overline{0}} \oplus [\mathfrak{s}_{\overline{0}}, N]^{\perp_B} \ni (X, Y) \to [X, s.u] + s.Y \in \mathfrak{s}_{\overline{1}}\,.$$

Therefore the range of the derivative of σ at (s, u) is equal to

$$[\mathfrak{s}_{\overline{0}}, s.u] + s.[\mathfrak{s}_{\overline{0}}, N]^{\perp_B} = s.\left([\mathfrak{s}_{\overline{0}}, u] + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}\right)\,. \tag{22}$$

Let

$$U = \{u \in N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B};\ [\mathfrak{s}_{\overline{0}}, u] + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B} = \mathfrak{s}_{\overline{1}}\}\,. \tag{23}$$

Then U is the maximal open neighborhood of N in $N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$ such that the map

$$\sigma : \mathrm{S} \times U \ni (s, u) \to s.u \in \mathfrak{s}_{\overline{1}} \tag{24}$$

is a submersion. Therefore $\sigma(\mathrm{S} \times U) \subseteq \mathfrak{s}_{\overline{1}}$ is an open S-invariant subset and

$$\sigma : \mathrm{S} \times U \ni (s, u) \to s.u \in \sigma(\mathrm{S} \times U) \tag{25}$$

is a surjective submersion. The title of this section refers to the set U and a nilpotent element $N \in \mathfrak{s}_{\overline{1}}$. Here, nilpotent means nilpotent as a matrix; see (10). Notice that $N \in \mathfrak{s}_{\overline{1}}$ is nilpotent if and only if $\tau(w_N) \in \mathfrak{g}$ is nilpotent, i.e. equal to 0 since G is compact. By (3), it follows that $w_N \in \mathcal{O}_k$ for some $k \in \{0, 1, \ldots, m\}$. We shall use the map (25) to study the S-orbital integrals in $\mathfrak{s}_{\overline{1}}$.

Lemma 5. *Keep the notation of Lemma 1, and let $N \in \mathfrak{s}_{\overline{1}}$ such that $w_N \in \mathcal{O}_k$. Then the map*

$$N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B} \ni u \to u^2 \in \mathfrak{s}_{\overline{0}} \tag{26}$$

is proper (i.e. the preimage of a compact set is compact).

Proof. We can choose the matrix F as follows:

$$F = \begin{pmatrix} 0 & 0 & I_k \\ 0 & F' & 0 \\ -I_k & 0 & 0 \end{pmatrix} \tag{27}$$

with $0 \leq k \leq m$, where m is the minimum of d and the Witt index of the form $(\cdot, \cdot)'$, as in Lemma 1, and F' is a suitable element in $\mathrm{GL}_{d'-2k}(\mathbb{D})$ satisfying $F' = -\overline{F'}^t$. Then, with the block decomposition of an element $M_{d',d}(\mathbb{D}) = M_{d',k}(\mathbb{D}) \oplus M_{d',d-k}(\mathbb{D})$ dictated by (27),

$$\begin{pmatrix} w_1 & w_4 \\ w_2 & w_5 \\ w_3 & w_6 \end{pmatrix}^* = \begin{pmatrix} -\overline{w}_3^t & \overline{w}_2^t F' & \overline{w}_1^t \\ -\overline{w}_6^t & \overline{w}_5^t F' & \overline{w}_4^t \end{pmatrix}.$$

By the assumptions, we may choose $N = \begin{pmatrix} 0 & w_N^* \\ w_N & 0 \end{pmatrix}$ where

$$w_N = \begin{pmatrix} I_k & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{28}$$

Notice that

$$[\mathfrak{s}_{\overline{0}}, N]^{\perp_B} = \theta\left([\mathfrak{s}_{\overline{0}}, N]^{\perp}\right) = \theta\left({}^N\mathfrak{s}_{\overline{1}}\right) = {}^{\theta N}\mathfrak{s}_{\overline{1}},$$

where " $\perp$ " is the orthogonal complement with respect to the symplectic form and the second equality is taken from [Prz06, Lemma 3.1]. Since,

$$w_{\theta N} = -F^{-1} w_N = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -I_k & 0 \end{pmatrix}$$

a straightforward computation using (18) shows that $[\mathfrak{s}_{\overline{0}}, N]^{\perp_B} = \left\{ x \in \mathfrak{s}_{\overline{1}}; w_x \in \mathrm{W}_{[\mathfrak{s}_{\overline{0}}, N]^{\perp_B}} \right\}$, where

$$\mathrm{W}_{[\mathfrak{s}_{\overline{0}}, N]^{\perp_B}} = \left\{ w = \begin{pmatrix} 0 & 0 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix} \in \mathrm{W};\ w_3 = -\overline{w}_3^t \right\}. \tag{29}$$

Let $x = x_w$ with w as in (29). Then the image of $N + x$ under the map (26) consists of pairs of matrices

$$\begin{pmatrix} I_k & 0 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix} \begin{pmatrix} I_k & 0 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix}^* = \begin{pmatrix} w_3 & 0 & I_k \\ -w_5\overline{w}_6^t & w_5\overline{w}_5^t F' & 0 \\ -w_3\overline{w}_3^t - w_6\overline{w}_6^t & w_6\overline{w}_5^t F' & w_3 \end{pmatrix} \in \mathfrak{g}' \tag{30}$$

and

$$\begin{pmatrix} I_k & 0 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix}^* \begin{pmatrix} I_k & 0 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix} = \begin{pmatrix} 2w_3 & w_6 \\ -\overline{w}_6^t & \overline{w}_5^t F' w_5 \end{pmatrix} \in \mathfrak{g}\,. \tag{31}$$

If the set of these pairs varies through a compact set, so do the w_3, w_6 and $w_5\overline{w}_5^t F'$. Hence the claim follows. □

The maps τ, τ' from (2) can be considered as maps $\tau : \mathfrak{s}_{\overline{1}} \to \mathfrak{g}$ and $\tau' : \mathfrak{s}_{\overline{1}} \to \mathfrak{g}'$ by setting

$$\tau(x_w) = \tau(w) = w^* w \quad \text{and} \quad \tau'(x_w) = \tau'(w) = w w^* \qquad (w \in \mathrm{W})\,, \tag{32}$$

or equivalently,

$$\tau(x) = x^2|_{\mathsf{V}_{\overline{0}}} \quad \text{and} \quad \tau'(x) = x^2|_{\mathsf{V}_{\overline{1}}} \qquad (x \in \mathfrak{s}_{\overline{1}})\,,$$

where $|_{\mathsf{V}_{\overline{0}}}$ and $|_{\mathsf{V}_{\overline{1}}}$ respectively indicate the selection of the upper diagonal block of size d or the lower diagonal block of size d'.

Corollary 6. *If $k = m$, then the restriction $\tau|_{N+[\mathfrak{s}_{\overline{0}}, N]^{\perp_B}}$ of $\tau : \mathfrak{s}_{\overline{1}} \to \mathfrak{g}$ to $N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$ is proper.*

Proof. This follows from the formula (31). Indeed, it is enough to see that the map

$$w_5 \to \overline{w}_5^t F' w_5$$

is proper. The variable w_5 does not exist unless $\mathbb{D} = \mathbb{C}$ and $d > m$. This means that m is the Witt index of the form $(\cdot, \cdot)'$. Hence iF' is a definite hermitian matrix. Therefore the above map is proper. □

Corollary 7. *Suppose $k = m$. If $E \subseteq \mathfrak{s}_{\overline{1}}$ is a subset such that $\tau(E) \subseteq \mathfrak{g}$ is bounded, then*

$$E \cap \left(N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B} \right)$$

is bounded.

Proof. This is immediate from Corollary 6. □

3. **Limits of orbital integrals.**

Since we are interested in S-invariant distributions, we want to see dilations by $t > 0$ in $\mathfrak{s}_{\overline{1}}$ as transformations in the slice U modulo the adjoint action of the group S. This will be accomplished in Lemma 8 below.

For $t > 0$ let

$$s_t = \begin{pmatrix} t^{-1}I_k & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & tI_k \end{pmatrix}$$

where the blocks are as in (27). Then $s_t \in \mathrm{G}'$. Define isomorphisms s_t, M_t, g_t of $\mathrm{W} = M_{d',d}(\mathbb{D})$ by

$$\begin{aligned} s_t(w) &= s_t w \qquad (w \in \mathrm{W})\,, \\ M_t(w) &= tw \qquad (w \in \mathrm{W})\,, \end{aligned}$$

and $g_t = M_t \circ s_t$, i.e.

$$g_t(w) = ts_t w \qquad (w \in \mathrm{W})\,.$$

Explicitly,

$$g_t \begin{pmatrix} w_1 & w_4 \\ w_2 & w_5 \\ w_3 & w_6 \end{pmatrix} = \begin{pmatrix} w_1 & w_4 \\ tw_2 & tw_5 \\ t^2w_3 & t^2w_6 \end{pmatrix}. \tag{33}$$

We denote by the same symbols the corresponding linear isomorphisms of $\mathfrak{s}_{\overline{1}}$, as in (17). In particular,

$$g_t(x) = ts_t.x \qquad (x \in \mathfrak{s}_{\overline{1}})\,.$$

Lemma 8. *The linear map $g_t \in \mathrm{GL}(\mathfrak{s}_{\overline{1}})$ preserves $[\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$, $N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$ and the subset $U \subseteq N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$ defined in* (23)*. In fact,*

$$\tau|_U \circ g_t|_U = M_{t^2} \circ \tau|_U\,. \tag{34}$$

Furthermore, for σ as in (24)*,*

$$g_t \circ \sigma = \sigma \circ (\mathrm{Ad}(s_t) \times g_t|_{N+[\mathfrak{s}_{\overline{0}},N]^{\perp_B}})\,, \tag{35}$$

where $g_t|_{N+[\mathfrak{s}_{\overline{0}},N]^{\perp_B}}$ on the right-hand side stands for the restriction of g_t to $N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$. In particular, the subset $\sigma(\mathrm{S} \times U) \subseteq \mathfrak{s}_{\overline{1}}$ is closed under multiplication by positive reals. Moreover, the determinant of the derivative g_t' of the map $g_t : \mathfrak{s}_{\overline{1}} \to \mathfrak{s}_{\overline{1}}$ is

$$\det(g_t') = t^{\dim \mathfrak{s}_{\overline{1}}}\,, \tag{36}$$

and

$$\det((g_t|_{N+[\mathfrak{s}_{\overline{0}},N]^{\perp_B}})') = t^{\dim \mathfrak{s}_{\overline{1}} - \dim \mathcal{O}_k'}\,. \tag{37}$$

Proof. The preservation of $[\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$ and $N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$ follows from (33), (28) and (29). The equality (34) follows from (33) and (31). Notice that

$$\Big[\begin{pmatrix} y & 0 \\ 0 & y' \end{pmatrix}, g_t u \Big] = g_t \Big[\begin{pmatrix} y & 0 \\ 0 & \mathrm{Ad}(s_t^{-1})y' \end{pmatrix}, u \Big] \qquad (y \in \mathfrak{g},\, y' \in \mathfrak{g}',\, t > 0,\, u \in U)\,.$$

So

$$[\mathfrak{s}_{\overline{0}}, g_t u] = g_t[\mathfrak{s}_{\overline{0}}, u] \qquad (t > 0,\, u \in U)\,.$$

Hence

$$[\mathfrak{s}_{\overline{0}}, g_t u] + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B} = [\mathfrak{s}_{\overline{0}}, g_t u] + g_t[\mathfrak{s}_{\overline{0}}, N]^{\perp_B} = g_t([\mathfrak{s}_{\overline{0}}, u] + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B})\,.$$

This implies that the set U is also preserved.

To verify (35), we notice that for $s \in \mathrm{S}$ and $u \in N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$ we have

$$\begin{aligned} g_t \circ \sigma(s,u) &= g_t(s.u) = t(s_t s).u = (s_t s s_t^{-1}).(t s_t.u) \\ &= \sigma(s_t s s_t^{-1}, g_t u) = \sigma \circ (\mathrm{Ad}(s_t) \times g_t|_{N+[\mathfrak{s}_{\overline{0}},N]^{\perp_B}})(s,u)\,. \end{aligned}$$

Fix $t > 0$. The conjugation by $s_{t^{-1}}$ preserves $\sigma(\mathrm{S} \times U)$ because $s_{t^{-1}} \in \mathrm{S}$. Since multiplication by t coincides with $g_t \circ s_{t^{-1}}$, (35) implies that $\sigma(\mathrm{S} \times U)$ is preserved under the multiplication by t.

Since $g_t' = (M_t \circ s_t)' = M_t \circ s_t$ and since $\det s_t = 1$, (36) is obvious.

In order to verify (37) we proceed as follows. The derivative of the map $g_t|_{N+[\mathfrak{s}_{\overline{0}},N]^{\perp_B}}$ coincides with the following linear map

$$\begin{pmatrix} 0 & 0 \\ 0 & w_5 \\ w_3 & w_6 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & t w_5 \\ t^2 w_3 & t^2 w_6 \end{pmatrix}\,.$$

By (29), the determinant of this map is equal to

$$t^{2\dim_{\mathbb{R}} \mathcal{SH}_k(\mathbb{D})} t^{d'(d-k)\dim_{\mathbb{R}} \mathbb{D}}\,.$$

Since, by (5),

$$2\dim_{\mathbb{R}} \mathcal{SH}_k(\mathbb{D}) + d'(d-k)\dim_{\mathbb{R}} \mathbb{D} = \dim \mathfrak{s}_{\overline{1}} - \dim \mathcal{O}_k'\,,$$

(37) follows. □

Next we consider an S-invariant distribution F on $\sigma(\mathrm{S} \times U)$. The following lemma proves that the restriction of F to U exists and that the restriction of the t-dilation of F is equal to $(g_t|_U)^*$ applied to $F|_U$.

Lemma 9. *Suppose $F \in \mathcal{D}'(\sigma(\mathrm{S} \times U))^{\mathrm{S}}$. Then the intersection of the wave front set of F with the conormal bundle to U is zero, so that the restriction $F|_U$ is well defined. Furthermore, $\sigma^* F = \mu_{\mathrm{S}} \otimes F|_U$, where μ_{S} is a Haar measure on* S. *Moreover, for $t > 0$,*

$$M_t^* F = g_t^* F\,, \tag{38}$$

and

$$(M_t^* F)|_U = (g_t|_U)^* F|_U\,. \tag{39}$$

Proof. Since $s_t^* F = F$ we see that $g_t^* F = M_t^* s_t^* F = M_t^* F$ and (38) follows.

The wave front set of F is contained in the union of the conormal bundles to the S-orbits through elements of $\mathfrak{s}_{\overline{1}}$. This is because the characteristic variety of the system of differential equations expressing the condition that this distribution is annihilated by the action of the Lie algebra $\mathfrak{s}_{\overline{0}}$ coincides with that set. The intersection of this set with the

conormal bundle to U is zero. Indeed, at each point $u \in U$, this intersection is equal to the annihilator of both, the tangent space to U at u and the tangent space to the S-orbit through u. Since by (24) the map σ is submersive, these tangent spaces add up to the whole tangent space to $\mathfrak{s}_{\overline{1}}$ at u. Hence the annihilator is zero. Therefore F restricts to U. The formula $\sigma^* F = \mu_{\mathrm{S}} \otimes F|_U$ follows from the diagram

$$U \longrightarrow \mathrm{S} \times U \xrightarrow{\sigma} \sigma(\mathrm{S} \times U), \quad u \to (1, u) \to u\,,$$

which shows that the restriction to U equals the composition of σ^* and the pullback via the embedding of U into $\mathrm{S} \times U$. By combining this with (38) we deduce (39). □

The following lemma shows that the computation of limits of weighted dilations of S-invariant distributions on W may be accomplished by computing weighted limits on the slice U.

Lemma 10. *Suppose $F, F_0 \in \mathcal{D}'(\sigma(\mathrm{S} \times U))^{\mathrm{S}}$ and $a \in \mathbb{C}$ are such that*

$$t^a (g_{t^{-1}}|_U)^* F|_U \underset{t \to 0+}{\longrightarrow} F_0|_U\,.$$

Then

$$t^a M_{t^{-1}}^* F \underset{t \to 0+}{\longrightarrow} F_0$$

in $\mathcal{D}'(\sigma(\mathrm{S} \times U))$.

Proof. Proposition B.1 shows that it suffices to see that

$$\sigma^* \left(t^a M_{t^{-1}}^* F\right) \underset{t \to 0+}{\longrightarrow} \sigma^* F_0\,.$$

But Lemma 9 implies

$$\sigma^* \left(t^a M_{t^{-1}}^* F\right) = \mu_{\mathrm{S}} \otimes t^a (g_{t^{-1}}|_U)^* F|_U \quad \text{and} \quad \sigma^* F_0 = \mu_{\mathrm{S}} \otimes F_0|_U\,.$$

Hence the claim follows. □

Now we are ready to compute the limit of the weighted dilatation of the unnormalized almost semisimple orbital integral $\mu_{\mathcal{O}}$.

Proposition 11. *Let $\mathcal{O} \subseteq \sigma(\mathrm{S} \times U)$ be an* S*-orbit and let $\mu_{\mathcal{O}} \in \mathcal{D}'(\mathfrak{s}_{\overline{1}})$ be the corresponding orbital integral. Then*

$$\lim_{t \to 0+} t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* \mu_{\mathcal{O}}|_{\sigma(\mathrm{S} \times U)} = \mu_{\mathcal{O}}|_U(U)\, \mu_{\mathcal{O}_m}|_{\sigma(\mathrm{S} \times U)}\,, \tag{40}$$

where $\mu_{\mathcal{O}_m} \in \mathcal{D}'(\sigma(\mathrm{S} \times U))$ is the orbital integral on the orbit $\mathcal{O}_m = \mathrm{S}.N$ normalized so that $\mu_{\mathcal{O}_m}|_U$ is the Dirac delta at N and the convergence is in $\mathcal{D}'(\sigma(\mathrm{S} \times U))$.

Before the proof, we make two remarks. First, the scalar $\mu_{\mathcal{O}}|_U(U)$ may be thought of as the volume of the intersection $\mathcal{O} \cap U$. This volume is finite because the restriction $\mu_{\mathcal{O}}|_U$ is a distribution on U with support equal to the closure of $\mathcal{O} \cap U$, which is compact by Corollary 7, since $\tau(\mathcal{O})$ is a G-orbit and therefore bounded. Hence $\mu_{\mathcal{O}}|_U$ applies to any smooth function on U, in particular to the indicator function $\mathbb{I}_U$, equal to 1 on U. Thus $\mu_{\mathcal{O}}|_U(U) = \mu_{\mathcal{O}}|_U(\mathbb{I}_U)$.

The second remark is that our normalization of $\mu_{\mathcal{O}_m}$ does not depend on the normalization of $\mu_{\mathcal{O}}$, which is absorbed by the factor $\mu_{\mathcal{O}}|_U(U)$.

Proof. By the definition of pull-back and (37)

$$\mu_{\mathcal{O}}|_U(\psi \circ g_t|_U) = t^{\dim \mathcal{O}'_m - \dim \mathfrak{s}_{\overline{1}}}(g_{t^{-1}}|_U)^* \mu_{\mathcal{O}}|_U(\psi)\,.$$

(Indeed, for a distribution equal to a function $f(x)$ times the Lebesgue measure,

$$g^*_{t^{-1}} f(\psi) = \int_{\mathfrak{s}_{\overline{1}}} f(g_{t^{-1}}x)\psi(x)\,dx = |\det(g'_t)| \int_{\mathfrak{s}_{\overline{1}}} f(x)\psi(g_t x)\,dx\,. \tag{41}$$

Since $\mu_{\mathcal{O}}|_U$ is a limit of such functions, it has the same transformation property.) We see from (33) that

$$\lim_{t\to 0} g_t u = N \qquad (u \in U)\,.$$

Hence, for any $\psi \in C_c^\infty(U)$,

$$\lim_{t\to 0} \mu_{\mathcal{O}}|_U(\psi \circ g_t) = \mu_{\mathcal{O}}|_U(\psi(N)\mathbb{I}_U) = \mu_{\mathcal{O}}|_U(\mathbb{I}_U)\psi(N) = \mu_{\mathcal{O}}|_U(U)\psi(N)\,. \tag{42}$$

Replacing $\mathcal{O}$ with $\mathcal{O}_m$ in (42) we see that the restriction of $\mu_{\mathcal{O}_m}$ to U is a multiple of the Dirac delta at N. Thus (40) follows from Lemma 10 with $F_0 = \mu_{\mathcal{O}_m}$. □

Next, we want to compute the limit of the weighted dilations of the normalized almost elliptic orbital integrals. We need some additional notation.

For $x, y \in \mathfrak{s}_{\overline{1}}$, let $\{x, y\} = xy + yx \in \mathfrak{s}_{\overline{0}}$ denote their anticommutator. Let $x \in \mathfrak{s}_{\overline{1}}$ be fixed. The anticommutant and the double anticommutant of x in $\mathfrak{s}_{\overline{1}}$ are

$$\begin{aligned} {}^x\mathfrak{s}_{\overline{1}} &= \{y \in \mathfrak{s}_{\overline{1}} : \{x, y\} = 0\}\,, \\ {}^{x\mathfrak{s}_{\overline{1}}}\mathfrak{s}_{\overline{1}} &= \bigcap_{y \in {}^x\mathfrak{s}_{\overline{1}}} {}^y\mathfrak{s}_{\overline{1}}\,, \end{aligned}$$

respectively. A semisimple element $x \in \mathfrak{s}_{\overline{1}}$ is said to be regular if it is nonzero and $\dim(S.x) \geq \dim(S.y)$ for all semisimple $y \in \mathfrak{s}_{\overline{1}}$. A Cartan subspace $\mathfrak{h}_{\overline{1}}$ of $\mathfrak{s}_{\overline{1}}$ is defined as the double anticommutant of a regular semisimple element $x \in \mathfrak{s}_{\overline{1}}$. The Cartan subspaces of $\mathfrak{s}_{\overline{1}}$ are classified in [Prz06, §6]. See also [MPP15, §4] and [MPP20, §2.2] for additional information. We denote by $\mathfrak{h}_{\overline{1}}^{reg}$ the set of regular elements in $\mathfrak{h}_{\overline{1}}$. As in [MPP20, (13)–(15)] the linear spans of $\tau(\mathfrak{h}_{\overline{1}})$ and $\tau'(\mathfrak{h}_{\overline{1}})$ will be identified and both denoted by $\mathfrak{h}$.

Let l and l' denote the ranks of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Then $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ if $l \leq l'$ and $\mathfrak{h} \subseteq \mathfrak{g}'$ is a Cartan subalgebra of $\mathfrak{g}'$ otherwise. One can check that $d > d'$ is equivalent to $l > l'$ except for $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'})$ with $l' = l$.

Let $\mathfrak{z} \subseteq \mathfrak{g}$ and $\mathfrak{z}' \subseteq \mathfrak{g}'$ be the centralizers of $\mathfrak{h}$. Suppose $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and fix a set of positive roots of $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$. Let $\pi_{\mathfrak{g}/\mathfrak{h}}$ denote the product of all positive roots and let $\pi_{\mathfrak{g}/\mathfrak{z}}$ denote the product of all positive roots such that the corresponding root spaces do not occur in $\mathfrak{z}_{\mathbb{C}}$. Similar notations will be used when $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}'$.

Harish-Chandra's almost elliptic orbital integral $F(y) \in \mathcal{S}'(\mathrm{W})^{\mathrm{S}}$ attached to the S-orbit $\mathcal{O}(w)$ was defined in [MPP20, Definition 3.2]. Here $y \in \cup_{\mathfrak{h}_{\overline{1}}} \tau(\mathfrak{h}_{\overline{1}}^{reg})$, the union being on the family of mutually non-S-conjugate Cartan subspaces of $\mathfrak{s}_{\overline{1}}$, and $w \in \mathrm{W}$ is such that

$x_w \in \mathfrak{h}_{\overline{1}}^{reg}$ and $y = \tau(x_w) = \tau(w)$. Observe that, by classification, [Prz06, §6], all Cartan subspaces $\mathfrak{h}_{\overline{1}} \subseteq \mathfrak{s}_{\overline{1}}$ are S-conjugate except when $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}_l, \mathrm{U}_{p,q})$ with $l < p + q$. Besides these exceptional cases, the above union reduces therefore to one term. Following Harish-Chandra's notation, we shall write $F_\phi(y)$ for $F(y)(\phi)$, where $\phi \in \mathcal{S}(\mathrm{W})$.

As indicated in Remark 2, in the following we will adopt the notation from [MPP20] (and references therein) and identify $\mathfrak{s}_{\overline{1}}$ and W by means of the isomorphism (13). So, for instance, $\mathcal{O}(w)$ means $\mathcal{O}(x_w) = \mathrm{S}.x_w$ and we write $w \in \mathfrak{h}_{\overline{1}}^{reg}$ instead of $x_w \in \mathfrak{h}_{\overline{1}}^{reg}$. Moreover, $\mathcal{S}'(\mathrm{W})^{\mathrm{S}} = \mathcal{S}'(\mathfrak{s}_{\overline{1}})^{\mathrm{S}}$, $\mathcal{S}'(\mathrm{W}) = \mathcal{S}'(\mathfrak{s}_{\overline{1}})$, and $C_c^\infty(\mathrm{W}) = C_c^\infty(\mathfrak{s}_{\overline{1}})$.

We refer to [MPP20, Theorems 3.4 and 3.6] for the differentiable extension and regularity properties of the map $y \to F(y)$. These properties of are different when $l > l'$ or $l \leq l'$. These two cases have therefore to be treated separately.

In fact, when $l > l'$, then $F(y)$ turns out to be a constant multiple of Harish-Chandra's orbital integral; see [MPP20, (39)]. When $l \leq l'$, then $F(y)$ can still be related to Harish-Chandra's orbital integral, but the situation is more involved: the differential extension of $F(y)$, up to a specific order, is on the set $\mathfrak{h} \cap \tau(\mathrm{W})$. We refer to [MPP20, Theorem 3.6 and (72)] for more details.

Corollary 12. *Let $l > l'$. Assume (for the construction of U) that $k = m$. Then,*

$$\lim_{t \to 0+} t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* F(y)|_{\sigma(\mathrm{S} \times U)} = F(y)|_U(U)\, \mu_{\mathcal{O}_m}|_{\sigma(\mathrm{S} \times U)}\,. \tag{43}$$

Proof. The statement (43) is immediate from Proposition 11. □

As in [Har57], we identify the symmetric algebra on $\mathfrak{g}$ with $\mathbb{C}[\mathfrak{g}]$, the algebra of the polynomials on $\mathfrak{g}$, using the invariant symmetric bilinear form B on $\mathfrak{g}$.

Lemma 13. *Assume that $l \leq l'$. Let $y \in \mathfrak{h} \cap \tau(\mathrm{W})$ and let $Q \in \mathbb{C}[\mathfrak{h}]$ be such that* $\deg(Q)$ *is small enough so that, by* [MPP20, Theorem 3.6]*, $\partial(Q)F(y)$ exists. Then*

$$t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* \partial(Q) F(y)|_{\sigma(\mathrm{S} \times U)} \underset{t \to 0+}{\longrightarrow} C \mu_{\mathcal{O}_m} \tag{44}$$

in $\mathcal{D}'(\sigma(\mathrm{S} \times U))$, where $C = \partial(Q)F(y)|_U(\mathbb{I}_U)$ is the value of the compactly supported distribution $\partial(Q)F(y)|_U$ on U applied to the indicator function $\mathbb{I}_U$.

Proof. We see from Lemma 10 that it suffices to prove the lemma with (44) replaced by

$$t^{\deg \mu_{\mathcal{O}_m}} \left(g_{t^{-1}}|_U\right)^* \partial(Q) F(y)|_U \underset{t \to 0+}{\longrightarrow} C \delta_N|_U\,, \tag{45}$$

Let $\psi \in C_c^\infty(U)$. Lemma 2, the argument of (41), and the equality (37) show that

$$t^{\deg \mu_{\mathcal{O}_m}} \left(g_{t^{-1}}|_U\right)^* \partial(Q) F(y)|_U(\psi) = \partial(Q) F(y)|_U(\psi \circ g_t)\,.$$

Since $\partial(Q)F(y)|_U$ is a compactly supported distribution on U,

$$\begin{aligned} \partial(Q) F(y)|_U(\psi \circ g_t) &\underset{t \to 0+}{\longrightarrow} \partial(Q) F(y)|_U(\psi(N)\mathbb{I}_U) \\ &= \partial(Q) F(y)|_U(\mathbb{I}_U)\delta_N(\psi)\,. \end{aligned}$$

□

Next we show that the convergence of Lemma 13 happens not only in distributions in $\mathcal{D}'(\sigma(\mathrm{S}\times U))$ but also in $\mathcal{S}'(\mathrm{W})$. This generalization will require Harish-Chandra's Regularity Theorem.

Proposition 14. *Let $y\in\mathfrak{h}\cap\tau(\mathrm{W})$. If $l\le l'$ let $Q\in\mathbb{C}[\mathfrak{h}]$ be such that $\deg(Q)$ is small enough so that, by* [MPP20, Theorem 3.6]*, $\partial(Q)F(y)$ exists. If $l>l'$ set $\partial(Q)F(y)=F(y)$ Then,*

$$t^{\deg\mu_{\mathcal{O}_m}}M^*_{t^{-1}}\partial(Q)F(y)\underset{t\to 0+}{\longrightarrow}C\mu_{\mathcal{O}_m} \tag{46}$$

in the topology of $\mathcal{S}'(\mathrm{W})$, where $C=\partial(Q)F(y)|_U(\mathbb{I}_U)$. Moreover, there is a seminorm q on $\mathcal{S}(\mathrm{W})$ and $N\ge 0$ such that

$$\left|t^{\deg\mu_{\mathcal{O}_m}}M^*_{t^{-1}}\partial(Q)F_\phi(y)\right|\le(1+|y|)^N q(\phi)$$
$$(0<t\le 1,\ y\in\mathfrak{h}\cap\tau(\mathrm{W}),\ \phi\in\mathcal{S}(\mathrm{W}))\,. \tag{47}$$

Proof. Since the pull-back

$$\mathcal{S}(\mathfrak{g}')\ni\psi\to\psi\circ\tau'\in\mathcal{S}(\mathrm{W})$$

is well defined and continuous, we have a push-forward of tempered distributions

$$\mathcal{S}'(\mathrm{W})\ni u\to\tau'_*u\in\mathcal{S}'(\mathfrak{g}')\,,\quad \tau'_*u(\psi)=u(\psi\circ\tau')\,,$$

see [Prz91, (6.1)]. If $l>l'$ then $\tau'_*(F(y))$ is a constant multiple of a semisimple orbital integral supported on the G'-orbit through y in $\mathfrak{g}'$; see [MPP20, (39)–(40)]. As a distribution, it is annihilated by the ideal in $\mathbb{C}[\mathfrak{g}']^{\mathrm{G}'}$ of the polynomials vanishing on that orbit. This is an ideal of finite codimension.

We shall prove a similar statement about $\tau'_*(\partial(Q)F(y))$ in the case $l\le l'$. According to [MPP20, (75) for $\mathrm{G}=\mathrm{O}_{2l+1}$ with $l\le l'$, and (72) otherwise], we may complete $\mathfrak{h}$ to an elliptic Cartan subalgebra $\mathfrak{h}'=\mathfrak{h}\oplus\mathfrak{h}''\subseteq\mathfrak{g}'$ and there is a positive constant C such that for $\psi\in\mathcal{S}(\mathfrak{g}')$

$$\begin{aligned}\tau'_*(\partial(Q)F(y))(\psi) &= \partial(Q)\tau'_*(F(y))(\psi) \qquad (48)\\ &= C\,\partial(Q\tilde\pi_{\mathfrak{z}'/\mathfrak{h}'})\left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+y'')\int_{\mathrm{G}'}\psi(g.(y+y''))\,dg\right)\Big|_{y''=0},\end{aligned}$$

where $y''\in\mathfrak{h}''$, $\tilde\pi_{\mathfrak{z}'/\mathfrak{h}'}=\pi^{short}_{\mathfrak{z}'/\mathfrak{h}'}$ (the product of the positive short roots) if $\mathrm{G}=\mathrm{O}_{2l+1}$ with $l<l'$, and $\tilde\pi_{\mathfrak{z}'/\mathfrak{h}'}=\pi_{\mathfrak{z}'/\mathfrak{h}'}$ otherwise. Let $P\in\mathbb{C}[\mathfrak{g}']^{\mathrm{G}'}$. Then

$$\begin{aligned}&\partial(Q\tilde\pi_{\mathfrak{z}'/\mathfrak{h}'})\left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+y'')\int_{\mathrm{G}'}(P\psi)(g.(y+y''))\,dg\right)\Big|_{y''=0} \qquad (49)\\ =\ &\partial(Q\tilde\pi_{\mathfrak{z}'/\mathfrak{h}'})\left(P(y+y'')\pi_{\mathfrak{g}'/\mathfrak{h}'}(y+y'')\int_{\mathrm{G}'}\psi(g.(y+y''))\,dg\right)\Big|_{y''=0}.\end{aligned}$$

By commuting the operators of multiplication by a polynomial with differentiation, we may write

$$\partial(Q\tilde\pi_{\mathfrak{z}'/\mathfrak{h}'})P(y+y'')=\sum_{|\alpha|\le\deg(Q\tilde\pi_{\mathfrak{z}'/\mathfrak{h}'})}P_\alpha(y+y'')\partial^\alpha\,,$$

where $\partial^\alpha = \prod_{j=1}^{l'} \partial(J'_j)^{\alpha_j}$ for $\alpha = (\alpha_1, \dots, \alpha_{l'})$. Hence, (49) is equal to

$$\sum_{|\alpha| \le \deg(Q\tilde{\pi}_{\mathfrak{z}'/\mathfrak{h}'})} P_\alpha(y) \partial^\alpha \left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y + y'') \int_{\mathrm{G}'} \psi(g.(y + y''))\, dg \right) \Big|_{y''=0} . \tag{50}$$

We see from (48)–(50) that the range of the map

$$\mathbb{C}[\mathfrak{g}']^{\mathrm{G}'} \ni P \to \tau'_*(\partial(Q)F(y)) \cdot P \in \mathcal{S}'(\mathfrak{g}') \tag{51}$$

is contained in the space spanned by the distributions

$$\partial^\alpha \left(\pi_{\mathfrak{g}'/\mathfrak{h}'}(y + y'') \int_{\mathrm{G}'} \psi(g.(y + y''))\, dg \right) \Big|_{y''=0} \qquad (|\alpha| \le \deg(Q\tilde{\pi}_{\mathfrak{z}'/\mathfrak{h}'})) .$$

In particular this range is finite dimensional. Therefore the distribution (48) is annihilated by an ideal of finite co-dimension in $\mathbb{C}[\mathfrak{g}']^{\mathrm{G}'}$.

Hence, in any case ($l > l'$ or $l \le l'$), the Fourier transform

$$(\tau'_*(\partial(Q)F(y)))^\wedge \in \mathcal{S}'(\mathfrak{g}') \tag{52}$$

is annihilated by an ideal of finite co-dimension in $\partial(\mathbb{C}[\mathfrak{g}']^{\mathrm{G}'})$. Here $\partial(\mathbb{C}[\mathfrak{g}']^{\mathrm{G}'})$ is the algebra of G'-invariant constant-coefficient differential operators on $\mathfrak{g}'$. Now Harish-Chandra Regularity Theorem [Har65, Theorem 1, page 11] implies that the distribution (52) is a locally integrable function whose restriction to the set of the regular semisimple elements has a known structure. Specifically, Harish-Chandra's formula for the radial component of a G'-invariant differential operator with constant coefficients on $\mathfrak{g}'$ together with [Har64, Lemma 19] shows that the restriction

$$\pi_{\mathfrak{g}'/\mathfrak{h}'} \, (\tau'_*(\partial(Q)F(y)))^\wedge \mid_{\mathfrak{h}'^{reg}}$$

is annihilated by an ideal of finite co-dimension in $\partial(\mathbb{C}[\mathfrak{h}'])$. Hence, for any connected component $C(\mathfrak{h}'^{reg}) \subseteq \mathfrak{h}'^{reg}$ there is an exponential polynomial $\sum_j p_j e^{\lambda_j}$ such that

$$(\tau'_*(\partial(Q)F(y)))^\wedge \mid_{C(\mathfrak{h}'^{reg})} = \frac{1}{\pi_{\mathfrak{g}'/\mathfrak{h}'}} \sum_j p_j e^{\lambda_j} ; \tag{53}$$

see e.g. [War72, Lemma 2, Appendix to 8.3.1]. Let

$$p(x) = \sum_j p_j(x) e^{\lambda_j(x)} \qquad (x \in C(\mathfrak{h}'^{reg})) .$$

This function extends analytically beyond the connected component and for any $k = 1, 2, 3, \dots$ we have Taylor's formula, as in [Hör83],

$$p(x) = \sum_{|\alpha|<k} \partial^\alpha p(0) \frac{x^\alpha}{\alpha!} + k \int_0^1 (1-s)^{k-1} \sum_{|\alpha|=k} \partial^\alpha p(sx)\, ds \frac{x^\alpha}{\alpha!} . \tag{54}$$

Since the distribution (52) is tempered, the real parts of the λ_j are non-positive on $C(\mathfrak{h}'^{reg})$. Furthermore, the λ_j depend linearly on y and the p_j depend polynomially on y. Therefore

a straightforward argument shows that there is $N > 0$ such that

$$|\partial^\alpha p(tx)| \le \text{constant} \cdot (1+|y|)^N (1+|x|)^N \sum_{|\alpha|=k} \left| \frac{x^\alpha}{\alpha!} \right| . \tag{55}$$

Hence (52) is a finite sum of homogeneous distributions, of possibly negative degrees, plus the error term which is bounded by (55). Thus there is an integer a such that the following limit exists in $\mathcal{S}'(\mathfrak{g}')$:

$$\lim_{t\to 0+} t^a M_t^* \left(\tau'_*(\partial(Q)F(y))\right)^\wedge . \tag{56}$$

Moreover, there is a seminorm q on $\mathcal{S}(\mathfrak{g}')$ and $N \ge 0$ such that

$$|t^a M_t^* \left(\tau'_*(\partial(Q)F(y))\right)^\vee(\psi)| \le (1+|y|)^N q(\psi)$$
$$(0 < t \le 1,\ y \in \mathfrak{h} \cap \tau(\mathrm{W}),\ \psi \in \mathcal{S}(\mathfrak{g}')) . \tag{57}$$

By taking the inverse Fourier transform we see that there is an integer b such that the following limit exists in $\mathcal{S}'(\mathfrak{g}')$:

$$\lim_{t\to 0+} t^b M_{t^{-1}}^* \tau'_*(\partial(Q)F(y)) . \tag{58}$$

Moreover, there is a seminorm q on $\mathcal{S}(\mathfrak{g}')$ and $N \ge 0$ such that

$$\left| \lim_{t\to 0+} t^b M_{t^{-1}}^* \tau'_*(\partial(Q)F(y))(\psi) \right| \le (1+|y|)^N q(\psi)$$
$$(0 < t \le 1,\ y \in \mathfrak{h} \cap \tau(\mathrm{W}),\ \psi \in \mathcal{S}(\mathfrak{g}')) . \tag{59}$$

Notice that the following equivalent formulas hold.

$$\begin{aligned} (\psi \circ \tau')_t &= t^{2\,\dim \mathfrak{g}' - \dim \mathrm{W}} \psi_{t^2} \circ \tau' \qquad (\psi \in \mathcal{S}(\mathfrak{g}')) , \\ \tau'_*(M_{t^{-1}}^* u) &= t^{\dim \mathrm{W} - 2\dim \mathfrak{g}'} M_{t^{-2}}^* \tau'_*(u) \qquad (u \in \mathcal{S}'(\mathrm{W})) . \end{aligned} \tag{60}$$

The injectivity of the map τ'_*, see Corollary [MPP20, (6)], and (60) imply that there is an integer n such that the following limit exists in $\mathcal{S}'(\mathrm{W})$,

$$\lim_{t\to 0+} t^n M_{t^{-1}}^* \partial(Q)F(y) . \tag{61}$$

Now Lemma 13 shows that $n = \deg \mu_{\mathcal{O}_m}$ and the proposition follows. □

4. **An integral over the slice through a nilpotent element.**

4.1. **Normalization of measures.** Recall from section 2 the positive definite symmetric bilinear form $B(\cdot,\cdot) = -\langle \theta\cdot, \cdot\rangle$ on $\mathfrak{s}$. We normalize the Lebesgue measure on $\mathfrak{s}$ so that the volume of unit cube, defined in terms of $B(\cdot,\cdot)$, is 1.

Let $\mathrm{G}^0 \subseteq \mathrm{G}$ denote the connected component of the identity and set $-\mathrm{G}^0 = \{-g; g \in \mathrm{G}^0\}$. Recall that for our compact group G, the Cayley transform $c(x) = (x+1)(x-1)^{-1}$ maps $\mathfrak{g}$ onto $-\mathrm{G}^0$. Notice that $\mathrm{G} = \mathrm{G}^0 = -\mathrm{G}^0$ if $\mathrm{G} = \mathrm{U}_d$ or Sp_d. Set $r = \frac{2\dim_{\mathbb{R}} \mathfrak{g}}{\dim_{\mathbb{R}} \mathrm{V}}$. Then,

as checked in [Prz91, (3.11)], one may normalize the Haar measure on the group G so that

$$dc(x) = |\det_{\mathbb{R}}(1-x)|^{-r}\, dx \qquad (x \in \mathfrak{g})\,.$$

The proof presented in [Prz91, (3.11)] is valid for $\mathrm{G} \neq \mathrm{O}_{2n+1}$. In the case $\mathrm{G} = \mathrm{O}_{2n+1}$ a parallel argument works too. This is different than the normalization given in [Hel84, Theorem 1.14].

Having normalized the measures, we may study the distributions on W, $\mathfrak{g}$ and G as "generalized functions", in the sense that they are derivatives of continuous functions multiplied by the corresponding measures, as in [Hör83, section 6.3].

4.2. **Some geometry of the moment map.** Fix an element $N \in \mathfrak{s}_{\overline{1}}$ such that $w_N \in \mathcal{O}_m$, see Lemma 1. Let $\mathrm{G}_N \subseteq \mathrm{G}$ be the stabilizer of N and let $\mathfrak{g}_N \subseteq \mathfrak{g}$ be the Lie algebra of G_N. Then we have a direct sum decomposition, orthogonal with respect to the form $B(\cdot,\cdot)$ of section 2,

$$\mathfrak{g} = \mathfrak{g}_N \oplus \mathfrak{g}_N^{\perp_B}\,.$$

Recall the subspaces

$$[\mathfrak{s}_{\overline{0}}, N]^{\perp_B} \subseteq \mathfrak{s}_{\overline{1}} \qquad \text{and} \qquad \mathrm{W}_{[\mathfrak{s}_{\overline{0}},N]^{\perp_B}} \subseteq \mathrm{W}$$

defined in (20) and (29). Let $R_N \subseteq \mathrm{W}_{[\mathfrak{s}_{\overline{0}},N]^{\perp_B}}$ denote the radical of the restriction of the symplectic form $\langle\cdot,\cdot\rangle$ to $\mathrm{W}_{[\mathfrak{s}_{\overline{0}},N]^{\perp_B}}$, and let $\mathrm{W}_N \subseteq \mathrm{W}_{[\mathfrak{s}_{\overline{0}},N]^{\perp_B}}$ denote the orthogonal complement of R_N with respect to the form $B(\cdot,\cdot)$. Then either $\mathrm{W}_N = 0$ or the restriction of the symplectic form $\langle\cdot,\cdot\rangle$ to W_N is non-degenerate and

$$\mathrm{W}_{[\mathfrak{s}_{\overline{0}},N]^{\perp_B}} = R_N \oplus \mathrm{W}_N\,.$$

In the notation of the proof of Lemma 5, we have

$$R_N = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ w_3 & w_6 \end{pmatrix} ;\ w_3 \in \mathcal{SH}_m(\mathbb{D})\,,\quad w_6 \in M_{m,d-m}(\mathbb{D}) \right\}$$

$$\mathrm{W}_N = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & w_5 \\ 0 & 0 \end{pmatrix} ;\ w_5 \in M_{d'-2m,d-m}(\mathbb{D}) \right\}$$

if $d > m$, and

$$R_N = \left\{ \begin{pmatrix} 0 \\ 0 \\ w_3 \end{pmatrix} ;\ w_3 \in \mathcal{SH}_m(\mathbb{D})\,,\quad w_6 \in M_{m,d-m}(\mathbb{D}) \right\}$$

$$\mathrm{W}_N = 0$$

if $d = m$.

Suppose $d > m$. Then $\mathrm{W}_N = 0$ if and only if $d' = 2m$, i.e. $(\cdot,\cdot)'$ is split. Hence $\mathrm{W}_N \neq 0$ if and only if $\mathrm{G}' = \mathrm{U}_{p,q}$ with $p = m < q = m + (d' - 2m)$.

Lemma 15. *The map*

$$R_N \ni v \to \tau(w_N + v) \in \mathfrak{g}_N^{\perp_B} \tag{62}$$

is an $\mathbb{R}$-linear bijection. The absolute value of the determinant of the matrix of this map defined in terms of any orthonormal basis is equal to

$$2^{\dim_{\mathbb{R}} \mathcal{SH}_m(\mathbb{D}) + \frac{1}{2}\dim_{\mathbb{R}} M_{m,d-m}(\mathbb{D})} = 2^{\frac{1}{2}\dim_{\mathbb{R}} \mathcal{SH}_m(\mathbb{D})} 2^{\frac{1}{2}\dim \mathfrak{g}_N^{\perp_B}} . \tag{63}$$

Proof. An orthonormal basis of R_N consists of the matrices

$$\begin{aligned} &\frac{1}{\sqrt{2}}\left(E_{p,q} - E_{q,p}\right), \quad 1 \le p < q \le m, \\ &E_{r,s}, \quad m < r, s \le d, \end{aligned}$$

if $\mathbb{D} = \mathbb{R}$, and of the matrices

$$\begin{aligned} &\frac{1}{\sqrt{2}}\left(E_{p,q} - E_{q,p}\right), \quad 1 \le p < q \le m, \\ &\gamma E_{p,p}, \quad 1 \le p \le m, \\ &\frac{\gamma}{\sqrt{2}}\left(E_{p,q} + E_{q,p}\right), \quad 1 \le p < q \le m, \\ &\gamma' E_{r,s}, \quad m < r, s \le d, \end{aligned}$$

where $\gamma = i$, $\gamma' = 1, i$ if $\mathbb{D} = \mathbb{C}$, and $\gamma = i, j, k$, $\gamma' = 1, i, j, k$, if $\mathbb{D} = \mathbb{H}$.

An orthonormal basis of $\mathfrak{g}$ consists of the matrices

$$\frac{1}{\sqrt{2}}\left(E_{p,q} - E_{q,p}\right), \quad 1 \le p < q \le d,$$

if $\mathbb{D} = \mathbb{R}$, and of the matrices

$$\begin{aligned} &\frac{1}{\sqrt{2}}\left(E_{p,q} - E_{q,p}\right), \quad 1 \le p < q \le d, \\ &\gamma E_{p,p}, \quad 1 \le p \le m, \\ &\frac{\gamma}{\sqrt{2}}\left(E_{p,q} + E_{q,p}\right), \quad 1 \le p \le q \le d, \end{aligned}$$

where $\gamma = i$ if $\mathbb{D} = \mathbb{C}$, and $\gamma = i, j, k$, if $\mathbb{D} = \mathbb{H}$.

As we have seen in (31), the map (62) is given by the formula

$$R_N \ni \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ w_3 & w_6 \end{pmatrix} \longrightarrow \begin{pmatrix} 2w_3 & w_6 \\ -\overline{w}_6^t & 0 \end{pmatrix} \in \mathfrak{g}.$$

Since

$$\mathfrak{g}_N = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x_{22} \end{pmatrix};\ x_{22} = -\overline{x_{22}}^t \in M_{d-m}(\mathbb{D}) \right\}$$

and

$$\mathfrak{g}_N^{\perp_B} = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ -\overline{x_{12}}^t & 0 \end{pmatrix};\ x_{11} = -\overline{x_{11}}^t \in M_m(\mathbb{D}),\ x_{12} \in M_{m,d-m}(\mathbb{D}) \right\}$$

the $\mathbb{R}$-linearity and bijectivity of the map (62) follows.

Also, this map sends an element of our orthonormal basis contained in the w_3 block to 2 times an element of the orthonormal basis contained in the x_{11} block. Furthermore, it sends an element of our orthonormal basis contained in the w_6 block to $\sqrt{2}$ times an element of the orthonormal basis contained in the $\begin{pmatrix} 0 & x_{12} \\ -\overline{x_{12}}^t & 0 \end{pmatrix}$ block. Hence, (63) follows. □

Lemma 16. *Let $\tau_N : \mathrm{W}_N \to \mathfrak{g}_N$ be the unnormalized moment map. Then*

$$\tau(w_N + v + w) = \tau(w_N + v) + \tau_N(w) \qquad (v \in R_N\,,\ w \in \mathrm{W}_N)\,, \tag{64}$$

where $\tau(w_N + v) \in \mathfrak{g}_N^{\perp_B}$. If $\mathrm{W}_N = 0$, then the map (64) coincides with the map (62).

Proof. This is immediate from the formulas (29) and (31). □

4.3. **The integral as a distribution on $\mathfrak{g}$.** Recall the character $\chi(t) = e^{2\pi i t}$, $t \in \mathbb{R}$, and the imaginary Gaussians

$$\chi_x(w) = \chi(\frac{1}{4}\langle xw, w\rangle) = \chi(\frac{1}{4}\operatorname{tr}_{\mathbb{D}/\mathbb{R}}(x\tau(w))) \qquad (x \in \mathfrak{g}\,,\ w \in \mathrm{W})\,. \tag{65}$$

As usual, by (14), we can consider χ_x as a function on $\mathfrak{s}_{\overline{1}}$ by setting

$$\chi_x(y) = \chi_x(w_y) \qquad (y \in \mathfrak{s}_{\overline{1}})\,.$$

Fix an element $\tilde{c}(0) \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ lifting $c(0) = -1$. Since $\mathfrak{g}$ is simply connected, there is a unique continuous (in fact real analytic) lift $\tilde{c} : \mathfrak{g} \to \widetilde{\mathrm{G}}$ passing through $\tilde{c}(0)$. Then $\tilde{c} : \mathfrak{g}_N \to \widetilde{\mathrm{G}}_N$. Since G is compact, the Cayley transform c maps $\mathfrak{g}$ onto the dense subset of $-\mathrm{G}^0$ consisting of the elements g such that $\det(g-1) \neq 0$. The fixed normalization of the measure on G is so that on $\widetilde{c(\mathfrak{g})} \subseteq \widetilde{-\mathrm{G}^0}$ we have

$$d\tilde{c}(x) = dc(x) \qquad (x \in \mathfrak{g})\,.$$

Lemma 17. *Recall the slice $U = N + [\mathfrak{s}_{\overline{0}}, N]^{\perp_B}$ through N, (23). As a distribution on $\mathfrak{g}$,*

$$\int_U \chi_x(u)\,dx\,du = C\delta_{\mathfrak{g}_N^{\perp_B}}(x^{\perp_B})\frac{\Theta_{\mathrm{W}_N}(\tilde{c}(0)\tilde{c}(x_N))}{\Theta_{\mathrm{W}_N}(\tilde{c}(0))\Theta_{\mathrm{W}_N}(\tilde{c}(x_N))}\,dx_N \qquad (x \in \mathfrak{g})\,, \tag{66}$$

where $C = 2^{\frac{3}{2}\dim \mathfrak{g}_N^{\perp_B}} 2^{-\frac{1}{2}\dim_{\mathbb{R}} \mathcal{SH}_m(\mathbb{D})}$, $x = x^{\perp_B} + x_N$, $x^{\perp_B} \in \mathfrak{g}_N^{\perp_B}$, $x_N \in \mathfrak{g}_N$, $\delta_{\mathfrak{g}_N^{\perp_B}}$ is Dirac delta at 0 on $\mathfrak{g}_N^{\perp_B}$, and Θ_{W_N} is the character of the Weil representation of $\widetilde{\mathrm{Sp}}(\mathrm{W}_N)$ attached to the same character χ. If $\mathrm{W}_N = 0$ then $\Theta_{\mathrm{W}_N} = 1$.

Proof. We see from Lemma 16 that

$$\int_U \chi_x(u)\,dx\,du = \int_{R_N} \chi_{x^{\perp_B}}(w_N + v)\,dx^{\perp_B}dv \int_{\mathrm{W}_N} \chi_{x_N}(w)\,dx_N\,dw\,.$$

Lemma 15 implies that

$$\int_{R_N} \chi_{x^{\perp_B}}(w_N + v)\, dx^{\perp_B}\, dv = 2^{-\frac{1}{2}\dim \mathfrak{g}_N^{\perp_B}} 2^{-\frac{1}{2}\dim_{\mathbb{R}} \mathcal{SH}_m(\mathbb{D})} \int_{\mathfrak{g}_N^{\perp_B}} \chi(\frac{1}{4}\operatorname{tr}_{\mathbb{D}/\mathbb{R}}(yx^{\perp_B}))\, dx^{\perp_B}\, dy$$
$$= 2^{-\frac{1}{2}\dim \mathfrak{g}_N^{\perp_B}} 2^{-\frac{1}{2}\dim_{\mathbb{R}} \mathcal{SH}_m(\mathbb{D})} \delta_{\mathfrak{g}_N^{\perp_B}}(\frac{1}{4}x^{\perp_B}) = 2^{\frac{3}{2}\dim \mathfrak{g}_N^{\perp_B}} 2^{-\frac{1}{2}\dim_{\mathbb{R}} \mathcal{SH}_m(\mathbb{D})} \delta_{\mathfrak{g}_N^{\perp_B}}(x^{\perp_B})\,.$$

Furthermore, by evaluating both sides of the equation [AP14, (139)] at $w = 0$ we see that

$$\Theta_{\mathrm{W}_N}(\tilde{c}(0))\Theta_{\mathrm{W}_N}(\tilde{c}(x_N)) \int_{\mathrm{W}_N} \chi_{x_N}(w)\, dx_N\, dw = \Theta_{\mathrm{W}_N}(\tilde{c}(0)\tilde{c}(x_N))\, dx_N\,.$$

(Here we are using the convention on "generalized functions" we introduced in subsection 4.1. So, with the notation of [AP14, (139)], $t(\tilde{c}(x))(w) = \chi_x(w)$ and $[t(\tilde{c}(0))\natural t(\tilde{c}(x_N))](w) = [1\natural\chi_{x_N}](w)$, where $\natural$ denotes the twisted convolution on W_N. Since χ_{x_N} is even, we conclude that $[t(\tilde{c}(0))\natural t(\tilde{c}(x_N))](0) = [1\natural\chi_{x_N}](0) = \int_{\mathrm{W}_N} \chi_{x_N}(w)\, dw$.) □

4.4. **The integral as a distribution on $\widetilde{-\mathrm{G}^0}$.** As in [AP14, (138)], we consider the embedding

$$T : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathcal{S}'(\mathrm{W}) \tag{67}$$

of the metaplectic group into the space of tempered distributions on the symplectic space. In particular,

$$T(\tilde{c}(x)) = \Theta(\tilde{c}(x))\chi_x(w)\, dw \qquad (x \in \mathfrak{g}\,,\ w \in \mathrm{W})\,, \tag{68}$$

where Θ denotes the character of the Weil representation of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ attached to the character χ.

Suppose $\mathrm{W}_N \neq 0$. The structure of our dual pair is such that the metaplectic covering

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \supseteq \widetilde{\mathrm{G}} \to \mathrm{G} \subseteq \mathrm{Sp}(\mathrm{W})$$

restricts to the metaplectic covering

$$\widetilde{\mathrm{Sp}}(\mathrm{W}_N) \supseteq \widetilde{\mathrm{G}}_N \to \mathrm{G}_N \subseteq \mathrm{Sp}(\mathrm{W}_N)\,.$$

Indeed, G_N consists of the elements of G of the block diagonal form $\begin{pmatrix} I_m & 0 \\ 0 & g \end{pmatrix}$. The dual pair $(\mathrm{G}_N, \mathrm{G}'_N)$ is of the same type as $(\mathrm{G}, \mathrm{G}')$, with G'_N consisting of elements of the form $\begin{pmatrix} I_m & 0 & 0 \\ 0 & g' & 0 \\ 0 & 0 & I_m \end{pmatrix}$. The dimension of the defining space V'_N of G'_N is $d' - 2m$, which has the same parity as d'. The claim therefore follows from [MPP21, Appendix D].

In particular we have an inclusion $\iota : \widetilde{\mathrm{G}}_N \to \widetilde{\mathrm{G}}$ and hence the pull-back of test functions $\iota^* : C_c^\infty(\widetilde{\mathrm{G}}) \to C_c^\infty(\widetilde{\mathrm{G}}_N)$ and push-forward of distributions $\iota_* : \mathcal{D}'(\widetilde{\mathrm{G}}_N) \to \mathcal{D}'(\widetilde{\mathrm{G}})$. By restriction, we get

$$\iota_* : \mathcal{D}'(\widetilde{-\mathrm{G}^0_N}) \to \mathcal{D}'(\widetilde{-\mathrm{G}^0})\,. \tag{69}$$

If $\mathrm{W}_N = 0$ and $d > m$ (and hence the form $(\cdot,\cdot)'$ is split), then we still have (69), where the coverings are in $\widetilde{\mathrm{Sp}}(\mathrm{W})$. It follows from [AP14, Proposition 4.28] that in the above two cases, the formula

$$\chi_+(\tilde{g}) = \frac{\Theta(\tilde{g})}{|\Theta(\tilde{g})|} \qquad (\tilde{g} \in \widetilde{\mathrm{G}}) \tag{70}$$

defines a group homomorphism $\chi_+ : \widetilde{\mathrm{G}} \to \mathbb{C}^\times$, because there is a complete polarization of W preserved by G. Indeed, such a polarization is $\mathrm{W} = \mathrm{X} \oplus \mathrm{Y}$, where X and Y are the spaces of the first m rows and of the last m rows of W, respectivement. In particular, χ_+ restricts to a character of $\widetilde{\mathrm{G}}_N$. Notice that χ_+ is a character of $\widetilde{\mathrm{G}}$ whenever there is a polarization is $\mathrm{W} = \mathrm{X} \oplus \mathrm{Y}$ such that G preserves X and Y to fit into [AP14, Proposition 4.28]. This is always the case when the form $(\cdot,\cdot)'$ is split.

If $\mathrm{W}_N = 0$ and $d = m$, then $\mathrm{G}_N = 1$. In this case we artificially enlarge G_N to be the center $\mathrm{Z} = \{1, -1\}$ of the symplectic group $\mathrm{Sp}(\mathrm{W})$. Then $\widetilde{\mathrm{G}}_N = \widetilde{\mathrm{Z}}$ and, as checked in [MPP21, (22)] the formula

$$\chi_+(\tilde{g}) = \frac{\Theta(\tilde{g})}{|\Theta(\tilde{g})|} \qquad (\tilde{g} \in \widetilde{\mathrm{G}}_N) \tag{71}$$

defines a group homomorphism $\chi_+ : \widetilde{\mathrm{G}}_N \to \mathbb{C}^\times$.

Lemma 18. *Suppose* $\mathrm{W}_N = 0$. *Then, as a distribution on* $\widetilde{-\mathrm{G}^0}$,

$$\int_U T(\tilde{g})(u)\, d\tilde{g}\, du = C 2^{-\frac{1}{2}\dim \mathrm{W}} \iota_*(\chi_+(\tilde{g}_N)\, d\tilde{g}_N)\,, \tag{72}$$

where $C = 2^{\frac{3}{2}\dim \mathfrak{g}_N^{\perp_B}} 2^{-\frac{1}{2}\dim_{\mathbb{R}} \mathcal{SH}_m(\mathbb{D})}$ *and* $d\tilde{g}_N$ *is the Haar measure on* $\widetilde{-\mathrm{G}^0_N}$.

Proof. We compute using Lemma 17,

$$\begin{aligned}
\int_U T(\tilde{c}(x))(u)\, d\tilde{c}(x)\, du &= \Theta(\tilde{c}(x)) \int_U \chi_x(u) |\det(1-x)|^{-r}\, dx\, du \\
&= C\Theta(\tilde{c}(x_N)) \delta_{\mathfrak{g}_N^{\perp_B}}(x^{\perp_B}) |\det(1-x_N)|^{-r}\, dx_N \\
&= C\delta_{\mathfrak{g}_N^{\perp_B}}(x^{\perp_B}) \chi_+(\tilde{c}(x_N)) |\Theta(\tilde{c}(x_N))| |\det(1-x_N)|^{-r}\, dx_N \\
&= C 2^{-\frac{1}{2}\dim \mathrm{W}} \delta_{\mathfrak{g}_N^{\perp_B}}(x^{\perp_B}) \chi_+(\tilde{c}(x_N)) |\det(1-x_N)|^{\frac{d'}{2}-r}\, dx_N \\
&= C 2^{-\frac{1}{2}\dim \mathrm{W}} \delta_{\mathfrak{g}_N^{\perp_B}}(x^{\perp_B}) \chi_+(\tilde{c}(x_N))\, d\tilde{c}(x_N)\,,
\end{aligned}$$

because (a straightforward computation shows that) $\frac{d'}{2} - r = \frac{2\dim_{\mathbb{R}} \mathfrak{g}_N}{\dim_{\mathbb{R}} \mathrm{V}_N}$, where $\mathrm{V}_N \subseteq \mathrm{V}$ is the defining module for G_N. □

Lemma 19. *Suppose* $\mathrm{W}_N \neq 0$. *(Equivalently,* $d > m$ *and the form* $(\cdot,\cdot)'$ *is not split.) Then, as a distribution on* $\widetilde{-\mathrm{G}^0}$,

$$\int_U T(\tilde{g})(u)\, d\tilde{g}\, du = C 2^{\frac{1}{2}\dim \mathrm{W} - \dim \mathrm{W}_N} \chi_+(\tilde{c}(0))^{-1} \iota_*(\Theta_{\mathrm{W}_N}(\tilde{c}(0)\tilde{g}_N)\, d\tilde{g}_N)\,, \tag{73}$$

where $C = 2^{\frac{3}{2}\dim \mathfrak{g}_N^{\perp_B}} 2^{-\frac{1}{2}\dim_{\mathbb{R}} \mathcal{SH}_m(\mathbb{D})}$ *and* $d\tilde{g}_N$ *is the Haar measure on* $\widetilde{-\mathrm{G}_N^0}$.

Proof. We compute using Lemma 17,

$$\int_U T(\tilde{c}(x))(u)\, d\tilde{c}(x)\, du = C\delta_{\mathfrak{g}_N^{\perp_B}}(x^{\perp_B})\Theta(\tilde{c}(x))\frac{\Theta_{\mathrm{W}_N}(\tilde{c}(0)\tilde{c}(x_N))}{\Theta_{\mathrm{W}_N}(\tilde{c}(0))\Theta_{\mathrm{W}_N}(\tilde{c}(x_N))}|\det(1-x_N)|^{-r}\, dx$$
$$= C\delta_{\mathfrak{g}_N^{\perp_B}}(x^{\perp_B})\frac{1}{\Theta(\tilde{c}(0))}\frac{\Theta(\tilde{c}(0))}{\Theta_{\mathrm{W}_N}(\tilde{c}(0))}\frac{\Theta(\tilde{c}(x_N))}{\Theta_{\mathrm{W}_N}(\tilde{c}(x_N))}\Theta_{\mathrm{W}_N}(\tilde{c}(0)\tilde{c}(x_N))|\det(1-x_N)|^{-r}\, dx\,.$$

Notice that

$$\mathfrak{g}_N \ni x_N \to \frac{\Theta(\tilde{c}(x_N))}{\Theta_{\mathrm{W}_N}(\tilde{c}(x_N))}\left|\frac{\Theta_{\mathrm{W}_N}(\tilde{c}(x_N))}{\Theta(\tilde{c}(x_N))}\right| = \frac{\Theta(\tilde{c}(x_N))}{|\Theta(\tilde{c}(x_N))|}\frac{|\Theta_{\mathrm{W}_N}(\tilde{c}(x_N))|}{\Theta_{\mathrm{W}_N}(\tilde{c}(x_N))} \in \mathbb{C}^\times \tag{74}$$

is a continuous function taking values in a finite set. The latter property is a consequence of [AP14, Proposition 4.28]: G may be considered as a subgroup of GL(X), where $\mathrm{X}\oplus\mathrm{Y}$ is the polarization of W. Then $\chi_+(\tilde{g})$ is written in terms of $\det_{\mathrm{X}}^{-1/2}(\tilde{g})$, which can assume a finite set of values because the image of $\det_{\mathrm{X}}|_{\tilde{\mathrm{G}}}$ is a compact subgroup of $\mathbb{R}^\times$. Hence (74) is constant, equal to its value at 0, which is 1. So

$$\frac{\Theta(\tilde{c}(x_N))}{\Theta_{\mathrm{W}_N}(\tilde{c}(x_N))} = \left|\frac{\Theta(\tilde{c}(x_N))}{\Theta_{\mathrm{W}_N}(\tilde{c}(x_N))}\right|\,.$$

Therefore

$$\frac{\Theta(\tilde{c}(0))}{\Theta_{\mathrm{W}_N}(\tilde{c}(0))}\frac{\Theta(\tilde{c}(x_N))}{\Theta_{\mathrm{W}_N}(\tilde{c}(x_N))} = \left|\frac{\Theta(\tilde{c}(0))}{\Theta_{\mathrm{W}_N}(\tilde{c}(0))}\frac{\Theta(\tilde{c}(x_N))}{\Theta_{\mathrm{W}_N}(\tilde{c}(x_N))}\right|\,.$$

The only dual pair that satisfies the assumptions of this Lemma is $(\mathrm{G},\mathrm{G}') = (\mathrm{U}_d, \mathrm{U}_{m,m+(d'-2m)})$ with $d'-2m>0$. In terms of matrices, as in the proof of Lemma 5,

$$\mathrm{G}_N = \mathrm{U}_{d-m}\,,\quad \mathrm{W}_N = M_{d'-2m,d-m}\,.$$

Hence,

$$r = \frac{2\dim_{\mathbb{R}}\mathfrak{g}}{\dim_{\mathbb{R}}\mathbb{C}^d} = d\,,\quad r_N = \frac{2\dim_{\mathbb{R}}\mathfrak{g}_N}{\dim_{\mathbb{R}}\mathbb{C}^{d-m}} = d-m$$

and therefore

$$\frac{d'}{2} - \frac{d'-2m}{2} - r = -r_N\,.$$

Thus

$$\left|\frac{\Theta(\tilde{c}(x_N))}{\Theta_{\mathrm{W}_N}(\tilde{c}(x_N))}\right||\det(1-x_N)|^{-r} = \left|\frac{\Theta(\tilde{c}(0))}{\Theta_{\mathrm{W}_N}(\tilde{c}(0))}\right||\det(1-x_N)|^{-r_N}\,.$$

Therefore

$$\int_U T(\tilde{c}(x))(u)\, d\tilde{c}(x)\, du = C\delta_{\mathfrak{g}_N^{\perp_B}}(x^{\perp_B})\frac{1}{\chi_+(\tilde{c}(0))}\frac{1}{|\Theta(\tilde{c}(0))|}\left|\frac{\Theta(\tilde{c}(0))}{\Theta_{\mathrm{W}_N}(\tilde{c}(0))}\frac{\Theta(\tilde{c}(x_N))}{\Theta_{\mathrm{W}_N}(\tilde{c}(x_N))}\right|$$
$$\times\, \Theta_{\mathrm{W}_N}(\tilde{c}(0)\tilde{c}(x_N))|\det(1-x_N)|^{-r}\, dx_N$$
$$= C\left|\frac{\Theta(\tilde{c}(0))}{\Theta_{\mathrm{W}_N}(\tilde{c}(0))^2}\right|\delta_{\mathfrak{g}_N^{\perp_B}}(x^{\perp_B})\frac{1}{\chi_+(\tilde{c}(0))}\Theta_{\mathrm{W}_N}(\tilde{c}(0)\tilde{c}(x_N))|\det(1-x_N)|^{-r_N}\, dx_N$$

and the formula follows. □

5. **Proof of the main theorem.**

Here we verify Theorem 3. We begin with an intermediate statement. Recall the connected identity component $\mathrm{G}^0 \subseteq \mathrm{G}$. Retain the notation of the previous subsection.

Theorem 20. *Let Π be an irreducible representation of $\widetilde{\mathrm{G}}$ that occurs in the restriction of the Weil representation to $\widetilde{\mathrm{G}}$. Then, in the topology of $\mathcal{S}'(\mathrm{W})$,*

$$t^{\deg \mu_{\mathcal{O}_m}} M^*_{t^{-1}} T(\check{\Theta}_\Pi|_{\widetilde{-\mathrm{G}^0}}) \underset{t\to 0+}{\longrightarrow} K\mu_{\mathcal{O}_m}\,, \tag{75}$$

where $K \neq 0$.

Suppose $d = m$ or $d > m$ and $(\cdot,\cdot)'$ is split. (Equivalently, suppose (G,G') is different from $(\mathrm{U}_d, \mathrm{U}_{m,d'-m})$ with $d' - 2m > 0$.) Then

$$K = C 2^{\frac{1}{2}\dim \mathrm{W}} \chi_{\Pi\otimes\chi_+^{-1}}(-1) \int_{\mathrm{G}_N^0} \Theta_{\Pi\otimes\chi_+^{-1}}(g_N)\, dg_N\,, \tag{76}$$

where C is as in Lemma 18 and χ_+ is the character defined in (70)*. The integral in* (76) *is equal to the multiplicity of the trivial representation of G_N^0 in the restriction of $\Pi\otimes\chi_+^{-1}$ to G_N^0.*

Suppose $(\mathrm{G},\mathrm{G}') = (\mathrm{U}_d, \mathrm{U}_{m,d'-m})$ with $d' - 2m > 0$. Then

$$K = C 2^{\frac{1}{2}\dim \mathrm{W} - \dim \mathrm{W}_N} \chi_\Pi(\tilde{c}(0)) \int_{\mathrm{G}_N} \Theta_\Pi(\widetilde{g_N}^{-1}) \Theta_{\mathrm{W}_N}(\widetilde{g_N})\, dg_N\,, \tag{77}$$

where C is as above, $\mathrm{G}_N = \mathrm{U}_{d-m}$, and Θ_{W_N} is the character of the Weil representation of $\widetilde{\mathrm{Sp}}(\mathrm{W}_N)$. The integral in (77) *is equal to the sum of multiplicities of the irreducible component of $\Pi|_{\widetilde{\mathrm{G}_N}}$ in the restriction of ω_N to $\widetilde{\mathrm{G}_N}$.*

Notice that if $\mathrm{G} = \mathrm{U}_d$ or Sp_d then $-\mathrm{G}^0 = \mathrm{G}^0 = \mathrm{G}$. Hence, in these cases, Theorem 20 is equivalent to Theorem 3.

Proof. We first prove that the limit in (75) exists and is a constant multiple of $\mu_{\mathcal{O}_m}$. For this, we use the expression of $T(\check{\Theta}_\Pi|_{\widetilde{-\mathrm{G}^0}})$ in terms of Harish-Chandra's almost elliptic orbital integrals $F(y) \in \mathcal{S}'(\mathrm{W})^{\mathrm{S}}$ determined in [MPP21]. We need some additional notation. If $l \leq l'$, let $(J_1,\dots,J_l)$ be the basis of $\mathfrak{h}$ introduced in [MPP21, (42)]. If $l > l'$, extend $\mathfrak{h}$ to the Cartan subalgebra $\mathfrak{h}(\mathfrak{g})$ of $\mathfrak{g}$, with basis $(J_1,\dots,J_l)$ defined as in [MPP21, (45)]. Then $(J_1,\dots,J_{l'})$ is a basis of $\mathfrak{h}$. We denote by $(y_1,\dots,y_l)$ (respectively, $(y_1,\dots,y_{l'})$) the coordinates of $y \in \mathfrak{h}$ with respect to these bases. Let $(J_1^*,\dots,J_l^*)$ be the dual basis of $\mathfrak{h}^*$ if $l \leq l'$ (respectively, of $\mathfrak{h}(\mathfrak{g})^*$ if $l > l'$), and set $e_j = iJ_j^*$ for $1 \leq j \leq l$. The Harish-Chandra parameter $\mu = \sum_{j=1}^l \mu_j e_j$ of Π is strictly dominant. In this paper, this means that $\mu_1 > \mu_2 > \cdots > \mu_l$.

For $1 \leq j \leq l$ set

$$a_j = -\mu_j + \delta - 1 \quad \text{and} \quad b_j = -\mu_j + \delta - 1\,,$$

where

$$\delta - 1 = \begin{cases} l' - l & \text{if } \mathrm{G} = \mathrm{O}_{2l} \\ l' - l - \frac{1}{2} & \text{if } \mathrm{G} = \mathrm{O}_{2l+1} \\ \frac{l'-l-1}{2} & \text{if } \mathrm{G} = \mathrm{U}_l \\ l' - l - 1 & \text{if } \mathrm{G} = \mathrm{Sp}_{2l}. \end{cases}$$

Furthermore, set $\beta = 4\pi$ if $\mathrm{G} = \mathrm{Sp}_l$ and $\beta = 2\pi$ otherwise.

Suppose first that $l \le l'$. Then, according to [MPP21, Theorem 2],

$$T(\check{\Theta}_{\Pi}|_{\widetilde{-\mathrm{G}^0}})(\phi) = C \int_{\mathfrak{h}\cap\tau(\mathrm{W})} \left(\prod_{j=1}^{l} \big(p_j(y_j) + q_j(\partial_{y_j})\delta_0(y_j)\big) \right) \cdot F(y)(\phi)\, dy \qquad (\phi \in \mathcal{S}(\mathrm{W}))\,, \tag{78}$$

where

$$p_j(y) = P_{a_j,b_j}(-\beta y) e^{-\beta|y|} \quad \text{and} \quad q_j(y) = \beta^{-1} Q_{a_j,b_j}(\beta^{-1} y)\,,$$

and P_{a_j,b_j} and Q_{a_j,b_j} are polynomial functions on $(-\infty, 0]$ and on $[0, +\infty)$. The explicit expression of P_{a_j,b_j} and Q_{a_j,b_j} does not play any role here, but one needs to notice that $P_{a_j,b_j} = 0$ if $a_j \le 0$ and $b_j \le 0$ (i.e. if $|\mu_j| \le \delta - 1$), and in this case $Q_{a_j,b_j} \ne 0$.

The domain of integration $\mathfrak{h}\cap\tau(\mathrm{W})$ is described in [MPP20, Lemma 3.5]. It agrees with $\mathfrak{h}$ unless $\mathrm{G} = \mathrm{U}_l$. If $\mathrm{G} = \mathrm{U}_l$, then $\mathfrak{h} \cap \tau(\mathrm{W})$ is a union of closed orthants associated with the fixed basis $(J_1, \dots, J_l)$ of $\mathfrak{h}$. In all cases, the right-hand-side of (78) is the constant C times a finite sum of integrals of the form

$$\int_{Y_I} \prod_{j\in I} p_j(y_j) \Big(\prod_{j\in I^c} q_j(\partial_{y_j}) F(y)(\phi) \Big)\Big|_{y_{I^c}=0} dy_I\,, \tag{79}$$

where $I = \{j_1, \dots, j_I\}$ is a (possibly empty) subset of $\{j \in \{1, 2, \dots, l\} : p_j \ne 0\}$, $I^c = \{1, 2, \dots, l\} \setminus I$, the integration domain is $Y_I = \prod_{j\in I} Y_j$ where Y_j can be $(-\infty, 0]$, $[0, +\infty)$ or $\mathbb{R}$, and $dy_I = dy_{j_1} \cdots dy_{j_l}$.

Proposition 14, the exponential decay of the p_j's in (79) and the Lebesgue Dominated Convergence Theorem imply that

$$\begin{aligned} &\lim_{t\to 0+} t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* T(\check{\Theta}_{\Pi}|_{\widetilde{-\mathrm{G}^0}}) \\ &\qquad = \left(C \int_{\mathfrak{h}\cap\tau(\mathrm{W})} \left(\prod_{j=1}^{l} \big(p_j(y_j) + q_j(\partial_{y_j})\delta_0(y_j)\big) \right) \cdot F(y)|_U(\mathbb{I}_U)\, dy \right) \mu_{\mathcal{O}_m}\,. \end{aligned} \tag{80}$$

Suppose now that $l > l'$. According to [MPP21, Theorem 3],

$$T(\check{\Theta}_{\Pi}|_{\widetilde{-\mathrm{G}^0}})(\phi) = C \int_{\tau'(\mathfrak{h}_{\overline{1}}{}^{reg})} \Big(\prod_{j\in I_0} p_j((s_0^{-1} y)_j) \Big) \cdot F(y)(\phi)\, dy \qquad (\phi \in \mathcal{S}(\mathrm{W}))\,, \tag{81}$$

where s_0 is a suitable element of $W(\mathrm{G}, \mathfrak{h}(\mathfrak{g}))$ and

$$I_0 = \begin{cases} \{1, \dots, q\} \cup \{l - p + 1, \dots, l\} & \text{if } \mathrm{G} = \mathrm{U}_l \\ \{1, \dots, l'\} & \text{otherwise}\,. \end{cases} \tag{82}$$

With respect to the fixed basis $(J_1, \dots, J_{l'})$ of $\mathfrak{h}$, the integration domain $\tau'(\mathfrak{h}_{\overline{1}}^{reg})$ is a dense subset of the positive orthant. As in the case $l \geq l'$, Proposition 14, the exponential decay of the p_j's and the Lebesgue Dominated Convergence Theorem imply that

$$\lim_{t\to 0+} t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* T(\check{\Theta}_\Pi|_{\widetilde{c(\mathfrak{g})}}) = \left(C \int_{\tau'(\mathfrak{h}_{\overline{1}}^{reg})} \Big(\prod_{j\in J_0} p_j((s_0^{-1}y)_j) \Big) F(y)|_U(\mathbb{I}_U)\, dy \right) \mu_{\mathcal{O}_m}\,. \quad (83)$$

Thus, in each case, the limit is a constant multiple of the measure $\mu_{\mathcal{O}_m}$. This constant is the term in parenthesis in (80) or in (83). It is equal to

$$T(\check{\Theta}_\Pi|_{\widetilde{-\mathrm{G}^0}})|_U(\mathbb{I}_U) = \int_U \int_{-\mathrm{G}^0} \Theta_\Pi(\tilde{g}^{-1}) T(\tilde{g})(u)\, dg\, du\,. \quad (84)$$

We need to prove that it is non-zero.

Suppose $d = m$ (stable range) or $d > m$ and the form $(\cdot,\cdot)'$ is split. Then Lemma 18 implies that (84) is equal to

$$C 2^{\frac{1}{2}\dim \mathrm{W}} \int_{-\mathrm{G}_N^0} \Theta_\Pi(\tilde{g}_N^{-1}) \chi_+(\tilde{g}_N)\, dg_N\,.$$

Furthermore,

$$\int_{-\mathrm{G}_N^0} \Theta_\Pi(\tilde{g}_N^{-1}) \chi_+(\tilde{g}_N)\, dg_N = \int_{-\mathrm{G}_N^0} \Theta_{\Pi\otimes\chi_+^{-1}}(\tilde{g}_N^{-1})\, dg_N = \int_{-\mathrm{G}_N^0} \Theta_{\Pi\otimes\chi_+^{-1}}(g_N)\, dg_N\,,$$

where in the last formula $\Pi\otimes\chi_+^{-1}$ is viewed as a representation of G_N. Thus

$$T(\check{\Theta}_\Pi|_{\widetilde{-\mathrm{G}^0}})|_U(\mathbb{I}_U) = C 2^{\frac{1}{2}\dim \mathrm{W}} \int_{-\mathrm{G}_N^0} \Theta_{\Pi\otimes\chi_+^{-1}}(g_N)\, dg_N\,. \quad (85)$$

Since -1 is in the center of $\mathrm{Sp}(\mathrm{W})$, it acts via multiplication by a scalar $\chi_{\Pi\otimes\chi_+^{-1}}(-1)$ on $\Pi\otimes\chi_+^{-1}$. Therefore

$$\int_{-\mathrm{G}_N^0} \Theta_{\Pi\otimes\chi_+^{-1}}(g_N)\, dg_N = \chi_{\Pi\otimes\chi_+^{-1}}(-1) \int_{\mathrm{G}_N^0} \Theta_{\Pi\otimes\chi_+^{-1}}(g_N)\, dg_N\,.$$

Hence, (76) follows.

The integral in (76) is the multiplicity of the trivial representation of G_N^0 in the restriction of $\Pi\otimes\chi_+^{-1}$ to G_N^0. If $\mathrm{G}_N = \{1\}$, i.e. $d = m$, then this multiplicity is equal to the degree of Π. Otherwise, there are three cases:

$$\begin{aligned} &\mathrm{G} = \mathrm{O}_d\,, && \mathrm{G}' = \mathrm{Sp}_{2m}(\mathbb{R}), && \mathrm{G}_N = \mathrm{O}_{d-m}\,,\\ &\mathrm{G} = \mathrm{Sp}_d\,, && \mathrm{G}' = \mathrm{O}^*_{2m}(\mathbb{R}), && \mathrm{G}_N = \mathrm{Sp}_{d-m}\,,\\ &\mathrm{G} = \mathrm{U}_d\,, && \mathrm{G}' = \mathrm{U}_{m,m}, && \mathrm{G}_N = \mathrm{U}_{d-m}\,,\,. \end{aligned}$$

Suppose that $\mathrm{G} = \mathrm{O}_d$ or Sp_d. Since Π occurs in Howe's correspondence, by [Prz96, (A.4.2.1) and (A.6.2)], the highest weight of $\Pi\otimes\chi_+^{-1}$ is $\lambda = (\lambda_1, \dots, \lambda_m, 0, \dots, 0)$, where the last $d-m$ entries are equal to 0 and $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$ are integers. We then recognize that the trivial representation of G_N^0 occurs in the restriction of $\Pi\otimes\chi_+^{-1}$ to G_N^0

by iterating the branching laws $\mathrm{SO}_n \downarrow \mathrm{SO}_{n-1}$ or $\mathrm{Sp}_n \downarrow \mathrm{Sp}_{n-1}$, see e.g. [Kna02, Theorems 9.16 and 9.18]. If $\mathrm{G} = \mathrm{U}_d$, then by [Prz96, (A.5.2)], the highest weight of $\Pi \otimes \chi_+^{-1}$ is $\lambda = (\mu_1, \dots, \mu_s, 0, \dots, 0, -\nu_r, \dots, -\nu_1)$, where $0 \le s \le m$, $0 \le r \le m$, $r + s \le d$, and $\mu_1 \ge \cdots \ge \mu_s > 0$ and $\nu_1 \ge \cdots \ge \mu_r > 0$ are integers. Notice that there are $d - (r + s)$ zero entries in the central part of λ. The highest weights of the irreducible representations occurring in the branching $\mathrm{U}_n \downarrow \mathrm{U}_{n-1}$ interleave λ, see e.g. [Kna02, Theorems 9.14]. Iterating these branching laws m times therefore allows the highest weight of all zero entries. Hence the trivial representation of G_N^0 occurs in the restriction of $\Pi \otimes \chi_+^{-1}$ to G_N^0 in this case too.

Let us now consider the remaining cases, i.e. when $\mathrm{W}_N \neq 0$. Lemma 19 implies that (84) is equal to

$$C 2^{\frac{1}{2}\dim \mathrm{W} - \dim \mathrm{W}_N} \int_{-\mathrm{G}_N^0} \Theta_\Pi(\tilde g_N^{-1}) \Theta_{\mathrm{W}_N}(\tilde c(0) \tilde g_N)\, d g_N \,.$$

Notice that

$$\begin{aligned} \int_{-\mathrm{G}_N^0} \Theta_\Pi(\tilde g_N^{-1}) \Theta_{\mathrm{W}_N}(\tilde c(0) \tilde g_N)\, d g_N &= \int_{\mathrm{G}_N^0} \Theta_\Pi(\tilde c(0) \tilde g_N^{-1}) \Theta_{\mathrm{W}_N}(\tilde g_N)\, d g_N \\ &= \chi_\Pi(\tilde c(0)) \int_{\mathrm{G}_N^0} \Theta_\Pi(\tilde g_N^{-1}) \Theta_{\mathrm{W}_N}(\tilde g_N)\, d g_N \,, \end{aligned}$$

where χ_Π is the central character of Π.

Notice that G_N is isomorphic to U_{d-m}. Hence $\mathrm{G}_N^0 = \mathrm{G}_N$ and the centralizer of G_N in $\mathrm{Sp}(\mathrm{W}_N)$ is compact, isomorphic to $\mathrm{U}_{d'-2m}$. Thus we have the dual pair $(\mathrm{U}_{d-m}, \mathrm{U}_{d'-2m})$ inside $\widetilde{\mathrm{Sp}}(\mathrm{W}_N)$. The restriction $\Pi|_{\widetilde{\mathrm{G}_N}}$ decomposes into a finite sum of irreducibles and the integral

$$\int_{\widetilde{\mathrm{G}_N}} \Theta_\Pi(\tilde g_N^{-1}) \Theta_{\mathrm{W}_N}(\tilde g_N)\, d \tilde g_N \tag{86}$$

is the sum of the multiplicities of those irreducibles that occur in the restriction of ω_N to $\widetilde{\mathrm{G}_N}$. Again, looking at the highest weight λ of Π, [Prz96, (A.5.2)], and the branching rules $\mathrm{U}_n \downarrow \mathrm{U}_{n-1}$, e.g. [Kna02, Theorems 9.14], we see that the irreducible representation of $\widetilde{\mathrm{U}_{d-m}}$ whose highest weight has the central $d - m$ components of λ is a representation of $\widetilde{\mathrm{U}_{d-m}}$ occurring in both the restriction of Π and the restriction of ω_N to $\widetilde{\mathrm{U}_{d-m}}$. Thus the number (86) is not zero. □

Now we consider the dual pairs $(\mathrm{G}, \mathrm{G}')$ for which $-\mathrm{G}^0 \neq \mathrm{G}$. They are isomorphic to $(\mathrm{O}_d, \mathrm{Sp}_{2l'}(\mathbb{R}))$. More precisely, $\mathrm{G} \setminus (-\mathrm{G}^0) = \mathrm{G}^0$ if $\mathrm{G} = \mathrm{O}_{2l+1}$, and $\mathrm{G} \setminus (-\mathrm{G}^0) = \mathrm{G} \setminus \mathrm{G}^0$ if $\mathrm{G} = \mathrm{O}_{2l}$. Here $\mathrm{G} \setminus (-\mathrm{G}^0)$ is the complement of $-\mathrm{G}^0$ in G. We need to know how to compute

$$\lim_{t \to 0+} t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* T(\check\Theta_\Pi|_{\widetilde{\mathrm{G} \setminus (-\mathrm{G}^0)}}) \,. \tag{87}$$

Suppose $d = 1$. Then $\mathrm{G} = \mathrm{O}_1$ and $\mathrm{G}^0 = \{1\}$. Hence $T(\check\Theta_\Pi|_{\widetilde{\mathrm{G}^0}}) = T(1) = \delta$. Also, $\mathcal{O}_m = \mathrm{W} \setminus \{0\}$ and $\mu_{\mathcal{O}_m}$ is the Lebesgue measure. Hence $\deg \mu_{\mathcal{O}_m} = 0$ and we see that

(87) is equal to

$$\lim_{t\to 0+} t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* T(\check{\Theta}_\Pi|_{\widetilde{\mathrm{G}\setminus(-\mathrm{G}^0)}}) = \lim_{t\to 0+} M_{t^{-1}}^* \delta = \lim_{t\to 0+} t^{\dim \mathrm{W}} \delta = 0\,. \tag{88}$$

Assume from now on that $d > 1$. As shown in [MPP21, section 4], there is a symplectic subspace $\mathrm{W}_s \subseteq \mathrm{W}$ such that the restriction of the dual pair $(\mathrm{G}, \mathrm{G}')$ to W_s is isomorphic to $(\mathrm{O}_{d-1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ and the following statements hold, where T_s is the map (67) for the dual pair $(\mathrm{G}_s, \mathrm{G}')$.

Theorem 21. *Let $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ with $l \geq 1$. Then for $\phi \in \mathcal{S}(\mathrm{W})$*

$$\int_{\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi)\, dg = \int_{\mathrm{G}_s^0} \check{\Theta}_\Pi(\tilde{g}) \det(1-g) T_s(\tilde{g})(\phi^{\mathrm{G}}|_{\mathrm{W}_s})\, dg\,, \tag{89}$$

where

$$\phi^{\mathrm{G}}(w) = \int_{\mathrm{G}} \phi(g.w)\, dw \qquad (w \in \mathrm{W})\,. \tag{90}$$

Theorem 22. *Let $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ and assume that the character Θ_Π is not supported on $\widetilde{\mathrm{G}^0}$.*

Suppose that $1 \leq l \leq l'$ and the pair $(\mathrm{O}_2, \mathrm{Sp}_2(\mathbb{R}))$ is excluded. Then for all $\phi \in \mathcal{S}(\mathrm{W})$

$$\int_{\mathrm{G}\setminus\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi)\, dg = C(\Pi) \int_{\mathrm{G}_s^0} \check{\Theta}_{\Pi_s}(\tilde{g}) T_s(\tilde{g})(\phi^{\mathrm{G}}|_{\mathrm{W}_s})\, dg\,, \tag{91}$$

where $C(\Pi)$ is a constant equal to ± 1, $\Pi_s = \Pi_{\lambda_s} \otimes \sigma$ is the tensor product of the irreducible representation of Spin_{2l-1} of highest weight $\lambda_s = \lambda + \frac{1}{2}\sum_{j=1}^{l-1} e_j$ and the spin representation σ. Recall that λ is the highest weight of Π.

If $l > l'$, then $\int_{\mathrm{G}\setminus\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\, dg = 0$.

If $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_2, \mathrm{Sp}_2(\mathbb{R}))$, then Θ_Π is not supported in $\widetilde{\mathrm{G}^0} = \widetilde{\mathrm{SO}}_2$ if and only if $\Pi = \nu^{-1}$ where $\nu(g,\xi) = \det(g)^{1/2}$ for $(g,\xi) \in \widetilde{\mathrm{O}}_2$. In this case, $\int_{\mathrm{G}\setminus\mathrm{G}^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})\, dg = 0$.

The following lemma will allow us to reduce the integral on the right hand-side of (89) to a linear combination of integrals as on the right hand-side of (91).

Lemma 23. *Suppose $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$. The function $\mathrm{G}_s \ni \tilde{g} \to \check{\Theta}_\Pi(\tilde{g}) \det(1-g) \in \mathbb{C}$ is a finite linear combination of irreducible characters of $\widetilde{\mathrm{G}}_s$.*

Proof. Let σ denote the spin representation of G_s^0 and let σ^c be its contragradient representation. Then, by [Lit06, Ch. XI, III, p. 254]

$$\det(1+g) = |\Theta_\sigma(g)|^2 = \Theta_{\sigma\otimes\sigma^c}(g) \qquad (g \in \mathrm{G}_s^0)\,. \tag{92}$$

Recall that for $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_d, \mathrm{Sp}_{2l'}(\mathbb{R}))$, χ_+ is a character of $\widetilde{\mathrm{G}}$. Write $\check{\Theta}_\Pi(\tilde{g}) \det(1-g) = \check{\Theta}_\Pi(\tilde{g})\chi_+(\tilde{g}) \det(1-g)\chi_+^{-1}(\tilde{g})$. Decomposing $(\Pi \otimes \chi_+^{-1})^c \otimes \sigma \otimes \sigma^c = \sum_j \sigma_j$ into a finite sum of irreducible representations σ_j of G_s, we then obtain

$$\check{\Theta}_\Pi(\tilde{g}) \det(1-g) = \Theta_{\Pi\otimes\chi_+^{-1}} \det(1-g)\chi_+^{-1}(\tilde{g}) = \sum_j \Theta_{\sigma_j}(g)\chi_+^{-1}(\tilde{g}) = \sum_j \check{\Theta}_{\sigma_j^c\otimes\chi_+}(\tilde{g})\,,$$

where $\check{\Theta}_{\sigma_j^c}(\tilde{g}) = \check{\Theta}_{\sigma_j^c}(g)$. □

Let $\mathcal{O}_{m,s} \subseteq \mathrm{W}_s$ denote the maximal nilpotent $\mathrm{G}_s \times \mathrm{G}'$ orbit with invariant measure $\mu_{\mathcal{O}_{m,s}} \in \mathcal{S}'(\mathrm{W}_s)$.

Lemma 24. *The sharp inequality*

$$\deg \mu_{\mathcal{O}_m} > \deg \mu_{\mathcal{O}_{m,s}} + (\dim \mathrm{W} - \dim \mathrm{W}_s) \tag{93}$$

holds, unless the dual pair $(\mathrm{G}, \mathrm{G}')$ *is isomorphic to* $(\mathrm{O}_d, \mathrm{Sp}_{2l'}(\mathbb{R}))$ *with* $d > l'$. *In this cases*

$$\deg \mu_{\mathcal{O}_m} = \deg \mu_{\mathcal{O}_{m,s}} + (\dim \mathrm{W} - \dim \mathrm{W}_s)\,. \tag{94}$$

Proof. We know from Lemmas 1 and 2 that

$$\deg \mu_{\mathcal{O}_m} = \dim \mathcal{O}'_m - \dim \mathrm{W} = 2l' \min\{d, l'\} - \min\{d, l'\}(\min\{d, l'\} - 1) - d\,2l'\,,$$

and similarly

$$\begin{aligned}\deg \mu_{\mathcal{O}_{m,s}} &= \dim \mathcal{O}'_{m,s} - \dim \mathrm{W}_s \\ &= 2l' \min\{d-1, l'\} - \min\{d-1, l'\}(\min\{d-1, l'\} - 1) - (d-1)2l'\,.\end{aligned}$$

Suppose $d \le l'$. Then

$$\deg \mu_{\mathcal{O}_m} = 2l'd - d(d-1) - d\,2l' = -d(d-1)$$

and, because $d - 1 < l'$,

$$\deg \mu_{\mathcal{O}_{m,s}} = -(d-1)(d-2) = -d(d-1) + 2(d-1)\,.$$

Also,

$$\dim \mathrm{W} - \dim \mathrm{W}_s = d2l' - (d-1)2l' = 2l' > 2(d-1)\,.$$

Thus (93) follows.

Suppose $d > l'$. Then

$$\deg \mu_{\mathcal{O}_m} = 2l'l' - l'(l'-1) - d2l'$$

and, because $d - 1 \ge l'$,

$$\deg \mu_{\mathcal{O}_{m,s}} = 2l'l' - l'(l'-1) - (d-1)2l' = \deg \mu_{\mathcal{O}_m} + 2l'\,.$$

Thus (94) follows. □

Lemma 25. *Suppose* $d = m$. *Then*

$$\lim_{t \to 0+} t^{\deg \mu_{\mathcal{O}_m}} M^*_{t^{-1}} T(\check{\Theta}_{\Pi}|_{\widetilde{\mathrm{G}\setminus(-\mathrm{G}^0)}}) = 0\,.$$

Proof. Recall that

$$M^*_{t^{-1}} T(\check{\Theta}_{\Pi}|_{\widetilde{\mathrm{G}\setminus(-\mathrm{G}^0)}})(\phi) = T(\check{\Theta}_{\Pi}|_{\widetilde{\mathrm{G}\setminus(-\mathrm{G}^0)}})(\phi_{t^{-1}})\,,$$

where

$$\phi_{t^{-1}}(w) = t^{\dim \mathrm{W}} \phi(tw)\,.$$

Suppose first we are in the situation described in Theorem 22. Then $T(\check{\Theta}_\Pi|_{\widetilde{\mathrm{G}\setminus(-\mathrm{G}^0)}})(\phi)$ is a constant multiple of $T_s(\check{\Theta}_{\Pi_s}|_{\widetilde{-\mathrm{G}_s^0}})(\phi|_{\mathrm{W}_s})$ because $-\mathrm{G}_s^0 = \mathrm{G}_s^0$. Notice that

$$(\phi|_{\mathrm{W}_s})_{t^{-1}}(w) = t^{\dim \mathrm{W}_s}(\phi|_{\mathrm{W}_s})(tw)$$

and as above

$$M_{t^{-1}}^* T_s(\check{\Theta}_{\Pi_s}|_{\widetilde{-\mathrm{G}_s^0}})(\phi|_{\mathrm{W}_s}) = T_s(\check{\Theta}_{\Pi_s}|_{\widetilde{-\mathrm{G}_s^0}})((\phi|_{\mathrm{W}_s})_{t^{-1}})\,,$$

Hence the decompostion of Π_s into irreducibles and Theorem 20 imply that

$$t^{\deg \mu_{\mathcal{O}_{m,s}}} M_{t^{-1}}^* T_s(\check{\Theta}_{\Pi_s}|_{\widetilde{-\mathrm{G}_s^0}})(\phi|_{\mathrm{W}_s}) \underset{t\to 0+}{\longrightarrow} K_s \mu_{\mathcal{O}_{m,s}}\,,$$

where K_s is a non-zero constant. Thus, for a constant C_s,

$$\begin{aligned}
&t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* T(\check{\Theta}_\Pi|_{\widetilde{\mathrm{G}\setminus(-\mathrm{G}^0)}})(\phi)\\
&\quad = C_s\, t^{\deg \mu_{\mathcal{O}_m} - \dim \mathrm{W} + \dim \mathrm{W}_s} M_{t^{-1}}^* T_s(\check{\Theta}_{\Pi_s}|_{\widetilde{-\mathrm{G}_s^0}})(\phi|_{\mathrm{W}_s})\\
&\quad = C_s\, t^{\deg \mu_{\mathcal{O}_m} - \dim \mathrm{W} + \dim \mathrm{W}_s - \deg \mu_{\mathcal{O}_{m,s}}} \left(t^{\deg \mu_{\mathcal{O}_{m,s}}} M_{t^{-1}}^* T_s(\check{\Theta}_{\Pi_s}|_{\widetilde{-\mathrm{G}_s^0}})(\phi|_{\mathrm{W}_s})\right)\\
&\quad \underset{t\to 0+}{\longrightarrow} C_s \cdot 0 \cdot K_s \mu_{\mathcal{O}_{m,s}} = 0
\end{aligned}$$

because, by Lemma 24,

$$\deg \mu_{\mathcal{O}_m} - \dim \mathrm{W} + \dim \mathrm{W}_s - \deg \mu_{\mathcal{O}_{m,s}} > 0\,.$$

Lemma 23 implies that a similar argument applies to the case of Theorem 89. □

Because of (88) and Lemma 25, it remains to compute (87) for $(\mathrm{O}_d, \mathrm{Sp}_{2l'}(\mathbb{R}))$ with $d > m = l'$. According to Theorem 22, we can also suppose $l' \geq l$ and $(l', l) \neq (1, 1)$ when $d = 2l$. This leads us to the cases $2l + 1 > l'$ for $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$, and $2l > l' \geq l$, with $(l', l) \neq (1, 1)$, for $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$.

Lemma 26. *Suppose* $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ *with* $2l + 1 > l'$. *Let* Π *be an irreducible representation of* $\widetilde{\mathrm{G}}$ *that occurs in the restriction of the Weil representation to* $\widetilde{\mathrm{G}}$. *Then, in the topology of* $\mathcal{S}'(\mathrm{W})$,

$$t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* T(\check{\Theta}_\Pi|_{\widetilde{\mathrm{G}^0}}) \underset{t\to 0+}{\longrightarrow} K^+ \mu_{\mathcal{O}_m}\,, \tag{95}$$

where

$$K^+ = |S^{2l}| C_s 2^{\frac{1}{2}\dim \mathrm{W}_s} \chi_{\Pi\otimes\chi_+^{-1}}(-1) \int_{(\mathrm{G}_s^0)_N} \Theta_{\Pi\otimes\chi_+^{-1}}(g^{-1}) \det(1+g)\, dg\,, \tag{96}$$

and $|S^{2l}|$ *is the area of the unit sphere,* C_s *is as in Lemma 18 for the group* G_s *acting on* W_s, *and* $(\mathrm{G}_s^0)_N$ *is the stabilizer of* N *in* G_s^0.

Proof. Recall the formula (91). We know from Lemma 23 that $\check{\Theta}_\Pi(\tilde{g}) \det(1-g)$ is a finite linear combination of irreducible characters of $\widetilde{\mathrm{G}}_s$. Since $\mathrm{G}_s^0 = -\mathrm{G}_s^0$, we apply the argument used in the proof of Theorem 20, together with (94), to each individual representation of $\widetilde{\mathrm{G}}_s$ and sum the results. This shows that for $\phi \in \mathcal{S}(\mathrm{W})$,

$$t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* T(\check{\Theta}_\Pi|_{\widetilde{\mathrm{G}^0}})(\phi) \underset{t\to 0+}{\longrightarrow} K_s \mu_{\mathcal{O}_{m,s}}(\phi^{\mathrm{G}}|_{\mathrm{W}_s})\,, \tag{97}$$

where $\mu_{\mathcal{O}_{m,s}}$ is the normalized measure on the maximal nilpotent $\mathrm{G}_s\mathrm{G}'$-orbit $\mathcal{O}_{m,s} \subseteq \mathrm{W}_s$ and

$$K_s = C_s 2^{\frac{1}{2}\mathrm{W}_s} \chi_{\Pi\otimes\chi_+^{-1}}(-1) \int_{(\mathrm{G}_s^0)_N} \Theta_{\Pi\otimes\chi_+^{-1}}(g^{-1}) \det(1+g)\, dg\,. \tag{98}$$

Since, by Corollary D.5

$$\mu_{\mathcal{O}_{m,s}}(\phi^{\mathrm{G}}|_{\mathrm{W}_s}) = |S^{2l}| \mu_{\mathcal{O}_m}(\phi)\,,$$

(95) follows. □

Lemma 27. *Suppose* $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ *with* $2l > l' \geq l$. *Let* Π *be an irreducible representation of* $\widetilde{\mathrm{G}}$ *that occurs in the restriction of the Weil representation to* $\widetilde{\mathrm{G}}$ *and whose character is not supported on* $\widetilde{\mathrm{G}^0}$. *Then, in the topology of* $\mathcal{S}'(\mathrm{W})$,

$$t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* T(\check{\Theta}_\Pi|_{\widetilde{\mathrm{G}\setminus\mathrm{G}^0}}) \underset{t\to 0+}{\longrightarrow} K^+ \mu_{\mathcal{O}_m}\,, \tag{99}$$

where $K^+ = 0$ *if* $2l = l' + 1$ *and*

$$K^+ = C(\Pi)|S^{2l-1}||S^{2l-2}| C_{ss} 2^{\frac{1}{2}\dim \mathrm{W}_{ss}} \times \chi_{\Pi\otimes\chi_+^{-1}}(-1) \int_{(\mathrm{G}_{ss}^0)_N} \Theta_{\Pi\otimes\chi_+^{-1}}(g^{-1}) \det(1+g)\, dg \tag{100}$$

for $2l > l' + 1$. *In* (100), $C(\Pi) = \pm 1$ *and* C_{ss} *is as in Lemma 18 for the group* G_{ss}, *isomorphic to* O_{2l-2}, *acting on* W_{ss}.

Proof. Formulas (91) and (94) imply that

$$\lim_{t\to 0+} t^{\deg \mu_{\mathcal{O}_m}} M_{t^{-1}}^* T(\check{\Theta}_\Pi|_{\widetilde{\mathrm{G}\setminus\mathrm{G}^0}})(\phi) = C(\Pi) \lim_{t\to 0+} t^{\deg \mu_{\mathcal{O}_{m,s}}} M_{t^{-1}}^* T_s(\check{\Theta}_{\Pi_s}|_{\widetilde{\mathrm{G}_s^0}})(\phi^{\mathrm{G}}|_{\mathrm{W}_s})\,. \tag{101}$$

Suppose first that $2l = l' + 1$. The defining space of G_s has dimension $d_s = 2l - 1 = l'$. Lemma 25 applies then to the dual pair $(\mathrm{G}_s, \mathrm{G}')$, yielding

$$\lim_{t\to 0+} t^{\deg \mu_{\mathcal{O}_{m,s}}} M_{t^{-1}}^* T_s(\check{\Theta}_{\Pi_s}|_{\widetilde{\mathrm{G}_s^0}})(\phi^{\mathrm{G}}|_{\mathrm{W}_s}) = 0.$$

Suppose now that $2l > l' + 1$. Then $2(l-1) + 1 = 2l - 1 > l'$ and Lemma 26 shows that

$$\lim_{t\to 0+} t^{\deg \mu_{\mathcal{O}_{m,s}}} M_{t^{-1}}^* T_s(\check{\Theta}_{\Pi_s}|_{\widetilde{\mathrm{G}_s^0}})(\phi^{\mathrm{G}}|_{\mathrm{W}_s}) = K_s^+ \mu_{\mathcal{O}_{m,s}}(\phi^{\mathrm{G}}|_{\mathrm{W}_s})\,, \tag{102}$$

where

$$K_s^+ = |S^{2l-2}| C_{ss} 2^{1+\frac{1}{2}\dim \mathrm{W}_{ss}} \chi_{\Pi\otimes\chi_+^{-1}}(-1) \int_{(\mathrm{G}_{ss}^0)_N} \Theta_{\Pi\otimes\chi_+^{-1}}(g^{-1}) \det(1+g)\, dg\,, \tag{103}$$

with C_{ss} as in Lemma 18 for the group G_{ss}, isomorphic to O_{2l-2}, acting on W_{ss}.

Since, by Corollary D.5

$$\mu_{\mathcal{O}_{m,s}}(\phi^{\mathrm{G}}|_{\mathrm{W}_s}) = |S^{2l-1}| \mu_{\mathcal{O}_m}(\phi)\,,$$

(99) follows. □

Lemma 28. *Let* K *be as in Theorem 20. With the notation and assumptions of Lemmas 26 and 27,*

$$K + K^+ \neq 0\,. \tag{104}$$

Proof. Recall that

$$K = C 2^{\frac{1}{2}\dim \mathrm{W}} \chi_{\Pi\otimes\chi_+^{-1}}(-1) \int_{\mathrm{G}_N^0} \Theta_{\Pi\otimes\chi_+^{-1}}(g_N)\, dg_N\,.$$

In the situation of Lemma 26,

$$K^+ = |S^{2l}| C_s 2^{\frac{1}{2}\dim \mathrm{W}_s} \chi_{\Pi\otimes\chi_+^{-1}}(-1) \int_{(\mathrm{G}_s^0)_N} \Theta_{\Pi\otimes\chi_+^{-1}}(g^{-1}) \det(1+g)\, dg\,.$$

The constants C and C_s as well as both integrals are integers, and $\chi_{\Pi\otimes\chi_+^{-1}}(-1) = \pm 1$. Moreover,

$$|S^{2l}| = \frac{2\pi^{\frac{2l+1}{2}}}{\Gamma(l+\frac{1}{2})} = \frac{2\cdot 4^l l!}{(2l)!}\,\pi^l\,,$$

which is an irrational number. Hence (104) follows.

In the situation of Lemma 27, with $2l > l'+1$,

$$K^+ = C(\Pi)\left(\frac{i}{2}\right)^{l'} |S^{2l-1}||S^{2l-2}| C_{ss} 2^{1+\frac{1}{2}\dim \mathrm{W}_{ss}} \times \chi_{\Pi\otimes\chi_+^{-1}}(-1) \int_{(\mathrm{G}_{ss}^0)_N} \Theta_{\Pi\otimes\chi_+^{-1}}(g^{-1}) \det(1+g)\, dg,$$

where both C_{ss} and the integral are integers. Since

$$|S^{2l-1}||S^{2l-2}| = \frac{2\pi^l}{(l-1)!}\,\frac{2\pi^{l-\frac{1}{2}}}{\Gamma((l-1)+\frac{1}{2})} = 4^l \frac{\pi^{2l-1}}{(2l-2)!}$$

is irrational, (104) follows. Finally, if $2l = l'+1$, then $K + K^+ = K \neq 0$. □

Now we easily deduce Theorem 3 from Theorem 20 and Lemmas 25 to 28.

6. **The wave front set of Π'.**

Recall from Theorem 3 that

$$t^{\deg \mu_{\mathcal{O}}} M_{t^{-1}}^* T(\check{\Theta}_\Pi) \underset{t\to 0+}{\to} C\,\mu_{\mathcal{O}_m}\,, \tag{105}$$

as tempered distributions on W, where C is a non-zero constant. Hence, in the topology of $\mathcal{S}'(\mathfrak{g}')$,

$$t^{\dim \mathcal{O}'_m} M_{t^2}^* \tau'_*\widehat{(T(\check{\Theta}_\Pi))} \underset{t\to 0+}{\longrightarrow} C\widehat{\mu_{\mathcal{O}'_m}}\,, \tag{106}$$

where

$$\tau'_*(T(\check{\Theta}_\Pi))(\psi) = T(\check{\Theta}_\Pi)(\psi\circ\tau')\,,$$

$\widehat{\tau'_*(T(\check{\Theta}_\Pi))}$ is a Fourier transform of the tempered distribution $\tau'_*(T(\check{\Theta}_\Pi))$ on $\mathfrak{g}'$, and similarly for $\mu_{\mathcal{O}'_m}$.

There is an easy-to-verify inclusion $WF(\Pi') \subseteq \overline{\mathcal{O}'}$, [Prz91, (6.14)] and a formula for the character $\Theta_{\Pi'}$ in terms of $\tau'_*(\widehat{T(\check{\Theta}_\Pi)})$, namely,

$$\frac{1}{\sigma} \cdot \tilde{c}_-^* \Theta_{\Pi'} = \tau'_*(\widehat{T(\check{\Theta}_\Pi)})\,, \tag{107}$$

where σ is a smooth function, [Prz91, Theorem 6.7]. By combining this with Lemma C.1 one completes the argument.

Appendix A

A.1. **Proof of Lemma 1.** The equality $w^*w = 0$ means that the pullback of the form $(\cdot,\cdot)'$ via $w \in \mathrm{W} = \mathrm{Hom}(\mathrm{V},\mathrm{V}')$ is zero. Equivalently, the range of w is an isotropic subspace of V'. Let us fix a maximal isotropic subspace $\mathrm{X}' \subseteq \mathrm{V}'$. We may assume that the range of w is contained in X'. Thus $w \in \mathrm{Hom}(\mathrm{V},\mathrm{X}')$. Under the action of G and $\mathrm{GL}(\mathrm{X}')$, the set $\mathrm{Hom}(\mathrm{V},\mathrm{X}')$ breaks down into a union of orbits. Each orbit consists of maps of rank $k \in \{0,1,2,\dots,m\}$. Since by Witt's Theorem $\mathrm{GL}(\mathrm{X}') \subseteq \mathrm{G}'$ and since the action of G' cannot change the rank of an element of $\mathrm{Hom}(\mathrm{V},\mathrm{V}')$, (4) will follow as soon as we compute the dimension of $\mathcal{O}_k$. We shall do it in terms of matrices. We keep the notation introduced in section 2. Let F, F' be as in (27) and choose

$$N = \begin{pmatrix} I_k & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{A.1}$$

as in (28). The Lie algebra $\mathfrak{g}$ consists of the skew-hermitian matrices of size d with coefficients in $\mathbb{D}$ and $\mathfrak{g}'$ of matrices of size d' and coefficients in $\mathbb{D}$, described in $(k, d'-2k, k)$ block-form as

$$x' = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & -F'^{-1}\overline{x_{12}}^t \\ x_{31} & -\overline{x_{21}}^t F' & -\overline{x_{11}}^t \end{pmatrix}\,, \quad x_{13} = \overline{x_{13}}^t\,, \quad x_{31} = \overline{x_{31}}^t\,, \quad \overline{x_{22}}^t F' + F' x_{22} = 0\,.$$

The Lie algebra of the stabilizer of N in $\mathrm{G}\times\mathrm{G}'$ consists of pairs of matrices $(x,x') \in \mathfrak{g}\times\mathfrak{g}'$ such that

$$x = \begin{pmatrix} y_{11} & 0 \\ 0 & y_{22} \end{pmatrix} \qquad x' = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & -F'^{-1}\overline{x_{12}}^t \\ 0 & 0 & -\overline{x_{11}}^t \end{pmatrix}\,, \quad x_{11} = y_{11}\,.$$

This implies the dimension formula in (4). Since

$$NN^* = \begin{pmatrix} 0 & 0 & I_k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,,$$

the stabilizer of NN^* in $\mathfrak{g}'$ consists of matrices of the form

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & -F'^{-1}\overline{x_{12}}^t \\ 0 & 0 & -\overline{x_{11}}^t \end{pmatrix}, \quad x_{11} = -\overline{x_{11}}^t,$$

and (5) follows.

Remark 3. *The fact that, for* G *compact,* $\tau^{-1}(0)$ *is the closure of a single* GG'*-orbit and a finite union of* GG'*-orbits was proved in* [Prz91, Lemma (2.16)]. *If in addition the pair* $(\mathrm{G}, \mathrm{G}')$ *is in the stable range with* G *the smaller member, then* [Prz91, Lemma (2.19)] *also computes the dimension of the maximal orbit. So, Lemma 1 is a generalization of these statements.*

As for other references in the literature, notice that given a dual pair $(\mathrm{G}, \mathrm{G}')$*, there are two moment maps one usually considers:*

$$\tau_{\mathfrak{g}'} : \mathrm{W} \to \mathfrak{g}'^* \qquad \text{and} \qquad \tau_{\mathfrak{s}'} : \mathrm{W} \to \mathfrak{s}'^*,$$

where $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$ *is a Cartan decomposition and the second map is obtained from the first one by composing with the restriction to* $\mathfrak{s}'$*. The first map leads to* G'*-orbits and the second to* $\mathrm{K}'_{\mathbb{C}}$*-orbits.*

Our Lemma 1 deals with the maps $\tau_{\mathfrak{g}'}$*, whereas the articles* [NZ04, NZ01] *deal with the map* $\tau_{\mathfrak{s}'}$ *only. Therefore they do not provide any direct proof of Lemma 1. Moreover, these references consider only dual pairs in the stable range. We do not have this assumption in our Lemma.*

Furthermore, these two moment maps are sort of "equivalent" in the stable range as was shown in [DKP05]*, but they are not "equivalent" beyond the stable range.*

A.2. **Proof of Lemma 2.** Let $N \in \mathcal{O}_k$ as in A.1. The stabilizer of the image of N in V' is a parabolic subgroup $\mathrm{P}' \subseteq \mathrm{G}'$ with Langlands decomposition $P' = \mathrm{GL}_k(\mathbb{D})\mathrm{G}''\mathrm{N}'$, where G'' is an isometry group of the same type as G' and N' is the unipotent radical. As a $\mathrm{GL}_k(\mathbb{D})$–module, $\mathfrak{n}'$, the Lie algebra of N', is isomorphic to $M_{k,d'-2k}(\mathbb{D}) \oplus \mathcal{H}_k(\mathbb{D})$, where $\mathcal{H}_k(\mathbb{D}) \subseteq M_k(\mathbb{D})$ stands for the space of the hermitian matrices. In the notation of A.1,

$$\begin{aligned}
\mathfrak{n}' &= \left\{ \begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & -F'^{-1}\overline{x_{12}}^t \\ 0 & 0 & 0 \end{pmatrix} ;\, x_{12} \in M_{k,d'-2k}(\mathbb{D}),\, x_{13} \in \mathcal{H}_d(\mathbb{D}) \right\}, \\
\mathfrak{g}'' &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} ;\, \overline{x_{22}}^t F' + F' x_{22} = 0 \right\}, \\
\mathrm{GL}_k(\mathbb{D}) &\equiv \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & I_{d'-2k} & 0 \\ 0 & 0 & (\overline{a}^t)^{-1} \end{pmatrix} ;\, a \in \mathrm{GL}_k(\mathbb{D}) \right\}.
\end{aligned}$$

Hence the absolute value of the determinant of the adjoint action of an element $a \in \mathrm{GL}_k(\mathbb{D})$ on the real vector space $\mathfrak{n}'$ is equal to

$$|\det_{\mathbb{R}} \mathrm{Ad}(a)_{\mathfrak{n}'}| = |\det_{\mathbb{R}}(a)|^{d'-2k+\frac{2\dim_{\mathbb{R}}\mathcal{H}_k(\mathbb{D})}{k\dim_{\mathbb{R}}\mathbb{D}}}.$$

Since $\mathrm{G}' = \mathrm{K}'\mathrm{P}'$, where K' is a maximal compact subgroup, the Haar measure on G' may be written as

$$dg' = |\det_{\mathbb{R}} \mathrm{Ad}(a)_{\mathfrak{n}'}|\, dk\, da\, dg''\, dn'.$$

Since the stabilizer of N in G' is equal to $\mathrm{G}''\mathrm{N}' \subseteq \mathrm{P}'$, the G' orbit of N defines a tempered distribution on W by

$$\int_{\mathrm{W}} \phi(w)\, d\mu_{\mathrm{G}'N}(w) = \int_{\mathrm{GL}_k(\mathbb{D})}\int_{\mathrm{K}'} \phi(kaN)|\det \mathrm{Ad}(a)_{\mathfrak{n}'}|\, dk\, da \qquad (\phi \in \mathcal{S}(\mathrm{W}))\,.$$

Since

$$t\begin{pmatrix} a & 0 & 0\\ 0 & I_{d'-2k} & 0\\ 0 & 0 & (\overline{a}^t)^{-1}\end{pmatrix}\begin{pmatrix} I_k & 0\\ 0 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix} ta & 0\\ 0 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix} ta & 0 & 0\\ 0 & I_{d'-2k} & 0\\ 0 & 0 & (\overline{ta}^t)^{-1}\end{pmatrix}\begin{pmatrix} I_k & 0\\ 0 & 0\\ 0 & 0\end{pmatrix}$$

and for $t > 0$

$$\begin{aligned}\int_{\mathrm{W}} \phi_t(w)\, d\mu_{\mathrm{G}'N}(w) &= t^{-\dim \mathrm{W}}\int_{\mathrm{GL}_k(\mathbb{D})}\int_{\mathrm{K}'} \phi(k(t^{-1}a)N)|\det \mathrm{Ad}(a)_{\mathfrak{n}'}|\, dk\, da\\ &= t^{-\dim \mathrm{W}}\int_{\mathrm{GL}_k(\mathbb{D})}\int_{\mathrm{K}'} \phi(kaN)|\det_{\mathbb{R}}(ta)|^{d'-2k+\frac{2\dim_{\mathbb{R}}\mathcal{H}_k(\mathbb{D})}{k\dim_{\mathbb{R}}\mathbb{D}}}\, dk\, da\\ &= t^{-\dim \mathrm{W}+\left(d'-2k+\frac{2\dim_{\mathbb{R}}\mathcal{H}_k(\mathbb{D})}{k\dim_{\mathbb{R}}\mathbb{D}}\right)k\dim_{\mathbb{R}}\mathbb{D}}\int_{\mathrm{W}} \phi(w)\, d\mu_{\mathrm{G}'N}(w)\,,\end{aligned}$$

this distribution is homogeneous of degree

$$(d'-2k+\frac{2\dim \mathcal{H}_k(\mathbb{D})}{k\dim_{\mathbb{R}}\mathbb{D}})\dim_{\mathbb{R}}\mathbb{D} - \dim \mathrm{W}\,.$$

Thus it remains to check that

$$(d'-2k+\frac{2\dim \mathcal{H}_k(\mathbb{D})}{k\dim_{\mathbb{R}}\mathbb{D}})\,k\dim_{\mathbb{R}}\mathbb{D} = d'k\dim_{\mathbb{R}}(\mathbb{D}) - 2\dim_{\mathbb{R}}\mathcal{SH}_k(\mathbb{D}),$$

which is easy, because $M_{k,k}(\mathbb{D}) = \mathcal{H}_k(\mathbb{D}) \oplus \mathcal{SH}_k(\mathbb{D})$. In order to conclude the proof we notice that the orbital integral on the $\mathrm{G}\times\mathrm{G}'$-orbit of N is (up to a positive multiple) the G-average of the orbital integral we just considered:

$$\int_{\mathrm{W}} \phi(w)\, d\mu_{\mathcal{O}_k}(w) = \int_{\mathrm{G}}\int_{\mathrm{GL}_k(\mathbb{D})}\int_{\mathrm{K}'} \phi(kaNg)\det \mathrm{Ad}(a)_{\mathfrak{n}'}\, dk\, da\, dg\,.$$

A.3. **A few facts about nilpotent orbits.** Let $\mathfrak{g}'$ be a semisimple Lie algebra over $\mathbb{C}$. Then there is a unique non-zero nilpotent orbit in $\mathfrak{g}'$ of minimal dimension which is contained in the closure of any non-zero nilpotent orbit, [CM93, Theorem 4.3.3, Remark 4.3.4]. The dimension of that orbit is equal to one plus the number of positive roots not orthogonal to the highest root, relative to a choice of a Cartan subalgebra and a choice of positive roots, [CM93, Lemma 4.3.5]. Thus in the case $\mathfrak{g}' = \mathfrak{sp}_{2l'}(\mathbb{C})$, the dimension of the minimal non-zero nilpotent orbit is equal to $2l'$. This is precisely the dimension of the defining module for the symplectic group $\mathrm{Sp}_{2l'}(\mathbb{C})$, which may be viewed as the symplectic space for the dual pair $(\mathrm{O}_1, \mathrm{Sp}_{2l'}(\mathbb{C}))$.

Consider the dual pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_1, \mathrm{Sp}_{2l'}(\mathbb{R}))$, with the symplectic space W and the unnormalized moment map $\tau' : \mathrm{W} \to \mathfrak{g}'$. Since $\mathrm{W} \setminus \{0\}$ is a single G'-orbit, so is $\tau'(\mathrm{W} \setminus \{0\})$. Further, $\dim(\tau'(\mathrm{W} \setminus \{0\})) = \dim(\mathrm{W}) = 2l'$. Hence, $\tau'(\mathrm{W} \setminus \{0\}) \subseteq \mathfrak{g}'$ is a minimal non-zero G'-orbit. In fact, there are only two such orbits, [CM93, Theorem 9.3.5]. In terms of dual pairs, the second one is obtained from the same dual pair, with the symplectic form replaced by its negative (or equivalently the symmetric form on the defining module for O_1 replaced by its negative).

Consider an irreducible dual pair $(\mathrm{G}, \mathrm{G}')$ with G compact. Denote by l the dimension of a Cartan subalgebra of $\mathfrak{g}$ and by l' the dimension of a Cartan subalgebra of $\mathfrak{g}'$. Let us identify the corresponding symplectic space W with $\mathrm{Hom}(\mathsf{V}_{\overline{1}}, \mathsf{V}_{\overline{0}})$ as in [Prz91, sec.2].

Recall that $\mathrm{W}_{\mathfrak{g}}$ denotes the maximal subset of W on which the restriction of the unnormalized moment map $\tau : \mathrm{W} \to \mathfrak{g}$ is a submersion. Then [Prz91, Lemma 2.6] shows that $\mathrm{W}_{\mathfrak{g}}$ consists of all the elements $w \in \mathrm{W}$ such that for any $x \in \mathfrak{g}$,

$$xw = 0 \text{ implies } x = 0\,. \tag{A.2}$$

The condition (A.2) means that x restricted to the image of w is zero. But in that case x preserves the orthogonal complement of that image. Thus we need to know that the Lie algebra of the isometries of that orthogonal complement is zero. This happens if w is surjective or if G is the orthogonal group and the dimension of the image of w in $\mathsf{V}_{\overline{0}}$ is $\geq \dim(\mathsf{V}_{\overline{0}}) - 1$. Thus

$$\mathrm{W}_{\mathfrak{g}} \neq \emptyset \text{ if and only if } l \leq l'\,. \tag{A.3}$$

Consider in particular the dual pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_3, \mathrm{Sp}_{2l'}(\mathbb{R}))$ with $1 \leq l'$. We see from the above discussion that $\mathrm{W}_{\mathfrak{g}}$ consists of elements of rank ≥ 2. Hence, $\mathrm{W} \setminus (\mathrm{W}_{\mathfrak{g}} \cup \{0\})$ consists of elements w of rank equal to 1. By replacing $\mathsf{V}_{\overline{0}}$ with the image of w, we may consider w as an element of the symplectic space for the pair $(\mathrm{O}_1, \mathrm{Sp}_{2l'})$. Hence the image of w under the moment map generates a minimal non-zero nilpotent orbit in $\mathfrak{g}'$.

If $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_2, \mathrm{Sp}_{2l'}(\mathbb{R}))$, with $1 \leq l'$, then $\mathrm{W}_{\mathfrak{g}}$ consists of elements of rank ≥ 1. Therefore $\mathrm{W} \setminus \mathrm{W}_{\mathfrak{g}} = \{0\}$.

Appendix B: Pull-back of a distribution via a submersion

We collect here some textbook results which are attributed to Ranga Rao in [BV80]. These results date back to the time before the textbook [Hör83] was available.

We shall use the definition of a smooth manifold and a distribution on a smooth manifold as described in [Hör83, sec. 6.3]. Thus, if M is a smooth manifold of dimension m and

$$M \supseteq M_\kappa \xrightarrow{\kappa} \widetilde{M_\kappa} \subseteq \mathbb{R}^m$$

is any coordinate system on M, then a distribution u on M is the collection of distributions $u_\kappa \in \mathcal{D}'(\widetilde{M_\kappa})$ such that

$$u_{\kappa_1} = (\kappa \circ \kappa_1^{-1})^* u_\kappa \,. \tag{B.1}$$

Suppose W is another smooth manifold of dimension n and v is a distribution on W. Thus for any coordinate system

$$W \supseteq W_\lambda \xrightarrow{\lambda} \widetilde{W_\lambda} \subseteq \mathbb{R}^n$$

we have a distribution $v_\lambda \in \mathcal{D}'(\widetilde{W_\lambda})$ such that the condition (B.1) holds. Suppose

$$\sigma : M \to W$$

is a submersion. Then for every κ there is a unique distribution $u_\kappa \in \mathcal{D}'(\widetilde{M_\kappa})$ such that

$$u_\kappa\big|_{(\lambda\circ\sigma\circ\kappa^{-1})^{-1}(\widetilde{W_\lambda})} = (\lambda \circ \sigma \circ \kappa^{-1})^* v_\lambda \,. \tag{B.2}$$

Since

$$(\kappa \circ \kappa_1^{-1})^*(\lambda \circ \sigma \circ \kappa^{-1})^* v_\lambda = (\lambda \circ \sigma \circ \kappa^{-1} \circ \kappa \circ \kappa_1^{-1})^* v_\lambda = (\lambda \circ \sigma \circ \kappa_1^{-1})^* v_\lambda$$

the u_κ satisfy the condition (B.1). The resulting distribution u is denoted by $\sigma^* v$ and is called the pullback of v from W to M via σ.

Proposition B.1. *Let M and W be smooth manifolds and let $\sigma : M \to W$ be a surjective submersion. Suppose $u_n \in \mathcal{D}'(W)$ is a sequence of distributions such that*

$$\lim_{n\to\infty} \sigma^* u_n = 0 \qquad \text{in } \mathcal{D}'(M)\,. \tag{B.3}$$

Then

$$\lim_{n\to\infty} u_n = 0 \qquad \text{in } \mathcal{D}'(W)\,. \tag{B.4}$$

In particular the map $\sigma^* : \mathcal{D}'(W) \to \mathcal{D}'(M)$ is injective.

More generally, if $u_n \in \mathcal{D}'(W)$ and $\tilde{u} \in \mathcal{D}'(M)$ are such that

$$\lim_{n\to\infty} \sigma^* u_n = \tilde{u} \qquad \text{in } \mathcal{D}'(M)\,, \tag{B.5}$$

then there is a distribution $u \in \mathcal{D}'(W)$ such that

$$\lim_{n\to\infty} u_n = u \qquad \text{in } D'(W) \tag{B.6}$$

and $\tilde{u} = \sigma^* u$.

Proof. By the definition of a distribution on a manifold, as in [Hör83, sec.6.3], we may assume that M is an open subset of $\mathbb{R}^m$ and W is an open subset of $\mathbb{R}^n$.

We recall the definition of the pull-back

$$\sigma^* : \mathcal{D}'(W) \to \mathcal{D}'(M) \tag{B.7}$$

from the proof of Theorem 6.1.2 in [Hör83]. Fix a point $x_0 \in M$ and a smooth map $g : M \to \mathbb{R}^{m-n}$ such that

$$\sigma \oplus g : M \to \mathbb{R}^n \times \mathbb{R}^{m-n}$$

has a bijective differential at x_0. By the Inverse Function Theorem there is an open neighborhood M_0 of x_0 in M such that

$$(\sigma \oplus g)\,|_{M_0} : M_0 \to Y_0$$

is a diffeomorphism onto an open neighborhood Y_0 of $(\sigma \oplus g)(x_0) = (\sigma(x_0), g(x_0))$ in $\mathbb{R}^n \times \mathbb{R}^{m-n}$. Let

$$h : Y_0 \to M_0$$

denote the inverse. For $\phi \in C_c^\infty(M_0)$ define $\Phi \in C_c^\infty(Y_0)$ by

$$\Phi(y) = \phi(h(y))|\det\, h'(y)| \qquad (y \in Y_0)\,. \tag{B.8}$$

Then

$$\sigma^* u(\phi) = u \otimes 1(\Phi) \qquad (u \in \mathcal{D}'(W),\ \phi \in C_c^\infty(M_0))\,. \tag{B.9}$$

By localization this gives the pull-back (B.7).

Let W_0 be an open neighborhood of $\sigma(x_0)$ in W and let X_0 be an open neighborhood of $g(x_0)$ in $\mathbb{R}^{m-n}$ such that

$$W_0 \times X_0 \subseteq Y_0\,.$$

Fix a function $\eta \in C_c^\infty(X_0)$ such that

$$\int_{X_0} \eta(x)\,dx = 1\,.$$

Given $\psi \in C_c^\infty(W_0)$ define

$$\Phi(x', x'') = \psi(x')\eta(x'') \qquad (x' \in W_0,\ x'' \in X_0)\,.$$

Then Φ defines ϕ via (B.8) and

$$\sigma^* u(\phi) = u(\psi)\,.$$

Hence the assumption (B.3) implies

$$\lim_{n\to\infty} u_n(\psi) = 0 \qquad (\psi \in C_c^\infty(W_0))\,.$$

Thus, by [Hör83, Theorem 2.1.8],

$$\lim_{n\to\infty} u_n|_{W_0} = 0$$

in $\mathcal{D}'(W_0)$. Since the point $x_0 \in M$ is arbitrary, the claim (B.4) follows by localization.

Similarly, the assumption (B.5) implies that for any $\psi \in C_c^\infty(W_0)$

$$\lim_{n\to\infty} u_n(\psi) = \lim_{n\to\infty} \sigma^* u_n(\phi) = \tilde{u}(\phi)$$

exists. Thus, by [Hör83, Theorem 2.1.8], there is $u \in \mathcal{D}'(W_0)$ such that

$$\lim_{n\to\infty} u_n|_{W_0} = u\,.$$

By the continuity of σ^*, $\sigma^* u = \tilde{u}$. Again, since the point $x_0 \in M$ is arbitrary, the claim follows by localization. □

Lemma B.2. *Let M and W be smooth manifolds and let $\sigma : M \to W$ be a surjective submersion. Then for any smooth differential operator D on W there is, not necessary unique, smooth differential operator $\sigma^* D$ on M such that*

$$\sigma^*(u \circ D) = (\sigma^* u) \circ (\sigma^* D) \qquad (u \in \mathcal{D}'(W)) .$$

If D annihilates constants then so does $\sigma^* D$. The operator $\sigma^* D$ is unique if σ is a diffeomorphism.

Proof. Suppose σ is a diffeomorphism between two open subsets of $\mathbb{R}^n$. Then

$$\sigma^* u(\phi) = u(\phi \circ \sigma^{-1} |\det((\sigma^{-1})')|) \qquad (\phi \in C_c^\infty(M)) .$$

Let

$$(\sigma^* D)(\phi) = (D(\phi \circ \sigma^{-1})) \circ \sigma \qquad (\phi \in C_c^\infty(M)) .$$

Hence

$$\begin{aligned} \sigma^*(u \circ D)(\phi) &= (u \circ D)(\phi \circ \sigma^{-1} |\det((\sigma^{-1})')|) \\ &= u(D(\phi \circ \sigma^{-1} |\det((\sigma^{-1})')|)) \\ &= u((D(\phi \circ \sigma^{-1}) \circ \sigma) \circ \sigma^{-1} |\det((\sigma^{-1})')|) . \end{aligned}$$

Using the local classification of the submersions modulo the diffeomorphism [Die71, 16.7.4], we may assume that σ is a linear projection

$$\sigma : \mathbb{R}^{m+n} \ni (x, y) \to x \in \mathbb{R}^n ,$$

in which case the lemma is obvious. □

Suppose M is a Lie group. Then there are functions $m_\kappa \in C^\infty(\widetilde{M}_\kappa)$ such that the formula

$$\int_M \phi \circ \kappa(y)\, d\mu_M(y) = \int_{\tilde{M}_\kappa} \phi(x) m_\kappa(x)\, dx \qquad (\phi \in C_c^\infty(\widetilde{M}_\kappa)) \tag{B.10}$$

defines a left-invariant Haar measure on M. We shall tie the normalization of the Haar measure $d\mu_M(y)$ on M to the normalization of the Lebesgue measure dx on $\mathbb{R}^m$ by requiring that near the identity,

$$m_{\exp^{-1}}(x) = \det\left(\frac{1 - e^{-\operatorname{ad}(x)}}{\operatorname{ad}(x)}\right) , \tag{B.11}$$

as in [Hel84, Theorem 1.14, page 96]. Collectively, the distributions $m_\kappa(x)\, dx \in \mathcal{D}'(\widetilde{M}_\kappa)$ form a distribution density on M. (See [Hör83, sec. 6.3] for the definition of a distribution density.)

Suppose W is another Lie group with left Haar measure given by

$$\int_W \psi \circ \lambda(y)\, d\mu_W(y) = \int_{\tilde{W}_\lambda} \phi(x) w_\lambda(x)\, dx \qquad (\psi \in C_c^\infty(\widetilde{W}_\lambda)) ,$$

and let $\sigma : M \to W$ be a submersion. Given any distribution density $v_\lambda \in \mathcal{D}'(\widetilde{W}_\lambda)$ we associate to it a distribution on W given by $\frac{1}{w_\lambda} v_\lambda \in \mathcal{D}'(\widetilde{W}_\lambda)$. We may pullback this

distribution to M and obtain another distribution. Then we multiply by the m_κ and obtain a distribution density. Thus, if $\sigma : M_\kappa \to W_\lambda$ then

$$(\sigma^* v)_\kappa = m_\kappa (\lambda \circ \sigma \circ \kappa^{-1})^* (\frac{1}{w_\lambda} v_\lambda) \,. \tag{B.12}$$

Distribution densities on W are identified with the continuous linear forms on $C_c^\infty(W)$ by

$$v(\psi \circ \lambda) = v_\lambda(\psi) \qquad (\psi \in C_c^\infty(\widetilde{W}_\lambda)) \,.$$

(Here v stands for the corresponding continuous linear form.) In particular if $F \in C(W)$. then $F\mu_W$ is a continuous linear form on $C_c^\infty(W)$ and for $\psi \in C_c^\infty(\widetilde{W}_\lambda)$,

$$\begin{aligned} (F\mu_W)_\lambda(\psi) &= (F\mu_W)(\psi \circ \lambda) = \int_W \psi \circ \lambda(y) F(y) \, d\mu_W(y) \\ &= \int_{\tilde{W}_\lambda} \psi(x) (F \circ \lambda^{-1})(x) w_\lambda(x) \, dx \,. \end{aligned}$$

Hence, for $\phi \in C_c^\infty(\widetilde{M}_\kappa)$, with $\sigma : M_\kappa \to W_\lambda$,

$$\begin{aligned} (\sigma^*(F\mu_W))_\kappa(\phi) &= (\lambda \circ \sigma \circ \kappa^{-1})^* (\frac{1}{w_\lambda} (F\mu_W)_\lambda)(m_\kappa \phi) \\ &= \int_{\tilde{M}_\kappa} m_\kappa(x) \phi(x) F \circ \lambda^{-1} \circ (\lambda \circ \sigma \circ \kappa^{-1})(x) \, dx \\ &= \int_{\tilde{M}_\kappa} \phi(x) (F \circ \sigma) \circ \kappa^{-1}(x) m_\kappa(x) \, dx \\ &= \int_M \phi \circ \kappa(y) (F \circ \sigma)(y) \, d\mu_M(y) \,. \end{aligned}$$

Thus

$$\sigma^*(F\mu_W) = (F \circ \sigma)\mu_M \,. \tag{B.13}$$

As explained above, we identify $\mathcal{D}'(M)$ with the space of the continuous linear forms on $C_c^\infty(M)$ and similarly for W and obtain

$$\sigma^* : \mathcal{D}'(M) \to \mathcal{D}'(W) \tag{B.14}$$

as the unique continuous extension of (B.13). Our identification of distribution densities with continuous linear forms on on the space of the smooth compactly supported functions applies also to submanifolds of Lie groups.

Let S be a Lie group acting on another Lie group W and let $U \subseteq W$ be a submanifold. (In our applications W is going to be a vector space.) We shall consider the following function

$$\sigma : \mathrm{S} \times U \ni (s, u) \to s.u \in W \,. \tag{B.15}$$

The following fact is easy to check.

Lemma B.3. *If $\mathcal{O} \subseteq W$ is an* S*-orbit then $\sigma^{-1}(\mathcal{O}) = \mathrm{S} \times (\mathcal{O} \cap U)$.*

Assume that the map (B.15) is submersive. Let us fix Haar measures on S and on W so that the pullback

$$\sigma^* : \mathcal{D}'(W) \to \mathcal{D}'(\mathrm{S} \times U)$$

is well defined, as in (B.14). Denote by $\mathrm{S}^U \subseteq \mathrm{S}$ the stabilizer of U.

Lemma B.4. *Assume that the map (B.15) is submersive and surjective. Let $\mathcal{O} \subseteq W$ be an* S*-orbit and let $\mu_{\mathcal{O}} \in \mathcal{D}'(W)$ be an* S*-invariant positive measure supported on the closure on $\mathcal{O}$. Let $\mu_{\mathcal{O}}|_U \in \mathcal{D}'(U)$ be the restriction of $\mu_{\mathcal{O}}$ to U in the sense of* [Hör83, Cor. 8.2.7]*. Then $\mu_{\mathcal{O}}|_U$ is a positive S^U-invariant measure supported on the closure of $\mathcal{O} \cap U$ in U. Moreover,*

$$\sigma^* \mu_{\mathcal{O}} = \mu_{\mathrm{S}} \otimes \mu_{\mathcal{O}}|_U \,. \tag{B.16}$$

Proof. Let $s \in \mathrm{S}^U$. Then

$$s^* \left(\mu_{\mathcal{O}}|_U\right) = \left(s^* \mu_{\mathcal{O}}\right)|_U = \mu_{\mathcal{O}}|_U \,.$$

Hence the distribution $\mu_{\mathcal{O}}|_U$ is S^U-invariant. Lemma B.1 implies that $\mu_{\mathcal{O}}|_U \neq 0$ and Lemma B.3 that $\mu_{\mathcal{O}}|_U$ is supported in the closure of $\mathcal{O} \cap U$ in U. Since the pullback of a positive measure is a non-negative measure, $\mu_{\mathcal{O}}|_U$ is a positive S^U-invariant measure supported on the closure of $\mathcal{O} \cap U$ in U.

Theorem 3.1.4' in [Hör83] implies that there is a positive measure $\mu_{\mathcal{O}\cap U}$ on U such that

$$\sigma^* \mu_{\mathcal{O}} = \mu_{\mathrm{S}} \otimes \mu_{\mathcal{O}\cap U} \,.$$

Consider the embedding

$$\sigma_1 : U \ni u \to (1, u) \in \mathrm{S} \times U \,.$$

Then $\sigma \circ \sigma_1 : U \to W$ is the inclusion of U into W. Hence,

$$(\sigma \circ \sigma_1)^* \mu_{\mathcal{O}} = \mu_{\mathcal{O}}|_U \,.$$

The conormal bundle to σ_1, as defined in [Hör83, Theorem 8.2.4], is equal to

$$N_{\sigma_1} = T^*(\mathrm{S}) \times 0|^*_{\{1\}\times U} \subseteq T^*(\mathrm{S}) \times 0 \subseteq T^*(\mathrm{S} \times U) \,.$$

By the S-invariance of $\sigma^* \mu_{\mathcal{O}}$,

$$WF(\mu_{\mathrm{S}} \otimes \mu_{\mathcal{O}\cap U}) \subseteq 0 \times T^*(U) \subseteq T^*(\mathrm{S} \times U) \,.$$

Hence

$$N_{\sigma_1} \cap WF(\mu_{\mathrm{S}} \otimes \mu_{\mathcal{O}\cap U}) = 0 \,.$$

Therefore

$$\mu_{\mathcal{O}}|_U = (\sigma \circ \sigma_1)^* \mu_{\mathcal{O}} = \sigma_1^* \circ \sigma^* \mu_{\mathcal{O}} = \sigma_1^* (\mu_{\mathrm{S}} \otimes \mu_{\mathcal{O}\cap U}) = \mu_{\mathcal{O}\cap U} \,.$$

This implies (B.16). □

Appendix C: Wave front set of an asymptotically homogeneous distribution

Let

$$\mathcal{F}f(x) = \int_{\mathbb{R}^n} F(y) e^{-2\pi i x\cdot y}\, dy$$

denote the usual Fourier transform on $\mathbb{R}^n$. Recall that for $t > 0$ the function $M_t : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $M_t(x) = tx$.

Lemma C.1. *Suppose $f, u \in \mathcal{S}'(\mathbb{R}^n)$, u is homogeneous of degree $d \in \mathbb{C}$, and*

$$t^d M_{t^{-1}}^* f(\psi) \underset{t\to 0+}{\longrightarrow} u(\psi) \qquad (\psi \in \mathcal{S}(\mathbb{R}^n))\,. \tag{C.1}$$

Then

$$WF_0(\mathcal{F}^{-1} f) \supseteq \operatorname{supp} u\,. \tag{C.2}$$

Proof. Suppose $\Phi \in C_c^\infty(\mathbb{R}^n)$ is such that $\Phi(0) \neq 0$. We need to show that the localized Fourier transform

$$\mathcal{F}((\mathcal{F}^{-1} f)\Phi)$$

is not rapidly decreasing in any open cone Γ which has a non-empty intersection with $\operatorname{supp} u$. (See [Hör83, Definition 8.1.2].) In order to do it, we will choose a function $\psi \in C_c^\infty(\Gamma)$ such that $u(\psi) \neq 0$ and show that

$$\int_{\mathbb{R}^n} (t^{-1})^{-d} \mathcal{F}((\mathcal{F}^{-1} f)\Phi)(t^{-1}x)\psi(x)\, dx \underset{t\to 0+}{\longrightarrow} u(\psi)\,, \tag{C.3}$$

assuming $\Phi(0) = 1$. Let $\phi = \mathcal{F}\Phi$. Then $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$. Notice that

$$t^d M_{t^{-1}}^*(f * \phi) = (t^d M_{t^{-1}}^* f) * (t^{-n} M_{t^{-1}}^* \phi)\,, \tag{C.4}$$

so that, by setting $\check{\psi}(x) = \psi(-x)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} (t^{-1})^{-d} \mathcal{F}((\mathcal{F}^{-1} f)\Phi)(t^{-1}x)\psi(x)\, dx \\ = t^d M_{t^{-1}}^*(f * \phi) * \check{\psi}(0) = (t^d M_{t^{-1}}^* f) * \left((t^{-n} M_{t^{-1}}^* \phi) * \check{\psi}\right)(0)\,.\end{aligned} \tag{C.5}$$

We will check that for an arbitrary $\psi \in \mathcal{S}(\mathbb{R}^n)$

$$(t^{-n} M_{t^{-1}}^* \phi) * \psi \underset{t\to 0+}{\longrightarrow} \psi \tag{C.6}$$

in the topology of $\mathcal{S}(\mathbb{R}^n)$. This, together with (C.5) and Banach-Steinhaus Theorem, [Rud91, Theorem 2.6], will imply (C.3). Explicitly,

$$\left((t^{-n} M_{t^{-1}}^* \phi) * \psi\right)(x) - \psi(x) = \int_{\mathbb{R}^n} \phi(y)(\psi(x - ty) - \psi(x))\, dy\,. \tag{C.7}$$

Fix $N = 0, 1, 2, \ldots$ and $\epsilon > 0$. Choose $R > 0$ so that

$$\int_{|y|\geq R} |\phi(y)|\, dy \cdot \left((|(1 + |y|)^N + 1\right) \sup_{x\in\mathbb{R}^n} (1 + |x|)^N |\psi(x)| < \epsilon\,. \tag{C.8}$$

Let $0 < t \leq 1$. Then

$$\begin{aligned}(1+|x|)^N \int_{|y|\geq R} |\phi(y)||\psi(x-ty)|\,dy \qquad (C.9)\\ \leq \int_{|y|\geq R} |\phi(y)|(1+|ty|)^N(1+|x-ty|)^N|\psi(x-ty)|\,dy\\ \leq \int_{|y|\geq R} |\phi(y)|(1+|y|)^N\,dy \cdot \sup_{x\in\mathbb{R}^n}(1+|x|)^N|\psi(x)|\end{aligned}$$

and

$$(1+|x|)^N \int_{|y|\geq R} |\phi(y)||\psi(x)|\,dy \leq \int_{|y|\geq R} |\phi(y)|\,dy \cdot \sup_{x\in\mathbb{R}^n}(1+|x|)^N|\psi(x)| \qquad (C.10)$$

so that, by (C.8),

$$(1+|x|)^N \left|\int_{|y|\geq R} \phi(y)(\psi(x-ty)-\psi(x))\,dy\right| < \epsilon \qquad (0<t\leq 1,\ x\in\mathbb{R}^n)\,. \qquad (C.11)$$

Choose $r > 0$ so that

$$(1+|x|)^N \left|\int_{|y|\leq R} \phi(y)(\psi(x-ty)-\psi(x))\,dy\right| < \epsilon \qquad (0<t\leq 1,\ |x|\geq r)\,. \qquad (C.12)$$

Since the function ψ is uniformly continuous,

$$\limsup_{t\longrightarrow 0+} \sup_{|x|\leq r} \left|\int_{|y|\leq R} \phi(y)(\psi(x-ty)-\psi(x))\,dy\right| = 0\,. \qquad (C.13)$$

Hence,

$$\limsup_{t\longrightarrow 0+} \sup_{x\in\mathbb{R}^n} (1+|x|)^N \left|\int_{|y|\leq R} \phi(y)(\psi(x-ty)-\psi(x))\,dy\right| \leq \epsilon\,. \qquad (C.14)$$

By combining (C.11) and (C.14), we see that

$$\limsup_{t\longrightarrow 0+} \sup_{x\in\mathbb{R}^n} (1+|x|)^N \left|\int_{\mathbb{R}^n} \phi(y)(\psi(x-ty)-\psi(x))\,dy\right| \leq 2\epsilon\,. \qquad (C.15)$$

Since the $\epsilon > 0$ is arbitrary, (C.15) and (C.7) show that

$$\limsup_{t\longrightarrow 0+} \sup_{x\in\mathbb{R}^n} (1+|x|)^N \left|(t^{-n}M^*_{t^{-1}}\phi) * \psi(x) - \psi(x)\right| = 0\,. \qquad (C.16)$$

Since the differentiation commutes with the convolution, (C.16) implies (C.6) and we are done. □

Appendix D: A restriction of a nilpotent orbital integral

Let W be a Euclidean space, isomorphic to $\mathbb{R}^M$ with the usual dot product. The Lebesgue measure on any subspace of W will be normalized so that the volume of the unit cube is 1. This is consistent with [Hör83].

Consider the following diagram

$$\begin{array}{ccc} \mathrm{W} & \xrightarrow{\kappa} & \mathrm{W} \\ {\scriptstyle \underline{\iota}}\uparrow & & \uparrow{\scriptstyle \iota} \\ \underline{\mathrm{V}} & \xrightarrow[\kappa]{} & \mathrm{V} \end{array} \tag{D.1}$$

where $\underline{\iota} : \underline{\mathrm{V}} \to \mathrm{W}$ and $\iota : \mathrm{V} \to \mathrm{W}$ are submanifolds and $\kappa(\underline{\mathrm{V}}) = \mathrm{V}$. Then we have the following formula for the pull-backs of distributions,

$$\underline{\iota}^* f = (\kappa|_{\underline{\mathrm{V}}})^* \iota^* (\kappa^{-1})^* f \tag{D.2}$$

[Hör83, Theorems 6.1.2 and 8.2.4], where $f \in C_c^\infty(\mathrm{W})^*$ is such that these pullbacks are well defined.

Assume further that W is the direct sum of two orthogonal subspaces

$$\mathrm{W} = \mathrm{U} \oplus \mathrm{V}\,, \tag{D.3}$$

that $\underline{\mathrm{V}} = \kappa^{-1}(\mathrm{V})$ and that

$$\kappa^{-1}(u+v) = \kappa^{-1}(u) + v \qquad (u \in \mathrm{U}, v \in \mathrm{V})\,.$$

Then

$$\underline{\mathrm{V}} = N + \mathrm{V}\,,$$

where $N = \kappa^{-1}(0)$. Let

$$\iota_{\mathrm{U}} : \mathrm{U} \to \mathrm{W}\,, \quad p_{\mathrm{U}} : \mathrm{W} \to \mathrm{U}$$

be the injection and the projection defined by the decomposition (D.3).

Lemma D.1. *Suppose $a \in C^\infty(\mathrm{U})$ and*

$$f(\phi) = \int_{\mathrm{U}} \big(\phi \circ \kappa^{-1}\big)(u) a(u)\, du \qquad (\phi \in C_c^\infty(\mathrm{W}))\,.$$

Then

$$\underline{\iota}^* f = |\det((p_{\mathrm{U}} \kappa^{-1} \iota_{\mathrm{U}})')(0)| a(0) \delta_N \in \mathcal{S}'(N + \mathrm{V})\,.$$

Proof. By taking the derivative of both sides of the equation $I = \kappa \circ \kappa^{-1}$ we see that

$$I = (\kappa \circ \kappa^{-1}) \circ (\kappa^{-1})'\,.$$

Hence,

$$\det(\kappa') \circ \kappa^{-1} \circ \iota_{\mathrm{U}} = \frac{1}{\det((p_{\mathrm{U}} \kappa^{-1} \iota_{\mathrm{U}})')}\,.$$

Therefore, by [Hör83, Theorems 6.1.2],

$$(\kappa^{-1})^* f(\phi) = f(\phi \circ \kappa |\det \kappa'|) = \int_{\mathrm{U}} \phi \circ \kappa \circ \kappa^{-1}(u) |\det(\kappa' \circ \kappa^{-1})(u)| a(u)\, du$$
$$= \int_{\mathrm{U}} \phi(u) |\det((p_{\mathrm{U}} \kappa^{-1} \iota_{\mathrm{U}})')(u)| a(u)\, du$$

and we deduce from [Hör83, Example 8.2.8] that

$$\iota^*(\kappa^{-1})^* f(\phi) = |\det((p_{\mathrm{U}} \kappa^{-1} \iota_{\mathrm{U}})')(0)| a(0)\phi(0)\,.$$

Now the claim follows from (D.2). □

From now on we specialize to $\mathrm{W} = M_{2m,n}(\mathbb{R})$ with $m \le n$. Let $\mathcal{O} \subseteq \mathrm{W}$ denote the $\mathrm{Sp}_{2m}(\mathbb{R}) \times \mathrm{O}_n$ - orbit through

$$N = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{W}\,.$$

Denote by $\mathcal{H}_m(\mathbb{R}) \subseteq M_m(\mathbb{R})$ the subspace of the symmetric matrices.

Lemma D.2. *The following formula*

$$f(\phi) = \int_{\mathcal{H}_m(\mathbb{R})} \int_{M_{m,n}(\mathbb{R})} \phi \begin{pmatrix} X \\ CX \end{pmatrix} |\det(XX^t)|^{\frac{m+1-n}{2}}\, dX\, dC$$

defines an invariant measure $f \in \mathcal{S}'(\mathrm{W})$ on the orbit $\mathcal{O}$.

Proof. Since for $g \in \mathrm{GL}_m(\mathbb{R})$ and $B, C \in SM_m(\mathbb{R})$,

$$\begin{pmatrix} I_m & 0 \\ C & I_m \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{pmatrix} \begin{pmatrix} I_m & B \\ 0 & I_m \end{pmatrix} \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} g & 0 \\ Cg & 0 \end{pmatrix}$$

we see that

$$\mathcal{O} = \left\{ \begin{pmatrix} X \\ CX \end{pmatrix};\ X \in M_{m,n}(\mathbb{R})\,,\ \mathrm{rank}(X) = m\,,\ C \in \mathcal{H}_m(\mathbb{R}) \right\}\,.$$

Furthermore the elements

$$\begin{pmatrix} I_m & 0 \\ C & I_m \end{pmatrix}\,, \quad \begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{pmatrix}\,, \quad \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

generate $\mathrm{Sp}_{2m}(\mathbb{R})$ and it is easy to check that f is invariant under the action of these elements, assuming the following two formulas:

$$\int_{\mathcal{H}_m(\mathbb{R})} \psi(gCg^t)\, dC = |\det g|^{m+1} \int_{\mathcal{H}_m(\mathbb{R})} \psi(C)\, dC\,,$$
$$\int_{\mathcal{H}_m(\mathbb{R})} \psi(C^{-1})\, dC = \int_{\mathcal{H}_m(\mathbb{R})} \psi(C) |\det C|^{-m-1}\, dC\,.$$

□

The space tangent to $\mathcal{O}$ at N may be identified with

$$\mathrm{U} = \left\{ \begin{pmatrix} u_{1,1} & u_{1,2} \\ B & 0 \end{pmatrix};\ u_{1,1} \in M_m(\mathbb{R}),\ u_{1,2} \in M_{m,n-m}(\mathbb{R}),\ B \in \mathcal{H}_m(\mathbb{R}) \right\}.$$

Then the orthogonal complement is equal to

$$\mathrm{V} = \left\{ \begin{pmatrix} 0 & 0 \\ D & u_{2,2} \end{pmatrix};\ D = -D^t \in M_m(\mathbb{R}),\ u_{2,2} \in M_{m,n-m}(\mathbb{R}) \right\}.$$

Set $\underline{\mathrm{V}} = N + \mathrm{V}$. Then we have the inclusion $\underline{\iota} : \underline{\mathrm{V}} \to \mathrm{W}$.

Lemma D.3. *Let f be as in Lemma D.2. Then*

$$\underline{\iota}^* f = \delta_N \in \mathcal{S}'(N + \mathrm{V}).$$

Proof. First we rewrite f as an integral over U. Let

$$N_1 = (I_m\ 0) \in M_{m,n}(\mathbb{R}).$$

Then

$$f(\phi) = \int_{\mathrm{U}} \phi \begin{pmatrix} u_1 + N_1 \\ B(u_1 + N_1) \end{pmatrix} |\det(u_1 + N_1)(u_1 + N_1)^t|^{\frac{m+1-n}{2}}\, du\, dB,$$

where

$$u = \begin{pmatrix} u_{1,1} & u_{1,2} \\ B & 0 \end{pmatrix}, \quad u_1 = (u_{1,1}\ u_{1,2}).$$

Next we introduce the diffeomorphism

$$\kappa^{-1}(u + v) = \begin{pmatrix} u_1 + N_1 \\ B(u_1 + N_1) \end{pmatrix} + v \qquad (u \in \mathrm{U},\ v \in \mathrm{V}).$$

Then

$$p_{\mathrm{U}} \kappa^{-1} \iota_{\mathrm{U}}(u) = \begin{pmatrix} u_{1,1} + N_1 & u_{1,2} \\ \frac{1}{2}(B(u_{1,1} + N_1) + (u_{1,1} + N_1)^t B) & 0 \end{pmatrix}.$$

Hence

$$(p_{\mathrm{U}} \kappa^{-1} \iota_{\mathrm{U}})'(0)(\Delta u) = \begin{pmatrix} \Delta u_{1,1} & \Delta u_{1,2} \\ \Delta B & 0 \end{pmatrix}$$

and consequently

$$\det((p_{\mathrm{U}} \kappa^{-1} \iota_{\mathrm{U}})'(0)) = 1.$$

Since

$$|\det(u_1 + N_1)(u_1 + N_1)^t|^{\frac{m+1-n}{2}}|_{u=0} = 1$$

the claim follows from Lemma D.1. □

Lemma D.4. *Suppose $m \le n$. For $\psi \in \mathcal{S}(M_{m,n}(\mathbb{R}))^{\mathrm{O}_n}$*

$$\int_{M_{m,n}(\mathbb{R})} \psi(X)\, dX = |S^{n-1}| \int_{M_{m,n-1}(\mathbb{R})} |\det(XX^t)|^{\frac{1}{2}} \psi|_{M_{m,n-1}}(X)\, dX.$$

Proof. By working in spherical coordinates of decreasing dimensions on the rows of X, we see that the left-hand side is equal to

$$\int_{\mathbb{R}^{\frac{m(m-1)}{2}}} \int_{(\mathbb{R}^+)^m} \int_{S^{n-1}} \cdots \int_{S^{n-m}} \psi \left(\begin{array}{ccccc|ccc} r_1\sigma_{1,1} & r_1\sigma_{1,2} & \cdots & \cdots & r_1\sigma_{1,m} & r_1\sigma_{1,m+1} & \cdots & r_1\sigma_{1,n} \\ x_{2,1} & r_2\sigma_{2,2} & \cdots & \cdots & r_2\sigma_{2,m} & r_2\sigma_{2,m+1} & \cdots & r_2\sigma_{2,n} \\ x_{3,1} & x_{3,2} & r_3\sigma_{3,3} & \cdots & r_3\sigma_{3,m} & r_3\sigma_{3,m+1} & \cdots & r_3\sigma_{3,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m-1} & r_m\sigma_{m,m} & r_m\sigma_{m,m+1} & \cdots & r_m\sigma_{m,n} \end{array} \right)$$
$$d\sigma_m \cdots d\sigma_1 \, r_1^{n-1} r_2^{n-2} \cdots r_m^{n-m} \, dr_m \cdots dr_2 \, dr_1 \, dx_{2,1} \cdots dx_{m,m-1} \,, \quad \text{(D.4)}$$

where

$$\begin{aligned} \sigma_1 &= (\sigma_{1,1}, \sigma_{1,2}, \dots, \sigma_{1,n}) \in S^{n-1} \,, \\ \sigma_2 &= (\sigma_{2,2}, \sigma_{2,3}, \dots, \sigma_{2,n}) \in S^{n-2} \,, \\ &\vdots \\ \sigma_m &= (\sigma_{m,m}, \sigma_{m,m+1}, \dots, \sigma_{m,n}) \in S^{n-m} \,. \end{aligned}$$

The O_n-invariance implies that (D.4) is equal to

$$\int_{\mathbb{R}^{\frac{m(m-1)}{2}}} \int_{(\mathbb{R}^+)^m} |S^{n-1}| \cdots |S^{n-m}| \psi \left(\begin{array}{ccccc|ccc} r_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ x_{2,1} & r_2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots & & \vdots \\ x_{m,1} & \dots & \dots & x_{m,m-1} & r_m & 0 & \cdots & 0 \end{array} \right)$$
$$\times \; r_1^{n-1} r_2^{n-2} \cdots r_m^{n-m} \, dr_m \cdots dr_2 \, dr_1 \, dx_{2,1} \cdots dx_{m,m-1}$$
$$= |S^{n-1}| \int_{\mathbb{R}^{\frac{m(m-1)}{2}}} \int_{(\mathbb{R}^+)^m} |S^{n-2}| \cdots |S^{n-m}| \psi \left(\begin{array}{ccccccccc|c} r_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 \\ x_{2,1} & r_2 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots & & \vdots & \vdots \\ x_{m,1} & \dots & \dots & x_{m,m-1} & r_m & 0 & \cdots & 0 & 0 \end{array} \right)$$
$$\times \, (r_1 r_2 \cdots r_m) \; r_1^{n-2} r_2^{n-3} \cdots r_m^{n-1-m} \, dr_m \cdots dr_2 \, dr_1 \, dx_{2,1} \cdots dx_{m,m-1}$$
$$= |S^{n-1}| \int_{M_{m,n-1}(\mathbb{R})} \psi|_{M_{m,n-1}(\mathbb{R})}(X) |\det(XX^t)|^{\frac{1}{2}} \, dX \,.$$

For the last equality, we consider spherical coordinates, as before, but on the first $n-1$ columns only, noticing that for

$$X = \left(\begin{array}{c|c} T & 0 \end{array} \right) = \left(\begin{array}{ccccc|ccc} r_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ x_{2,1} & r_2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots & & \vdots \\ x_{m,1} & \dots & \dots & x_{m,m-1} & r_m & 0 & \cdots & 0 \end{array} \right) \in M_{m,n-1}(\mathbb{R}) \,,$$

we have

$$XX^t = \left(\begin{array}{c|c} T & 0 \end{array} \right) \left(\frac{T^t}{0} \right) = TT^t \,.$$

Hence

$$\det(XX^t) = \det(TT^t) = \det(T)^2 = (r_1 r_2 \cdots r_m)^2 \,.$$

□

Corollary D.5. *Let us denote the measure $f \in \mathcal{S}'(M_{m,n})$ defined in Lemma D.2 by f_n and assume $n > m$. Then for $\phi \in \mathcal{S}(M_{m,n}(\mathbb{R}))^{\mathrm{O}_n}$*

$$f_n(\phi) = |S^{n-1}| f_{n-1}(\phi|_{M_{m,n-1}(\mathbb{R})}) \,,$$

where

$$\phi|_{M_{m,n-1}(\mathbb{R})}(X) = \phi(\begin{array}{c|c} X & 0 \end{array}) \qquad (X \in M_{m,n-1}(\mathbb{R})) \,.$$

Proof. This is clear from Lemmas D.2 and D.4. □

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