



HAL
open science

A family of constacyclic codes over a class of non-chain rings $A_{q,r}$ and new quantum codes

Habibul Islam, Shikha Patel, Om Prakash, Patrick Solé

► **To cite this version:**

Habibul Islam, Shikha Patel, Om Prakash, Patrick Solé. A family of constacyclic codes over a class of non-chain rings $A_{q,r}$ and new quantum codes. *Journal of Applied Mathematics and Computing*, 2021. hal-03324685

HAL Id: hal-03324685

<https://hal.science/hal-03324685>

Submitted on 23 Aug 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A family of constacyclic codes over a class of non-chain rings $\mathcal{A}_{q,r}$ and new quantum codes

Habibul Islam · Shikha Patel · Om Prakash ·
Patrick Solé

Received: date / Accepted: date

Abstract For a prime p , let \mathbb{F}_{q^2} be the finite field of q^2 elements where $q = p^m$ and $m \geq 1$ is an integer. In this paper, we study constacyclic and skew constacyclic codes of length n over a class of finite commutative non-chain rings $\mathcal{A}_{q,r} = \mathbb{F}_{q^2}[u_1, u_2, \dots, u_r] / \langle u_i^2 - \gamma_i u_i, u_i u_j = u_j u_i = 0 \rangle$ where $1 \leq i \neq j \leq r$, $\gamma_i \in \mathbb{F}_{q^2}^*$, and $r \geq 1$ is an integer. For a unit $\Lambda = \sum_{i=0}^r \kappa_i \Lambda_i$ in the ring $\mathcal{A}_{q,r}$, we show that a (skew) Λ -constacyclic code of length n is a direct sum of (skew) Λ_i -constacyclic codes of length n over \mathbb{F}_{q^2} . Also, we derive the necessary and sufficient conditions for such codes (constacyclic and skew constacyclic) to contain their Hermitian duals. By applying the Hermitian construction on the Gray images of dual containing codes, many MDS and new quantum codes with parameters better than the best-known codes are constructed.

Keywords Non-chain ring · Constacyclic code · Skew constacyclic code · Gray map · Hermitian construction · Quantum code.

Mathematics Subject Classification (2000) 94B05 · 94B15 · 94B35 · 94B60

1 Introduction

Constacyclic codes form one of the most studied and fascinating classes of codes. Their rich algebraic properties offer easy encoding and decoding methods. Also, their duality properties have been successfully applied in the construction of quantum error-correcting codes. In fact, after the discovery of first quantum code in 1995 by Shor [43], a large amount of quantum codes have been obtained based on the structure of constacyclic codes [11, 24]. Besides, many new quantum codes by using cyclic codes over different finite commutative rings [12, 18, 23, 25, 29, 31, 39, 40] are reported in the literature. In that context, finite commutative non-chain rings are most studied alphabets. In 2018, $(1 - 2v)$ -constacyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$ in [34], u -constacyclic codes over $\mathbb{F}_p + u\mathbb{F}_p$, $u^2 = u$ in [17], constacyclic codes over $\mathbb{F}_p + v\mathbb{F}_p + v^2\mathbb{F}_p$, $v^3 = v$ in [35] are studied to obtain quantum codes. Later in 2019, Ma et al. [36] determined many new quantum codes from constacyclic codes over the non-chain ring $\mathbb{F}_q[u, v] / \langle u^2 - 1, v^2 - v, uv - vu \rangle$. Recently, Islam et al. [27] considered a new class of non-chain rings $R_{k,m} = \mathbb{F}_{p^m}[u_1, u_2, \dots, u_k] / \langle u_i^2 - 1, u_i u_j - u_j u_i \rangle_{1 \leq i \neq j \leq k}$, ($k, m \geq 1$) and studied constacyclic codes to explore quantum codes with better parameters. Further, Alahmadi et al. [2] studied constacyclic codes over the family of non-chain rings $R_m = \mathbb{F}_p[u] / \langle u^m - 1 \rangle$ whereas Islam and Prakash [28] studied these codes over a non-chain ring $\mathbb{F}_q[u, v] / \langle u^2 - \gamma u, v^2 - \delta v, uv =$

Habibul Islam
Shikha Patel
Om Prakash
Department of Mathematics
Indian Institute of Technology Patna, Patna 801 106, India
E-mail: habibul.pma17@iitp.ac.in, shikha.1821ma05@iitp.ac.in, om@iitp.ac.in,

Patrick Solé
I2M, (CNRS, Aix-Marseille University, Centrale Marseille)
Marseilles, France
E-mail: sole@enst.fr

$vu = 0$) ($\gamma, \delta \in \mathbb{F}_q^*$) and obtained many new quantum codes from them. Therefore, the above literature leads to conclude that constacyclic codes over non-chain rings are one of the richest sources of good quantum codes. Hence, it makes sense to study these codes over different and new non-chain rings in that context. Towards this end, we consider the class of non-chain rings $\mathcal{A}_{q,r} = \mathbb{F}_{q^2}[u_1, u_2, \dots, u_r] / \langle u_i^2 - \gamma_i u_i, u_i u_j = u_j u_i = 0 \rangle$ where $1 \leq i \neq j \leq r$, $\gamma_i \in \mathbb{F}_{q^2}^*$, where q is a prime power and $r \geq 1$ is an integer. Note that for $r = 2$, it is a non-chain ring that appeared in [28] with Euclidean inner product, however, here we investigate with Hermitian inner product.

Linear codes acted upon noncommutative rings (skew polynomial rings) have been studied intensively for the last two decades. In 2007, Boucher et al. [6] introduced the notion of skew cyclic code (a generalization of cyclic code) by using the skew polynomial ring $\mathbb{F}_q[x; \theta]$, where θ is an automorphism on \mathbb{F}_q . From the application point of view, they have provided many codes with parameters better than previously best-known codes [6, 8]. Later, skew constacyclic codes over Galois rings were introduced in [7]. Subsequently, skew codes have been studied over many finite chain and non-chain rings. For instance, we can cite the rings $\mathbb{F}_2 + v\mathbb{F}_2, v^2 = v$ in [1], $\mathbb{F}_p + v\mathbb{F}_p, v^2 = v$ in [15], $\mathbb{F}_q + v\mathbb{F}_q, v^2 = v$ in [16, 20], $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$ where $u^2 = u, v^2 = v, uv = vu = 0$ in [3, 22], $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$ where $u^2 = u, v^2 = v, uv = vu$ in [21, 45]. On the other hand, skew constacyclic codes over finite commutative chain rings were studied in [30]. Like codes over commutative rings, quantum codes can also be constructed from dual containing skew cyclic and constacyclic codes. In fact, after the derivation of dual containing properties of such codes, the Hermitian construction (Lemma 2) provides quantum codes. It is well known that the number of quantum codes directly depends on the richness in the factorization of the polynomial $x^n \pm 1$ (see Theorem 6 and Theorem 12). As a skew polynomial ring is not constrained by the unicity of factorization, it is logical to study skew codes in the context of quantum code construction. To do so, some articles [4, 33, 37] have studied skew constacyclic codes over non-chain rings and obtained many new quantum codes. Due to the strong motivation discussed above, we have studied skew constacyclic codes over a class of non-chain ring $R_{e,q} = \mathbb{F}_q[u] / \langle u^e - 1 \rangle$ in [38] and explored many better codes compared to the best-known codes. Recently, the work of [27] has extended to skew constacyclic codes in [44] and obtained several superior quantum codes.

In continuation of that line of work, in the present paper we extend our study to skew constacyclic codes over $\mathcal{A}_{q,r} = \mathbb{F}_{q^2}[u_1, u_2, \dots, u_r] / \langle u_i^2 - \gamma_i u_i, u_i u_j = u_j u_i = 0 \rangle$. Note that for $r = 2, \gamma_1 = \gamma_2 = 1$, skew constacyclic codes are studied in [22]. First, we derive the structural properties of skew Λ -constacyclic codes where Λ is a unit in $\mathcal{A}_{q,r}$. Then by applying the Hermitian construction on the Gray images of dual containing skew Λ -constacyclic codes, we determine many better quantum codes compared to the best-known codes in the literature. It is important to mention that for $3 \leq n \leq q$, there always exists (see, [19]) a quantum MDS code with parameters $[[n, n - 2d + 2, d]]_q$ where $1 \leq d \leq \frac{n}{2} + 1$. Therefore, in this paper, we try to obtain those quantum codes whose lengths are larger than q . Two primary contributions of the paper are

1. It derives Λ -constacyclic codes and their Hermitian duals of length n over $\mathcal{A}_{q,r}$, and presents many new quantum codes with better parameters.
2. The paper investigates the algebraic properties of skew Λ -constacyclic codes of length n over $\mathcal{A}_{q,r}$ and their successful application to construct quantum codes. In this case, we also validate the novelty of obtained codes by comparing them with the best-known codes.

2 Preliminary

Let $\mathcal{A}_{q,r} := \mathbb{F}_{q^2}[u_1, u_2, \dots, u_r] / \langle u_i^2 - \gamma_i u_i, u_i u_j = u_j u_i = 0 \rangle$ where $1 \leq i \neq j \leq r$, $\gamma_i \in \mathbb{F}_{q^2}^*$, q is prime power and $r \geq 1$ is an integer. Thus $\mathcal{A}_{q,r}$ is a class of finite commutative rings (with unity). Moreover, $\mathcal{A}_{q,r}$ has another representation $\mathcal{A}_{q,r} = \mathbb{F}_{q^2} + u_1 \mathbb{F}_{q^2} + \dots + u_r \mathbb{F}_{q^2}$, where $u_i^2 = \gamma_i u_i, u_i u_j = u_j u_i = 0$. So, it is a non-chain semi-local Frobenius ring with $r + 1$ maximal ideals. For instance, there are three maximal ideals $\langle u_1 + u_2 \rangle$, $\langle \gamma_1 - u_1 \rangle$ and $\langle \gamma_2 - u_2 \rangle$ in the non-chain ring $\mathcal{A}_{q,2}$, see [28]. Define the $r + 1$ ring elements

$$\kappa_0 = \prod_{i=1}^r \left(1 - \frac{u_i}{\gamma_i} \right), \text{ and } \kappa_j = \frac{u_j}{\gamma_j} \text{ for } i \leq j \leq r.$$

Then following easy but useful results show that the κ_i s form a fundamental system of idempotents of $\mathcal{A}_{q,r}$:

1. $\kappa_0 + \kappa_1 + \cdots + \kappa_r = 1$,
2. $\kappa_i \kappa_j = 0$, for $i \neq j$ and $\kappa_i^2 = \kappa_i$ for all i ,
3. $\mathcal{A}_{q,r} = \kappa_0 \mathcal{A}_{q,r} \oplus \kappa_1 \mathcal{A}_{q,r} \oplus \cdots \oplus \kappa_r \mathcal{A}_{q,r} \cong \kappa_0 \mathbb{F}_{q^2} \oplus \kappa_1 \mathbb{F}_{q^2} \oplus \cdots \oplus \kappa_r \mathbb{F}_{q^2}$.

In the light of above discussion, we conclude that every element $\mathbf{a} \in \mathcal{A}_{q,r}$ has a unique representation $\mathbf{a} = \kappa_0 a_0 + \kappa_1 a_1 + \cdots + \kappa_r a_r$, where $a_i \in \mathbb{F}_{q^2}$. In addition, \mathbf{a} is a unit in $\mathcal{A}_{q,r}$ if and only if $a_i \in \mathbb{F}_{q^2}^*$, for all i .

Now, we recall some basic facts and results. A non-empty subset \mathfrak{C} of $\mathcal{A}_{q,r}^n$ is said to be a *linear code* of length n if it is an $\mathcal{A}_{q,r}$ -submodule of $\mathcal{A}_{q,r}^n$, and each member of \mathfrak{C} is called *codeword*. For a codeword $s = (s_0, s_1, \dots, s_{n-1}) \in \mathfrak{C}$, the *Hamming weight* $w_H(s)$ is defined as the number of non-zero places in s . Also, the Hamming distance between two codewords $s = (s_0, s_1, \dots, s_{n-1})$, and $t = (t_0, t_1, \dots, t_{n-1})$ is given by $d_H(s, t) = w_H(s - t)$. In this way, the Hamming distance for a linear code \mathfrak{C} is defined by $d_H(\mathfrak{C}) = \min\{d_H(s, t) : s \neq t, s, t \in \mathfrak{C}\}$. The Hermitian inner product between two codewords s and t is $\langle s, t \rangle_h = \sum_{i=0}^{n-1} s_i t_i^q$. Therefore, the dual $\mathfrak{C}^{\perp_h} = \{s \in \mathcal{A}_{q,r}^n : \langle s, t \rangle_h = 0, \text{ for all } t \in \mathfrak{C}\}$ is also a linear code over $\mathcal{A}_{q,r}$. Further, \mathfrak{C} is said to be *self-dual* if $\mathfrak{C} = \mathfrak{C}^{\perp_h}$, *self-orthogonal* if $\mathfrak{C} \subseteq \mathfrak{C}^{\perp_h}$ and dual containing if $\mathfrak{C}^{\perp_h} \subseteq \mathfrak{C}$.

Let $GL_{r+1}(\mathbb{F}_{q^2})$ denotes the group of all $(r+1) \times (r+1)$ invertible matrices over \mathbb{F}_{q^2} . For a matrix $M \in GL_{r+1}(\mathbb{F}_{q^2})$ satisfying $MM^T = \alpha I_{r+1}$, where $\alpha \in \mathbb{F}_{q^2}^*$, consider the Gray map (see, [18, 27] for similar mapping) $\varphi : \mathcal{A}_{q,r} \rightarrow \mathbb{F}_{q^2}^{r+1}$ defined by

$$\varphi(\mathbf{a}) = (a_0, a_1, \dots, a_{r+1})M$$

where $\mathbf{a} = \sum_{i=0}^r \kappa_i a_i \in \mathcal{A}_{q,r}$. Then φ is a linear bijection and can be extended over $\mathcal{A}_{q,r}$ componentwise. Now, as in [27], we define the Gray weight of $\mathbf{a} \in \mathcal{A}_{q,r}$ as $w_G(\mathbf{a}) = w_H(\varphi(\mathbf{a}))$. The Gray weight for $s = (s_0, s_1, \dots, s_{n-1}) \in \mathfrak{C}$ is $w_G(s) = \sum_{i=0}^{n-1} w_G(\varphi(s_i))$ and the Gray distance for \mathfrak{C} is $d_G(\mathfrak{C}) = \min\{w_G(s) : 0 \neq s \in \mathfrak{C}\}$. In this way, it is seen that φ is a linear isometric map from $\mathcal{A}_{q,r}^n$ (Gray distance) to $\mathbb{F}_{q^2}^{n(r+1)}$ (Hamming distance). The map φ preserves the orthogonality of a linear code over $\mathcal{A}_{q,r}$ as shown in the next result. The proof of result being analogous to ([18], Theorem 5) is omitted here.

Theorem 1 *Let \mathfrak{C} be a dual containing linear code of length n over $\mathcal{A}_{q,r}$. Then $\varphi(\mathfrak{C})$ is also a dual containing linear code of length $n(r+1)$ over \mathbb{F}_{q^2} . Moreover, $\varphi(\mathfrak{C})$ is a self-dual code if \mathfrak{C} is self-dual.*

Let \mathfrak{C} be a linear code of length n over $\mathcal{A}_{q,r}$. Define $\mathfrak{C}_i := \{s_i \in \mathbb{F}_{q^2}^n \mid \text{there exists } s_0, s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_r \in \mathbb{F}_{q^2}^n \text{ such that } \sum_{i=0}^r \kappa_i s_i \in \mathfrak{C}\}$ for $0 \leq i \leq r$. Then \mathfrak{C}_i is a linear code of length n over \mathbb{F}_{q^2} , and \mathfrak{C} can be written as $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$. In this case, the generator matrices of \mathfrak{C} and $\varphi(\mathfrak{C})$ are

$$M = \begin{bmatrix} \kappa_0 M_0 \\ \kappa_1 M_1 \\ \vdots \\ \kappa_r M_r \end{bmatrix}$$

and

$$\varphi(M) = \begin{bmatrix} \varphi(\kappa_0 M_0) \\ \varphi(\kappa_1 M_1) \\ \vdots \\ \varphi(\kappa_r M_r) \end{bmatrix},$$

respectively, where M_i is a generator matrix of \mathfrak{C}_i for $0 \leq i \leq r$. Moreover, $\mathfrak{C}^{\perp_h} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i^{\perp_h}$ and hence \mathfrak{C} is a self-dual code if and only if \mathfrak{C}_i is a self-dual code for $0 \leq i \leq r$.

3 Constacyclic codes and associated quantum codes

Here, following [11], we recall some basic facts. Let $\gcd(n, q) = 1$, $\Lambda \in \mathbb{F}_{q^2}^*$ and $\text{ord}_{q^2}(\Lambda) = l$. Then there exists a primitive ln -th root of unity w in some extension of \mathbb{F}_{q^2} such that $w^n = \Lambda$. Thus, $x^n - \Lambda = \prod_{i=0}^{n-1} (x - w^{1+il})$. Let $\Sigma = \{1 + il : 0 \leq i \leq n-1\}$ and for each $t \in \Sigma$, C_t be a q^2 -cyclotomic coset mod ln containing t , defined by $C_t = \{t, tq^2, \dots, tq^{2(m-1)}\}$ where m is the smallest positive integer such that $tq^{2m} \equiv t \pmod{ln}$. Let $f(x) = \prod_{i \in \mathcal{D}} (x - w^i)$. Then $f(x)$ is a factor of $x^n - \Lambda$ and $\mathfrak{C} = \langle f(x) \rangle$ is a Λ -constacyclic code of length n over \mathbb{F}_{q^2} with defining set \mathcal{D} and dimension $n - |\mathcal{D}|$. It is evident that \mathfrak{C}^{\perp_h} is a Λ^{-q} -constacyclic code with defining set $\mathcal{D}_\perp = \{t \in \Sigma : -qt \pmod{rn} \notin \mathcal{D}\}$.

Definition 1 Let $\Lambda \in \mathcal{A}_{q,r}$ be a unit. Then a linear code \mathfrak{C} is said to be a Λ -constacyclic code of length n if $\tau_\Lambda(s) = (\Lambda s_{n-1}, s_0, \dots, s_{n-2}) \in \mathfrak{C}$ whenever $s = (s_0, s_1, \dots, s_{n-1}) \in \mathfrak{C}$. The operator τ_Λ is called Λ -constacyclic shift. In particular, for $\Lambda = 1$, it is cyclic and for $\Lambda = -1$, it is a negacyclic code.

Let $\Lambda = \lambda_0 + u_1 \lambda_1 + \dots + u_r \lambda_r$ be a unit in $\mathcal{A}_{q,r}$ such that $\Lambda = \sum_{i=0}^r \kappa_i \Lambda_i$. Then $\Lambda_0 = \lambda_0, \Lambda_i = \lambda_0 + \gamma_i \lambda_i$ are units in \mathbb{F}_{q^2} , for $1 \leq i \leq r$. In this case, $\Lambda^{-1} = \sum_{i=0}^r \kappa_i \Lambda_i^{-1}$. Here, for such a unit Λ , we study Λ -constacyclic codes of length n over $\mathcal{A}_{q,r}$. Later on, these codes help to obtain quantum codes. Now, we identify each codeword $s = (s_0, s_1, \dots, s_{n-1}) \in \mathfrak{C}$ to a polynomial $s(x) = s_0 + s_1 x + \dots + s_{n-1} x^{n-1} \in \frac{\mathcal{A}_{q,r}[x]}{\langle x^n - \Lambda \rangle}$ under the corresponding $s = (s_0, s_1, \dots, s_{n-1}) \mapsto s(x) = s_0 + s_1 x + \dots + s_{n-1} x^{n-1}$. As a result, we have the following theorem.

Theorem 2 Let \mathfrak{C} be a linear code of length n over $\mathcal{A}_{q,r}$. Then \mathfrak{C} is a Λ -constacyclic code if and only if it is an ideal of $\frac{\mathcal{A}_{q,r}[x]}{\langle x^n - \Lambda \rangle}$ under the above identification.

The cyclicity of a code over the ring translates into cyclicity of its component codes over the base field.

Theorem 3 Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a linear code of length n over $\mathcal{A}_{q,r}$. Then \mathfrak{C} is a Λ -constacyclic code if and only if \mathfrak{C}_i is a Λ_i -constacyclic code of length n over \mathbb{F}_{q^2} for $0 \leq i \leq r$.

Proof Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a Λ -constacyclic code of length n . Let $s^i = (s_0^i, s_1^i, \dots, s_{n-1}^i) \in \mathfrak{C}_i$ for $0 \leq i \leq r$. Take $\mathbf{a}_j = \sum_{i=0}^r \kappa_i s_j^i$ for $0 \leq j \leq n-1$. Then $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{n-1}) \in \mathfrak{C}$, and hence $\tau_\Lambda(\mathbf{a}) = (\Lambda \mathbf{a}_{n-1}, \mathbf{a}_0, \dots, \mathbf{a}_{n-2}) \in \mathfrak{C}$. Also, $\tau_\Lambda(\mathbf{a}) = \sum_{i=0}^r \kappa_i \tau_{\Lambda_i}(s^i) \in \mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$. Therefore, $\tau_{\Lambda_i}(s^i) \in \mathfrak{C}_i$ for $0 \leq i \leq r$. Thus, \mathfrak{C}_i is a Λ_i -constacyclic code of length n over \mathbb{F}_{q^2} , for $0 \leq i \leq r$.

Conversely, let \mathfrak{C}_i be a Λ_i -constacyclic code of length n over \mathbb{F}_{q^2} , for $0 \leq i \leq r$. Let $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{n-1}) \in \mathfrak{C}$ where $\mathbf{a}_j = \sum_{i=0}^r \kappa_i s_j^i$ for $0 \leq j \leq n-1$. Then $s^i = (s_0^i, s_1^i, \dots, s_{n-1}^i) \in \mathfrak{C}_i$ and hence $\tau_{\Lambda_i}(s^i) \in \mathfrak{C}_i$ for $0 \leq i \leq r$. Therefore, $\tau_\Lambda(\mathbf{a}) = \sum_{i=0}^r \kappa_i \tau_{\Lambda_i}(s^i) \in \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i = \mathfrak{C}$. In this way, \mathfrak{C} is a Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$.

Theorem 4 Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$. Then $\mathfrak{C} = \langle \sum_{i=0}^r \kappa_i f_i(x) \rangle = \langle f(x) \rangle$ where $x^n - \Lambda_i = f_i(x)h_i(x)$ in $\mathbb{F}_{q^2}[x]$. Moreover, $f(x)$ is a factor of $x^n - \Lambda$.

Proof Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a Λ -constacyclic code of length n . Then, by Theorem 3, \mathfrak{C}_i is a Λ_i -constacyclic code of length n over \mathbb{F}_{q^2} for $0 \leq i \leq r$. In other words, \mathfrak{C}_i is a principally generated ideal of $\frac{\mathbb{F}_{q^2}[x]}{\langle x^n - \Lambda_i \rangle}$. Let $\mathfrak{C}_i = \langle f_i(x) \rangle$ such that $x^n - \Lambda_i = f_i(x)h_i(x)$ for $0 \leq i \leq r$. Then $\mathfrak{C} = \langle \kappa_0 f_0(x), \kappa_1 f_1(x), \dots, \kappa_r f_r(x) \rangle$. Let $f(x) = \sum_{i=0}^r \kappa_i f_i(x)$. Then $\langle f(x) \rangle \subseteq \mathfrak{C}$. On the other hand,

$\kappa_i f_i(x) = \kappa_i f(x) \in \langle f(x) \rangle$ for $0 \leq i \leq r$. Hence, $\langle f(x) \rangle \subseteq \mathfrak{C}$. Now, combining both sides, we conclude that $\mathfrak{C} = \langle f(x) \rangle$.

Further, $[\sum_{i=0}^r \kappa_i h_i(x)]f(x) = \sum_{i=0}^r \kappa_i h_i(x) f_i(x) = \sum_{i=0}^r \kappa_i (x^n - \Lambda_i) = x^n - \Lambda$. This implies that $f(x)$ is a factor of $x^n - \Lambda$.

Corollary 1 *The ring $\frac{\mathcal{A}_{q,r}[x]}{\langle x^n - \Lambda \rangle}$ is a principal ideal ring.*

Theorem 5 *Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$, where $\mathfrak{C}_i = \langle f_i(x) \rangle$ such that $x^n - \Lambda_i = f_i(x)h_i(x)$ in $\mathbb{F}_{q^2}[x]$. Then $\mathfrak{C}^{\perp h} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i^{\perp h}$ is a Λ^{-q} -constacyclic code such that $\mathfrak{C}^{\perp h} = \langle \sum_{i=0}^r \kappa_i h_i^*(x) \rangle$ where $h_i^*(x)$ is the reciprocal polynomial of $h_i(x)$, for $i = 0, 1, \dots, r$.*

Proof Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$. By Theorem 3, we have \mathfrak{C}_i is a Λ_i -constacyclic code of length n over \mathbb{F}_{q^2} for $0 \leq i \leq r$. Therefore, $\mathfrak{C}_i^{\perp h}$ is a Λ_i^{-q} -constacyclic code of length n , for $0 \leq i \leq r$. Since $\Lambda^{-q} = \sum_{i=0}^r \kappa_i \Lambda_i^{-q}$ and $\mathfrak{C}^{\perp h} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i^{\perp h}$, by Theorem 3, we conclude that $\mathfrak{C}^{\perp h}$ is a Λ^{-q} -constacyclic code of length n . In addition, let $\mathfrak{C}_i^{\perp h} = \langle h_i^*(x) \rangle$, where $h_i^*(x)$ is the reciprocal polynomial of $h_i(x)$ for $0 \leq i \leq r$. Therefore, by Theorem 4, we have $\mathfrak{C}^{\perp h} = \langle \sum_{i=0}^r \kappa_i h_i^*(x) \rangle$.

Let \mathbb{C}^q be a complex Hilbert space of dimension q . Then the n -fold tensor product $(\mathbb{C}^q)^{\otimes n} = \underbrace{\mathbb{C}^q \otimes \mathbb{C}^q \otimes \dots \otimes \mathbb{C}^q}_{n \text{ times}}$ is also a q^n -dimensional Hilbert space. We recall that a q -ary quantum code of length n is a K -dimensional subspace of $(\mathbb{C}^q)^{\otimes n}$ denoted by $[[n, k, d]]_q$ where d is the minimum distance and $k = \log_q K$. Every quantum code with parameters $[[n, k, d]]_q$ satisfies the quantum singleton bound $k + 2d \leq n + 2$, and is known as a quantum MDS (maximum-distance-separable) code when that bound is attained. Here, we obtain quantum codes (Theorem 7) from dual containing Λ -constacyclic codes of length n over $\mathcal{A}_{q,r}$. Before that, we develop the necessary and sufficient conditions (Theorem 6) of these codes to contain their duals. The next result (Lemma 1) recalls the same conditions for constacyclic codes over \mathbb{F}_{q^2} .

Lemma 1 *Let $\Lambda \in \mathbb{F}_{q^2}^*$ such that $l = \text{ord}_{q^2}(\Lambda)|(q+1)$ and \mathfrak{C} be a Λ -constacyclic code with defining set \mathcal{D} . Then $\mathfrak{C}^{\perp h} \subseteq \mathfrak{C}$ if and only if $\mathcal{D} \cap \mathcal{D}^{-q} = \{0\}$ where $\mathcal{D}^{-q} = \{-qa \pmod{ln} : a \in \mathcal{D}\}$.*

This condition for dual containment translates at the component level as follows.

Theorem 6 *Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$. Also, let \mathcal{D}_i be the defining set of \mathfrak{C}_i and $l_i = \text{ord}_{q^2}(\Lambda_i)|(q+1)$ for $0 \leq i \leq r$. Then $\mathfrak{C}^{\perp h} \subseteq \mathfrak{C}$ if and only if $\mathcal{D}_i \cap \mathcal{D}_i^{-q} = \{0\}$ for $0 \leq i \leq r$.*

Proof Let $\mathfrak{C}^{\perp h} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i^{\perp h} \subseteq \mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$. Since \mathfrak{C}_i is a q^2 -ary linear code such that $\kappa_i \mathfrak{C}_i \equiv \mathfrak{C} \pmod{\kappa_i}$ for $0 \leq i \leq r$, we have $\mathfrak{C}_i^{\perp h} \subseteq \mathfrak{C}_i$ for $0 \leq i \leq r$. Then, by Lemma 1, we have $\mathcal{D}_i \cap \mathcal{D}_i^{-q} = \{0\}$ for $0 \leq i \leq r$.

Conversely, let $\mathcal{D}_i \cap \mathcal{D}_i^{-q} = \{0\}$ for $0 \leq i \leq r$. Then, by Lemma 1, $\mathfrak{C}_i^{\perp h} \subseteq \mathfrak{C}_i$, for $0 \leq i \leq r$. Thus, $\mathfrak{C}^{\perp h} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i^{\perp h} \subseteq \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i = \mathfrak{C}$.

We recall the Hermitian construction of quantum codes from classical codes.

Lemma 2 [32] *(Hermitian Construction) Let \mathfrak{C} be an $[[n, k, d]]_{q^2}$ linear code of length n such that $\mathfrak{C}^{\perp h} \subseteq \mathfrak{C}$. Then there exists a quantum code with parameters $[[n, 2k - n, \geq d]]_q$.*

The consequence for constacyclic codes over $\mathcal{A}_{q,r}$ is as follows.

Theorem 7 Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$ and $\varphi(\mathfrak{C})$ has the parameters $[n(r+1), k, d_H]$. If $\mathfrak{C}^{\perp_h} \subseteq \mathfrak{C}$, then there exists a quantum code with parameters $[[n(r+1), 2k - n(r+1), \geq d_H]]_q$.

Proof Let \mathfrak{C} be a Λ -constacyclic code of length n such that $\mathfrak{C}^{\perp_h} \subseteq \mathfrak{C}$. Since $\varphi(\mathfrak{C}^{\perp_h}) = (\varphi(\mathfrak{C}))^{\perp_h}$, we have $(\varphi(\mathfrak{C}))^{\perp} \subseteq \varphi(\mathfrak{C})$. Therefore, $\varphi(\mathfrak{C})$ is a dual containing (Hermitian) linear code with parameters $[n(r+1), k, d_H]$. Now, by Lemma 2, there exists a quantum code with parameters $[[n(r+1), 2k - n(r+1), \geq d_H]]_q$.

Corollary 2 Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$ and \mathcal{D}_i be the defining set of \mathfrak{C}_i such that $l_i = \text{ord}_{q^2}(\Lambda_i) | (q+1)$ for $0 \leq i \leq r$. If $\mathcal{D}_i \cap \mathcal{D}_i^{-q} = \{0\}$ for $0 \leq i \leq r$, then there exists a quantum code with parameters $[[n(r+1), 2k - n(r+1), \geq d_H]]_q$.

Proof If $\mathcal{D}_i \cap \mathcal{D}_i^{-q} = \{0\}$ for $0 \leq i \leq r$, then by Theorem 6, we have $\mathfrak{C}^{\perp_h} \subseteq \mathfrak{C}$. Now, the result follows by Theorem 7.

4 New quantum codes and comparison

Example 1 Let $r = 2, q = 9$ and $\mathcal{A}_{9,2} = \mathbb{F}_{81}[u_1, u_2] / \langle u_1^2 - \gamma_1 u_1, u_2^2 - \gamma_2 u_2, u_1 u_2 = u_2 u_1 = 0 \rangle$. Let $\Lambda = w^{16}$ and $n = 8$. Then $\Lambda_0 = \Lambda_1 = \Lambda_2 = w^{16}$ and $l_i = \text{ord}_{q^2}(\Lambda_i) = 5 | 10 = q+1$. Now, we calculate $81 (= q^2)$ -cyclotomic coset modulo $40 (= ln)$. In fact, let $\mathcal{D}_0 = \{1\} \pmod{40}$, $\mathcal{D}_1 = \{6\} \pmod{40}$, and $\mathcal{D}_2 = \{11\} \pmod{40}$. Then $\mathcal{D}_0^{-9} = \{31\}$, $\mathcal{D}_1^{-9} = \{26\}$, $\mathcal{D}_2^{-9} = \{21\}$. Let $\mathbb{F}_{81}^* = \langle w \rangle$. Then $f_0(x) = x - w^2, f_1(x) = x - w^{12}, f_2(x) = x - w^{22}$ are factors of $x^8 - w^{16}$ whose defining sets are \mathcal{D}_i , for $i = 0, 1, 2$, respectively. Therefore, $\mathfrak{C} = \langle \sum_{i=0}^2 \kappa_i f_i(x) \rangle$ is a Λ -constacyclic code of length 8 over $\mathcal{A}_{9,2}$. Now, let

$$M = \begin{bmatrix} w & w^4 & 2 \\ w^4 & 2 & w \\ 2 & w & w^4 \end{bmatrix} \in GL_3(\mathbb{F}_{81})$$

satisfying $MM^T = (2w^2 + w)I_3$. Thus, the Gray image $\varphi(\mathfrak{C})$ has the parameters $[24, 21, 4]$. Since $\mathcal{D}_i \cap \mathcal{D}_i^{-9} = \{0\}$ for $i = 0, 1, 2$, by Theorem 6, we have $\mathfrak{C}^{\perp_h} \subseteq \mathfrak{C}$. Thus, by Theorem 7, there exists a quantum code with parameters $[[24, 18, \geq 4]]_9$. It is a quantum MDS code satisfying $n - k + 2 - 2d = 0$.

Example 2 Let $r = 2, q = 7$ and $\mathcal{A}_{7,2} = \mathbb{F}_{49}[u_1, u_2] / \langle u_1^2 - \gamma_1 u_1, u_2^2 - \gamma_2 u_2, u_1 u_2 = u_2 u_1 = 0 \rangle$. Let $n = 12$ and $\Lambda = -1 + 2\gamma_1^{-1} u_1$. Then $\Lambda_0 = -1, \Lambda_1 = 1, \Lambda_2 = -1$, and $l_0 = 2, l_1 = 1, l_2 = 2$ where $l_i = \text{ord}_{49}(\Lambda_i)$. Now, from 49-cyclotomic coset modulo 24 and 12, let $\mathcal{D}_0 = \{1\} \pmod{24}$, $\mathcal{D}_1 = \{1\} \pmod{12}$ and $\mathcal{D}_2 = \{5\} \pmod{24}$. Then $\mathcal{D}_0^{-7} = \{17\}$, $\mathcal{D}_1^{-7} = \{5\}$, $\mathcal{D}_2^{-7} = \{13\}$. In this way, $f_0(x) = x - w^2, f_1(x) = x - w^4, f_2(x) = x - w^{10}$ are factors of $x^{12} - \Lambda_i$ whose defining sets are \mathcal{D}_i , for $i = 0, 1, 2$, respectively. Therefore, $\mathfrak{C} = \langle \sum_{i=0}^2 \kappa_i f_i(x) \rangle$ is a Λ -constacyclic code of length 12 over $\mathcal{A}_{7,2}$. Let

$$M = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 2 & 1 \\ 1 & 2 & 5 \end{bmatrix} \in GL_3(\mathbb{F}_{49})$$

satisfying $MM^T = 2I_3$. Then the Gray image $\varphi(\mathfrak{C})$ is a $[36, 33, 3]$ linear code over \mathbb{F}_{49} . Further, $\mathcal{D}_i \cap \mathcal{D}_i^{-7} = \{0\}$ for $i = 0, 1, 2$. Therefore, by Theorem 6, $\mathfrak{C}^{\perp_h} \subseteq \mathfrak{C}$. Consequently, by Theorem 7, we have a quantum code $[[36, 30, \geq 3]]_7$. It has same length and distance but larger dimension compared to the best-known code $[[36, 29, 3]]_7$ given by [13]. Therefore, our code has better code rate than the best-known code. Moreover, the obtained code satisfies $n - k + 2 - 2d = 2$.

Example 3 Let $r = 3, q = 11$ and $\mathcal{A}_{11,3} = \mathbb{F}_{121}[u_1, u_2, u_3] / \langle u_1^2 - \gamma_1 u_1, u_2^2 - \gamma_2 u_2, u_3^2 - \gamma_3 u_3, u_1 u_2 = u_2 u_1 = u_2 u_3 = u_3 u_2 = u_3 u_1 = u_1 u_3 = 0 \rangle$. Let $n = 15$ and $\Lambda = -1 + 2\gamma_2^{-1} u_2 + 2\gamma_3^{-1} u_3$. Then $\Lambda_0 = \Lambda_1 = -1, \Lambda_2 = \Lambda_3 = 1$ and $l_0 = l_1 = 2, l_2 = l_3 = 1$, where $l_i = \text{ord}_{121}(\Lambda_i)$ for $i = 0, 1, 2, 3$. Now, among 121-cyclotomic cosets, let $\mathcal{D}_0 = \{1\} \pmod{30}$, $\mathcal{D}_1 = \{3\} \pmod{30}$, $\mathcal{D}_2 = \{1\} \pmod{15}$, $\mathcal{D}_3 = \{2\} \pmod{15}$. Then $\mathcal{D}_0^{-11} = \{21\}$, $\mathcal{D}_1^{-11} = \{27\}$, $\mathcal{D}_2^{-11} = \{4\}$, $\mathcal{D}_3^{-11} = \{8\}$. If

$\mathbb{F}_{121}^* = \langle w \rangle$, then $f_0(x) = x - w^4, f_1(x) = x - w^{12}, f_2(x) = x - w^8, f_3(x) = x - w^{16}$ are factors of $x^{15} - A_i$ whose defining sets are \mathcal{D}_i for $i = 0, 1, 2, 3$, respectively. Thus, $\mathfrak{C} = \langle \sum_{i=0}^3 \kappa_i f_i(x) \rangle$ is a Λ -constacyclic code of length 15 over $\mathcal{A}_{11,3}$. Let

$$M = \begin{bmatrix} 10 & 1 & 1 & 1 \\ 1 & 1 & 1 & 10 \\ 1 & 10 & 1 & 1 \\ 1 & 1 & 10 & 1 \end{bmatrix} \in GL_4(\mathbb{F}_{121})$$

satisfying $MM^T = I_4$. Therefore, the Gray image $\varphi(\mathfrak{C})$ is a $[60, 56, 4]$ linear code over \mathbb{F}_{121} . Also, $\mathcal{D}_i \cap \mathcal{D}_i^{-11} = \{0\}$ for $i = 0, 1, 2, 3$. By Theorem 6, we have $\mathfrak{C}^{\perp_h} \subseteq \mathfrak{C}$. Finally, by Theorem 7, we have an associated quantum code $[[60, 52, \geq 4]]_{11}$, which has larger minimum distance than the best-known code $[[60, 52, 3]]_{11}$ obtained in [36]. In addition, our code satisfies $n - k + 2 - 2d = 2$.

Example 4 Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$ where $\gcd(n, q) = 1$ and $l_i = \text{ord}_{q^2}(\Lambda_i) | (q+1)$. Let $\mathfrak{C}_i = \langle f_i(x) \rangle$ where \mathcal{D}_i is a defining set of $f_i(x)$ such that $\mathcal{D}_i \cap \mathcal{D}_i^{-q} = \{0\}$. Therefore, by Theorem 6, we have $\mathfrak{C}^{\perp_h} \subseteq \mathfrak{C}$. We compute the Gray images of such constacyclic codes under the map φ using the matrices M_1, M_2, M_3 in Table 1 (for $r = 2$) and M_4 in Table 3 (for $r = 3$), where

$$M_1 = \begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix} \in GL_3(\mathbb{F}_q), \text{ satisfying } M_1 M_1^T = 9I_3, \text{ for } q = 5, 7, 13, 17,$$

$$M_2 = \begin{bmatrix} w & 2w^7 & 1 \\ 2w^7 & 1 & w \\ 1 & w & 2w^7 \end{bmatrix} \in GL_3(\mathbb{F}_9), \text{ satisfying } M_2 M_2^T = I_3,$$

$$M_3 = \begin{bmatrix} w & w^4 & 2 \\ w^4 & 2 & w \\ 2 & w & w^4 \end{bmatrix} \in GL_3(\mathbb{F}_{81}), \text{ satisfying } M_3 M_3^T = (2w^2 + w)I_3,$$

$$M_4 = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \in GL_4(\mathbb{F}_q), \text{ satisfying } M_4 M_4^T = 4I_4.$$

Finally, by Theorem 7, we obtain new quantum codes $[[n, k, \geq d]]_q$ which have better parameters compared to the best-known codes $[[n', k', d']]_q$. Moreover, in Table 2 (resp. Table 4), we enlist the units and q^2 -cyclotomic cosets for the construction of Λ -constacyclic of Table 1 (resp. Table 3). It is worth mentioning that all obtained codes in Table 1 and 3 satisfy the condition $n - k + 2 - 2d = 2$. Therefore, all these codes are close to quantum MDS codes. In Table 1 and 3, to represent the generator polynomial $f_i(x)$, we write the coefficients of the power of x in decreasing order. For example, we write $1w^{46}w^{36}$ to represent the polynomial $x^2 + w^{46}x + w^{36}$. Moreover, in third column of Table 4, we use the following notations:

$$\begin{aligned} A_1 &= w^{16} - (1 + w^{16})\gamma_1^{-1}u_1 - (1 + w^{16})\gamma_2^{-1}u_2 - (1 + w^{16})\gamma_3^{-1}u_3, \\ A_2 &= -1 + (1 + w^{12})\gamma_1^{-1}u_1 + (1 + w^{12})\gamma_2^{-1}u_2 + (1 + w^{12})\gamma_3^{-1}u_3, \\ A_3 &= w^6 - (1 + w^6)\gamma_2^{-1}u_2 - (1 + w^6)\gamma_3^{-1}u_3 \\ A_4 &= -1 + 2\gamma_2^{-1}u_2 + 2\gamma_3^{-1}u_3. \end{aligned}$$

Table 1: New quantum codes $[[n, k, \geq d]]_q$ satisfying $n - k + 2 - 2d = 2$

Label	(n, q)	$[f_0(x), f_1(x), f_2(x)]$	$\varphi(\mathcal{C})$	$[[n, k, \geq d]]_q$	$[[n', k', d']]_q$
1	(10, 9)	$[1(2w^4), 1(2w^{12}), 1(2w^{20})]$	$[30, 27, 3]_{81}$	$[[30, 24, \geq 3]]_9$	$[[29, 24, 3]]_9[13]$
2	(8, 9)	$[1(2w^2), 1(2w^{12}), 1(2w^{22})]$	$[24, 21, 4]_{81}$	$[[24, 18, \geq 4]]_9$	MDS
3	(16, 9)	$[1(2w^5), 1(2w^{10}), 1(2w^{15})]$	$[48, 45, 3]_{81}$	$[[48, 42, \geq 3]]_9$	$[[51, 39, 3]]_9[13]$
4	(10, 9)	$[1w^{51}w^{32}, 1(2w^{12}), 1(2w^{20})]$	$[30, 26, 4]_{81}$	$[[30, 22, \geq 4]]_9$	$[[30, 20, 4]]_9[28]$
5	(12, 7)	$[1(6w^2), 1(6w^4), 1(6w^{10})]$	$[36, 33, 3]_{49}$	$[[36, 30, \geq 3]]_7$	$[[36, 29, 3]]_7[13]$
6	(24, 7)	$[1(6w), 1(6w^4), 1(6w^5)]$	$[72, 69, 3]_{49}$	$[[72, 66, \geq 3]]_7$	$[[73, 65, 3]]_7[13]$
7	(6, 7)	$[1(6w^4), 1(6w^{28}), 1(6w^{20})]$	$[18, 15, 3]_{49}$	$[[18, 12, \geq 3]]_7$	$[[14, 6, 3]]_7[27]$
8	(12, 5)	$[1(4w), 1(4w^2), 1(4w^5)]$	$[36, 33, 3]_{25}$	$[[36, 30, \geq 3]]_5$	$[[37, 29, 3]]_5[13]$
9	(6, 5)	$[1(4w^2), 1(4w^6), 1(4w^{10})]$	$[18, 15, 3]_{25}$	$[[18, 12, \geq 3]]_5$	$[[22, 12, 3]]_5[13]$
10	(24, 5)	$[1(4w), 1(4w^3), 1(4w^7)]$	$[72, 69, 3]_{25}$	$[[72, 66, \geq 3]]_5$	$[[72, 65, 3]]_5[13]$
11	(4, 5)	$[1(4w^3), 1w^3, 1(4w^6)]$	$[12, 8, 4]_{25}$	$[[12, 4, \geq 4]]_5$	$[[11, 1, 4]]_5[13]$
12	(6, 13)	$[1(12w^{14}), 1(12w^{56}), 1(12w^{70})]$	$[18, 15, 3]_{169}$	$[[18, 12, \geq 3]]_{13}$	$[[18, 10, 3]]_{13}[27]$
13	(6, 13)	$[1(12w^{14}), 1w^{46}w^{36}, 1(12w^{70})]$	$[18, 14, 4]_{169}$	$[[18, 10, \geq 4]]_{13}$	$[[18, 8, 4]]_{13}[28]$
14	(16, 17)	$[1(16w^9), 1(16w^{56}), 1(16w^{45})]$	$[48, 45, 3]_{289}$	$[[48, 42, \geq 3]]_{17}$	$[[48, 40, 3]]_{17}[27]$
15	(16, 17)	$[1(16w^9), 1w^{122}w^{84}, 1(16w^{45})]$	$[48, 44, 4]_{289}$	$[[48, 40, \geq 4]]_{17}$	$[[48, 38, 4]]_{17}[27]$

Table 2: List of units and q^2 -cyclotomic cosets used in Table 1

Label	(n, q)	A	(A_0, A_1, A_2)	(l_0, l_1, l_2)	$[\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2]$	$[\mathcal{D}_0^{-q}, \mathcal{D}_1^{-q}, \mathcal{D}_2^{-q}]$
1	(10, 9)	-1	$(-1, -1, -1)$	$(2, 2, 2)$	$\{\{1\}, \{3\}, \{5\}\}$	$\{\{11\}, \{13\}, \{15\}\}$
2	(8, 9)	w^{16}	(w^{16}, w^{16}, w^{16})	$(5, 5, 5)$	$\{\{1\}, \{6\}, \{11\}\}$	$\{\{31\}, \{26\}, \{21\}\}$
3	(16, 9)	1	$(1, 1, 1)$	$(1, 1, 1)$	$\{\{1\}, \{2\}, \{3\}\}$	$\{\{7\}, \{14\}, \{5\}\}$
4	(10, 9)	-1	$(-1, -1, -1)$	$(2, 2, 2)$	$\{\{1, 7\}, \{3\}, \{5\}\}$	$\{\{11, 17\}, \{13\}, \{15\}\}$
5	(12, 7)	$-1 + 2\gamma_1^{-1}u_1$	$(-1, 1, -1)$	$(2, 1, 2)$	$\{\{1\}, \{1\}, \{5\}\}$	$\{\{17\}, \{5\}, \{13\}\}$
6	(24, 7)	1	$(1, 1, 1)$	$(1, 1, 1)$	$\{\{1\}, \{4\}, \{5\}\}$	$\{\{17\}, \{20\}, \{13\}\}$
7	(6, 7)	-1	$(-1, -1, -1)$	$(2, 2, 2)$	$\{\{1\}, \{7\}, \{5\}\}$	$\{\{5\}, \{11\}, \{1\}\}$
8	(12, 5)	$-1 + 2\gamma_1^{-1}u_1$	$(-1, 1, -1)$	$(2, 1, 2)$	$\{\{1\}, \{1\}, \{5\}\}$	$\{\{19\}, \{7\}, \{23\}\}$
9	(6, 5)	-1	$(-1, -1, -1)$	$(2, 2, 2)$	$\{\{1\}, \{3\}, \{5\}\}$	$\{\{7\}, \{9\}, \{11\}\}$
10	(24, 5)	1	$(1, 1, 1)$	$(1, 1, 1)$	$\{\{1\}, \{3\}, \{7\}\}$	$\{\{19\}, \{9\}, \{17\}\}$
11	(4, 5)	$-1 + 2\gamma_2^{-1}u_2$	$(-1, -1, 1)$	$(2, 2, 1)$	$\{\{1\}, \{1, 7\}, \{1\}\}$	$\{\{3\}, \{3, 5\}, \{3\}\}$
12	(6, 13)	$-1 + 2\gamma_1^{-1}u_1$	$(-1, 1, -1)$	$(2, 1, 2)$	$\{\{1\}, \{2\}, \{5\}\}$	$\{\{11\}, \{4\}, \{7\}\}$
13	(6, 13)	$-1 + 2\gamma^{-1}(w^{24} + 1)u_1$	$(-1, w^{24}, -1)$	$(2, 7, 2)$	$\{\{1\}, \{1, 8\}, \{5\}\}$	$\{\{11\}, \{29, 22\}, \{7\}\}$
14	(16, 17)	$-1 + 2\gamma_1^{-1}u_1$	$(-1, 1, -1)$	$(2, 1, 2)$	$\{\{1\}, \{2\}, \{5\}\}$	$\{\{15\}, \{14\}, \{11\}\}$
15	(16, 17)	$-1 + 2\gamma_1^{-1}u_1$	$(-1, 1, -1)$	$(2, 1, 2)$	$\{\{1\}, \{1, 2\}, \{5\}\}$	$\{\{15\}, \{15, 14\}, \{11\}\}$

Table 3: New quantum codes $[[n, k, \geq d]]_q$ satisfying $n - k + 2 - 2d = 2$

Label	(n, q)	$[f_0(x), f_1(x), f_2(x), f_3(x)]$	$\varphi(\mathcal{C})$	$[[n, k, \geq d]]_q$	$[[n', k', d']]_q$
1	(16, 9)	$[1(2w^5), 1(2w^{10}), 1(2w^{15}), 1(2w^{20})]$	$[64, 60, 4]_{81}$	$[[64, 56, \geq 4]]_9$	$[[65, 51, 4]]_9[13]$
2	(4, 9)	$[1(2w^4), 1(2w^{10}), 1(2w^{30}), 1(2w^{30})]$	$[16, 12, 4]_{81}$	$[[16, 8, \geq 4]]_9$	$[[18, 8, 4]]_9[28]$
3	(5, 9)	$[1(2w^4), 1(2w^{12}), 1(2w^{20}), 1(2w^{20})]$	$[20, 16, 4]_{81}$	$[[20, 12, \geq 4]]_9$	$[[24, 12, 4]]_9[28]$
4	(6, 7)	$[1(6w^4), 1(6w^{10}), 1(6w^2), 1(6w^{26})]$	$[24, 20, 4]_{49}$	$[[24, 16, \geq 4]]_7$	$[[20, 12, 4]]_7[13]$
5	(3, 7)	$[1(6w^2), 1(6w^4), 1(6w^8), 1(6w^{10})]$	$[12, 8, 4]_{49}$	$[[12, 4, \geq 4]]_7$	$[[14, 2, 4]]_7[27]$
6	(2, 3)	$[1(2w), 1(2w^5), 1(2w^5), 1(2w^5)]$	$[8, 4, 4]_9$	$[[8, 0, \geq 4]]_3$	$[[8, 0, 4]]_3[14]$
7	(15, 11)	$[1(10w^4), 1(10w^{12}), 1(10w^8), 1(10w^{16})]$	$[60, 56, 4]_{121}$	$[[60, 52, \geq 4]]_{11}$	$[[60, 52, 3]]_{11}[36]$

Table 4: List of units and q^2 -cyclotomic cosets used in Table 3

Label	(n, q)	Λ	$(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)$	(l_0, l_1, l_2, l_3)	$[\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3]$	$[\mathcal{D}_0^{-q}, \mathcal{D}_1^{-q}, \mathcal{D}_2^{-q}, \mathcal{D}_3^{-q}]$
1	(16, 9)	1	(1, 1, 1, 1)	(1, 1, 1, 1)	[[1], {2}, {3}, {4}]	[[7], {14}, {5}, {12}]
2	(4, 9)	A_1	$(w^{16}, -1, -1, -1)$	(5, 2, 2, 2)	[[1], {1}, {3}, {3}]	[[11], {7}, {5}, {5}]
3	(5, 9)	w^{20}	$(w^{20}, w^{20}, w^{20}, w^{20})$	(4, 4, 4, 4)	[[1], {3}, {5}, {5}]	[[11], {13}, {15}, {15}]
4	(6, 7)	A_2	$(-1, w^{12}, w^{12}, w^{12})$	(2, 4, 4, 4)	[[1], {5}, {1}, {13}]	[[5], {13}, {17}, {5}]
5	(3, 7)	A_3	$(w^6, w^6, -1, -1)$	(8, 8, 2, 2)	[[1], {17}, {1}, {5}]	[[17], {1}, {5}, {1}]
6	(2, 3)	w^2	(w^2, w^2, w^2, w^2)	(4, 4, 4, 4)	[[1], {5}, {5}, {5}]	[[5], {1}, {1}, {1}]
7	(15, 11)	A_4	$(-1, -1, 1, 1)$	(2, 2, 1, 1)	[[1], {3}, {1}, {2}]	[[21], {27}, {4}, {8}]

5 Skew constacyclic codes and associated quantum codes

In this section, we extend our study to noncommutative structure (skew polynomial ring). Let us define an automorphism $\theta : \mathbb{F}_{q^2} \rightarrow \mathbb{F}_{q^2}$ given by $\theta(a) = a^q$, for all $a \in \mathbb{F}_{q^2}$. Now, we extend the above map over the ring $\mathcal{A}_{q,r}$ by $\Theta(r) = \sum_{i=0}^r \kappa_i \theta(a_i)$ where $r = \sum_{i=0}^r \kappa_i a_i \in \mathcal{A}_{q,r}$. Evidently, Θ is an automorphism on $\mathcal{A}_{q,r}$ of order 2. The set $\mathcal{A}_{q,r}[x; \Theta] = \{f(x) = f_0 + f_1 x + \dots + f_{n-1} x^{n-1} : f_i \in \mathcal{A}_{q,r} \text{ for all } i\}$ is a ring under the usual addition of polynomials and multiplication subject to the condition $(ax^i)(bx^j) = a\theta^i(b)x^{i+j}$. Clearly, it is a noncommutative ring (unless Θ is identity), called skew polynomial ring. Let n be an even integer. Then $\mathcal{A}_{q,r,n} = \frac{\mathcal{A}_{q,r}[x; \Theta]}{\langle x^n - \Lambda \rangle}$ is a ring and $\langle x^n - \Lambda \rangle$ is a two sided ideal in $\mathcal{A}_{q,r}[x; \Theta]$. But, if n is an odd integer, then $\mathcal{A}_{q,r,n}$ fails to be a ring. However, in both cases, $\mathcal{A}_{q,r,n}$ is a left $\mathcal{A}_{q,r}[x; \Theta]$ module under the scalar multiplication defined by $f(x)r(x) + \langle x^n - \Lambda \rangle = f(x)r(x) + \langle x^n - \Lambda \rangle$ where $f(x), r(x) \in \mathcal{A}_{q,r}[x; \Theta]$.

Definition 2 [26] Let Λ be a unit in $\mathcal{A}_{q,r}$. Then a linear code \mathfrak{C} of length n over $\mathcal{A}_{q,r}$ is a skew Λ -constacyclic code if and only if $\tau_{\Lambda, \Theta}(s) = (\Lambda\theta(s_{n-1}), \theta(s_0), \dots, \theta(s_{n-2})) \in \mathfrak{C}$ whenever $s = (s_0, s_1, \dots, s_{n-1}) \in \mathfrak{C}$. For $\Lambda = 1$, it is skew cyclic and for $\Lambda = -1$, it is a skew negacyclic code over $\mathcal{A}_{q,r}$.

Similar to the commutative case, skew Λ -constacyclic codes also afford a polynomial representation under the correspondence $s = (s_0, s_1, \dots, s_{n-1}) \mapsto s(x) = s_0 + s_1 x + \dots + s_{n-1} x^{n-1} \in \mathcal{A}_{q,r,n}$. In fact, a linear code \mathfrak{C} is a skew Λ -constacyclic if and only if it is a left $\mathcal{A}_{q,r}[x; \Theta]$ -submodule of $\mathcal{A}_{q,r,n}$, where every codeword is a polynomial in $\mathcal{A}_{q,r}$ of degree at most $n-1$. Now, we discuss the algebraic structure of skew Λ -constacyclic codes (Theorem 8 and Theorem 10) and their Hermitian duals (Theorem 11). These structures are useful to construct quantum codes from them subsequently.

Theorem 8 Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a linear code of length n over $\mathcal{A}_{q,r}$. Then \mathfrak{C} is a skew Λ -constacyclic code if and only if \mathfrak{C}_i is a skew Λ_i -constacyclic code of length n over \mathbb{F}_{q^2} for $0 \leq i \leq r$.

Proof Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a skew Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$. Let $s^i = (s_0^i, s_1^i, \dots, s_{n-1}^i) \in \mathfrak{C}_i$ for $0 \leq i \leq r$. Let $a_j = \sum_{i=0}^r \kappa_i s_j^i$ for $0 \leq j \leq n-1$. Then $a = (a_0, a_1, \dots, a_{n-1}) \in \mathfrak{C}$ and hence $\tau_{\Lambda, \Theta}(a) = (\Lambda\theta(a_{n-1}), \theta(a_0), \dots, \theta(a_{n-2})) \in \mathfrak{C}$. Now, $\tau_{\Lambda, \Theta}(a) = \sum_{i=0}^r \kappa_i \tau_{\Lambda_i, \theta}(s^i) \in \mathfrak{C}$. Therefore, $\tau_{\Lambda_i, \theta}(s^i) \in \mathfrak{C}_i$ for $0 \leq i \leq r$. This implies that \mathfrak{C}_i is a skew Λ_i -constacyclic code of length n over \mathbb{F}_{q^2} for $0 \leq i \leq r$.

On the other hand, let \mathfrak{C}_i be a skew Λ_i -constacyclic code of length n over \mathbb{F}_{q^2} for $0 \leq i \leq r$. Let $a = (a_0, a_1, \dots, a_{n-1}) \in \mathfrak{C}$ where $a_j = \sum_{i=0}^r \kappa_i s_j^i$ for $0 \leq j \leq n-1$. Then $s^i = (s_0^i, s_1^i, \dots, s_{n-1}^i) \in \mathfrak{C}_i$, and hence $\tau_{\Lambda_i, \theta}(s^i) \in \mathfrak{C}_i$ for $0 \leq i \leq r$. Therefore, $\tau_{\Lambda, \Theta}(a) = \sum_{i=0}^r \kappa_i \tau_{\Lambda_i, \theta}(s^i) \in \mathfrak{C}$. Thus, \mathfrak{C} is a skew Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$.

Theorem 9 [16] *Let \mathfrak{C} be a skew Λ -constacyclic code of length n over \mathbb{F}_{q^2} . Then $\mathfrak{C} = \langle f(x) \rangle$ is a left $\mathbb{F}_{q^2}[x; \theta]$ -submodule of $\mathbb{F}_{q^2}[x; \theta]/\langle x^n - \Lambda \rangle$, where $f(x)$ is a right divisor of $x^n - \Lambda$.*

Theorem 9 gives the structure of skew Λ -constacyclic code over the finite field \mathbb{F}_{q^2} . Here, we derive the algebraic properties of these codes over $\mathcal{A}_{q,r}$ by using Theorem 9 in the next result.

Theorem 10 *Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a skew Λ -constacyclic code of length n . Then $\mathfrak{C} = \langle \sum_{i=0}^r \kappa_i f_i(x) \rangle = \langle f(x) \rangle$ where $x^n - \Lambda_i = h_i(x)f_i(x)$ in $\mathbb{F}_{q^2}[x; \theta]$. Moreover, $f(x)$ is a right divisor of $x^n - \Lambda$.*

Proof Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a skew Λ -constacyclic code of length n . Then, by Theorem 8, we have \mathfrak{C}_i is a skew Λ_i -constacyclic code of length n over \mathbb{F}_{q^2} for $0 \leq i \leq r$. Following Theorem 9, let $\mathfrak{C}_i = \langle f_i(x) \rangle$ where $x^n - \Lambda_i = h_i(x)f_i(x)$ for $0 \leq i \leq r$. Therefore, $\kappa_0 f_0(x), \kappa_1 f_1(x), \dots, \kappa_r f_r(x)$ are generators of \mathfrak{C} . Let $f(x) = \sum_{i=0}^r \kappa_i f_i(x)$. Then $\langle f(x) \rangle \subseteq \mathfrak{C}$. Again $\kappa_i f_i(x) = \kappa_i f(x) \in \langle f(x) \rangle$ for $0 \leq i \leq r$. Hence, $\mathfrak{C} \subseteq \langle f(x) \rangle$, and $\mathfrak{C} = \langle f(x) \rangle$. Further, $\left(\sum_{i=0}^r \kappa_i h_i(x) \right) f(x) = \sum_{i=0}^r \kappa_i h_i(x) f_i(x) = \sum_{i=0}^r \kappa_i (x^n - \Lambda_i) = x^n - \Lambda$. Thus, $f(x)$ is a right divisor of $x^n - \Lambda$.

Corollary 3 *Every left $\mathcal{A}_{q,r}[x; \theta]$ -submodule of $\mathcal{A}_{q,r,n} = \frac{\mathcal{A}_{q,r}[x; \theta]}{\langle x^n - \Lambda \rangle}$ is principally generated.*

We recall from [30] that for a skew Λ -constacyclic code of length n over \mathbb{F}_{q^2} given by $\mathfrak{C} = \langle f(x) \rangle$ where $x^n - \Lambda = h(x)f(x)$, its dual code $\mathfrak{C}^{\perp h}$ is a skew Λ^{-1} -constacyclic code given by $\mathfrak{C}^{\perp h} = \langle \overline{h(x)} \rangle$, where $h(x) = \theta(h_t) + \theta^2(h_{t-1})x + \dots + \theta^t(h_1)x^{t-1} + \theta^{t+1}(h_0)x^t$ if $h(x) = h_0 + h_1x + \dots + h_t x^t$. Now, we derive the analogue results of skew Λ -constacyclic codes over the ring $\mathcal{A}_{q,r}$ as below.

Theorem 11 *Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a skew Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$, where $\mathfrak{C}_i = \langle f_i(x) \rangle$ such that $x^n - \Lambda_i = h_i(x)f_i(x)$ in $\mathbb{F}_{q^2}[x; \theta]$ for $i = 0, 1, \dots, r$. Then $\mathfrak{C}^{\perp h} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i^{\perp h}$ is a skew Λ^{-1} -constacyclic code such that $\mathfrak{C}^{\perp h} = \langle \sum_{i=0}^r \kappa_i \overline{h_i(x)} \rangle$.*

Proof Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a skew Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$, where $\mathfrak{C}_i = \langle f_i(x) \rangle$ such that $x^n - \Lambda_i = h_i(x)f_i(x)$ in $\mathbb{F}_{q^2}[x; \theta]$. Then \mathfrak{C}_i is a skew Λ_i -constacyclic code of length n over \mathbb{F}_{q^2} for $0 \leq i \leq r$. Therefore, $\mathfrak{C}_i^{\perp h}$ is a skew Λ_i^{-1} -constacyclic code of length n such that $\mathfrak{C}_i^{\perp h} = \langle \overline{h_i(x)} \rangle$ for $0 \leq i \leq r$. As $\Lambda^{-1} = \sum_{i=0}^r \kappa_i \Lambda_i^{-1}$, then by Theorem 8, we conclude that $\mathfrak{C}^{\perp h} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i^{\perp h}$ is a skew Λ^{-1} -constacyclic code of length n over $\mathcal{A}_{q,r}$. Further, by Theorem 10, we have $\mathfrak{C}^{\perp h} = \langle \sum_{i=0}^r \kappa_i \overline{h_i(x)} \rangle$.

It is evident that under the Hermitian construction, quantum codes can be constructed once there are some dual containing linear codes. Therefore, we first need to develop the necessary and sufficient conditions for the existence of such codes. Note that $\text{ord}(\langle \theta \rangle) = 2$. Therefore, a skew Λ -constacyclic code of odd length n is equivalent to a Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$ (similar way of Theorem 22 in [21]). Consequently, non-commutativity of the multiplication is no longer valid. Also, quantum codes from Λ -constacyclic codes are well studied in Section 3. Therefore, we investigate these codes in the other case, i.e., when n is an even integer. In fact, when n is even, the dual containing property of a skew Λ -constacyclic code over \mathbb{F}_{q^2} is presented by Lemma 3. We therefore extend this property for a skew Λ -constacyclic codes over $\mathcal{A}_{q,r}$ (Theorem 12). Finally, utilizing these structures, we construct quantum codes by Theorem 13.

Lemma 3 ([33], Lemma 6) *Let \mathfrak{C} be a skew Λ -constacyclic code of even length n over \mathbb{F}_{q^2} such that $\mathfrak{C} = \langle f(x) \rangle$ where $x^n - \Lambda = h(x)f(x)$ and $\Lambda = \pm 1$. Then $\mathfrak{C}^{\perp h} \subseteq \mathfrak{C}$ if and only if $x^n - \Lambda$ is a right divisor of $\overline{h(x)}h(x)$.*

Theorem 12 *Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a skew Λ -constacyclic code of even length n over $\mathcal{A}_{q,r}$ where $\mathfrak{C}_i = \langle f_i(x) \rangle$ such that $x^n - \Lambda_i = h_i(x)f_i(x)$ in $\mathbb{F}_{q^2}[x; \theta]$ and $\Lambda_i = \pm 1$ for $0 \leq i \leq r$. Then $\mathfrak{C}^{\perp h} \subseteq \mathfrak{C}$ if and only if $x^n - \Lambda_i$ is a right divisor of $\overline{h_i(x)}h_i(x)$, for $0 \leq i \leq r$.*

Proof Let $\mathfrak{C}^{\perp h} \subseteq \mathfrak{C}$. Therefore, $\mathfrak{C}^{\perp h} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i^{\perp h} \subseteq \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i = \mathfrak{C}$. As \mathfrak{C}_i is a q^2 -ary linear code such that $\kappa_i \mathfrak{C}_i \equiv \mathfrak{C} \pmod{\kappa_i}$, for $0 \leq i \leq r$, $\mathfrak{C}^{\perp h} \subseteq \mathfrak{C}_i$ for $0 \leq i \leq r$. Then by Lemma 3, $x^n - \Lambda_i$ is a right divisor of $\overline{h_i(x)h_i(x)}$, for $0 \leq i \leq r$.

Conversely, let $x^n - \Lambda_i$ be a right divisor of $\overline{h_i(x)h_i(x)}$, for $0 \leq i \leq r$. Then by Lemma 3, $\mathfrak{C}_i^{\perp h} \subseteq \mathfrak{C}_i$ for $0 \leq i \leq r$. Therefore, $\mathfrak{C}^{\perp h} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i^{\perp h} \subseteq \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i = \mathfrak{C}$.

Theorem 13 Let $\mathfrak{C} = \bigoplus_{i=0}^r \kappa_i \mathfrak{C}_i$ be a skew Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$ where $\mathfrak{C}_i = \langle f_i(x) \rangle$ such that $x^n - \Lambda_i = h_i(x)f_i(x)$ in $\mathbb{F}_{q^2}[x; \theta]$ and $\Lambda_i^2 = \pm 1$ for $0 \leq i \leq r$. Also, let $\varphi(\mathfrak{C})$ has the parameters $[n(r+1), k, d_H]$ and $x^n - \Lambda_i$ be a right divisor of $\overline{h_i(x)h_i(x)}$, for $0 \leq i \leq r$. Then there exists a quantum code $[[n(r+1), 2k - n(r+1), \geq d_H]]_q$.

Proof Let $x^n - \Lambda_i$ be a right divisor of $\overline{h_i(x)h_i(x)}$, for $0 \leq i \leq r$. Then by Theorem 12 we have $\mathfrak{C}^{\perp h} \subseteq \mathfrak{C}$. Therefore, $(\varphi(\mathfrak{C}))^{\perp h} = \varphi(\mathfrak{C}^{\perp h}) \subseteq \varphi(\mathfrak{C})$. Hence, $\varphi(\mathfrak{C})$ is a dual containing linear code with parameters $[[n(r+1), k, d_H]$ over \mathbb{F}_{q^2} . Now, by Lemma 2, there exists a quantum code $[[n(r+1), 2k - n(r+1), \geq d_H]]_q$.

6 New quantum codes and comparison

Example 5 Let $r = 3, q = 3, n = 8$ and $\mathcal{A}_{3,3} = \mathbb{F}_9[u_1, u_2, u_3] / \langle u_i^2 - \gamma_i u_i, u_i u_j = u_j u_i = 0 \rangle_{i \neq j=1,2,3}$. Let $\Lambda = 1 - 2\gamma_2^{-1}u_2 - 2\gamma_3^{-1}u_3$ and \mathfrak{C} be a skew Λ -constacyclic code of length 8 over $\mathcal{A}_{3,3}$. Then $\Lambda_0 = \Lambda_1 = 1, \Lambda_2 = \lambda_3 = -1$. Hence, $\mathfrak{C}_0, \mathfrak{C}_1$ are skew cyclic codes and $\mathfrak{C}_2, \mathfrak{C}_3$ are skew negacyclic codes of length 8 over \mathbb{F}_9 , respectively. Now, under the automorphism $\theta : \mathbb{F}_9 \rightarrow \mathbb{F}_9$ defined by $\theta(a) = a^3$ for all $a \in \mathbb{F}_9$, we have

$$\begin{aligned} x^8 - 1 &= (w^5 x^7 + x^6 + w x^5 + 2x^4 + w^5 x^3 + x^2 + w x + 2)(w x + 1) \\ &= h_0(x) f_0(x) \in \mathbb{F}_9[x; \theta], \\ x^8 - 1 &= (w^7 x^7 + x^6 + w^3 x^5 + 2x^4 + w^7 x^3 + x^2 + w^3 x + 2)(w^3 x + 1) \\ &= h_1(x) f_1(x) \in \mathbb{F}_9[x; \theta], \\ x^8 + 1 &= (w^3 x^6 + w^2 x^4 + w x^2 + 1)(w^5 x^2 + 1) = h_3(x) f_3(x) \in \mathbb{F}_9[x; \theta]. \end{aligned}$$

Then $\mathfrak{C} = \langle f(x) \rangle = \langle \sum_{i=0}^3 \kappa_i f_i(x) \rangle$, where $f_0(x) = w x + 1, f_1(x) = w^3 x + 1$ are right divisors of $x^8 - 1$ and $f_2(x) = 1, f_3(x) = w^5 x^2 + 1$ are right divisors of $x^4 + 1$, respectively. Again,

$$\begin{aligned} h_0(x) &= w^5 x^7 + x^6 + w x^5 + 2x^4 + w^5 x^3 + x^2 + w x + 2, \\ \overline{h_0(x)} &= 2x^7 + w^3 x^6 + x^5 + w^7 x^4 + 2x^3 + w^3 x^2 + x + w^7, \\ \overline{h_0(x)} h_0(x) &= (w^3 x^6 + w^7 x^4 + w^3 x^2 + w^7)(x^8 - 1), \\ h_1(x) &= w^7 x^7 + x^6 + w^3 x^5 + 2x^4 + w^7 x^3 + x^2 + w^3 x + 2, \\ \overline{h_1(x)} &= 2x^7 + w x^6 + x^5 + w^5 x^4 + 2x^3 + w x^2 + x + w^5, \\ \overline{h_1(x)} h_1(x) &= (w x^6 + 2w x^4 + w x^2 + 2w)(x^8 - 1), \\ h_2(x) &= x^8 + 1, \\ \overline{h_2(x)} h_2(x) &= \overline{h_2(x)}(x^8 + 1) \\ h_3(x) &= w^3 x^6 + w^2 x^4 + w x^2 + 1, \\ \overline{h_3(x)} &= x^6 + w x^4 + w^2 x^2 + w^3, \\ \overline{h_3(x)} h_3(x) &= (w^3 x^4 + w x^2 + w^3)(x^8 + 1). \end{aligned}$$

In this way, $x^8 - 1$ is a right divisor of both $\overline{h_0(x)} h_0(x)$ and $\overline{h_1(x)} h_1(x)$, and also $x^8 + 1$ is a right divisor of $\overline{h_2(x)} h_2(x)$ and $\overline{h_3(x)} h_3(x)$. Consequently, by Theorem 12, we have $\mathfrak{C}^{\perp h} \subseteq \mathfrak{C}$. Now, let

$$M = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix} \in GL_4(\mathbb{F}_9),$$

satisfying $MM^T = 4I_3$. Thus, the Gray image $\varphi(\mathfrak{C})$ is a dual containing $[32, 28, 3]$ linear code over \mathbb{F}_9 . Also, by Theorem 13, there exists a quantum code $[[32, 24, \geq 3]]_3$, which has better code rate than the known code $[[33, 23, 3]]_3$ given by [13]. In addition, our obtained code $[[32, 24, \geq 3]]_3$ satisfies $n - k + 2 - 2d = 4$.

Example 6 Let $\mathfrak{C} = \langle \sum_{i=0}^r \kappa_i f_i(x) \rangle$ be a skew Λ -constacyclic code of length n over $\mathcal{A}_{q,r}$ such that $\text{ord}(\langle \Theta \rangle) | n$ and $x^n - \Lambda_i = h_i(x)f_i(x)$ for $0 \leq i \leq r$. Also, let $x^n - \Lambda_i$ be a right divisor of $\overline{h_i(x)}h_i(x)$ for $0 \leq i \leq r$. Hence, by Theorem 12, $\mathfrak{C}^{\perp_h} \subseteq \mathfrak{C}$. Now, by Theorem 13, we obtain quantum codes in Table 5. In fact, Table 5 includes some better codes $[[n, k, \geq d]]_q$ (in seventh column) compared to the best-known codes $[[n', k', d']]_q$ (in eighth column). To compute the Gray images $\varphi(\mathfrak{C})$ (in sixth column), we used the matrices given in Example 4 for $r = 2, 3$ and

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \in GL_2(\mathbb{F}_{q^2})$$

satisfying $MM^T = 2I_2$ for $r = 1$. In order to make the table precise, we write their generator polynomials (f_0, f_1, \dots, f_r) in which each polynomial is represented by a string containing their coefficients in decreasing order. For example, we write the string w^5ww to present the polynomial $w^5x^2 + wx + w$. Also, in Table 5 we have used the following notations:

$$\begin{aligned} B_1 &= -1 + 2\gamma_1^{-1}u_1 + 2\gamma_2^{-1}u_2 \\ B_2 &= -1 + 2\gamma_1^{-1}u_1 + 2\gamma_2^{-1}u_2 + 2\gamma_3^{-1}u_3 \\ B_3 &= 1 - 2\gamma_2^{-1}u_2 - 2\gamma_3^{-1}u_3. \end{aligned}$$

Remark 1 In Table 5, codes $[[32, 24, \geq 3]]_3$, $[[30, 20, \geq 4]]_5$ satisfy $n - k + 2 - 2d = 4$, and $[[32, 20, \geq 4]]_4$ satisfies $n - k + 2 - 2d = 6$ whereas remaining codes satisfy $n - k + 2 - 2d = 2$. It is worth mentioning that $[[12, 6, \geq 3]]_3$, $[[16, 8, \geq 4]]_3$ are best-known record holder codes appeared in Ezerman et al.[14].

Table 5: New quantum codes from dual containing skew Λ -constacyclic codes of length n over $\mathcal{A}_{q,r}$

(r, n)	q^2	Λ	$(\Lambda_0, \Lambda_1, \dots, \Lambda_r)$	(f_0, f_1, \dots, f_r)	$\varphi(\mathfrak{C})$	$[[n, k, \geq d]]_q$	$[[n', k', d]]_q$
(1, 4)	9	$-1 + 2\gamma_1^{-1}u_1$	$(-1, 1)$	(w^5ww, w^31)	$[8, 5, 4]_9$	$[[8, 2, \geq 4]]_3$	MDS
(1, 6)	9	$1 - 2\gamma_1^{-1}u_1$	$(1, -1)$	(w^21, w^211)	$[12, 9, 3]_9$	$[[12, 6, \geq 3]]_3$	$[[12, 6, 3]]_3[14]$
(1, 8)	9	$1 - 2\gamma_1^{-1}u_1$	$(1, -1)$	$(w1, w^201)$	$[16, 13, 3]_9$	$[[16, 10, \geq 3]]_3$	$[[16, 10, 3]]_3[13]$
(2, 8)	9	B_1	$(-1, 1, 1)$	(w^5w^3w, w^31, w^31)	$[12, 8, 4]_9$	$[[12, 4, \geq 4]]_3$	$[[13, 4, 4]]_3[13]$
(3, 4)	9	B_2	$(-1, 1, 1, 1)$	$(w^5ww, w^31, w^31, 1)$	$[16, 12, 4]_9$	$[[16, 8, \geq 4]]_3$	$[[17, 7, 4]]_3[13]$
(3, 8)	9	B_3	$(1, 1, -1, -1)$	$(w1, w^31, 1, w^501)$	$[32, 28, 3]_9$	$[[32, 24, \geq 3]]_3$	$[[33, 23, 3]]_3[13]$
(3, 8)	9	B_3	$(1, 1, -1, -1)$	$(w1, w^31, w^301, w^501)$	$[32, 26, 4]_9$	$[[32, 20, \geq 4]]_3$	$[[36, 20, 4]]_3[27]$
(1, 4)	25	$1 - 2\gamma_1^{-1}u_1$	$(1, -1)$	$(4w1, 421)$	$[12, 8, 4]_{25}$	$[[12, 4, \geq 4]]_5$	$[[11, 1, 4]]_5[13]$
(1, 8)	25	1	$(1, 1)$	$(w1, w^311)$	$[16, 13, 3]_{25}$	$[[16, 10, \geq 3]]_5$	$[[16, 10, 3]]_5[13]$
(1, 8)	25	$-1 + 2\gamma_1^{-1}u_1$	$(-1, 1)$	(w^301, w^311)	$[16, 12, 4]_{25}$	$[[16, 8, \geq 4]]_5$	$[[16, 8, 4]]_5[13]$
(1, 10)	25	$-1 + 2\gamma_1^{-1}u_1$	$(-1, 1)$	(w^8w1, w^41)	$[20, 17, 3]_{25}$	$[[20, 14, \geq 3]]_5$	$[[22, 12, 3]]_5[13]$
(2, 10)	25	$-1 + 2\gamma_2^{-1}u_2$	$(-1, -1, 1)$	(w^21, w^8w1, w^811)	$[30, 25, 4]_{25}$	$[[30, 20, \geq 4]]_5$	$[[30, 20, 3]]_5[35]$

Remark 2 Let $r = 3, n = 2, q = 3, \Lambda = 1$ and the automorphism $\theta : \mathbb{F}_9 \rightarrow \mathbb{F}_9$, we have the factorization

$$\begin{aligned} x^2 - 1 &= (w^2x + 2)(w^2x + 1) = h_0(x)f_0(x) \in \mathbb{F}_9[x; \theta] \\ &= (w^6x + 2)(w^6x + 1) = h_1(x)f_1(x) \in \mathbb{F}_9[x; \theta] \\ &= (w^5x + w)(wx + w^3) = h_2(x)f_2(x) \in \mathbb{F}_9[x; \theta] \\ &= (w^7x + w^3)(w^3x + w) = h_3(x)f_3(x) \in \mathbb{F}_9[x; \theta]. \end{aligned}$$

Since $x^2 - 1$ is a right divisor of $\overline{h_i(x)}h_i(x)$, by Theorem 12, $\mathfrak{C}^{\perp_h} \subseteq \mathfrak{C}$, where $\mathfrak{C} = \langle \kappa_0 f_0(x) + \dots + \kappa_3 f_3(x) \rangle$ is a skew cyclic code of length 2 over $\mathcal{A}_{3,3}$. In this way, by applying Theorem 13 on its Gray image $\varphi(\mathfrak{C})$, we obtained a quantum code $[[8, 0, \geq 4]]_3$, which is better than the known code $[[8, 0, 3]]_3$ in [13].

On the other hand, if we consider the factorization of $x^2 - 1$ over $\mathbb{F}_9[x]$, we see that there does not exist any factor satisfying Theorem 6; in other words, there does not exist any dual containing code, given the factorization of $x^2 - 1$. Therefore, it is not possible to construct a quantum code for $r = 3, n = 2, q = 3, \Lambda = 1$. Again, it is easy to see that there does not exist any dual containing codes for the factorization of $x^2 + 1$ over $\mathbb{F}_9[x]$ and hence, it is not possible to construct a quantum code in this case too. In conclusion, we must say that the quantum code $[[8, 0, \geq 4]]_3$ can be constructed by using the concept of skew polynomial ring but not a commutative structure for $r = 3$. This example explains the advantage of skew cyclic codes over classical cyclic codes. In fact, it is one of the major motivations of studying skew codes in the context of quantum code.

Remark 3 From [19] it is well known that for $3 \leq n \leq q$, there exists a quantum MDS code with parameters $[[n, n - 2d + 2, d]]_q$ where $1 \leq d \leq \frac{n}{2} + 1$. Therefore, here we attempt to construct new quantum codes of length n where $n \geq q + 1$. In fact, all obtained codes in Tables 1, 3 and 5 are satisfying $n \geq q + 1$.

7 Conclusion

In this paper, we first studied Λ -constacyclic codes and then extended to skew Λ -constacyclic codes of length n over $\mathcal{A}_{q,r}$. From an application point of view, we determined many quantum codes via these codes by using Hermitian construction. Also, we have shown the novelty of our obtained codes by comparing them with the best-known codes available in the literature.

Acknowledgement

First author is thankful to the University Grants Commission (UGC) (under Sr. No. 2121540952, Ref. No. 20/12/2015(ii)EU-V dated 31/08/2016) and Second and third authors are thankful to the Department of Science and Technology (DST) (under Ref No.: DST/INSPIRE/03/2016/001445 and CRG/2020/005927, respectively), Govt. of India. The authors would like to thank the anonymous referee(s) and the Editor for their valuable comments to improve the presentation of the manuscript.

References

1. Abualrub, T., Aydin, N., Seneviratne, P.: θ -cyclic codes over $\mathbb{F}_2 + v\mathbb{F}_2$. *Australas. J. Combin.* **54**, 115-126 (2012).
2. Alahmadi, A., Islam, H., Prakash, O., Solé, P., Alkenani, A., Muthana, N., Hijazi, R.: New quantum codes from constacyclic codes over a non-chain ring. *Quantum Inf. Process.* **20**(2), Paper number: 60, 17pp (2021)
3. Ashraf, M., Mohammad, G.: Skew cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$. *Asian-Eur. J. Math.* **11**(05), 1850072 (2018).
4. Bag, T., Dinh, H. Q., Upadhyay, A. K., Bandi, R., Yamaka, W.: Quantum codes from skew constacyclic codes over the ring $\mathbb{F}_q[u, v]/\langle u^2 - 1, v^2 - 1, uv - vu \rangle$. *Discrete Math.* **343**(3), 111737 (2020).
5. Bhardwaj, S., Raka, M.: Skew constacyclic codes over a non-chain ring $\mathbb{F}_q[u, v]/\langle f(u), g(v), uv - vu \rangle$. *Appl. Algebra Engrg. Comm. Comput.* **31** (3-4), 173-194 (2020).
6. Boucher, D., Geiselmann, W., Ulmer, F.: Skew cyclic codes, *Appl. Algebra Engrg. Comm. Comput.* **18**, 379-389 (2007).
7. Boucher, D., Ulmer, F.: Coding with skew polynomial rings. *J. Symbolic Comput.* **44**(12), 1644-1656 (2009).
8. Boucher, D., Solé, P., Ulmer, F.: Skew constacyclic codes over Galois rings. *Adv. Math. Commun.* **2**, 273-292 (2008).
9. Bosma, W., Cannon, J.: *Handbook of Magma Functions*. Univ. of Sydney (1995).
10. Calderbank, A. R., Rains, E. M., Shor, P. M., Sloane, N. J. A.: Quantum error correction via codes over $GF(4)$. *IEEE Trans. Inform. Theory* **44**, 1369-1387 (1998).
11. Chen, B., Ling, S., Zhang, G.: Application of constacyclic codes to quantum MDS codes. *IEEE Trans. Inform. Theory* **61**(3), 1474-1484 (2015).
12. Dertli, A., Cengellenmis, Y., Eren, S.: On quantum codes obtained from cyclic codes over A_2 . *Int. J. Quantum Inf.* **13**, 1550031, 9 pp (2015).
13. Edel, Y.: Some good quantum twisted codes. <https://www.mathi.uni-heidelberg.de/~yves/Matritzen/QT BCH/QT BCHIndex.html>.
14. Ezerman, M. F., Ling, S., Qzkaya, B., Solé, P.: Good stabilizer codes from quasi-cyclic codes over \mathbb{F}_4 and \mathbb{F}_9 . *IEEE International Symposium on Information Theory (ISIT)*, Paris, France, 2898-2902 (2019).
15. Gao, J.: Skew cyclic codes over $\mathbb{F}_p + v\mathbb{F}_p$. *J. Appl. Math. Inform.* **31**, 337-342 (2013).
16. Gao, J., Ma, F., Fu, F.: Skew constacyclic codes over the ring $\mathbb{F}_q + v\mathbb{F}_q$. *Appl. Comput. Math.* **16**(3), 286-295 (2017).
17. Gao, J., Wang, Y.: u -Constacyclic codes over $\mathbb{F}_p + u\mathbb{F}_p$ and their applications of constructing new non-binary quantum codes. *Quantum Inf. Process.* **17**, Art. 4, 9 pp (2018).
18. Gao, Y., Gao, J., Fu, F. W.: On Quantum codes from cyclic codes over the ring $\mathbb{F}_q + v_1\mathbb{F}_q + \dots + v_r\mathbb{F}_q$. *Appl. Algebra Engrg. Comm. Comput.* **30**(2), 161-174 (2019).
19. Grassl, M., Beth, T., Roetteler, M.: On optimal quantum codes. *Int. J. Quantum Inf.* **2**, 55-64 (2004).
20. Gursoy, F., Siap, I., Yildiz, B.: Construction of skew cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$. *Adv. Math. Commun.* **8**(3), 313-322 (2014).
21. Islam, H., Prakash, O.: Skew cyclic and skew $(\alpha_1 + u\alpha_2 + v\alpha_3 + uv\alpha_3)$ -constacyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$. *Int. J. Inf. Coding Theory.* **5**(2), 101-116 (2019).
22. Islam, H., Prakash, O.: A note on constacyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$. *Discrete Math. Algorithms Appl.* **11**(3), 1950030 (2019).
23. Islam, H., Prakash, O.: Quantum codes from the cyclic codes over $\mathbb{F}_p[u, v, w]/\langle u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wv - uv \rangle$. *J. Appl. Math. Comput.* **60**(1-2), 625-635 (2019).
24. Islam, H., Prakash, O., Bhunia, D. K.: Quantum codes obtained from constacyclic codes. *Internat. J. Theoret. Phys.* **58**(11), 3945-3951 (2019).
25. Islam, H., Prakash, O., Verma, R. K.: Quantum codes from the cyclic codes over $\mathbb{F}_p[v, w]/\langle v^2 - 1, w^2 - 1, vw - wv \rangle$. *Springer Proceedings in Mathematics & Statistics* **307**, <https://doi.org/10.1007/978-981-15-1157-8-6> (2019).
26. Islam, H., Prakash, O., Verma, R. K.: A family of constacyclic codes over $\mathbb{F}_p^m[v, w]/\langle v^2 - 1, w^2 - 1, vw - wv \rangle$. *Int. J. Inf. Coding Theory.* **5**(3-4), 198-210 (2020).
27. Islam, H., Prakash, O., Verma, R. K.: New quantum codes from constacyclic codes over the ring $R_{k,m}$. *Adv. Math. Commun.* doi:10.3934/amc.2020097 (2020).
28. Islam, H., Prakash, O.: New quantum codes from constacyclic and additive constacyclic codes. *Quantum Inf. Process.* **19**(9), Paper number: 319, 17pp (2020).
29. Islam, H., Prakash, O.: Construction of LCD and new quantum codes from cyclic codes over a finite non-chain ring. *Cryptogr. Commun.* <https://doi.org/10.1007/s12095-021-00516-9> (2021).
30. Jitman, S., Ling, S., Udomkavanich, P.: Skew constacyclic codes over finite chain rings. *Adv. Math. Commun.* **6**(1), 29-63 (2012).
31. Kai, X., Zhu, S.: Quaternary construction of quantum codes from cyclic codes over $\mathbb{F}_4 + u\mathbb{F}_4$. *Int. J. Quantum Inf.* **9**, 689-700 (2011).
32. Ketkar, A., Klappenecker, A., Kumar, S., Sarvepalli, P.: Nonbinary stabilizer codes over finite fields. *IEEE Trans. Inform. Theory.* **52** (11), 4892-4914 (2006)
33. Li, J., Gao, J., Fu, F. W., Ma, F.: $\mathbb{F}_q R$ -linear skew constacyclic codes and their application of constructing quantum codes. *Quantum Inf. Process.* **19**, Art. 193, 23pp (2020).
34. Li, J., Gao, J., Wang, Y.: Quantum codes from $(1 - 2v)$ -constacyclic codes over the ring $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$. *Discrete Math. Algorithms Appl.* **10**, 1850046, 8 pp (2018).
35. Ma, F., Gao, J., Fu, F. W.: Constacyclic codes over the ring $\mathbb{F}_p + v\mathbb{F}_p + v^2\mathbb{F}_p$ and their applications of constructing new non-binary quantum codes. *Quantum Inf. Process.* **17**(6), Art. 122, 19 pp (2018).
36. Ma, F., Gao, J., Fu, F. W.: New non-binary quantum codes from constacyclic codes over $\mathbb{F}_q[u, v]/\langle u^2 - 1, v^2 - v, uv - vu \rangle$. *Adv. Math. Commun.* **13**(2), 421-434 (2019).
37. Ozen, M., Ozzaim, T., Ince, H.: Skew quasi cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$. *J. Algebra Appl.* **18**(4), 1950077, 16 pp (2019).
38. Prakash, O., Islam, H., Patel, S., Solé, P.: New quantum codes from skew constacyclic codes over a class of non-chain rings Re,q . *Internat. J. Theoret. Phys.* <https://doi.org/10.1007/s10773-021-04910-0> (2021).
39. Qian, J., Ma, W., Gou, W.: Quantum codes from cyclic codes over finite ring. *Int. J. Quantum Inf.* **7**, 1277-1283 (2009).

40. Sari, M., Siap, I.: On quantum codes from cyclic codes over a class of nonchain rings. *Bull. Korean Math. Soc.* **53**, 1617-1628 (2016).
41. Shi, M., Qian, L., Sok, L., Aydin, N., Solé, P.: On constacyclic codes over $\mathbb{Z}_4[u]/\langle u^2 - 1 \rangle$ and their Gray images. *Finite Fields Appl.* **45**, 86-95 (2017).
42. Siap, I., Abualrub, T., Aydin, N., Seneviratne, P.: Skew cyclic codes of arbitrary length, *Int. J. Inform. Coding Theory* **2**(1), 10-20 (2011).
43. Shor, P. W.: Scheme for reducing decoherence in quantum memory. *Phys. Rev. A.* **52**, 2493-2496 (1995).
44. Verma, R. K., Prakash, O., Singh, A., Islam, H.: New quantum codes from skew constacyclic codes. *Adv. Math. Commun.* doi: 10.3934/amc.2021028 (2021).
45. Yao, T., Shi, M., Solé, P.: Skew cyclic codes over $\mathbb{F}_q + \mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$. *J. Algebra Comb. Discrete Struct. Appl.* **2**(3), 163-168 (2015).