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To cite this version:

Julien Bensmail, Hervé Hocquard, Dimitri Lajou, Éric Sopena. A proof of the Multiplicative 1-2-3 Conjecture. Combinatorica, 2023, 43, pp.37-55. 10.1007/s00493-023-00003-0 . hal-03324458v2

HAL Id: hal-03324458

https://hal.science/hal-03324458v2

Submitted on 16 May 2022

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A proof of the Multiplicative 1-2-3 Conjecture

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Abstract

We prove that the product version of the 1-2-3 Conjecture, raised by Skowronek-Kaziów in 2012, is true. Namely, for every connected graph with order at least 3, we prove that we can assign labels $1, 2, 3$ to the edges in such a way that no two adjacent vertices are incident to the same product of labels.

Keywords: 1-2-3 Conjecture; product version; labels $1, 2, 3$.

1. Introduction

Let G be a graph. A *k-labelling* $\ell : E(G) \to \{1, \ldots, k\}$ is an assignment of labels $1, \ldots, k$ to the edges of G. From ℓ, we can compute different parameters of interest for all vertices v, such as the *sum* $\sigma_\ell(v)$ of incident labels (being formally $\sigma_\ell(v) = \Sigma_{u \in N(v)} \ell(uv)$), or similarly the *multiset* $\mu_\ell(v)$ of labels incident to v or the *product* $\rho_\ell(v)$ of labels incident to v. We say that ℓ is *s-proper* if σ_ℓ is a proper vertex-colouring of G, *i.e.*, we have $\sigma_\ell(u) \neq \sigma_\ell(v)$ for every edge $uv \in E(G)$. Similarly, we say that ℓ is *m-proper* and *p-proper*, if μ_ℓ and ρ_ℓ, respectively, form proper vertex-colourings of G.

In the context of so-called *distinguishing labellings*, the goal is generally to not only distinguish vertices within some distance according to some parameter computed from labellings (such as the parameters σ_ℓ, μ_ℓ and ρ_ℓ above, to name a few), but also to construct such k-labellings with k being as small as possible. We refer the interested reader to [6], in which hundreds of such labelling techniques are listed.

Regarding s-proper, m-proper and p-proper labellings, which are the main focus in this work, we are thus interested, as mentioned above, in finding such k-labellings with k as small as possible, for a given graph G. In other words, we are interested in the parameters $\chi_{\mathrm{S}}(G)$, $\chi_{\mathrm{M}}(G)$ and $\chi_{\mathrm{P}}(G)$ which denote the smallest $k \geq 1$ such that s-proper, m-proper and p-proper, respectively, k-labellings exist (if any). Actually, through greedy labelling arguments, it can be observed that the only connected graph G for which $\chi_{\mathrm{S}}(G)$, $\chi_{\mathrm{M}}(G)$ or $\chi_{\mathrm{P}}(G)$ is not defined, is K_2, the complete graph on 2 vertices. Consequently, these three parameters are generally investigated for so-called *nice graphs*, which are those graphs with no connected component isomorphic to K_2.

S-proper, m-proper and p-proper labellings form a subfield of distinguishing labellings, which has been attracting attention due to the so-called 1-2-3 Conjecture, raised, in [11], by Karoński, Łuczak and Thomason in 2004:

1-2-3 Conjecture (sum version). *If G is a nice graph, then $\chi_{\mathrm{S}}(G) \leq 3$.*

Later on, counterparts of the 1-2-3 Conjecture were raised for m-proper and p-proper labellings. Addario-Berry, Aldred, Dalal and Reed first raised, in 2005, the following in [1]:

1-2-3 Conjecture (multiset version). *If G is a nice graph, then $\chi_{\mathrm{M}}(G) \leq 3$.*

while Skowronek-Kaziów then raised, in 2012, the following in [14]:

1-2-3 Conjecture (product version). *If G is a nice graph, then $\chi_{\mathrm{P}}(G) \leq 3$.*

It is worth mentioning that all three conjectures above, if true, would be tight, as attested for instance by complete graphs. Note also that the multiset version of the 1-2-3 Conjecture is, out of the three variants above, the easiest one in a sense, as every s-proper or p-proper labelling is also m-proper (thus, proving the sum or product variant of the 1-2-3 Conjecture would prove the multiset variant).

To date, the best result towards the sum version of the 1-2-3 Conjecture, proved by Kalkowski, Karoński and Pfender in [8], is that $\chi_{\mathrm{S}}(G) \leq 5$ holds for every nice graph G. Another significant result is due to Przybyło, who recently proved in [12] that even $\chi_{\mathrm{S}}(G) \leq 4$ holds for every nice regular graph G. Karoński, Łuczak and Thomason themselves also proved in [11] that $\chi_{\mathrm{S}}(G) \leq 3$ holds for nice 3-colourable graphs. Regarding the multiset version, for long the best result was the one proved by Addario-Berry, Aldred, Dalal and Reed in [1], stating that $\chi_{\mathrm{M}}(G) \leq 4$ holds for every nice graph G. Building on that result, Skowronek-Kaziów later proved in [14] that $\chi_{\mathrm{P}}(G) \leq 4$ holds for every nice graph G. She also proved that $\chi_{\mathrm{P}}(G) \leq 3$ holds for every nice 3-colourable graph G.

A breakthrough result was recently obtained by Vučković, as he totally proved the multiset version of the 1-2-3 Conjecture in [15]. Due to connections between m-proper and p-proper 3-labellings, we observed in [4] that this result directly implies that $\chi_{\mathrm{P}}(G) \leq 3$ holds for every nice regular graph G. Inspired by Vučković's proof scheme, we were also able to prove that $\chi_{\mathrm{P}}(G) \leq 3$ holds for nice 4-chromatic graphs G, and to prove related results that are very close to what is stated in the product version of the 1-2-3 Conjecture.

Building on these results, we prove the following throughout the rest of this paper.

Theorem 1.1. *The product version of the 1-2-3 Conjecture is true. That is, every nice graph admits p-proper 3-labellings.*

2. Proof of Theorem 1.1

Let us start by introducing some terminology and recalling some properties of p-proper labellings, which will be used throughout the proof. Let G be a graph, and ℓ be a 3-labelling of G. For a vertex $v \in V(G)$ and a label $i \in \{1, 2, 3\}$, we denote by $d_i(v)$ the *i-degree* of v by ℓ, being the number of edges incident to v that are assigned label i by ℓ. Note then that $\rho_\ell(v) = 2^{d_2(v)}3^{d_3(v)}$. We say that v is *1-monochromatic* if $d_2(v) = d_3(v) = 0$, while we say that v is *2-monochromatic* (*3-monochromatic*, resp.) if $d_2(v) > 0$ and $d_3(v) = 0$ ($d_3(v) > 0$ and $d_2(v) = 0$, resp.). In case v has both 2-degree and 3-degree at least 1, we say that v is *bichromatic*. We also define the *$\{2, 3\}$-degree* of v as the sum $d_2(v) + d_3(v)$ of its 2-degree and its 3-degree. Thus, if v is bichromatic, then its $\{2, 3\}$-degree is at least 2.

Because ℓ assigns labels $1, 2, 3$, and, in particular, because 2 and 3 are coprime, note that, for every edge uv of G, we have $\rho_\ell(u) \neq \rho_\ell(v)$ as soon as u and v have different 2-degrees, 3-degrees, or $\{2, 3\}$-degrees. In particular, u and v cannot be in conflict, *i.e.*, verify $\rho_\ell(u) = \rho_\ell(v)$, if u and v are i-monochromatic and j-monochromatic, respectively, for $i \neq j$, or if u is monochromatic while v is bichromatic.

Before going into the proof of Theorem 1.1, let us start by giving an overview of it. Let G be a nice graph. Our goal is to build a p-proper 3-labelling ℓ of G. We can clearly assume that G is connected. We also set $t = \chi(G)$, where, recall, $\chi(G)$ refers to the chromatic

number[1] of G. In particular, $t \geq 2$. We could even assume that $t \geq 5$, due to the product version of the 1-2-3 Conjecture being true for 4-colourable graphs (recall [4]), though this is not needed throughout the proof.

In what follows, we construct ℓ through three main steps. First, we need to partition the vertices of G in a way verifying specific cut properties, forming what we call a valid partition of $V(G)$ (see later Definition 2.1 for a more formal definition). A *valid partition* $\mathcal{V} = (V_1, \ldots, V_t)$ is a partition of $V(G)$ into t independent sets $V_1, \ldots, V_t$ fulfilling two main properties, being, roughly put, that 1) every vertex v in some part V_i with $i > 1$ has an incident *upward edge* to every part V_j with $j < i$, and 2) for every connected component of $G[V_1 \cup V_2]$ having only one edge, we can freely swap its two vertices in V_1 and V_2 while preserving the properties of a valid partition.

Once we have this valid partition $\mathcal{V}$ in hand, we can then start constructing ℓ. The main part of the labelling process, Step 2 below, consists in starting from all edges of G being assigned label 1 by ℓ, and then processing the vertices of $V_3, \ldots, V_t$ one after another, possibly changing the labels assigned by ℓ to some of their incident edges, so that certain product types are achieved by ρ_ℓ. These desired product types can be achieved due to the many upward edges that some vertices are incident to (in particular, the deeper a vertex lies in $\mathcal{V}$, the more upward edges it is incident to). The product types we achieve for the vertices depend on the part V_i of $\mathcal{V}$ they belong to. In particular, the modifications we make on ℓ guarantee that all vertices in $V_3, \ldots, V_t$ are bichromatic, every two vertices in V_i and V_j with $i, j \in \{3, \ldots, t\}$ and $i \neq j$ have distinct 2-degrees or 3-degrees, all vertices in V_2 are 1-monochromatic or 2-monochromatic, and all vertices in V_1 are 1-monochromatic or 3-monochromatic. By itself, achieving these product types makes ℓ almost p-proper, in the sense that the only possible conflicts are between 1-monochromatic vertices in V_1 and V_2. An important point also, is that, through these label modifications, we will make sure that all edges of $G[V_1 \cup V_2]$ remain assigned label 1, and no vertex in $V_3 \cup \cdots \cup V_t$ has 3-degree 1, 2-degree at least 2, and odd $\{2, 3\}$-degree; in last Step 3 below, we will use that last fact to remove remaining conflicts by allowing some vertices of $V_1 \cup V_2$ to become *special*, *i.e.*, make their product realising these exact label conditions.

Step 3 is designed to get rid of the last conflicts between the adjacent 1-monochromatic vertices of V_1 and V_2 without introducing new ones in G. To that end, we will consider the set $\mathcal{H}$ of the connected components of $G[V_1 \cup V_2]$ having conflicting vertices, and, if needed, modify the labels assigned by ℓ to some of their incident edges so that no conflicts remain, and no new conflicts are created in G. To make sure that no new conflicts are created between vertices in $V_1 \cup V_2$ and vertices in $V_3 \cup \cdots \cup V_t$, we will modify labels while making sure that all vertices in $V_1 \cup V_2$ are monochromatic or special. An important point also, is that the fixing procedures we introduce require the number of edges in a connected component of $\mathcal{H}$ to be at least 2. Because of that, once Step 2 ends, we must make sure that $\mathcal{H}$ does not contain a connected component with only one edge incident to two 1-monochromatic vertices. To guarantee this, we will also make sure, during Step 2, to modify labels and the partition $\mathcal{V}$ slightly so that $\mathcal{H}$ has no such configuration.

Step 1: Constructing a valid partition

Let $\mathcal{V} = (V_1, \ldots, V_t)$ be a partition of $V(G)$ where each V_i is an independent set. Note that such a partition exists, as, for instance, any proper t-vertex-colouring of G forms such

[1]Recall that a *proper k-vertex-colouring* of a graph G is a partition $(V_1, \ldots, V_k)$ of $V(G)$ where all V_i's are independent. The *chromatic number* $\chi(G)$ of G is the smallest $k \geq 1$ such that proper k-vertex-colourings of G exist. We say that G is *k-colourable* if $\chi(G) \leq k$.

a partition of $V(G)$. For every vertex $u \in V_i$, an incident *upward edge* (*downward edge*, resp.) is an edge uv for which v belongs to some V_j with $j < i$ ($j > i$, resp.). Note that all vertices in V_1 have no incident upward edges, while all vertices in V_t have no incident downward edges.

We denote by $M_0(\mathcal{V})$ (also denoted M_0 when the context is clear) the set of isolated edges in the subgraph $G[V_1 \cup V_2]$ of G induced by the vertices of $V_1 \cup V_2$. That is, M_0 contains the edges of the connected components of $G[V_1 \cup V_2]$ that consist of one edge only. To lighten the exposition, whenever referring to the vertices of M_0, we mean the vertices of G incident to the edges in M_0.

For an edge $uv \in M_0$ with $u \in V_1$ and $v \in V_2$, *swapping* uv consists in modifying the partition $\mathcal{V}$ by removing u from V_1 (v from V_2, resp.) and adding it to V_2 (V_1, resp.). In other words, we exchange the parts to which u and v belong. Note that if V_1 and V_2 are independent sets before the swap, then, because $uv \in M_0$, by definition the resulting new V_1 and V_2 remain independent. Also, the set M_0 is unchanged by the swap operation.

We can now give a formal definition for the notion of valid partition.

Definition 2.1 (Valid partition)**.** *For a t-colourable graph G, a partition $\mathcal{V} = (V_1, \ldots, V_t)$ of $V(G)$ is a* valid partition *(of G) if $\mathcal{V}$ verifies the following properties.*

($\mathcal{I}$) *Every V_i is an independent set.*

($\mathcal{P}_1$) *Every vertex in every V_i with $i \geq 2$ has a neighbour in V_j for every $j < i$.*

($\mathcal{S}$) *For every set $\{e_1, \ldots, e_p\}$ of edges of $M_0(\mathcal{V})$, successively swapping every e_i (in any order) results in a partition $\mathcal{V}'$ verifying Properties ($\mathcal{I}$) and ($\mathcal{P}_1$).*

Note that Property ($\mathcal{S}$) in Definition 2.1 implies that any valid partition $\mathcal{V}$ also verifies the following additional property:

($\mathcal{P}_2$) Successively swapping any number of edges of $M_0(\mathcal{V})$ results in a valid partition $\mathcal{V}'$.

In order to prove Theorem 1.1, as mentioned earlier, to start constructing ℓ we first need to have a valid partition of G in hand. The following result guarantees its existence.

Lemma 2.2. *Every nice t-colourable graph G admits a valid partition.*

Proof. For a partition $\mathcal{V} = (V_1, \ldots, V_t)$ of $V(G)$ where each V_i is an independent set (note that such a partition exists, as every proper t-vertex-colouring of G is one such partition), set $f(\mathcal{V}) = \sum_{k=1}^{t} k \cdot |V_k|$. Among all possible $\mathcal{V}$'s, we consider a $\mathcal{V}$ that minimises $f(\mathcal{V})$.

Suppose that there is a vertex $u \in V_i$ with $i \geq 2$ for which Property ($\mathcal{P}_1$) does not hold, *i.e.*, there is a $j < i$ such that u has no incident upward edge to V_j. By moving u to V_j, we obtain another partition $\mathcal{V}'$ of $V(G)$ where every part is an independent set. However, note that $f(\mathcal{V}') = f(\mathcal{V}) + j - i < f(\mathcal{V})$, a contradiction to the minimality of $\mathcal{V}$. From this, we deduce that every partition $\mathcal{V}$ minimising f must verify Property ($\mathcal{P}_1$).

Let now $\mathcal{V}'$ be the partition of $V(G)$ obtained by successively swapping edges of $M_0(\mathcal{V})$. Recall that the swapping operation preserves Property ($\mathcal{I}$) and observe that $f(\mathcal{V}) = f(\mathcal{V}')$. Hence, $\mathcal{V}'$ minimises f and thus verifies Properties ($\mathcal{I}$) and ($\mathcal{P}_1$). Thus Property ($\mathcal{S}$) also holds, and $\mathcal{V}$ is a valid partition of G. □

From here, we thus assume that we have a valid partition $\mathcal{V} = (V_1, \ldots, V_t)$ of G.

Step 2: Labelling the upward edges of $V_3, \dots, V_t$

From G and $\mathcal{V}$, our goal now is to construct a 3-labelling ℓ of G achieving certain properties, the most important of which being that the only possible conflicts are between pairs of vertices of V_1 and V_2 that do not form an edge of M_0. The following result sums up the exact conditions we want ℓ to fulfil. Recall that a vertex v is special by ℓ, if $d_3(v) = 1$, $d_2(v) \geq 2$ and $d_2(v) + d_3(v)$ is odd. Note that special vertices are bichromatic.

Lemma 2.3. *For every nice graph G and every valid partition $(V_1, \dots, V_t)$ of G, there exists a 3-labelling ℓ of G such that:*

1. *all vertices of V_1 are either 1-monochromatic or 3-monochromatic,*
2. *all vertices of V_2 are either 1-monochromatic or 2-monochromatic,*
3. *all vertices of $V_3 \cup \dots \cup V_t$ are bichromatic,*
4. *no vertex is special,*
5. *if $u \in V_1$ and $v \in V_2$ are adjacent, then $\ell(uv) = 1$,*
6. *if two vertices u and v are in conflict, then $u \in V_1$ and $v \in V_2$ (or vice versa), and at least one of them has a neighbour w in $V_1 \cup V_2 \setminus \{u, v\}$.*

Proof. From now on, we fix the valid partition $\mathcal{V} = (V_1, \dots, V_t)$ of G. During the construction of ℓ, we may have, however, to swap some edges of M_0, resulting in a different valid partition of G. Abusing the notations, for simplicity we will still denote by $\mathcal{V}$ any valid partition of G obtained this way, through swapping edges. Recall that valid partitions are closed under swapping edges of M_0 (by Property $(\mathcal{P}_2)$).

Our goal is to design ℓ so that it not only verifies the four colour properties of Items 1 to 4 of the statement, but also achieves the following refined product types, for every vertex v in a part V_i of $\mathcal{V}$:

- $v \in V_1$: v is 1-monochromatic or 3-monochromatic;
- $v \in V_2$: v is 1-monochromatic or 2-monochromatic;
- $v \in V_3$: v is bichromatic with 2-degree 1 and even $\{2, 3\}$-degree;
- $v \in V_4$: v is bichromatic with 3-degree 2 and odd $\{2, 3\}$-degree;
- $v \in V_5$: v is bichromatic with 2-degree 2 and even $\{2, 3\}$-degree;
- ...
- $v \in V_{2n}$, $n \geq 3$: v is bichromatic with 3-degree n and odd $\{2, 3\}$-degree;
- $v \in V_{2n+1}$, $n \geq 3$: v is bichromatic with 2-degree n and even $\{2, 3\}$-degree;
- ...

We start from ℓ assigning label 1 to all edges of G. Let us now describe how to modify ℓ so that the conditions above are met for all vertices. We consider the vertices of $V_t, \dots, V_3$ following that order, from "bottom to top", and modify labels assigned to upward edges. An important condition we will maintain, is that every vertex in an odd part V_{2n+1} $(n \geq 0)$ has all its incident downward edges (if any) labelled 3 or 1, while every vertex in an even

part V_{2n} ($n \geq 1$) has all its incident downward edges (if any) labelled 2 or 1. Note that this is trivially verified for the vertices in V_t, since they have no incident downward edges.

At any point in the process, let M be the set of edges of M_0 for which both ends are 1-monochromatic (initially, $M = M_0$). When treating a vertex $u \in V_3 \cup \cdots \cup V_t$, we define M_u as the subset of edges of M having an end that is a neighbour of u. For every edge $e \in M_u$, we choose one end of e that is a neighbour of u and we add it to a set S_u. Note that $|S_u| = |M_u|$. Another goal during the labelling process, to fulfil Item 6, is to label the edges incident to u so that at least one end of every edge in M_u is no longer 1-monochromatic. Note that the set M_u considered when labelling the edges incident to u is not necessarily the set of edges of M_0 incident to a neighbour of u, as, during the whole process, some of these edges might be removed from M when dealing with previous vertices in $V_3 \cup \cdots \cup V_t$.

Let us now consider the vertices in $V_t, \ldots, V_3$ one by one, following that order. Let thus $u \in V_i$ be a vertex that has not been treated yet, with $i \geq 3$. Recall that every vertex belonging to some V_j with $j > i$ was treated earlier on, and thus has its desired product. Suppose that $i = 2n$ with $n \geq 2$ ($i = 2n + 1$ with $n \geq 1$, resp.). Recall also that u is assumed to have all its incident downward edges labelled 1 or 2 (3, resp.), due to how vertices in V_j's with $j > i$ have been treated earlier on. Also, all upward edges incident to u are currently assigned labelled 1 by ℓ.

If $M_u \neq \varnothing$, then we swap edges of M_u, if necessary, so that every vertex in S_u belongs to V_2 (V_1, resp.). This does not invalidate any of our invariants since both ends of an edge in S_u are 1-monochromatic.

In any case, by Property ($\mathcal{P}_1$), we know that, for every $j < i$, there is a vertex $x_j \in V_j$ which is a neighbour of u. In particular, the vertex x_1 (x_2, resp.) does not belong to S_u (but may be the other end of an edge in M_u). We label the edges $ux_3, ux_5, \ldots, ux_{2n-1}$ with 3 ($ux_4, ux_6, \ldots, ux_{2n}$ with 2, resp.). Note that, at this point, $d_3(u) = n - 1$ ($d_2(u) = n - 1$, resp.). To finish dealing with u, we need to distinguish two cases depending on whether M_u is empty or not.

- Suppose first that $M_u = \varnothing$. Label ux_1 with 3 (ux_2 with 2, resp.). Now u has the desired 3-degree (2-degree, resp.). If $i > 3$, then label ux_{i-2} with 2 (3, resp.) so that u is sure to be bichromatic. If $i > 3$ and the $\{2, 3\}$-degree of u does not have the desired parity, then label ux_2 with 2 (ux_1 with 3, resp.). If $u \in V_3$ and the $\{2, 3\}$-degree of u is even, then u is already bichromatic since $d_2(u) = 1$. If $u \in V_3$ and the $\{2, 3\}$-degree of u is odd, then label ux_1 with 3 to adjust the parity of the $\{2, 3\}$-degree of u and make u bichromatic. In all cases, at this point u is bichromatic with 3-degree n (2-degree n, resp.) and odd $\{2, 3\}$-degree (even $\{2, 3\}$-degree, resp.), which is precisely what is desired for u.

- Suppose now that $M_u \neq \varnothing$. Let $z \in S_u$ and let e be the edge of M_u containing z. For every vertex $w \in S_u \setminus \{z\}$, we label the edge uw with 2 (3, resp.). Then:

  - If $d_2(u) + d_3(u)$ is odd (even, resp.), then label uz with 2 (3, resp.) and ux_1 with 3 (ux_2 with 2, resp.). In this case, every edge in M_u is incident to at least one vertex which is not 1-monochromatic, while u is bichromatic with 3-degree n (2-degree n, resp.) and odd $\{2, 3\}$-degree (even $\{2, 3\}$-degree, resp.).
  - If $d_2(u) + d_3(u)$ is even (odd, resp.) and $d_2(u) > 0$ ($d_3(u) > 0$, resp.), then swap e and label uz with 3 (2, resp.). Note that, after the swap of e, we have $z \in V_1$ ($z \in V_2$, resp.). In this case, every edge in M_u is incident to at least one vertex which is not 1-monochromatic, while u is bichromatic with 3-degree n (2-degree n, resp.) and odd $\{2, 3\}$-degree (even $\{2, 3\}$-degree, resp.).

– The last case is when $d_2(u)+d_3(u)$ is even (odd, resp.) and $d_2(u) = 0$ ($d_3(u) = 0$, resp.). If $i > 4$, then we can label ux_{i-2} with 2 (3, resp.) and fall back into one of the previous cases. If $i = 4$, then the only edge labelled 3 is the edge ux_3 which implies that $d_3(u) = 1$, which is impossible since $d_2(u) = 0$ and thus $d_2(u) + d_3(u)$ is odd which contradicts our hypothesis. If $i = 3$, then the conditions of this case imply that $d_2(u) \geq 1$ while every upward edge incident to u is labelled 1 or 3 and similarly for every incident downward edge; this case thus cannot occur.

To finish, we remove the edges of M_u from M since their two ends are not both 1-monochromatic anymore.

At the end of this process, all vertices in V_1 are 1-monochromatic or 3-monochromatic, while all vertices in V_2 are 1-monochromatic or 2-monochromatic. Every vertex in $V_3 \cup \cdots \cup V_t$ is bichromatic and there are no conflicts involving any pair of these vertices. Indeed if $a \in V_i$ and $b \in V_j$ are adjacent with $i > j \geq 3$, then either i and j do not have the same parity, in which case a and b do not have the same $\{2, 3\}$-degree; or both i and j are even (odd, resp.) and $d_3(a) = \frac{i}{2} \neq \frac{j}{2} = d_3(b)$ ($d_2(a) = \frac{i-1}{2} \neq \frac{j-1}{2} = d_2(b)$, resp.). Note also that no vertex in G is special, as, by definition, special vertices have 3-degree 1, 2-degree at least 2, and odd $\{2, 3\}$-degree. Moreover, we did not change the label of any edge in the cut (V_1, V_2).

Finally, suppose that there is a conflict between two vertices u and v. Previous remarks imply that $u \in V_1$ and $v \in V_2$ (or *vice versa*) and that both u and v are 1-monochromatic. If none of u and v has another neighbour w in $V_1 \cup V_2$, then the edge uv belongs to the set M_0. Since G is nice, one of u or v must have a neighbour in $V_3 \cup \cdots \cup V_t$. Hence $uv \in M_z$ for one such neighbour z. Recall also that we relabelled the edges incident to z in such a way that, for every edge of M_z, at least one incident vertex became 2-monochromatic or 3-monochromatic, a contradiction to the existence of u and v. Hence, all properties of the lemma hold. □

Step 3: Labelling the edges between V_1 and V_2

From now on, we will modify a 3-labelling ℓ of G obtained by applying Lemma 2.3. We denote by $\mathcal{H}$ the set of the connected components of $G[V_1 \cup V_2]$ that contain two adjacent vertices $u \in V_1$ and $v \in V_2$ having the same product by ℓ. By Items 1 and 2 of Lemma 2.3, such u and v are 1-monochromatic. Also, by Item 6 of Lemma 2.3, recall that every connected component of $\mathcal{H}$ has at least two edges. In what follows, we only relabel edges of some connected components $H \in \mathcal{H}$ while making sure that their vertices (in $V_1 \cup V_2$) are monochromatic or special. This ensures that only vertices of H have their product affected, thus no new conflicts involving vertices in $V_3 \cup \cdots \cup V_t$ are created.

For a subgraph X of $H \in \mathcal{H}$ (possibly $X = H$), if, after having relabelled edges of X, no conflict remains between vertices of X and all vertices of X are either monochromatic or special, then we say that X verifies Property $(\mathcal{P}_3)$.

Lemma 2.4. *If we can relabel the edges of every $H \in \mathcal{H}$ so that every H verifies Property $(\mathcal{P}_3)$, then the resulting 3-labelling is p-proper.*

Proof. This is because if we get rid of all conflicts in $\mathcal{H}$, then the only possible remaining conflicts are between vertices in $V_1 \cup V_2$ and in $V_3 \cup \cdots \cup V_t$. In particular, recall that any two vertices of two distinct connected components $H_1, H_2 \in G[V_1 \cup V_2]$ cannot be adjacent. Note also that, because we only relabelled edges in $\mathcal{H}$, the vertices in $V_3 \cup \cdots \cup V_t$ retain the product types described in Lemma 2.3. In particular, they remain bichromatic and none of them is special. Thus, they cannot be in conflict with the vertices in $V_1 \cup V_2$. □

In order to show that we can relabel the edges of every $H \in \mathcal{H}$ so that it fulfils Property $(\mathcal{P}_3)$, the following result will be particularly handy.

Lemma 2.5. *For every integer $s \in \{2, 3\}$, every connected bipartite graph H whose edges are labelled 1 or s, and any vertex v in any part $V_i \in \{V_1, V_2\}$ of H, we can relabel the edges of H with 1 and s so that $d_s(u)$ is odd (even, resp.) for every $u \in V_i \setminus \{v\}$, and $d_s(u)$ is even (odd, resp.) for every $u \in V_{3-i}$.*

Proof. As long as H has a vertex u different from v that does not verify the desired condition, apply the following. Choose any path P from u to v, which exists by the connectedness of H. Now follow P from u to v, and change the labels of the traversed edges from 1 to s and *vice versa*. It can be noted that this alters the parity of the s-degrees of u and v, while this does not alter that parity for any of the other vertices of H. Thus, this makes u satisfy the desired condition, while the situation did not change for the other vertices different from u and v. Thus, once this process ends, all vertices of H different from v have their s-degree being as desired by the resulting labelling. □

We are now ready to treat the connected components $H \in \mathcal{H}$ independently, so that they all meet Property $(\mathcal{P}_3)$. To ease the reading, we distinguish several cases depending on the types and on the degrees of the vertices that H includes. In each of the successive cases we consider, it is implicitly assumed that H does not meet the conditions of any previous case.

Claim 2.6. *If $H \in \mathcal{H}$ contains a 3-monochromatic vertex $v \in V_1$, or a 1-monochromatic vertex $v_1 \in V_1$ having two 1-monochromatic neighbours $u_1, u_2 \in V_2$ with degree 1 (in H), then we can relabel edges of H so that H verifies Property $(\mathcal{P}_3)$.*

Proof. Recall that all edges of H (and thus in $\mathcal{H}$) are assigned label 1; thus, if a vertex of H is 3-monochromatic, then it must be due to incident downward edges to $V_3, \ldots, V_t$.

If H has a 1-monochromatic vertex $v_1 \in V_1$ having two degree-1 1-monochromatic neighbours $u_1, u_2 \in V_2$, then we set $\ell(v_1u_1) = \ell(v_1u_2) = 3$. Note that u_1 and u_2 become 3-monochromatic with 3-degree 1, and are thus no longer in conflict with v_1, as it becomes 3-monochromatic with 3-degree 2. Note that either we got rid of all conflicts in H and H now verifies Property $(\mathcal{P}_3)$ as desired, or conflicts between other 1-monochromatic vertices of H remain. In the latter case, we continue with the following arguments.

Assume H has remaining conflicts, and that H has a 3-monochromatic vertex $v \in V_1$ (and, due to the previous process, perhaps 3-monochromatic vertices u_1 and u_2 in V_2, in which case their 3-degree (and degree in H) is precisely 1, while their unique neighbour v in $V_1 \cap V(H)$ is 3-monochromatic with 3-degree 2). Let X be the set of all 3-monochromatic vertices of H belonging to V_1. Let $C_1, \ldots, C_q$ denote the $q \geq 1$ connected components of $H - X$ that do not contain any 3-monochromatic vertex of V_2 (the vertices u_1 and u_2 we dealt with earlier on). For every C_i, we choose arbitrarily a vertex $x_i \in X$ and a vertex $y_i \in C_i$ such that x_i and y_i are adjacent in H. Note that the vertices of C_i are either 1-monochromatic or 2-monochromatic (in which case they belong to V_2), since all 3-monochromatic vertices of H are part of X (or are the vertices u_1 and u_2 dealt with earlier on, which we have omitted for now and are not part of the C_i's).

By Lemma 2.5, in every C_i we can relabel the edges with 1 and 2 so that all vertices in $(V_2 \cap V(C_i)) \setminus \{y_i\}$ are 2-monochromatic with odd 2-degree, while all vertices in $V_1 \cap V(C_i)$ are 2-monochromatic with even 2-degree or possibly 1-monochromatic if their even 2-degree is 0. In particular, recall that y_i must be 1-monochromatic or 2-monochromatic. If y_i has odd 2-degree, then there are no conflicts between vertices of C_i. If y_i has even non-zero 2-degree, then we set $\ell(x_iy_i) = 3$, thereby making y_i special.

Let Y be the set containing all 1-monochromatic y_i's having a 1-monochromatic neighbour w_i in C_i. Let H' be the subgraph of H induced by $Y \cup X$. Note that every edge of H' is labelled 1. Let now $Q_1, \ldots, Q_p$ denote the connected components of H' and choose $x_k \in X \cap V(Q_k)$ for every $k \in \{1, \ldots, p\}$. For every k, we apply Lemma 2.5 with labels 1 and 3 so that all vertices in $V_2 \cap V(Q_k)$ get 3-monochromatic with odd 3-degree, while all vertices in $V_1 \cap V(Q_k) \setminus \{x_k\}$ get 3-monochromatic with even 3-degree or possibly 1-monochromatic if their 3-degree is 0.

If x_k is involved in a conflict with a vertex $y_i \in V_2 \cap V(Q_k)$, then this is because x_k has odd 3-degree. Then:

- If $\ell(x_k y_i) = 3$, then $d_3(y_i) = d_3(x_k) \geq 3$ since $x_k \in X$ (x_k must thus be incident to at least one other edge labelled 3, either a downward edge to $V_3, \ldots, V_t$ or an edge incident to u_1 (and similarly an edge incident to u_2)). We here assign label 1 to the edge $x_k y_i$ and label 3 to the edge $y_i w_i$. This way, x_k gets even 3-degree while the 3-degree of y_i does not change. Note that y_i and w_i are not in conflict since $d_3(w_i) = 1$ and $d_3(y_i) \geq 3$.

- Otherwise, if $\ell(x_k y_i) = 1$, then we assign label 3 to the edge $x_k y_i$ and label 3 to the edge $y_i w_i$. This way, x_k gets even 3-degree while the 3-degree of y_i remains odd and must be at least 3. Again y_i and w_i are not in conflict since $d_3(w_i) = 1$ and $d_3(y_i) \geq 3$.

We claim that we got rid of all conflicts in H. Indeed, consider two adjacent vertices $a \in V_1 \cap V(H)$ and $b \in V_2 \cap V(H)$. Suppose first that a and b belong to some C_i. Note that, with the exception of y_i and maybe of the vertex w_i (if it exists and $y_i \in Y$), every vertex of C_i is 1-monochromatic or 2-monochromatic, the vertices of $V_1 \cap V(C_i)$ having even 2-degree and the vertices of $V_2 \cap V(C_i)$ having odd 2-degree. Thus, no conflict involves two of these vertices. Suppose now that $b = y_i$. If y_i is 2-monochromatic with odd 2-degree, then there is no conflict involving y_i in C_i since all of its neighbours in C_i have even 2-degree. If y_i is special, then it is the only special vertex of C_i, so, here again, it cannot be involved in a conflict. If $y_i \notin Y$ and y_i is 1-monochromatic, then y_i has no other 1-monochromatic neighbour in C_i by definition of Y. If $y_i \in Y$, then y_i is 3-monochromatic with odd 3-degree, the only other possible 3-monochromatic neighbour of y_i in C_i being w_i, but we showed previously that their 3-degrees differ. Thus, in all cases, there cannot be conflicts between vertices of C_i.

We are left with the case where a and b do not belong to the same C_i. In particular, this implies that $a \in X$ and that a is 3-monochromatic. The only possible 3-monochromatic vertices in V_2 are the vertices of Y, which have odd 3-degree, and the 3-monochromatic vertices u_1 and u_2 with 3-degree 1 and degree 1 in H which might have been created at the very beginning of the proof. If $b \in Y$, then, due to the application of Lemma 2.5 above, the only vertex of X which can have odd 3-degree is some x_k, but for this vertex we either ensured that it was involved in no conflict, or we tweaked the labelling so that it got even 3-degree without modifying the labelling properties obtained through Lemma 2.5. If b is u_1 or u_2, then b has only one neighbour v. Note that the edges vu_1 and vu_2 are still labelled 3 as they are not part of the Q_i's, and, thus, $d_3(b) = 1$ and $d_3(v) \geq 2$. Hence, there is no conflict between vertices of X and other vertices of H. This implies that H verifies Property $(\mathcal{P}_3)$. □

Claim 2.7. *If H contains a 1-monochromatic vertex $u \in V_2$ with at least two neighbours in H, then we can relabel edges of H so that H verifies Property $(\mathcal{P}_3)$.*

Proof. Let $v_1, \ldots, v_p$ denote the neighbours of u in H. Due to Lemma 2.3 and because Claim 2.6 does not apply on H, for every vertex v of H we have $d_3(v) = 0$. In particular, none of the v_i's is 3-monochromatic, implying that they are all 1-monochromatic. Let $C_1, \ldots, C_q$ be the $q \geq 1$ connected components of $H - u$. Every C_i contains at least one of the v_i's. Up to renaming the v_i's, we can suppose w.l.o.g. that $v_i \in V(C_i)$ if $i \leq q$. The vertices v_i with $i > q$ (if any) can belong to any of the C_i's.

Let us focus on one component C_i. Let $J_1^i, \ldots, J_r^i$ denote the r connected components of $C_i - v_i$. If C_i has order 1, then by convention we set $r = 0$. In every J_j^i, choose a neighbour x_j^i of v_i. By Lemma 2.5, we can relabel edges of J_j^i with 1 and 2 so that every vertex of $V_1 \cap V(J_j^i)$ has even 2-degree, while every vertex of $V_2 \cap V(J_j^i)$, except possibly x_j^i, has odd 2-degree. Let X_i be the set containing all x_j^i's with even 2-degree. Note that v_i has even 2-degree, being precisely 0 since it is 1-monochromatic; thus the only possible conflicts in C_i involve vertices of X_i as they are the only ones not following the parity rule on their 2-degree (that is, they have even 2-degree).

If $|X_i| = 0$, $|X_i| \geq 2$ or if $X_i = \{w_i\}$ and $d_2(w_i) \geq 1$ for some vertex w_i, then we say that C_i is *nice*. In this case, we can relabel edges of C_i so that C_i verifies Property $(\mathcal{P}_3)$. If $|X_i| = 0$, then C_i already verifies Property $(\mathcal{P}_3)$. If $|X_i| \geq 2$, then, for every $z \in X_i$, set $\ell(v_i z) = 3$. If $X_i = \{w_i\}$ and $d_2(w_i) \geq 1$, then set $\ell(v_i w_i) = 3$. In the last two cases, all vertices of X_i either become special while they have no special neighbours; or they become 3-monochromatic with 3-degree 1 in which case v_i is their only 3-monochromatic neighbour and $d_3(v_i) \geq 2$. Moreover, in both cases, $d_3(v_i) \geq 1$ and all the neighbours of v_i in C_i which are not in X_i have 3-degree 0. Thus, v_i cannot be in conflict with its neighbours. Because the products of the other vertices of C_i were not altered by these labelling modifications, C_i verifies Property $(\mathcal{P}_3)$.

If $X_i = \{w_i\}$ and w_i is 1-monochromatic with no such neighbours in $C_i - v_i$, then we say that C_i is *bad*. In such a bad component C_i, the only current conflict is between v_i and w_i. If $X_i = \{w_i\}$ and w_i is 1-monochromatic with at least one 1-monochromatic neighbour y_i in $C_i - v_i$, then we say that C_i is *tricky*. We denote by N_n the number of nice components, by N_b the number of bad components, and by N_t the number of tricky components.

In what follows, we consider several cases. In each case, we implicitly assume that none of the previous cases applies.

- **Case 1.** $N_t > 0$.

  Let C_i be a tricky component. For every bad or tricky component C_j with $j \neq i$, set $\ell(v_j w_j) = 2$ and $\ell(u v_j) = 2$. In C_j, every vertex of V_1 now has even 2-degree since $d_2(v_j) = 2$ and every vertex of V_2 has odd 2-degree since $d_2(w_j) = 1$.

  Now, at this point:

  - If $d_2(u)$ is even, then set $\ell(v_i w_i) = 2$ and $\ell(u v_i) = 2$. Here, C_i behaves exactly like the other bad or tricky components and thus contains no conflicts.
  - If $d_2(u)$ is odd, then set $\ell(v_i w_i) = \ell(w_i y_i) = 3$. Recall that all conflicts of C_i involved w_i. Note that w_i is now 3-monochromatic with 3-degree 2 and no such neighbours. The vertices y_i and v_i are now 3-monochromatic with 3-degree 1 and no such neighbours (in particular, they are not adjacent since they both belong to V_1). Hence C_i does not contain any conflict.

  In both cases, note that u is 2-monochromatic with odd 2-degree. To summarise, we have reached the following situation. Special vertices (which were only created when dealing with nice components) only belong to V_2. 3-monochromatic vertices

are involved in no conflicts inside their component C_j and have no 3-monochromatic neighbours outside C_j since $d_3(u) = 0$. All the other vertices of H are either 1-monochromatic or 2-monochromatic: in particular, they have even 2-degree if they belong to V_1, while they have odd 2-degree if they belong to V_2. Hence, there is no conflict in H, and H thus verifies Property $(\mathcal{P}_3)$.

From now on, we can thus suppose that none of the C_i's is tricky.

- **Case 2.** $N_n = 0$.

  In this case, all C_i's are bad. We consider two cases:

  - If $N_b = 1$, *i.e.*, H contains only one (bad) component C_1, then set $\ell(v_1w_1) = 2$ and $\ell(uv_1) = 2$. Then every vertex of H in V_1 is 1-monochromatic or 2-monochromatic with even 2-degree, while every vertex in V_2 is 2-monochromatic with odd 2-degree. In particular, $d_2(w_1) = 1$, $d_2(v_1) = 2$ and $d_2(u) = 1$.
  - If $N_b > 1$, then, for every (bad) component C_i, set $\ell(uv_i) = 3$. Note that this makes all vertices of H be monochromatic. Every neighbour z of u verifies $d_3(z) \leq 1$ and, because $d_3(u) \geq 2$, the vertex u cannot be in conflict with any of its neighbours in H. The vertices v_i with $i \leq q$ are 3-monochromatic with 3-degree 1 and have no such neighbours. The w_i's are 1-monochromatic and have no 1-monochromatic neighbours since the C_i's were bad and their v_i's (with $i \leq q$) are no longer 1-monochromatic. The other 1-monochromatic vertices and 2-monochromatic vertices raise no conflicts since, for every such vertex z in $V_j \cap V(H)$ (where $j \in \{1, 2\}$), we have $d_2(z) \equiv j - 1 \bmod 2$.

  Hence H verifies Property $(\mathcal{P}_3)$ in both cases. Thus, we can now assume $N_n > 0$.

- **Case 3.** $N_b > 0$.

  Suppose now that at least one of the C_i's is bad. Since $N_n \geq 1$, not all C_i's are bad. So, since $N_t = 0$, we can thus suppose that C_1 is nice. For every bad component C_j, set $\ell(v_jw_j) = 2$ and $\ell(uv_j) = 2$. In C_j, every vertex of V_1 has even 2-degree (since $d_2(v_j) = 2$) while every vertex of V_2 has odd 2-degree (since $d_2(w_j) = 1$).

  Let us now analyse the 2-degree of u, which is at least 1 since $N_b > 0$.

  - If $d_2(u)$ is odd, then we claim that we have no conflicts in H. First, we saw earlier that any two vertices in a nice C_i cannot be in conflict. Next, in every bad C_j, every vertex of V_1 has even 2-degree, while every vertex of V_2 has odd 2-degree; hence, any two vertices of C_j cannot be in conflict. Thus, every possible conflict in H must involve u. Note that u is 2-monochromatic with odd 2-degree while no vertex of $V_1 \cap V(H)$ is 2-monochromatic with odd 2-degree. Thus u cannot be in conflict with a vertex of H.
  - If $d_2(u)$ is even (and thus at least 2 since $N_b > 0$), then set $\ell(uv_1) = 3$. Again, for the same reasons as earlier, any two vertices in a C_i with $i > 1$ cannot be in conflict. Since only v_1 had its product changed in C_1, then, if there is a conflict between two vertices of C_1, then it must involve v_1. Note that v_1 is 3-monochromatic. If $d_3(v_1) \geq 2$, then it is the only vertex of C_1 with this property. If $d_3(v_1) = 1$, then v_1 was 1-monochromatic before uv_1 was assigned labelled 3, in which case v_1, now, still has no 3-monochromatic neighbours in C_1 by construction. Thus, in both cases, v_1 cannot be in conflict with any other

vertex of C_1. Thus, any conflict in H must involve u. Note that u is special and that every other special vertex of H must belong to some nice component C_i, and must be a neighbour of v_i. In other words, all special vertices of H must belong to V_2, and thus u cannot be involved in a conflict.

Thus, in both cases, H verifies Property $(\mathcal{P}_3)$, and, from now on, we can assume $N_b = 0$. That is, all C_i's are nice, since also $N_t = 0$.

- **Case 4.** $N_n = 1$.

  Since $N_b = N_t = 0$, we have that $H - u$ is connected, *i.e.*, $q = 1$ and C_1 is the only (nice) component. As we assumed that $d(u) \geq 2$, vertex u has at least one other neighbour v_2 (in V_1) in C_1. Since C_1 is nice, recall that any two adjacent vertices of C_1 cannot be in conflict, due to how ℓ was modified so far.

  Let us analyse the possible situations, with respect to v_1.

  - If v_1 is 1-monochromatic, then set $\ell(uv_1) = \ell(uv_2) = 3$. In this case, u has 3-degree 2 while no other vertex of H has 3-degree at least 2. In C_1, the vertices of V_1 are either 2-monochromatic with even 2-degree, 1-monochromatic, special (only v_2 can verify this, and this is only if $d_2(v_2) > 0$ since $d_2(v_2)$ is even), or 3-monochromatic with 3-degree 1 (only v_1 and v_2 can verify this, and, for the latter vertex, this is only if $d_2(v_2) = 0$). Also, in C_1, the vertices of V_2 are 2-monochromatic with odd 2-degree. Hence, there are no conflicts.
  - If v_1 is 3-monochromatic, then set $\ell(uv_1) = 3$. In this case, in H, the vertices of V_1 are either 2-monochromatic with even 2-degree, 1-monochromatic, or 3-monochromatic with 3-degree at least 2 (only v_1 verifies this). The vertices of V_2 are either 2-monochromatic with odd 2-degree, special, or 3-monochromatic with 3-degree 1 (in particular, u verifies this). Hence, again there are no conflicts.

  Thus, in both cases, H eventually verifies Property $(\mathcal{P}_3)$. From now on, in the next cases, we can thus assume that $N_n > 1$.

- **Case 5.** $N_n \geq 2$ and there is some nice C_i with $d_3(v_i) \geq 2$ that contains another neighbour x of u (*i.e.*, u has at least two neighbours in C_i).

  Assume C_i does verify these properties. Let us start by modifying ℓ, by changing to 2 the label assigned to every edge incident to v_i assigned label 3. Note then that, in C_i, due to why we originally assigned label 3 to edges incident to v_i in the first place, now every vertex of V_2 is 2-monochromatic with odd 2-degree while every vertex of $V_1 \setminus \{v_i\}$ is 2-monochromatic with even 2-degree. Also, due to our assumption on v_i, we have $d_2(v_i) \geq 2$.

  Let us now focus on v_i.

  - If $d_2(v_i)$ is odd, then set $\ell(uv_i) = 2$. This makes v_i become 2-monochromatic with even 2-degree with no such neighbours, while u becomes 2-monochromatic with odd 2-degree with no such neighbours.
  - Assume now $d_2(v_i)$ is even. Let C be a shortest cycle containing u, v_i and x (note that C must exist since C_i is connected). Now relabel every edge of C so that 1's becomes 2's and *vice versa*. Note that, as a result, we get $d_2(u) = 2$, and, in C_i, every vertex of V_2 is 2-monochromatic with odd 2-degree while every vertex of V_1 is 1-monochromatic or 2-monochromatic with even 2-degree. Hence,

there is no conflict in C_i. Also, since every C_j with $j \neq i$ is nice and we did not modify labels incident to vertices of C_j, there are still no conflicts in C_j.

If no conflicts remain, then H now verifies Property $(\mathcal{P}_3)$. So assume some conflicts remain. All these conflicts must involve u, but, now, we have that $d_2(u) = 2$. Since $N_n \geq 2$, there exists $v_j \notin C_i$ and $j \leq q$. Set $\ell(uv_j) = 3$, so that u becomes special. Note that this increases $d_3(v_j)$. If v_j had 3-degree 0, then v_j was 1-monochromatic and C_j had no 3-monochromatic vertices, and, hence, now, there is no conflict in C_j. If v_j had non-zero 3-degree, then every neighbour of v_j in C_j still has 3-degree at most 1 while v_j has 3-degree at least 2. Hence, there is no conflict in C_j.

From here, it can be checked that no conflicts remain at all in H. In particular, all special vertices, including u, lie in V_2, and they are thus not in conflict. Thus, u is not in conflict. Also, there is still no conflict in a C_k with $k \notin i, j$ since C_k is nice and the products of their vertices did not change. Also, there is no conflict in C_i and C_j by our previous remarks.

Thus, in both cases, H verifies Property $(\mathcal{P}_3)$. We now deal with a final case.

- **Case 6.** $N_n \geq 2$.

  Let $A = \{a_1, \ldots, a_r\}$ be the subset of neighbours of u having 2-degree 0. Note that $r \geq N_n \geq 2$. Note also that some of these a_i's are v_i's with $i \leq q$ (all of which are in nice components, since $N_t = N_b = 0$), in which case, by how the nice components were treated earlier, they can be 3-monochromatic. Furthermore, A may contain more than N_n vertices since it may also contain 1-monochromatic v_i's with 2-degree 0 and $i > q$. However, since the previous case does not apply, if some v_i verifies $d_3(v_i) \geq 2$ (thus $i \leq q$), then u cannot neighbour any other vertex of C_i.

  For every $a_i \in A$, we define n_i as the current value of $d_3(a_i)$, at the beginning of this case (*i.e.*, before modifying labels below). Recall that we can have $d_3(a_i) > 0$, in which case a_i is a v_j in a (nice) C_j for which we had to remove some conflicts. Also, by the choice of A, at this point, $\ell(ua_i) = 1$. The goal now, is to relabel some ua_i's with 3 in such a way that u is not in conflict with the vertices of A. To show this can be achieved, we use the Combinatorial Nullstellensatz [2].

  Theorem 2.8 (Combinatorial Nullstellensatz [2])**.** *Let $\mathbb{F}$ be an arbitrary field, and $P = P(Z_1, \ldots, Z_p)$ be a polynomial in $\mathbb{F}[Z_1, \ldots, Z_p]$. Suppose that the coefficient of a monomial $Z_1^{k_1} \ldots Z_p^{k_p}$, where every k_i is a non-negative integer, is non-zero in P and the degree of P equals $\sum_{i=1}^{p} k_i$. If $S_1, \ldots, S_p$ are subsets of $\mathbb{F}$ with $|S_i| > k_i$ for every $i \in \{1, \ldots, p\}$, then there are $z_1 \in S_1, \ldots, z_p \in S_p$ so that $P(z_1, \ldots, z_p) \neq 0$.*

  For every $i \in \{1, \ldots, r\}$, let Z_i be a variable belonging to $S_i = \{0, 1\}$ and representing whether ua_i is assigned label 3 ($Z_i = 1$) or not ($Z_i = 0$). Let P be the following polynomial:

  $$P(Z_1, \ldots, Z_r) = \prod_{i=1}^{r} \left(\sum_{\substack{j=1 \\ j \neq i}}^{r} Z_j - n_i \right).$$

  Since $r \geq N_n \geq 2$, note that P has degree r at least 2. Furthermore, the monomial $\prod_{i=1}^{r} Z_i$ has non-zero coefficient (since every Z_i has positive coefficient in the description of P). Hence the Combinatorial Nullstellensatz applies and there is a way to choose values $z_1, \ldots, z_r$ in $\{0, 1\}$ for $Z_1, \ldots, Z_r$ so that $P(z_1, \ldots, z_r) \neq 0$.

Now, for every $i \in \{1, \ldots, r\}$ for which $z_i = 1$, set $\ell(ua_i) = 3$. Note that $d_3(u) = \sum_{j=1}^{r} z_j$ and $d_2(u) = 0$. We claim that H now verifies Property $(\mathcal{P}_3)$. Assume this is wrong, and suppose that there is a conflict in H between two vertices $x \in V_1$ and $y \in V_2$. For now, suppose that u is not one of these two vertices.

- If x and y are 2-monochromatic, then, because we did not modify 2-degrees when we modified ℓ above, and all C_i's are nice, then $d_2(x)$ is even while $d_2(y)$ is odd, a contradiction to the fact that x and y are in conflict.
- Similarly, the modifications above did not introduce new 1-monochromatic vertices. Thus, x and y cannot be both 1-monochromatic, since all 1-monochromatic vertices of H (different from u) belong to V_1.
- Similarly, x and y cannot be special. This is because, since the a_i's have 2-degree 0, the modifications did not introduce new special vertices. So, all special vertices are adjacent to v_i's (with $i \leq q$), and thus lie in V_2.
- If x and y are 3-monochromatic, then y must be a neighbour of some v_i (with $i \leq q$) and y thus verifies $d_3(y) = 1$. In this case, v_i verified $d_3(v_i) \geq 2$ at the beginning of this case (by how ℓ was constructed in C_i, and, in particular, because y is not special), and thus $x \neq v_i$. Thus, x became 3-monochromatic because ux was relabelled with label 3 through the Combinatorial Nullstellensatz. So we deduce that u has two neighbours in C_i, where we had $d_3(v_i) \geq 2$ at the beginning of this case. This is not possible, as this configuration is forbidden due to previous **Case 5** not applying.

Hence, every possible conflict must involve u. Vertex u has two types of neighbours: those with non-zero 2-degree, and the vertices of A. Since $d_2(u) = 0$, the first group of neighbours cannot be in conflict with u. Suppose now that $a_i \in A$ is in conflict with u. Note that $d_3(a_i) = n_i + z_i$ and $d_3(u) = \sum_{j=1}^{r} z_j$. Since $d_3(a_i) = d_3(u)$, we have $\sum_{\substack{j=1 \\ j \neq i}}^{r} z_j - n_i = 0$ and thus $P(z_1, \ldots, z_r) = 0$, a contradiction.

Hence there is no conflict in H, and H verifies Property $(\mathcal{P}_3)$. □

We are now ready to get rid of the last possible conflicts in $\mathcal{H}$.

Claim 2.9. *For every remaining H, we can relabel edges so that H verifies Property $(\mathcal{P}_3)$.*

Proof. Let $v \in V_1$ and $u \in V_2$ be two adjacent 1-monochromatic vertices of H (which must exist as otherwise H would verify Property $(\mathcal{P}_3)$). Because H has at least two edges (as otherwise it would belong to M, not to $\mathcal{H}$), at least one of v and u must have another neighbour in H. Since Claim 2.6 does not apply, the neighbours of u are 1-monochromatic and since Claim 2.7 does not apply, u must have degree 1 in H. So v is also adjacent to $k \geq 1$ vertices $x_1, \ldots, x_k \in V_2$ different from u. Still by Claim 2.7, note that if an x_i is 1-monochromatic, then it must be of degree 1 in H, since v is a neighbour of x_i; but then Claim 2.6 would apply, as v is 1-monochromatic and neighbours u and x_1, which are 1-monochromatic and of degree 1 in H. Thus, we can assume all x_i's are 2-monochromatic (because of incident downward edges to $V_3, \ldots, V_t$; recall that all edges of H are labelled 1).

Set $H' = H - u$. According to Lemma 2.5, we can relabel edges in H' with 1 and 2 so that all vertices in $(V_1 \cap V(H')) \setminus \{v\}$ have odd 2-degree, while all vertices in $V_2 \cap V(H')$ have even 2-degree. Recall that u is 1-monochromatic. Thus, if also v is 2-monochromatic with odd 2-degree, then we are done. Assume thus that v is 2-monochromatic with even 2-degree.

- Assume first that the 2-degree of v is even at least 2. In that case, set $\ell(vu) = 3$. This way, u becomes 3-monochromatic, while v becomes special.
- Assume now v is 1-monochromatic. This implies that $\ell(vx_1) = 1$. Change $\ell(vx_1)$ to 3. This way, x_1 becomes special (recall its 2-degree is even and at least 1, due to incident downward edges), while v becomes 3-monochromatic. Note that u remains 1-monochromatic.

In both cases, it can be checked that H now fulfils Property $(\mathcal{P}_3)$. □

At this point, we dealt with all connected components of $\mathcal{H}$, and the resulting labelling ℓ of G is p-proper by Lemma 2.4. The whole proof is thus complete.

3. Conclusion

Although we provide a solution to the product version of the 1-2-3 Conjecture, our investigations and our proof methodology actually open the way to several appealing directions for further research on the topic. In particular:

- Distinguishing labellings, generally speaking, is a field with many interconnections between more or less distant problems, and, as a result, any major breakthrough on one particular distinguishing labelling problem can have drastic consequences on related others. A perfect illustration for that claim, is through the example of a brilliant algorithm designed by Kalkowski in [7] to get very close to a full verification of the total version of the 1-2-3 Conjecture (where vertices are also labelled, the label assigned to every vertex taking part to its sum) from [13]. Since its introduction, Kalkowski's Algorithm has indeed been revisited in numerous works, which, sometimes, allowed to improve significantly the best results that were known for long. In particular, the upper bound, from [8], of 5 on $\chi_{\mathrm{S}}(G)$ for every nice graph G results from straight modifications of Kalkowski's Algorithm. Very interesting results for generalisations of the 1-2-3 Conjecture to hypergraphs were also established through modifications of Kalkowski's approach [10]. In [9], new bounds on the irregularity strength of graphs (which is, roughly put, a generalisation of the 1-2-3 Conjecture where all vertices, not only the adjacent ones, are required to be distinguished through their sums by a labelling) were established, and the proof arguments were, again, strongly influenced by Kalkowski's Algorithm. Distinguishing labellings really form a field where making significant progress relies on the introduction of novel ideas, which might lead to many appealing perspectives for the whole field.

  As seen through this work, the product version of the 1-2-3 Conjecture, and in particular p-proper labellings, rely on very peculiar properties. Yet, proving it required quite some efforts, the resulting proof being rather technical at times. As mentioned in the introductory section, we were highly influenced by Vučković result from [15], which, we believe, is another one of these major results that can lead to many interesting accomplishments, as our main result in this work just showcases.

  According to these thoughts, one can naturally wonder whether our proof scheme could in turn be modified to deal with problems that are close to the product version of the 1-2-3 Conjecture. A few candidates come immediately to mind. In particular, one could wonder whether we can get new results on the product irregularity strength of graphs [3] (in which all vertices must be distinguished through their products by a labelling). One could also wonder about consequences for the list version of the

product version of the 1-2-3 Conjecture (introduced in [5], in which labellings must be constructed by assigning labels from dedicated lists of three labels). We are not sure exactly what one could expect, but these questions would definitely be worth considering.

- Note that an m-proper 3-labelling is similar, when a, b, c are pairwise coprime labels, to a p-proper $\{a, b, c\}$-labelling. Thus, for any three pairwise coprime labels a, b, c, the result of Vučković from [15] implies that every nice graph admits a p-proper $\{a, b, c\}$-labelling. An intermediate question lying in between the product version of the 1-2-3 Conjecture and its list variant would thus be about the existence of p-proper $\{a, b, c\}$-labellings for any nice graph and any three fixed labels a, b, c.

- Other directions of interest would deal with the connections between the sum version and the product version of the 1-2-3 Conjecture. Note indeed that there are definitely connections, as, by labellings, label 0 in the sum version plays the same role as label 1 in the product version. For this reason, s-proper $\{0, a\}$-labellings and p-proper $\{1, b\}$-labellings are similar objects for any $a, b \neq 0$. When considering three labels, note that the situation is not as obvious, as the equivalence between an s-proper $\{0, a_1, a_2\}$-labelling and a p-proper $\{1, b_1, b_2\}$-labelling is not guaranteed (as two sums of a_1's and a_2's might be different while the corresponding two products of b_1's and b_2's might not be, and *vice versa*). However, there are situations where this is guaranteed, for instance when $a_i = \log(b_i)$ for every $i \in \{1, 2\}$, or when the a_i's (and b_i's) are such that we can infer the coefficients of a sum (and product) from said sum (and product).

  These thoughts relate to an interesting question related to the 1-2-3 Conjecture. By the arguments above, it can be checked that from an m-proper 3-labelling of some graph G, we can obtain an s-proper $\{1, \Delta(G), \Delta(G)^2\}$-labelling of G. So, for G, there indeed exist three labels a_G, b_G, c_G for which we know s-proper $\{a_G, b_G, c_G\}$-labellings exist. Note however that these a_G, b_G, c_G are functions of G, and thus, for a graph H different from G, we might have $\{a_G, b_G, c_G\} \neq \{a_H, b_H, c_H\}$. The question is whether we can provide three labels a^*, b^*, c^* that would work for all nice graphs. The 1-2-3 Conjecture asserts that $1, 2, 3$ would be an example of three such labels a^*, b^*, c^*. As mentioned earlier, the result of Vučković implies that there is an s-proper $\{1, b_G, c_G\}$-labelling of any nice graph G where b_G and c_G are functions of G. By earlier arguments, our proof of the product version of the 1-2-3 Conjecture implies that that there is an s-proper $\{0, 1, c_G\}$-labelling of every nice graph G, where c_G is a function of G. Thus, in some sense, we are now just one step away from providing three labels a^*, b^*, c^* as described above.

Acknowledgement

The authors are grateful to the three anonymous referees for their careful reading of a previous version of the current work, which allowed to improve the general quality and correctness not only of the main proof, but also of the whole paper.

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