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A proof of the Multiplicative 1-2-3 Conjecture

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Abstract

We prove that the product version of the 1-2-3 Conjecture, raised by Skowronek-Kaziów in 2012, is true. Namely, for every connected graph with order at least 3, we prove that we can assign labels 1, 2, 3 to the edges in such a way that no two adjacent vertices are incident to the same product of labels.

Keywords: 1-2-3 Conjecture; product version; labels 1, 2, 3.

1. Introduction

Let G be a graph. A k -labelling $\ell : E(G) \rightarrow \{1, \dots, k\}$ is an assignment of labels $1, \dots, k$ to the edges of G . From ℓ , we can compute different parameters of interest for all vertices v , such as the *sum* $\sigma_\ell(v)$ of incident labels (being formally $\sigma_\ell(v) = \sum_{u \in N(v)} \ell(uv)$), or similarly the *multiset* $\mu_\ell(v)$ of labels incident to v or the *product* $\rho_\ell(v)$ of labels incident to v . We say that ℓ is *s-proper* if σ_ℓ is a proper vertex-colouring of G , *i.e.*, we have $\sigma_\ell(u) \neq \sigma_\ell(v)$ for every edge $uv \in E(G)$. Similarly, we say that ℓ is *m-proper* and *p-proper*, if μ_ℓ and ρ_ℓ , respectively, form proper vertex-colourings of G .

In the context of so-called *distinguishing labellings*, the goal is generally to not only distinguish vertices within some distance according to some parameter computed from labellings (such as the parameters σ_ℓ , μ_ℓ and ρ_ℓ above, to name a few), but also to construct such k -labellings with k being as small as possible. We refer the interested reader to [4], in which hundreds of such labelling techniques are listed.

Regarding s-proper, m-proper and p-proper labellings, which are the main focus in this work, we are thus interested, as mentioned above, in finding such k -labellings with k as small as possible, for a given graph G . In other words, we are interested in the parameters $\chi_S(G)$, $\chi_M(G)$ and $\chi_P(G)$ which denote the smallest $k \geq 1$ such that s-proper, m-proper and p-proper, respectively, k -labellings exist (if any). Actually, through greedy labelling arguments, it can be observed that the only connected graph G for which $\chi_S(G)$, $\chi_M(G)$ or $\chi_P(G)$ is not defined, is K_2 , the complete graph on 2 vertices. Consequently, these three parameters are generally investigated for so-called *nice graphs*, which are those graphs with no connected component isomorphic to K_2 .

S-proper, m-proper and p-proper labellings form a subfield of distinguishing labellings, which has been attracting attention due to the so-called 1-2-3 Conjecture, raised, in [6], by Karoński, Łuczak and Thomason in 2004:

1-2-3 Conjecture (sum version). *If G is a nice graph, then $\chi_S(G) \leq 3$.*

Later on, counterparts of the 1-2-3 Conjecture were raised for m-proper and p-proper labellings. Addario-Berry, Aldred, Dalal and Reed first raised, in 2005, the following in [2]:

1-2-3 Conjecture (multiset version). *If G is a nice graph, then $\chi_M(G) \leq 3$.*

while Skowronek-Kaziów then raised, in 2012, the following in [8]:

1-2-3 Conjecture (product version). *If G is a nice graph, then $\chi_P(G) \leq 3$.*

It is worth mentioning that all three conjectures above, if true, would be tight, as attested for instance by complete graphs. Note also that the multiset version of the 1-2-3 Conjecture is, out of the three variants above, the easiest one in a sense, as every s-proper or p-proper labelling is also m-proper (thus, proving the sum or product variant of the 1-2-3 Conjecture would prove the multiset variant).

To date, the best result towards the sum version of the 1-2-3 Conjecture, proved by Kalkowski, Karoński and Pfender in [5], is that $\chi_S(G) \leq 5$ holds for every nice graph G . Another significant result is due to Przybyło, who recently proved in [7] that even $\chi_S(G) \leq 4$ holds for every nice regular graph G . Karoński, Łuczak and Thomason themselves also proved in [6] that $\chi_S(G) \leq 3$ holds for nice 3-colourable graphs. Regarding the multiset version, for long the best result was the one proved by Addario-Berry, Aldred, Dalal and Reed in [2], stating that $\chi_M(G) \leq 4$ holds for every nice graph G . Building on that result, Skowronek-Kaziów later proved in [8] that $\chi_P(G) \leq 4$ holds for every nice graph G . She also proved that $\chi_P(G) \leq 3$ holds for every nice 3-colourable graph G .

A breakthrough result was recently obtained by Vučković, as he totally proved the multiset version of the 1-2-3 Conjecture in [9]. Due to connections between m-proper and p-proper 3-labellings, we observed in [3] that this result directly implies that $\chi_P(G) \leq 3$ holds for every nice regular graph G . Inspired by Vučković's proof scheme, we were also able to prove that $\chi_P(G) \leq 3$ holds for nice 4-chromatic graphs G , and to prove related results that are very close to what is stated in the product version of the 1-2-3 Conjecture.

Building on these results, we prove the following throughout the rest of this paper.

Theorem 1.1. *The product version of the 1-2-3 Conjecture is true. That is, every nice graph admits p-proper 3-labellings.*

2. Proof of Theorem 1.1

Let us start by introducing some terminology and recalling some properties of p-proper labellings, which will be used throughout the proof. Let G be a graph, and ℓ be a 3-labelling of G . For a vertex $v \in V(G)$ and an $i \in \{1, 2, 3\}$, we denote by $d_i(v)$ the i -degree of v by ℓ , being the number of edges incident to v that are assigned label i by ℓ . Note then that $\rho_\ell(v) = 2^{d_2(v)}3^{d_3(v)}$. We say that v is *1-monochromatic* if $d_2(v) = d_3(v) = 0$, while we say that v is *2-monochromatic* (*3-monochromatic*, resp.) if $d_2(v) > 0$ and $d_3(v) = 0$ ($d_3(v) > 0$ and $d_2(v) = 0$, resp.). In case v has both 2-degree and 3-degree at least 1, we say that v is *bichromatic*. We also define the $\{2, 3\}$ -degree of v as the sum $d_2(v) + d_3(v)$ of its 2-degree and its 3-degree. Thus, if v is bichromatic, then its $\{2, 3\}$ -degree is at least 2.

Because ℓ assigns labels 1, 2, 3, and, in particular, because 2 and 3 are coprime, note that, for every edge uv of G , we have $\rho_\ell(u) \neq \rho_\ell(v)$ as soon as u and v have different 2-degrees, 3-degrees, or $\{2, 3\}$ -degrees. In particular, u and v cannot be in conflict, *i.e.*, verify $\rho_\ell(u) = \rho_\ell(v)$, if u and v are i -monochromatic and j -monochromatic, respectively, for $i \neq j$, or if u is monochromatic while v is bichromatic.

Before going into the proof of Theorem 1.1, let us start by giving an overview of it. Let G be a nice graph. Our goal is to build a p-proper 3-labelling ℓ of G . We can clearly assume that G is connected. An important point also, is that, because the product version

of the 1-2-3 Conjecture was verified *e.g.* for 3-colourable graphs (see [8]), we can assume that G has chromatic number¹ $\chi(G)$ at least 4. We set $t = \chi(G) \geq 4$.

In what follows, we construct ℓ through three main steps. First, we need to partition the vertices of G in a way verifying specific cut properties, forming what we call a valid partition of $V(G)$ (see later Definition 2.1 for a more formal definition). A *valid partition* $\mathcal{V} = (V_1, \dots, V_t)$ is a partition of $V(G)$ into t independent sets V_1, \dots, V_t fulfilling two main properties, being, roughly put, that 1) every vertex v in some part V_i with $i > 1$ has an incident *upward edge* to every part V_j with $j < i$, and 2) for every connected component of $G[V_1 \cup V_2]$ having only one edge, we can freely swap its two vertices in V_1 and V_2 while preserving the properties of a valid partition.

Once we have this valid partition \mathcal{V} in hand, we can then start constructing ℓ . The main part of the labelling process, Step 2 below, consists in starting from all edges of G being assigned label 1 by ℓ , and then processing the vertices of V_3, \dots, V_t one after another, possibly changing the labels by ℓ assigned to some of their incident edges, so that certain product types are achieved by ρ_ℓ . These desired product types can be achieved due to the many upward edges that some vertices are incident to (in particular, the deeper a vertex lies in \mathcal{V} , the more upward edges it is incident to). The product types we achieve for the vertices depend on the part V_i of \mathcal{V} they belong to. In particular, the modifications we make on ℓ guarantee that all vertices in V_3, \dots, V_t are bichromatic, every two vertices in V_i and V_j with $i, j \in \{3, \dots, t\}$ and $i \neq j$ have different 2-degrees or 3-degrees, all vertices in V_2 are 1-monochromatic or 2-monochromatic, and all vertices in V_1 are 1-monochromatic or 3-monochromatic. By itself, achieving these product types makes ℓ almost p-proper, in the sense that the only possible conflicts are between 1-monochromatic vertices in V_1 and V_2 . An important point also, is that, through these label modifications, we will make sure that all edges of $G[V_1 \cup V_2]$ remain assigned label 1, and no vertex in $V_3 \cup \dots \cup V_t$ has 3-degree 1, 2-degree at least 2, and odd $\{2, 3\}$ -degree; in last Step 3 below, we will use that last fact to remove remaining conflicts by allowing some vertices of $V_1 \cup V_2$ to become *special*, *i.e.*, make their product realising these exact label conditions.

Step 3 is designed to get rid of the last conflicts between the adjacent 1-monochromatic vertices of V_1 and V_2 without introducing new ones in G . To that end, we will consider the set \mathcal{H} of the connected components of $G[V_1 \cup V_2]$ having conflicting vertices, and, if needed, modify the labels assigned by ℓ to some of their incident edges so that no conflicts remain, and no new conflicts are created in G . To make sure that no new conflicts are created between vertices in $V_1 \cup V_2$ and vertices in $V_3 \cup \dots \cup V_t$, we will modify labels while making sure that all vertices in $V_1 \cup V_2$ are monochromatic or special. An important point also, is that the fixing procedures we introduce require the number of edges in a connected component of \mathcal{H} to be at least 2. Because of that, once Step 2 ends, we must make sure that \mathcal{H} does not contain a connected component with only one edge incident to two 1-monochromatic vertices. To guarantee this, we will also make sure, during Step 2, to modify labels and the partition \mathcal{V} slightly so that \mathcal{H} has no such configuration.

Step 1: Constructing a valid partition

Let $\mathcal{V} = (V_1, \dots, V_t)$ be a partition of $V(G)$ where each V_i is an independent set. Note that such a partition exists, as, for instance, any proper t -vertex-colouring of G forms such a partition of $V(G)$. For every vertex $u \in V_i$, an incident *upward edge* (*downward edge*,

¹Recall that a *proper k -vertex-colouring* of a graph G is a partition (V_1, \dots, V_k) of $V(G)$ where all V_i 's are independent. The *chromatic number* $\chi(G)$ of G is the smallest $k \geq 1$ such that proper k -vertex-colourings of G exist. We say that G is *k -colourable* if $\chi(G) \leq k$.

resp.) is an edge uv for which v belongs to some V_j with $j < i$ ($j > i$, resp.). Note that all vertices in V_1 have no incident upward edges, while all vertices in V_t have no incident downward edges.

We denote by $M_0(\mathcal{V})$ (also denoted M_0 when the context is clear) the set of isolated edges in the subgraph $G[V_1 \cup V_2]$ of G induced by the vertices of $V_1 \cup V_2$. That is, M_0 contains the edges of the connected components of $G[V_1 \cup V_2]$ that consist in one edge only. To lighten the exposition, whenever referring to the vertices of M_0 , we mean the vertices of G incident to the edges in M_0 .

For an edge $uv \in M_0$ with $u \in V_1$ and $v \in V_2$, *swapping* uv consists in modifying the partition \mathcal{V} by removing u from V_1 (v from V_2 , resp.) and adding it to V_2 (V_1 , resp.). In other words, we exchange the parts to which u and v belong. Note that if V_1 and V_2 are independent sets before the swap, then, because $uv \in M_0$, by definition the resulting new V_1 and V_2 remain independent. Also, the set M_0 is unchanged by the swap operation.

We can now give a formal definition for the notion of valid partition.

Definition 2.1 (Valid partition). *For a t -colourable graph G , a partition $\mathcal{V} = (V_1, \dots, V_t)$ of $V(G)$ is a valid partition (of G) if \mathcal{V} verifies the following properties.*

(\mathcal{I}) *Every V_i is an independent set.*

(\mathcal{P}_1) *Every vertex in some V_i with $i \geq 2$ has a neighbour in V_j for every $j < i$.*

(\mathcal{S}) *For every sequence $(e_i)_i$ of edges of $M_0(\mathcal{V})$, successively swapping every e_i (in any order) results in a partition \mathcal{V}' verifying Properties (\mathcal{I}) and (\mathcal{P}_1).*

Note that Property (\mathcal{S}) implies the following property:

(\mathcal{P}_2) *Successively swapping any number of edges of $M_0(\mathcal{V})$ results in a valid partition \mathcal{V}' .*

In order to prove Theorem 1.1, as mentioned earlier, to start constructing ℓ we first need to have a valid partition of G in hand. The following result guarantees its existence.

Lemma 2.2. *Every nice t -colourable graph G admits a valid partition.*

Proof. For a partition $\mathcal{V} = (V_1, \dots, V_t)$ of $V(G)$ where each V_i is an independent set (note that a such partition exists, as every proper t -vertex-colouring of G is one such partition), set $f(\mathcal{V}) = \sum_{k=1}^t k \cdot |V_k|$. Among all possible \mathcal{V} 's, we consider a \mathcal{V} that minimises $f(\mathcal{V})$.

Suppose that there is a vertex $u \in V_i$ with $i \geq 2$ for which Property (\mathcal{P}_1) does not hold, *i.e.*, there is a $j < i$ such that u has no incident upward edge to V_j . By moving u to V_j , we obtain another partition \mathcal{V}' of $V(G)$ where every part is an independent set. However, note that $f(\mathcal{V}') = f(\mathcal{V}) + j - i < f(\mathcal{V})$, a contradiction to the minimality of \mathcal{V} . From this, we deduce that every partition \mathcal{V} minimising f must verify Property (\mathcal{P}_1).

Let now \mathcal{V}' be the partition of $V(G)$ obtained by successively swapping edges of $M_0(\mathcal{V})$. Recall that the swapping operation preserves Property (\mathcal{I}) and observe that $f(\mathcal{V}) = f(\mathcal{V}')$. Hence, \mathcal{V}' minimises f and thus verifies Properties (\mathcal{I}) and (\mathcal{P}_1). Thus Property (\mathcal{S}) also holds, and \mathcal{V} is a valid partition of G . \square

From here, we thus assume that we have a valid partition $\mathcal{V} = (V_1, \dots, V_t)$ of G .

Step 2: Labelling the upward edges of V_3, \dots, V_t

From G and \mathcal{V} , our goal now is to construct a 3-labelling ℓ of G achieving certain properties, the most important of which being that the only possible conflicts are between pairs of vertices of V_1 and V_2 that do not form an edge of M_0 . The following result sums up the exact conditions we want ℓ to fulfil. Recall that a vertex v is special by ℓ , if $d_3(v) = 1$, $d_2(v) \geq 2$ and $d_2(v) + d_3(v)$ is odd. Note that special vertices are bichromatic.

Lemma 2.3. *For every nice graph G and every valid partition (V_1, \dots, V_t) of G , there exists a 3-labelling ℓ such that:*

1. *all vertices of V_1 are either 1-monochromatic or 3-monochromatic,*
2. *all vertices of V_2 are either 1-monochromatic or 2-monochromatic,*
3. *all vertices of $V_3 \cup \dots \cup V_t$ are bichromatic,*
4. *no vertex is special,*
5. *if $u \in V_1$ and $v \in V_2$ are adjacent, then $\ell(uv) = 1$,*
6. *if two vertices u and v are in conflict, then $u \in V_1$ and $v \in V_2$ (or vice versa), and at least one of u or v has a neighbour w in $V_1 \cup V_2$.*

Proof. From now on, we fix the valid partition $\mathcal{V} = (V_1, \dots, V_t)$ of G . During the construction of ℓ , we may have, however, to swap some edges of M_0 , resulting in a different valid partition of G . Abusing the notations, for simplicity we will still denote by \mathcal{V} any valid partition of G obtained this way, through swapping edges. Recall that valid partitions are closed under swapping edges of M_0 (Property (\mathcal{P}_2) of Definition 2.1).

Our goal is to design ℓ so that it not only verifies the four colour properties of Items 1. to 4. of the statement, but also achieves the following refined product types, for every vertex v in a part V_i of \mathcal{V} :

- $v \in V_1$: 1-monochromatic or 3-monochromatic;
- $v \in V_2$: 1-monochromatic or 2-monochromatic;
- $v \in V_3$: bichromatic, 2-degree 1, and even $\{2, 3\}$ -degree;
- $v \in V_4$: bichromatic, 3-degree 2, and odd $\{2, 3\}$ -degree;
- $v \in V_5$: bichromatic, 2-degree 2, and even $\{2, 3\}$ -degree;
- ...
- $v \in V_{2n}$, $n \geq 3$: bichromatic, 3-degree n , and odd $\{2, 3\}$ -degree;
- $v \in V_{2n+1}$, $n \geq 3$: bichromatic, 2-degree n , and even $\{2, 3\}$ -degree;
- ...

We start from ℓ assigning label 1 to all edges of G . Let us now describe how to modify ℓ so that the conditions above are met for all vertices. We consider the vertices of V_t, \dots, V_3 following that order, from “bottom to top”, and modify labels assigned to upward edges. An important condition we will maintain, is that every vertex in an odd part V_{2n+1} ($n \geq 0$) has all its incident downward edges (if any) labelled 3 or 1, while every vertex in an even

part V_{2n} ($n \geq 1$) has all its incident downward edges (if any) labelled 2 or 1. Note that this is trivially verified for the vertices in V_i , since they have no incident downward edges.

At any point in the process, let M be the set of edges of M_0 for which both ends are 1-monochromatic (initially, $M = M_0$). When treating a vertex $u \in V_3 \cup \dots \cup V_t$, we define M_u as the subset of edges of M having an end that is a neighbour of u . For every edge $e \in M_u$, we choose one end of e that is a neighbour of u and we add it to a set S_u . Note that $|S_u| = |M_u|$. Another goal during the labelling process, to fulfil Item 6., is to label the edges incident to u so that at least one end of every edge in M_u is no longer 1-monochromatic. Note that the set M_u considered when labelling the edges incident to u is not necessarily the set of edges of M_0 incident to a neighbour of u , as, during the whole process, some of these edges might be removed from M when dealing with previous vertices in $V_3 \cup \dots \cup V_t$.

Let us now consider the vertices in V_t, \dots, V_3 one by one, following that order. Let thus $u \in V_i$ be a vertex that has not be treated yet, with $i \geq 3$. Recall that every vertex belonging to some V_j with $j > i$ was treated earlier on, and thus has its desired product. Suppose that $i = 2n$ with $n \geq 2$ ($i = 2n + 1$ with $n \geq 1$, resp.). Recall also that u is assumed to have all its incident downward edges labelled 1 or 2 (3, resp.), due to how vertices in V_j 's with $j > i$ have been treated earlier on. Also, all upward edges incident to u are currently assigned labelled 1 by ℓ .

If $M_u \neq \emptyset$, then we swap edges of M_u , if necessary, so that every vertex in S_u belongs to V_2 (V_1 , resp.). This does not invalidate any of our invariants since both ends of an edge in S_u are 1-monochromatic.

In any case, by Property (\mathcal{P}_1), we know that, for every $j < i$, there is a vertex $x_j \in V_j$ which is a neighbour of u . In particular, the vertex x_1 (x_2 , resp.) does not belong to S_u (but may be the other end of an edge in M_u). We label the edges $ux_3, ux_5, \dots, ux_{2n-1}$ with 3 ($ux_4, ux_6, \dots, ux_{2n}$ with 2, resp.). Note that, at this point, $d_3(u) = n - 1$ ($d_2(u) = n - 1$, resp.). To finish dealing with u , we need to distinguish two cases depending on whether M_u is empty or not.

- Suppose first that $M_u = \emptyset$. Label ux_1 with 3 (ux_2 with 2, resp.). Now u has the desired 3-degree (2-degree, resp.). If $i > 3$, then label ux_{i-2} with 2 (3, resp.) so that u is sure to be bichromatic. If $i > 3$ and the $\{2, 3\}$ -degree of u does not have the desired parity, then label ux_2 with 2 (ux_1 with 3, resp.). If $u \in V_3$ and the $\{2, 3\}$ -degree of u is even, then u is already bichromatic since $d_2(u) = 1$. If $u \in V_3$ and the $\{2, 3\}$ -degree of u is odd, then label ux_1 with 3 to adjust the parity of the $\{2, 3\}$ -degree of u and making u bichromatic. In all cases, at this point u is bichromatic with 3-degree n (2-degree n , resp.) and odd $\{2, 3\}$ -degree (even $\{2, 3\}$ -degree, resp.), which is precisely what is desired for u .
- Suppose now that $M_u \neq \emptyset$. Let $z \in S_u$ and let e be the edge of M_u containing z . For every vertex $w \in S_u \setminus \{z\}$, we label the edge uw with 2 (3, resp.). Then:
 - If $d_2(u) + d_3(u)$ is odd (even, resp.), then label uz with 2 (3, resp.) and ux_1 with 3 (ux_2 with 2, resp.). In this case, every edge in M_u is incident to at least one vertex which is not 1-monochromatic, while u is bichromatic with 3-degree n (2-degree n , resp.) and odd $\{2, 3\}$ -degree (even $\{2, 3\}$ -degree, resp.).
 - If $d_2(u) + d_3(u)$ is even (odd, resp.) and $d_2(u) > 0$ ($d_3(u) > 0$, resp.), then swap e and label uz with 3 (2, resp.). Note that, after the swap of e , we have $z \in V_1$ ($z \in V_2$, resp.). In this case, every edge in M_u is incident to at least one vertex

which is not 1-monochromatic, while u is bichromatic with 3-degree n (2-degree n , resp.) and odd $\{2, 3\}$ -degree (even $\{2, 3\}$ -degree, resp.).

- The last case is when $d_2(u) + d_3(u)$ is even (odd, resp.) and $d_2(u) = 0$ ($d_3(u) = 0$, resp.). If $i > 4$, then we can label ux_{i-2} with 2 (3, resp.) and fall back into one of the previous cases. If $i = 4$, then the only edge labelled 3 is the edge ux_3 which implies that $d_3(u) = 1$, which is impossible since $d_2(u) = 0$ ($d_3(u) = 0$, resp.) and $d_2(u) + d_3(u)$ is odd. If $i = 3$, then the conditions of this case imply that $d_2(u) = 1$ while every upward edge incident to u is labelled 1 or 3 and similarly for every incident downward edge; this case thus cannot occur.

To finish, we remove the edges of M_u from M since their two ends are not both 1-monochromatic any more.

At the end of this process, all vertices in V_1 are 1-monochromatic or 3-monochromatic, while all vertices in V_2 are 1-monochromatic or 2-monochromatic. Every vertex in $V_3 \cup \dots \cup V_t$ is bichromatic and there are no conflicts involving any pair of these vertices. Indeed if $a \in V_i$ and $b \in V_j$ are adjacent with $i > j \geq 3$, then either i and j do not have the same parity, in which case a and b do not have the same $\{2, 3\}$ -degree; or both i and j are even (odd, resp.) and $d_3(a) = \frac{i}{2} \neq \frac{j}{2} = d_3(b)$ ($d_2(a) = \frac{i-1}{2} \neq \frac{j-1}{2} = d_2(b)$, resp.). Note also that no vertex in G is special, as, by definition, special vertices have 3-degree 1, 2-degree at least 2, and odd $\{2, 3\}$ -degree. Moreover, we did not change the label of any edge in the cut (V_1, V_2) .

Finally, suppose that there is a conflict between two vertices u and v . Previous remarks imply that $u \in V_1$ and $v \in V_2$ (or *vice versa*) and that both u and v are 1-monochromatic. If none of u and v has another neighbour w in $V_1 \cup V_2$, then the edge uv belongs to the set M_0 . Since G is nice, one of u or v must have a neighbour z in $V_3 \cup \dots \cup V_t$. Hence $uv \in M_z$. Recall also that we relabelled the edges incident to z in such a way that, for every edge of M_z , at least one incident vertex became 2-monochromatic or 3-monochromatic, a contradiction to the existence of u and v . Hence, all properties of the lemma hold. \square

Step 3: Labelling the edges between V_1 and V_2

From now on, we will modify a 3-labelling ℓ of G obtained by applying Lemma 2.3. We denote by \mathcal{H} the set of the connected components of $G[V_1 \cup V_2]$ that contain two adjacent vertices $u \in V_1$ and $v \in V_2$ having the same product by ℓ . By Items 1. and 2. of Lemma 2.3, such u and v are 1-monochromatic. Also, by Item 6. of Lemma 2.3, recall that every connected component of \mathcal{H} has at least two edges. In what follows, we only relabel edges of some connected components $H \in \mathcal{H}$ with making sure that their vertices (in $V_1 \cup V_2$) are monochromatic or special. This ensures that only vertices of H have their product affected, thus that no new conflicts involving vertices in $V_3 \cup \dots \cup V_t$ are created.

For a subgraph X of $H \in \mathcal{H}$ (possibly $X = H$), if, after having relabelled edges of X , no conflict remains between vertices of X and all vertices of X are either monochromatic or special, then we say that X verifies Property (\mathcal{P}_3) .

Lemma 2.4. *If we can relabel the edges of every $H \in \mathcal{H}$ so that every H verifies Property (\mathcal{P}_3) , then the resulting 3-labelling is p -proper.*

Proof. This is because if we get rid of all conflicts in \mathcal{H} , then the only possible remaining conflicts are between vertices in $V_1 \cup V_2$ and in $V_3 \cup \dots \cup V_t$. In particular, recall that any two vertices of two distinct connected components $H_1, H_2 \in \mathcal{H}$ cannot be adjacent. Note also that, because we only relabelled edges in \mathcal{H} , the vertices in $V_3 \cup \dots \cup V_t$ retain the

product types described in Lemma 2.3. In particular, they remain bichromatic and none of them is special. Thus, they cannot be in conflict with the vertices in $V_1 \cup V_2$. \square

In order to show that we can relabel the edges of every $H \in \mathcal{H}$ so that it fulfils Property (\mathcal{P}_3) , the following result will be particularly handy.

Lemma 2.5. *For every integer $s \in \{2, 3\}$, every connected bipartite graph H whose edges are labelled 1 or s , and any vertex v in any part $V_i \in \{V_1, V_2\}$ of H , we can relabel the edges of H with 1 and s so that $d_s(u)$ is odd (even, resp.) for every $u \in V_i \setminus \{v\}$, and $d_s(u)$ is even (odd, resp.) for every $u \in V_{3-i}$.*

Proof. As long as H has a vertex u different from v that does not verify the desired condition, apply the following. Choose P any path from u to v , which exists by the connectedness of H . Now follow P from u to v , and change the labels of the traversed edges from 1 to s and *vice versa*. It can be noted that this alters the parity of the s -degrees of u and v , while this does not alter that parity for any of the other vertices of H . Thus, this makes u satisfy the desired condition, while the situation did not change for the other vertices different from u and v . Thus, once this process ends, all vertices of H different from v have their s -degree being as desired by the resulting labelling. \square

We are now ready to treat the connected components $H \in \mathcal{H}$ independently, so that they all meet Property (\mathcal{P}_3) . To ease the reading, we distinguish several cases depending on the types and on the degrees of the vertices that H includes. In each of the successive cases we consider, it is implicitly assumed that H does not meet the conditions of any previous case.

Claim 2.6. *If $H \in \mathcal{H}$ contains a 3-monochromatic vertex $v \in V_1$, or a 1-monochromatic vertex $v_1 \in V_1$ having two 1-monochromatic neighbours $u_1, u_2 \in V_2$ with degree 1 (in H), then we can relabel edges of H so that H verifies Property (\mathcal{P}_3) .*

Proof. Recall that all edges of H (and thus in \mathcal{H}) are assigned label 1; thus, if a vertex of H is 3-monochromatic, then it must be due to incident downward edges to V_3, \dots, V_t .

If H has a 1-monochromatic vertex $v_1 \in V_1$ having two degree-1 1-monochromatic neighbours $u_1, u_2 \in V_2$, then we set $\ell(v_1 u_1) = \ell(v_1 u_2) = 3$. Note that u_1 and u_2 become 3-monochromatic with 3-degree 1, and are thus no longer in conflict with v_1 , as it becomes 3-monochromatic with 3-degree 2. Note that either we got rid of all conflicts in H and H now verifies Property (\mathcal{P}_3) as desired, or conflicts between other 1-monochromatic vertices of H remain. In the latter case, we continue with the following arguments.

Assume H has remaining conflicts, and that H has a 3-monochromatic vertex $v \in V_1$ (and, due to the previous process, perhaps 3-monochromatic vertices u_1 and u_2 in V_2 , in which case their 3-degree (and degree) is precisely 1, while their unique neighbour v in $V_1 \cap V(H)$ is 3-monochromatic with 3-degree 2). Let X be the set of all 3-monochromatic vertices of H belonging to V_1 . Let C_1, \dots, C_q denote the $q \geq 1$ connected components of $H - X$ that do not consist in a 3-monochromatic vertex of V_2 (the vertices u_1 and u_2 we dealt with earlier on). For every C_i , we choose arbitrarily a vertex $x_i \in X$ and a vertex $y_i \in C_i$ such that x_i and y_i are adjacent in H . Note that the vertices of C_i are either 1-monochromatic or 2-monochromatic (in which case they belong to V_2), since all 3-monochromatic vertices of H are part of X (or are the vertices u_1 and u_2 dealt with earlier on, which we have omitted for now and are not part of the C_i 's).

By Lemma 2.5, in every C_i we can relabel the edges with 1 and 2 so that all vertices in $(V_2 \cap V(C_i)) \setminus \{y_i\}$ are 2-monochromatic with odd 2-degree, while all vertices in $V_1 \cap V(C_i)$ are 2-monochromatic with even 2-degree or possibly 1-monochromatic if their even 2-degree

is 0. In particular, recall that y_i must be 1-monochromatic or 2-monochromatic. If y_i has odd 2-degree, then there are no conflicts between vertices of C_i . If y_i has even non-zero 2-degree, then we set $\ell(x_i y_i) = 3$, thereby making u_i special.

Let Y be the set containing all 1-monochromatic y_i 's having a 1-monochromatic neighbour w_i in C_i . Let H' be the subgraph of H induced by $Y \cup X$. Note that every edge of H' is labelled 1. Let now Q_1, \dots, Q_p denote the connected components of H' and choose $x_k \in X \cap V(Q_k)$ for every $k \in \{1, \dots, p\}$. For every k , we apply Lemma 2.5 with labels 1 and 3 so that all vertices in $V_2 \cap V(Q_k)$ get 3-monochromatic with odd 3-degree, while all vertices in $V_1 \cap V(Q_k) \setminus \{x_k\}$ get 3-monochromatic with even 3-degree or possibly 1-monochromatic if their 3-degree is 0.

If x_k is involved in a conflict with a vertex $y_i \in V_2 \cap V(Q_k)$, then this is because x_k has odd 3-degree. Then:

- If $\ell(x_k y_i) = 3$, then $d_3(y_i) = d_3(x_k) \geq 3$ since $x_k \in X$ (it must thus be incident to at least one other edge labelled 3, either a downward edge to V_3, \dots, V_t or an edge incident to u_1 (and similarly an edge incident to u_2)). We here assign label 1 to the edge $x_k y_i$ and label 3 to the edge $y_i w_i$. This way, x_k gets even 3-degree while the 3-degree of y_i does not change. Note that y_i and w_i are not in conflict since $d_3(w_i) = 1$ and $d_3(y_i) \geq 3$.
- Otherwise, if $\ell(x_k y_i) = 1$, then we assign label 3 to the edge $x_k y_i$ and label 3 to the edge $y_i w_i$. This way, x_k gets even 3-degree while the 3-degree of y_i remains odd and must be at least 3. Again y_i and w_i are not in conflict since $d_3(w_i) = 1$ and $d_3(y_i) \geq 3$.

We claim that we got rid of all conflicts in H . Indeed, consider two adjacent vertices $a \in V_1 \cap V(H)$ and $b \in V_2 \cap V(H)$. Suppose first that a and b belong to some C_i . Note that, with the exception of y_i and maybe of the vertex w_i (if it exists and $y_i \in Y$), every vertex of C_i is 1-monochromatic or 2-monochromatic, the vertices of $V_1 \cap V(C_i)$ having even 2-degree and the vertices of $V_2 \cap V(C_i)$ having odd 2-degree. Thus, no conflict involves two of these vertices. Suppose now that $b = y_i$. If y_i is 2-monochromatic with odd 2-degree, then there is no conflict involving y_i in C_i since all of its neighbours in C_i have even 2-degree. If y_i is special, then it is the only special vertex of C_i , so, here again, it cannot be involved in a conflict. If $y_i \notin Y$ and y_i is 1-monochromatic, then y_i has no other 1-monochromatic neighbour in C_i by definition of Y . If $y_i \in Y$, then y_i is 3-monochromatic with odd 3-degree, the only other possible 3-monochromatic neighbour of y_i in C_i being w_i , but we showed previously that their 3-degrees differ. Thus, in all cases, there cannot be conflicts between vertices of C_i .

We are left with the case where a and b do not belong to the same C_i . In particular, this implies that $a \in X$ and that a is 3-monochromatic. The only possible 3-monochromatic vertices in V_2 are the vertices of Y , which have odd 3-degree, and the 3-monochromatic vertices u_1 and u_2 with 3-degree 1 and degree 1 in H which might have been created at the very beginning of the proof. If $b \in Y$, then, due to the application of Lemma 2.5 above, the only vertex of X which can have odd 3-degree is some x_k , but for this vertex we either ensured that it was involved in no conflict, or we tweaked the labelling so that it got even 3-degree without modifying the labelling properties obtained through Lemma 2.5. If b is u_1 or u_2 , then b has only one neighbour v . Note that the edges vu_1 and vu_2 are still labelled 3 as they are not part of the Q_i 's, and, thus, $d_3(b) = 1$ and $d_3(v) \geq 2$. Hence, there is no conflict between vertices of X and other vertices of H . This implies that H verifies Property (\mathcal{P}_3). \square

Claim 2.7. *If H contains a 1-monochromatic vertex $u \in V_2$ with at least two neighbours in H , then we can relabel edges of H so that H verifies Property (\mathcal{P}_3) .*

Proof. Let v_1, \dots, v_p denote the neighbours of u in H . Due to Lemma 2.3 and because Claim 2.6 does not apply on H , for every vertex v of H we have $d_3(v) = 0$. In particular, none of the v_i 's is 3-monochromatic, implying that they are all 1-monochromatic. Let C_1, \dots, C_q be the $q \geq 1$ connected components of $H - u$. Every C_i contains at least one of the v_i 's. Up to renaming the v_i 's, we can suppose w.l.o.g. that $v_i \in V(C_i)$ if $i \leq q$. The vertices v_i with $i > q$ (if any) can belong to any of the C_i 's.

Let us focus on one component C_i . Let J_1^i, \dots, J_r^i denote the r connected components of $C_i - v_i$. If C_i has order 1, then by convention we set $r = 0$. In every J_j^i , choose a neighbour x_j^i of v_i . By Lemma 2.5, we can relabel edges of J_j^i with 1 and 2 so that every vertex of $V_1 \cap V(J_j^i)$ has even 2-degree, while every vertex of $V_2 \cap V(J_j^i)$, except possibly x_j^i , has odd 2-degree. Let X_i be the set containing all x_j^i 's with even 2-degree. Note that v_i has even 2-degree, being precisely 0 since it is 1-monochromatic; thus the only possible conflicts in C_i involve vertices of X_i as they are the only ones not following the parity rule on their 2-degree (that is, they have even 2-degree).

If $|X_i| = 0$, $|X_i| \geq 2$ or if $X_i = \{w_i\}$ and $d_2(w_i) \geq 1$ for some vertex w_i , then we say that C_i is *nice*. In this case, we can relabel edges of C_i so that C_i verifies Property (\mathcal{P}_3) . If $|X_i| = 0$, then C_i already verifies Property (\mathcal{P}_3) . If $|X_i| \geq 2$, then, for every $z \in X_i$, set $\ell(v_i z) = 3$. If $X_i = \{w_i\}$ and $d_2(w_i) \geq 1$, then set $\ell(v_i w_i) = 3$. In the last two cases, all vertices of X_i either become special while they have no special neighbours; or they become 3-monochromatic with 3-degree 1 in which case v_i is their only 3-monochromatic neighbour and $d_3(v_i) \geq 2$. Moreover, in both cases, $d_3(v_i) \geq 1$ and all the neighbours of v_i which are not in X_i have 3-degree 0. Thus, v_i cannot be in conflict with its neighbours. Because the products of the other vertices of C_i were not altered by these labelling modifications, C_i verifies Property (\mathcal{P}_3) .

If $X_i = \{w_i\}$ and w_i is 1-monochromatic with no such neighbours in $C_i - v_i$, then we say that C_i is *bad*. In such a bad component C_i , the only current conflict is between v_i and w_i . If $X_i = \{w_i\}$ and w_i is 1-monochromatic with at least one 1-monochromatic neighbour y_i in $C_i - v_i$, then we say that C_i is *tricky*. We denote by N_n the number of nice components, by N_b the number of bad components, and by N_t the number of tricky components. Let also N_{an} be the number of neighbours of u with 2-degree 0 belonging to nice components. Note that $N_{an} \geq N_n$ as, in particular, for every nice component C_i , vertex v_i is a neighbour of u with 2-degree 0.

In what follows, we consider several cases. In each case, we implicitly assume that none of the previous cases applies.

- **Case 1.** $N_t > 0$.

Let C_i be a tricky component. For every bad or tricky component C_j with $j \neq i$, set $\ell(v_j w_j) = 2$ and $\ell(u w_j) = 2$. In C_j , every vertex of V_1 now has even 2-degree since $d_2(v_j) = 2$ and every vertex of V_2 has odd 2-degree since $d_2(w_j) = 1$.

At this point, if $d_2(u)$ is even, then set $\ell(v_i w_i) = 2$ and $\ell(u v_i) = 2$. Here, C_i behaves exactly like the other bad or tricky components and thus contains no conflicts.

If $d_2(u)$ is odd, then set $\ell(v_i w_i) = \ell(w_i y_i) = 3$. Recall that all conflicts of C_i involved w_i . Note that w_i is now 3-monochromatic with 3-degree 2 and no such neighbours. The vertices y_i and v_i are now 3-monochromatic with 3-degree 1 and no such neighbours (in particular, they are not adjacent since they both belong to V_1). Hence C_i does not contain any conflict.

In both cases, note that u is 2-monochromatic with odd 2-degree. To summarise, we have reached the following situation. Special vertices (which were only created when dealing with nice components) only belong to V_2 . 3-monochromatic vertices are involved in no conflicts inside their component C_j and have no 3-monochromatic neighbours outside C_j since $d_3(u) = 0$. All the other vertices of H are either 1-monochromatic or 2-monochromatic: in particular, they have even 2-degree if they belong to V_1 , while they have odd 2-degree if they belong to V_2 . Hence, there is no conflict in H , and H thus verifies Property (\mathcal{P}_3) .

From now on, we can thus suppose that none of the C_i 's is tricky.

• **Case 2.** $N_{an} = N_n = 0$.

Note that we must indeed have $N_{an} = N_n$ when $N_n = 0$. In this case, all C_i 's are bad. If $N_b = 1$, *i.e.*, H contains only one (bad) component C_1 , then set $\ell(v_1w_1) = 2$ and $\ell(uv_1) = 2$. Then every vertex of H in V_1 is 1-monochromatic or 2-monochromatic with even 2-degree, while every vertex of H in V_2 is 2-monochromatic with odd 2-degree. Indeed $d_2(w_1) = 1$, $d_2(v_i) = 2$ and $d_2(u) = 1$.

If $N_b > 1$, then, for every (bad) component C_i , set $\ell(uv_i) = 3$. Note that this makes all vertices of H be monochromatic. Every neighbour z of u verifies $d_3(z) \leq 1$ and, because $d_3(u) \geq 2$, vertex u cannot be in conflict with any of its neighbours in H . The vertices v_i with $i \leq q$ are 3-monochromatic with 3-degree 1 and have no such neighbours. The w_i 's are 1-monochromatic and have no 1-monochromatic neighbours since the C_i 's were bad and their v_i 's are no longer 1-monochromatic. The other 1-monochromatic vertices and 2-monochromatic vertices raise no conflicts since, for every such vertex z in $V_j \cap V(H)$ (where $j \in \{1, 2\}$), we have $d_2(z) \equiv j - 1 \pmod{2}$. Hence H verifies Property (\mathcal{P}_3) in both cases.

• **Case 3.** $N_{an} = 1$.

Recall that this implies that $N_n = 1$. Let thus C_1 be the nice component. Suppose first that $H - u$ is connected, *i.e.*, $q = 1$ and C_1 is the only component. As we assumed that $d(u) \geq 2$, vertex u has at least one other neighbour v_2 in C_1 . Since $N_{an} = 1$ and $d_2(v_1) = 0$, vertex v_2 cannot have 2-degree 0, meaning that v_2 has even 2-degree with $d_2(v_2) \geq 2$ (since, in a nice component, every vertex of V_1 has even 2-degree). If v_1 is 3-monochromatic, then there are no conflicts involving u in H . If v_1 is 1-monochromatic, then H contains no 3-monochromatic vertex and no special vertex (due to how we modified labels in nice components). Thus, setting $\ell(uv_2) = 3$ makes u become 3-monochromatic with no such neighbour and v_2 become special with no such neighbour. The 1-monochromatic vertices and 2-monochromatic vertices raise no conflicts, as, for every such vertex z in $V_j \cap V(H)$, $j \in \{1, 2\}$, we have $d_2(z) \equiv j - 1 \pmod{2}$. Hence H verifies Property (\mathcal{P}_3) .

Suppose now that there are at least two connected components in $H - u$. Since $N_t = 0$ and $N_n \leq N_{an} = 1$, all the components different from C_1 must be bad. For every bad component C_j , set $\ell(v_iw_i) = 2$ and $\ell(uv_i) = 2$. In C_j , every vertex of V_1 has even 2-degree since $d_2(v_j) = 2$ and every vertex of V_2 has odd 2-degree since $d_2(w_j) = 1$. Now:

- If $d_2(u)$ is odd, then we claim that we have no conflicts in H . First, we saw earlier that any two vertices of C_1 cannot be in conflict. Next, in every C_i with $i > 1$, every vertex of V_1 has even 2-degree, while every vertex of V_2 has

odd 2-degree; hence any two vertices of C_i cannot be in conflict. Thus, every possible conflict in H must involve u . Note that u is 2-monochromatic with odd 2-degree while every vertex of $V_1 \cap V(H)$ has even 2-degree. Thus u cannot be in conflict with a vertex of H , and H verifies Property (\mathcal{P}_3) .

- If $d_2(u)$ is even, then set $\ell(uv_1) = 3$. Again there is no conflict between two vertices in any C_i with $i > 1$. Since only v_1 changed product in C_1 , if there is a conflict between two vertices of C_1 , then it must involve v_1 . Note that v_1 is 3-monochromatic. If $d_3(v_1) \geq 2$, then it is the only vertex of C_1 with this property. If $d_3(v_1) = 1$, then v_1 was 1-monochromatic before the labelling of uv_1 , in which case v_1 , now, has no 3-monochromatic neighbours in C_1 by construction. In both cases, v_1 cannot be in conflict with any other vertex of C_1 . The last possibility for a conflict involves u . Note that u is special and that every special vertex of H belongs to V_2 (more precisely, to the nice component C_1); thus, once again, u cannot be involved in a conflict. Hence H verifies Property (\mathcal{P}_3) .

Thus, in both cases, H verifies Property (\mathcal{P}_3) .

- **Case 4.** $N_{an} \geq 2$.

Thus, $N_n \geq 1$. Let $A = \{a_1, \dots, a_r\}$ be the subset of neighbours of u with 2-degree 0 that are not a v_i for which $i \leq q$ and C_i is a bad component. In other words, the a_i 's are all neighbours of u with 2-degree 0 that are not a v_i with C_i bad. Note that some of these a_i 's are v_i 's (thus in nice components), in which case, by how the nice components were treated earlier, they can be 3-monochromatic. Note also that A may contain more than N_{an} vertices since it may also contain 1-monochromatic v_i 's with 2-degree 0 and $i > q$ and which belong to a bad component.

For every bad component C_i , set $\ell(uv_i) = 3$. Then, for every $a_i \in A$, we define n_i as the value of $d_3(a_i)$ at the beginning of this case. Recall that we can have $d_3(a_i) > 0$, in case a_i is a v_j for which C_j is nice. Also, by the choice of A , at this point, $\ell(ua_i) = 1$. The goal now, is to relabel some of the ua_i 's with 3 in such a way that u is not in conflict with the vertices of A . To show this can be achieved, we use the Combinatorial Nullstellensatz [1].

Theorem 2.8 (Combinatorial Nullstellensatz). *Let \mathbb{F} be an arbitrary field, and $P = P(X_1, \dots, X_n)$ be a polynomial in $\mathbb{F}[X_1, \dots, X_p]$. Suppose that the coefficient of a monomial $X_1^{k_1} \dots X_p^{k_p}$, where every k_i is a non-negative integer, is non-zero in P and the degree of P equals $\sum_{i=1}^p k_i$. If S_1, \dots, S_p are subsets of \mathbb{F} with $|S_i| > k_i$ for every $i \in \{1, \dots, p\}$, then there are $s_1 \in S_1, \dots, s_p \in S_p$ so that $P(s_1, \dots, s_p) \neq 0$.*

For every $i \in \{1, \dots, r\}$, let X_i be a variable belonging to $S_i = \{0, 1\}$ and representing whether ua_i is assigned label 3 ($X_i = 1$) or not ($X_i = 0$). Let P be the following polynomial:

$$P(X_1, \dots, X_r) = \prod_{i=1}^r \left(\sum_{\substack{j=1 \\ j \neq i}}^r X_j + N_b - n_i \right).$$

Since $r \geq N_{an} \geq 2$, note that P has degree r . Furthermore, the monomial $\prod_{i=1}^r X_i$ has non-zero coefficient (since every X_i has positive coefficient in the description of

P). Hence the Combinatorial Nullstellensatz applies and there is a way to choose values x_1, \dots, x_r in $\{0, 1\}$ for X_1, \dots, X_r so that $P(x_1, \dots, x_r) \neq 0$.

For every $i \in \{1, \dots, r\}$ for which $x_i = 1$, set $\ell(ua_i) = 3$. Note that $d_3(u) = N_b + \sum_{j=1}^r x_j$ and $d_2(u) = 0$. Now, if $d_3(v_i) = d_3(u)$ (in which case this value is 1) for some bad component C_i , then set $\ell(v_i w_i) = 3$. This makes u and v_i have different 3-degrees, and similarly for v_i and w_i .

Suppose that there is a conflict in H between two vertices $x \in V_1$ and $y \in V_2$. For now, suppose that u is not one of these two vertices. If $d_2(x) > 0$, then x and y are 2-monochromatic but we did not modify their 2-degrees in this case, and, hence, $d_2(x)$ is even while $d_2(y)$ odd, a contradiction. If x and y are 1-monochromatic, then y must be some w_i in a bad component C_i . By definition of a bad component, w_i can only be in conflict with v_i , which is now 3-monochromatic, a contradiction. Special vertices only belong to V_2 (in nice components), and, hence, they cannot be involved in conflicts. If x and y are 3-monochromatic, then, again, the only vertex of V_2 which might be 3-monochromatic is some w_i in a bad component C_i . Thus, for $y = w_i$, we have $d_3(w_i) = 1$, and hence x cannot be v_i (since $d_3(v_i) = 2$) and must instead be some other neighbour z of u which belongs to A . In particular, z was 1-monochromatic at the beginning of this case, and, thus, w_i and z were already in conflict before, meaning that C_i was a tricky component and not a bad component, which is impossible.

Hence, every possible conflict must involve u . Vertex u has three types of neighbours: those with non-zero 2-degree, the vertices of A , and the v_i 's with $i \leq q$ such that C_i is a bad component. Since $d_2(u) = 0$, the first group of neighbours cannot be in conflict with u . The third group of neighbours cannot be in conflict with u since we made sure that they do not have the same 3-degree as u through the application of the Combinatorial Nullstellensatz. This leaves us with the second group of neighbours, the vertices of A . Suppose that $a_i \in A$ is in conflict with u . Note that $d_3(a_i) = n_i + x_i$. Since $d_3(a_i) = d_3(u) = N_b + \sum_{j=1}^r x_j$, we have $\sum_{j=1, j \neq i}^r x_j + N_b - n_i = 0$ and thus

$P(x_1, \dots, x_r) = 0$, a contradiction.

Hence there is no conflict in H , and H verifies Property (\mathcal{P}_3) . \square

We are now ready to get rid of the last possible conflicts in \mathcal{H} .

Claim 2.9. *For every remaining H , we can relabel edges so that H verifies Property (\mathcal{P}_3) .*

Proof. Let $v \in V_1$ and $u \in V_2$ be two adjacent 1-monochromatic vertices of H (which must exist as otherwise H would verify Property (\mathcal{P}_3)). Because H has at least two edges (as otherwise it would belong to M , not to \mathcal{H}), at least one of v and u must have another neighbour in H . Since Claim 2.7 does not apply, note that u must have degree 1 in H (since all neighbours of u in H must be 1-monochromatic due to Claim 2.6 not applying). So v is also adjacent to $k \geq 1$ vertices $x_1, \dots, x_k \in V_2$ different from u , which must all be 2-monochromatic (because of incident downward edges to V_3, \dots, V_i ; recall that all edges of H are labelled 1) as otherwise Claim 2.7 would apply.

Set $H' = H - u$. According to Lemma 2.5, we can relabel edges in H' with 1 and 2 so that all vertices in $(V_1 \cap V(H')) \setminus \{v\}$ have odd 2-degree, while all vertices in $V_2 \cap V(H')$ have even 2-degree. Recall that u is 1-monochromatic. Thus, if also v is 2-monochromatic with odd 2-degree, then we are done. Assume thus that v is 2-monochromatic with even 2-degree.

- Assume first that the 2-degree of v is even at least 2. In that case, set $\ell(vu) = 3$. This way, u becomes 3-monochromatic, while v becomes special.
- Assume now v is 1-monochromatic. This implies that $\ell(vx_1) = 1$. Change $\ell(vx_1)$ to 3. This way, x_1 becomes special (recall its 2-degree is even and at least 1, due to incident downward edges), while v becomes 3-monochromatic. Note that u remains 1-monochromatic.

In both cases, it can be checked that H now fulfils Property (\mathcal{P}_3). □

At this point, we dealt with all connected components of \mathcal{H} , and the resulting labelling ℓ of G is p-proper by Lemma 2.4. The whole proof is thus complete.

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