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# A proof of the Multiplicative 1-2-3 Conjecture

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## Abstract

We prove that the product version of the 1-2-3 Conjecture, raised by Skowronek-Kaziów in 2012, is true. Namely, for every connected graph with order at least 3, we prove that we can assign labels 1, 2, 3 to the edges in such a way that no two adjacent vertices are incident to the same product of labels.

*Keywords:* 1-2-3 Conjecture; product version; labels 1, 2, 3.

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## 1. Introduction

Let  $G$  be a graph. A  $k$ -labelling  $\ell : E(G) \rightarrow \{1, \dots, k\}$  is an assignment of labels  $1, \dots, k$  to the edges of  $G$ . From  $\ell$ , we can compute different parameters of interest for all vertices  $v$ , such as the *sum*  $\sigma_\ell(v)$  of incident labels (being formally  $\sigma_\ell(v) = \sum_{u \in N(v)} \ell(uv)$ ), or similarly the *multiset*  $\mu_\ell(v)$  of labels incident to  $v$  or the *product*  $\rho_\ell(v)$  of labels incident to  $v$ . We say that  $\ell$  is *s-proper* if  $\sigma_\ell$  is a proper vertex-colouring of  $G$ , *i.e.*, we have  $\sigma_\ell(u) \neq \sigma_\ell(v)$  for every edge  $uv \in E(G)$ . Similarly, we say that  $\ell$  is *m-proper* and *p-proper*, if  $\mu_\ell$  and  $\rho_\ell$ , respectively, form proper vertex-colourings of  $G$ .

In the context of so-called *distinguishing labellings*, the goal is generally to not only distinguish vertices within some distance according to some parameter computed from labellings (such as the parameters  $\sigma_\ell$ ,  $\mu_\ell$  and  $\rho_\ell$  above, to name a few), but also to construct such  $k$ -labellings with  $k$  being as small as possible. We refer the interested reader to [4], in which hundreds of such labelling techniques are listed.

Regarding s-proper, m-proper and p-proper labellings, which are the main focus in this work, we are thus interested, as mentioned above, in finding such  $k$ -labellings with  $k$  as small as possible, for a given graph  $G$ . In other words, we are interested in the parameters  $\chi_S(G)$ ,  $\chi_M(G)$  and  $\chi_P(G)$  which denote the smallest  $k \geq 1$  such that s-proper, m-proper and p-proper, respectively,  $k$ -labellings exist (if any). Actually, through greedy labelling arguments, it can be observed that the only connected graph  $G$  for which  $\chi_S(G)$ ,  $\chi_M(G)$  or  $\chi_P(G)$  is not defined, is  $K_2$ , the complete graph on 2 vertices. Consequently, these three parameters are generally investigated for so-called *nice graphs*, which are those graphs with no connected component isomorphic to  $K_2$ .

S-proper, m-proper and p-proper labellings form a subfield of distinguishing labellings, which has been attracting attention due to the so-called 1-2-3 Conjecture, raised, in [6], by Karoński, Łuczak and Thomason in 2004:

**1-2-3 Conjecture (sum version).** *If  $G$  is a nice graph, then  $\chi_S(G) \leq 3$ .*

Later on, counterparts of the 1-2-3 Conjecture were raised for m-proper and p-proper labellings. Addario-Berry, Aldred, Dalal and Reed first raised, in 2005, the following in [2]:

**1-2-3 Conjecture (multiset version).** *If  $G$  is a nice graph, then  $\chi_M(G) \leq 3$ .*

while Skowronek-Kaziów then raised, in 2012, the following in [8]:

**1-2-3 Conjecture (product version).** *If  $G$  is a nice graph, then  $\chi_P(G) \leq 3$ .*

It is worth mentioning that all three conjectures above, if true, would be tight, as attested for instance by complete graphs. Note also that the multiset version of the 1-2-3 Conjecture is, out of the three variants above, the easiest one in a sense, as every s-proper or p-proper labelling is also m-proper (thus, proving the sum or product variant of the 1-2-3 Conjecture would prove the multiset variant).

To date, the best result towards the sum version of the 1-2-3 Conjecture, proved by Kalkowski, Karoński and Pfender in [5], is that  $\chi_S(G) \leq 5$  holds for every nice graph  $G$ . Another significant result is due to Przybyło, who recently proved in [7] that even  $\chi_S(G) \leq 4$  holds for every nice regular graph  $G$ . Karoński, Łuczak and Thomason themselves also proved in [6] that  $\chi_S(G) \leq 3$  holds for nice 3-colourable graphs. Regarding the multiset version, for long the best result was the one proved by Addario-Berry, Aldred, Dalal and Reed in [2], stating that  $\chi_M(G) \leq 4$  holds for every nice graph  $G$ . Building on that result, Skowronek-Kaziów later proved in [8] that  $\chi_P(G) \leq 4$  holds for every nice graph  $G$ . She also proved that  $\chi_P(G) \leq 3$  holds for every nice 3-colourable graph  $G$ .

A breakthrough result was recently obtained by Vučković, as he totally proved the multiset version of the 1-2-3 Conjecture in [9]. Due to connections between m-proper and p-proper 3-labellings, we observed in [3] that this result directly implies that  $\chi_P(G) \leq 3$  holds for every nice regular graph  $G$ . Inspired by Vučković's proof scheme, we were also able to prove that  $\chi_P(G) \leq 3$  holds for nice 4-chromatic graphs  $G$ , and to prove related results that are very close to what is stated in the product version of the 1-2-3 Conjecture.

Building on these results, we prove the following throughout the rest of this paper.

**Theorem 1.1.** *The product version of the 1-2-3 Conjecture is true. That is, every nice graph admits p-proper 3-labellings.*

## 2. Proof of Theorem 1.1

Let us start by introducing some terminology and recalling some properties of p-proper labellings, which will be used throughout the proof. Let  $G$  be a graph, and  $\ell$  be a 3-labelling of  $G$ . For a vertex  $v \in V(G)$  and an  $i \in \{1, 2, 3\}$ , we denote by  $d_i(v)$  the  $i$ -degree of  $v$  by  $\ell$ , being the number of edges incident to  $v$  that are assigned label  $i$  by  $\ell$ . Note then that  $\rho_\ell(v) = 2^{d_2(v)}3^{d_3(v)}$ . We say that  $v$  is *1-monochromatic* if  $d_2(v) = d_3(v) = 0$ , while we say that  $v$  is *2-monochromatic* (*3-monochromatic*, resp.) if  $d_2(v) > 0$  and  $d_3(v) = 0$  ( $d_3(v) > 0$  and  $d_2(v) = 0$ , resp.). In case  $v$  has both 2-degree and 3-degree at least 1, we say that  $v$  is *bichromatic*. We also define the  $\{2, 3\}$ -degree of  $v$  as the sum  $d_2(v) + d_3(v)$  of its 2-degree and its 3-degree. Thus, if  $v$  is bichromatic, then its  $\{2, 3\}$ -degree is at least 2.

Because  $\ell$  assigns labels 1, 2, 3, and, in particular, because 2 and 3 are coprime, note that, for every edge  $uv$  of  $G$ , we have  $\rho_\ell(u) \neq \rho_\ell(v)$  as soon as  $u$  and  $v$  have different 2-degrees, 3-degrees, or  $\{2, 3\}$ -degrees. In particular,  $u$  and  $v$  cannot be in conflict, *i.e.*, verify  $\rho_\ell(u) = \rho_\ell(v)$ , if  $u$  and  $v$  are  $i$ -monochromatic and  $j$ -monochromatic, respectively, for  $i \neq j$ , or if  $u$  is monochromatic while  $v$  is bichromatic.

Before going into the proof of Theorem 1.1, let us start by giving an overview of it. Let  $G$  be a nice graph. Our goal is to build a p-proper 3-labelling  $\ell$  of  $G$ . We can clearly assume that  $G$  is connected. An important point also, is that, because the product version

of the 1-2-3 Conjecture was verified *e.g.* for 3-colourable graphs (see [8]), we can assume that  $G$  has chromatic number<sup>1</sup>  $\chi(G)$  at least 4. We set  $t = \chi(G) \geq 4$ .

In what follows, we construct  $\ell$  through three main steps. First, we need to partition the vertices of  $G$  in a way verifying specific cut properties, forming what we call a valid partition of  $V(G)$  (see later Definition 2.1 for a more formal definition). A *valid partition*  $\mathcal{V} = (V_1, \dots, V_t)$  is a partition of  $V(G)$  into  $t$  independent sets  $V_1, \dots, V_t$  fulfilling two main properties, being, roughly put, that 1) every vertex  $v$  in some part  $V_i$  with  $i > 1$  has an incident *upward edge* to every part  $V_j$  with  $j < i$ , and 2) for every connected component of  $G[V_1 \cup V_2]$  having only one edge, we can freely swap its two vertices in  $V_1$  and  $V_2$  while preserving the properties of a valid partition.

Once we have this valid partition  $\mathcal{V}$  in hand, we can then start constructing  $\ell$ . The main part of the labelling process, Step 2 below, consists in starting from all edges of  $G$  being assigned label 1 by  $\ell$ , and then processing the vertices of  $V_3, \dots, V_t$  one after another, possibly changing the labels by  $\ell$  assigned to some of their incident edges, so that certain product types are achieved by  $\rho_\ell$ . These desired product types can be achieved due to the many upward edges that some vertices are incident to (in particular, the deeper a vertex lies in  $\mathcal{V}$ , the more upward edges it is incident to). The product types we achieve for the vertices depend on the part  $V_i$  of  $\mathcal{V}$  they belong to. In particular, the modifications we make on  $\ell$  guarantee that all vertices in  $V_3, \dots, V_t$  are bichromatic, every two vertices in  $V_i$  and  $V_j$  with  $i, j \in \{3, \dots, t\}$  and  $i \neq j$  have different 2-degrees or 3-degrees, all vertices in  $V_2$  are 1-monochromatic or 2-monochromatic, and all vertices in  $V_1$  are 1-monochromatic or 3-monochromatic. By itself, achieving these product types makes  $\ell$  almost p-proper, in the sense that the only possible conflicts are between 1-monochromatic vertices in  $V_1$  and  $V_2$ . An important point also, is that, through these label modifications, we will make sure that all edges of  $G[V_1 \cup V_2]$  remain assigned label 1, and no vertex in  $V_3 \cup \dots \cup V_t$  has 3-degree 1, 2-degree at least 2, and odd  $\{2, 3\}$ -degree; in last Step 3 below, we will use that last fact to remove remaining conflicts by allowing some vertices of  $V_1 \cup V_2$  to become *special*, *i.e.*, make their product realising these exact label conditions.

Step 3 is designed to get rid of the last conflicts between the adjacent 1-monochromatic vertices of  $V_1$  and  $V_2$  without introducing new ones in  $G$ . To that end, we will consider the set  $\mathcal{H}$  of the connected components of  $G[V_1 \cup V_2]$  having conflicting vertices, and, if needed, modify the labels assigned by  $\ell$  to some of their incident edges so that no conflicts remain, and no new conflicts are created in  $G$ . To make sure that no new conflicts are created between vertices in  $V_1 \cup V_2$  and vertices in  $V_3 \cup \dots \cup V_t$ , we will modify labels while making sure that all vertices in  $V_1 \cup V_2$  are monochromatic or special. An important point also, is that the fixing procedures we introduce require the number of edges in a connected component of  $\mathcal{H}$  to be at least 2. Because of that, once Step 2 ends, we must make sure that  $\mathcal{H}$  does not contain a connected component with only one edge incident to two 1-monochromatic vertices. To guarantee this, we will also make sure, during Step 2, to modify labels and the partition  $\mathcal{V}$  slightly so that  $\mathcal{H}$  has no such configuration.

#### *Step 1: Constructing a valid partition*

Let  $\mathcal{V} = (V_1, \dots, V_t)$  be a partition of  $V(G)$  where each  $V_i$  is an independent set. Note that such a partition exists, as, for instance, any proper  $t$ -vertex-colouring of  $G$  forms such a partition of  $V(G)$ . For every vertex  $u \in V_i$ , an incident *upward edge* (*downward edge*,

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<sup>1</sup>Recall that a *proper  $k$ -vertex-colouring* of a graph  $G$  is a partition  $(V_1, \dots, V_k)$  of  $V(G)$  where all  $V_i$ 's are independent. The *chromatic number*  $\chi(G)$  of  $G$  is the smallest  $k \geq 1$  such that proper  $k$ -vertex-colourings of  $G$  exist. We say that  $G$  is  *$k$ -colourable* if  $\chi(G) \leq k$ .

resp.) is an edge  $uv$  for which  $v$  belongs to some  $V_j$  with  $j < i$  ( $j > i$ , resp.). Note that all vertices in  $V_1$  have no incident upward edges, while all vertices in  $V_t$  have no incident downward edges.

We denote by  $M_0(\mathcal{V})$  (also denoted  $M_0$  when the context is clear) the set of isolated edges in the subgraph  $G[V_1 \cup V_2]$  of  $G$  induced by the vertices of  $V_1 \cup V_2$ . That is,  $M_0$  contains the edges of the connected components of  $G[V_1 \cup V_2]$  that consist in one edge only. To lighten the exposition, whenever referring to the vertices of  $M_0$ , we mean the vertices of  $G$  incident to the edges in  $M_0$ .

For an edge  $uv \in M_0$  with  $u \in V_1$  and  $v \in V_2$ , *swapping*  $uv$  consists in modifying the partition  $\mathcal{V}$  by removing  $u$  from  $V_1$  ( $v$  from  $V_2$ , resp.) and adding it to  $V_2$  ( $V_1$ , resp.). In other words, we exchange the parts to which  $u$  and  $v$  belong. Note that if  $V_1$  and  $V_2$  are independent sets before the swap, then, because  $uv \in M_0$ , by definition the resulting new  $V_1$  and  $V_2$  remain independent. Also, the set  $M_0$  is unchanged by the swap operation.

We can now give a formal definition for the notion of valid partition.

**Definition 2.1** (Valid partition). *For a  $t$ -colourable graph  $G$ , a partition  $\mathcal{V} = (V_1, \dots, V_t)$  of  $V(G)$  is a valid partition (of  $G$ ) if  $\mathcal{V}$  verifies the following properties.*

( $\mathcal{I}$ ) *Every  $V_i$  is an independent set.*

( $\mathcal{P}_1$ ) *Every vertex in some  $V_i$  with  $i \geq 2$  has a neighbour in  $V_j$  for every  $j < i$ .*

( $\mathcal{S}$ ) *For every sequence  $(e_i)_i$  of edges of  $M_0(\mathcal{V})$ , successively swapping every  $e_i$  (in any order) results in a partition  $\mathcal{V}'$  verifying Properties ( $\mathcal{I}$ ) and ( $\mathcal{P}_1$ ).*

*Note that Property ( $\mathcal{S}$ ) implies the following property:*

( $\mathcal{P}_2$ ) *Successively swapping any number of edges of  $M_0(\mathcal{V})$  results in a valid partition  $\mathcal{V}'$ .*

In order to prove Theorem 1.1, as mentioned earlier, to start constructing  $\ell$  we first need to have a valid partition of  $G$  in hand. The following result guarantees its existence.

**Lemma 2.2.** *Every nice  $t$ -colourable graph  $G$  admits a valid partition.*

*Proof.* For a partition  $\mathcal{V} = (V_1, \dots, V_t)$  of  $V(G)$  where each  $V_i$  is an independent set (note that a such partition exists, as every proper  $t$ -vertex-colouring of  $G$  is one such partition), set  $f(\mathcal{V}) = \sum_{k=1}^t k \cdot |V_k|$ . Among all possible  $\mathcal{V}$ 's, we consider a  $\mathcal{V}$  that minimises  $f(\mathcal{V})$ .

Suppose that there is a vertex  $u \in V_i$  with  $i \geq 2$  for which Property ( $\mathcal{P}_1$ ) does not hold, *i.e.*, there is a  $j < i$  such that  $u$  has no incident upward edge to  $V_j$ . By moving  $u$  to  $V_j$ , we obtain another partition  $\mathcal{V}'$  of  $V(G)$  where every part is an independent set. However, note that  $f(\mathcal{V}') = f(\mathcal{V}) + j - i < f(\mathcal{V})$ , a contradiction to the minimality of  $\mathcal{V}$ . From this, we deduce that every partition  $\mathcal{V}$  minimising  $f$  must verify Property ( $\mathcal{P}_1$ ).

Let now  $\mathcal{V}'$  be the partition of  $V(G)$  obtained by successively swapping edges of  $M_0(\mathcal{V})$ . Recall that the swapping operation preserves Property ( $\mathcal{I}$ ) and observe that  $f(\mathcal{V}) = f(\mathcal{V}')$ . Hence,  $\mathcal{V}'$  minimises  $f$  and thus verifies Properties ( $\mathcal{I}$ ) and ( $\mathcal{P}_1$ ). Thus Property ( $\mathcal{S}$ ) also holds, and  $\mathcal{V}$  is a valid partition of  $G$ .  $\square$

From here, we thus assume that we have a valid partition  $\mathcal{V} = (V_1, \dots, V_t)$  of  $G$ .

*Step 2: Labelling the upward edges of  $V_3, \dots, V_t$*

From  $G$  and  $\mathcal{V}$ , our goal now is to construct a 3-labelling  $\ell$  of  $G$  achieving certain properties, the most important of which being that the only possible conflicts are between pairs of vertices of  $V_1$  and  $V_2$  that do not form an edge of  $M_0$ . The following result sums up the exact conditions we want  $\ell$  to fulfil. Recall that a vertex  $v$  is special by  $\ell$ , if  $d_3(v) = 1$ ,  $d_2(v) \geq 2$  and  $d_2(v) + d_3(v)$  is odd. Note that special vertices are bichromatic.

**Lemma 2.3.** *For every nice graph  $G$  and every valid partition  $(V_1, \dots, V_t)$  of  $G$ , there exists a 3-labelling  $\ell$  such that:*

1. *all vertices of  $V_1$  are either 1-monochromatic or 3-monochromatic,*
2. *all vertices of  $V_2$  are either 1-monochromatic or 2-monochromatic,*
3. *all vertices of  $V_3 \cup \dots \cup V_t$  are bichromatic,*
4. *no vertex is special,*
5. *if  $u \in V_1$  and  $v \in V_2$  are adjacent, then  $\ell(uv) = 1$ ,*
6. *if two vertices  $u$  and  $v$  are in conflict, then  $u \in V_1$  and  $v \in V_2$  (or vice versa), and at least one of  $u$  or  $v$  has a neighbour  $w$  in  $V_1 \cup V_2$ .*

*Proof.* From now on, we fix the valid partition  $\mathcal{V} = (V_1, \dots, V_t)$  of  $G$ . During the construction of  $\ell$ , we may have, however, to swap some edges of  $M_0$ , resulting in a different valid partition of  $G$ . Abusing the notations, for simplicity we will still denote by  $\mathcal{V}$  any valid partition of  $G$  obtained this way, through swapping edges. Recall that valid partitions are closed under swapping edges of  $M_0$  (Property  $(\mathcal{P}_2)$  of Definition 2.1).

Our goal is to design  $\ell$  so that it not only verifies the four colour properties of Items 1. to 4. of the statement, but also achieves the following refined product types, for every vertex  $v$  in a part  $V_i$  of  $\mathcal{V}$ :

- $v \in V_1$ : 1-monochromatic or 3-monochromatic;
- $v \in V_2$ : 1-monochromatic or 2-monochromatic;
- $v \in V_3$ : bichromatic, 2-degree 1, and even  $\{2, 3\}$ -degree;
- $v \in V_4$ : bichromatic, 3-degree 2, and odd  $\{2, 3\}$ -degree;
- $v \in V_5$ : bichromatic, 2-degree 2, and even  $\{2, 3\}$ -degree;
- ...
- $v \in V_{2n}$ ,  $n \geq 3$ : bichromatic, 3-degree  $n$ , and odd  $\{2, 3\}$ -degree;
- $v \in V_{2n+1}$ ,  $n \geq 3$ : bichromatic, 2-degree  $n$ , and even  $\{2, 3\}$ -degree;
- ...

We start from  $\ell$  assigning label 1 to all edges of  $G$ . Let us now describe how to modify  $\ell$  so that the conditions above are met for all vertices. We consider the vertices of  $V_t, \dots, V_3$  following that order, from “bottom to top”, and modify labels assigned to upward edges. An important condition we will maintain, is that every vertex in an odd part  $V_{2n+1}$  ( $n \geq 0$ ) has all its incident downward edges (if any) labelled 3 or 1, while every vertex in an even

part  $V_{2n}$  ( $n \geq 1$ ) has all its incident downward edges (if any) labelled 2 or 1. Note that this is trivially verified for the vertices in  $V_i$ , since they have no incident downward edges.

At any point in the process, let  $M$  be the set of edges of  $M_0$  for which both ends are 1-monochromatic (initially,  $M = M_0$ ). When treating a vertex  $u \in V_3 \cup \dots \cup V_t$ , we define  $M_u$  as the subset of edges of  $M$  having an end that is a neighbour of  $u$ . For every edge  $e \in M_u$ , we choose one end of  $e$  that is a neighbour of  $u$  and we add it to a set  $S_u$ . Note that  $|S_u| = |M_u|$ . Another goal during the labelling process, to fulfil Item 6., is to label the edges incident to  $u$  so that at least one end of every edge in  $M_u$  is no longer 1-monochromatic. Note that the set  $M_u$  considered when labelling the edges incident to  $u$  is not necessarily the set of edges of  $M_0$  incident to a neighbour of  $u$ , as, during the whole process, some of these edges might be removed from  $M$  when dealing with previous vertices in  $V_3 \cup \dots \cup V_t$ .

Let us now consider the vertices in  $V_t, \dots, V_3$  one by one, following that order. Let thus  $u \in V_i$  be a vertex that has not be treated yet, with  $i \geq 3$ . Recall that every vertex belonging to some  $V_j$  with  $j > i$  was treated earlier on, and thus has its desired product. Suppose that  $i = 2n$  with  $n \geq 2$  ( $i = 2n + 1$  with  $n \geq 1$ , resp.). Recall also that  $u$  is assumed to have all its incident downward edges labelled 1 or 2 (3, resp.), due to how vertices in  $V_j$ 's with  $j > i$  have been treated earlier on. Also, all upward edges incident to  $u$  are currently assigned labelled 1 by  $\ell$ .

If  $M_u \neq \emptyset$ , then we swap edges of  $M_u$ , if necessary, so that every vertex in  $S_u$  belongs to  $V_2$  ( $V_1$ , resp.). This does not invalidate any of our invariants since both ends of an edge in  $S_u$  are 1-monochromatic.

In any case, by Property ( $\mathcal{P}_1$ ), we know that, for every  $j < i$ , there is a vertex  $x_j \in V_j$  which is a neighbour of  $u$ . In particular, the vertex  $x_1$  ( $x_2$ , resp.) does not belong to  $S_u$  (but may be the other end of an edge in  $M_u$ ). We label the edges  $ux_3, ux_5, \dots, ux_{2n-1}$  with 3 ( $ux_4, ux_6, \dots, ux_{2n}$  with 2, resp.). Note that, at this point,  $d_3(u) = n - 1$  ( $d_2(u) = n - 1$ , resp.). To finish dealing with  $u$ , we need to distinguish two cases depending on whether  $M_u$  is empty or not.

- Suppose first that  $M_u = \emptyset$ . Label  $ux_1$  with 3 ( $ux_2$  with 2, resp.). Now  $u$  has the desired 3-degree (2-degree, resp.). If  $i > 3$ , then label  $ux_{i-2}$  with 2 (3, resp.) so that  $u$  is sure to be bichromatic. If  $i > 3$  and the  $\{2, 3\}$ -degree of  $u$  does not have the desired parity, then label  $ux_2$  with 2 ( $ux_1$  with 3, resp.). If  $u \in V_3$  and the  $\{2, 3\}$ -degree of  $u$  is even, then  $u$  is already bichromatic since  $d_2(u) = 1$ . If  $u \in V_3$  and the  $\{2, 3\}$ -degree of  $u$  is odd, then label  $ux_1$  with 3 to adjust the parity of the  $\{2, 3\}$ -degree of  $u$  and making  $u$  bichromatic. In all cases, at this point  $u$  is bichromatic with 3-degree  $n$  (2-degree  $n$ , resp.) and odd  $\{2, 3\}$ -degree (even  $\{2, 3\}$ -degree, resp.), which is precisely what is desired for  $u$ .
- Suppose now that  $M_u \neq \emptyset$ . Let  $z \in S_u$  and let  $e$  be the edge of  $M_u$  containing  $z$ . For every vertex  $w \in S_u \setminus \{z\}$ , we label the edge  $uw$  with 2 (3, resp.). Then:
  - If  $d_2(u) + d_3(u)$  is odd (even, resp.), then label  $uz$  with 2 (3, resp.) and  $ux_1$  with 3 ( $ux_2$  with 2, resp.). In this case, every edge in  $M_u$  is incident to at least one vertex which is not 1-monochromatic, while  $u$  is bichromatic with 3-degree  $n$  (2-degree  $n$ , resp.) and odd  $\{2, 3\}$ -degree (even  $\{2, 3\}$ -degree, resp.).
  - If  $d_2(u) + d_3(u)$  is even (odd, resp.) and  $d_2(u) > 0$  ( $d_3(u) > 0$ , resp.), then swap  $e$  and label  $uz$  with 3 (2, resp.). Note that, after the swap of  $e$ , we have  $z \in V_1$  ( $z \in V_2$ , resp.). In this case, every edge in  $M_u$  is incident to at least one vertex

which is not 1-monochromatic, while  $u$  is bichromatic with 3-degree  $n$  (2-degree  $n$ , resp.) and odd  $\{2, 3\}$ -degree (even  $\{2, 3\}$ -degree, resp.).

- The last case is when  $d_2(u) + d_3(u)$  is even (odd, resp.) and  $d_2(u) = 0$  ( $d_3(u) = 0$ , resp.). If  $i > 4$ , then we can label  $ux_{i-2}$  with 2 (3, resp.) and fall back into one of the previous cases. If  $i = 4$ , then the only edge labelled 3 is the edge  $ux_3$  which implies that  $d_3(u) = 1$ , which is impossible since  $d_2(u) = 0$  ( $d_3(u) = 0$ , resp.) and  $d_2(u) + d_3(u)$  is odd. If  $i = 3$ , then the conditions of this case imply that  $d_2(u) = 1$  while every upward edge incident to  $u$  is labelled 1 or 3 and similarly for every incident downward edge; this case thus cannot occur.

To finish, we remove the edges of  $M_u$  from  $M$  since their two ends are not both 1-monochromatic any more.

At the end of this process, all vertices in  $V_1$  are 1-monochromatic or 3-monochromatic, while all vertices in  $V_2$  are 1-monochromatic or 2-monochromatic. Every vertex in  $V_3 \cup \dots \cup V_t$  is bichromatic and there are no conflicts involving any pair of these vertices. Indeed if  $a \in V_i$  and  $b \in V_j$  are adjacent with  $i > j \geq 3$ , then either  $i$  and  $j$  do not have the same parity, in which case  $a$  and  $b$  do not have the same  $\{2, 3\}$ -degree; or both  $i$  and  $j$  are even (odd, resp.) and  $d_3(a) = \frac{i}{2} \neq \frac{j}{2} = d_3(b)$  ( $d_2(a) = \frac{i-1}{2} \neq \frac{j-1}{2} = d_2(b)$ , resp.). Note also that no vertex in  $G$  is special, as, by definition, special vertices have 3-degree 1, 2-degree at least 2, and odd  $\{2, 3\}$ -degree. Moreover, we did not change the label of any edge in the cut  $(V_1, V_2)$ .

Finally, suppose that there is a conflict between two vertices  $u$  and  $v$ . Previous remarks imply that  $u \in V_1$  and  $v \in V_2$  (or *vice versa*) and that both  $u$  and  $v$  are 1-monochromatic. If none of  $u$  and  $v$  has another neighbour  $w$  in  $V_1 \cup V_2$ , then the edge  $uv$  belongs to the set  $M_0$ . Since  $G$  is nice, one of  $u$  or  $v$  must have a neighbour  $z$  in  $V_3 \cup \dots \cup V_t$ . Hence  $uv \in M_z$ . Recall also that we relabelled the edges incident to  $z$  in such a way that, for every edge of  $M_z$ , at least one incident vertex became 2-monochromatic or 3-monochromatic, a contradiction to the existence of  $u$  and  $v$ . Hence, all properties of the lemma hold.  $\square$

### *Step 3: Labelling the edges between $V_1$ and $V_2$*

From now on, we will modify a 3-labelling  $\ell$  of  $G$  obtained by applying Lemma 2.3. We denote by  $\mathcal{H}$  the set of the connected components of  $G[V_1 \cup V_2]$  that contain two adjacent vertices  $u \in V_1$  and  $v \in V_2$  having the same product by  $\ell$ . By Items 1. and 2. of Lemma 2.3, such  $u$  and  $v$  are 1-monochromatic. Also, by Item 6. of Lemma 2.3, recall that every connected component of  $\mathcal{H}$  has at least two edges. In what follows, we only relabel edges of some connected components  $H \in \mathcal{H}$  with making sure that their vertices (in  $V_1 \cup V_2$ ) are monochromatic or special. This ensures that only vertices of  $H$  have their product affected, thus that no new conflicts involving vertices in  $V_3 \cup \dots \cup V_t$  are created.

For a subgraph  $X$  of  $H \in \mathcal{H}$  (possibly  $X = H$ ), if, after having relabelled edges of  $X$ , no conflict remains between vertices of  $X$  and all vertices of  $X$  are either monochromatic or special, then we say that  $X$  verifies Property  $(\mathcal{P}_3)$ .

**Lemma 2.4.** *If we can relabel the edges of every  $H \in \mathcal{H}$  so that every  $H$  verifies Property  $(\mathcal{P}_3)$ , then the resulting 3-labelling is  $p$ -proper.*

*Proof.* This is because if we get rid of all conflicts in  $\mathcal{H}$ , then the only possible remaining conflicts are between vertices in  $V_1 \cup V_2$  and in  $V_3 \cup \dots \cup V_t$ . In particular, recall that any two vertices of two distinct connected components  $H_1, H_2 \in \mathcal{H}$  cannot be adjacent. Note also that, because we only relabelled edges in  $\mathcal{H}$ , the vertices in  $V_3 \cup \dots \cup V_t$  retain the



product types described in Lemma 2.3. In particular, they remain bichromatic and none of them is special. Thus, they cannot be in conflict with the vertices in  $V_1 \cup V_2$ .  $\square$

In order to show that we can relabel the edges of every  $H \in \mathcal{H}$  so that it fulfils Property  $(\mathcal{P}_3)$ , the following result will be particularly handy.

**Lemma 2.5.** *For every integer  $s \in \{2, 3\}$ , every connected bipartite graph  $H$  whose edges are labelled 1 or  $s$ , and any vertex  $v$  in any part  $V_i \in \{V_1, V_2\}$  of  $H$ , we can relabel the edges of  $H$  with 1 and  $s$  so that  $d_s(u)$  is odd (even, resp.) for every  $u \in V_i \setminus \{v\}$ , and  $d_s(u)$  is even (odd, resp.) for every  $u \in V_{3-i}$ .*

*Proof.* As long as  $H$  has a vertex  $u$  different from  $v$  that does not verify the desired condition, apply the following. Choose  $P$  any path from  $u$  to  $v$ , which exists by the connectedness of  $H$ . Now follow  $P$  from  $u$  to  $v$ , and change the labels of the traversed edges from 1 to  $s$  and *vice versa*. It can be noted that this alters the parity of the  $s$ -degrees of  $u$  and  $v$ , while this does not alter that parity for any of the other vertices of  $H$ . Thus, this makes  $u$  satisfy the desired condition, while the situation did not change for the other vertices different from  $u$  and  $v$ . Thus, once this process ends, all vertices of  $H$  different from  $v$  have their  $s$ -degree being as desired by the resulting labelling.  $\square$

We are now ready to treat the connected components  $H \in \mathcal{H}$  independently, so that they all meet Property  $(\mathcal{P}_3)$ . To ease the reading, we distinguish several cases depending on the types and on the degrees of the vertices that  $H$  includes. In each of the successive cases we consider, it is implicitly assumed that  $H$  does not meet the conditions of any previous case.

**Claim 2.6.** *If  $H \in \mathcal{H}$  contains a 3-monochromatic vertex  $v \in V_1$ , or a 1-monochromatic vertex  $v_1 \in V_1$  having two 1-monochromatic neighbours  $u_1, u_2 \in V_2$  with degree 1 (in  $H$ ), then we can relabel edges of  $H$  so that  $H$  verifies Property  $(\mathcal{P}_3)$ .*

*Proof.* Recall that all edges of  $H$  (and thus in  $\mathcal{H}$ ) are assigned label 1; thus, if a vertex of  $H$  is 3-monochromatic, then it must be due to incident downward edges to  $V_3, \dots, V_t$ .

If  $H$  has a 1-monochromatic vertex  $v_1 \in V_1$  having two degree-1 1-monochromatic neighbours  $u_1, u_2 \in V_2$ , then we set  $\ell(v_1 u_1) = \ell(v_1 u_2) = 3$ . Note that  $u_1$  and  $u_2$  become 3-monochromatic with 3-degree 1, and are thus no longer in conflict with  $v_1$ , as it becomes 3-monochromatic with 3-degree 2. Note that either we got rid of all conflicts in  $H$  and  $H$  now verifies Property  $(\mathcal{P}_3)$  as desired, or conflicts between other 1-monochromatic vertices of  $H$  remain. In the latter case, we continue with the following arguments.

Assume  $H$  has remaining conflicts, and that  $H$  has a 3-monochromatic vertex  $v \in V_1$  (and, due to the previous process, perhaps 3-monochromatic vertices  $u_1$  and  $u_2$  in  $V_2$ , in which case their 3-degree (and degree) is precisely 1, while their unique neighbour  $v$  in  $V_1 \cap V(H)$  is 3-monochromatic with 3-degree 2). Let  $X$  be the set of all 3-monochromatic vertices of  $H$  belonging to  $V_1$ . Let  $C_1, \dots, C_q$  denote the  $q \geq 1$  connected components of  $H - X$  that do not consist in a 3-monochromatic vertex of  $V_2$  (the vertices  $u_1$  and  $u_2$  we dealt with earlier on). For every  $C_i$ , we choose arbitrarily a vertex  $x_i \in X$  and a vertex  $y_i \in C_i$  such that  $x_i$  and  $y_i$  are adjacent in  $H$ . Note that the vertices of  $C_i$  are either 1-monochromatic or 2-monochromatic (in which case they belong to  $V_2$ ), since all 3-monochromatic vertices of  $H$  are part of  $X$  (or are the vertices  $u_1$  and  $u_2$  dealt with earlier on, which we have omitted for now and are not part of the  $C_i$ 's).

By Lemma 2.5, in every  $C_i$  we can relabel the edges with 1 and 2 so that all vertices in  $(V_2 \cap V(C_i)) \setminus \{y_i\}$  are 2-monochromatic with odd 2-degree, while all vertices in  $V_1 \cap V(C_i)$  are 2-monochromatic with even 2-degree or possibly 1-monochromatic if their even 2-degree

is 0. In particular, recall that  $y_i$  must be 1-monochromatic or 2-monochromatic. If  $y_i$  has odd 2-degree, then there are no conflicts between vertices of  $C_i$ . If  $y_i$  has even non-zero 2-degree, then we set  $\ell(x_i y_i) = 3$ , thereby making  $u_i$  special.

Let  $Y$  be the set containing all 1-monochromatic  $y_i$ 's having a 1-monochromatic neighbour  $w_i$  in  $C_i$ . Let  $H'$  be the subgraph of  $H$  induced by  $Y \cup X$ . Note that every edge of  $H'$  is labelled 1. Let now  $Q_1, \dots, Q_p$  denote the connected components of  $H'$  and choose  $x_k \in X \cap V(Q_k)$  for every  $k \in \{1, \dots, p\}$ . For every  $k$ , we apply Lemma 2.5 with labels 1 and 3 so that all vertices in  $V_2 \cap V(Q_k)$  get 3-monochromatic with odd 3-degree, while all vertices in  $V_1 \cap V(Q_k) \setminus \{x_k\}$  get 3-monochromatic with even 3-degree or possibly 1-monochromatic if their 3-degree is 0.

If  $x_k$  is involved in a conflict with a vertex  $y_i \in V_2 \cap V(Q_k)$ , then this is because  $x_k$  has odd 3-degree. Then:

- If  $\ell(x_k y_i) = 3$ , then  $d_3(y_i) = d_3(x_k) \geq 3$  since  $x_k \in X$  (it must thus be incident to at least one other edge labelled 3, either a downward edge to  $V_3, \dots, V_t$  or an edge incident to  $u_1$  (and similarly an edge incident to  $u_2$ )). We here assign label 1 to the edge  $x_k y_i$  and label 3 to the edge  $y_i w_i$ . This way,  $x_k$  gets even 3-degree while the 3-degree of  $y_i$  does not change. Note that  $y_i$  and  $w_i$  are not in conflict since  $d_3(w_i) = 1$  and  $d_3(y_i) \geq 3$ .
- Otherwise, if  $\ell(x_k y_i) = 1$ , then we assign label 3 to the edge  $x_k y_i$  and label 3 to the edge  $y_i w_i$ . This way,  $x_k$  gets even 3-degree while the 3-degree of  $y_i$  remains odd and must be at least 3. Again  $y_i$  and  $w_i$  are not in conflict since  $d_3(w_i) = 1$  and  $d_3(y_i) \geq 3$ .

We claim that we got rid of all conflicts in  $H$ . Indeed, consider two adjacent vertices  $a \in V_1 \cap V(H)$  and  $b \in V_2 \cap V(H)$ . Suppose first that  $a$  and  $b$  belong to some  $C_i$ . Note that, with the exception of  $y_i$  and maybe of the vertex  $w_i$  (if it exists and  $y_i \in Y$ ), every vertex of  $C_i$  is 1-monochromatic or 2-monochromatic, the vertices of  $V_1 \cap V(C_i)$  having even 2-degree and the vertices of  $V_2 \cap V(C_i)$  having odd 2-degree. Thus, no conflict involves two of these vertices. Suppose now that  $b = y_i$ . If  $y_i$  is 2-monochromatic with odd 2-degree, then there is no conflict involving  $y_i$  in  $C_i$  since all of its neighbours in  $C_i$  have even 2-degree. If  $y_i$  is special, then it is the only special vertex of  $C_i$ , so, here again, it cannot be involved in a conflict. If  $y_i \notin Y$  and  $y_i$  is 1-monochromatic, then  $y_i$  has no other 1-monochromatic neighbour in  $C_i$  by definition of  $Y$ . If  $y_i \in Y$ , then  $y_i$  is 3-monochromatic with odd 3-degree, the only other possible 3-monochromatic neighbour of  $y_i$  in  $C_i$  being  $w_i$ , but we showed previously that their 3-degrees differ. Thus, in all cases, there cannot be conflicts between vertices of  $C_i$ .

We are left with the case where  $a$  and  $b$  do not belong to the same  $C_i$ . In particular, this implies that  $a \in X$  and that  $a$  is 3-monochromatic. The only possible 3-monochromatic vertices in  $V_2$  are the vertices of  $Y$ , which have odd 3-degree, and the 3-monochromatic vertices  $u_1$  and  $u_2$  with 3-degree 1 and degree 1 in  $H$  which might have been created at the very beginning of the proof. If  $b \in Y$ , then, due to the application of Lemma 2.5 above, the only vertex of  $X$  which can have odd 3-degree is some  $x_k$ , but for this vertex we either ensured that it was involved in no conflict, or we tweaked the labelling so that it got even 3-degree without modifying the labelling properties obtained through Lemma 2.5. If  $b$  is  $u_1$  or  $u_2$ , then  $b$  has only one neighbour  $v$ . Note that the edges  $vu_1$  and  $vu_2$  are still labelled 3 as they are not part of the  $Q_i$ 's, and, thus,  $d_3(b) = 1$  and  $d_3(v) \geq 2$ . Hence, there is no conflict between vertices of  $X$  and other vertices of  $H$ . This implies that  $H$  verifies Property ( $\mathcal{P}_3$ ).  $\square$

**Claim 2.7.** *If  $H$  contains a 1-monochromatic vertex  $u \in V_2$  with at least two neighbours in  $H$ , then we can relabel edges of  $H$  so that  $H$  verifies Property  $(\mathcal{P}_3)$ .*

*Proof.* Let  $v_1, \dots, v_p$  denote the neighbours of  $u$  in  $H$ . Due to Lemma 2.3 and because Claim 2.6 does not apply on  $H$ , for every vertex  $v$  of  $H$  we have  $d_3(v) = 0$ . In particular, none of the  $v_i$ 's is 3-monochromatic, implying that they are all 1-monochromatic. Let  $C_1, \dots, C_q$  be the  $q \geq 1$  connected components of  $H - u$ . Every  $C_i$  contains at least one of the  $v_i$ 's. Up to renaming the  $v_i$ 's, we can suppose w.l.o.g. that  $v_i \in V(C_i)$  if  $i \leq q$ . The vertices  $v_i$  with  $i > q$  (if any) can belong to any of the  $C_i$ 's.

Let us focus on one component  $C_i$ . Let  $J_1^i, \dots, J_r^i$  denote the  $r$  connected components of  $C_i - v_i$ . If  $C_i$  has order 1, then by convention we set  $r = 0$ . In every  $J_j^i$ , choose a neighbour  $x_j^i$  of  $v_i$ . By Lemma 2.5, we can relabel edges of  $J_j^i$  with 1 and 2 so that every vertex of  $V_1 \cap V(J_j^i)$  has even 2-degree, while every vertex of  $V_2 \cap V(J_j^i)$ , except possibly  $x_j^i$ , has odd 2-degree. Let  $X_i$  be the set containing all  $x_j^i$ 's with even 2-degree. Note that  $v_i$  has even 2-degree, being precisely 0 since it is 1-monochromatic; thus the only possible conflicts in  $C_i$  involve vertices of  $X_i$  as they are the only ones not following the parity rule on their 2-degree (that is, they have even 2-degree).

If  $|X_i| = 0$ ,  $|X_i| \geq 2$  or if  $X_i = \{w_i\}$  and  $d_2(w_i) \geq 1$  for some vertex  $w_i$ , then we say that  $C_i$  is *nice*. In this case, we can relabel edges of  $C_i$  so that  $C_i$  verifies Property  $(\mathcal{P}_3)$ . If  $|X_i| = 0$ , then  $C_i$  already verifies Property  $(\mathcal{P}_3)$ . If  $|X_i| \geq 2$ , then, for every  $z \in X_i$ , set  $\ell(v_i z) = 3$ . If  $X_i = \{w_i\}$  and  $d_2(w_i) \geq 1$ , then set  $\ell(v_i w_i) = 3$ . In the last two cases, all vertices of  $X_i$  either become special while they have no special neighbours; or they become 3-monochromatic with 3-degree 1 in which case  $v_i$  is their only 3-monochromatic neighbour and  $d_3(v_i) \geq 2$ . Moreover, in both cases,  $d_3(v_i) \geq 1$  and all the neighbours of  $v_i$  which are not in  $X_i$  have 3-degree 0. Thus,  $v_i$  cannot be in conflict with its neighbours. Because the products of the other vertices of  $C_i$  were not altered by these labelling modifications,  $C_i$  verifies Property  $(\mathcal{P}_3)$ .

If  $X_i = \{w_i\}$  and  $w_i$  is 1-monochromatic with no such neighbours in  $C_i - v_i$ , then we say that  $C_i$  is *bad*. In such a bad component  $C_i$ , the only current conflict is between  $v_i$  and  $w_i$ . If  $X_i = \{w_i\}$  and  $w_i$  is 1-monochromatic with at least one 1-monochromatic neighbour  $y_i$  in  $C_i - v_i$ , then we say that  $C_i$  is *tricky*. We denote by  $N_n$  the number of nice components, by  $N_b$  the number of bad components, and by  $N_t$  the number of tricky components. Let also  $N_{an}$  be the number of neighbours of  $u$  with 2-degree 0 belonging to nice components. Note that  $N_{an} \geq N_n$  as, in particular, for every nice component  $C_i$ , vertex  $v_i$  is a neighbour of  $u$  with 2-degree 0.

In what follows, we consider several cases. In each case, we implicitly assume that none of the previous cases applies.

- **Case 1.**  $N_t > 0$ .

Let  $C_i$  be a tricky component. For every bad or tricky component  $C_j$  with  $j \neq i$ , set  $\ell(v_j w_j) = 2$  and  $\ell(u w_j) = 2$ . In  $C_j$ , every vertex of  $V_1$  now has even 2-degree since  $d_2(v_j) = 2$  and every vertex of  $V_2$  has odd 2-degree since  $d_2(w_j) = 1$ .

At this point, if  $d_2(u)$  is even, then set  $\ell(v_i w_i) = 2$  and  $\ell(u v_i) = 2$ . Here,  $C_i$  behaves exactly like the other bad or tricky components and thus contains no conflicts.

If  $d_2(u)$  is odd, then set  $\ell(v_i w_i) = \ell(w_i y_i) = 3$ . Recall that all conflicts of  $C_i$  involved  $w_i$ . Note that  $w_i$  is now 3-monochromatic with 3-degree 2 and no such neighbours. The vertices  $y_i$  and  $v_i$  are now 3-monochromatic with 3-degree 1 and no such neighbours (in particular, they are not adjacent since they both belong to  $V_1$ ). Hence  $C_i$  does not contain any conflict.

In both cases, note that  $u$  is 2-monochromatic with odd 2-degree. To summarise, we have reached the following situation. Special vertices (which were only created when dealing with nice components) only belong to  $V_2$ . 3-monochromatic vertices are involved in no conflicts inside their component  $C_j$  and have no 3-monochromatic neighbours outside  $C_j$  since  $d_3(u) = 0$ . All the other vertices of  $H$  are either 1-monochromatic or 2-monochromatic: in particular, they have even 2-degree if they belong to  $V_1$ , while they have odd 2-degree if they belong to  $V_2$ . Hence, there is no conflict in  $H$ , and  $H$  thus verifies Property  $(\mathcal{P}_3)$ .

From now on, we can thus suppose that none of the  $C_i$ 's is tricky.

- **Case 2.**  $N_{an} = N_n = 0$ .

Note that we must indeed have  $N_{an} = N_n$  when  $N_n = 0$ . In this case, all  $C_i$ 's are bad. If  $N_b = 1$ , *i.e.*,  $H$  contains only one (bad) component  $C_1$ , then set  $\ell(v_1 w_1) = 2$  and  $\ell(u v_1) = 2$ . Then every vertex of  $H$  in  $V_1$  is 1-monochromatic or 2-monochromatic with even 2-degree, while every vertex of  $H$  in  $V_2$  is 2-monochromatic with odd 2-degree. Indeed  $d_2(w_1) = 1$ ,  $d_2(v_i) = 2$  and  $d_2(u) = 1$ .

If  $N_b > 1$ , then, for every (bad) component  $C_i$ , set  $\ell(u v_i) = 3$ . Note that this makes all vertices of  $H$  be monochromatic. Every neighbour  $z$  of  $u$  verifies  $d_3(z) \leq 1$  and, because  $d_3(u) \geq 2$ , vertex  $u$  cannot be in conflict with any of its neighbours in  $H$ . The vertices  $v_i$  with  $i \leq q$  are 3-monochromatic with 3-degree 1 and have no such neighbours. The  $w_i$ 's are 1-monochromatic and have no 1-monochromatic neighbours since the  $C_i$ 's were bad and their  $v_i$ 's are no longer 1-monochromatic. The other 1-monochromatic vertices and 2-monochromatic vertices raise no conflicts since, for every such vertex  $z$  in  $V_j \cap V(H)$  (where  $j \in \{1, 2\}$ ), we have  $d_2(z) \equiv j - 1 \pmod{2}$ . Hence  $H$  verifies Property  $(\mathcal{P}_3)$  in both cases.

- **Case 3.**  $N_{an} = 1$ .

Recall that this implies that  $N_n = 1$ . Let thus  $C_1$  be the nice component. Suppose first that  $H - u$  is connected, *i.e.*,  $q = 1$  and  $C_1$  is the only component. As we assumed that  $d(u) \geq 2$ , vertex  $u$  has at least one other neighbour  $v_2$  in  $C_1$ . Since  $N_{an} = 1$  and  $d_2(v_1) = 0$ , vertex  $v_2$  cannot have 2-degree 0, meaning that  $v_2$  has even 2-degree with  $d_2(v_2) \geq 2$  (since, in a nice component, every vertex of  $V_1$  has even 2-degree). If  $v_1$  is 3-monochromatic, then there are no conflicts involving  $u$  in  $H$ . If  $v_1$  is 1-monochromatic, then  $H$  contains no 3-monochromatic vertex and no special vertex (due to how we modified labels in nice components). Thus, setting  $\ell(u v_2) = 3$  makes  $u$  become 3-monochromatic with no such neighbour and  $v_2$  become special with no such neighbour. The 1-monochromatic vertices and 2-monochromatic vertices raise no conflicts, as, for every such vertex  $z$  in  $V_j \cap V(H)$ ,  $j \in \{1, 2\}$ , we have  $d_2(z) \equiv j - 1 \pmod{2}$ . Hence  $H$  verifies Property  $(\mathcal{P}_3)$ .

Suppose now that there are at least two connected components in  $H - u$ . Since  $N_t = 0$  and  $N_n \leq N_{an} = 1$ , all the components different from  $C_1$  must be bad. For every bad component  $C_j$ , set  $\ell(v_i w_i) = 2$  and  $\ell(u v_i) = 2$ . In  $C_j$ , every vertex of  $V_1$  has even 2-degree since  $d_2(v_j) = 2$  and every vertex of  $V_2$  has odd 2-degree since  $d_2(w_j) = 1$ . Now:

- If  $d_2(u)$  is odd, then we claim that we have no conflicts in  $H$ . First, we saw earlier that any two vertices of  $C_1$  cannot be in conflict. Next, in every  $C_i$  with  $i > 1$ , every vertex of  $V_1$  has even 2-degree, while every vertex of  $V_2$  has

odd 2-degree; hence any two vertices of  $C_i$  cannot be in conflict. Thus, every possible conflict in  $H$  must involve  $u$ . Note that  $u$  is 2-monochromatic with odd 2-degree while every vertex of  $V_1 \cap V(H)$  has even 2-degree. Thus  $u$  cannot be in conflict with a vertex of  $H$ , and  $H$  verifies Property  $(\mathcal{P}_3)$ .

- If  $d_2(u)$  is even, then set  $\ell(uv_1) = 3$ . Again there is no conflict between two vertices in any  $C_i$  with  $i > 1$ . Since only  $v_1$  changed product in  $C_1$ , if there is a conflict between two vertices of  $C_1$ , then it must involve  $v_1$ . Note that  $v_1$  is 3-monochromatic. If  $d_3(v_1) \geq 2$ , then it is the only vertex of  $C_1$  with this property. If  $d_3(v_1) = 1$ , then  $v_1$  was 1-monochromatic before the labelling of  $uv_1$ , in which case  $v_1$ , now, has no 3-monochromatic neighbours in  $C_1$  by construction. In both cases,  $v_1$  cannot be in conflict with any other vertex of  $C_1$ . The last possibility for a conflict involves  $u$ . Note that  $u$  is special and that every special vertex of  $H$  belongs to  $V_2$  (more precisely, to the nice component  $C_1$ ); thus, once again,  $u$  cannot be involved in a conflict. Hence  $H$  verifies Property  $(\mathcal{P}_3)$ .

Thus, in both cases,  $H$  verifies Property  $(\mathcal{P}_3)$ .

- **Case 4.**  $N_{an} \geq 2$ .

Thus,  $N_n \geq 1$ . Let  $A = \{a_1, \dots, a_r\}$  be the subset of neighbours of  $u$  with 2-degree 0 that are not a  $v_i$  for which  $i \leq q$  and  $C_i$  is a bad component. In other words, the  $a_i$ 's are all neighbours of  $u$  with 2-degree 0 that are not a  $v_i$  with  $C_i$  bad. Note that some of these  $a_i$ 's are  $v_i$ 's (thus in nice components), in which case, by how the nice components were treated earlier, they can be 3-monochromatic. Note also that  $A$  may contain more than  $N_{an}$  vertices since it may also contain 1-monochromatic  $v_i$ 's with 2-degree 0 and  $i > q$  and which belong to a bad component.

For every bad component  $C_i$ , set  $\ell(uv_i) = 3$ . Then, for every  $a_i \in A$ , we define  $n_i$  as the value of  $d_3(a_i)$  at the beginning of this case. Recall that we can have  $d_3(a_i) > 0$ , in case  $a_i$  is a  $v_j$  for which  $C_j$  is nice. Also, by the choice of  $A$ , at this point,  $\ell(ua_i) = 1$ . The goal now, is to relabel some of the  $ua_i$ 's with 3 in such a way that  $u$  is not in conflict with the vertices of  $A$ . To show this can be achieved, we use the Combinatorial Nullstellensatz [1].

**Theorem 2.8** (Combinatorial Nullstellensatz). *Let  $\mathbb{F}$  be an arbitrary field, and  $P = P(X_1, \dots, X_n)$  be a polynomial in  $\mathbb{F}[X_1, \dots, X_p]$ . Suppose that the coefficient of a monomial  $X_1^{k_1} \dots X_p^{k_p}$ , where every  $k_i$  is a non-negative integer, is non-zero in  $P$  and the degree of  $P$  equals  $\sum_{i=1}^p k_i$ . If  $S_1, \dots, S_p$  are subsets of  $\mathbb{F}$  with  $|S_i| > k_i$  for every  $i \in \{1, \dots, p\}$ , then there are  $s_1 \in S_1, \dots, s_p \in S_p$  so that  $P(s_1, \dots, s_p) \neq 0$ .*

For every  $i \in \{1, \dots, r\}$ , let  $X_i$  be a variable belonging to  $S_i = \{0, 1\}$  and representing whether  $ua_i$  is assigned label 3 ( $X_i = 1$ ) or not ( $X_i = 0$ ). Let  $P$  be the following polynomial:

$$P(X_1, \dots, X_r) = \prod_{i=1}^r \left( \sum_{\substack{j=1 \\ j \neq i}}^r X_j + N_b - n_i \right).$$

Since  $r \geq N_{an} \geq 2$ , note that  $P$  has degree  $r$ . Furthermore, the monomial  $\prod_{i=1}^r X_i$  has non-zero coefficient (since every  $X_i$  has positive coefficient in the description of

$P$ ). Hence the Combinatorial Nullstellensatz applies and there is a way to choose values  $x_1, \dots, x_r$  in  $\{0, 1\}$  for  $X_1, \dots, X_r$  so that  $P(x_1, \dots, x_r) \neq 0$ .

For every  $i \in \{1, \dots, r\}$  for which  $x_i = 1$ , set  $\ell(ua_i) = 3$ . Note that  $d_3(u) = N_b + \sum_{j=1}^r x_j$  and  $d_2(u) = 0$ . Now, if  $d_3(v_i) = d_3(u)$  (in which case this value is 1) for some bad component  $C_i$ , then set  $\ell(v_i w_i) = 3$ . This makes  $u$  and  $v_i$  have different 3-degrees, and similarly for  $v_i$  and  $w_i$ .

Suppose that there is a conflict in  $H$  between two vertices  $x \in V_1$  and  $y \in V_2$ . For now, suppose that  $u$  is not one of these two vertices. If  $d_2(x) > 0$ , then  $x$  and  $y$  are 2-monochromatic but we did not modify their 2-degrees in this case, and, hence,  $d_2(x)$  is even while  $d_2(y)$  odd, a contradiction. If  $x$  and  $y$  are 1-monochromatic, then  $y$  must be some  $w_i$  in a bad component  $C_i$ . By definition of a bad component,  $w_i$  can only be in conflict with  $v_i$ , which is now 3-monochromatic, a contradiction. Special vertices only belong to  $V_2$  (in nice components), and, hence, they cannot be involved in conflicts. If  $x$  and  $y$  are 3-monochromatic, then, again, the only vertex of  $V_2$  which might be 3-monochromatic is some  $w_i$  in a bad component  $C_i$ . Thus, for  $y = w_i$ , we have  $d_3(w_i) = 1$ , and hence  $x$  cannot be  $v_i$  (since  $d_3(v_i) = 2$ ) and must instead be some other neighbour  $z$  of  $u$  which belongs to  $A$ . In particular,  $z$  was 1-monochromatic at the beginning of this case, and, thus,  $w_i$  and  $z$  were already in conflict before, meaning that  $C_i$  was a tricky component and not a bad component, which is impossible.

Hence, every possible conflict must involve  $u$ . Vertex  $u$  has three types of neighbours: those with non-zero 2-degree, the vertices of  $A$ , and the  $v_i$ 's with  $i \leq q$  such that  $C_i$  is a bad component. Since  $d_2(u) = 0$ , the first group of neighbours cannot be in conflict with  $u$ . The third group of neighbours cannot be in conflict with  $u$  since we made sure that they do not have the same 3-degree as  $u$  through the application of the Combinatorial Nullstellensatz. This leaves us with the second group of neighbours, the vertices of  $A$ . Suppose that  $a_i \in A$  is in conflict with  $u$ . Note that  $d_3(a_i) = n_i + x_i$ . Since  $d_3(a_i) = d_3(u) = N_b + \sum_{j=1}^r x_j$ , we have  $\sum_{j=1, j \neq i}^r x_j + N_b - n_i = 0$  and thus

$P(x_1, \dots, x_r) = 0$ , a contradiction.

Hence there is no conflict in  $H$ , and  $H$  verifies Property  $(\mathcal{P}_3)$ .  $\square$

We are now ready to get rid of the last possible conflicts in  $\mathcal{H}$ .

**Claim 2.9.** *For every remaining  $H$ , we can relabel edges so that  $H$  verifies Property  $(\mathcal{P}_3)$ .*

*Proof.* Let  $v \in V_1$  and  $u \in V_2$  be two adjacent 1-monochromatic vertices of  $H$  (which must exist as otherwise  $H$  would verify Property  $(\mathcal{P}_3)$ ). Because  $H$  has at least two edges (as otherwise it would belong to  $M$ , not to  $\mathcal{H}$ ), at least one of  $v$  and  $u$  must have another neighbour in  $H$ . Since Claim 2.7 does not apply, note that  $u$  must have degree 1 in  $H$  (since all neighbours of  $u$  in  $H$  must be 1-monochromatic due to Claim 2.6 not applying). So  $v$  is also adjacent to  $k \geq 1$  vertices  $x_1, \dots, x_k \in V_2$  different from  $u$ , which must all be 2-monochromatic (because of incident downward edges to  $V_3, \dots, V_i$ ; recall that all edges of  $H$  are labelled 1) as otherwise Claim 2.7 would apply.

Set  $H' = H - u$ . According to Lemma 2.5, we can relabel edges in  $H'$  with 1 and 2 so that all vertices in  $(V_1 \cap V(H')) \setminus \{v\}$  have odd 2-degree, while all vertices in  $V_2 \cap V(H')$  have even 2-degree. Recall that  $u$  is 1-monochromatic. Thus, if also  $v$  is 2-monochromatic with odd 2-degree, then we are done. Assume thus that  $v$  is 2-monochromatic with even 2-degree.

- Assume first that the 2-degree of  $v$  is even at least 2. In that case, set  $\ell(vu) = 3$ . This way,  $u$  becomes 3-monochromatic, while  $v$  becomes special.
- Assume now  $v$  is 1-monochromatic. This implies that  $\ell(vx_1) = 1$ . Change  $\ell(vx_1)$  to 3. This way,  $x_1$  becomes special (recall its 2-degree is even and at least 1, due to incident downward edges), while  $v$  becomes 3-monochromatic. Note that  $u$  remains 1-monochromatic.

In both cases, it can be checked that  $H$  now fulfils Property ( $\mathcal{P}_3$ ). □

At this point, we dealt with all connected components of  $\mathcal{H}$ , and the resulting labelling  $\ell$  of  $G$  is p-proper by Lemma 2.4. The whole proof is thus complete.

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