



HAL
open science

Local and implied volatilities with the Mixed-Modified-Fractional-Dupire Model

Eric Djeutcha, Jules Sadefo-Kamdem

► **To cite this version:**

Eric Djeutcha, Jules Sadefo-Kamdem. Local and implied volatilities with the Mixed-Modified-Fractional-Dupire Model. *Chaos, Solitons & Fractals*, 2021, 152, pp.111328. <10.1016/j.chaos.2021.111328>. <hal-03324320>

HAL Id: hal-03324320

<https://hal.science/hal-03324320v1>

Submitted on 16 Oct 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY-NC 4.0 - Attribution - Non-commercial use - International License

Local and implied volatilities with the Mixed-Modified-Fractional-Dupire Model

Eric Djeutcha^a Jules Sadefo Kamdem^b

^a Département de Mathématique, Université de Maroua, Cameroun

^b MRE EA 7491 et Faculté d'Economie (Université de Montpellier)
site Richter av Dugrand, 34960 Montpellier cedex 2, France.

^b Corresponding author : jules.sadefo-kamdem@umontpellier.fr

June 16, 2021

Abstract

In this paper, we use the Mellin transform to obtain the analytical formulas of European option (call or put) values, when the evolution of the underlying asset return is governed by a mixed modified fractional stochastic process. As an extension of the Dupire Model [12], we also introduce the so-called ‘Mixed-Modified-Fractional-Dupire model’, by giving the expression of its local volatility and its sensitivity in relation to the Hurst coefficient H . Finally, in the same vein, we highlight an analytical relationship between local volatility and implied volatility.

Key Words: Hurst Coefficient, Option Pricing, Surface Volatility, Mellin Transform, Local Volatility.

1 Introduction

A European option on an underlying asset is generally priced according to the model introduced by Black and Scholes [3]. Since its publication in the 1970s, the Black-Scholes formula has become the most popular pricing of option method, and its generalized version have provided mathematically correct and interesting results on option pricing. This framework assumes, among other things, that the logarithmic returns of an underlying asset are normally distributed. However, after empirical analyses of time series of assets that may be underlying certain options, many researchers have shown that this assumption is violated in practice. Cont [10] has put forward several ‘stylized facts’ of time series ex-post returns of assets, defined as ‘statistical properties that are common to a wide range of instruments, markets and time periods’. These include properties such as playtokurticity or even leptokurticity of the distribution of returns, clustering of return volatility, short memory, but

especially long memory (depending on the behavior of the autocorrelation function of returns). Mandelbrot and Van Ness [25] was the first paper to define a fractional Brownian motion as a long memory process characterized by a Hurst parameter exponent, noted H , allowing to classify time series according to their dependency structure: short or no memory, long positive memory (persistence) and anti-persistence. Lardic and Mignon[21] propose to forecast some forex exchange price using an ARFIMA model.

While many authors have studied the implications of these stylized facts through a variety of market applications, some authors have proposed option pricing models under the assumption that the log-returns of the underlying asset do not follow a normal probability distribution. More recently, other authors have proposed pricing models likely to take into account other stylized facts such as the short or long memory of the processes that generate log-returns, thanks to an estimate of the Hurst coefficient from an analysis of the auto-correlations of the returns of the underlying asset. Berg and Lyhagen [4], Lo [22], Hsieth [18] and Huang and Yang [19] have shown that the returns of financial time series can be linked to short-term or long-term dependence. Lo and Mackinlay [23], Elton and Gruber [14], Frennberg and Hansson [16], Fama and French [17] as well as Poterba and Summer [29] reported a positive auto-correlation for the short-term memory stock price process and a negative auto-correlation for the long-term memory stock price process. The process of generating a time series of returns on the asset underlying a European option is self-similar, if it is a type of stochastic process presenting the phenomenon of self-similarity. A self-similar phenomenon behaves in the same way when observed at different degrees of magnification or at different scales over one dimension (space or time). Self-similar processes can sometimes be described using long-tailed distributions. This type of self-similar process can be addictive in the long run. In addition to having introduced fractional Brownian processes, a pioneer study of long-memory processes was in the work of Mandelbrot [24] on self-similar and other stationary stochastic processes that exhibit long-range dependence. Of great importance is the problem of statistically estimating auto-similarity and/or the long memory parameter (the Hurst coefficient denoted by H). This parameter H determines the mathematical properties of the model and therefore very helpful to describe the behavior of the underlying asset return. The problem of the statistical estimation of the self-similarity and/or long memory parameter H is of great importance (see Hurst[20] that introduced the relevant rescaled adjusted range or R/S statistic and suggested a graphical methodology in order to estimate H). Recently, Chronopoulou et al. [8] has applied Malliavin calculus to long-memory parameter estimation H for non-Gaussian processes.

The logarithmic price of the stock is often defined as a self-similar process with the Hurst exponent $H \in]0, 1[$. From [35], many facts show that the Hurst exponent H varies in $]0, 1[$, for different stock returns or commodity prices or various exchange rates. These prompt us to consider option pricing with an exponent of Hurst $H \in]0, 1[$. Takahashi [34], Cutland et al. [9] and Ren et al.[30] examined the options pricing problems for $H \in]\frac{1}{2}, 1[$) and the case of $H \in]0, \frac{1}{2}[$, as we have known, it was studied by Wang et al. [35]. In the same way, Aliou et al.[1] have studied the Black-Scholes option pricing equations described by the Caputo generalized fractional derivative, Ndolane et al.[28] studied the Novel approaches for getting the solution of the fractional Black-Scholes equation described by Mittag-Leffler

fractional derivative, Stelios et al.[32] have studied the economic uncertainty, stock market predictability and nonlinear spillover effects and Stelios and Dimitris [33] have studied the direction of change forecasting using a volatility based recurrent neural network. For option pricing using integral transforms in fractional Brownian motion framework, see [2] and many references therein.

In order to add the property of long-term dependence in the financial market, a mixed Brownian fractional model has also been developed by Cheridito [7, 6] and Mishura and Valkeila [27]. However, Cheridito [6] shows that for $H \in]\frac{3}{4}, 1[$ there is no arbitrage and that the arbitrage problem persists for values of $H \in]\frac{1}{2}, \frac{3}{4}[$. To resolve this arbitrage problem, Djoutcha et al.[13] have proposed a modified fractional mixed model. Additionally, we know generally that, volatility is a measure of how the price of a certain financial instrument changes over time in finance, and there are many types of volatilities classified by different standards (historical volatility, constant volatility and implied volatility). Nevertheless, there are still facts that contradict the key assumptions of the Black-Scholes fractional mixed model, in particular the constant volatility assumption. Proof of this contradiction is an implied volatility model, in which at-the-money options tend to have lower implied volatilities than in-or out-of-the-money options. One explanation for this phenomenon is that in reality the volatility of an underlying asset is not really a constant value throughout the life of the derivative. This is why the volatility curve plotted by using the implied volatility values inferred by the modified Black-Scholes mixed fractional model does not appear to be horizontal, but displays a ‘volatility smile’ in the charts. This, however, motivates us to wonder if such a model can be found, which gives a series of volatility values quite close to the volatility values in the volatility smile, (ie. implied volatility); or more precisely what is the difference between the volatility given by this alternative model and the corresponding implied value the volatility deduced by the mixed fractional Black-Scholes model, if any. With this thought in mind, we can also apply this scheme of finding suitable test models among different types of models.

Our contribution goes in this direction, by making it possible to propose a pricing model for European options, with the hypothesis that the evolution of the return on the asset underlying the option is governed by a mixed modified fractional Brownian stochastic process. To do this, we use for the first time in the literature, the Mellin transform in order to obtain the price of a European option under the hypothesis mentioned above. Specifically, this article examines several theoretical and practical issues in pricing conditional claims when the underlying is expected to display a long memory. We go further in our contributions, by analyzing the implicit and local volatilities of the option price, when the evolution of the return of the asset underlying the option is governed by a mixed fractional Brownian stochastic process. Precisely, our approach in this paper uses the local volatility model and the Dupire volatility. The historical aspect of local volatility models has been extensively studied in the literature by Breeden and Litzenberger [5]. Their work shows that there is a close relationship between stock density probability and the price of options. Dupire [12], Derman and Kani [11] have showed that under the risk-neutral probability, there is a unique diffusion constant with this distribution. In the Dupire model [12], the volatility is

constructed in a discrete form using tree methods.

In this paper, we study the option pricing problem of a mixed modified fractional Black-Scholes model with Hurst exponent $H \in]\frac{1}{2}, 1[$ in general; using the same method [35]. As in [31], we obtain the European option pricing closed form formulas using Mellin transform, when the underlying asset return evolution is governed by a mixed modified fractional Brownian. Special attention is given to the extension of the formula of the Dupire's Model, in continuous time. In order to avoid the arbitrage opportunity, the random part of our 'extended Dupire model' will be governed by the mixed modified fractional process. We therefore introduce the 'Mixed-Modified -Fractional-Dupire (MMFD)' model. In this new MMFD model, we adopt a more realistic and flexible assumption that, the underlying asset return evolution is governed by a mixed modified fractional Brownian motion.

The rest of this paper is organized as follows. In section 2, we introduce the MMFD model framework. In section 3, we present the Mixed fractional Dupire model. In section 4, we present the MFDD model and the particular case of the Mixed Fractional Dupire (MFD) model. In section 5, the implied volatility of the MMFD model is studied and conclusions are presented in the last section.

2 The MMFD Model Framework

Definition 1. [12] *The Dupire model is an asset process model and the corresponding Stochastic Differential Equation (SDE) is defined as follows,*

$$dS_t = S_t \sigma dW_t. \quad (1)$$

σ is a deterministic positive constant, $(W_t)_{t \geq 0}$ denotes the standard Wiener process and S_t denotes the asset price at time t . The local volatility given by Dupire(1994)[12] is expressed as a function of the strike K and the maturity T , It is written as follows:

$$\sigma(T, K) = \sqrt{\frac{2 \frac{\partial C}{\partial T}}{K^2 \frac{\partial^2 C}{\partial K^2}}}. \quad (2)$$

Definition 2. [13] *A Mixed Modified Fractional Brownian Motion process whose parameters are a, b, ε and H is a linear combination of Brownian Motion B_t and an independent semimartingale process $B_t^{H, \varepsilon}$, defined on a probability space (Ω, F, \mathbb{P}) by:*

$$M_t^{H, a, b, \varepsilon} = M_t^{H, \varepsilon} = aB_t + bB_t^{H, \varepsilon}, \quad \forall (a, b) \in \mathbb{R}_+^* \times \mathbb{R}_+, \quad t \in [0, T], \quad (3)$$

where $H \in]\frac{1}{2}, 1[$ denotes the Hurst parameter and B_t denotes a standard Brownian motion. The symbol $B_t^{H, \varepsilon}$ is a fractional Brownian motion defined by

$$B_t^{H, \varepsilon} = \int_0^t (t + \varepsilon - s)^{H - \frac{1}{2}} dB_s. \quad (4)$$

From Eq.(3). We define

$$dM_t^{H,\varepsilon} = adB_t + bdB_t^{H,\varepsilon}. \quad (5)$$

We know that

$$dB_t^{H,\varepsilon} = \phi_t^\varepsilon dt + \varepsilon^{H-\frac{1}{2}} dB_t, \quad (6)$$

where

$$\phi_t^\varepsilon = b \int_0^t (H - \frac{1}{2})(t + \varepsilon - s)^{H-\frac{3}{2}} dB_s. \quad (7)$$

By letting $\chi_\varepsilon = a + b\varepsilon^{H-\frac{1}{2}}$, Eq.(5) helps to define the following expression:

$$dM_t^{H,\varepsilon} = \phi_t^\varepsilon dt + \chi_\varepsilon dB_t. \quad (8)$$

In this paper, our aim is to establish an extension of the Dupire model Eq.(1), by substituting in Eq.(1) the Brownian motion also known as the standard Wiener process W_t , by the Mixed Modified Fractional Brownian process Eq.(3). We thus consider the following model known as Mixed-Modified- Fractional-Dupire(MMFD) model given in the following definition:

Definition 3. *The Mixed-Modified-Fractional-Dupire model is an asset process model and the corresponding Stochastic Differential Equation (SDE) is defined as follows,*

$$dS_t^\varepsilon = S_t^\varepsilon \sigma dM_t^{H,\varepsilon}. \quad (9)$$

From Eq.(8), Eq.(9) becomes

$$dS_t^\varepsilon = S_t^\varepsilon \sigma \phi_t^\varepsilon dt + S_t^\varepsilon \sigma \chi_\varepsilon dB_t. \quad (10)$$

By setting

$$\mu(t, S_t^\varepsilon) = S_t^\varepsilon \sigma \phi_t^\varepsilon; \sigma(t, S_t^\varepsilon) = S_t^\varepsilon \sigma \chi_\varepsilon, \quad (11)$$

the equation Eq.(10) can be written as follows:

$$dS_t^\varepsilon = \mu(t, S_t^\varepsilon) dt + \sigma(t, S_t^\varepsilon) dB_t. \quad (12)$$

We then obtain our main result on the local volatility, as follows:

Theorem 1. *The local volatility of the MMFD model Eq.(9) is defined by:*

$$\sigma(T, K, a, b) = \frac{\phi_T^\varepsilon}{K \chi_\varepsilon^2} \left[\frac{\partial C}{\partial K} \right] + \sqrt{\frac{(\phi_T^\varepsilon)^2}{K^2 \chi_\varepsilon^4} \left[\frac{\partial C}{\partial K} \right]^2 + \frac{2}{K^2 \chi_\varepsilon^2} \left[\frac{\partial C}{\partial T} \right]}, \quad (13)$$

where $\chi_\varepsilon = a + b\varepsilon^{H-\frac{1}{2}}$ and $\phi_T^\varepsilon = \int_0^T b(H - \frac{1}{2})(T + \varepsilon - s)^{H-\frac{3}{2}} dB_s$.

Proof of Theo 2.1. *To prove Theorem 1, we apply the approach of Dupire[12] to express the volatility σ as a function of the strike price K and maturity T . An expression of the Fokker-Planck equation for the diffusion S^ε is defined as follows:*

$$\frac{\partial g_T}{\partial t} = -\frac{\partial}{\partial S^\varepsilon} (\mu(t, S^\varepsilon) g_T) + \frac{1}{2} \frac{\partial^2}{\partial (S^\varepsilon)^2} (\sigma^2(t, S^\varepsilon) g_T), \quad (14)$$

where g_T is the neutral risk density of the stock price S^ε . We know that, the call price of the European option at initial time $t = 0$, with the maturity T and the strike K is given by

$$C(T, K) = \int_K^\infty (s - K)g_T(s)ds. \quad (15)$$

The density probability g_T of Eq.(15) have to check Eq.(14) and then, we have:

$$\frac{\partial g_T}{\partial T} = -\frac{\partial}{\partial S^\varepsilon} (\mu(T, S^\varepsilon)g_T) + \frac{1}{2} \frac{\partial^2}{\partial (S^\varepsilon)^2} (\sigma^2(T, S^\varepsilon)g_T). \quad (16)$$

The double derivation of Eq.(15) with respect to the strike price K gives

$$g_T(K) = \frac{\partial^2 C}{\partial K^2}. \quad (17)$$

By replacing Eq.(17) in Eq.(16) and taking $S^\varepsilon = K$, we have

$$\frac{\partial}{\partial T} \left[\frac{\partial^2 C}{\partial K^2} \right] = -\frac{\partial}{\partial K} \left[\mu(T, K) \frac{\partial^2 C}{\partial K^2} \right] + \frac{1}{2} \frac{\partial^2}{\partial K^2} \left[\sigma^2(T, K) \frac{\partial^2 C}{\partial K^2} \right]. \quad (18)$$

By integrating Eq.(18) with respect to K , we obtain

$$\frac{\partial}{\partial T} \left[\frac{\partial C}{\partial K} \right] = -\left[\mu(T, K) \frac{\partial^2 C}{\partial K^2} \right] + \frac{1}{2} \frac{\partial}{\partial K} \left[\sigma^2(T, K) \frac{\partial^2 C}{\partial K^2} \right]. \quad (19)$$

By re-integrating Eq.(19) with respect to K , we obtain

$$\frac{\partial C}{\partial T} = -\left[\mu(T, K) \frac{\partial C}{\partial K} \right] + \frac{1}{2} \left[\sigma^2(T, K) \frac{\partial^2 C}{\partial K^2} \right]. \quad (20)$$

While we set $S^\varepsilon = K$ and $t = T$, Eq.(11) becomes

$$\mu(T, K) = K\sigma\phi_T^\varepsilon; \sigma(T, K) = K\sigma\chi_\varepsilon. \quad (21)$$

Substituting Eq.(21) into Eq.(20), we obtain

$$\frac{\partial C}{\partial T} = -\left[K\sigma\phi_T^\varepsilon \frac{\partial C}{\partial K} \right] + \frac{1}{2} \left[K^2\sigma^2\chi_\varepsilon^2 \frac{\partial^2 C}{\partial K^2} \right]. \quad (22)$$

By considering Eq.(22) as a quadratic equation in $\sigma = \sigma(T, K, a, b)$, the discriminant denoted by Δ of this second order equation gives

$$\Delta = K^2 \left[\left(\phi_T^\varepsilon \frac{\partial C}{\partial K} \right)^2 + 2\chi_\varepsilon^2 \left(\frac{\partial^2 C}{\partial K^2} \frac{\partial C}{\partial T} \right) \right].$$

We assume that $\frac{\partial^2 C}{\partial K^2} \frac{\partial C}{\partial T} > 0$, however this assumption makes the discriminant Δ positive. The roots of Eq.(22) are defined for all $i = 1, 2$. as follows:

$$\sigma_i(T, K, a, b) = \frac{\phi_T^\varepsilon \frac{\partial C}{\partial K} + (-1)^i \sqrt{\left(\phi_T^\varepsilon \frac{\partial C}{\partial K}\right)^2 + 2\chi_\varepsilon^2 \left(\frac{\partial^2 C}{\partial K^2} \frac{\partial C}{\partial T}\right)}}{K\chi_\varepsilon^2 \frac{\partial^2 C}{\partial K^2}}. \quad (23)$$

The local volatility market is always positive, this assumption leads to choosing the value defined for $i = 2$ which is given by

$$\sigma(T, K, a, b) = \frac{\phi_T^\varepsilon}{K\chi_\varepsilon^2} \left[\frac{\partial C}{\partial K} \right] + \sqrt{\frac{(\phi_T^\varepsilon)^2}{K^2\chi_\varepsilon^4} \left[\frac{\partial C}{\partial K} \right]^2 + \frac{2}{K^2\chi_\varepsilon^2} \left[\frac{\partial C}{\partial T} \frac{\partial^2 C}{\partial K^2} \right]}.$$

Remark 1. The particular case of Eq.(13) is obtained for $a = 1$ and $b = 0$, this implies that $\chi_\varepsilon^2 = 1$ and $\phi_T^\varepsilon = 0$. Therefore,

$$\sigma(T, K, 1, 0) = \sqrt{\frac{2\frac{\partial C}{\partial T}}{K^2 \frac{\partial^2 C}{\partial K^2}}}. \quad (24)$$

We observe that $\sigma(T, K, 1, 0)$ is equal to $\sigma(T, K)$ given by Dupire Eq.(2).

3 Mixed fractional Dupire model

We first recall the definition of the mixed fractional Brownian motion.

Definition 4. [36] The mixed fractional Brownian motion with a, b and H parameters is a linear combination of Brownian motion and fractional Brownian motion with parameter H , defined for any $t \in [0, T]$ by:

$$M_t^H = aB_t + bB_t^H, \forall a, b > 0, \quad (25)$$

where B_t^H denotes the fractional Brownian motion with Hurst parameter exponent $H \in (0, 1)$.

In this section, our aim is to establish an extension of the Dupire model Eq.(1), by substituting in Eq.(1) the Brownian motion or the standard Wiener process W_t by the Mixed Fractional Brownian process Eq.(25). We thus consider the model given in the following definition as the Mixed-Fractional-Dupire(MFD) model:

Definition 5. The Mixed-Fractional-Dupire model is an asset process model and the corresponding Stochastic Differential Equation (SDE) is defined as follows,

$$dS_t = \mu(t, S_t) dt + \sigma(t, S_t) dM_t^H. \quad (26)$$

where we have set

$$\mu(t, S_t) = 0; \sigma(t, S_t) = S_t \sigma. \quad (27)$$

Now, we apply the approach of Dupire[12] to express the volatility σ as a function of the strike K and maturity T . An expression of the Fokker-Planck equation for the diffusion S is defined as follows:

$$\frac{\partial g_T}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial (S^\varepsilon)^2} (\tilde{\sigma}^2(t, S) g_T), \quad (28)$$

where g_T is the neutral risk density of the stock price S and

$$\tilde{\sigma}^2(t, S) = (a^2 + 2Hb^2t^{2H-1})\sigma^2(t, S). \quad (29)$$

The call price of the European option at $t = 0$, with the maturity T and the strike K is given by Eq.(15). The density probability g_T of Eq.(15) have to check Eq.(28) and then, we have:

$$\frac{\partial g_T}{\partial T} = \frac{1}{2} \frac{\partial^2}{\partial S^2} (\tilde{\sigma}^2(T, S) g_T) \quad (30)$$

Differentiating Eq.(15) twice with respect to strike K gives the value of $g_T(K)$ as seen in Eq.(17). By putting Eq.(17) in Eq.(30) and taking $S = K$, we have

$$\frac{\partial}{\partial T} \left[\frac{\partial^2 C}{\partial K^2} \right] = \frac{1}{2} \frac{\partial^2}{\partial K^2} \left[\tilde{\sigma}^2(T, K) \frac{\partial^2 C}{\partial K^2} \right]. \quad (31)$$

Integrating Eq.(31) with respect to K , we obtain

$$\frac{\partial}{\partial T} \left[\frac{\partial C}{\partial K} \right] = \frac{1}{2} \frac{\partial}{\partial K} \left[\tilde{\sigma}^2(T, K) \frac{\partial^2 C}{\partial K^2} \right]. \quad (32)$$

By re-integrating Eq.(32) with respect to K , we obtain

$$\frac{\partial C}{\partial T} = \frac{1}{2} \left[\tilde{\sigma}^2(T, K) \frac{\partial^2 C}{\partial K^2} \right]. \quad (33)$$

While setting $S = K$ and $t = T$ in Eq.(29), we get

$$\tilde{\sigma}(T, K) = K \sigma(T, K, a, b, H) (a^2 + 2Hb^2T^{2H-1}). \quad (34)$$

Substituting Eq.(34) into Eq.(33), we obtain

$$\frac{\partial C}{\partial T} = \frac{1}{2} \left[K^2 \sigma^2(T, K, a, b, H) (a^2 + 2Hb^2T^{2H-1}) \frac{\partial^2 C}{\partial K^2} \right]. \quad (35)$$

The local volatility is therefore obtained by

$$\sigma(T, K, a, b, H) = \sqrt{\frac{2}{K^2 (a^2 + 2Hb^2T^{2H-1}) \frac{\frac{\partial C}{\partial T}}{\frac{\partial^2 C}{\partial K^2}}}}. \quad (36)$$

Remark 2. The particular case of Eq.(36) is obtained for $a = 1$ and $b = 0$, this implies that:

$$\sigma(T, K, 1, 0, H) = \sqrt{\frac{2 \frac{\partial C}{\partial T}}{K^2 \frac{\partial^2 C}{\partial K^2}}}. \quad (37)$$

We observe that $\sigma(T, K, 0, 1, H)$ obtained by Eq.(37) is equal to $\sigma(T, K)$ given by Dupire Eq.(2).

3.1 Fractional Dupire model

Definition 6. The Fractional-Dupire model is an asset process model and the corresponding Stochastic Differential Equation (SDE) is defined as follows,

$$dS_t = S_t \sigma dB_t^H, \quad (38)$$

where B_t^H is a fractional brownian motion.

Remark 3. The local volatility of fractional Dupire model is obtained from Eq.(36) by taking $a = 0$ and $b = 1$. Therefore, we have

$$\sigma(T, K, 0, 1) = \sqrt{\frac{2}{2HK^2T^{2H-1}} \frac{\frac{\partial C}{\partial T}}{\frac{\partial^2 C}{\partial K^2}}}. \quad (39)$$

Moreover, for $H = \frac{1}{2}$, we have

$$\sigma(T, K, 0, 1, \frac{1}{2}) = \sqrt{\frac{2}{K^2} \frac{\frac{\partial C}{\partial T}}{\frac{\partial^2 C}{\partial K^2}}} = \sigma(T, K). \quad (40)$$

We deduce that the local volatility of fractional Dupire model obtained from Eq.(39) is also equal to $\sigma(T, K)$ given by Dupire Eq.(2).

4 Implied volatility of the MMFD model

In this section, we set-out to find the value of a European call option under the Mixed-Modified -Fractional-Black-Scholes(MMFBS) model by using the Mellin transform. Our approach in this section uses the model of Fadugba and Ogunrinde [15]. We also recall the properties of the Mellin transform (see [26] for more details). For a function f , and for $x \in (0, \infty)$, the Mellin transform is defined as the complex function

$$\mathcal{M}\{f(x); w\} = \hat{f}(w) = \int_0^\infty f(x)x^{w-1}dx, \quad (41)$$

and the inversion of the Mellin transform \hat{f} is given by

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{f}(w)x^{-w}dw. \quad (42)$$

The scale property of the Mellin transform is defined by

$$\mathcal{M}\{x^\alpha f(x); w\} = \widehat{f}(w + \alpha) \quad (43)$$

and

$$\mathcal{M}\left\{x^k \frac{d^k}{dx^k} f(x); w\right\} = (-1)^k \frac{\Gamma(w + k)}{\Gamma(w)} \widehat{f}(w), \quad (44)$$

where $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$ is the gamma function, and s a complex number with a positive real part. For the given functions f and g , such that their Mellin transforms are well defined, the convolution property of the Mellin transform is given by

$$\mathcal{M}\{(f \star g)(x); w\} = \mathcal{M}\{f(x); w\} \mathcal{M}\{g(x); w\}, \quad (45)$$

with

$$(f \star g)(x) = \int_0^\infty f(y) g\left(\frac{x}{y}\right) \frac{dy}{y}. \quad (46)$$

4.1 Value of European options under the MMFBS model.

Definition 7. *The dynamic diffusion S^ε of the Black-Scholes under the Mixed Modified Fractional Brownian Motion is given by:*

$$dS_t^\varepsilon = S_t^\varepsilon r dt + \sigma S_t^\varepsilon dM_t^{H,\varepsilon}. \quad (47)$$

Let $\widetilde{C}(t, S_t^\varepsilon)$ be a value of the european call option, S_t^ε the stock price at the current time t . We know that $\widetilde{C}(t, S_t^\varepsilon)$ satisfy:

$$\begin{cases} \frac{\partial \widetilde{C}(t, S_t^\varepsilon)}{\partial t} + \frac{1}{2} \xi^2 (S_t^\varepsilon)^2 \frac{\partial^2 \widetilde{C}(t, S_t^\varepsilon)}{\partial (S_t^\varepsilon)^2} + r S_t^\varepsilon \frac{\partial \widetilde{C}(t, S_t^\varepsilon)}{\partial S_t^\varepsilon} - r \widetilde{C}(t, S_t^\varepsilon) = 0, \\ C(T, S^\varepsilon) = \max(S^\varepsilon - K, 0), S^\varepsilon \in]0, +\infty[\end{cases} \quad (48)$$

where $\xi = \left(a + b\varepsilon^{H-\frac{1}{2}}\right) \sigma$.

The value of the European call option price is given by the following theorem:

Theorem 2. *Let $\widetilde{C}(t, S_t^\varepsilon)$ be a value of the European call option satisfying Eq.(48), S_t^ε be the price of the underlying asset, K be the strike price, r be the risk interest rate and T be the time to maturity. Applying the convolution property of the Mellin transform Eq.(46), the Mixed Modified Fractional Black-Scholes formula for the valuation of European power call option is defined by*

$$\widetilde{C}(t, S_t^\varepsilon) = S_t^\varepsilon \Phi(d_2^H(a, b)) - K e^{-r(T-t)} \Phi(d_1^H(a, b)) \quad (49)$$

where the parameters $d_1^H(a, b)$ and $d_2^H(a, b)$ are given by

$$d_1^H(a, b) = \frac{\ln\left(\frac{S_t^\varepsilon}{K}\right) + r(T-t) + \frac{\sigma^2(a+b\varepsilon^{H-\frac{1}{2}})^2}{2}(T-t)}{\sigma(a+b\varepsilon^{H-\frac{1}{2}})\sqrt{T-t}} \quad (50)$$

$$d_2^H(a, b) = \frac{\ln\left(\frac{S_t^\varepsilon}{K}\right) + r(T-t) - \frac{\sigma^2(a+b\varepsilon^{H-\frac{1}{2}})^2}{2}(T-t)}{\sigma(a+b\varepsilon^{H-\frac{1}{2}})\sqrt{T-t}} \quad (51)$$

with Φ denoting the normal distribution.

Proof of Theo 4.1. Now, we use the Mellin transform to solve equation (48). We have the European call option of the form

$$\frac{\partial \tilde{C}(t, S_t^\varepsilon)}{\partial t} + \frac{1}{2} \xi^2 (S_t^\varepsilon)^2 \frac{\partial^2 \tilde{C}(t, S_t^\varepsilon)}{\partial (S_t^\varepsilon)^2} + r S_t^\varepsilon \frac{\partial \tilde{C}(t, S_t^\varepsilon)}{\partial S_t^\varepsilon} - r \tilde{C}(t, S_t^\varepsilon) = 0, \quad (52)$$

with the boundary conditions

$$\tilde{C}(t, S^\varepsilon) = h(S^\varepsilon) = \max(S^\varepsilon - K, 0), \quad \text{on } [0, \infty[. \quad (53)$$

Let $\mathcal{M} \left\{ \tilde{C}(t, S^\varepsilon; w) \right\} = \hat{C}(t, w)$ be the Mellin transform of $\tilde{C}(t, S^\varepsilon)$. Applying the Mellin Transform Eq.(41), using Eq.(44), linearity, and independence of the time derivative yields,

$$\frac{\partial \hat{C}(t, w)}{\partial t} + \frac{1}{2} \xi^2 G(w) \hat{C}(t, w) = 0, \quad (54)$$

where

$$G(w) = (w + \alpha_2)^2 - \alpha_2^2 - z_1, \quad (55)$$

with

$$\begin{cases} \xi^2 = \sigma^2 (a + b \varepsilon^{H-\frac{1}{2}})^2 \\ z_1 = \frac{2r}{\xi^2} \\ \alpha_2 = \frac{1-z_1}{2} \end{cases} \quad (56)$$

with the final condition $\hat{C}(T, w) = \hat{h}(S^\varepsilon)$ given by Eq.(53). This first order Eq.(54) has a solution of the form

$$\hat{C}(t, w) = \hat{h}(w) \exp \left\{ \frac{1}{2} \xi^2 G(w) (T - t) \right\}. \quad (57)$$

Applying the inverse Mellin transform Eq.(42) yields,

$$\tilde{C}(t, S^\varepsilon) = h(S^\varepsilon) \star \mathcal{M}^{-1} \left\{ \exp \left[\frac{1}{2} \xi^2 (T - t) ((w + \alpha_2)^2 - \alpha_2^2 - z_1) \right]; w \right\} \quad (58)$$

$$= h(S^\varepsilon) e^{\alpha_1 (T-t)} \star \mathcal{M}^{-1} \left\{ \exp \left[\frac{1}{2} \xi^2 (T - t) (w + \alpha_2)^2 \right]; w \right\} \quad (59)$$

$$= h(S^\varepsilon) e^{\alpha_1 (T-t)} \star R(t, S^\varepsilon), \quad (60)$$

where $\alpha_1 = -\frac{1}{2} \xi^2 (\alpha_2^2 + z_1)$. Substituting $w + \alpha_2 = x$ and using the scale property of the Mellin transform Eq.(43), we have

$$R(t, S^\varepsilon) = (S^\varepsilon)^{\alpha_2} \mathcal{M}^{-1} \left\{ \exp \left[\frac{1}{2} \xi^2 (T - t) (w + \alpha_2)^2 \right]; x \right\} \quad (61)$$

$$= \frac{(S^\varepsilon)^{\alpha_2}}{\sqrt{2\pi \xi^2 (T - t)}} \exp \left[-\frac{1}{2} \frac{1}{\xi^2 (T - t)} (\ln(S^\varepsilon))^2 \right]. \quad (62)$$

From Eq.(60), Eq.(62), and Eq.(46), we obtain:

$$\begin{aligned}
\tilde{C}(t, S^\varepsilon) &= h(S^\varepsilon) \star \mathcal{M}^{-1} \left\{ \exp \left[\frac{1}{2} \chi^2 (T-t) \left((w + \alpha_2)^2 - \alpha_2^2 - z_1 \right) \right]; w \right\} \\
&= \frac{e^{\alpha_1(T-t)}}{\xi \sqrt{2\pi(T-t)}} \int_K^\infty \frac{(u-K)}{u} \left(\frac{S^\varepsilon}{u} \right)^{\alpha_2} \exp \left[-\frac{1}{2} \left(\frac{\ln\left(\frac{S^\varepsilon}{u}\right)}{\xi \sqrt{T-t}} \right)^2 \right] du \\
&= \frac{e^{\alpha_1(T-t)}}{\xi \sqrt{2\pi(T-t)}} \int_K^\infty \left[\frac{(S^\varepsilon)^{\alpha_2}}{u^{\alpha_2}} - K \frac{(S^\varepsilon)^{\alpha_2}}{u^{\alpha_2+1}} \right] \exp \left[-\frac{1}{2} \left(\frac{\ln\left(\frac{S^\varepsilon}{u}\right)}{\xi \sqrt{T-t}} \right)^2 \right] du \\
&= \frac{e^{\alpha_1(T-t)} (S^\varepsilon)^{\alpha_2}}{\xi \sqrt{2\pi(T-t)}} J_1 - K \frac{e^{\alpha_1(T-t)} (S^\varepsilon)^{\alpha_2}}{\chi \sqrt{2\pi(T-t)}} J_2
\end{aligned} \tag{63}$$

where

$$J_1 = \int_K^\infty \frac{1}{u^{\alpha_2}} \exp \left[-\frac{1}{2} \left(\frac{\ln\left(\frac{S^\varepsilon}{u}\right)}{\xi \sqrt{T-t}} \right)^2 \right] du \tag{64}$$

$$J_2 = \int_K^\infty \frac{1}{u^{\alpha_2+1}} \exp \left[-\frac{1}{2} \left(\frac{\ln\left(\frac{S^\varepsilon}{u}\right)}{\xi \sqrt{T-t}} \right)^2 \right] du. \tag{65}$$

For J_1 , we set

$$y(u) = \frac{1}{\xi \sqrt{(T-t)}} \left[\ln \left(\frac{S_t^\varepsilon}{u} \right) - (\alpha_2 - 1) \xi^2 (T-t) \right] \tag{66}$$

for J_2 , we set

$$z(u) = \frac{1}{\xi \sqrt{(T-t)}} \left[\ln \left(\frac{S_t^\varepsilon}{u} \right) - (\alpha_2) \xi^2 (T-t) \right] \tag{67}$$

Then

$$\begin{aligned}
J_1 &= e^{-\alpha_1(T-t)} \frac{\xi \sqrt{2\pi(T-t)}}{(S^\varepsilon)^{\alpha_2-1}} \int_{-\infty}^{d_2^H(a,b)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= \frac{\Phi(d_2^H(a,b))}{(S^\varepsilon)^{\alpha_2-1} \frac{e^{\alpha_1(T-t)}}{\xi \sqrt{2\pi(T-t)}}}
\end{aligned} \tag{68}$$

and

$$\begin{aligned}
J_2 &= e^{-\alpha_1(T-t)} \frac{\xi \sqrt{2\pi(T-t)}}{(S^\varepsilon)^{\alpha_2}} \int_{-\infty}^{d_1^H(a,b)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= e^{-r(T-t)} \frac{\Phi(d_1^H(a,b))}{(S^\varepsilon)^{\alpha_2} \frac{e^{\alpha_1(T-t)}}{\xi \sqrt{2\pi(T-t)}}}
\end{aligned} \tag{69}$$

where

$$d_1^H(a, b) = \frac{\ln(\frac{S_t^\varepsilon}{K}) + r(T-t) + \frac{\xi^2}{2}(T-t)}{\xi\sqrt{T-t}} \quad (70)$$

$$d_2^H(a, b) = \frac{\ln(\frac{S_t^\varepsilon}{K}) + r(T-t) - \frac{\xi^2}{2}(T-t)}{\xi\sqrt{T-t}}. \quad (71)$$

Substituting Eq.(68) and Eq.(69) into Eq.(63), we obtain

$$\tilde{C}(t, S^\varepsilon) = S^\varepsilon \Phi(d_2^H(a, b)) - Ke^{-r(T-t)} \Phi(d_1^H(a, b)). \quad (72)$$

From eq.(72), we deduce the value of European call option as

$$\tilde{C}(S^\varepsilon, 0) = S^\varepsilon \Phi(d_1^H(a, b)) - Ke^{-rT} \Phi(d_2^H(a, b)), \quad (73)$$

where for $t = 0$, the parameters $d_1^H(a, b)$ and $d_2^H(a, b)$ are given by

$$d_1^H(a, b) = \frac{\ln(\frac{S^\varepsilon}{K}) + rT - \frac{\sigma^2(a+b\varepsilon^{H-\frac{1}{2}})^2 T}{2}}{\sigma(a+b\varepsilon^{H-\frac{1}{2}})\sqrt{T}} \quad (74)$$

$$d_2^H(a, b) = \frac{\ln(\frac{S^\varepsilon}{K}) + rT + \frac{\sigma^2(a+b\varepsilon^{H-\frac{1}{2}})^2 T}{2}}{\sigma(a+b\varepsilon^{H-\frac{1}{2}})\sqrt{T}}. \quad (75)$$

Let $\tilde{P}(t, S_t^\varepsilon)$ be a value of the european put option, S_t^ε the stock price at the current time t . We know that $\tilde{P}(t, S_t^\varepsilon)$ satisfy:

$$\begin{cases} \frac{\partial \tilde{P}(t, S_t^\varepsilon)}{\partial t} + \frac{1}{2}\xi^2(S_t^\varepsilon)^2 \frac{\partial^2 \tilde{P}(t, S_t^\varepsilon)}{\partial (S_t^\varepsilon)^2} + rS_t^\varepsilon \frac{\partial \tilde{P}(t, S_t^\varepsilon)}{\partial S_t^\varepsilon} - r\tilde{P}(t, S_t^\varepsilon) = 0, \\ \tilde{P}(T, S^\varepsilon) = \max(K - S^\varepsilon, 0), S^\varepsilon \in]0, +\infty[\end{cases} \quad (76)$$

where $\xi = (a + b\varepsilon^{H-\frac{1}{2}})\sigma$.

By using the put-call parity formula, the value of the European put option price is given by the following result:

Theorem 3. Let $\tilde{P}(t, S_t^\varepsilon)$ be a value of the European put option satisfying Eq.(76), S_t^ε be the price of the underlying asset at time t , K be the option strike price, r be the risk-free interest rate and T the time to maturity date. Using the parity put-call relation, the Mixed Modified Fractional Black-Scholes formula for the valuation of European put option is defined by

$$\tilde{P}(t, S_t^\varepsilon) = -S_t^\varepsilon \Phi(-d_1^H(a, b)) + Ke^{-r(T-t)} \Phi(-d_2^H(a, b)) \quad (77)$$

where the parameters $d_1^H(a, b)$ and $d_2^H(a, b)$ are respectively given by equations (70) and (71).

The Hurst parameter H plays an important role in the Mixed Modified Fractional Black Scholes Model Eq.(47). The sensitivity of our MMFD model in relation with the Hurst parameter coefficient H , can be obtained via the following result:

Theorem 4. Let $\tilde{C}(t, S_t^\varepsilon)$ be the value of European call option under the Mixed Modified Fractional Black-Scholes Model Eq.(47). Then, $\tilde{C}(t, S_t^\varepsilon)$ satisfies Eq.(48) of Theorem 2. The impact of the Hurst parameter H is given by

$$\frac{\partial \tilde{C}}{\partial H} = S^\varepsilon \sigma b \ln(\varepsilon) \varepsilon^{H-\frac{1}{2}} \Phi'(d_1^H(a, b)).$$

Proof of Theo 4.2. Let z be one of the influence factors. Thus we have

$$\begin{aligned} \frac{\partial \tilde{C}}{\partial H} &= \frac{\partial}{\partial z}(S^\varepsilon) \Phi(d_1^H(a, b)) + S^\varepsilon \frac{\partial}{\partial z}(\Phi(d_1^H(a, b))) \\ &\quad - \frac{\partial}{\partial z}(K e^{-r(T-t)}) \Phi(d_2^H(a, b)) - K e^{-r(T-t)} \frac{\partial}{\partial z}(\Phi(d_2^H(a, b))), \end{aligned}$$

Moreover,

$$\frac{\partial \tilde{C}(\Phi(d_2^H(a, b)))}{\partial H} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(d_1^H(a, b))^2}{2}} \frac{S^\varepsilon}{K} e^{-r(T-t)} \frac{\partial}{\partial z}(d_2^H(a, b)).$$

Therefore, we have

$$\begin{aligned} \frac{\partial \tilde{C}}{\partial H} &= \frac{\partial}{\partial z}(S^\varepsilon) \Phi(d_1^H(a, b)) + S^\varepsilon \frac{\partial}{\partial z}(\Phi(d_1^H(a, b))) \\ &\quad - \frac{\partial}{\partial z}(K e^{-r(T-t)}) \Phi(d_2^H(a, b)) + S^\varepsilon \Phi'(d_1^H(a, b)) \frac{\partial}{\partial z}(\sigma(a + b\varepsilon^{H-\frac{1}{2}})\sqrt{T-t}). \end{aligned}$$

We deduce that for $Z = H$, an obtain

$$\begin{aligned} \frac{\partial \tilde{C}}{\partial H} &= S^\varepsilon \Phi'(d_1^H(a, b)) \frac{\partial}{\partial H}(\sigma(a + b\varepsilon^{H-\frac{1}{2}})\sqrt{T-t}) \\ &= S^\varepsilon \Phi'(d_1^H(a, b)) \sigma b \varepsilon^{H-\frac{1}{2}} \ln(\varepsilon) \sqrt{T-t}. \end{aligned}$$

The following figures 1,2 and 3 show the influence of the Hurst parameter H on the volatility surface by using the simulated parameter $\varepsilon = 0.0001$; $a = 1$; $b = 100$; $K = 50$; $\sigma = 0.2$; $r = 0.06$. We observe that, when H increases, the volatility surface varies and is smoother.

$\eta = \frac{\partial \sigma}{\partial H}$: *DJEUSADE for a European Call* : $H = 0.7$; *Strike* $K = 50$

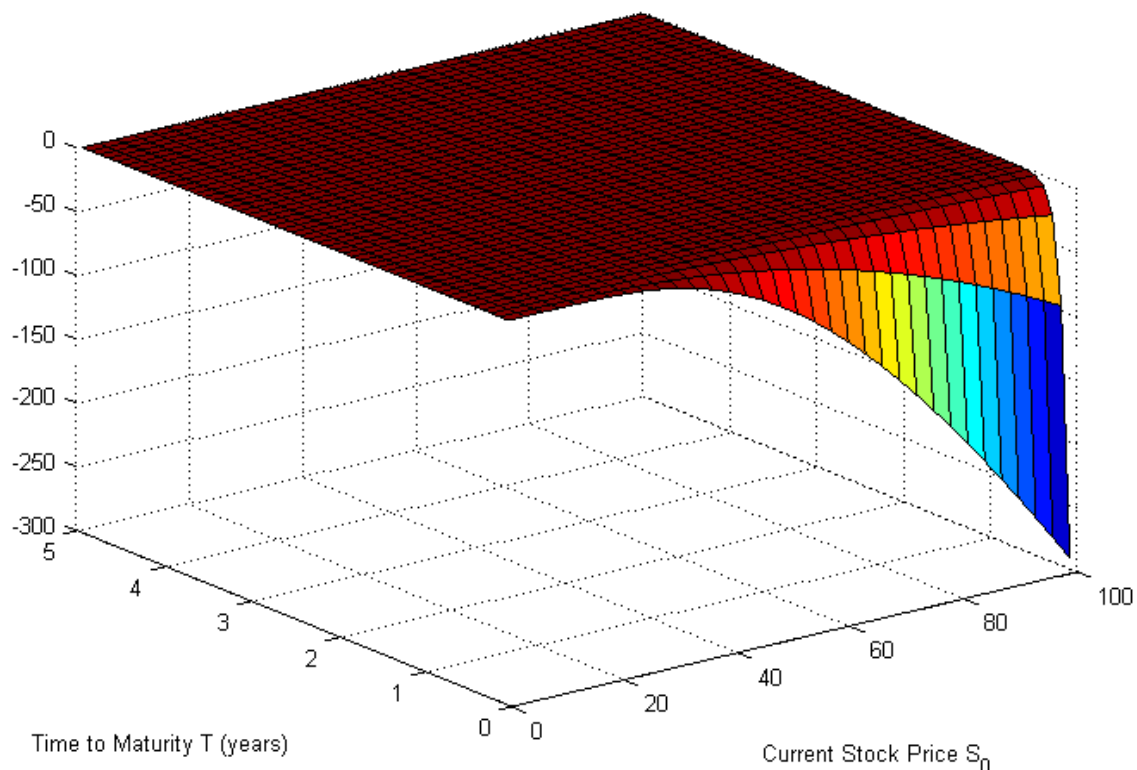


Figure 1: Djeutcha-Sadefo for European call: $H = 0.7$

$\eta = \frac{\partial C}{\partial H}$: DJEUSADE for a European Call : $H = 0.7$; Strike $K = 50$

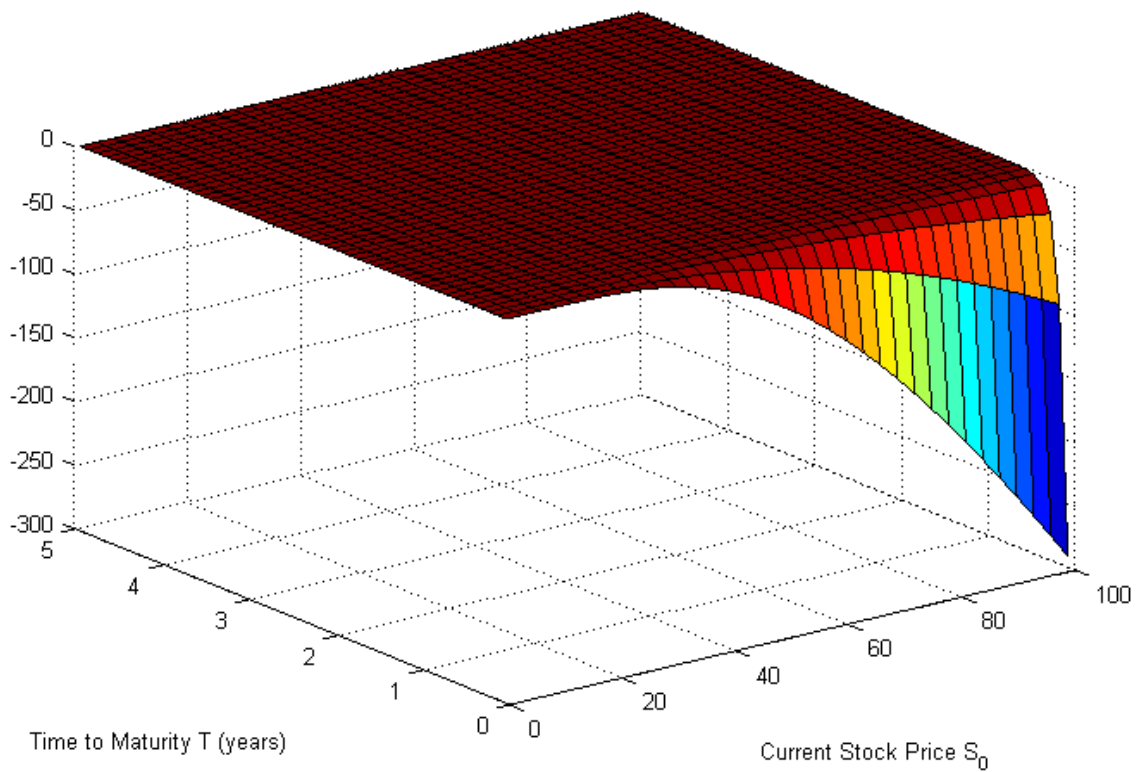


Figure 2: Djeutcha-Sadefo for European call: $H = 0.8$

$\eta = \frac{\partial C}{\partial H}$: *DJEUSADE for a European Call* : $H = 0.9$; *Strike $K = 50$*

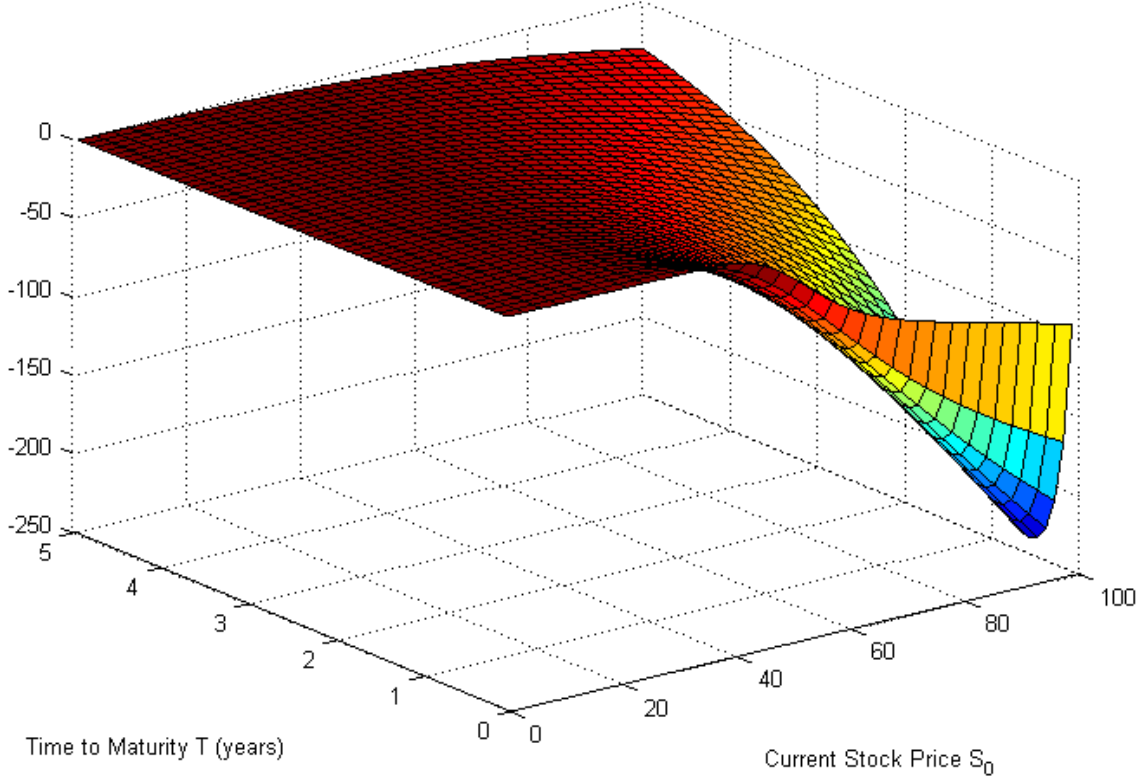


Figure 3: **Djeutcha-Sadefo for European call:** $H = 0.9$

We believe that it is possible to link local volatility with implied volatility. The call option prices are the real local volatility market prices. So they must correspond to the call option price calculated with implied volatility. Let $C(T, K) = \tilde{C}$ be the call option price under the Mixed-Modified-Fractional-Black-Scholes model, where I is the implied volatility. By applying the Chain derivation formula, we obtain:

$$\frac{\partial C}{\partial T} = \frac{\partial \tilde{C}}{\partial T} + \frac{\partial \tilde{C}}{\partial \sigma} \cdot \frac{\partial I}{\partial T} \quad (78)$$

$$\frac{\partial C}{\partial K} = \frac{\partial \tilde{C}}{\partial K} + \frac{\partial \tilde{C}}{\partial \sigma} \frac{\partial I}{\partial K} \quad (79)$$

$$\frac{\partial^2 C}{\partial K^2} = \frac{\partial^2 \tilde{C}}{\partial K^2} + 2 \frac{\partial^2 \tilde{C}}{\partial K \partial \sigma} \frac{\partial I}{\partial K} + \frac{\partial^2 \tilde{C}}{\partial \sigma^2} \left(\frac{\partial I}{\partial K} \right)^2 + \frac{\partial I}{\partial \sigma} \frac{\partial^2 I}{\partial K^2}. \quad (80)$$

By using Eq.(78),Eq.(79) and Eq.(80) we obtain the following expression:

$$\frac{\frac{\partial C}{\partial T}}{\frac{\partial^2 C}{\partial K^2}} = \frac{\frac{\partial \tilde{C}}{\partial T} + \frac{\partial \tilde{C}}{\partial \sigma} \cdot \frac{\partial I}{\partial T}}{\frac{\partial^2 \tilde{C}}{\partial K^2} + 2 \frac{\partial^2 \tilde{C}}{\partial K \partial \sigma} \frac{\partial I}{\partial K} + \frac{\partial^2 \tilde{C}}{\partial C^2} \left(\frac{\partial I}{\partial K} \right)^2 + \frac{\partial I}{\partial \sigma} \frac{\partial^2 I}{\partial K^2}} \quad (81)$$

$$\frac{\frac{\partial C}{\partial K}}{\frac{\partial^2 C}{\partial K^2}} = \frac{\frac{\partial \tilde{C}}{\partial K} + \frac{\partial \tilde{C}}{\partial \sigma} \frac{\partial I}{\partial K}}{\frac{\partial^2 \tilde{C}}{\partial K^2} + 2 \frac{\partial^2 \tilde{C}}{\partial K \partial \sigma} \frac{\partial I}{\partial K} + \frac{\partial^2 \tilde{C}}{\partial C^2} \left(\frac{\partial I}{\partial K} \right)^2 + \frac{\partial I}{\partial \sigma} \frac{\partial^2 I}{\partial K^2}} \quad (82)$$

$$\left[\frac{\frac{\partial C}{\partial K}}{\frac{\partial^2 C}{\partial K^2}} \right]^2 = \left[\frac{\frac{\partial \tilde{C}}{\partial K} + \frac{\partial \tilde{C}}{\partial \sigma} \frac{\partial I}{\partial K}}{\frac{\partial^2 \tilde{C}}{\partial K^2} + 2 \frac{\partial^2 \tilde{C}}{\partial K \partial \sigma} \frac{\partial I}{\partial K} + \frac{\partial^2 \tilde{C}}{\partial C^2} \left(\frac{\partial I}{\partial K} \right)^2 + \frac{\partial I}{\partial \sigma} \frac{\partial^2 I}{\partial K^2}} \right]^2. \quad (83)$$

Substituting Eq.(81),Eq.(82) and Eq.(83) into Eq.(13), the local volatility becomes

$$\begin{aligned} \sigma(T, K, a, b) &= \frac{\phi_T^\varepsilon}{K \chi_\varepsilon^2} \frac{\frac{\partial \tilde{C}}{\partial K} + \frac{\partial \tilde{C}}{\partial \sigma} \frac{\partial I}{\partial K}}{\frac{\partial^2 \tilde{C}}{\partial K^2} + 2 \frac{\partial^2 \tilde{C}}{\partial K \partial \sigma} \frac{\partial I}{\partial K} + \frac{\partial^2 \tilde{C}}{\partial C^2} \left(\frac{\partial I}{\partial K} \right)^2 + \frac{\partial I}{\partial \sigma} \frac{\partial^2 I}{\partial K^2}} \\ &+ \sqrt{\frac{(\phi_T^\varepsilon)^2}{K^2 \chi_\varepsilon^4} \left[\frac{\frac{\partial \tilde{C}}{\partial K} + \frac{\partial \tilde{C}}{\partial \sigma} \frac{\partial I}{\partial K}}{\frac{\partial^2 \tilde{C}}{\partial K^2} + 2 \frac{\partial^2 \tilde{C}}{\partial K \partial \sigma} \frac{\partial I}{\partial K} + \frac{\partial^2 \tilde{C}}{\partial C^2} \left(\frac{\partial I}{\partial K} \right)^2 + \frac{\partial I}{\partial \sigma} \frac{\partial^2 I}{\partial K^2}} \right]^2} + \sigma_{11}^\varepsilon. \end{aligned} \quad (84)$$

Where

$$\sigma_{11}^\varepsilon = \frac{2}{K^2 \chi_\varepsilon^2} \frac{\frac{\partial \tilde{C}}{\partial T} + \frac{\partial \tilde{C}}{\partial \sigma} \cdot \frac{\partial I}{\partial T}}{\frac{\partial^2 \tilde{C}}{\partial K^2} + 2 \frac{\partial^2 \tilde{C}}{\partial K \partial \sigma} \frac{\partial I}{\partial K} + \frac{\partial^2 \tilde{C}}{\partial C^2} \left(\frac{\partial I}{\partial K} \right)^2 + \frac{\partial I}{\partial \sigma} \frac{\partial^2 I}{\partial K^2}}. \quad (85)$$

From eq.(73), we have the following partial derivatives

$$\begin{cases} \frac{\partial \tilde{C}}{\partial K} = e^{-rT} \Phi(d_1^H(a, b)), \\ \frac{\partial^2 \tilde{C}}{\partial K^2} = \frac{e^{-rT} \Phi(d_1^H(a, b))}{KI \chi_\varepsilon \sqrt{T}}, \\ \frac{\partial \tilde{C}}{\partial K} = KI \chi_\varepsilon \sqrt{T}, \\ \frac{\partial \tilde{C}}{\partial \sigma} = S^\varepsilon \Phi(d_2^H(a, b)) \sqrt{T} \\ \frac{\partial \tilde{C}}{\partial T} = \frac{S^\varepsilon I \Phi(d_2^H(a, b))}{2\sqrt{T}} + rK e^{-rT} \Phi(d_1^H(a, b)). \end{cases} \quad (86)$$

By replacing Eq.(86) in Eq.(84) and considering that the implied volatility I does not depend directly of the strike K ($\frac{\partial I}{\partial K} = 0$), we have

$$\sigma(T, K, a, b) = \phi_T^\varepsilon \chi_\varepsilon I \sqrt{T} + \sqrt{\frac{(\phi_T^\varepsilon)^2}{\chi_\varepsilon^2} I^2 T + \sigma_{12}^\varepsilon}. \quad (87)$$

where

$$\begin{aligned} \sigma_{12}^\varepsilon &= \frac{2}{K^2 \chi_\varepsilon^2} \frac{\left[\frac{S^\varepsilon I \Phi(d_2^H(a, b))}{2\sqrt{T}} + rK e^{-rT} \Phi(d_1^H(a, b)) \right] + S^\varepsilon \Phi(d_2^H(a, b)) \sqrt{T} \cdot \frac{\partial I}{\partial T}}{\frac{e^{-rT} \Phi(d_1^H(a, b))}{KI \chi_\varepsilon \sqrt{T}}} \\ &= I^2(T) + 2TI(T) \frac{\partial I}{\partial T} \end{aligned} \quad (88)$$

and then Eq.(87) gives

$$\sigma(T, a, b) = \phi_T^\varepsilon \chi_\varepsilon I(T) \sqrt{T} + \sqrt{\frac{(\phi_T^\varepsilon)^2}{\chi_\varepsilon^2} T I^2(T) + I^2(T) + 2T I(T) \frac{\partial I}{\partial T}}. \quad (89)$$

We deduce the following result:

Theorem 5. *The implied volatility I solution of Eq.(89) is a mean quadratic of the local volatility $\sigma^2(T, a, b)$ of the MMFD model defined by*

$$I^2(T) = \frac{1}{2} \int_0^T \sigma^2(\nu, a, b) d\nu.$$

5 Conclusion

Thanks to a recourse to the integral transform of Mellin [26], this paper proposes closed formulas for the European option (call or put) values, in order to guarantee at the same time the absence of arbitrage opportunity, but also to take into account the short or long memory of the processes generating the price time series of the underlying asset of the option. Thus, the closed form formulas of the European options (call or put) values are obtained, with the assumption that the evolution of the logarithmic return of the asset underlying the option is governed by a mixed modified fractional Brownian motion. We therefore introduce the Mixed-Modified-Fractional-Black-Scholes model Eq.(47) with a closed form solution formula. If we select the driftless of Eq.(47) as equal to zero, we obtain the Mixed-Modified-Fractional-Dupire model Eq.(13). We give an expression of the local volatility and analyze its sensitivity in relation with the Hurst parameter coefficient H , via our Mixed-Modified Fractional-Black-Scholes model. Finally, we give an explicit relation between implied volatility of Eq.(13) and the local volatility of Eq.(13) .

As a perspective for this work, we intend to use real data from the markets, in order to provide reliable and realistic estimates of local or implied volatilities.

References

- [1] Aliou, N.F., Seydou N.N., Ndolane, S.(2019), Black-Scholes option pricing equations described by the Caputo generalized fractional derivative. *Chaos, Solitons & Fractals* 125, 108-118.
- [2] Akgul, E.K., Akgul,A., Yavuz, M.(2021): New Illustrative Applications of Integral Transforms to Financial Models with Different Fractional Derivatives. *Chaos, Solitons & Fractals*, Volume 146, 1108-77.
- [3] Black, F., and Scholes, M.(1973), The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:(1973), 637-654.
- [4] Berg E,Lyhagen J.(1998), Short and long-run dependence in Swedish stock returns. *Appl Financial Econ*;8:435-443.
- [5] Breeden,D.T., Litzenberger,R.H. (1978), Prices of State-Contingent Claims Implicit in Option Prices, *Journal of Business* 51, 621-651.
- [6] Cheridito, P. (2003), *Arbitrage in fractional Brownian motion models*, *Finance Stoch.* **7**(4),533-553.
- [7] Cheridito J.(2001) Regularizing fractional Brownian motion with a view toward stock pricing modeling. PhD dissertation, ETH zurich.
- [8] Chronopoulou,A., Tudor, C.A. and Viens, F. (2009): Application of Malliavin calculus to long-memory parameter estimation for non-Gaussian processes. *Comptes rendus Mathematique* 347, 663-666.
- [9] Cutland N, Kopp P, Willinger W.(1993), Stock price returns and the Joseph effect: a fractional version of the Black-Scholes model. In: *Proceedings of the Monte Vertia Conference*, Ascona, Switzerland.
- [10] Cont, R., (2001), Empirical properties of asset returns: Stylized facts and statistical issues, *Quantitative Finance* 1(2), 223-236. <https://doi.org/10.1080/713665670>
- [11] Derman,E., Kani,I., Riding on a Smile, *Risk*, 7(2):(1994), 32-39.
- [12] Dupire,B., Pricing with a Smile, *Risk*,(1994), 18-20.
- [13] Djeutcha, E., Njamen Njomen, D.A., and Fono, L.A. (2019), Solving Arbitrage Problem on the Financial Market Under the Mixed Fractional Brownian Motion With Hurst Parameter $H \in]1/2, 3/4[$.*Journal of Mathematics Research*, vol. 11(1), 76-92.
- [14] Elton E J, Gruber M J. (1995), *Modern portfolio theory and investment analysis*. New York: Wiley.
- [15] Fadugba,S. E., Ogunrinde, R.B.(2014), Black-Scholes Partial Differential Equation In The Mellin Transform Domain,*International Journal of Scientific and Technologic Research*, 3(12), ISSN: 2277-8616, 200-206.
- [16] Frennberg P, Hansson B.(1993), Testing the random walk hypothesis on Swedish stock prices: 1919-1990. *J Bank Finan* ;17:175-91.

- [17] Fama E, French K.(1988), Permanent and temporary components of stock prices. *J Polit Econ* ;96:245-73.
- [18] Hsieth D A.(1991), Chaos and non-linear dynamics: applications to financial market. *Journal of Finance*;46:1839-77.
- [19] Huang B, Yang C W. (1995), The fractal structure in multinational stock returns. *Appl Econ Lett*;2:67-71.
- [20] Hurst, H. (1951): Long Term Storage Capacity of Reservoirs, *Transactions of the American Society of Civil Engineers*, 116, 770-799.
- [21] Lardic, S. et Mignon, V. (1999), PrÉvision ARFIMA des taux de change: Les modÉlisateurs doivent-ils encore exhorter Á la naivetÉ des prÉvisions ? . *Annales d'Économie et de Statistique* , No. 54, pp. 47-68
- [22] Lo, A.W. (1991), Long-term memory in stock market prices. *Econometrica*;59:1279-313.
- [23] Lo, A.W., Mackinlay, A.C.(1988), Stock market prices do not follow random walks: evidence from a simple specification test. *Rev Finan Stud* 1988;1:41-66.
- [24] Mandelbrot, B.B. (1975): Limit theorems of the self-normalized range for weakly and strongly dependent processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 31,271-285.
- [25] Mandelbrot B.B. et Van Ness J. (1968). 'Fractional Brownian Motion, Fractional Noises and Applications', *SIAM Review*, vol. 10, pp. 422-437.
- [26] H. Mellin, H. (1896): 'Ueber die fundamentelle Wichtigkeit des Satzes von Cauchy far die Theorie der Gamma- und hypergeometrischen Funktionen' *Acta Soc. Sci. Fennica* , 21 : 1, pp. 1-115
- [27] Mishura Yu, Valkeila E.(2001), On arbitrage in the mixed Brownian-fractional Brownian market model. Department of Mathematics, University of Helsinki, Preprint 261.
- [28] Ndolane, S., Babacar, S., Seydou N.N., and Awa, T.(2020), Novel Approaches for Getting the Solution of the Fractional Black-Scholes Equation Described by Mittag-Leffler Fractional Derivative, *Discrete Dynamics in Nature and Society*, vol. 2020, Article ID 8047347, 11 pages,(2020) . <https://doi.org/10.1155/2020/8047347>.
- [29] Poterba J, Summer L.(1988): Mean reversion in stock returns: evidence and implications. *J Finan Econ* 1988;22:27-60.
- [30] Ren F-Y, Wang X-T, Lang J-R.(2001), A proof for French's empirical formula on option pricing. *Chaos, Solitons & Fractals* 2000;12:2441-53.
- [31] Sadefo Kamdem, J., (2005), Option pricing with Levy process using Mellin Transform. *Bachelier Congress Society (2006), Tokyo Japan*.
- [32] Stelios, B., Rangan, G., Clement, K.(2015), On economic uncertainty, stock market predictability and nonlinear spillover effects. *The North American journal of economics and finance* 36, 184-191.

- [33] Stelios, B., and Dimitris, G.(2008), Direction of change forecasting using a volatility based recurrent neural network. *Journal of Forecasting* 27 (5),(2008), 407-417.
- [34] Takahashi M. Evaluation methods for derivative securities using the Hurst exponent. Master thesis, Tsukuba University [in Japanese].
- [35] Wang X-T, Qiu W-Y, Ren F-Y.(2001), Option pricing of fractional version of the Black-Scholes model with Hurst exponent H being in $]1/3; 1/2[$. *Chaos, Solitons & Fractals*;12:599-608.
- [36] Zili, M.(2006), On the mixed fractional Brownian motion, *Journal of Applied Mathematics and Stochastic Analysis*.ID 324-35,1-9.