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Abstract

We introduce and study an extension of the classical elapsed time equation in the context of neuron populations that are described by the elapsed time since last discharge. In this extension, we incorporate the elapsed time since the penultimate discharge and we obtain a more complex system of integro-differential equations. For this new system, we prove convergence with exponential rate to stationary state by means of Doeblin's theory in the case of weak non-linearities using an appropriate functional setting, inspired by the case of the classical elapsed time equation. Moreover, we present some numerical simulations to observe how different firing rates can give different types of behaviors and to contrast them with theoretical results of both the classical and extended models.

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1 Introduction

In the study and modelling of neural processes, population density models have proved to be a useful approach to understand brain phenomena at different scales. Among these models, we can mention for example the Wilson-Cowan model [46] and Amari model [1], which describe dynamics

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of interactions between inhibitory and excitatory neurons, and they have been widely studied in different extensions with many theoretical results and simulations.

Another class of population-based models is the elapsed time model, which has attracted the attention of many researchers. In this model we consider a neural network where neurons are described by their elapsed time since the last discharge as the key variable. After receiving some stimulation, neurons spike and interact with other neurons leading them to spike as well.

The elapsed time model is closely related to the well-known integrate-and-fire model. This system describes the dynamics of the membrane potential and has been studied by several authors such as Carrillo et al. [4, 7], Perthame et al. [39, 38, 37] and Zhou et al. [27] in different variants and approaches. Like the integrate-and-fire model [10, 11], the elapsed time model is related to the limit of stochastic processes at microscopic scale and the connection with Poisson processes was established in the works of Chevalier et al. [9, 8]. Other important works on spiking neurons include Brunel [3], Gerstner et al. [21], Ly et al. in [29] and Pham et al. [40] and by Schwalger et al. [41, 43]. The relation between integrate-and-fire and the elapsed time model was studied in Dumont et al. [15, 16].

From a mathematical and analytical point of view, the elapsed-time model has been studied by several authors using different techniques such as Cañizo et al. in [6], Kang et al. [26], Mischler et al. in [32, 31] and the pioneering works of Pakdaman et al. in [33, 34, 35]. Moreover, different extensions of the elapsed time model have been considered by incorporating new elements such as spatial dependence with connectivity kernel in Salort et al. [45], a leaky memory variable in Fonte et al. [18] and a stochastic extension that accounts for finite population size in Schmutz et al. [42] and Schwalger et al. [44].

The aim of the present work is to extend the classical elapsed time model by taking into account the elapsed time since the penultimate discharge in addition to the last one. This extension is modeled through a multiple time renewal equation, whose study is more complex than that of the classical model. We aim to analyze how the incorporation of a second elapsed time changes dynamics of the classical model and what type of new patterns may arise.

From a biophysical point of view, the main motivation is to represent the dependence on the previous history of spikes in neural dynamics. Examples of models incorporating this approach include the works of Gerstner et al. [22] on the spike-response-model with escape noise, where the sum over the spike history is truncated at the second last spike. Among stochastic models, Chevallier et al. [9] consider Wold processes, which are point processes for which the next occurrence of a spike depends on the previous occurrence but also on the previous inter-spike interval. In their work, a connection with Wold processes and multiple-time renewal equations is established by deriving the PDE for the corresponding density. They extended the homogeneous Markovian case, which is study via the forward Kolmogorov's equations, to more general non-Markovian systems with a method called Ogata's thinning algorithm. They also included examples of other stochastic models like Hawkes processes.

Furthermore, multiple time renewal equations have been investigated in Fournier et al. [19] in the linear case, where a non-expanding distance was introduced via a coupling argument. Moreover, these type of equations have multiple potential applications in other domains such as epidemiology, where the evolution of individuals under possible secondary infections has been studied. For an example, see the work of Ferretti et al. [17].

The extended model is described as follows. Let $n = n(t, s, a)$ the probability density of finding a neuron at time t, such that the elapsed times since its last and penultimate discharge are respectively s

and a. Hence, for all $t \geq 0$, the domain of definition of n in the elapsed time variables is

$$\mathcal{D} \coloneqq \{(s,a) \in \mathbb{R}^2 \colon 0 \leq s \leq a\}.$$

In this setting, neural dynamics are modeled through the following nonlinear renewal system

$$\begin{cases} \partial_t n + \partial_s n + \partial_a n + p(s,a,X(t))n = 0 & t > 0,\ a > s > 0, \\ n(t, s=0, a) = N(t,a) \text{ with } N(t,a) \coloneqq \int_a^\infty p(a,u,X(t)) n(t,a,u)\, du & t > 0, a > 0, \\ X(t) = \int_0^\infty N(t,a)\, da & t > 0, \\ n(t=0,s,a) = n_0(s,a) & a > s > 0. \end{cases} \tag{1}$$

We assume that the initial data $n_0 \in L^1(\mathcal{D})$ is a probability density so that System (1) formally verifies

$$\iint n(t,s,a)\, da\, ds = \iint n_0(s,a)\, da\, ds = 1, \quad n(t,s,a) \geq 0 \qquad \forall t \geq 0. \tag{2}$$

As in the classical elapsed time model the function $p \colon \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ is the firing rate of neurons, which depends on the total activity $X(t)$. Furthermore, for the firing rate function p, we assume that there exist $\sigma, p_0, p_\infty > 0$ such that

$$p_0 \mathbb{1}_{\{a > s \geq \sigma\}} \leq p \leq p_\infty. \tag{3}$$

Thus, we get

$$0 \leq X(t) \leq p_\infty, \qquad \forall t \geq 0. \tag{4}$$

We assume for simplicity that $p \in W^{1,\infty}(\mathcal{D} \times \mathbb{R})$, although most of the theoretical results are also valid for firing rates with simple jump discontinuities and the behavior of solutions does not depend on this regularity assumption as we show in the numerical simulations. Furthermore, we say that the network is *inhibitory* if p is decreasing with respect to the total activity X and *excitatory* if p is increasing. If in addition $\|\partial_X p\|_\infty$ is small, we say that System (1) is under a *weak interconnection* regime.

The function $N(t,a)$ represents the flux of discharging neurons, conditioned to the elapsed time since last discharge with value $a > 0$. The boundary condition of n at $s = 0$ states that, when a neurone discharges, its last elapsed time resets to the penultimate elapsed time. The function X models the total activity received by the network, and we assume that it is given by the integral of $N(t,a)$ with respect to all the times $a > 0$.

Finally, we also remark that when p does not depend on a, the probability density

$$m(t,s) \coloneqq \int_s^\infty n(t,s,a)\, da$$

satisfies the equation

$$\begin{cases} \partial_t m + \partial_s m + p(s,X(t))m = 0 & t > 0,\ s > 0, \\ m(t, s=0) = X(t) = \int_0^\infty p(u,X(t)) m(t,u)\, du & t > 0, \\ m(t=0,s) = \int_s^\infty n_0(s,a)\, da & s > 0. \end{cases} \tag{5}$$

In other words, the probability with respect to the last elapsed time is a solution of the classical elapsed time equation. In particular, if we consider firing rates of the form

$$p = \varphi(X(t))\mathbb{1}_{\{s>\sigma\}},$$

with $\varphi \in W^{1,\infty}(\mathbb{R})$ strictly positive and $\sigma > 0$, we know from Torres et al. [45] that periodic solutions and jump discontinuities may occur. Based on this choice of p, we discuss the impact of taking into account the penultimate discharge time regarding to the emergence of periodic solutions of System (1).

Equation (1) represents the probability distribution of neurons in a large (infinite) all-to-all network which can be described as follows. For each neuron, labeled by $i = 1, ..., N$, the last two spike times $S_i(t)$, $A_i(t)$ are recorded. They follow a Piecewise Deterministic Markov Process where, between the spike times $\tau_{i,j}$, the 'ages' progress as

$$\dot{S}_i(t) = 1, \qquad \dot{A}_i(t) = 1, \qquad \tau_{i,j-1} < t < \tau_{i,j}, \qquad i = 1, ..., N,$$

and when a discharge occurs, the penultimate time spike is renewed:

$$A_i(\tau_{i,j}^+) = 0, \qquad S_i(\tau_{i,j}^+) = A_i(\tau_{i,j}^-).$$

The $\tau_{i,j}$ are chosen randomly according to a Poisson distribution of parameter $p(S_i(t), A_i(t), X_\delta(t))$, where the network activity is computed as

$$X_\delta(t) = \frac{1}{N}\sum_{i,j} \omega_\delta(t - \tau_{i,j}),$$

whith ω_δ a smoothing kernel of parameter δ which takes into account the transmission delay. In the limit $N \to \infty$ and $\delta \to 0$, one expects the neurons become independent, driven by the average discharge rate $\mathbb{E}[X_\delta(t)] \approx \iint p(s, a, X(t)n(t, s, a)\, ds\, da$.

In the study of age-structured models, the entropy method has been a useful tool for proving convergence to the steady state. The main idea consists in finding a Lyapunov functional $\mathcal{H}[n]$ and a dissipation functional $D_{\mathcal{H}}[n]$ such that the solutions of the system satisfy

$$\frac{d}{dt}\mathcal{H}[n] = -D_{\mathcal{H}}[n] \leq 0.$$

Thus if we can find a Poincaré inequality of the type $\lambda\mathcal{H}[n] \leq D_{\mathcal{H}}[n]$ for some $\lambda > 0$, we can deduce the exponential decay of $\mathcal{H}[n]$ by using the classical Gronwall inequality, which eventually allows to deduce convergence to the steady state in some convenient norm. This method was developed in the works of [30, 36] with extensions to measure initial data in [23], and it has been applied to different types of models. However, when such inequalities are not available the study of asymptotic behavior becomes more complex.

Another important approach, that we use in this paper, is Doeblin's theory, which was first introduced in the context of Markov chains [13, 12] and later developed in the work of Harris [25] which has inspired several works such as Bansaye et al. [2], Cañizo et al. [5] and Hairer & Mattingly [24]. It ensures that exponential convergence in L^1 to the stationary state follows from the existence of a probability measure ν, a time t_0, and $0 < \alpha$, such that for all initial data which is a probability measure, the solution at time t_0 is bounded from bellow by $\alpha\nu$. Several works in PDE, also consider

this approach, see for example the works of Gabriel et al. [20], B. Perthame et al. [37], Dumont et al. [14]. In particular, this theory has already been an alternative to the classical entropy methods to prove exponential convergence to the steady state for a large class of time elapsed model since the last discharge (see Cañizo et al. [6]).

The article is organized as follows. In Section 2 we prove that System (1) is well-posed in a suitable space for weak non-linearities. Starting with the asymptotic analysis for the linear case, we prove in Section 3 the existence of a stationary state and exponential convergence via Doeblin's theory. For the non-linear problem in the case of weak interconnections, we show in Section 4 the uniqueness of the steady state and in Section 5 we prove the exponential convergence via a perturbation argument. Finally in Section 6 we present some examples of numerical simulations for different initial data and class of firing rates.

2 Well-posedness for weak non-linearities

We prove that System (1) is well-posed in the weak interconnection regime. To do so, we start by studying an auxiliary linear problem where the total activity is fixed and then we proceed to prove well-posedness of System (1) via the Picard fixed point contraction theorem.

This requires some function analytic definitions. For a Banach space $\mathcal{X}$ we denote $\mathcal{C}_b([0,\infty),\mathcal{X})$ as the Banach space given by

$$\mathcal{C}_b([0,\infty),\mathcal{X}) \coloneqq \{f\colon [0,\infty)\to\mathcal{X} \mid f \text{ is continuous and bounded}\}.$$

In this setting we look for weak solutions satisfying $n\in\mathcal{C}_b([0,\infty),L^1(\mathcal{D}))$, so that $N\in\mathcal{C}_b([0,\infty),L^1(0,\infty))$ and $X\in\mathcal{C}_b[0,\infty)$.

2.1 The linear problem

Given $X\in\mathcal{C}_b[0,\infty)$, we consider the following linear problem

$$\begin{cases} \partial_t n+\partial_s n+\partial_a n+p(s,a,X(t))n=0 & t>0, a>s>0,\\ n(t,s=0,a)=N(t,a)\coloneqq\int_a^\infty p(a,u,X(t))n(t,a,u)\,du & t>0, a>0,\\ n(t=0,s,a)=n_0(s,a)\geq 0 & a>s>0. \end{cases} \tag{6}$$

Even though the boundary condition holds only for $t>0$, we may define $N(t,a)$ at $t=0$ by the same formula

$$N(t,a)=\int_0^\infty p(a,u,X(t))n(t,a,u)\,du, \qquad \text{for almost all } a>0,\forall t\geq 0,$$

which implies that n may have discontinuities along the set $\{(t,s,a)\in[0,\infty)\times\mathcal{D}\colon t=s\}$ as a consequence of the method of characteristics (7) explained below.

Lemma 1. *Assume that $n_0\in L^1(\mathcal{D})$ is a probability density and $p\in W^{1,\infty}(\mathcal{D}\times\mathbb{R})$ satisfies (3). Then for a given $X\in\mathcal{C}_b[0,\infty)$, Equation (6) has a unique weak solution $n\in\mathcal{C}_b([0,\infty),L^1(\mathcal{D}))$ with $N\in\mathcal{C}_b([0,\infty),L^1(0,\infty))$. Moreover n is non-negative and verifies the property (2).*

In particular this lemma proves the property (2) for the non-linear System (1).

Proof. From the method of characteristics, we start by noticing that a solution of the linear System (6) satisfies the following fixed point equation

$$\begin{aligned} n(t,s,a) = \Psi[n](t,s,a) \coloneqq n_0(s-t,a-t)e^{-\int_0^t p(t'+s-t,t'+a-t,X(t'))dt'}\mathbb{1}_{\{a>s>t\}} \\ + N(t-s,a-s)e^{-\int_0^s p(s',s'+a-s,X(s'+t-s))ds'}\mathbb{1}_{\{t,a>s\}}. \end{aligned} \tag{7}$$

Let $T > 0$ and $\mathcal{X}_T \coloneqq \{n \in \mathcal{C}_b([0,T], L^1(\mathcal{D})), n(0) = n_0\}$, it readily follows that Ψ maps $\mathcal{X}_T \to \mathcal{X}_T$. We prove by the Picard contraction theorem that Ψ has a unique fixed point in $\mathcal{X}_T$ for $T > 0$ small enough, i.e., there exists a unique weak solution of (6) defined on $[0,T]$. Consider $n_1, n_2 \in \mathcal{X}_T$, we compute

$$\begin{aligned} \iint |\Psi[n_1] - \Psi[n_2]|(t,s,a)\, ds\, da &\leq \int_0^t \int_s^\infty |N_1 - N_2|(t-s,a-s)\, da\, ds \\ &\leq T \sup_{t\in[0,T]} \int_0^\infty |N_1 - N_2|(t,a)\, da \\ &\leq T\, p_\infty \sup_{t\in[0,T]} \|n_1(t,s,a) - n_2(t,s,a)\|_{L^1(\mathcal{D})}, \end{aligned} \tag{8}$$

thus for $T < \frac{1}{p_\infty}$, we have proved that Ψ is a contraction and there exists a unique $n \in X_T$ such that $\Psi[n] = n$. Since the choice of T is independent of n_0, we can reiterate this argument to get a unique solution of (6), which is defined for all $t \geq 0$.

From Formula (7) we can extend the notion of a weak solution for Equation (6) for an initial data $n_0 \in (\mathcal{M}(\mathcal{D}), \|\cdot\|_{M^1})$, the space of finite regular measures on $\mathcal{D}$ endowed with the total variation norm (see section 3 for the definition). Therefore we can use the same argument to prove existence and uniqueness of a weak solution $n \in \mathcal{C}_b([0,\infty), \mathcal{M}(\mathcal{D}))$ with $N \in \mathcal{C}_b([0,\infty), \mathcal{M}(0,\infty))$ and $X \in \mathcal{C}_b[0,\infty)$.

Next, we prove the mass conservation property. For all $t \geq 0$, consider $\mathcal{S}_t \colon \mathcal{M}(\mathcal{D}) \to \mathcal{M}(\mathcal{D})$ the semi-group given by

$$\mathcal{S}_t[f](s,a) = f(s-t,a-t)\mathbb{1}_{\{a>s>t\}},$$

whose infinitesimal generator is the operator $\mathcal{L}f = -\partial_s f - \partial_a f$. From Duhamel's formula, the solution of the fixed point problem (7) also verifies the following equality

$$n(t,s,a) = \mathcal{S}_t[n_0](s,a) + \int_0^t \mathcal{S}_{t-\tau}[\delta_{\{s=0\}}(s,a)N(\tau,a)]\, d\tau - \int_0^t \mathcal{S}_{t-\tau}[p(s,a,X(\tau))n(\tau,s,a)]\, d\tau, \tag{9}$$

where $\delta_{\{s=0\}}(s,a)$ is the measure along the line $\{(0,a) \colon a \geq 0\}$. This formula is translated as

$$\begin{aligned} n(t,s,x) = n_0(s-t,a-t)\mathbb{1}_{\{a>s>t\}} + N(t-s,a-s)\mathbb{1}_{\{t,a>s\}} \\ - \int_0^t p(s-t+\tau,a-t+\tau,X(\tau))n(\tau,s-t+\tau,a-t+\tau)\mathbb{1}_{\{a>s>t-\tau\}}\, d\tau, \end{aligned} \tag{10}$$

and we get the mass conservation property by integrating with respect to (s,a) on the domain $\mathcal{D}$.

Finally, since n_0 is non-negative then Ψ preserves positivity, so by uniqueness of the fixed point, the corresponding solution n is also non-negative. □

2.2 The non-linear problem

Like the classical elapsed-time model, studying well-posedness for a general firing rate is a difficult problem. Indeed, examples of multiple solutions for a same initial data has been observed in Torres et al. [45] and some ill-posed examples have been remarked on Cañizo et al. [6].

As in the linear problem, and with the assumptions below, we may define the quantities

$$N(0,a) = \int_0^\infty p(a,u,X(0))\, n_0(a,u)\, du$$

with $X(0)$ the solution of

$$X(0) = \int_0^\infty \int_a^\infty p(a,u,X(0))\, n_0(a,u)\, du\, da. \tag{11}$$

But again, the boundary condition is not imposed at $t=0$ and in general $X(0) \neq \int_0^\infty n_0(0,a)\, da$.

By using the results of the linear problem, we are now ready to prove that System (1) is well-posed in the case of weak interconnection.

Theorem 1 (Well-posedness for weak interconnections)**.** *Assume that $n_0 \in L^1(\mathcal{D})$ is a probability density and that $p \in W^{1,\infty}(\mathcal{D}\times\mathbb{R})$ satisfies (3). Then for*

$$\|\partial_X p\|_\infty < 1, \tag{12}$$

System (1) has a unique solution with $n \in \mathcal{C}_b([0,\infty), L^1(\mathcal{D}))$, $N \in \mathcal{C}_b([0,\infty), L^1(0,\infty))$ and $X \in \mathcal{C}_b[0,\infty)$. Moreover n verifies Condition (2) for all $t>0$.

Proof. Consider $T>0$. We fix a function $X \in \mathcal{C}_b[0,\infty)$ and define the functions $n \in \mathcal{C}_b([0,\infty), L^1(\mathcal{D}))$ and $N \in \mathcal{C}_b([0,\infty), L^1(0,\infty))$ which are solutions of System (6) by Lemma 1. Furthermore, the solution of this linear equation satisfies (2).

So we have a solution of System (1) defined on $[0,T]$ if X satisfies for all $0 \le t \le T$ and $x \in \Omega$, the following fixed point condition

$$X(t) = \mathcal{T}[X](t) \coloneqq \int_0^\infty N[X](t,a)\, da. \tag{13}$$

We prove that $\mathcal{T}$ defines for all $T>0$ an operator that maps $\mathcal{X}_T \to \mathcal{X}_T$ with $\mathcal{X}_T \coloneqq \mathcal{C}_b([0,T])$. First, we observe the following estimate

$$\left| \int N(t,a)\, da \right| \le p_\infty, \quad \forall t \in [0,T], \tag{14}$$

and it is immediate that $\mathcal{T}[X] \in \mathcal{X}_T$.

We now prove that for T small enough, $\mathcal{T}$ is a contraction. Let $X_1, X_2 \in \mathcal{X}_T$ with their respective solutions $(n_1,N_1),(n_2,N_2)$ of System (6). For the difference between N_1 and N_2 we have

$$\begin{aligned}
\int |N_1 - N_2|(t,a)\, da &\le \iint |p(a,u,X_1)\, n_1(t,a,u) - p(a,u,X_2)\, n_2(t,a,u)|\, du\, da \\
&\le \iint |p(a,u,X_1) - p(a,u,X_2)|\, n_1\, du\, da + \iint p(a,u,X_2)|n_1 - n_2|(t,a,u)\, du\, da \\
&\le \|\partial_X p\|_\infty \|X_1 - X_2\|_\infty + p_\infty \|n_1 - n_2\|_{L^1(\mathcal{D})}.
\end{aligned} \tag{15}$$

To estimate the difference between n_1 and n_2, we combine this inequality with (10) and get

$$\|n_1 - n_2\|_{L^1(\mathcal{D})} \leq 2T \|\partial_X p\|_\infty \|X_1 - X_2\|_\infty + 2Tp_\infty \|n_1 - n_2\|_{L^1(\mathcal{D})}.$$

Then, for $T < \frac{1}{2p_\infty}$, we obtain

$$\|n_1 - n_2\|_{L^1(\mathcal{D})} \leq \frac{2T\|\partial_X p\|_\infty}{1 - 2Tp_\infty} \|X_1 - X_2\|_\infty. \tag{16}$$

Finally, by using again estimate (15), the operator $\mathcal{T}$ satisfies

$$\|\mathcal{T}[X_1] - \mathcal{T}[X_2]\|_\infty \leq \|\partial_X p\|_\infty \left(1 + \frac{2Tp_\infty}{1 - 2Tp_\infty}\right) \|X_1 - X_2\|_\infty. \tag{17}$$

Therefore, for $\|\partial_X p\|_\infty < 1$ and T small enough, $\mathcal{T}$ is a contraction.

From Picard's fixed point we get a unique $X \in \mathcal{X}_T$ such that $\mathcal{T}[X] = X$, and this implies the existence of a unique solution of (1) defined on $[0, T]$. Since estimate (14) is uniform in T, we can iterate this argument to get a unique solution of (1) defined for all $t > 0$.

Furthermore, we conclude from this construction that the non-linear System (1) satisfies (2) like the linear System (6). □

3 Asymptotic behavior for the linear case

In order to study the behavior of System (1), we start by studying the case when $X \geq 0$ is a fixed constant. Thus we consider the linear problem given by

$$\begin{cases} \partial_t n + \partial_s n + \partial_a n + p(s, a, X)n = 0 & t > 0,\ a > s > 0, \\ n(t, s = 0, a) = N(t, a) \coloneqq \int_a^\infty p(a, u, X)n(t, a, u)\, du & t > 0, a > 0, \\ n(t = 0, s, a) = n_0(s, a) & a > s > 0. \end{cases} \tag{18}$$

To determine the behavior of System (18), we consider (n_X, N_X) as the solution of the steady state problem given by

$$\begin{cases} \partial_s n + \partial_a n + p(s, a, X)n = 0 & a > s > 0, \\ n(s = 0, a) = N(a) \coloneqq \int_a^\infty p(a, u, X)n(a, u)\, du & a > 0, \end{cases} \tag{19}$$

In the classical elapsed time model the generalized relative entropy inequality for the linear problem is a well-known property of this class of age-structured models. For a reference on this method, see Perthame et al. [36, 30] and for some applications in the non-linear elapsed time model see for instance [33, 26]. Following these ideas, we can prove this property for the linear System (18).

Proposition 1 (Generalized relative entropy). *Assume there exists a steady state solution of the linear System* (18) *with* $n_X, N_X > 0$. *Then for all convex functions* $H\colon [0, \infty) \to [0, \infty)$ *with* $H(0) = 0$, *the*

solution n of the linear System (18) satisfies

$$\begin{aligned} \frac{d}{dt}\iint n_X(s,a)H\left(\frac{n(t,s,a)}{n_X(s,a)}\right)\,da\,ds = -D_H[n(t,s,a)] \leq 0 \qquad \forall t \geq 0, \\ D_H[n(t,s,a)] = \iint p(s,a,X)n_X(s,a)H\left(\frac{n(t,s,a)}{n_X(s,a)}\right)\,da\,ds - \int N_X(a)H\left(\frac{N(t,a)}{N_X(a)}\right)\,da, \end{aligned} \tag{20}$$

and in particular the steady state is unique.

Proof. In order to prove the relative entropy property, we follow the arguments in [4]. We start by noticing the following identities

$$\partial_s n = n_X \partial_s\left(\frac{n}{n_X}\right) + \frac{n}{n_X}\partial_s n_X, \qquad \partial_a n = n_X \partial_a\left(\frac{n}{n_X}\right) + \frac{n}{n_X}\partial_a n_X, \tag{21}$$

and for simplicity we reformulate Equation (18) as follows

$$\begin{cases} \partial_t n + \partial_s n + \partial_a n + p(s,a,X)n = \delta_{\{s=0\}}(s,a)N(t,a) & t>0,\ a>s>0, \\ n(t,s=0,a) = 0 & t>0, a>0, \\ n(t=0,s,a) = n_0(s,a) & a>s>0, \end{cases} \tag{22}$$

where $\delta_{\{s=0\}}(s,a)$ is the measure along the line $\{(0,a)\colon a \geq 0\}$. In the same way, we reformulate the corresponding steady state problem (19).

$$\begin{cases} \partial_s n_X + \partial_a n_X + p(s,a,X)n_X = \delta_{\{s=0\}}(s,a)N_X(a) & a>s>0, \\ n_X(s=0,a) = 0 & a>0. \end{cases} \tag{23}$$

Hence by using the identities (21) along with Equations (22) and (23), we get the following equation for $\frac{n}{n_X}$

$$\partial_t\left(\frac{n}{n_X}\right) + \partial_s\left(\frac{n}{n_X}\right) + \partial_a\left(\frac{n}{n_X}\right) = \delta_{\{s=0\}}(s,a)\frac{N_X}{n_X}\left(\frac{N}{N_X} - \frac{n}{n_X}\right)$$

and if we multiply this equality by $H'\left(\frac{n}{n_X}\right)$, we get

$$\partial_t H\left(\frac{n}{n_X}\right) + \partial_s H\left(\frac{n}{n_X}\right) + \partial_a H\left(\frac{n}{n_X}\right) = \delta_{\{s=0\}}(s,a)\frac{N_X}{n_X}\left(\frac{N}{N_X} - \frac{n}{n_X}\right)H'\left(\frac{n}{n_X}\right).$$

Therefore, by multiplying the latter equality by n_X and using Equation (23), we have the corresponding equation for $u = n_X H\left(\frac{n}{n_X}\right)$

$$\partial_t u + \partial_s u + \partial_a u + p(s,a,X)u = \delta_{\{s=0\}}(s,a)N_X\left[\left(\frac{N}{N_X} - \frac{n}{n_X}\right)H'\left(\frac{n}{n_X}\right) + H\left(\frac{n}{n_X}\right)\right]. \tag{24}$$

Finally, by noticing the following limit

$$\lim_{s\to 0}\frac{n(t,s,a)}{n_X(s,a)} = \frac{N(t,a)}{N_X(a)}, \qquad \text{for a.e.} \quad t,a>0,$$

we conclude the generalized relative entropy property (20) by integrating Equation (24) with respect to (s, a) on the domain $\mathcal{D}$. Moreover, we observe that $D_H[\cdot]$ is non-negative by applying Jensen's inequality with the probability measure $d\mu = p(a, y)\frac{n_X(a,y)}{N_X(a)}\, dy$ for each $a > 0$. In particular when H is strictly convex and $D_H[n] = 0$, we deduce that $\frac{n}{n_X}$ is constant and subsequently we get $n = n_X$, since both n, n_X are probability densities. Therefore, the steady state is unique. □

If we consider the entropy method to prove exponential convergence for the linear Equation (18) in $L^1(\mathcal{D})$, we have the following equality for $H(\cdot) = |\cdot|$

$$\frac{d}{dt}\iint |n - n_X|\, da\, ds = \int \left|\int p(n - n_X)\, da\right| ds - \iint p|n - n_X|\, da\, ds \leq 0.$$

We observe that for the right-hand side it is not evident to find a L^1 Poincaré inequality involving the entropy in order to conclude exponential convergence, unlike the classical elapsed time model where this approach has proved to be useful (see for example Perthame [36]).

Furthermore, in Proposition (1) we assumed that n_X and N_X are strictly positive, which is not necessarily true. Unlike the classical elapsed time model, there exist solutions where n_X and N_X vanish for some values of (s, a). For example, consider $p(s, a, X) = \mathbb{1}_{\{s>1\}}$ which satisfies the bounds (3). From the method of characteristics we get the following equality

$$n_X(s, a) = N_X(a - s)e^{-(s-1)_+}, \qquad a > s,$$

so that form the boundary condition at $s = 0$, we observe that N_X satisfies

$$N_X(a) = \mathbb{1}_{\{a>1\}}e^{-(a-1)}\int_0^\infty N_X(u)\, du,$$

which implies that $N_X(a)$ vanishes for $a < 1$ and subsequently $n_X(s, a)$ vanishes for $a - s < 1$.

Due to the limitations of the entropy method approach we will make use of Doeblin's theory, which will be the key ingredient in proving convergence to steady state. In this context we start by recalling the useful concepts in order to apply Doeblin's theorem. Consider $(\mathcal{M}(\mathcal{X}), \|\cdot\|_{M^1})$ the space of finite signed measures on the space $\mathcal{X}$ with the norm of the total variation

$$\|\mu\|_{M^1} := \mu_+(\mathcal{X}) + \mu_-(\mathcal{X}), \tag{25}$$

where $\mu = \mu_+ - \mu_-$ is the Hahn–Jordan decomposition of the measure μ into its positive and negative parts. For simplicity of the computations, we will treat measures as if they were L^1 functions and we simply write the L^1-norm instead of M^1-norm.

We now recall the definition of a Markov semigroup and Doeblin's condition.

Definition 1 (Markov semi-group)**.** *Let $(\mathcal{X}, \mathcal{A})$ be a measure space and $P_t\colon \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathcal{X})$ be a linear semi-group. We say that P_t is a Markov semi-group if $P_t\mu \geq 0$ for all $\mu \geq 0$ and $\int_{\mathcal{X}} dP_t\mu = \int_{\mathcal{X}} d\mu$ for all $\mu \in \mathcal{M}(\mathcal{X})$. In other words, (P_t) preserves the subset of probability measures $\mathcal{P}(\mathcal{X})$.*

Definition 2 (Doeblin's condition)**.** *Let $P_t\colon \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathcal{X})$ be a Markov semi-group. We say that (P_t) satisfies Doeblin's condition if there exist $t_0 > 0$, $\alpha \in (0, 1)$ and $\nu \in \mathcal{P}(\mathcal{X})$ such that*

$$P_{t_0}\mu \geq \alpha\nu \quad \forall \mu \in \mathcal{P}(\mathcal{X}).$$

Under this functional setting, we are now ready to state Doeblin's theorem as follows.

Theorem 2 (Doeblin's Theorem)**.** *Let $P_t\colon \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathcal{X})$ be a Markov semi-group that satisfies Doeblin's condition. Then the semigroup has a unique equilibrium $\mu^* \in \mathcal{P}(\mathcal{X})$. Moreover, for all $\mu \in \mathcal{M}(\mathcal{X})$ we have*

$$\|P_t\mu - \langle\mu\rangle\mu^*\|_{M^1} \leq \frac{1}{1-\alpha}e^{-\lambda t}\|\mu - \langle\mu\rangle\mu^*\|_{M^1} \quad \forall t \geq 0,$$

with $\langle\mu\rangle = \int_{\mathcal{X}} d\mu$ and $\lambda = -\frac{\ln(1-\alpha)}{t_0} > 0$.

For a proof of Doeblin's Theorem, see for example [20].

From Lemma 1, the solution of the linear problem (18) determines a Markov semi-group acting on $L^1(\mathcal{D})$. By means of Doeblin's theory, the solutions of linear Equation (18) converge exponentially to a unique steady state, as we assert in the following theorem.

Theorem 3. *Let $n_0 \in L^1(\mathcal{D})$ be a probability density and assume that p smooth satisfies Assumption (3). Then for a fixed $X > 0$, there exists a unique stationary solution $n_X(s,a) \in L^1(\mathcal{D})$ of the linear Equation (18) satisfying $\iint n_X(s,a)\,da\,ds = 1$. Moreover, the corresponding solution of Equation (18) satisfies*

$$\|n(t) - n_X\|_{L^1_{s,a}} \leq \frac{1}{1-\alpha}e^{-\lambda t}\|n_0 - n_X\|_{L^1_{s,a}} \qquad \forall t \geq 0,$$

with $\alpha = \frac{1}{2}p_0^2\sigma^2 e^{-3p_\infty\sigma}$ and $\lambda = -\frac{\log(1-\alpha)}{3\sigma} > 0$.

In order to obtain the result, we show that after some time the solution of the linear problem is uniformly bounded from below for all probability densities. Thus from Doeblin's theorem we get the exponential convergence to equilibrium.

Lemma 2. *Assume (2) and (3). Let $n(t,s,a)$ be a solution of (18), then there exist $t_0 > 0$, $\alpha \in (0,1)$ and a probability density $\nu \in L^1$ such that*

$$n(t_0,s,a) \geq \alpha\nu(s,a).$$

Proof. The main idea of the proof is to control the mass transported along the lines of direction $(1,1)$. Firstly, we observe the transport of the initial data n_0. From Assumption (3) and the characteristics Formula (7) the following inequality holds

$$\int_t^\infty \int_s^\infty n(t,s,a)\,da\,ds \geq e^{-p_\infty t}, \qquad \forall t \geq \sigma. \tag{26}$$

Secondly, we see the mass that returns at $s = 0$. From (26) we get for all $t \geq \sigma$

$$\int_t^\infty n(t,s=0,a)\,da = \int_t^\infty N(t,a)\,da \geq p_0 \int_t^\infty \int_a^\infty n(t,a,u)\,du\,da \geq p_0 e^{-p_\infty t}. \tag{27}$$

This means that we reduced by one dimension the problem of finding the uniform lower bound. For $t \geq \sigma$ the mass of the region $\{(s,a)\colon a > s > t\}$ concentrates in the line $\{(0,a)\colon a \geq t\}$, as we see in Figure 1.

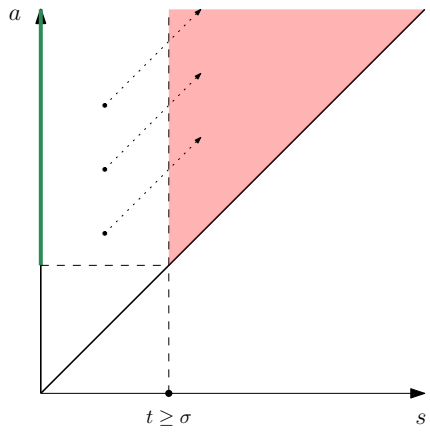


Figure 1: First reduction of dimension. For a $t \geq \sigma$, all points in $\mathcal{D}$ are transported to the red region, which has a total mass of at least $e^{-p_\infty t}$. Then a mass of at least $p_0 e^{-p_\infty t}$ returns to the green line.

Thirdly, in order to control the point values of $n(t,s,a)$, we regard the values of $N(t,a)$. Observe that from Formula (7) we have

$$\begin{aligned} n(t,s,a) &\geq N(t-s,a-s)e^{-\int_0^s p(s',s'+a-s,X)ds'}\mathbb{1}_{\{t,a>s\}} \\ &\geq N(t-s,a-s)e^{-p_\infty s}\mathbb{1}_{\{t,a>s\}}, \end{aligned} \tag{28}$$

thus for $a \geq \sigma$ and $t-a > \sigma$, we obtain by using again Assumption (3) that

$$\begin{aligned} N(t,a) &\geq p_0 \int_a^\infty n(t,a,u)\,du \\ &\geq p_0 e^{-p_\infty a}\int_a^\infty N(t-a,u-a)\,du = p_0 e^{-p_\infty a}\int_0^\infty N(t-a,u)\,du \\ &\geq p_0 e^{-p_\infty a}\int_{t-a}^\infty N(t-a,u)\,du \geq p_0^2 e^{-p_\infty t}. \end{aligned} \tag{29}$$

This means we reduced the problem of finding the uniform lower bound by one dimension again, as we see in Figure 2.

Finally, once we have estimated $N(t,a)$ from below, we come back to estimate (28) to conclude that for $a-s \geq \sigma$ and $t-a > \sigma$ we have

$$\begin{aligned} n(t,a,s) &\geq N(t-s,a-s)e^{-p_\infty s}\mathbb{1}_{\{t,a>s\}} \\ &\geq p_0^2 e^{-p_\infty t}\mathbb{1}_{\{t-a,a-s>\sigma\}}, \end{aligned} \tag{30}$$

so that we can choose $t = 3\sigma$ and conclude that

$$n(3\sigma,a,s) \geq p_0^2 e^{-3p_\infty \sigma}\mathbb{1}_{\{2\sigma>a>s+\sigma\}}.$$

Therefore we get the desired result with $t_0 = 3\sigma$, $\alpha = \frac{1}{2}\sigma^2 p_0^2 e^{-3p_\infty\sigma} \in (0,1)$ and ν given by

$$\nu(s,a) = \frac{2}{\sigma^2}\mathbb{1}_{\{2\sigma>a>s+\sigma\}},$$

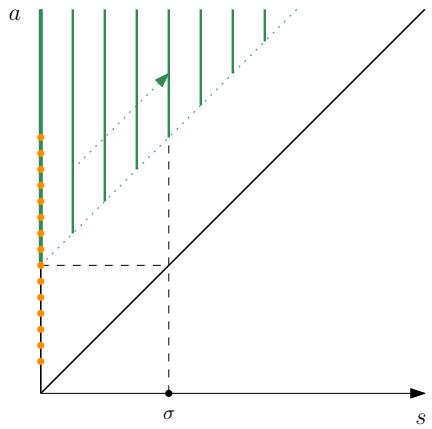


Figure 2: Second reduction of dimension. For $t \in [\sigma, 2\sigma]$ the green lines are transported to the region where $s \geq \sigma$ and their mass is of at least $p_0 e^{-2p_\infty \sigma}$. Then the mass of each green line is concentrated in the orange points, whose values are at least $p_0^2 e^{-2p_\infty \sigma}$.

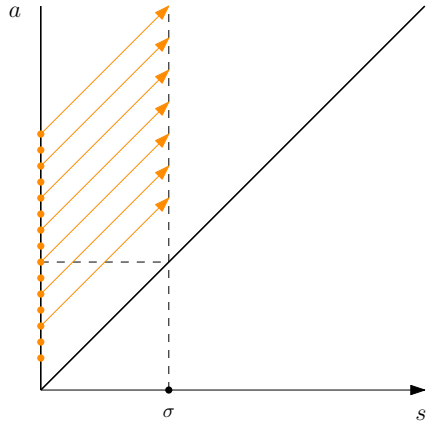


Figure 3: Finally for $t \in [2\sigma, 3\sigma]$ the orange dots are transported to region where $s \geq \sigma$, which allows to construct a minorization function for Doeblin's Theorem.

whose support is contained in orange region of Figure 3.

□

From Lemma 2 the hypothesis of Doeblin's theorem are verified and Theorem 3 readily follows.

Concerning the conditioned activity N in System (19), we conclude from Theorem 3 that for X fixed, there is a unique stationary $N_X \in L^1(0, \infty)$ determined by the method of characteristics through the formula

$$n_X(s, a) = N_X(a - s) \exp\left(- \int_0^s p(s', a - s + s', X)\, ds' \right), \qquad a > s. \tag{31}$$

By replacing this expression in the boundary condition at $s = 0$, we obtain the following integral

equation for $N_X(a)$

$$N_X(a) = \mathcal{T}_X[N_X](a), \tag{32}$$

with $\mathcal{T}_X \colon L^1(0,\infty) \to L^1(0,\infty)$ given by

$$\begin{aligned}\mathcal{T}_X[N](a) &\coloneqq \int_0^\infty p(a, u+a, X) \exp\left(-\int_0^a p(s', u+s', X)\, ds'\right) N(u)\, du \\ &= -\frac{\partial}{\partial a} \int_0^\infty \exp\left(-\int_0^a p(s', u+s', X)\, ds'\right) N(u)\, du.\end{aligned}$$

Moreover, by integrating Equation (31) we get

$$\int_0^\infty \int_0^\infty N_X(a) \exp\left(-\int_0^s p(s', a+s', X)\, ds'\right) da\, ds = 1. \tag{33}$$

Therefore we conclude that finding a function $N \in L^1(0,\infty)$ satisfying Equation (32) and Condition (33) is equivalent to finding a steady state $n_X(s,a)$ in Equation (19). The integral Equation (32) will play an important role in the analysis of the non-linear System (1), thus we prove the following two lemmas on the operator $\mathcal{T}_X$ that will be useful in the sequel.

Lemma 3. *Assume that p Lipschitz satisfies Assumption (3). For each $X > 0$ the operator $\mathcal{T}_X$ is compact and it satisfies that $\dim\ker(I - \mathcal{T}_X) = 1$, which is generated by a non-negative function, and*

$$ran(I - \mathcal{T}_X) = \left\{ f \in L^1(0,\infty) \colon \int_0^\infty f(x)\, dx = 0 \right\}.$$

Proof. The first step is to prove that $\mathcal{T}_X$ is a compact operator. This means we have to prove that the set $A = \{\mathcal{T}_X[f] \colon ||f||_1 \le 1\}$ is relatively compact in L^1.

First observe that $\|\mathcal{T}_X[f]\|_1 \le p_\infty$ for all f with $||f||_1 \le 1$, so A is bounded.

Second, we prove that

$$\int_r^\infty |\mathcal{T}_X[f](a)| da \to 0 \qquad \text{uniformly when } r \to \infty.$$

Indeed for $r > \sigma$ we have

$$\begin{aligned}\int_r^\infty |\mathcal{T}_X[f](a)| da &\le p_\infty \int_r^\infty \int_0^\infty |f(u)| e^{-\int_0^a p(s', u+s', X)\, ds'}\, du\, da \\ &\le p_\infty \int_r^\infty \int_\sigma^\infty |f(u)| e^{-p_0(a-\sigma)}\, du\, da \\ &\le p_\infty e^{p_0\sigma} \int_\sigma^\infty |f(u)| du \int_r^\infty e^{-p_0 a} da \\ &\le p_\infty e^{p_0\sigma} \frac{e^{-p_0 r}}{p_0} \to 0.\end{aligned}$$

Now we prove the equicontinuity property. Observe that

$$\begin{aligned}\frac{d}{da}\mathcal{T}_X[f](a) = &\int_0^\infty (\partial_s p + \partial_a p)(a, u+a, X) e^{-\int_0^a p(s', u+s', X)\, ds'} f(u) du \\ &- \int_0^\infty p(a, u+a, X)^2 e^{-\int_0^a p(s', u+s', X)\, ds'} f(u) du,\end{aligned}$$

thus for f with $\|f\|_1 \leq 1$ we have

$$\int_0^\infty \left| \frac{d}{da} \mathcal{T}_X[f](a) \right| da \leq \|\nabla p\|_\infty + p_\infty^2.$$

Therefore by the Kolmogorov-Frechet theorem we conclude that A is relatively compact so the operator $\mathcal{T}_X$ is.

Furthermore, since n_X is the unique steady state of Equation (18) that is a probability density, from the linearity we deduce that any other function in $\ker(I-\mathcal{T}_X)$ is a multiple of N_X and thus $\dim\ker(I-\mathcal{T}_X) = 1$.

Next, we proceed to determine $\operatorname{ran}(I-\mathcal{T}_X)$. Observe that adjoint operator $\mathcal{T}_X^*\colon L^\infty \to L^\infty$ is given by

$$\mathcal{T}_X^*[g](a) = \int_0^\infty p(u, u+a, X) \exp\left(- \int_0^u p(s', a+s', X)\, ds' \right) g(u)\, du,$$

and from Fredholm's alternative we get $\dim\ker(I-\mathcal{T}_X^*) = \dim\ker(I-\mathcal{T}_X) = 1$. Since $\mathcal{T}_X^*[g] \equiv 0$ for any constant function, we deduce that $\ker(I-\mathcal{T}_X^*)$ is the subspace of constant functions. Finally from orthogonality conditions we conclude that

$$\operatorname{ran}(I-\mathcal{T}_X) = \left\{ f \in L^1(0,\infty)\colon \int_0^\infty f(x)\, dx = 0 \right\}.$$

□

A direct consequence of Lemma 3 is the following result

Lemma 4. *Assume that p is smooth respect to variable X, then $N_X(a)$ is also smooth with respect to X.*

Proof. Define the $F\colon L^1(0,\infty)\times(0,\infty) \to \operatorname{ran}(I-\mathcal{T}_X)\times\mathbb{R}$ given by

$$F(N, X) = \left((I-\mathcal{T}_X)[N]\,, \quad \iint N(a) e^{-\int_0^a p(s', u+s', X)\, ds'}\, da\, ds - 1 \right),$$

so that for each X we have $F(N_X(a), X) = 0$. Observe that $D_N F$ is given by

$$D_N F[h] = \left((I-\mathcal{T}_X)[h]\,, \quad \iint h(a) e^{-\int_0^a p(s', u+s', X)\, ds'}\, da\, ds \right)$$

Thus by Lemma 3 this operator is an isomorphism and from the implicit function theorem we conclude that $N_X(a)$ depends smoothly on X. □

Remark 1. *The lower bound condition (3) on the firing rate p is important to verify the existence of a steady state for System (1) and Doeblin's condition. For example, when we consider a fixed $X > 0$ and*

$$p(s, a, X) = \mathbb{1}_{\{a-s>X\}},$$

then there are no steady states of the linear Equation (18), besides the zero solution. Indeed, from Equation (32) we deduce that the discharging flux N should satisfy

$$N_X(a) = e^{-a} \int_X^\infty N_X(u)\, du,$$

thus N_X is an exponential function and by integrating with respect to a we get $\int_0^\infty N_X(a)\, da = \int_X^\infty N_X(u)\, du$. Since N is non-negative, we conclude N vanishes on $(0, X)$, implying that $N \equiv 0$.

4 Steady states

Consider $n^* = n^*(s,a)$ with support in the set $\{s \le a\}$. We are interested in the stationary solutions of the non-linear System (1) given by

$$\begin{cases} \partial_s n + \partial_a n + p(s,a,X)n = 0 & a > s > 0, \\ n(s=0,a) = N(a) \coloneqq \int_0^\infty p(a,u,X)n(a,u)\,du & a > 0, \\ X = \int_0^\infty N(a)\,da, & \\ \iint n(s,a)\,da\,ds = 1, \quad n(s,a) \ge 0. & \end{cases} \tag{34}$$

We define N_X as the respective conditional activity in terms of X. In order to have a steady state of the non-linear Problem (1), we must find $X > 0$ such that

$$X = \Phi(X) \coloneqq \int_0^\infty N_X(a)\,da. \tag{35}$$

In the general case this equation has always a solution since the right-hand side is uniformly bounded thanks to estimate (4) and $N_X(a)$ depends continuously on X. By using the properties of the operator $\mathcal{T}_X$, we prove that under the weak interconnections regime the non-linear System (1) has a unique steady state.

Theorem 4. *Assume (2) and that p smooth satisfies Assumption (3). Then for $\|\partial_X p\|_\infty$ small enough, System (1) has a unique steady state (n^*, N^*, X^*).*

Proof. The goal is to prove that Φ is a contraction in order to obtain a unique fixed point. In order to estimate $\partial_X N$ we make use of the implicit function theorem. By differentiating Equation (33) we get

$$\iint \partial_X(N_X) e^{-\int_0^s p(s',a+s',X)\,ds'}\,da\,ds = \iint N_X e^{-\int_0^s p(s',a+s',X)\,ds'} \left(\int_0^s \partial_X p(s',u'+s',X)\,ds'\right) da\,ds. \tag{36}$$

Furthermore, if we differentiate with respect to X the Equation (32), we get

$$\begin{aligned} \partial_X N_X(a) = &\int_0^\infty \left(-\partial_a e^{-\int_0^a p(s',u+s',X)\,ds'}\right) \partial_X N_X(u)\,du \\ &+ \int_0^\infty \left(-\partial_a \partial_X e^{-\int_0^a p(s',u+s',X)\,ds'}\right) N_X(u)\,du, \end{aligned}$$

i.e. $\partial_X N_X(a)$ satisfies the equation

$$\begin{aligned} (I - \mathcal{T}_X)[\partial_X N](a) = &\int_0^\infty p(a,u+a,X) e^{-\int_0^a p(s',u+s',X)\,ds'} N_X(u)\,du \\ &- \int_0^\infty p(a,u+a,X) \left(\int_0^a \partial_X p(s',u+s',X)\,ds'\right) e^{-\int_0^a p(s',u+s',X)\,ds'} N_X(u)\,du. \end{aligned} \tag{37}$$

By using the implicit function theorem and the Condition (36) we can define an inverse of $I - \mathcal{T}_X$ which depends continuously on X. Observe that $\|(I - \mathcal{T}_X)^{-1}\|$ is uniformly bounded on X in the

operator norm, since X is uniformly bounded. Thus, for the function Φ we get

$$\begin{aligned}
|\Phi'(X)| &= \left| \int \partial_X N_X(a)\, da \right| \\
&\leq \|(I - \mathcal{T}_X)^{-1}\| \, \|\partial_X p\|_\infty \left(\iint (1 + p_\infty a) e^{-\int_0^a p(s', u+s', X)\, ds} N_X(u) da\, du \right) \\
&\leq C \|\partial_X p\|_\infty \left(\iint (1 + p_\infty a) e^{-p_0 a} N_X(u)\, da\, du \right) \\
&\leq C \|\partial_X p\|_\infty p_\infty \left(\int (1 + p_\infty a) e^{-p_0 a}\, da \right),
\end{aligned}$$

so that for $\|\partial_X p\|_\infty$ small enough Φ is a contraction and we conclude the result. □

5 Convergence to equilibrium

After studying the linear case, we are now ready to prove convergence to the steady state under the weak interconnection regime, i.e. $\|\partial_X p\|_\infty$ small enough, by a perturbation argument.

Theorem 5 (Convergence to equilibrium)**.** *Assume that $n_0 \in L^1(\mathcal{D})$ satisfies Assumption (2) and that p Lipschitz satisfies Assumption (3). For $\|\partial_X p\|_\infty$ small enough, let (n^*, N^*, X^*) be the corresponding stationary state of System (1). Then there exist $C, \lambda > 0$ such that the solution n of System (1) satisfies*

$$\|n(t) - n^*\|_{L^1_{s,a}} \leq C e^{-\lambda t} \|n_0 - n^*\|_{L^1_{s,a}},\ \forall t \geq 0.$$

Moreover $\|N(t) - N^*\|_{L^1}$ and $|X(t) - X^*|$ converge exponentially to 0 when $t \to \infty$.

Proof. Observe that n satisfies the evolution equation

$$\partial_t n = \mathcal{L}_X[n] \coloneqq -\partial_s n - \partial_a n - p(s, a, X(t)) n + \delta_{\{s=0\}}(s, a) \int_0^\infty p(a, u, X(t)) n(t, a, u)\, du,$$

where $\delta_{\{s=0\}}(s, a)$ is the measure along the line $\{(0, a) \colon a \geq 0\}$. We can rewrite the evolution equation as

$$\partial_t n = \mathcal{L}_{X^*}[n] + (\mathcal{L}_X[n] - \mathcal{L}_{X^*}[n]) = \mathcal{L}_{X^*}[n] + h. \tag{38}$$

with $h(t, s, a)$ given by

$$h = \big(p(s, a, X^*) - p(s, a, X(t))\big) n(t, s, a) + \delta_{\{s=0\}}(s, a) \int_0^\infty \big(p(a, u, X(t)) - p(a, u, X^*)\big) n(t, a, u)\, du. \tag{39}$$

Let $P_t \colon L^1(\mathcal{D}) \to L^1(\mathcal{D})$ be the linear semi-group associated to operator $\mathcal{L}_{X^*}$. As in the proof of Lemma 1, P_t is extended to space $(\mathcal{M}(\mathcal{D}), \|\cdot\|_{M^1})$ in order to be able to evaluate at the measure h. Since $P_t n^* = n^*$ for all $t \geq 0$, we get that n satisfies

$$n - n^* = P_t(n_0 - n^*) + \int_0^t P_{t-\tau} h(\tau, s, a)\, d\tau, \tag{40}$$

so we need find an estimate for the function h. Observe that we have the following inequalities:

$$\|h(t)\|_{L^1_{s,a}} \leq 2\|\partial_X p\|_\infty |X(t) - X^*|,$$
$$|X(t) - X^*| \leq \|N(t) - N^*\|_1,$$
$$\|N(t) - N^*\|_1 \leq \|\tfrac{\partial p}{\partial X}\|_\infty |X(t) - X^*| + p_\infty \|n(t) - n^*\|_{L^1_{s,a}},$$

and since $\|\partial_X p\|_\infty < 1$ we get

$$\|h(t)\|_{L^1_{s,a}} \leq \tfrac{2p_\infty \|\partial_X p\|_\infty}{1-\|\partial_X p\|_\infty} \|n(t) - n^*\|_{L^1_{s,a}},$$
$$|X(t) - X^*| \leq \tfrac{p_\infty}{1-\|\partial_X p\|_\infty} \|n(t) - n^*\|_{L^1_{s,a}},$$
$$\|N(t) - N^*\|_1 \leq p_\infty \left(\tfrac{\|\partial_X p\|_\infty}{1-\|\partial_X p\|_\infty} + 1 \right) \|n(t) - n^*\|_{L^1_{s,a}},$$

thus by taking norm in Equality (40) and applying Doeblin's Theorem we obtain

$$\begin{aligned}
\|n(t) - n_*\|_{L^1_{s,a}} &\leq \|P_t(n_0 - n_*)\|_{L^1_{s,a}} + \int_0^t \|P_{t-\tau} h(\tau)\|_{L^1_{s,a}} \, d\tau \\
&\leq \frac{e^{-\lambda t}}{1-\alpha} \|n_0 - n_*\|_{L^1_{s,a}} + \frac{1}{1-\alpha} \int_0^t e^{-\lambda(t-\tau)} \|h(\tau)\|_{L^1_{s,a}} \, d\tau \\
&\leq \frac{e^{-\lambda t}}{1-\alpha} \|n_0 - n_*\|_{L^1_{s,a}} + C \int_0^t e^{-\lambda(t-\tau)} \|n(\tau) - n^*\|_{L^1_{s,a}} \, d\tau,
\end{aligned}$$

with $C := \frac{1}{1-\alpha} \frac{2p_\infty \|\partial_X p\|_\infty}{1-\|\partial_X p\|_\infty}$. By using Gronwall's inequality with respect to the function $e^{\lambda t}\|n(t) - n^*\|_{L^1_{s,a}}$ we conclude

$$\|n(t) - n^*\|_{L^1_{s,a}} \leq \frac{e^{-(\lambda - C)t}}{1-\alpha} \|n_0 - n^*\|_{L^1_{s,a}},$$

so that for $\|\partial_X p\|_\infty$ small enough we have $C < \lambda$ and we deduce the exponential convergence of $n(t,\cdot,\cdot), N(t,\cdot)$ and $X(t)$ when $t \to \infty$. □

6 Numerical simulations

In order to illustrate the theoretical long time results and other possible behaviors of System (1), we present numerical simulations for different firing rates and initial data. The numerical illustrations below are obtained by solving the equation (1) with a classical first-order upwind scheme, while the initial activity $X(0)$ is computed by solving numerically the fixed-point equation (11).

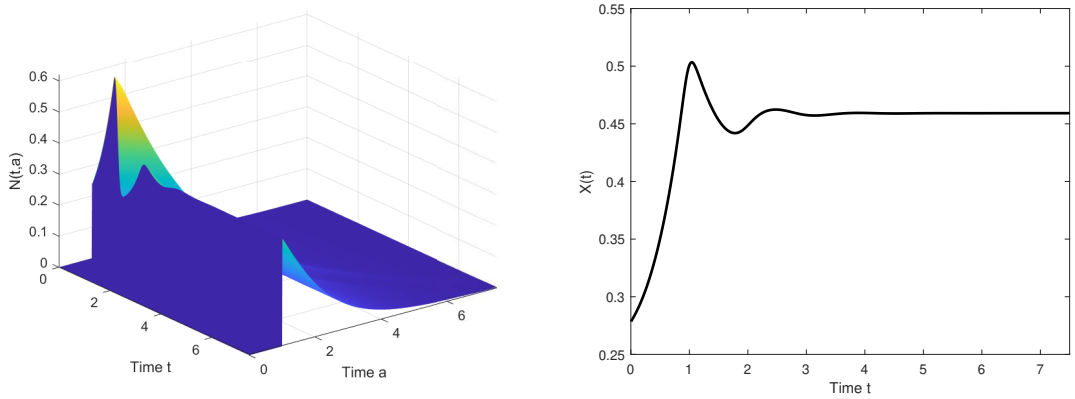
We focus in displaying the discharging flux $N(t,a)$ and the total activity $X(t)$ since these two elements determine the general behavior of system (1).

6.1 Example 1: Convergence to equilibrium

For our first example, we consider a firing rate given by

$$p = \varphi(X - e^{-s} - e^{-a}) \mathbb{1}_{\{s>1\}}, \qquad \varphi(u) = \frac{1.5}{1+e^{-u}},$$

which corresponds to an excitatory case, modulated by a sigmoid function φ with an absolute refractory period $\sigma = 1$. By taking as initial data $n_0(s,a) = e^{-a}$, we observe in Figure 4 that the solution



(a) Discharging Flux $N(t,a)$.

(b) Total Activity $X(t)$.

Figure 4: **Example 1.1.** Case $n_0(s,a) = e^{-a}$ and $p = \varphi(X - e^{-s} - e^{-a})\mathbb{1}_{\{s>1\}}$.

converges to the steady state for both discharging flux N and the total activity X, following the results of Theorem 5. As consequence of the refractory period in the firing rate, we observe in Figure 4a that steady states vanishes for $a < 1$.

From Example 1.1, we can compute a numerical approximation of the stationary flux N^* and the respective total activity X^*. In this setting, consider now as initial data the stationary density conditioned to a spike immediately before $s = 0$, i.e.

$$n_0(s,a) = e^{-a}\delta_0(s)$$

which corresponds to a synchronization of all neurons at $t = 0$. The theoretical results on well-posedness in Section 2 allows us to consider probability measures as initial data and we approximate the Dirac mass as

$$\delta_0(s) \approx \frac{1}{\Delta s}\mathbb{1}_{\{s\in[0,\Delta s]\}}, \qquad \Delta s \text{ small enough.}$$

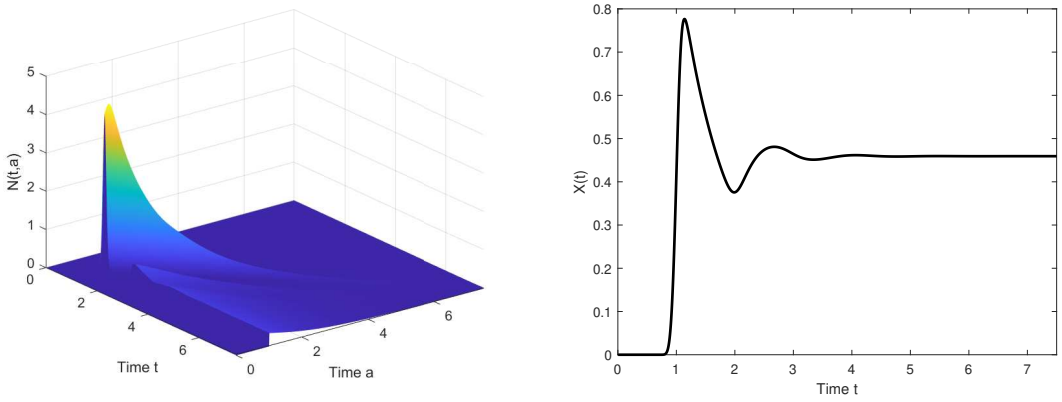
As in Example 1.1, we observe in Figure 5 that the solution converges to the steady state and it stabilizes at similar times. This numerical result is compatible with Theorem (5) extended for initial data in the space of measures $(\mathcal{M}(\mathcal{D}), \|\cdot\|_{M^1})$. Moreover, as effect of the refractory period and the fact that initial data is supported on the line $\{s = 0\}$ of the domain $\mathcal{D}$, the total activity vanishes for small times and then it becomes increasing when the time approaches to $t = 1$.

We continue with some numerical examples beyond the scope of the theory of weak interconnections, in order to explore the possible behaviors of a strongly interconnected regime.

6.2 Example 2: Jump discontinuities

We now consider a compactly supported initial data $n_0(s,a) = 2 \cdot \mathbb{1}_{\{2>a>s+1\}}$, with a firing rate given by

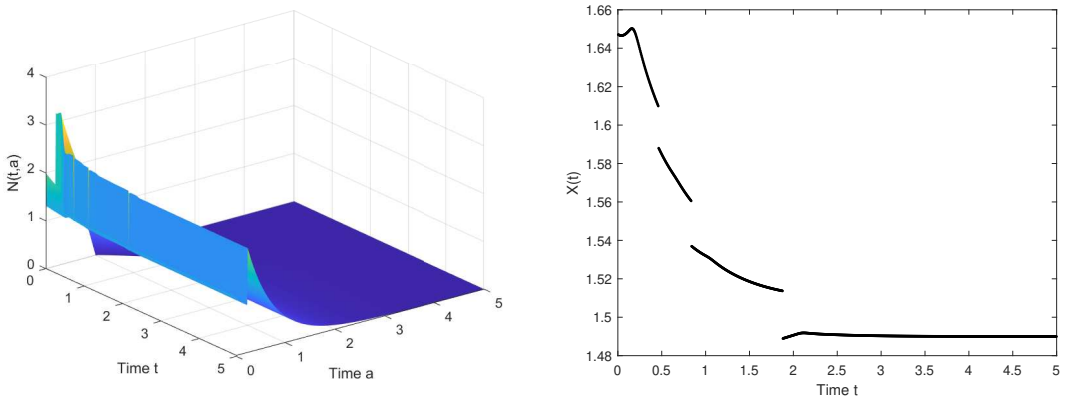
$$p = \mathbb{1}_{\{s>e^{-x}\}} + \mathbb{1}_{\{a-s>e^{-x}\}},$$



(a) Discharging Flux $N(t,a)$. (b) Total Activity $X(t)$.

Figure 5: **Example 1.2.** Case $n_0(s,a) = e^{-a}\delta_0(s)$ and $p = \varphi(X - e^{-s} - e^{-a})\mathbb{1}_{\{s>1\}}$.

which corresponds to an excitatory case since p is increasing with respect to X.



(a) Discharging flux $N(t,a)$. (b) Total Activity $X(t)$.

Figure 6: **Example 2.** Case $n_0(s,a) = 2 \cdot \mathbb{1}_{\{2>a>s+1\}}$ and $p = \mathbb{1}_{\{s>e^{-x}\}} + \mathbb{1}_{\{a-s>e^{-x}\}}$.

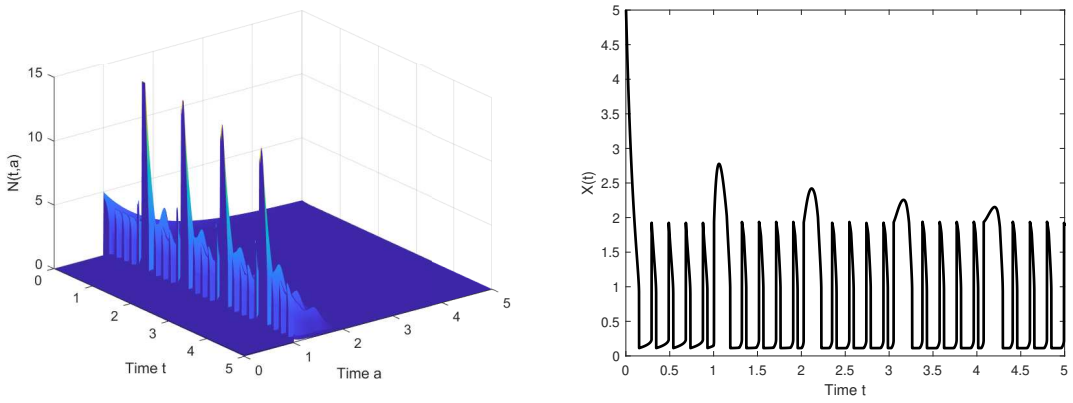
Like the previous examples, the solution converges to the steady state, but the total activity X shows three jump discontinuities as we see in Figure 6. The multiple jump discontinuities are consequence of the contribution of the term depending on the difference between the two elapsed times. Furthermore, converging solutions with a single jump discontinuity were already observed in Torres et al. [45] for the classical elapsed time model. The phenomenon of multiple jumps discontinuities in Figure 6 is an extension for the case of Equation (1).

6.3 Example 3: Periodic solutions and stabilization

Next, we choose initial data $n_0(s,a) = \frac{1}{2}e^{-(a-1)}\mathbb{1}_{\{a>\max(s,1)\}}$ and the firing rate is given by

$$p = \varphi(X)\mathbb{1}_{\{s>1\}}, \qquad \varphi(u) = \frac{10u^2}{u^2+1} + 0.5,$$

which corresponds to an excitatory case since $\varphi'(u) > 0$. Since p does not depend on a, we take advantage by solving the classical elapsed time Equation (5) after integrating with respect to a, as we remarked in the introduction.



(a) Discharging Flux $N(t,a)$.

(b) Total Activity $X(t)$.

Figure 7: **Example 3.1.** Case $n_0(s,a) = \frac{1}{2}e^{-(a-1)}\mathbb{1}_{\{a>\max(s,1)\}}$ and $p = \varphi(X)\mathbb{1}_{\{s>1\}}$.

For these data, both the discharging flux N and the total activity X are asymptotic to a periodic pattern as we see in Figure 7. Similar examples on periodic solutions were found in Torres et al. [45] in the classical elapsed time model for the same type of firing rates. For periodic solutions of Fokker-Planck type models, see [28] and the references therein.

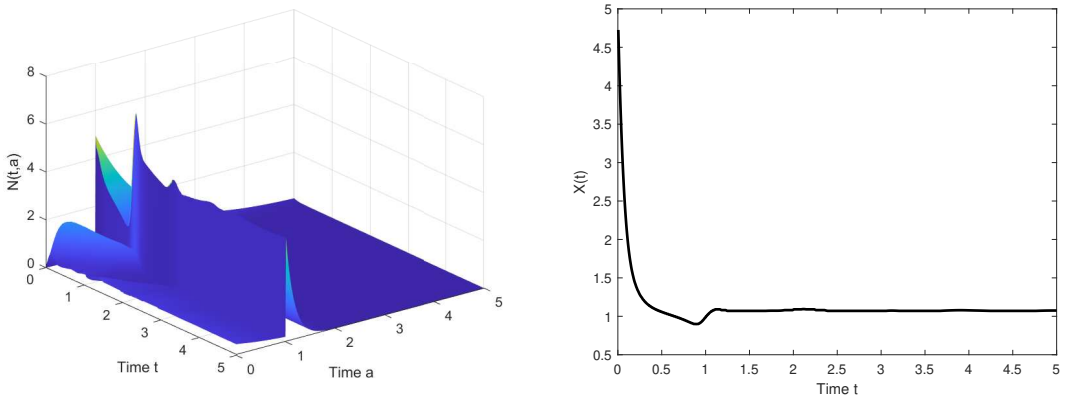
However, when we incorporate the effects of the difference between the elapsed times, the periodic regime changes. For the same initial data and

$$p = \varphi(X)\mathbb{1}_{\{s>1\}} + \mathbb{1}_{\{a-s>X\}},$$

we observe in Figure 8 that the solution of System (1) converges to the steady state. Like the classical model, we can find examples of continuous solutions of System (1) converging to the steady state under a general strongly interconnected regime.

7 Perspectives

By means of Doeblin's theory applied to linear Equation (18) than the classical elapsed time model, we managed to understand the dynamics of System (1) for weak non-linearities by adapting the ideas



(a) Discharging Flux $N(t,a)$. (b) Total Activity $X(t)$.

Figure 8: **Example 3.2.** Case $n_0(s,a) = \frac{1}{2}\, e^{-(a-1)} \mathbb{1}_{\{a>\max(s,1)\}}$ and $p = \varphi(X)\mathbb{1}_{\{s>1\}} + \mathbb{1}_{\{a-s>X\}}$.

of Cañizo et al. [6]. However, aspects such as well-posedness and the asymptotic behavior for strong interconnections are still an open problem as in the classical elapsed time model, where multiple solutions and discontinuities in the total activity are possible.

Concerning the strongly inhibitory case, it remains pending to prove uniqueness of the steady state for the non-linear Equation (1). Whilst in the classical elapsed time model this problem is reduced to solving a simple equation, for the model with two elapsed times we have to prove uniqueness for the integral Equation (35), which is not straightforward to deal with. Moreover, we conjecture in the general case that the speed of convergence to a steady state must be exponential like it is expected for the classical elapsed equation.

With respect to the existence of periodic solutions, an interesting problem is to find or construct non-trivial examples relying on dynamics for two elapsed times. The only examples we have found so far are adaptations of solutions of the classical elapsed time equation that were obtained in Torres et al. [45] and these types of solutions presents jump discontinuities, making them difficult to analyze. Furthermore, it would be interesting to find some class of firing rates for the System (1) leading to delay differential equation with two discrete delays. Following the ideas in Torres et al. [45] for the classical elapsed time model and differential equations with a single delay, we expect to construct specific periodic solutions for the multiple-time renewal equation.

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