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The exact meaning of the angular-momentum and spin operators in quantum mechanics

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Abstract. The theory of angular momentum and spin in quantum mechanics seems to defy common-sense intuition. We render the theory intelligible again by pointing out that this apparent impenetrability merely stems from an *undue* parallel interpretation of the algebraic expressions for the angular-momentum and spin operators in the group representation theory of $SO(3)$ and $SU(2)$. E.g. the correct meaning of the quantum operator $\hat{L}_z = \frac{\hbar}{i} (x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x})$ is not that it is the operator for the z -component L_z of the angular momentum \mathbf{L} , but rather the expression of the operator for the angular momentum \mathbf{L} when it is aligned with the z -axis. Hence what we are used to note (erroneously) as L_z is not a scalar but a vector operator. The same applies *mutatis mutandis* for the spin operators. In the correct interpretation, the whole algebraic formalism is just the group representation theory for the rotations of three-dimensional Euclidean geometry. It is thus mere, elementary high-school mathematics (in a less usual, more technical guise) and as such totally exempt of any physics, let alone quantum mysteries. The change of interpretation has no impact on the algebraic results, such that they remain in agreement with experimental data and the algebra quantum mechanics remains correct. It is all only a matter of the correct geometrical meaning of the algebra. All these statements are proved within the framework of the group representation theory for $SO(3)$ and $SU(2)$ which is the basic tool used to describe rotational motion in quantum mechanics.

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1 Introduction

Quantum mechanics (QM) deduces the expressions for the angular-momentum operators from de Broglie's ansatz for a wave function¹:

$$\psi(\mathbf{r}, t) = e^{-\frac{i}{\hbar}(Et - \mathbf{p} \cdot \mathbf{r})}, \quad (1)$$

and then using the substitutions:

$$\hat{E} = -\frac{\hbar}{i} \frac{\partial}{\partial t}, \quad \hat{\mathbf{p}} = \frac{\hbar}{i} \nabla. \quad (2)$$

This follows the same philosophy as Schrödinger used to derive the Schrödinger equation. By operating $\hat{\mathbf{L}} = \hat{\mathbf{r}} \wedge \hat{\mathbf{p}} = \mathbf{r} \wedge \frac{\hbar}{i} \nabla$, with components:

$$\hat{L}_z = \frac{\hbar}{i} \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] \text{ (cycl)}, \quad (3)$$

¹ Throughout the article the symbol \hat{A} will be used to design an operator. In Section 1 this will be a quantum operator which corresponds to the classical quantity A . But from Subsection 2.2 on we will use the symbol also to design tangent vectors to a curved group manifold (“infinitesimal generators”). Ultimately we will show that the two operator concepts coincide.

on ψ one obtains then indeed $\hat{\mathbf{L}}\psi = \mathbf{L}\psi$, where $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$. The notation (cycl) indicates cyclic permutation. From this, we can derive the commutation relations:

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z \text{ (cycl)}, \quad (4)$$

as a straightforward calculation shows. This entails stunning, highly counter-intuitive results, which render the subject of angular momentum in traditional QM completely mysterious and conceptually inscrutable. It tells us e.g. that the values of L_x and L_y cannot be defined simultaneously because according to Eq. 4 their operators do not commute.² Following the same methods, textbooks also show that:

$$\hat{\mathbf{L}}^2\psi = \hbar^2\ell(\ell + 1)\psi, \quad (5)$$

where ℓ is a quantum number. This is also confusing because, based on the methodology used, one may think that $\hat{\mathbf{L}}^2$ is the operator for the square L^2 . This would then seem to imply that in QM a square (i.e. L^2) is not a square (i.e. $\hbar^2\ell(\ell + 1)$). We will explain below how this stunning result is only due to a wrong interpretation of the algebra. In textbooks this problem is just swept under the carpet by introducing *ad hoc* the concept of expectation value. The expectation value for L^2 would be $\hbar^2\ell(\ell + 1)$.

This becomes all the more perplexing once one has noticed that the quantum operators are the spitting image of the operators we encounter in the Lie algebra for the group representation theory of SU(2) or SO(3). The only difference resides in a multiplicative pre-factor. Now, rotations are just part of Euclidean geometry. The topic does not contain any physics, let alone “quantum magic”. How can it possibly be true that in the group representation theory of SO(3), a square would not be a square, while *by its very definition* a rotation is an isometry and therefore preserves squares. How come that it is possible to draw from the Lie algebra of the rotation group the conclusion that it is impossible to define L_x and L_y , two components of a vector, simultaneously?

The solution for these conceptual difficulties is that Schrödinger’s heuristics based on Eq. 1 and 2 are based on an intuition that there must be a correspondence between operators for physical quantities and the physical quantities themselves. As we will explain below, that intuition was wrong, because the correspondence proposed also gives rise to results that are completely bogus.

In other words, despite the fact that these two equations have stood the test of comparing them with experimental data with flying colours such that their validity is beyond questioning, the way they have been discovered must be qualified as a fluke because they are correct results obtained by using misleading flawed logic.

In fact, the true derivation of operators must be based on group theory as explained in [2] for Eq. 2 and below for Eqs. 3-5. The lack of reliability of Schrödinger’s heuristic approach based on the intuition of a correspondence was already pointed out by Messiah (see [3], p.70), who illustrated with an example that the derivations based on Eq. 1 could lead to ambiguous results. He suggested that we can obtain the correct result by trial and error. But this is conceptually unsatisfactory. We cannot seriously pretend to build a theory on such methods. The origin of the problem is of course that the algebra for operators is different from the algebra for the quantities they intend to be the operators for. With products of operators we can this way run into problems.

Following Messiah, the conclusion we might have drawn from Eq. 5 is actually that in our method of trial and error we failed because $\hat{\mathbf{L}}^2$ does not yield a square (as in fact, $\widehat{[\mathbf{L}^2]} \neq [\hat{\mathbf{L}}]^2$, due to the difference between the algebras for operators and for physical quantities already mentioned). The true solution of the conceptual problems evoked above will consist in showing that we can avoid them by deriving the operators from group theory rather than a prescription based on correspondence according to Schrödinger’s intuition, which must be abandoned. Using the group theory the author has even been able to derive the Dirac equation deductively rather than inductively (as Schrödinger and Dirac did). Many other puzzles of QM are solved by using the correct geometrical meaning of the algebra.

2 Group representation theory restores the conceptual clarity

2.1 Context and caveats

We will render the theory of angular momentum in QM intuitive again. In fact, all the conceptual difficulties with angular momentum in QM are a consequence of a completely undue parallel interpretation of the algebra of the group theory of SO(3) and SU(2).

These remarks must be seen within the context of a larger framework. In [2] we have reconstructed QM from scratch. We were able to derive the Dirac equation by only using the representation theories for the rotation group

² It is this mystifying result which has clearly been Einstein’s motivation for formulating the EPR paradox [1] by arguing how in spite of the commutation relations L_x and L_y could be measured simultaneously by doing it on two identical particles. Our results will dissipate this debate by showing how it is wrong-footed by a misinterpretation of the meaning of the operators.

and the homogeneous Lorentz group. The starting point for this derivation is to express that the electron is spinning, by making the substitution $\varphi = \omega_0\tau$ in the Rodrigues formula in Eq. 7 below. Here τ is the proper time. Introducing such a time dependence to describe rotational motion is completely analogous to describing displacements along an orbit by introducing a time dependence $\mathbf{r}(t)$ of the position in classical mechanics. The whole further derivation is carried out with the rigour of a mathematical proof and does not require the introduction of supplementary stunning assumptions as one might expect based on the fact that QM is full of mysteries which the theory has to account for. As the Schrödinger and Pauli equations can be derived from the Dirac equation, this derivation from scratch offers a broad platform from which we can start to study the foundations of QM. We obtain this way a better understanding of what the calculus of QM physically means. To be very clear and avoid all possible confusion, the whole work is derived from the basic postulate that QM must be entirely correct as far as its algebra and agreement with experiment is concerned, but that it is the parallel interpretation given to this algebra that is wrong and the origin of our puzzlement. Anything else would amount to attacking QM and this would be preposterous because QM reproduces perfectly all known experimental data.

The parallel interpretation given to the algebra in traditional QM is often at variance with its correct geometrical meaning, which is already provided by the group theory itself in a completely natural way. Mathematics are not subject to interpretation, because they are self-explanatory. The undue over-interpretations are responsible for many of the conceptual difficulties we encounter in QM. In our new approach which uses the correct geometrical meaning of the algebra, a lot of the counterintuitive interpretations we qualify as quantum mysteries disappear, while the agreement of the algebra with the experimental results remains rigorously preserved. Our alternative approach cannot be criticized on the basis of its differences with the traditional approach, because these differences do not occur in the theoretical predictions of the experimental data. There is a thorough discussion of how our approach preserves the algebra and only differs from the standard approach in using the correct geometrical meaning of that algebra in [2]. It is in this context of correcting for the over-interpretations in the traditional theory that the present paper must be situated.

It is with the harvest of these results in store that we feel entitled to ask the reader to be tolerant and prepared to admit that whenever he/she thinks that the present approach can only be wrong, it could be his/her own viewpoint that might be wrong because it is based on over-interpretations which are absent from our approach, which has been outlined in [2]. After getting over his/her initial nervousness the reader will find out that the conclusions we reach are unassailable. There is actually nothing to get nervous about as the algebra and the theoretical predictions are preserved. Only our understanding of the formalism will be improved.

In fact, some of our statements may disturb the reader, because they fly in the face of what he/she has learned such that his/her first reaction might be one of fierce resistance. In order to avoid misunderstandings and futile polemics, reference [2] should therefore be consulted before reading the present paper. This advice should not be taken lightly: the contents of [2] are beyond guessing. With the correct interpretation of the operators, the counter-intuitive textbook results about the spin and angular-momentum operators just disappear.

The results reported in the present paper are only the illustration on a specific case of what is a recurrent theme in QM. There is a long laundry list of similar unpleasant truths about other over-interpretations [2]. To the readers who are upset, I suggest to travel mentally backwards in time and try to remember their bewilderment when they were agonizing over the meaning of these angular-momentum and spin operators in their tender years. Perhaps the flashbacks will revive some long-forgotten repressed feelings of alienation and frustration. The pain the reader may have experienced in trying to get his/her head around the subject matter is actually telltale of a cognitive dissonance provoked by a brainwash. It is from this brainwash and its “die-hard certainties” that we must free ourselves by deconstructing it [2].

2.2 Infinitesimal generators

2.2.1 Definition

Let G be a Lie group. Let us note a representation matrix $\mathbf{D}(g)$ of a group element $g \in G$ as $\mathbf{D}(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2 \cdots \lambda_{\dim(G)}) \in \mathbb{R}^{\dim(G)}$ is a set of independent real group parameters which define g . Here $\dim(G)$ is the dimension of the group, which is of course different from the dimension of the representation, which is the rank of the matrix $\mathbf{D}(\boldsymbol{\lambda})$. E.g. in $SU(2)$ the dimension of the representation is 2, while the dimension of the group is 3. These two dimensions are in general still different from the dimension of the vector space the transformations might be acting on. E.g. the homogeneous Lorentz group is six-dimensional, it acts on four-dimensional space-time, and the dimension of the representation $SL(2, \mathbb{C})$ is two. In the group $SO(3)$, the three different types of dimensions all take the same value three, such that we may think that there is only one concept of dimension. The resulting absence of disambiguation can stir confusion in one’s first contact with the group theory.

In Lie groups G one uses so-called “infinitesimal generators”. Textbooks [4,5] explain then the following. We consider a neighbourhood of the identity element $\mathbf{1}$. In this neighbourhood, all parameters λ_j are small. We now consider the group elements, whereby only one parameter λ_j is allowed to vary and to be different from zero. We let

λ_j vary between 0 and 1. The matrices $\mathbf{D}(0, 0, \dots, 0, \lambda_j, 0, \dots, 0, 0)$ will then constitute one-parameter sets of group elements. They will describe a one-dimensional curve on the group manifold. An infinitesimal generator is then defined as:

$$\hat{\mathbf{D}}_j = \iota \left[\frac{\partial}{\partial \lambda_j} \mathbf{D}(0, 0, \dots, 0, \lambda_j, 0, \dots, 0, 0) \right]_{\lambda_j=0}. \quad (6)$$

The infinitesimal generators belong to the tangent space to the group manifold at the identity element. They are thus tangent vectors. The aim of defining the infinitesimal generators and the Lie algebra is to construct a basis for the tangent space to the manifold of the Lie group. The elements of the Lie algebra, the tangent vectors, are thus objects that are completely different from the elements of the Lie group which are group elements. As they belong to the tangent space rather than to the group itself, “infinitesimal generators” are not generators of the Lie group. (They are generators of the tangent space). Furthermore, the matrices $\mathbf{D}(0, 0, \dots, 0, d\lambda_j, 0, \dots, 0, 0) - \mathbb{1}$ are infinitesimal, but the quantities defined by Eq. 6 are not. The “infinitesimal generators” are thus also not infinitesimal. These infinitesimal generators could also be calculated at another point of the Lie group, but choosing the identity element enhances the simplicity of the expressions obtained.

Within a broader context, we must consider Eq. 6 as an example, not a definition. In fact, it is in general not spelled out in textbooks which propose Eq. 6 that the infinitesimal generators are vectors of the tangent space to a given point of the Lie group, and that the aim is to obtain a complete set of basis vectors for the tangent space in that point. The examples do not explain how to define an appropriate choice for the one-parameter families in order to obtain such a complete basis. One only discovers this difficulty when one gets stuck in trying to work out an example, e.g. for $SU(3)$, for which textbooks only give the final result in the form of the Gell-Mann matrices, rather than their detailed derivation based on Eq. 6. We also run into such difficulties when we try to find a complete orthogonal basis of tangent vectors to $SU(2)$ by starting from the expression $\mathbf{R}(\alpha, \beta, \gamma)$ for a rotation in terms of its Euler angles (α, β, γ) . With a complete basis we can carry out calculations in the tangent space with the Lie algebra. We can thus make our calculations in the Lie group (Schrödinger, Dirac) or in the Lie algebra (Heisenberg).

2.2.2 Note for the sake of mathematical rigour

The method to define tangent vectors, tangent space and the tangent bundle outlined in this subsection is “quick and dirty” from the viewpoint of differential geometry. The non-abelian Lie group G is a manifold M rather than a vector space and linear combinations of group elements are not defined. All the group axioms define are the products of group elements. Therefore the algebra we carry out on the representation matrices can be translated back to the group elements for matrix multiplication, but not for linear combinations of representation matrices. Consequently, the method to define a tangent vector to a manifold M used in Eq. 6 becomes non-sense if we want to define that tangent vector *intrinsically*. The procedure described in Subsection 2.2.1 only makes sense if we consider the manifold M as embedded in a vector space V . In the intrinsic description of a manifold M , the points of $V \setminus M$ are not defined.

A nice example of this is curved space-time in general relativity. We can embed the curved space-time manifold in a five-dimensional vector space \mathbb{R}^5 just like we can embed a two-dimensional sphere in the three-dimensional vector space \mathbb{R}^3 but the points of \mathbb{R}^5 that do not belong to space-time are physically meaningless. This is the reason why general relativity has been formulated intrinsically. Similarly, in a non-intrinsic approach to the sphere, we can embed the sphere in \mathbb{R}^3 but the points of \mathbb{R}^3 that do not belong to the sphere are also not open to observation by imaginary inhabitants of the sphere. This is also why we have always stressed that spinors do not build a vector space [6, 7], a notion that shocks the many physicists who are unaware of these fundamental mathematical issues.

Without the additional structure of a vector space V provided by an embedding, the operations of subtracting and dividing in the expression $\lim_{\lambda_j \rightarrow 0} [(g(\lambda_j) - g(0))/\lambda_j]$ that would correspond by isomorphism to the algebra which intervenes in the calculation of the derivative in Eq. 6 are just not defined for the group elements $g(\lambda_j)$ and $g(0)$. The calculations can only be made meaningful by providing an embedding of the group G within a vector space V which provides then the additional operations. Within an embedding, it is easy to visualize a tangent vector and a tangent space, and it corresponds then to the intuitive idea behind the definition used in Eq. 6. However, this intuitive approach illustrates also very clearly that except for the tangent point, the points of the tangent space do not belong to the manifold, such that the definition no longer makes sense in the intrinsic approach. The intrinsic approach teaches us also that two tangent spaces which may intersect in the embedding do not really intersect intrinsically. Consequently, correctly defining tangent vectors and tangent spaces becomes much more technical and abstract in the intrinsic approach, where the definitions must be justified very differently as e.g. explained in the lectures of Schuller [8, 9, 10]. Of course, this abstraction does not help in rendering the concepts used more physically intuitive, which is why we have adopted nevertheless the approach based on embedding in this paper. In fact, the aim of the construction in the intrinsic approach is that in the end its result must agree with the result based on the approach based on embedding, and it has been proved that this is indeed the case. That is why we can qualify Eq. 6 as “quick and dirty”. It provides the correct result, but the procedure is not intrinsic as it ideally ought to be.

2.3 The Rodrigues formula in SU(2)

The Rodrigues formula (see [2], p.11) for the SU(2) representation matrix $\mathbf{R}(\mathbf{s}, \varphi)$ of a rotation by an angle φ around an axis d defined by the unit vector $\mathbf{s} \parallel d$ is:

$$\mathbf{R}(\mathbf{s}, \varphi) = \cos(\varphi/2) \mathbf{1} - \iota \sin(\varphi/2) [\mathbf{s} \cdot \boldsymbol{\sigma}]. \quad (7)$$

Here $\mathbf{1}$ is the 2×2 unit matrix, and σ_j are the Pauli matrices:

$$\sigma_x = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} & -\iota \\ \iota & \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \quad (8)$$

We will use j as a general notation for the indices x, y, z . The term $\mathbf{s} \cdot \boldsymbol{\sigma}$ is not a true scalar product but a purely formal notation for $s_x \sigma_x + s_y \sigma_y + s_z \sigma_z$ (see [2,6]). The matrices σ_j represent the unit vectors \mathbf{e}_j and $\mathbf{s} \cdot \boldsymbol{\sigma}$ the vector \mathbf{s} . Hence $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is not a vector but represents the triad of basis vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. The Pauli matrices σ_j do not correspond to components of vectors but to vectors in their own right.³

2.4 Infinitesimal generators for SU(2)

Let us consider the one-parameter family of rotations over an angle φ around a fixed axis $d \parallel \mathbf{s}$. This is a one-parameter family because only φ is allowed to vary. By applying the definition given in Eq. 6 we obtain the infinitesimal generator:

$$\hat{\mathbf{R}}_{\mathbf{s}} = \iota \left[\frac{\partial}{\partial \varphi} \mathbf{R}(\mathbf{s}, \varphi) \right]_{\varphi=0} = \frac{1}{2} [\mathbf{s} \cdot \boldsymbol{\sigma}], \quad (9)$$

For $\mathbf{s} = \mathbf{e}_x$ one obtains thus $\hat{\mathbf{R}}_x = \sigma_x/2$. In general, for $\mathbf{s} = \mathbf{e}_j$ one obtains $\hat{\mathbf{R}}_j = \sigma_j/2$.

Important conclusion: The result $\hat{\mathbf{R}}_x = \sigma_x/2$ is not an operator for the x -component of a general rotation, but the infinitesimal generator for a rotation when its rotation axis is the x -axis, because we have derived all this by considering the one-parameter family of rotations around the x -axis (or in general the axis $d \parallel \mathbf{s}$). The same is true *mutatis mutandis* for $\hat{\mathbf{R}}_y = \sigma_y/2$ and $\hat{\mathbf{R}}_z = \sigma_z/2$. Furthermore:

$$\begin{aligned} \hat{\mathbf{R}}_x \hat{\mathbf{R}}_y - \hat{\mathbf{R}}_y \hat{\mathbf{R}}_x &= \frac{1}{4} (\sigma_x \sigma_y - \sigma_y \sigma_x) = \frac{\iota}{2} \sigma_z \\ &= \iota \hat{\mathbf{R}}_z \text{ (cycl)}. \end{aligned} \quad (10)$$

Any set of operators that satisfies these commutation relations can be used to represent the Lie algebra. Examples are the the spin operators and the angular-momentum operators. In fact, by defining $\hat{S}_j = \hbar \hat{\mathbf{R}}_j = \frac{\hbar}{2} \sigma_j$. we can transform $\frac{1}{4} (\sigma_x \sigma_y - \sigma_y \sigma_x) = \frac{\iota}{2} \sigma_z$ into the commutation relations for the spin operators $[\hat{S}_x, \hat{S}_y] = \frac{i\hbar}{2} \hat{S}_z$. By putting $\hat{L}_j = \hbar \hat{\mathbf{R}}_j$ we can transform $\hat{\mathbf{R}}_x \hat{\mathbf{R}}_y - \hat{\mathbf{R}}_y \hat{\mathbf{R}}_x = \iota \hat{\mathbf{R}}_z$, into the commutation relations for the angular-momentum operators $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$. Both commutation relations are thus realizations of the commutation relation $[\hat{\mathbf{R}}_x, \hat{\mathbf{R}}_y] = \iota \hat{\mathbf{R}}_z$. In other words \hat{L}_j and \hat{S}_j are just two valid disguises we can choose for the infinitesimal generators of the rotation group. We can do our Lie algebra with both choices. The quantities \hbar and $\hbar/2$ are *ad hoc* add-ons dictated by the physics [2].

It may strike the reader that we are using boldface characters to note the infinitesimal generators as $\hat{\mathbf{R}}_j$ while we are noting the related quantum operators as \hat{L}_j and \hat{S}_j . The inconsistency in the notation is due to our choice to preserve the traditional notations from QM, which - as we now have shown - are based on a wrong interpretation of vector operators in terms of scalar operators.

³ As explained in [2], the Pauli matrices occur with three different meanings in the theory of SU(2). (1) First of all, the rotations of \mathbb{R}^3 can be generated by reflections. Those reflections can be defined by unit vectors perpendicular to their reflection planes. The Pauli matrices are representations of the three reflections defined by the normals $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. (2) They also represent the unit vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ of \mathbb{R}^3 . This is the case in the Rodrigues equation. (3) Finally they constitute an orthonormal basis for the tangent space to the group manifold.

2.5 Infinitesimal generators for SO(3)

One can actually show directly that Eq. 3 is an infinitesimal generator for the rotation group in the following way given in [11]. We consider a function $\psi : (x, y, z) \in \mathbb{R}^3 \rightarrow \psi(x, y, z) \in \mathbb{K}$, where \mathbb{K} can be \mathbb{R} or \mathbb{C} . We suppose that under a rotation R we transform $\psi(x, y, z)$ to $\psi'(x', y', z')$ and $\mathbf{r} = (x, y, z)$ to $\mathbf{r}' = (x', y', z')$. We have then:

$$\psi'(\mathbf{r}') = \psi(R^{-1}(\mathbf{r}')). \quad (11)$$

From now on we will write $\mathbf{r}' = (x', y', z')$ as $\mathbf{r} = (x, y, z)$ For an infinitesimal rotation $R(d\varphi)$ over an angle $d\varphi$ around the z -axis we have:

$$[R(d\varphi)]^{-1}(x, y, z) = (x + y d\varphi, y - x d\varphi, z), \quad (12)$$

which just follows from the Taylor series expansion to the first order of the rotation matrix $\mathbf{R}(d\varphi)$:

$$\begin{bmatrix} \cos(d\varphi) & -\sin(d\varphi) \\ \sin(d\varphi) & \cos(d\varphi) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -d\varphi \\ d\varphi & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad (13)$$

where of course $[\mathbf{R}(d\varphi)]^{-1} = \mathbf{R}(-d\varphi)$. Therefore to first order in $d\varphi$:

$$\begin{aligned} \psi'(x, y, z) &= \psi(x + y d\varphi, y - x d\varphi, z) \\ &= \psi(x, y, z) - id\varphi \cdot \frac{1}{i} \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] \psi(x, y, z). \end{aligned} \quad (14)$$

Now $\psi' = R(d\varphi)\psi$ and $\psi = R(0)\psi$, where $R(0) = \mathbb{1}$ is the identity element, such that:

$$i[R(d\varphi) - R(0)]\psi = d\varphi \cdot \frac{1}{i} \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] R(0)\psi, \quad (15)$$

which is true for all functions ψ . Here ψ can e.g. be a harmonic polynomial. Hence in terms of operators acting on those functions ψ we have:

$$i \left[\frac{\partial R}{\partial \varphi} \right]_{\varphi=0} = \frac{1}{i} \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right], \quad (16)$$

such that (up to the factor \hbar) the operator \hat{L}_z is the infinitesimal generator associated with the one-parameter family of rotations R around the z -axis. It corresponds therefore to the angular-momentum vector rather than to one of its scalar components. This conclusion is inevitable and nevertheless its lacklustre consequence, viz. that traditional QM uses a wrong interpretation of \hat{L}_z , has tantalizingly been overlooked. Rather than the textbook “derivations” based on the de Broglie ansatz, it is this calculation that is correct, because the geometrical meaning of the algebraic formalism is firmly established prior to any use of these mathematics in physics. It proves that \hat{L}_j are infinitesimal generators, which we can call angular momentum operators after incorporating \hbar . An alternative proof is given by the verification that the expressions for \hat{L}_j given in Eq. 3 satisfy the commutation relations for the rotation group. The parameters (x, y, z) in the expressions for \hat{L}_j do not need to be position coordinates, because the algebra that intervenes in checking Eq. 4 for the operators defined by Eq. 3 can be carried out abstractly without specifying what (x, y, z) mean (see Eq. 17).

3 Discussion

As the infinitesimal generators $\hat{\mathbf{R}}_j$ form a basis for tangent space, both \hat{S}_j and \hat{L}_j form a basis for tangent space. The only difference is that infinitesimal operators are unit vectors, while the quantum operators are not of unit length. Whatever the meaning one gives to (x, y, z) the fact that \hat{L}_j satisfy the commutation relations is sufficient. The commutation relations for the angular-momentum operators are just defined by the group theory, nothing else.

We have thus shown that the quantum operators are not the operators for the components of the angular momentum or spin, but the operators for the angular momentum or spin when it is aligned with a given axis. The algebra of the angular-momentum and spin operators is therefore not mysterious. It is just the algebra of rotations in Euclidean geometry. All quantum mysteries surrounding angular momentum and spin in QM are only due to wrong interpretations of this algebra. This illustrates our thesis upheld in [2] that group theory is the key to making sense of spinor wave functions in QM.

If the interpretation of \hat{L}_j truly would be that it is the operator for the component L_j of the angular momentum, then this would be in flagrant contradiction with the group theory, which is just based on Euclidean geometry. Scalar quantities can only commute. In Euclidean geometry, nothing impedes L_x and L_y to exist simultaneously. This shows that the traditional QM interpretation must be abandoned. The correct interpretation for $\hat{L}_z = \frac{\hbar}{i} \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right]$ in Eq. 3 is the one that follows from the group theory: It is the operator for the angular momentum when it is aligned with the z -axis, with the analogous interpretations for \hat{L}_x and \hat{L}_y following by cyclic permutation.

We must therefore conclude that the derivation of the operators \hat{L}_j from Eq. 1 accidentally yields correct expressions, but with a wrong interpretation for them. We may note that the phase of ψ in Eq. 1 can be obtained from the time part $\tau = \gamma(t - \mathbf{v} \cdot \mathbf{r}/c^2)$ of the Lorentz transformation by multiplying it by $m_0 c^2/\hbar$ (see [2], p.36). The Dirac equation is derived by putting $\varphi = \omega_0 \tau$ in the Rodriguez formula, leading to a spinor that contains the factor $e^{-i\omega_0 \tau/2}$. The de Broglie wave is obtained from this by putting $m_0 c^2 = \hbar \omega_0/2$. Here τ is the proper time, ω_0 the angular frequency of the spinning motion and m_0 the rest mass. Like a Bloch wave $e^{-i\mathbf{k} \cdot \mathbf{r}}$ in solid-state physics, $\psi(\mathbf{r}, t) = e^{-i(\omega/c, \mathbf{k}) * (ct, \mathbf{r})}$ is just a one-dimensional representation expressing translational invariance along one dimension. Here $*$ is the scalar product in space-time. We are expressing here not translation symmetry in space, but in time because we are describing the electron's spinning motion in proper time. This wave function is defined for just one value of \mathbf{v} . If we had taken $\mathbf{v} \parallel \mathbf{e}_x$, we would just have written $\psi(x, y, z, t) = e^{-\frac{i}{\hbar}(Et - p_x x)}$. Obviously, one cannot derive \hat{L}_j from this expression by Schrödinger's method. Furthermore, the abelian group $\text{SO}(1,1)$ of boosts along a given direction does not contain $\text{SO}(3)$, which is non-abelian. It is therefore not possible that some commutation relations for $\text{SO}(3) \subset \text{SO}(1,3)$ could be derived from a development that belongs strictly to a context of $\text{SO}(1,1)$, even if this group can be embedded in $\text{SO}(1,3)$. Surely enough, the non-abelian group $\text{SO}(1,3)$ contains $\text{SO}(3)$ but its representations are not one-dimensional. The algebra $\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z$ used in Eq. 1 does not apply to the composition of boosts in $\text{SO}(1,3)$. It just corresponds to the decomposition of one specific boost vector in a basis. Therefore, we cannot use the one-dimensional representation ψ of the abelian group $\text{SO}(1,1)$ within the full extended context of the representations of $\text{SO}(1,3)$ wherein the commutation relations of $\text{SO}(3) \subset \text{SO}(1,3)$ become part of the theory.

All these group-theoretical observations force us to the conclusion that despite it's success in deriving the Schrödinger equation, the intuition that the algebra of the operators could be derived from the algebra of the classical observables by substitutions is not correct. We must therefore admit that Schrödinger obtained a correct equation by logically flawed methods. In fact, in [2] the Dirac equation is not derived by using the rules in Eq. 2. It is the other way around (see [2], p. 35) and the rules are therefore a consequence of the group theory, not of some cleverly guessed substitution principle. If the intuition used to formulate the substitution principle were justified, it would be generally applicable. It is Schrödinger's apparent success that has produced the confusion about the physical meaning of the angular-momentum operators, because it led naturally to the false expectation that one could generalize Schrödinger's methods to the definition of angular momentum operators. As we can see now, this is not the case. What further may have contributed to the confusion is that the operators obtained by these methods are perfectly useful quantities that indeed occur in the algebra of QM. Everything in this algebra is fine, such that also its comparison with experimental data is fine. It is only the meaning attributed to this algebra, i.e. its interpretation, which is wrong. We can conclude that Schrödinger's approach presented in textbooks must be considered as a purely historical account of how he discovered his equation. But the logically correct development of QM, which we ought to present to our students without confusing them, should be built on deriving the equations from group representation theory, as we have done for the Dirac equation in [2].

Correct proofs that \hat{L}_j are good angular momentum operators for $\text{SO}(3)$ have been given above. The operators are defined in the Lie algebra. The one-parameter families can be parameterized by elements of \mathbb{R}^3 or \mathbb{R}^4 , like $(\gamma, \gamma \mathbf{v}/c) \propto (E, \mathbf{c}\mathbf{p}) = m_0 c^2 (\gamma, \gamma \mathbf{v}/c)$ for the Lorentz boosts.

The culprit for this embarrassing situation is the sobering fact that the whole formalism of QM has not been derived but obtained by educated guessing and then validating the guesses by comparing the results of the algebra with experimental data. The algebra itself has been used as a blackbox in the sense that the calculations have been carried out in total unawareness of the insight that could be gained from figuring out what the geometry is that corresponds to the algebra. The experimental data validate the algebra, but not its interpretation. Consequently the algebra can be right, while its interpretation is flawed. The bottom line is that we should not worry about the validity of QM, but that we must revise its interpretation. The bonus of this revision is that some very hard conceptual puzzles, considered hitherto as impenetrable quantum mysteries, can be solved.

In the present case it is even not possible to check certain details of the algebra because one cannot consider simultaneously several operators \hat{L}_j . Considering $\hat{L}_x, \hat{L}_y, \hat{L}_z$ simultaneously is taboo in traditional QM because the operators do not commute, while in our approach the operators \hat{L}_j are just different forms which the operator $\hat{\mathbf{L}}$ takes on various disjoint definition domains. For each direction in space defined by an axis d , there is a different form of the operator $\hat{\mathbf{L}}$ with a different definition domain, which is the one-parameter family of rotations around d .

Our derivation of QM from scratch in [2] is not based on guesses. It is a mathematically rigorous derivation. Deriving the Dirac equation from group theory would not have been feasible without sticking to the firm belief that

the correctness of the algebra of QM is unassailable and that any problem that might exist can only reside in its interpretation. That is then the incentive for trying to figure out the geometry behind the algebra of the group theory. In this reconstruction of QM many quantum mysteries just disappeared. These are the features that have kept us on the right track. The quantum mysteries are due to overhauling the natural geometrical meaning of the algebra of the group theory with guesstimate, parallel interpretations, based on intuition. Of course, this verdict may sound harsh or unfair because it is always easy to criticize an achievement from an *a posteriori* perspective. Schrödinger could not know that the formalism he was developing would turn out to introduce the language of group representation theory. But in any case we can now see with hindsight that providing interpretations for the formalism as is done in traditional presentations is not in order because the meaning of the algebra is already determined by the group theory itself before one applies it to physics

We can now also see that it is not true that the components L_x and L_y of the angular momentum cannot be defined simultaneously due to the fact that their operators do not commute. All the commutation relation confirms is that when the angular momentum is aligned with the x -axis, it cannot simultaneously be aligned with the y -axis! When the angular momentum \mathbf{L} is aligned with the x -axis, we must use the operator \hat{L}_x and we have $\mathbf{L} = (L, 0, 0)$. The use of the operator \hat{L}_x is the only one valid for this situation $\mathbf{L} = (L, 0, 0)$. We have then $L_x = L$ and $[\hat{\mathbf{L}}^2] = \hat{L}_x^2$. Contrary to what the standard interpretation of QM claims, the components $L_x = L$, $L_y = 0$ and $L_z = 0$ are then defined simultaneously. The same applies, *mutatis mutandis*, for the cases $\mathbf{L} = L\mathbf{e}_j$.

Just as the correct spin operator for a spin $\frac{\hbar}{2}\mathbf{s}$ would be $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z = \frac{\hbar}{2}[\mathbf{s}\cdot\boldsymbol{\sigma}]$, the correct operator for the angular momentum would be $l_x\hat{L}_x + l_y\hat{L}_y + l_z\hat{L}_z$, where the unit vector $\mathbf{l} = (l_x, l_y, l_z) \parallel \mathbf{L}$. These expressions $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z$ and $l_x\hat{L}_x + l_y\hat{L}_y + l_z\hat{L}_z$ just serve to decompose \mathbf{s} and \mathbf{l} in the bases of their Lie algebras. This is in conformity with the fact that both the sets \hat{S}_j and \hat{L}_j form a basis. It is a fundamental result of differential geometry that such a tangent basis exists such that the decomposition is possible. When the spin is parallel to \mathbf{s} only $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z$ is a meaningful operator. An isolated component, e.g. $s_z\hat{S}_z$ has then no meaning, because the eigenfunction ψ of $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z$ is not an eigenfunction of \hat{S}_z . The definition domains of $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z$ and \hat{S}_z are disjoint. Every direction of space has its own operator. Just as $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is not a single vector operator but a basis of three vector operators, $\hat{\mathbf{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$ is not a single vector operator but a basis of three vector operators.

All this shows that, despite its successes, guessing operators from their action on $e^{-\frac{i}{\hbar}(Et-\mathbf{P}\cdot\mathbf{r})}$ was not a valid procedure, because it leads to the erroneous conclusion that \hat{L}_j would be the operator for the j -component of the angular momentum \mathbf{L} and the confusion that $\hat{L}^2\psi = \hbar^2\ell(\ell+1)\psi$ would define the operator for the square L^2 of the angular momentum. We should never lose sight that SO(3) and SU(2) are mere Euclidean geometry. When one sticks to the correct geometrical interpretation of the algebra of the rotation group the classical intuition is restored. The caveats one must observe in using the operators in the treatment of a physical problem are illustrated in [12].

4 Other possible meanings we can attribute to the parameters (x, y, z)

We have shown (see [2], p. 21), that in SU(2):

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = \begin{bmatrix} \pm\sqrt{\frac{x-iy}{2}} \\ \pm\sqrt{\frac{-x-iy}{2}} \end{bmatrix}, \quad (17)$$

represents a spinor corresponding to a rotation \mathbf{R} . Here $(x, y, z) = \mathbf{e}'_x + i\mathbf{e}'_y \in \mathbb{C}^3$, where $(\mathbf{e}'_x, \mathbf{e}'_y, \mathbf{e}'_z)$ is the canonical triad of basis vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of \mathbb{R}^3 after rotation by \mathbf{R} . Here $(x, y, z) \in \mathcal{S} \subset \mathbb{C}^3$ is a so-called isotropic vector, which belongs to the isotropic cone $\mathcal{S} = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 = 0\}$. The idea behind introducing isotropic vectors is that from $\mathbf{e}'_x + i\mathbf{e}'_y$ we can calculate the whole triad \mathbf{e}'_j of rotated basis vectors, and that this triad defines the rotation unambiguously. The coordinates (x, y, z) of the isotropic vector can therefore be used as coordinates for the rotation. They are therefore not position coordinates but true group parameters, in contrast with the parameters (x, y, z) and (p_x, p_y, p_z) in Eq. 1. They function as coordinates for rotations. The identity in Eq. 17 and its derivation lead to the notion that a spinor is a “square root of an isotropic vector”. This is of course really hard to make sense of, which the reason why Michael Atiyah stated that nobody understands spinors. However, we have shown (see [6], [2], p.7 and [13] p.166, Eq. [5.52]) that the spinors used in QM just represent group elements (i.e. rotations and Lorentz transformations). That is the key which really unlocks the door to understanding the meaning of spinors and their use in QM. Let us leave out the \pm signs from the algebra and calculate $\hat{L}_z\boldsymbol{\xi}$:

$$\begin{aligned}
\hat{L}_z \boldsymbol{\xi} &= \frac{\hbar}{i} \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] \begin{bmatrix} \sqrt{\frac{x-iy}{2}} \\ \sqrt{\frac{-x-iy}{2}} \end{bmatrix} = \frac{\hbar}{2i} \begin{bmatrix} \frac{-ix-y}{2\sqrt{\frac{x-iy}{2}}} \\ \frac{-ix+y}{2\sqrt{\frac{-x-iy}{2}}} \end{bmatrix} \\
&= \frac{\hbar}{2i} \begin{bmatrix} \frac{-i(x-iy)/2}{\sqrt{\frac{x-iy}{2}}} \\ \frac{i(-x-iy)/2}{\sqrt{\frac{-x-iy}{2}}} \end{bmatrix} = -\frac{\hbar}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \boldsymbol{\xi} = -\frac{\hbar}{2} \sigma_z \boldsymbol{\xi} = -\hat{S}_z \boldsymbol{\xi}.
\end{aligned} \tag{18}$$

Here the rotation coordinates (x, y, z) of the isotropic vector have nothing to do with the position coordinates in Eq. 1, but we can imagine to generalize the definition domain of the algebra by analytic continuation from the isotropic cone \mathcal{S} to \mathbb{C}^3 and then to restrict it to \mathbb{R}^3 . This will not change the algebraic calculations, but the algebraic extension from \mathcal{S} to \mathbb{C}^3 has no geometrical counterpart in the form of a parallel extension of the geometrical meaning of the parameters. We therefore have lost all contact with the initial geometrical meaning of the parameters (x, y, z) when we finally land the algebra in \mathbb{R}^3 .

This is important for the harmonic polynomials which are components of tensor products $\boldsymbol{\xi} \otimes \boldsymbol{\xi} \otimes \dots \otimes \boldsymbol{\xi}$ of rank 2ℓ of the spinor $\boldsymbol{\xi}$ (see [14], p.53 and [13], p. 68), and in which (x, y, z) are subsequently used as position coordinates in QM (see [13], p. 85). If one were to stick to the original definition of $(x, y, z) \in \mathcal{S}$ this would just be incomprehensible (see [13], p. 5). We might start to think that the identification of $(x, y, z) \in \mathcal{S}$ with position coordinates $(x, y, z) \in \mathbb{R}^3$ is balderdash and that the calculations are therefore absolute nonsense. However, the apparently completely illogical procedure is justified by the fact that the validity of the algebra of the group theory does not depend on the meaning we give to the parameters (x, y, z) .

Thus what saves us here is that any meaning we give to the parameters (x, y, z) can be used in the algebra, because it is formally just a calculus of differentiations carried out on polynomials. The meaning of the variables (x, y, z) does not intervene in the calculations, e.g. those that lead from Eq. 3 to Eq. 4. This amazing justification, completely passed under silence by Cartan [14] who must have thought it was trivial, illustrates how difficult it can be to make sense of QM, even after figuring out that it is entirely written in the language of group theory. Of course the link between rotation coordinates and position coordinates is provided by the isotropic vector.

Note that in Eq. 18 we obtain $-\hat{S}_z$ rather than \hat{S}_z . It is easy to see where this minus sign comes from. In fact $\varphi = \omega_0 \tau$ ends via $E = \hbar\omega/2$ up in Et , while to obtain \hat{L}_j we have used $-\mathbf{p} \cdot \mathbf{r}$. We may further note that the spinor:

$$\boldsymbol{\chi} = \begin{bmatrix} e^{-i\varphi/2} \\ 0 \end{bmatrix}, \tag{19}$$

in $SU(2)$ corresponds to a counterclockwise rotation around the z -axis in $SU(2)$, while the normal convention in $SO(3)$ uses the positive sign for counterclockwise rotation. This way, everything is consistent, because the two differences in sign coincide in the formalism.

The quantum numbers $\ell = (d_0 + d_1)/2$ and $m = (d_1 - d_0)/2$ are actually characterizing the degrees d_0 and d_1 of the polynomials $\xi_0^{d_0} \xi_1^{d_1}$ in ξ_0 and ξ_1 , which are components of the tensor $\boldsymbol{\xi} \otimes \boldsymbol{\xi} \otimes \dots \otimes \boldsymbol{\xi}$ (see [13], Section 3.10.5, p. 75). They are therefore labels we can use to identify various finite-dimensional representations of the rotation group. This is their true group-theoretical meaning. The choice of the representation will be dictated by the symmetry of the physical problem we want to treat.

It goes without saying that correcting the errors we have pointed out is of fundamental importance. The fact that nobody has noticed these errors for such a long time is not flattering but this cannot justify certain punitive retributions inflicted on me. The present paper has been censored and stonewalled with a whole panoply of underhanded tactics by Valentina Parigi, although she and her anonymous referee (who claims that a two-dimensional sphere is a circle) knew all too well that its contents are rigorously exact. Within the same spirit of cancel culture, the paper was also desk-rejected by Matteo Paris, with the usual completely gratuitous and unsubstantiated depreciative statements of such a non-committing generality that they can serve as a pretext for rejecting every single paper. He carefully made sure not to address the slightest factual content of the paper. The honours list of people who have also ostracized this article further comprises Gui-Lu Long and Vira Pobyhz. To their great satisfaction the truth will never be published, or it will be published by the right person: somebody entitled to present my work as his own.

References

1. Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **47**, 777 (1935).
2. Coddens, G.: The geometrical meaning of spinors lights the way to make sense of quantum mechanics, *Symmetry* **13**, 659 (2021).
3. Messiah, A.: *Quantum Mechanics, Volume I*. North Holland, Amsterdam, The Netherlands (1965).
4. Hladik, J., Cole, J.M.: *Spinors in Physics*. Springer, New York (2012).
5. Jones, H.F.: *Groups, Representations and Physics*. Adam Hilger, Bristol (1990).
6. Coddens, G.: Spinors for everyone.
<https://hal-cea.archives-ouvertes.fr/cea-01572342>.
7. Coddens, G.: Why spinors do not form a vector space.
<https://hal.archives-ouvertes.fr/hal-03289828>.
8. Schuller, F.: Differential structures, the pivotal concept of tangent vector spaces - Lecture 09.
<https://www.youtube.com/watch?v=UPGoXBfm6Js>.
9. Schuller, F.: Construction of the tangent bundel - Lecture 10.
<https://www.youtube.com/watch?v=XZcKSoI17r0>.
10. Schuller, F.: Application: Spin structures - Lecture 27.
<https://www.youtube.com/watch?v=Way8FfcMpf0>.
11. Del Debbio, L.: *Symmetries of Quantum Mechanics*, University of Edinburg, Chapter 4, 2006.
<https://www2.ph.ed.ac.uk/~ldeldebb/docs/QM2/chap4.pdf>.
12. Coddens, G.: The exact theory of the Stern-Gerlach experiment and why it does not imply that a fermion can only have its spin up or down, *Symmetry* **13**, 134 (2021).
13. Coddens, G.: *From Spinors to Quantum Mechanics*. Imperial College Press, London (2015).
14. Cartan, E.: *The Theory of Spinors*. Dover, New York (1981).