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# The exact meaning of the angular-momentum and spin operators in quantum mechanics

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**Abstract.** The theory of angular momentum and spin in quantum mechanics seems to defy common-sense intuition. We render the theory intelligible again by pointing out that this apparent impenetrability merely stems from an *undue* parallel interpretation of the algebraic expressions for the angular-momentum and spin operators in the group representation theory of  $SO(3)$  and  $SU(2)$ . E.g. the correct meaning of  $\hat{L}_z = \frac{\hbar}{i} (x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x})$  is not that it is the operator for the  $z$ -component  $L_z$  of the angular momentum  $\mathbf{L}$ , but rather the expression of the operator for the angular momentum  $\mathbf{L}$  when it is aligned with the  $z$ -axis. Hence what we are used to note (erroneously) as  $\hat{L}_z$  is not a scalar but a vector operator. The same applies *mutatis mutandis* for the spin operators. In the correct interpretation, the whole algebraic formalism is just the group representation theory for the rotations of three-dimensional Euclidean geometry. It is thus mere, elementary high-school mathematics (in a less usual, more technical guise) and as such totally exempt of any physics, let alone quantum mysteries. The change of interpretation has no impact on the algebraic results, such that they remain in agreement with experimental data. It is all only a matter of the correct geometrical meaning of the algebra. All these statements are proved within the framework of the group representation theory for  $SO(3)$  and  $SU(2)$  which is the basic tool used to describe rotational motion in quantum mechanics.

**PACS.** 02.20.-a, 03.65.Ta, 03.65.Ca Group theory, Angular Momentum, Quantum Mechanics

## 1 Textbook angular momentum

Quantum mechanics (QM) deduces the expressions for the angular-momentum operators from de Broglie's ansatz for a wave function:

$$\psi(\mathbf{r}, t) = e^{-\frac{i}{\hbar}(Et - \mathbf{p} \cdot \mathbf{r})}. \quad (1)$$

In fact, by operating  $\hat{\mathbf{L}} = \frac{\hbar}{i} \mathbf{r} \wedge \nabla$ , with components:

$$\hat{L}_z = \frac{\hbar}{i} \left[ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] \text{ (cycl)}, \quad (2)$$

on  $\psi$  one obtains indeed  $\hat{\mathbf{L}}\psi = \mathbf{L}\psi$ , where  $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ . The notation (cycl) indicates cyclic permutation. From this, we can derive the commutation relations:

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z \text{ (cycl)}, \quad (3)$$

as a straightforward calculation shows. This entails stunning, highly counter-intuitive results, which render the subject of angular momentum in traditional QM completely mysterious and conceptually inscrutable. It tells us e.g. that the values of  $L_x$  and  $L_y$  cannot be defined simultaneously because according to Eq. 3 their operators do not commute. Textbooks also show that:

$$\hat{\mathbf{L}}^2 \psi = \hbar^2 l(l+1) \psi, \quad (4)$$

where  $l$  is a quantum number. This is also puzzling because it seems to tell us in a sense that a square (i.e.  $L^2$ ) is not a square (i.e.  $\hbar^2 l(l+1)$ ). This is explained in textbooks by introducing the concept of expectation value. The expectation value for  $L^2$  would be  $\hbar^2 l(l+1)$ . But all this still remains puzzling because the operators also occur in the group representation theory of  $SU(2)$  or  $SO(3)$ , which are just Euclidean geometry. How can it possibly be true that in the group  $SO(3)$ , whose elements preserve the squares of vectors by definition, a square would not be a square? And how can it be true that in Euclidean geometry  $L_x$  and  $L_y$  could not be simultaneously defined?

The solution for these conceptual difficulties is that the heuristics based on Eq. 1 are wrong. Their lack of reliability was already pointed out by Messiah (see [1], p.70), who illustrated with some examples that the derivations based on Eq. 1 could lead to ambiguous results. He suggested that we can obtain the correct result by trial and error. But this is conceptually unsatisfactory. We cannot seriously pretend to build a theory on such methods. Following Messiah, the conclusion we should have drawn from Eq. 4 is actually that in our method of trial and error we failed because  $\hat{\mathbf{L}}^2$  does not yield a square (as in fact,  $[\hat{\mathbf{L}}^2] \neq [\hat{\mathbf{L}}]^2$ ).

## 2 Group representation theory restores the conceptual clarity

### 2.1 Context and caveats

We will render the theory of angular momentum in QM intuitive again. In fact, all the conceptual difficulties with angular momentum in QM are a consequence of an undue parallel interpretation of the algebra of the group theory of  $SO(3)$  and  $SU(2)$ . These remarks must be seen within the context of a larger framework. In [2] we have reconstructed QM from scratch. We were able to derive the Dirac equation by only using the representation theories for the rotation group and the homogeneous Lorentz group. The starting point for this derivation is to express that the electron is spinning, by making the substitution  $\varphi = \omega_0 \tau$  in the Rodrigues formula in Eq. 6 below. Here  $\tau$  is the proper time. The whole further derivation is carried out with the rigour of a mathematical proof and does not require the introduction of supplementary stunning assumptions as one might expect based on the fact that QM is full of mysteries which the theory has to account for. As the Schrödinger and Pauli equations can be derived from the Dirac equation, this derivation from scratch offers a broad platform from which we can start to study the foundations of QM. We obtain this way a better understanding of what the calculus of QM physically means.

The parallel interpretation of the algebra in traditional QM is often at variance with the correct geometrical meaning of it and which is naturally provided by the group theory. It is these over-interpretations which are responsible for many of the conceptual difficulties we encounter in QM. In our new approach which uses the correct geometrical meaning of the algebra, a lot of the perplexing puzzles we qualify as quantum mysteries disappear, while the agreement of the algebra with the experimental results remains rigorously preserved. Our alternative approach cannot be criticized on the basis of its differences with the traditional approach, because these differences do not occur in the theoretical predictions of the experimental data. There is a thorough discussion of how our approach preserves the algebra and only differs from the standard approach in using the correct geometrical meaning of that algebra in [2]. It is in this context of correcting for the over-interpretations in the traditional theory that the present paper must be situated.

It is with the harvest of these results in store that we feel entitled to ask the reader to be tolerant and prepared to admit that whenever he/she thinks that the present approach can only be wrong, it could be his/her own viewpoint that might be wrong because it is based on over-interpretations which are absent from our approach, which has been outlined in [2]. After getting over his/her initial nervousness the reader will find out that the conclusions we reach are unassailable.

In fact, they may disturb the reader, because they fly in the face of what he/she has learned such that his/her first reaction might be one of fierce resistance. In order to avoid misunderstandings and futile polemics, reference [2] should therefore be consulted before reading the present paper. In the correct interpretation of the operators, the counter-intuitive textbook results about the spin and angular momentum operators just disappear. Similar situations are encountered on several other occasions within the broader context of [2].

To the readers who are upset, I suggest to travel mentally backwards in time and try to remember their bewilderment when they were agonizing over the meaning of these angular-momentum and spin operators in their tender years. Perhaps the flashbacks will revive some long-forgotten repressed feelings of alienation and frustration. The pain the reader may have experienced in trying to get his/her head around the subject matter is actually telltale of a cognitive dissonance provoked by a brainwash. It is from this brainwash and its “die-hard certainties” that we must free ourselves by deconstructing it [2].

### 2.2 Infinitesimal generators

Let  $G$  be a Lie group. Let us note a representation matrix  $\mathbf{D}(g)$  of a group element  $g \in G$  as  $\mathbf{D}(\boldsymbol{\lambda})$ , where  $\boldsymbol{\lambda} = (\lambda_1, \lambda_2 \dots \lambda_{\dim(G)}) \in \mathbb{R}^{\dim(G)}$  is a set of independent real group parameters which define  $g$ . Here  $\dim(G)$  is the dimension of the

group, which is of course different from the dimension of the representation, which is the rank of the matrix  $\mathbf{D}(\boldsymbol{\lambda})$ . E.g. in  $SU(2)$  the dimension of the representation is 2, while the dimension of the group is 3. These two dimensions are in general still different from the dimension of the vector space the transformations might be acting on. E.g. the homogeneous Lorentz group is six-dimensional, it acts on four-dimensional space-time, and the dimension of the representation  $SL(2, \mathbb{C})$  is two. In the group  $SO(3)$ , the three different types of dimensions all take the same value three, such that we may think that there is only one concept of dimension. The resulting absence of disambiguation can stir confusion in one's first contact with the group theory.

In Lie groups  $G$  one uses so-called ‘‘infinitesimal generators’’. Textbooks [3,4] explain then the following. We consider a neighbourhood of the identity element  $\mathbf{1}$ . In this neighbourhood, all parameters  $\lambda_j$  are small. We now consider the group elements, whereby only one parameter  $\lambda_j$  is allowed to vary and to be different from zero. We let  $\lambda_j$  vary between 0 and 1. The matrices  $\mathbf{M}(0, 0, \dots, 0, \lambda_j, 0, \dots, 0, 0)$  will then constitute one-parameter sets of group elements. They will describe a one-dimensional curve on the group manifold. An infinitesimal generator is then defined as:

$$\overset{\circ}{\mathbf{D}}_j = \iota \left[ \frac{\partial}{\partial \lambda_j} \mathbf{D}(0, 0, \dots, 0, \lambda_j, 0, \dots, 0, 0) \right]_{\lambda_j=0}. \quad (5)$$

The infinitesimal generators belong to the tangent space to the group manifold at the identity element. The aim of defining the infinitesimal generators and the Lie algebra is to construct a basis for the tangent space to the manifold of the Lie group. The elements of the Lie algebra, the tangent vectors, are thus objects that are completely different from the elements of the Lie group which are group elements. As they belong to the tangent space rather than to the group itself, ‘‘infinitesimal generators’’ are not generators of the Lie group. Furthermore, the matrices  $\mathbf{M}(0, 0, \dots, 0, d\lambda_j, 0, \dots, 0, 0) - \mathbf{1}$  are infinitesimal, but the quantities defined by Eq. 5 are not. The ‘‘infinitesimal generators’’ are thus also not infinitesimal. These infinitesimal generators could also be calculated at another point of the Lie group, but choosing the identity element enhances the simplicity of the expressions obtained.

Within a broader context, we must consider Eq. 5 as an example, not a definition. In fact, it is in general not spelled out in textbooks which propose Eq. 5 that the infinitesimal generators are vectors of the tangent space to a given point of the Lie group, and that the aim is to obtain a complete set of basis vectors for the tangent space in that point. The examples do not explain how to define an appropriate choice for the one-parameter families in order to obtain such a complete basis. One only discovers this difficulty when one gets stuck in trying to work out an example, e.g. for  $SU(3)$ , for which textbooks only give the final result, rather than its detailed derivation based on Eq. 5. We also run into such difficulties when we try to find a complete orthogonal basis of tangent vectors to  $SU(2)$  by starting from the expression  $\mathbf{R}(\alpha, \beta, \gamma)$  for a rotation in terms of its Euler angles  $(\alpha, \beta, \gamma)$ . With a complete basis we can carry out calculations in the tangent space with the Lie algebra. We can thus make our calculations in the Lie group (Schrödinger, Dirac) or in the Lie algebra (Heisenberg).

### 2.3 The Rodrigues formula in $SU(2)$

The Rodrigues formula (see [2], p.11) for the  $SU(2)$  representation matrix  $\mathbf{R}(\mathbf{s}, \varphi)$  of a rotation by an angle  $\varphi$  around an axis  $d$  defined by the unit vector  $\mathbf{s} \parallel d$  is:

$$\mathbf{R}(\mathbf{s}, \varphi) = \cos(\varphi/2) \mathbf{1} - \iota \sin(\varphi/2) [\mathbf{s} \cdot \boldsymbol{\sigma}]. \quad (6)$$

Here  $\mathbf{1}$  is the  $2 \times 2$  unit matrix, and  $\sigma_j$  are the Pauli matrices:

$$\sigma_x = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} & -\iota \\ \iota & \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \quad (7)$$

We will use  $j$  as a general notation for the indices  $x, y, z$ . The term  $\mathbf{s} \cdot \boldsymbol{\sigma}$  is not a true scalar product but a purely formal notation for  $s_x \sigma_x + s_y \sigma_y + s_z \sigma_z$  (see [2,5]). The matrices  $\sigma_j$  represent the unit vectors  $\mathbf{e}_j$  and  $\mathbf{s} \cdot \boldsymbol{\sigma}$  the vector  $\mathbf{s}$ . Hence  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  is not a vector but represents the triad of basis vectors  $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ . The Pauli matrices  $\sigma_j$  do not correspond to components of vectors but to vectors in their own right.

### 2.4 Infinitesimal generators for $SU(2)$

We consider the one-parameter family of rotations over an angle  $\varphi$  around a fixed axis  $d \parallel \mathbf{s}$ . This is a one-parameter family because only  $\varphi$  is allowed to vary. By applying the definition given in Eq. 5 we obtain the infinitesimal generator:

$$\overset{\circ}{\mathbf{R}}_{\mathbf{s}} = \iota \left[ \frac{\partial}{\partial \varphi} \mathbf{R}(\mathbf{s}, \varphi) \right]_{\varphi=0} = \frac{1}{2} [\mathbf{s} \cdot \boldsymbol{\sigma}], \quad (8)$$

For  $\mathbf{s} = \mathbf{e}_x$  one obtains thus  $\overset{\circ}{\mathbf{R}}_x = \sigma_x/2$ . In general, for  $\mathbf{s} = \mathbf{e}_j$  one obtains  $\overset{\circ}{\mathbf{R}}_j = \sigma_j/2$ .

*Important conclusion:* The result  $\overset{\circ}{\mathbf{R}}_x = \sigma_x/2$  is not an operator for the  $x$ -component of a general rotation, but the infinitesimal generator for a rotation when its rotation axis is the  $x$ -axis, because we have derived all this by considering the one-parameter family of rotations around the  $x$ -axis (or in general the axis  $d \parallel \mathbf{s}$ ). The same is true *mutatis mutandis* for  $\overset{\circ}{\mathbf{R}}_y = \sigma_y/2$  and  $\overset{\circ}{\mathbf{R}}_z = \sigma_z/2$ . Furthermore:

$$\begin{aligned} \overset{\circ}{\mathbf{R}}_x \overset{\circ}{\mathbf{R}}_y - \overset{\circ}{\mathbf{R}}_y \overset{\circ}{\mathbf{R}}_x &= \frac{1}{4}(\sigma_x \sigma_y - \sigma_y \sigma_x) = \frac{i}{2} \sigma_z \\ &= i \overset{\circ}{\mathbf{R}}_z \text{ (cycl)}. \end{aligned} \quad (9)$$

Any set of operators that satisfies these commutation relations can be used to represent the Lie algebra. Examples are the the spin operators and the angular-momentum operators. In fact, by defining  $\hat{S}_j = \hbar \overset{\circ}{\mathbf{R}}_j = \frac{\hbar}{2} \sigma_j$ , we can transform  $\frac{1}{4}(\sigma_x \sigma_y - \sigma_y \sigma_x) = \frac{i}{2} \sigma_z$  into the commutation relations for the spin operators  $[\hat{S}_x, \hat{S}_y] = \frac{i\hbar}{2} \hat{S}_z$ . By putting  $\hat{L}_j = \hbar \overset{\circ}{\mathbf{R}}_j$  we can transform  $\overset{\circ}{\mathbf{R}}_x \overset{\circ}{\mathbf{R}}_y - \overset{\circ}{\mathbf{R}}_y \overset{\circ}{\mathbf{R}}_x = \overset{\circ}{\mathbf{R}}_z$ , into the commutation relations for the angular-momentum operators  $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$ . Both commutation relations are thus realizations of the commutation relation  $[\overset{\circ}{\mathbf{R}}_x, \overset{\circ}{\mathbf{R}}_y] = \overset{\circ}{\mathbf{R}}_z$ . In other words  $\hat{L}_j$  and  $\hat{S}_x$  are just two valid disguises we can choose for the infinitesimal generators of the rotation group. We can do our Lie algebra with both choices. The quantities  $\hbar$  and  $\hbar/2$  are *ad hoc* add-ons dictated by the physics [2].

## 2.5 Infinitesimal generators for SO(3)

One can actually show directly that Eq. 2 is an infinitesimal generator for the rotation group in the following way given in [6]. We consider a function  $\psi : (x, y, z) \in \mathbb{R}^3 \rightarrow \psi(x, y, z) \in \mathbb{K}$ , where  $\mathbb{K}$  can be  $\mathbb{R}$  or  $\mathbb{C}$ . We suppose that under a rotation  $R$  we transform  $\psi(x, y, z)$  to  $\psi'(x', y', z')$  and  $\mathbf{r} = (x, y, z)$  to  $\mathbf{r}' = (x', y', z')$ . We have then:

$$\psi'(\mathbf{r}') = \psi(R^{-1}(\mathbf{r}')). \quad (10)$$

From now on we will write  $\mathbf{r}' = (x', y', z')$  as  $\mathbf{r} = (x, y, z)$  For an infinitesimal rotation  $R(d\varphi)$  over an angle  $d\varphi$  around the  $z$ -axis we have:

$$[R(d\varphi)]^{-1}(x, y, z) = (x + y d\varphi, y - x d\varphi, z), \quad (11)$$

which just follows from the Taylor series expansion to the first order of the rotation matrix  $\mathbf{R}(d\varphi)$ :

$$\begin{bmatrix} \cos(d\varphi) & -\sin(d\varphi) \\ \sin(d\varphi) & \cos(d\varphi) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -d\varphi \\ d\varphi & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad (12)$$

where of course  $[\mathbf{R}(d\varphi)]^{-1} = \mathbf{R}(-d\varphi)$ . Therefore to first order in  $d\varphi$ :

$$\begin{aligned} \psi'(x, y, z) &= \psi(x + y d\varphi, y - x d\varphi, z) \\ &= \psi(x, y, z) - id\varphi \cdot \frac{1}{i} \left[ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] \psi(x, y, z). \end{aligned} \quad (13)$$

Now  $\psi' = R(d\varphi) \psi$  and  $\psi = R(0) \psi$ , where  $R(0) = \mathbb{1}$  is the identity element, such that:

$$i[R(d\varphi) - R(0)]\psi = d\varphi \cdot \frac{1}{i} \left[ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] R(0)\psi, \quad (14)$$

which is true for all functions  $\psi$ . Here  $\psi$  can e.g. be a harmonic polynomial. Hence in terms of operators acting on those functions  $\psi$  we have:

$$\left[ \frac{\partial R}{\partial \varphi} \right]_{\varphi=0} = \frac{1}{i} \left[ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right], \quad (15)$$

such that (up to the factor  $\hbar$ ) the operator  $\hat{L}_z$  is the infinitesimal generator associated with the one-parameter family of rotations  $R$  around the  $z$ -axis. It corresponds therefore to the angular-momentum vector rather than to one of

its scalar components. This conclusion is inevitable and nevertheless its lackluster consequence, viz. that traditional QM uses a wrong interpretation of  $\hat{L}_z$ , has tantalizingly been overlooked. Rather than the textbook “derivations” based on the de Broglie ansatz, it is this calculation that is correct, because the geometrical meaning of the algebraic formalism is firmly established prior to any use of these mathematics in physics. It proves that  $\hat{L}_j$  are infinitesimal generators, which we can call angular momentum operators after incorporating  $\hbar$ . An alternative proof is given by the verification that the expressions for  $\hat{L}_j$  given in Eq. 2 satisfy the commutation relations for the rotation group. The parameters  $(x, y, z)$  in the expressions for  $\hat{L}_j$  do not need to be position coordinates, because the algebra that intervenes in checking Eq. 3 for the operators defined by Eq. 2 can be carried out abstractly without specifying what  $(x, y, z)$  mean (see Eq. 16).

### 3 Discussion

As the infinitesimal generators  $\hat{\mathbf{R}}_j$  form a basis for tangent space, both  $\hat{S}_j$  and  $\hat{L}_j$  form a basis for tangent space. Whatever the meaning one gives to  $(x, y, z)$  the fact that  $\hat{L}_j$  satisfy the commutation relations is sufficient. The commutation relations for the angular-momentum operators are just defined by the group theory, nothing else.

We have thus shown that the quantum operators are not the operators for the components of the angular momentum or spin, but the operators for the angular momentum or spin when it is aligned with a given axis. The algebra of the angular-momentum and spin operators is therefore not mysterious. It is just the algebra of rotations in Euclidean geometry. All quantum mysteries surrounding angular momentum and spin in QM are only due to wrong interpretations of this algebra. This illustrates our thesis upheld in [2] that group theory is the key to making sense of QM.

If the interpretation of  $\hat{L}_j$  truly would be that it is the operator for the component  $L_j$  of the angular momentum, that would be in flagrant contradiction with the group theory, which is just based on Euclidean geometry. Scalar quantities can only commute. In Euclidean geometry, nothing impedes  $L_x$  and  $L_y$  to exist simultaneously. This shows that the traditional QM interpretation must be abandoned. The correct interpretation for  $\hat{L}_z = \frac{\hbar}{i} \left[ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right]$  in Eq. 2 is the one that follows from the group theory: It is the operator for the angular momentum when it is aligned with the  $z$ -axis, with the interpretations for  $\hat{L}_x$  and  $\hat{L}_y$  following by cyclic permutation.

We must therefore conclude that the derivation of the operators  $\hat{L}_j$  from Eq. 1 accidentally yields correct expressions, but with a wrong interpretation for them. We may note that the phase of  $\psi$  in Eq. 1 can be obtained from the time part  $\tau = \gamma(t - \mathbf{v} \cdot \mathbf{r}/c^2)$  of the Lorentz transformation by multiplying it by  $m_0 c^2/\hbar$  (see [2], p.36). The Dirac equation is derived by putting  $\varphi = \omega_0 \tau$  in the Rodriguez formula, leading to a spinor that contains the factor  $e^{-i\omega_0 \tau/2}$ . The de Broglie wave is obtained from this by putting  $m_0 c^2 = \hbar \omega_0/2$ . Here  $\tau$  is the proper time,  $\omega_0$  the angular frequency of the spinning motion and  $m_0$  the rest mass. Like a Bloch wave  $e^{-i\mathbf{k} \cdot \mathbf{r}}$  in solid-state physics,  $\psi(\mathbf{r}, t) = e^{-i(\omega/c, \mathbf{k}) \cdot (ct, \mathbf{r})}$  is just a one-dimensional representation expressing translational invariance along one dimension. Here  $\cdot$  is the scalar product in space-time. We are expressing here not translation symmetry in space, but in time because we are describing the electron’s spinning motion in proper time. This wave function is defined for just one value of  $\mathbf{v}$ . If we had taken  $\mathbf{v} \parallel \mathbf{e}_x$ , we would just have written  $\psi(x, y, z, t) = e^{-\frac{i}{\hbar}(Et - p_x x)}$ . One cannot derive  $\hat{L}_j$  from this expression. The abelian group  $\text{SO}(1,1)$  does not contain  $\text{SO}(3)$ , which is non-abelian. It is not possible that some commutation relations for  $\text{SO}(3)$  could be derived from a development that belongs strictly to a context of  $\text{SO}(1,1)$  embedded in  $\text{SO}(1,3)$ . Surely enough, the non-abelian group  $\text{SO}(1,3)$  contains  $\text{SO}(3)$  but its representations are not one-dimensional and the algebra  $\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z$  used in Eq. 1 does not apply to the composition of boosts in  $\text{SO}(1,3)$ . The fact that we obtain correct expressions for  $\hat{L}_j$  must therefore be considered as a fluke. It is based on calculations on the phase of  $\psi$  without knowing what it means. Correct proofs that  $\hat{L}_j$  are good angular momentum operators for  $\text{SO}(3)$  have been given above. The operators are defined in the Lie algebra. The one-parameter families can be parameterized by elements of  $\mathbb{R}^3$  or  $\mathbb{R}^4$ , like  $(\gamma, \gamma \mathbf{v}/c) \propto (E, \mathbf{c}\mathbf{p}) = m_0 c^2 (\gamma, \gamma \mathbf{v}/c)$  for the Lorentz boosts.

The culprit for this embarrassing situation is the sobering fact that the whole formalism of QM has not been derived but obtained by educated guessing and then validating the guesses by comparing the results of the algebra with experimental data. The algebra itself is used as a blackbox. The calculations are carried without bothering about the insight that could be gained from figuring out what the geometry is that corresponds to the algebra. The data validate the algebra, but not its interpretation. Consequently the algebra can be right, while its interpretation is flawed. In the present case it is even not possible to check certain details of the algebra because one cannot consider simultaneously several operators  $\hat{L}_j$ . Considering  $\hat{L}_x, \hat{L}_y, \hat{L}_z$  simultaneously is taboo in traditional QM because the operators do not commute, while in our approach the operators  $\hat{L}_j$  are just different forms which the operator  $\hat{\mathbf{L}}$  takes on various disjoint definition domains. For each direction in space defined by an axis  $d$ , there is a different form of the operator  $\hat{\mathbf{L}}$  with a different definition domain, which is the one-parameter family of rotations around  $d$ .

Our derivation of QM from scratch in [2] is not based on guesses. It is a mathematically rigorous derivation. In our reconstruction of QM many quantum mysteries just disappear. These are the features that have kept us on the

right track. The quantum mysteries are due to overhauling the natural geometrical meaning of the algebra of the group theory with guesstimate, parallel interpretations, based on “physical intuition”. Such interpretations are not due because the meaning of the algebra is already provided by the group theory itself.

It is not true that the components  $L_x$  and  $L_y$  of the angular momentum cannot be defined simultaneously due to the fact that their operators do not commute. All the commutation relation tells us is that when the angular momentum is aligned with the  $x$ -axis, it cannot simultaneously be aligned with the  $y$ -axis! When the angular momentum  $\mathbf{L}$  is aligned with the  $x$ -axis, we must use the operator  $\hat{L}_x$  and we have  $\mathbf{L} = (L, 0, 0)$ . The use of the operator  $\hat{L}_x$  is the only one valid for this situation  $\mathbf{L} = (L, 0, 0)$ . We have then  $L_x = L$  and  $[\hat{\mathbf{L}}^2] = \hat{L}_x^2$ . Contrary to what standard QM claims, the components  $L_x = L$ ,  $L_y = 0$  and  $L_z = 0$  are then defined simultaneously. The same applies, *mutatis mutandis*, for the cases  $\mathbf{L} = L\mathbf{e}_j$ .

Just as the correct spin operator for a spin  $\frac{\hbar}{2}\mathbf{s}$  would be  $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z = \frac{\hbar}{2}[\mathbf{s}\cdot\boldsymbol{\sigma}]$ , the correct operator for the angular momentum would be  $\ell_x\hat{L}_x + \ell_y\hat{L}_y + \ell_z\hat{L}_z$ , where the unit vector  $\boldsymbol{\ell} = (\ell_x, \ell_y, \ell_z) \parallel \mathbf{L}$ . These expressions  $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z$  and  $\ell_x\hat{L}_x + \ell_y\hat{L}_y + \ell_z\hat{L}_z$  just serve to decompose  $\mathbf{s}$  and  $\boldsymbol{\ell}$  in the bases of their Lie algebras. This is in conformity with the fact that both the sets  $\hat{S}_j$  and  $\hat{L}_j$  form a basis. When the spin is parallel to  $\mathbf{s}$  only  $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z$  is a meaningful operator. An isolated component, e.g.  $s_z\hat{S}_z$  has then no meaning, because the eigenfunction  $\psi$  of  $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z$  is not an eigenfunction of  $\hat{S}_z$ . The definition domains of  $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z$  and  $\hat{S}_z$  are disjoint. Every direction of space has its own operator. Just as  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  is not a single vector operator but a basis of three vector operators,  $\hat{\mathbf{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$  is not a single vector operator but a basis of three vector operators.

All this shows that guessing operators from their action on  $e^{-\frac{i}{\hbar}(Et-\mathbf{P}\cdot\mathbf{r})}$  is not a valid procedure, because it leads to the erroneous conclusions that  $\hat{L}_j$  would be the operator for the  $j$ -component of the angular momentum  $\mathbf{L}$  and that  $\hat{L}^2\psi = \hbar^2l(l+1)\psi$  would define the operator for the square  $L^2$  of the angular momentum. We should never lose sight that SO(3) and SU(2) are mere Euclidean geometry. When one sticks to the correct geometrical interpretation of the algebra of the rotation group the classical intuition is restored. The caveats one must observe in using the operators in the treatment of a physical problem are illustrated in [7].

#### 4 Other possible meanings we can attribute to the parameters $(x, y, z)$

We have shown (see [2], p. 21), that in SU(2):

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = \begin{bmatrix} \pm\sqrt{\frac{x-iy}{2}} \\ \pm\sqrt{\frac{-x-iy}{2}} \end{bmatrix}, \quad (16)$$

represents a spinor corresponding to a rotation  $\mathbf{R}$ . Here  $(x, y, z) = \mathbf{e}'_x + i\mathbf{e}'_y \in \mathbb{C}^3$ , where  $(\mathbf{e}'_x, \mathbf{e}'_y, \mathbf{e}'_z)$  is the canonical triad of basis vectors  $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$  of  $\mathbb{R}^3$  after rotation by  $\mathbf{R}$ . Here  $(x, y, z) \in \mathcal{I} \subset \mathbb{C}^3$  is a so-called isotropic vector, which belongs to the isotropic cone  $\mathcal{I} = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 = 0\}$ . The idea behind introducing isotropic vectors is that from  $\mathbf{e}'_x + i\mathbf{e}'_y$  we can calculate the whole triad  $\mathbf{e}'_j$  of rotated basis vectors, and that this triad defines the rotation unambiguously. The coordinates  $(x, y, z)$  of the isotropic vector are thus true group parameters, which is not the case for the parameters  $(x, y, z)$  and  $(p_x, p_y, p_z)$  in Eq. 1. They function as coordinates for rotations. The identity in Eq. 16 and its derivation lead to the notion that a spinor is a “square root of an isotropic vector”, but we have shown (see [2], p.7) that the spinors used in QM just represent group elements (i.e. rotations and Lorentz transformations). Let us leave out the  $\pm$  signs from the algebra and calculate  $\hat{L}_z\boldsymbol{\xi}$ :

$$\begin{aligned} \hat{L}_z\boldsymbol{\xi} &= \frac{\hbar}{i} \left[ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] \begin{bmatrix} \sqrt{\frac{x-iy}{2}} \\ \sqrt{\frac{-x-iy}{2}} \end{bmatrix} = \frac{\hbar}{2i} \begin{bmatrix} \frac{-ix-y}{2\sqrt{\frac{x-iy}{2}}} \\ \frac{-ix+y}{2\sqrt{\frac{-x-iy}{2}}} \end{bmatrix} = \frac{\hbar}{2i} \begin{bmatrix} \frac{-i(x-iy)/2}{\sqrt{\frac{x-iy}{2}}} \\ \frac{i(-x-iy)/2}{\sqrt{\frac{-x-iy}{2}}} \end{bmatrix} \\ &= -\frac{\hbar}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \boldsymbol{\xi} = -\frac{\hbar}{2}\sigma_z\boldsymbol{\xi} = -\hat{S}_z\boldsymbol{\xi}. \end{aligned} \quad (17)$$

Here the rotation coordinates  $(x, y, z)$  of the isotropic vector have nothing to do with the position coordinates in Eq. 1, but we can imagine to generalize the definition domain of the algebra by analytic continuation from the isotropic cone  $\mathcal{I}$  to  $\mathbb{C}^3$  and then to restrict it to  $\mathbb{R}^3$ . This will not change the algebraic calculations.

This is important for the harmonic polynomials which are components of tensor products  $\xi \otimes \xi \otimes \dots \otimes \xi$  of the spinor  $\xi$  (see [8], p.53 and [9], p. 68), and in which  $(x, y, z)$  are subsequently used as position coordinates in QM (see [9], p. 85). If one were to stick to the original definition of  $(x, y, z) \in \mathcal{S}$  this would be just incomprehensible (see [9], p. 5). The quantum numbers  $\ell$  and  $m$  are actually degrees within the polynomials in terms of  $\xi_0$  and  $\xi_1$  (see [9], Section 3.10.5, p. 75).

What saves us is that any meaning we give to the parameters  $(x, y, z)$  can be used in the algebra, because it is formally just a calculus of differentiations carried out on polynomials. The meaning of the variables  $(x, y, z)$  does not intervene in the calculations, e.g. those that lead from Eq. 2 to Eq. 3. Note that we obtain  $-\hat{S}_z$  rather than  $\hat{S}_z$ . It is easy to see where this minus sign comes from. In fact  $\varphi = \omega_0\tau$  ends via  $E = \hbar\omega/2$  up in  $Et$ , while to obtain  $\hat{L}_j$  we have used  $-\mathbf{p} \cdot \mathbf{r}$ . We may further note that the spinor:

$$\chi = \begin{bmatrix} e^{-i\varphi/2} \\ 0 \end{bmatrix}, \quad (18)$$

in SU(2) corresponds to a counterclockwise rotation around the  $z$ -axis in SU(2), while the normal convention in SO(3) uses the positive sign for counterclockwise rotation. This way, everything is consistent, because the two differences in sign coincide in the formalism.

It goes without saying that these results are absolutely fundamental. They are not flattering for our egos, but they have to be disclosed because science is about the truth, nothing but the truth. Nevertheless, this paper was desk-rejected by Matteo Paris acting as an editor for Elsevier, with the usual completely gratuitous and unsubstantiated depreciative statements of such a non-committing generality that they can serve as a pretext for rejecting every single paper. He carefully made sure not to address the slightest factual content of the paper. The honors list of people who have also censored this article further comprises Gui-Lu Long and Vira Pobyhz.

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