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# Clarifying issues about rotational motion in classical and quantum mechanics

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**Abstract.** It is proved in physics textbooks that angular frequencies can be treated as vectors. However this is subject to a caveat that in general is not pointed out, and this can lead to confusion. We also point out that the textbook interpretations of the angular-momentum and spin operators in quantum mechanics are wrong. E.g.  $\hat{L}_z = \frac{\hbar}{i} (x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x})$  is not the general expression of the operator for the  $z$ -component  $L_z$  of the

angular momentum  $\mathbf{L}$ , but the expression of the operator  $\hat{\mathbf{L}}$  for the angular momentum  $\mathbf{L}$  when it is aligned with the  $z$ -axis. Hence it corresponds to a full vector quantity  $\mathbf{L}$  rather than to a scalar component  $L_z$ . The same applies *mutatis mutandis* for the spin operators. With the correct interpretations the conceptual difficulties we encounter in making sense of angular momentum in quantum mechanics disappear, such that it is no longer mysterious. The change of interpretation has no impact on the algebraic results, such that they remain in agreement with experimental data. It is just a matter of the correct geometrical meaning of the algebra. All these statements are proved within the framework of the group theory of  $SU(2)$  which is the basic language on which descriptions of rotational motion in quantum mechanics are founded.

**PACS.** 02.20.-a, 03.65.Ta, 03.65.Ca Group theory, Angular Momentum, Quantum Mechanics

## 1 Addition of angular-frequency vectors

### 1.1 Intuitive approach in $SO(3)$

In many physics textbooks or lecture notes one gives proofs to show that angular-frequency vectors can be added [1, 2]. They may explain that whereas rotations are not vectors because rotations do not commute, infinitesimal rotations do commute and therefore can be added. But the lack of nuance with which this is formulated render these textbook presentations very misleading. To explain what this is all about we propose to think about a spinning top whose rotation axis  $d_1$  is defined by the unit vector  $\mathbf{s}_1 \parallel d_1$  and which is precessing around the vertical axis  $d_2 \parallel \mathbf{s}_2 = \mathbf{e}_z$ .

The spinning motion of the top in absence of precession is described by  $\boldsymbol{\omega}_1 = \omega_1 \mathbf{s}_1$ . The precession is described by  $\boldsymbol{\omega}_2 = \omega_2 \mathbf{e}_z$ . Now, textbooks state that the angular frequency  $\boldsymbol{\Omega}$  of the composed motion is given by:

$$\boldsymbol{\Omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = \boldsymbol{\omega}_2 + \boldsymbol{\omega}_1. \quad (1)$$

The textbook results are proved by representing the rotations as displacements on the surface of a sphere of radius  $R$  and then making a drawing of the situation in a point  $P$  of this sphere. One can then argue that infinitesimal displacements  $R \Delta\varphi$  can be added as vectors in the tangent plane to the sphere in  $P$ , and therefore also the angular frequencies  $\boldsymbol{\omega} = \frac{d\varphi}{dt}$ . The problem is that the notation suggests that the top spins at the constant value  $\boldsymbol{\Omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ . A simple inspection of what happens in the precession of the spinning top shows that this is not true. When one calculates the vector sums at two different moments in time one obtains two different results. The sum  $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$  is turning just like the spinning top is turning. One may then even wonder if one has to do a vector addition again. Nevertheless, the whole argument may look correct on the drawing showing the situation in  $P$ . The top is spinning at a constant frequency  $\omega_1$  and  $\omega_2$  is a constant. The error in the reasoning is that  $\boldsymbol{\omega}_1(t)$  is no longer a constant when it is turned by  $\boldsymbol{\omega}_2$ . Therefore also  $\boldsymbol{\omega}_2 + \boldsymbol{\omega}_1(t)$  is not constant. In fact, the tangent plane is turning under the action of  $\boldsymbol{\omega}_2$ , such that at some ulterior time, the point  $P$  where we must sum will have moved to a different point  $Q$ . Hence  $\omega_1 = |\boldsymbol{\omega}_1(t)|$  is a constant but  $\boldsymbol{\omega}_1(t)$  is not. But we do not notice this when we reason on a drawing which represents

an instantaneous snap-shot of the motion at the point  $P$ . The reasoning on the drawing is instantaneously and locally correct, but globally wrong in that it overlooks the fact that  $\boldsymbol{\omega}_1(t)$  varies. Eq. 1 is therefore inaccurate because it neglects pointing out the time dependence. What one has to write to describe the motion correctly is:

$$\boldsymbol{\Omega}(t) = \boldsymbol{\omega}_2 + \boldsymbol{\omega}_1(t). \quad (2)$$

This way the instantaneous character of the addition is then expressed correctly. Textbooks fail to point out this caveat. It is a kind of a nuisance not to point out the time dependence explicitly in a course. Moreover, if the top were spinning at the angular frequency  $\boldsymbol{\omega}_2$  and the precession were taking place at the angular frequency  $\boldsymbol{\omega}_1$ , we would obtain:

$$\boldsymbol{\Omega}(t) = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2(t). \quad (3)$$

This would not be the same motion as described in Eq. 2. It follows a different order of the spinning motions. Therefore  $\boldsymbol{\omega}_1$  and  $\boldsymbol{\omega}_2$  also do not really commute. They only commute instantaneously, while globally there is an order in the rotations, i.e. Eq. 2 corresponds to a product of rotations  $R_2 \circ R_1$  while Eq. 3 corresponds to a product of rotations  $R_1 \circ R_2$ .

The reason for the error in Eq. 1 is that the calculations are made in the instantaneous tangent space to the sphere or to the rotation group (see below) and that this tangent space does not remain fixed. It wanders with the precession. But in a co-rotating frame Eq. 1 would look exact.

## 1.2 Rigorous algebraic treatment in SU(2)

We will now give a much better purely algebraic proof in SU(2). A good introduction to SU(2) is given in [3,4,5]. The Rodrigues formula (see [3], p.11) for the SU(2) representation matrix  $\mathbf{R}(\mathbf{s}, \varphi)$  of a rotation by an angle  $\varphi$  around an axis  $d$  defined by the unit vector  $\mathbf{s} \parallel d$  is:

$$\mathbf{R}(\mathbf{s}, \varphi) = \cos(\varphi/2) \mathbf{1} - \imath \sin(\varphi/2) [\mathbf{s} \cdot \boldsymbol{\sigma}]. \quad (4)$$

Here  $\mathbf{1}$  is the  $2 \times 2$  unit matrix, and  $\sigma_j$  are the Pauli matrices:

$$\sigma_x = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} & -\imath \\ \imath & \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \quad (5)$$

We use  $j$  as a general notation for the indices  $x, y, z$ . The term  $\mathbf{s} \cdot \boldsymbol{\sigma}$  is not a true scalar product but a purely formal notation for  $s_x \sigma_x + s_y \sigma_y + s_z \sigma_z$ . The matrices  $\sigma_j$  represent the unit vectors  $\mathbf{e}_j$  and  $\mathbf{s} \cdot \boldsymbol{\sigma}$  the vector  $\mathbf{s}$ . Hence  $\boldsymbol{\sigma}$  is not a vector but represents the triad of basis vectors  $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ . As explained in [3] we can transform this Rodrigues formula into the description of a spinning or rotational motion by putting  $\varphi = \omega t$ . We consider now the two rotational motions:

$$\left| \begin{array}{lcl} \mathbf{R}_1(t) & = & \cos(\omega_1 t/2) - \imath \sin(\omega_1 t/2) [\mathbf{s}_1 \cdot \boldsymbol{\sigma}], \\ \mathbf{R}_2(t) & = & \cos(\omega_2 t/2) - \imath \sin(\omega_2 t/2) [\mathbf{s}_2 \cdot \boldsymbol{\sigma}]. \end{array} \right. \quad (6)$$

Derivation with respect to  $t$  yields:

$$\left| \begin{array}{lcl} \frac{d}{dt} \mathbf{R}_1(t) & = & -\imath(\omega_1/2) [\mathbf{s}_1 \cdot \boldsymbol{\sigma}] \mathbf{R}_1(t) = -(\imath/2) [\boldsymbol{\omega}_1 \cdot \boldsymbol{\sigma}] \mathbf{R}_1(t), \\ \frac{d}{dt} \mathbf{R}_2(t) & = & -\imath(\omega_2/2) [\mathbf{s}_2 \cdot \boldsymbol{\sigma}] \mathbf{R}_2(t) = -(\imath/2) [\boldsymbol{\omega}_2 \cdot \boldsymbol{\sigma}] \mathbf{R}_2(t). \end{array} \right. \quad (7)$$

Here  $\boldsymbol{\omega}_j$  are constants. Now we consider  $\mathbf{R}(t) = \mathbf{R}_2(t) \mathbf{R}_1(t)$ . In this composition of rotations,  $\mathbf{R}_1(t)$  could e.g. describe the spinning motion of an electron, and  $\mathbf{R}_2(t)$  the precession imposed on this spinning motion by a magnetic field. Or  $\mathbf{R}_1(t)$  could describe a spinning top and  $\mathbf{R}_2(t)$  its precession.

Let us do the calculation properly within the representation SU(2). When we express the combined motion with the aid of the representation matrices of SU(2) we obtain:

$$\frac{d\mathbf{R}}{dt} = \frac{d}{dt} [\mathbf{R}_2(t) \mathbf{R}_1(t)] = -(\imath/2) [\boldsymbol{\omega}_2 \cdot \boldsymbol{\sigma}] \mathbf{R}_2(t) \mathbf{R}_1(t) - (\imath/2) \mathbf{R}_2(t) [\boldsymbol{\omega}_1 \cdot \boldsymbol{\sigma}] \mathbf{R}_1(t). \quad (8)$$

The origin of the troubles evoked in Subsection 1.1 resides in the second term on the right-hand side. In the first term  $\mathbf{R}_2(t)$  and  $\mathbf{R}_1(t)$  occur nicely in juxtaposition together behind the vector  $-(\imath/2) [\boldsymbol{\omega}_2 \cdot \boldsymbol{\sigma}]$ , but in the second term  $\mathbf{R}_2(t)$  and  $\mathbf{R}_1(t)$  are not in juxtaposition. We see that Eq. 1 is not correct because  $\boldsymbol{\omega}_1 \cdot \boldsymbol{\sigma}$  does not commute with  $\mathbf{R}_2(t)$ , such that  $\boldsymbol{\omega}_1 \cdot \boldsymbol{\sigma}$  cannot be dragged in front of  $\mathbf{R}_2(t)$ . In fact, in order to obtain a juxtaposition  $\mathbf{R}_2(t) \mathbf{R}_1(t) = \mathbf{R}(t)$  we must rewrite Eq. 8 as:

$$\mathbf{R}_2(t)[\boldsymbol{\omega}_1 \cdot \boldsymbol{\sigma}] \mathbf{R}_1(t) = \mathbf{R}_2(t)[\boldsymbol{\omega}_1 \cdot \boldsymbol{\sigma}] \mathbf{R}_2^{-1}(t) \cdot \mathbf{R}_2(t) \mathbf{R}_1(t). \quad (9)$$

Let us put:

$$\mathbf{R}_2(t)[\boldsymbol{\omega}_1 \cdot \boldsymbol{\sigma}] \mathbf{R}_2^{-1}(t) = [\boldsymbol{\omega}_1(t) \cdot \boldsymbol{\sigma}]. \quad (10)$$

Here  $\boldsymbol{\omega}_1(t)$  is the constant vector  $\boldsymbol{\omega}_1$  rotated by  $\mathbf{R}_2(t)$ . In fact, as explained in [3] (in Subsection 2.6, starting on p. 15), in  $SU(2)$  a vector  $\mathbf{v} \in \mathbb{R}^3$  is represented by  $[\mathbf{v} \cdot \boldsymbol{\sigma}]$ , while turning the vector  $\mathbf{v}$  by a rotation  $\mathbf{R}$ :  $\mathbf{v} \rightarrow \mathbf{R} \mathbf{v}$  corresponds to the transformation:  $[\mathbf{v} \cdot \boldsymbol{\sigma}] \rightarrow \mathbf{R}[\mathbf{v} \cdot \boldsymbol{\sigma}] \mathbf{R}^{-1}$ . This shows that  $\boldsymbol{\omega}_1$  must be transported from the tangent space at the point  $P \leftrightarrow \mathbf{R}_2(t_1) \mathbf{R}_1(t_1)$  to the tangent space at the point  $Q \leftrightarrow \mathbf{R}_2(t_2) \mathbf{R}_1(t_2)$ . The algebra reflects exactly the geometrical argument we sketched above. We can conclude then that:

$$\frac{d\mathbf{R}}{dt}(t) = -(\imath/2) [(\boldsymbol{\omega}_2 + \boldsymbol{\omega}_1(t)) \cdot \boldsymbol{\sigma}] \mathbf{R}(t). \quad (11)$$

We obtain this way the rigorous result:

$$\frac{d\mathbf{R}}{dt}(t) = -(\imath/2) [\boldsymbol{\Omega}(t) \cdot \boldsymbol{\sigma}] \mathbf{R}(t), \quad (12)$$

where  $\boldsymbol{\Omega}(t)$  is given by Eq. 2. The algebraic proof avoids the pitfall of relying on a drawing on a sheet of paper which only represents an instantaneous snapshot of the tangent plane at some moment in time  $t$ . This tangent plane moves in time, traversing the sheet of paper at time  $t$ , while the sheet of paper with the drawing on a desk remains fixed in time.

Imagine that we mount a bicycle wheel with its axis within a frame that can be suspended to the ceiling. The wheel is then hanging completely vertically within the frame wherein we can make the wheel spin in the vertical plane around its horizontal axis. What happens now when we apply a torque to the frame to make the whole assembly spin also around the vertical axis? Well, you will see Eq. 2 at work, because the wheel will exhibit a tilt angle as a consequence of  $|\boldsymbol{\Omega}(t)| = |\boldsymbol{\omega}_2 + \boldsymbol{\omega}_1(t)|$ , which is a constant. But the wheel will also exhibit precession, such that we end up with Eq. 2. For a spinning top, the torque is applied by the gravitational pull of the gravitational field of the Earth.

## 2 Angular momentum in quantum mechanics

### 2.1 What the textbooks are telling us

With the definition  $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$  we can in quantum mechanics (QM) deduce the expression for the angular-momentum operators from the expression of the de Broglie wave:

$$\psi(\mathbf{r}, t) = e^{-\frac{\imath}{\hbar}(Et - \mathbf{p} \cdot \mathbf{r})}. \quad (13)$$

We obtain:

$$\hat{\mathbf{L}} = \frac{\hbar}{\imath} \mathbf{r} \wedge \nabla, \quad \hat{L}_z = \frac{\hbar}{\imath} \left[ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] \quad (\text{cycl}), \quad (14)$$

because this leads indeed to  $\hat{\mathbf{L}}\psi = \mathbf{L}\psi$ . A somewhat tedious but straightforward calculation shows then that:

$$[\hat{L}_x, \hat{L}_y] = \imath \hbar \hat{L}_z \quad (\text{cycl}). \quad (15)$$

This entails stunning, highly counter-intuitive results, which render the subject of angular momentum in traditional QM completely mysterious and conceptually impenetrable. It tells us e.g. that  $L_x$  and  $L_y$  cannot be defined simultaneously because according to Eq. 15 their operators do not commute. We also learn that:

$$\hat{\mathbf{L}}^2 \psi = \hbar^2 l(l+1) \psi, \quad (16)$$

where  $l$  is a quantum number. This is also puzzling because it tells us in a sense that a square (i.e.  $L^2$ ) is not a square (i.e.  $\hbar^2 l(l+1)$ ). This is explained away in textbooks by introducing the concept of expectation value. The expectation value for  $L^2$  would be  $\hbar^2 l(l+1)$ . But all this still remains puzzling because the operators also occur in the group theory of  $SU(2)$  or  $SO(3)$ . How can it possibly be true that in the group  $SO(3)$ , whose elements preserve the squares of vectors *by definition*, a square would not be a square? The answer is that the heuristics based on Eq. 13 are wrong. This lack of reliability was already pointed out by Messiah [6], who illustrated with some examples that the derivations

based on Eq. 13 could lead to ambiguous results. He suggested that we can obtain the correct result by trial and error. But this is conceptually unsatisfactory. A good rule should always work. Else we cannot seriously pretend to build a theory on such methods. And if the heuristics fail, we must not only be able to explain why it fails. We must also be able to explain why it does not fail in the other situations where it works.

In fact, the conclusion we should have drawn from Eq. 16 is that our method of trial and error tells us that the heuristics are failing because  $\hat{\mathbf{L}}^2$  does not yield the desired value  $L^2$ . The only operator that we can accept as a correct operator for  $L^2$  should be one that yields  $L^2\psi$  in operating on  $\psi$ . The way out of this paradox is of course that  $[\hat{\mathbf{L}}^2] \neq [\hat{\mathbf{L}}]^2$ .

## 2.2 Restoring conceptual clarity

We will now render the theory of angular momentum in QM intelligible again. We will show that the interpretation of  $\hat{L}_z$  as the operator for the  $z$ -component of the angular momentum is wrong. With the correct interpretation of the operators, all the mysterious counter-intuitive results from the textbooks will disappear.

In [3] we have reconstructed QM from scratch. The aim was to get a better understanding of what the calculus of QM physically means. We were able to derive the Dirac equation just from the group theory for the rotation group and the Lorentz group. The starting point is to express that the electron is spinning exactly by the substitution  $\varphi = \omega t$  in the Rodrigues formula we described above. In a sense the formalism of QM is therefore just classical (relativistic) mechanics expressed in the language of group theory. The reader may think that this just cannot be true, but we invite him to read the rigorous proofs in [3].

The conclusions we will reach in the following may equally disturb the reader, because they fly in the face of what we all have learned. Of course we will run here into Brandolini's law and meet a lot of resistance. It is such a watershed that the general attitude will be that the conclusions we reach can only be wrong. But unfortunately, there is truly no escape from the conclusions, exactly due what we proved in [3]. To readers who are upset, I can only suggest to try to remember their bewilderment about these angular-momentum operators when they had to learn the stuff as a student. It is of course a hefty blow to our egos to have to admit that for a whole century every single physics student has been brainwashed and force-fed with nonsense. After getting over the trauma of not having noticed this and after it all will have sunk in, the reader will notice that in the new approach a lot of the quantum mysteries do indeed disappear, while the agreement of the algebra with the experimental results is rigorously preserved. There is a thorough discussion of how our approach preserves the algebra and only points out the correct geometrical meaning of that algebra in [3].

## 2.3 "Infinitesimal generators"

Let  $G$  be a Lie group. Let us note a representation matrix  $[\mathbf{D}(g)](\boldsymbol{\lambda})$  of a group element  $g \in G$  as  $\mathbf{M}(\boldsymbol{\lambda})$ , where  $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{\dim(G)}) \in \mathbb{R}^{\dim(G)}$  is a set of independent real group parameters. Here  $\dim(G)$  is the dimension of the group, which is of course different from the dimension of the representation, which is the rank of the matrix  $\mathbf{M}$ . E.g. in  $SU(2)$  the dimension of the representation is 2, while the dimension of the group is 3. These two dimensions are in general still different from the dimension of the vector space the transformations might be acting on. E.g. the homogeneous Lorentz group is six-dimensional, it acts on four-dimensional space-time, and the dimension of the representation  $SL(2, \mathbb{C})$  is two. In the group  $SO(3)$ , the three different types of dimensions all take the same value three, such that we may think that there is only one concept of dimension. The resulting absence of disambiguation can stir confusion in one's first contact with the group theory.

In Lie groups  $G$  one uses so-called "infinitesimal generators". Textbooks [5,7] explain then the following. We consider a neighbourhood of the identity element  $\mathbf{1}$ . In this neighbourhood, all parameters  $\lambda_j$  are small. We now consider the group elements, whereby only one parameter  $\lambda_j$  is allowed to vary and to be different from zero. We let  $\lambda_j$  vary between 0 and 1. The matrices  $\mathbf{M}(0, 0, \dots, 0, \lambda_j, 0, \dots, 0, 0)$  will then be one-parameter sets of group elements. They will describe a one-dimensional curve on the group manifold. An infinitesimal generator is then:

$$i \left[ \frac{\partial}{\partial \lambda_j} \mathbf{M}(0, 0, \dots, 0, \lambda_j, 0, \dots, 0, 0) \right]_{\lambda_j=0}. \quad (17)$$

The infinitesimal generators belong to the tangent space of the group manifold at the identity element. The aim of defining the infinitesimal generators and the Lie algebra is to construct a basis for the tangent space to the manifold of the Lie group. The elements of the Lie algebra, the tangent vectors, are thus objects that are completely different from the elements of the Lie group which are group elements. As they belong to the tangent space rather than to the group itself, "infinitesimal generators" are not generators. Furthermore, the matrices  $\mathbf{M}(0, 0, \dots, 0, d\lambda_j, 0, \dots, 0, 0) - \mathbf{1}$  are infinitesimal, but the quantities defined by Eq. 17 are not. The "infinitesimal generators" are thus also not infinitesimal.

Within a broader context, we must consider Eq. 17 as an example, not a definition. In fact, it is in general not spelled out in textbooks which propose Eq. 17 that the infinitesimal generators are basis vectors for the tangent space to a given point of the Lie group, and that the aim is to obtain a complete basis for the tangent space in that point. The examples do not explain how we can obtain such a complete basis. One only discovers this when one tries to work out an example, e.g. for  $SU(3)$ . After obtaining a complete basis for the tangent space we can carry out calculations in the tangent space with the Lie algebra. We can thus make our calculations in the Lie group (Schrödinger, Dirac) or in the Lie algebra (Heisenberg).

## 2.4 Debunking the urban legend about angular momentum operators in QM

We start again from the Rodrigues formula given in Eq. 4. Partial differentiation of Eq. 4 yields:

$$\frac{\partial}{\partial \varphi} \mathbf{R} = -\frac{\imath}{2} [\mathbf{s} \cdot \boldsymbol{\sigma}] \mathbf{R}. \quad (18)$$

This also proves that:

$$\mathbf{R}(\mathbf{s}, \varphi) = e^{-\frac{\imath}{2} \varphi [\mathbf{s} \cdot \boldsymbol{\sigma}]} \mathbf{R}(\mathbf{s}, 0) = e^{-\frac{\imath}{2} \varphi [\mathbf{s} \cdot \boldsymbol{\sigma}]}, \quad (19)$$

as  $\mathbf{R}(\mathbf{s}, 0) = \mathbb{1}$ . Now fixing  $\mathbf{s}$  and varying  $\varphi$  defines a one-parameter family, such that we can calculate the infinitesimal generator for rotations around the axis defined by  $\mathbf{s}$  by applying the definition given in Eq. 17:

$$\overset{\circ}{\mathbf{R}}_{\mathbf{s}} = \imath \left[ \frac{\partial}{\partial \varphi} \mathbf{R}(\mathbf{s}, \varphi) \right]_{\varphi=0} = \frac{1}{2} [\mathbf{s} \cdot \boldsymbol{\sigma}], \quad (20)$$

as  $\lim_{\varphi \rightarrow 0} \mathbf{R}(\mathbf{s}, \varphi) = \mathbb{1}$ . For  $\mathbf{s} = \mathbf{e}_x$  one obtains thus  $\overset{\circ}{\mathbf{R}}_x = \sigma_x/2$ . This is not the infinitesimal generator for the  $x$ -component of a general rotation, but the infinitesimal generator for a rotation when its rotation axis is the  $x$ -axis, because we have derived all this by considering the one-parameter family of rotations around the  $x$ -axis (or in general the axis  $d \parallel \mathbf{s}$ ). The same is true *mutatis mutandis* for  $\overset{\circ}{\mathbf{R}}_y = \sigma_y/2$  and  $\overset{\circ}{\mathbf{R}}_z = \sigma_z/2$ . Furthermore:

$$\overset{\circ}{\mathbf{R}}_x \overset{\circ}{\mathbf{R}}_y - \overset{\circ}{\mathbf{R}}_y \overset{\circ}{\mathbf{R}}_x = \frac{1}{4} (\sigma_x \sigma_y - \sigma_y \sigma_x) = \frac{\imath}{2} \sigma_z = \imath \overset{\circ}{\mathbf{R}}_z \text{ (cycl)}. \quad (21)$$

Any set of operators that satisfies these commutation relations can be used to represent the Lie algebra. Examples are the the spin operators and the angular-momentum operators. In fact, by defining  $\hat{S}_j = \hbar \overset{\circ}{\mathbf{R}}_j = \frac{\hbar}{2} \sigma_j$ , we can transform  $\frac{1}{4} (\sigma_x \sigma_y - \sigma_y \sigma_x) = \frac{\imath}{2} \sigma_z$  into the commutation relations for the spin operators  $[\hat{S}_x, \hat{S}_y] = \frac{\imath \hbar}{2} \hat{S}_z$ . By putting  $\hat{L}_j = \hbar \overset{\circ}{\mathbf{R}}_j$  we can transform  $\overset{\circ}{\mathbf{R}}_x \overset{\circ}{\mathbf{R}}_y - \overset{\circ}{\mathbf{R}}_y \overset{\circ}{\mathbf{R}}_x = \imath \overset{\circ}{\mathbf{R}}_z$ , into the commutation relations for the angular-momentum operators  $[\hat{L}_x, \hat{L}_y] = \imath \hbar \hat{L}_z$ .

Both commutation relations are thus realizations of the commutation relation  $[\overset{\circ}{\mathbf{R}}_x, \overset{\circ}{\mathbf{R}}_y] = \imath \overset{\circ}{\mathbf{R}}_z$ . In other words  $\hat{L}_j$  and  $\hat{S}_j$  are just two isomorphic choices we can make for the infinitesimal generators of the rotation group. We can do our Lie algebra with both choices. The quantities  $\hbar$  and  $\hbar/2$  are *ad hoc* add-ons. One can actually show directly that Eq. 14 is an infinitesimal generator for the rotation group in the following way given in [8]. We consider a function  $\psi : (x, y, z) \in \mathbb{R}^3 \rightarrow \psi(x, y, z) \in \mathbb{K}$ , where  $\mathbb{K}$  can be  $\mathbb{R}$  or  $\mathbb{C}$ . We suppose that under a rotation  $R$  we transform  $\psi(x, y, z)$  to  $\psi'(x', y', z')$  and  $\mathbf{r} = (x, y, z)$  to  $\mathbf{r}' = (x', y', z')$ . We have then:

$$\psi'(\mathbf{r}') = \psi(R^{-1}(\mathbf{r}')). \quad (22)$$

From now on we will write  $\mathbf{r}' = (x', y', z')$  as  $\mathbf{r} = (x, y, z)$  For an infinitesimal rotation  $R$  over an angle  $d\varphi$  around the  $z$ -axis we have:

$$R^{-1}(x, y, z) = (x + y d\varphi, y - x d\varphi, z), \quad (23)$$

and therefore to first order in  $d\varphi$ :

$$\psi'(x, y, z) = \psi(x + y d\varphi, y - x d\varphi, z) = \psi(x, y, z) - \imath d\varphi \cdot \frac{1}{\imath} \left[ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right], \quad (24)$$

which clearly shows that the operator  $\hat{L}_z$  is an infinitesimal generator associated with the one-parameter family of rotations  $R^{-1}$  around the  $z$ -axis. Rather than the textbook “derivations” based on the de Broglie ansatz, it is this

calculation that proves that  $\hat{L}_j$  are infinitesimal generators, which we can call angular momentum operators after incorporating  $\hbar$ . An alternative proof is given by the verification that the expressions for  $\hat{L}_j$  given in Eq. 14 satisfy the commutation relations for the rotation group. The parameters  $(x, y, z)$  in the expressions for  $\hat{L}_j$  do not need to be position coordinates, because the algebra that intervenes in checking Eq. 15 for the operators defined by Eq. 14 can be carried out abstractly without specifying what  $(x, y, z)$  mean (see also Subsection 2.5).

As the infinitesimal generators  $\hat{\mathbf{R}}_j$  form a basis for tangent space, both  $\hat{S}_j$  and  $\hat{L}_j$  form a basis for tangent space. Whatever the meaning one gives to  $(x, y, z)$  the fact that  $\hat{L}_j$  satisfy the commutation relations is sufficient. The commutation relations for the angular-momentum operators are just defined by the group theory, nothing else. This also shows then that these operators are not the operators for the components of the angular momentum, but operators for the angular momentum when it is aligned with a given axis. The algebra of the angular-momentum operators is therefore not mysterious. It is just the algebra of rotations in Euclidean geometry. All quantum mysteries surrounding angular momentum in QM are only due to wrong interpretations of this algebra. It illustrates our thesis upheld in [3] that group theory is the key to making sense of QM.

If the interpretation of  $\hat{L}_j$  truly would be that it is the operator for the component  $L_j$  of the angular momentum, that would be in flagrant contradiction with the group theory, which is just based on Euclidean geometry. In Euclidean geometry, nothing impedes  $L_x$  and  $L_y$  to exist simultaneously. This shows that the QM interpretation is wrong. The correct interpretation for  $\hat{L}_z = \frac{\hbar}{i} \left[ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right]$  in Eq. 14 is the one that follows from the group theory: It is the operator for the angular momentum when it is aligned with the  $z$ -axis, with the interpretations for  $\hat{L}_x$  and  $\hat{L}_y$  following by cyclic permutation.

We must therefore conclude that the derivation of the operators  $\hat{L}_j$  from Eq. 13 accidentally yields correct expressions, but with a wrong interpretation for them. We may note that the phase of  $\psi$  in Eq. 13 can be obtained from the time part  $\tau = \gamma(t - \mathbf{v} \cdot \mathbf{r}/c^2)$  of the Lorentz transformation by multiplying it by  $m_0 c^2/\hbar$  (see [3], p.36). Here  $\tau$  is the proper time and  $m_0$  the rest mass. Like a Bloch wave in solid-state physics,  $\psi(\mathbf{r}, t)$  is just a one-dimensional representation of the abelian group  $\text{SO}(1,1)$  of one-dimensional Lorentz transformations whose boost parameters are  $\mathbf{v}' \parallel \mathbf{v}$ . If we had taken  $\mathbf{v} \parallel \mathbf{e}_x$ , we would just have written  $\psi(x, y, z, t) = e^{-\frac{i}{\hbar}(Et - p_x x)}$ . How would one derive  $\hat{L}_j$  from this expression? The abelian group  $\text{SO}(1,1)$  does not contain  $\text{SO}(3)$ , which is a non-abelian. How could it be possible that some commutation relations for  $\text{SO}(3)$  are derived from a development that belongs strictly to the context of  $\text{SO}(1,1)$ ? Surely enough, the non-abelian group  $\text{SO}(1,3)$  contains  $\text{SO}(3)$  but its representations are not one-dimensional and the algebra  $\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z$  used in Eq. 13 does not apply to the composition of boosts in  $\text{SO}(1,3)$ . The fact that we obtain correct expressions for  $\hat{L}_j$  must therefore be considered as a fluke. Correct proofs that  $\hat{L}_j$  are good angular momentum operators for  $\text{SO}(3)$  have been given above. The operators are defined in the Lie algebra. The elements of the Lie group can be parameterized by elements of  $\mathbb{R}^3$  or  $\mathbb{R}^4$ , like  $(\gamma, \gamma \mathbf{v}/c) \propto (E, c\mathbf{p})$  for the Lorentz boosts.

The fundamental reason for this horrible catastrophe is that the whole formalism of QM has been obtained by guessing and then validating the guesses by comparing the results of the algebra with experimental data. The algebra itself is used as a blackbox. The calculations are carried out without any insight about the corresponding geometry. The data validate the algebra, but not its geometrical interpretation. Consequently the algebra can be right, while its interpretation is wrong. In the present case it is even not possible to check the algebra because one cannot make linear sums of  $\hat{L}_x, \hat{L}_y, \hat{L}_z$ . Considering  $\hat{L}_x, \hat{L}_y, \hat{L}_z$  simultaneously is declared taboo in traditional QM because the operators do not commute, while in our approach the operators  $\hat{L}_j$  are just different forms which the operator  $\hat{\mathbf{L}}$  takes on various disjoint definition domains. For each direction in space defined by an axis  $d$ , there is a different form of the operator  $\hat{\mathbf{L}}$  with a different definition domain, which is the one-parameter family of rotations around  $d$ . Our derivation of QM from scratch in [3] is not based on guesses. It is a mathematically rigorous derivation. In our reconstruction of QM many quantum mysteries just disappear. The quantum mysteries are due to wrong interpretations of the algebra. An interpretation of the algebra is not due because the meaning of the algebra is already provided by the group theory itself. This cannot be paralleled by an alternative truth based on guesstimates proposed by physicists.

Of course, for  $\hat{S}_z \psi$  we can only find  $\pm \frac{\hbar}{2} \psi$  because  $\hat{S}_z$  applies by definition only to rotations about the  $z$ -axis. Here  $\psi = (\mathbb{1} + \sigma_z) \chi$ , whereby the spinor  $\chi$  is the first column of  $\cos(\varphi/2) \mathbb{1} - i \sin(\varphi/2) \sigma_z$  as explained in Remark 28 on p. 33 of [3]. Therefore there is no mysterious two-valuedness of the spin described by the spin formalism, and it does not explain the result of the Stern-Gerlach experiment, because this experiment measures precession-up and precession-down states instead of spin-up and spin-down states, as explained in [9].

Similarly, it is not true that the components  $L_x$  and  $L_y$  of the angular momentum cannot be defined simultaneously due to the fact that their operators do not commute. All the commutation relation tells us is that when the angular momentum is aligned with the  $x$ -axis, it cannot simultaneously be aligned with the  $y$ -axis! When the angular momentum  $\mathbf{L}$  is aligned with the  $x$ -axis, we must use the operator  $\hat{L}_x$  and we have  $\mathbf{L} = (L, 0, 0)$ . The use of the operator  $\hat{L}_x$  is strictly reserved for this situation  $\mathbf{L} = (L, 0, 0)$ . We have then  $L_x = L$  and  $[\mathbf{L}^2] = \hat{L}_x^2$ . Contrary to

what standard QM claims, the components  $L_x = L$ ,  $L_y = 0$  and  $L_z = 0$  are defined simultaneously. Similarly, the use of the operator  $\hat{L}_y$  is strictly reserved for the situation  $\mathbf{L} = (0, L, 0)$ . Just as the correct spin operator for a spin  $\frac{\hbar}{2}\mathbf{s}$  would be  $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z = \frac{\hbar}{2}[\mathbf{s} \cdot \boldsymbol{\sigma}]$ , the correct operator for the angular momentum would be  $\ell_x\hat{L}_x + \ell_y\hat{L}_y + \ell_z\hat{L}_z$ , where the unit vector  $\boldsymbol{\ell} = (\ell_x, \ell_y, \ell_z) \parallel \mathbf{L}$ . These expressions  $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z$  and  $\ell_x\hat{L}_x + \ell_y\hat{L}_y + \ell_z\hat{L}_z$  just serve to express  $\mathbf{s}$  and  $\boldsymbol{\ell}$  in the bases of their Lie algebras. This is in conformity with the fact that both the sets  $\hat{S}_j$  and  $\hat{L}_j$  form a basis. When the spin is parallel to  $\mathbf{s}$  only  $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z$  is a meaningful operator. An isolated component, e.g.  $s_z\hat{S}_z$  has then no meaning, because the eigenfunction  $\psi$  of  $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z$  is not an eigenfunction of  $\hat{S}_z$ . The definition domains of  $s_x\hat{S}_x + s_y\hat{S}_y + s_z\hat{S}_z$  and  $\hat{S}_z$  are disjoint. Every direction of space has its own operator. Just as  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  is not one vector operator but a basis of three vector operators,  $\hat{\mathbf{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$  is not one vector operator but a basis of three vector operators.

All this shows that guessing operators from their action on  $e^{-\frac{i}{\hbar}(Et - \mathbf{p} \cdot \mathbf{r})}$  is flawed, because it leads to the wrong conclusions that  $\hat{L}_j$  would be the general expression for the  $j$ -component of the angular momentum  $\mathbf{L}$  and that  $\hat{L}^2\psi = \hbar^2 l(l+1)\psi$  would define the operator for the square  $L^2$  of the angular momentum. We should never lose sight that SO(3) and SU(2) are mere Euclidean geometry. When one sticks to the correct geometrical interpretation of the algebra of the rotation group the classical intuition is restored.

## 2.5 Other possible meanings we can attribute to the parameters $(x, y, z)$

We have shown (see [3], p. 21), that in SU(2):

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = \begin{bmatrix} \pm \sqrt{\frac{x-iy}{2}} \\ \pm \sqrt{\frac{-x-iy}{2}} \end{bmatrix}, \quad (25)$$

represents a spinor corresponding to a rotation  $\mathbf{R}$ . Here  $(x, y, z) = \mathbf{e}'_x + i\mathbf{e}'_y \in \mathbb{C}^3$ , where  $(\mathbf{e}'_x, \mathbf{e}'_y, \mathbf{e}'_z)$  is the canonical triad of basis vectors  $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$  of  $\mathbb{R}^3$  after rotation by  $\mathbf{R}$ . Here  $(x, y, z) \in \mathcal{I} \subset \mathbb{C}^3$  is a so-called isotropic vector, which belongs to the isotropic cone  $\mathcal{I}$ . The idea behind introducing isotropic vectors is that from  $\mathbf{e}'_x + i\mathbf{e}'_y$  we can calculate the whole triad of rotated basis vectors of  $\mathbf{e}'_j$ , and that this triad defines the rotation unambiguously. The coordinates  $(x, y, z)$  of the isotropic vector are thus true group parameters, which is not the case for the parameters  $(x, y, z)$  and  $(p_x, p_y, p_z)$  in Eq. 13. The identity in Eq. 25 and its derivation lead to the notion that a spinor is a “square root of an isotropic vector”, but we have shown (see [3], p.7) that the spinors used in QM just represent group elements (i.e. rotations and Lorentz transformations). Let us calculate  $\hat{L}_z\boldsymbol{\xi}$ :

$$\begin{aligned} \hat{L}_z\boldsymbol{\xi} &= \frac{\hbar}{i} \left[ x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] \begin{bmatrix} \sqrt{\frac{x-iy}{2}} \\ \sqrt{\frac{-x-iy}{2}} \end{bmatrix} = \frac{\hbar}{2i} \begin{bmatrix} \frac{-ix-y}{2\sqrt{\frac{x-iy}{2}}} \\ \frac{-ix+y}{2\sqrt{\frac{-x-iy}{2}}} \end{bmatrix} = \frac{\hbar}{2i} \begin{bmatrix} \frac{-i(x-iy)/2}{\sqrt{\frac{x-iy}{2}}} \\ \frac{i(-x-iy)/2}{\sqrt{\frac{-x-iy}{2}}} \end{bmatrix} \\ &= -\frac{\hbar}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \boldsymbol{\xi} = -\frac{\hbar}{2} \sigma_z \boldsymbol{\xi} = -\hat{S}_z \boldsymbol{\xi}. \end{aligned} \quad (26)$$

Here the coordinates  $(x, y, z)$  of the isotropic vector have nothing to do with the position coordinates in Eq. 13, but we can imagine to generalize the definition domain of the algebra by analytic continuation from the isotropic cone  $\mathcal{I}$  to  $\mathbb{C}^3$  and then to restrict it to  $\mathbb{R}^3$ . This will not change the algebraic calculations. This is important for the harmonic polynomials which are tensor products  $\boldsymbol{\xi} \otimes \boldsymbol{\xi} \otimes \dots \otimes \boldsymbol{\xi}$  of the spinor  $\boldsymbol{\xi}$  (see [10], p.53 and [11], p. 68), and where  $(x, y, z)$  are subsequently used as position coordinates in QM (see [11], p. 85). If one were to stick to the original definition of  $(x, y, z) \in \mathcal{I}$  this would be just incomprehensible (see [11], p. 5). What saves us is that any meaning we give to the parameters  $(x, y, z)$  can be used in the algebra. The meaning does not intervene in the calculations. Note that we obtain  $-\hat{S}_z$  rather than  $\hat{S}_z$ . It is easy to see where this minus sign comes from. In fact  $\varphi = \omega\tau$  ends via  $E = \hbar\omega/2$  up in  $Et$ , while to obtain  $\hat{L}_j$  we have used  $-\mathbf{p} \cdot \mathbf{r}$ . The difference in sign is therefore not an issue. In fact, if we had started from  $\psi = e^{-\frac{i}{\hbar}(Et + \mathbf{p} \cdot \mathbf{r})}$  to define  $\hat{L}_z$ , we would have found  $\hat{L}_z\boldsymbol{\xi} = \hat{S}_z\boldsymbol{\xi}$ . Also changing the choice of the isotropic vector by the substitution  $y \rightarrow -y$  in  $\boldsymbol{\xi}$  would lead to  $\hat{L}_z\boldsymbol{\xi} = \hat{S}_z\boldsymbol{\xi}$ , as is easily checked. As another coincidence we can note that after introducing  $\boldsymbol{\psi} = (1 + \sigma_z)\boldsymbol{\chi}$ , Eq. 18 leads to  $\hat{S}_z\boldsymbol{\psi} = -\frac{\hbar}{i} \frac{\partial}{\partial \varphi} \boldsymbol{\psi}$ , which is again up to a sign analogous to the definition for  $\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \boldsymbol{\psi}$ , although  $\phi \neq \varphi$  is now the azimuthal angle of a set of spherical coordinates  $(\theta, \phi)$ .

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