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# Global observers with finite-time and fixed-time convergence for a class of uncertain mechanical systems

Amina BENARAB and W. Alejandro APAZA-PEREZ

**Abstract**—This paper proposes global sliding mode observers for a class mechanical systems of one degree of freedom. For the observer design, besides the usual Coriolis and centrifugal forces, we consider uncertain dry frictions and disturbances. Moreover, the system is not required to be bounded-input bounded-state stable, rendering the observer design problem challenging. The observer design exploits the specific relationships between the inertia and Coriolis terms providing a sliding-mode observer, with global theoretically exact finite-time and fixed-time convergence to the velocities of the mechanical system. The efficiency of proposed observer is validated through simulations on an inverted pendulum.

## I. INTRODUCTION

Control of mechanical systems usually requires the information about position and velocity. Often only the position is available for which an estimation of the velocity is required, this has given rise to interesting challenges in the area of control and observation [1], [2]. The main challenge in building observers to estimate the velocity in mechanical systems is the presence of nonlinear terms (e.g. from Coriolis and centrifugal forces) and disturbances (e.g. unmodeled dynamics). The sliding-mode observers/differentiators [3], [4], [5], [6], [7], [8] provide theoretically exact finite-time and fixed-time convergence to the real system's states despite the presence of bounded disturbances when the system has bounded-input bounded-state (BIBS) stability, consequently they are semi-global. If the BIBS property is not satisfied, then finite-time/fixed-time convergence is not guaranteed, see [9].

In this paper we will consider mechanical systems of one degree of freedom (1-DOF) with disturbances, which could not have the BIBS property.

### *State of art and motivation:*

1) If the dynamical systems with disturbances satisfying the conditions for the existence of an observer, among them the relative-degree-one condition [10], [11], the dissipative observers [12], [13], which include the classical high-gain observers, result to be efficient to deal with BIBS restriction. They are able to estimate globally and exponentially the real states using dissipative properties which could have the nonlinear terms. But mechanical systems with disturbances have relative degree greater than one with respect to (w.r.t.)

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the measured position.

2) There are many works (e.g. see [14], [15], [16], [17]) dealing with the Coriolis and centrifugal forces, which provide global observers when the model of the mechanical system is completely known. Through state transformations, [16], [17] propose observers with fairly high dimension, namely  $3n + 1$  and  $2n + 2$  respectively, where  $n$  is the dimension of the unmeasured velocity. [14], [15] propose observers with the same dimension of the system for a class of mechanical systems. However, the challenge of dealing with frictions, disturbances, and obtaining an estimation theoretically exact of velocity was not considered.

3) To overcome the BIBS restriction and the presence of Coriolis and centrifugal forces some recent works have been presented. [9] and [18] propose a finite-time sliding-mode observer for 1-DOF mechanical systems with disturbances and uncertain inertial matrix, respectively, using state transformations based in [14]. A extension to the 2-DOF was proposed in [19] using a transformation of states.

*Main contribution.* A global sliding mode observer for a class of mechanical systems with Coriolis and centrifugal forces, dry frictions, and bounded disturbances is proposed. The proposed observer does not require a prior state transformation and provides finite-time and fixed-time convergence to the position and velocity of the mechanical system. BIBS stability of mechanical system is not required, and its gains can be obtained from a feasible matrix inequality.

The rest of the paper is organized as follows. The problem statement is presented in section 2. Section 3 presents an observer design and the main results. Then, section 4 illustrates the proposed observer effectiveness through simulation results.

## II. PRELIMINARIES

*Notations.* Symbols  $\mathbb{R}$ ,  $\mathbb{R}_{\geq 0}$ ,  $\mathbb{R}_{> 0}$  denote the set of real, of nonnegative real numbers, of positive real numbers, respectively.  $|\cdot|$  denotes the absolute value in  $\mathbb{R}$  and  $\|\cdot\|$  denotes the Euclidean norm on  $\mathbb{R}^n$ . Let  $\lambda_{\min}(P)$  and  $\lambda_{\max}(P)$  be respectively the minimum and the maximum eigenvalues of the positive definite symmetric matrix  $P \in \mathbb{R}^{n \times n}$  and  $P^T$  defines the transposed matrix of  $P$ .

Define the signed power function  $[x]^r$ , for any  $x \in \mathbb{R}$  and  $r \in \mathbb{R}_{\geq 0}$ , as an increasing odd function defined as  $[x]^r = |x|^r \text{sign}(x)$ , where  $\text{sign}(x)$  is the standard sign function. The function  $[x]^r$  satisfies following properties:

- For each  $r \in [0, 1)$ ,  $[x]^r$  is differentiable for all  $x \neq 0$ ,  $[x]^r \in C^1$  if  $r \in [1, 2]$  and  $[x]^r \in C^2$  if  $r \in (2, \infty)$ .

- $\frac{d}{dx}[x]^r = r|x|^{r-1}$  and  $\frac{d}{dx}|x|^r = r[x]^{r-1}$ , for all  $x \neq 0$  and  $r \neq 0$ .
- For any  $r, s \in \mathbb{R}_{\geq 0}$ ,  $[x]^r [x]^s = |x|^{r+s}$  and  $|x|^r [x]^s = [x]^{r+s}$  and  $[x]^0 = \text{sign}(x)$ .

#### A. Finite-time and fixed-time stability

Consider the following system

$$\dot{x}(t) = f(t, x(t)), \quad t > t_0 \quad x(t_0) = x_0, \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state vector and  $f: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a nonlinear continuous vector field. Assume that the origin is an equilibrium point of (1), i.e.  $f(0) = 0$ .

*Definition 1 ([20]):* The origin  $x = 0$  of system (1) is said to be globally Finite-Time Stable if it is globally asymptotically stable and any solution  $x(t, x_0)$  of (1) reaches the equilibria at some finite-time moment, i.e.  $x(t, x_0) = 0 \forall t \geq T(x_0)$  where  $T: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is the so-called the settling-time function.

*Definition 2 ([21]):* The origin  $x = 0$  of system (1) is said to be globally Fixed-Time Stable if it is globally finite-time stable and the settling time function  $T$  is bounded by some positive number  $T_{\max} > 0$ , i.e.  $T(x_0) \leq T_{\max}$  for  $\forall x_0 \in \mathbb{R}^n$ .

We recall some technical results and well known properties of continuous homogeneous functions.

*Lemma 3 (Young's inequality [22]):* For any positive real numbers  $a > 0, b > 0, c > 0, p > 1$  and  $q > 1$ , with  $\frac{1}{p} + \frac{1}{q} = 1$ , the following inequality holds

$$ab \leq \frac{c^p}{p} a^p + \frac{c^{-q}}{q} b^q$$

and equality holds if and only if  $a^p = b^q$ .

*Lemma 4 (Jensen's inequality [22]):* For any real numbers  $a_1, \dots, a_n$ , if  $0 < p < q$

$$\left( \sum_{i=1}^n |a_i|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^n |a_i|^q \right)^{\frac{1}{q}}.$$

### III. PROBLEM STATEMENT

For convenience notations, in the following we will consider  $x, q, e, u, \delta$  instead of  $x(t), q(t), e(t), u(t), \delta(t)$ , respectively.

Consider the following 1-DOF mechanical systems

$$m(q)\ddot{q} + c(q)\dot{q}^2 + \ell \cdot \text{sign}(\dot{q}) + g(q) = u + \tilde{\delta} \quad (2)$$

where  $q \in \mathbb{R}$  denotes the (measured) generalized position,  $\dot{q}$  is the generalized velocity,  $m(q)$  denotes the inertia term,  $c(q)\dot{q}^2$  is Coriolis and centrifugal force,  $\ell \text{sign}(\dot{q})$  is a dry friction,  $u$  is the measured torque,  $g(q)$  denotes gravitational forces and  $\tilde{\delta}$  represents the disturbance in the system.

The objective is to design an observer for the system (2) with theoretically exact global finite-time convergence to the real values of position and velocity.

Consider the system (2), which additionally satisfies the following assumptions:

(A-1) There exist constants  $\zeta_m > 0$  and  $\zeta_M > 0$  such that

$$0 < \zeta_m \leq m(q) \leq \zeta_M, \quad \text{for all } q,$$

is satisfied.

(A-2) The inertia and Coriolis functions satisfy

$$\dot{m}(q) = 2c(q)\dot{q}$$

(A-3) The disturbance  $\tilde{\delta}(t)$  is bounded, i.e. there exists a constant  $L_{\tilde{\delta}} > 0$  such that  $\|\tilde{\delta}(t)\| \leq L_{\tilde{\delta}}$  is satisfied.

Assumptions (A-1) and (A-2) are assumed in most of mechanical systems [1]. Assumption (A-3) is a standard condition for the construction of the observer due to the fact that the system has relative degree greater than one [10]

### IV. OBSERVER DESIGN AND RESULTS

Considering  $q_1 = q, q_2 = \dot{q}$ , the state space representation of (2) is given by

$$\begin{cases} \dot{q}_1 = q_2 \\ \dot{q}_2 = \frac{1}{m(q_1)}(u - g(q_1) - c(q_1)q_2^2 + \delta) \\ y = q_1 \end{cases} \quad (3)$$

where  $\delta = \tilde{\delta} - \ell \cdot \text{sign}(\dot{q})$ .

We propose the following observer<sup>1</sup> for system (3):

$$\begin{cases} \dot{\hat{q}}_1 = \frac{1}{\sqrt{m(\hat{q}_1)}} \left( \sqrt{m(\hat{q}_1)} \hat{q}_2 + k_1 \psi_1(q_1 - \hat{q}_1) \right) \\ \dot{\hat{q}}_2 = \frac{1}{\sqrt{m(\hat{q}_1)m(q_1)}} (u - g(q_1) + k_2 \psi_2(q_1 - \hat{q}_1)) \\ \quad \quad \quad - \frac{1}{m(\hat{q}_1)} c(\hat{q}_1) \hat{q}_2^2 \end{cases} \quad (4)$$

where the correction functions  $\psi_1$  and  $\psi_2$  are defined by

$$\psi_1(\cdot) = \alpha[\cdot]^{\frac{1}{2}} + \beta[\cdot] + \gamma[\cdot]^{\frac{3}{2}} \quad (5)$$

$$\psi_2(\cdot) = \frac{\alpha^2}{2}[\cdot]^0 + \frac{3\alpha\beta}{2}[\cdot]^{\frac{1}{2}} + \beta^2[\cdot] + \alpha\gamma[\cdot] + \frac{5}{2}\beta\gamma[\cdot]^{\frac{3}{2}} + \frac{3}{2}\gamma^2[\cdot]^2 \quad (6)$$

#### A. Finite-time observer

In this section, consider the observer (4), and the nonlinear functions  $\psi_1$  and  $\psi_2$  given in (5) and (6), respectively, with  $\alpha > 0$  and  $\beta, \gamma \geq 0$ .

*Lemma 5:* Let  $C = [1 \ 0]$ ,  $B = [0 \ 1]^T$  and  $L_{\tilde{\delta}} > 0$ . Then, there exist  $\rho > 0, \alpha > 0$ , a constant matrix  $A = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}$  and  $P \in \mathbb{R}^{2 \times 2}$  a symmetric and positive definite matrix, such that the following nonlinear matrix inequality holds.

$$\begin{bmatrix} A^T P + PA + C^T L_{\tilde{\delta}}^2 C + \rho P & PB \\ B^T P & -\frac{\alpha^4}{4} \end{bmatrix} < 0. \quad (7)$$

*Proof:* For the symmetric matrix  $C^T L_{\tilde{\delta}}^2 C + I$  ( $I = I_{2 \times 2}$ ), there is a positive definite symmetric solution  $P = P^T > 0$  for the next inequality (quadratic Lyapunov inequality)

$$A^T P + PA + C^T L_{\tilde{\delta}}^2 C + I < 0. \quad (8)$$

In other hand, one has

$$\lambda_{\min}^2(P)I \leq P \leq \lambda_{\max}^2(P)I \quad (9)$$

<sup>1</sup>The solutions of the system (3) are defined in Filippov's sense (see [23]).

which implies that

$$I \geq \frac{P}{\lambda_{\max}^2(P)}. \quad (10)$$

Substituting inequality (10) in (8), one gets

$$PA + A^T P + C^T L_\delta^2 C + \rho P < 0$$

where  $\rho = \frac{1}{\lambda_{\max}^2(P)}$ . For  $\alpha > 0$  sufficiently large the following inequality is satisfied

$$-\frac{\alpha^4}{4} + B^T P [PA + A^T P + C^T L_\delta^2 C + \rho P]^{-1} P B < 0.$$

Finally, by Schur complement one can deduce that (7) holds.  $\blacksquare$

*Theorem 6:* The system (4) is a global finite-time observer for the system (3) in the sense that the estimation states  $\hat{q}_1$  and  $\hat{q}_2$  converge exactly in finite to the position  $q_1$  and the velocity  $q_2$  of system (3).

*Proof:* Define the error estimation of the systems (3) and (4) as:

$$\begin{aligned} e_1 &= q_1 - \hat{q}_1 \\ e_2 &= \sqrt{m(q_1)} q_2 - \sqrt{m(\hat{q}_1)} \hat{q}_2 \end{aligned}$$

and its dynamic error is given by

$$\begin{cases} \dot{e}_1 &= \frac{e_2 - k_1 \psi_1(e_1)}{\sqrt{m(q_1)}} \\ \dot{e}_2 &= \frac{1}{\sqrt{m(q_1)}} (-k_2 \psi_2(e_1) + \delta) \end{cases} \quad (11)$$

where  $\dot{m}(q_1) = 2c(q_1)q_2$  and  $\dot{m}(\hat{q}_1) = 2c(\hat{q}_1)\hat{q}_2$  are satisfied by Assumption (A-2).

Consider a homeomorphism  $\eta$  (from  $[e_1 \ e_2]$ ) defined as  $\eta = [\psi_1(e_1) \ e_2]^T$  and a quadratic Lyapunov function

$$V_1(\eta) = \eta^T P \eta$$

where  $P = P^T > 0$  is obtained from the matrix inequality (7). The time derivative of  $\eta$  with  $e_1 \neq 0$  is given by

$$\begin{aligned} \dot{\eta} &= [\psi_1'(e_1) \dot{e}_1 \quad \dot{e}_2]^T \\ &= \frac{\psi_1'(e_1)}{\sqrt{m(q_1)}} \left( A \eta + B \frac{\delta}{\psi_1'(e_1)} \right) \end{aligned} \quad (12)$$

where  $\psi_1'(e_1)$  denotes the derivative with respect to  $e_1$ ,  $\psi_2(e_1) = \psi_1'(e_1) \psi_1(e_1)$ ,  $A = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

The time derivative of  $V_1$  along trajectories (11):

$$\begin{aligned} \dot{V}_1(\eta) &= \dot{\eta}^T P \eta + \eta^T P \dot{\eta} \\ &= \frac{\psi_1'(e_1)}{\sqrt{m(q_1)}} \left( \eta^T (A^T P + P A) \eta + \eta^T P B \frac{\delta}{\psi_1'(e_1)} \right. \\ &\quad \left. + \frac{\delta}{\psi_1'(e_1)} B^T P \eta \right) \\ &= \frac{\psi_1'(e_1)}{\sqrt{m(q_1)}} \left[ \frac{\eta}{\psi_1'(e_1)} \right]^T \begin{bmatrix} A^T P + P A & P B \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \frac{\delta}{\psi_1'(e_1)} \end{bmatrix} \end{aligned} \quad (13)$$

Note that nonlinear correction term  $\psi_2$  defined in (6) satisfies

$$\begin{aligned} |\psi_2(e_1)| &= \left| [e_1]^0 \left( \frac{\alpha^2}{2} + \frac{3\alpha\beta}{2} |e_1|^{\frac{1}{2}} + \beta^2 |e_1| + \frac{3}{2} \gamma^2 |e_1|^2 \right. \right. \\ &\quad \left. \left. + \alpha \gamma |e_1| + \frac{5}{2} \beta \gamma |e_1|^{\frac{3}{2}} \right) \right| \\ &\geq \frac{\alpha^2}{2} \end{aligned}$$

which implies

$$\left( \frac{\delta}{\psi_1'(e_1)} \right)^2 = \left( \frac{\delta}{\psi_2(e_1)} \psi_1(e_1) \right)^2 \leq \left( \frac{2L_\delta}{\alpha^2} \right)^2 \psi_1^2(e_1) \quad (14)$$

with  $|\delta| \leq L_\delta$ . The inequality (14) can be written as a matrix inequality in terms of  $\eta$  and  $\frac{\delta}{\psi_1'(e_1)}$

$$0 \leq \begin{bmatrix} \frac{\eta}{\frac{\delta}{\psi_1'(e_1)}} \end{bmatrix}^T \begin{bmatrix} C^T L_\delta^2 C & 0 \\ 0 & -\left(\frac{\alpha^2}{2}\right)^2 \end{bmatrix} \begin{bmatrix} \frac{\eta}{\frac{\delta}{\psi_1'(e_1)}} \end{bmatrix}. \quad (15)$$

Adding the inequality (15) in the time derivative of Lyapunov function  $V_1(\eta)$ , we obtain

$$\begin{aligned} \dot{V}_1(\eta) &\leq \frac{\psi_1'(e_1)}{\sqrt{m(q_1)}} \begin{bmatrix} \frac{\eta}{\frac{\delta}{\psi_1'(e_1)}} \end{bmatrix}^T \begin{bmatrix} A^T P + P A + C^T L_\delta^2 C & P B \\ B^T P & -\left(\frac{\alpha^2}{2}\right)^2 \end{bmatrix} \begin{bmatrix} \frac{\eta}{\frac{\delta}{\psi_1'(e_1)}} \end{bmatrix} \\ &\leq -\rho \frac{\psi_1'(e_1)}{\sqrt{m(q_1)}} V_1(\eta) \quad (\text{from (7)}) \\ &\leq -\frac{\rho}{\sqrt{m(q_1)}} \left( \frac{\alpha}{2} |e_1|^{-\frac{1}{2}} + \beta \right) V_1(\eta). \end{aligned} \quad (16)$$

Note that

$$\begin{aligned} \alpha |e_1|^{\frac{1}{2}} &\leq |[e_1]^0 (\alpha |e_1|^{\frac{1}{2}} + \beta |e_1| + \gamma |e_1|^{\frac{3}{2}})| \\ &\leq |\psi_1(e_1)| \leq \|\eta\| \leq \frac{V_1^{\frac{1}{2}}(\eta)}{\lambda_{\min}(P)} \end{aligned} \quad (17)$$

holds, which is given from (9) and implies in (16) the next inequality holds

$$\dot{V}_1(\eta) \leq -\frac{\rho \alpha^2 \lambda_{\min}(P)}{2\sqrt{m(q_1)}} V_1^{\frac{1}{2}}(\eta) - \frac{\rho \beta}{\sqrt{m(q_1)}} V_1(\eta) \quad (18)$$

Note that trajectories of the estimation error dynamics cannot stay on the set  $S = \{(e_1, e_2) \in \mathbb{R}^2 \setminus \{0\} | e_1 = 0\}$ . This means  $V_1$  is a continuously decreasing function by using a Lyapunov's Theorem for Differential Inclusions [24, Prop. 14.1 p. 205]; (that does not require differentiability of the Lyapunov function). Since the solution of the differential equation  $\dot{v} = -\gamma_1 v^{1/2} - \gamma_2 v$  is given by  $v(t) = \exp(-\gamma_2 t) \left[ v_0^{1/2} + \frac{\gamma_1}{\gamma_2} (1 - \exp(\frac{\gamma_2}{2} t)) \right]^2$ , we can conclude that the equilibrium point  $(e_1, e_2) = 0$  is reached in finite time from every initial condition.  $\blacksquare$

### B. Fixed-time observer

In this section, consider the observer (4) with the nonlinear functions  $\psi_1$  and  $\psi_2$  given in (5) and (6) respectively, where  $\alpha, \gamma > 0$  and  $\beta = 0$  for simplicity of the proof, which is inspired in the Lyapunov function proposed in [7].

*Theorem 7:* The system (4) is a global fixed-time observer for the system (3) in the sense that the estimation states  $\hat{q}_1$

and  $\hat{q}_2$  converge uniformly w.r.t. the initial condition and in fixed-time to the position  $q_1$  and the velocity  $q_2$  of system (3).

*Proof:* Consider a robust Lyapunov-Like function defined as

$$V_2(e) = a|e_1|^3 - b|e_1| |e_2|^{\frac{4}{3}} + c|e_2|^2.$$

We use the following form inequalities, derived from the Young's inequality in Lemma 3:

$$\begin{aligned} (a) \quad & |e_1||e_2| \leq \frac{1}{3}\omega_0^3|e_1|^3 + \frac{2}{3}\omega_0^{-\frac{3}{2}}|e_2|^2, \quad \forall \omega_0 > 0, \\ (b) \quad & |e_1|^{\frac{1}{2}}|e_2|^{\frac{4}{3}} \leq \frac{1}{5}\omega_1^5|e_1|^{\frac{5}{2}} + \frac{4}{5}\omega_1^{-\frac{5}{4}}|e_2|^{\frac{5}{3}}, \quad \forall \omega_1 > 0, \\ (c) \quad & |e_1|^{\frac{3}{2}}|e_2|^{\frac{4}{3}} \leq \frac{3}{7}\omega_2^{\frac{7}{2}}|e_1|^{\frac{7}{2}} + \frac{4}{7}\omega_2^{-\frac{7}{4}}|e_2|^{\frac{7}{3}}, \quad \forall \omega_2 > 0, \\ (d) \quad & |e_1||e_2|^{\frac{1}{3}} \leq \frac{2}{3}\omega_3^{\frac{3}{2}}|e_1|^{\frac{3}{2}} + \frac{1}{3}\omega_3^{-3}|e_2|, \quad \forall \omega_3 > 0, \\ (e) \quad & |e_1|^2|e_2|^{\frac{1}{3}} \leq \frac{4}{5}\omega_4^{\frac{5}{4}}|e_1|^{\frac{5}{2}} + \frac{1}{5}\omega_4^{-5}|e_2|^{\frac{5}{3}}, \quad \forall \omega_4 > 0, \\ (f) \quad & |e_1|^3|e_2|^{\frac{1}{3}} \leq \frac{6}{7}\omega_5^{\frac{7}{2}}|e_1|^{\frac{7}{2}} + \frac{1}{7}\omega_5^{-7}|e_2|^{\frac{7}{3}}, \quad \forall \omega_5 > 0, \\ (g) \quad & |e_1||e_2| \leq \frac{2}{5}\omega_6^{\frac{5}{2}}|e_1|^{\frac{5}{2}} + \frac{3}{5}\omega_6^{-\frac{5}{3}}|e_2|^{\frac{5}{3}}, \quad \forall \omega_6 > 0, \\ (h) \quad & |e_1|^2|e_2| \leq \frac{4}{7}\omega_7^{\frac{7}{2}}|e_1|^{\frac{7}{2}} + \frac{3}{7}\omega_7^{-7}|e_2|^{\frac{7}{3}}, \quad \forall \omega_7 > 0. \end{aligned}$$

From inequality (a), the function  $V_2$  satisfies

$$\tilde{a}_1|e_1|^3 + \tilde{a}_2|e_2|^2 \leq V_2(e) \leq \tilde{b}_1|e_1|^3 + \tilde{b}_2|e_1|^2 \quad (19)$$

where

$$\tilde{a}_1 = \left(a - \frac{b\omega_0^3}{3}\right), \quad \tilde{a}_2 = \left(a - \frac{2b\omega_0^{-\frac{3}{2}}}{3}\right), \quad \tilde{b}_1 = \left(a + \frac{b\omega_0^3}{3}\right),$$

$$\text{and } \tilde{b}_2 = \left(a + \frac{2b\omega_0^{-\frac{3}{2}}}{3}\right).$$

The derivative of  $V_2$  along the trajectories of (11) is given by

$$\begin{aligned} \dot{V}_2(e) = & \frac{1}{\sqrt{m(q_1)}} \left\{ -3ak_1\alpha^2|e_1|^{5/2} - 3a\alpha k_1\gamma|e_1|^{7/2} \right. \\ & - b|e_2|^{\frac{7}{3}} + 3a\alpha|e_1|^2e_2 + bk_1\alpha|e_1|^{\frac{1}{2}}|e_2|^{\frac{4}{3}} \\ & + bk_1\gamma|e_1|^{\frac{3}{2}}|e_2|^{\frac{4}{3}} + \frac{2}{3}bk_2\alpha^2|e_1||e_2|^{\frac{1}{3}} \\ & + \frac{4}{3}bk_2\alpha\gamma|e_1|^2|e_2|^{\frac{1}{3}} + 2bk_2\gamma^2|e_1|^3|e_2|^{\frac{1}{3}} \\ & - ck_2\alpha^2|e_1|^0e_2 - 2ck_2\alpha\gamma e_1 e_2 \\ & \left. - 3ck_2\gamma^2|e_1|^2e_2 - \frac{4}{3}b\delta|e_1||e_2|^{\frac{1}{3}} + 2c\delta e_2 \right\} \end{aligned}$$

which can be bounded as the following

$$\begin{aligned} \dot{V}_2(e) \leq & \frac{1}{\sqrt{m(q_1)}} \left\{ -3ak_1\alpha^2|e_1|^{5/2} - 3a\alpha k_1\gamma|e_1|^{7/2} \right. \\ & - b|e_2|^{\frac{7}{3}} + bk_1\alpha|e_1|^{\frac{1}{2}}|e_2|^{\frac{4}{3}} + bk_1\gamma|e_1|^{\frac{3}{2}}|e_2|^{\frac{4}{3}} \\ & + \left(\frac{2}{3}bk_2\alpha^2 + \frac{4}{3}bL_\delta\right)|e_1||e_2|^{\frac{1}{3}} + \frac{4}{3}bk_2\alpha\gamma|e_1|^2|e_2|^{\frac{1}{3}} \\ & + 2bk_2\gamma^2|e_1|^3|e_2|^{\frac{1}{3}} + (ck_2\alpha^2 + 2cL_\delta)|e_2| \\ & \left. + 2ck_2\alpha\gamma|e_1||e_2| + (3ck_2\gamma^2 + 3a\alpha)|e_1|^2|e_2| \right\}. \end{aligned}$$

Using inequalities (b)-(h), the previous inequality leads to

$$\begin{aligned} \dot{V}_2(e) \leq & \frac{1}{\sqrt{m(q_1)}} \left\{ -\vartheta_1|e_1|^{\frac{7}{2}} - \vartheta_2|e_2|^{\frac{7}{3}} + \vartheta_3(|e_1|^{\frac{5}{2}} + |e_2|^{\frac{5}{3}}) \right. \\ & \left. + \vartheta_4(|e_1|^{\frac{3}{2}} + |e_2|) \right\} \end{aligned}$$

where

$$\begin{aligned} \vartheta_1 = & 3a\alpha k_1\gamma - \frac{3}{7}bk_1\gamma\omega_2^{\frac{7}{2}} - \frac{12}{5}bk_2\gamma^2\omega_5^{\frac{7}{2}} \\ & - \frac{12}{7}(ck_2\gamma^2 + 3a\alpha)\omega_7^{\frac{7}{2}}, \\ \vartheta_2 = & b - \frac{4}{7}bk_1\gamma\omega_2^{-\frac{7}{4}} - \frac{2}{7}bk_2\gamma^2\omega_5^{-7} - \frac{9}{7}(ck_2\gamma^2 + a\alpha)\omega_7^{-\frac{7}{3}}, \\ \vartheta_3 = & \max\{\zeta_3, \xi_3\}, \\ \vartheta_4 = & \max\{\zeta_4, \xi_4\}, \\ \zeta_3 = & -3ak_1\alpha^2 + \frac{1}{5}bk_1\alpha\omega_1^5 + \frac{16}{15}bk_2\alpha\gamma\omega_4^{\frac{5}{4}} + \frac{4}{5}ck_2\alpha\gamma\omega_6^{\frac{5}{2}}, \\ \xi_3 = & \frac{4}{5}bk_1\alpha\omega_1^{-\frac{5}{4}} + \frac{4}{15}bk_2\alpha\gamma\omega_4^{-5} + \frac{6}{5}ck_2\alpha\gamma\omega_6^{-\frac{5}{3}}, \\ \zeta_4 = & \frac{4}{9}(bk_2\alpha^2 + 2bL_\delta)\omega_3^{\frac{3}{2}}, \\ \xi_4 = & c(k_2\alpha^2 + 2L_\delta) + \frac{2}{9}b(k_2\alpha^2 + 2L_\delta)\omega_4^{-3}. \end{aligned}$$

We recall the fundamental inequality proposed in [25, Thm. 16, Section 2.9]

$$(\tilde{\sigma}|x_1|^{s_1} + (1-\tilde{\sigma})|x_2|^{s_1})^{\frac{1}{s_1}} \leq (\tilde{\sigma}|x_1|^{s_2} + (1-\tilde{\sigma})|x_2|^{s_2})^{\frac{1}{s_2}} \quad (20)$$

$0 < \tilde{\sigma} < 1$ ,  $\forall x \in \mathbb{R}^2$ ,  $0 < s_1 \leq s_2$ . This inequality implies

$$\begin{aligned} \frac{1}{2}(|e_1|^{\frac{5}{2}} + |e_2|^{\frac{5}{3}}) & \leq \frac{1}{2^{\frac{5}{7}}} \left(|e_1|^{\frac{7}{2}} + |e_2|^{\frac{7}{3}}\right)^{\frac{5}{7}} \\ \frac{1}{2}(|e_1|^{\frac{3}{2}} + |e_2|) & \leq \frac{1}{2^{\frac{3}{7}}} \left(|e_1|^{\frac{7}{2}} + |e_2|^{\frac{7}{3}}\right)^{\frac{3}{7}} \end{aligned}$$

which imply that  $\dot{V}_2$  satisfies

$$\begin{aligned} \dot{V}_2(e) \leq & -\vartheta_5\|e\|_h^{\frac{7}{2}} + \vartheta_6\|e\|_h^{\frac{5}{2}} + \vartheta_7\|e\|_h^{\frac{3}{2}} \\ = & -\sigma\vartheta_5\|e\|_h^{\frac{7}{2}} - \|e\|_h^{\frac{3}{2}}[(1-\sigma)\vartheta_5\|e\|_h^2 - \vartheta_6\|e\|_h - \vartheta_7] \end{aligned}$$

where  $0 < \sigma < 1$ ,  $\|e\|_h = \left(|e_1|^{\frac{7}{2}} + |e_2|^{\frac{7}{3}}\right)^{\frac{2}{7}}$  and  $\vartheta_5 = \min\{\vartheta_1, \vartheta_2\}$ ,  $\vartheta_6 = 2^{\frac{2}{7}}\vartheta_3$ ,  $\vartheta_7 = 2^{\frac{4}{7}}\vartheta_4$ . We can ensure that:

$$\dot{V}_2(e) \leq -\sigma\vartheta_5\|e\|_h^{\frac{7}{2}} \leq -\vartheta_8V_2^{\frac{7}{6}}(e), \quad (21)$$

for all  $e \in \mathcal{C} = \left\{ e \in \mathbb{R}^2 \mid \|e\|_h \geq \frac{\vartheta_6 + \sqrt{\vartheta_6^2 + 4(1-\sigma)\vartheta_7}}{2(1-\sigma)} \right\}$  and

$$\vartheta_8 = 2^{-\frac{1}{6}} \sigma \vartheta_5 \tilde{c}^{-\frac{7}{6}} \text{ with } \tilde{c} = \max\{\tilde{b}_1, \tilde{b}_2\}.$$

Since the solution of the differential equation  $\dot{v} = -\gamma v^{\frac{7}{6}}$  is given by  $v(t) = (v_0^{-\frac{1}{6}} + \frac{\gamma}{6}t)^{-6}$ , then, from (21) and comparison principle,  $V_2(e(t)) \leq (V_2(e(0))^{-\frac{1}{6}} + \frac{\vartheta_8}{6}t)^{-6}$  which implies  $V_2(e(t)) \leq V_2(e(0))$ . Put  $V_2(e(t)) = \tilde{k} \geq \frac{\vartheta_6 + \sqrt{\vartheta_6^2 + 4(1-\sigma)\vartheta_7}}{2(1-\sigma)}$ , the time  $T_{\tilde{k}}$  to go from  $V_2(e(0))$  to  $\tilde{k}$  is bounded by  $\frac{6}{\vartheta_8} \left( \tilde{k}^{-\frac{1}{6}} - V_2(e(0))^{-\frac{1}{6}} \right)$ , which can be bounded independently of the initial condition  $V_2(e(0))$ , i.e.  $T_{\tilde{k}} \leq \frac{6}{\vartheta_8} \tilde{k}^{-\frac{1}{6}}$ . This implies that the trajectories of the system (11) reach the compact set  $\tilde{\mathcal{C}} = \{e \in \mathbb{R}^2 \mid \|e\|_h \geq \tilde{k}\}$  in a time bounded by  $\frac{6}{\vartheta_8} \tilde{k}^{-\frac{1}{6}}$  independently of the initial condition. The time to reach  $e = 0$  with initial conditions in  $\tilde{\mathcal{C}}$  can be bounded by the compactness of  $\tilde{\mathcal{C}}$ . ■

## V. SIMULATION EXAMPLE

Consider an inverted pendulum on a cart which is adapted from [26]

$$\begin{aligned} \left(1 - \frac{3\varepsilon}{4} \cos^2(q)\right) \ddot{q} + \frac{3\varepsilon}{8} \dot{q}^2 \sin(2q) \\ - \sin(q) - u \cos(q) + \delta = 0. \end{aligned} \quad (22)$$

where  $m$  and  $M$  are the pendulum mass and cart mass respectively, and  $\varepsilon = \frac{m}{m+M}$  is the relative mass,  $u$  is the control law that is the horizontal driving force and  $\delta$  represents the disturbance. Assumption (A-1) is satisfied with  $\zeta_m = 1 - \frac{3\varepsilon}{4}$  and  $\zeta_M = 1$ , assumption (A-2) can be straightforwardly verified from trigonometric properties.

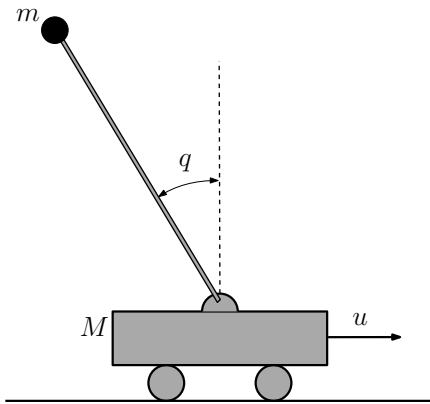


Fig. 1. Inverted pendulum on a cart.

For the numerical results reported below, the following parameters were chosen solving the matrix inequality (7) with disturbance bound  $L_\delta = 0.5$ :  $P = \begin{bmatrix} 0.78 & -0.4 \\ -0.4 & 0.37 \end{bmatrix}$ ,  $A = \begin{bmatrix} -1.956 & 1 \\ -1.87 & 0 \end{bmatrix}$ ,  $\rho = 1$ , and  $\alpha = 1.1$ . The pendulum mass and cart mass are chosen as  $m = 0.2$ ,  $M = 1$ . The proposed observer

is given by

$$\begin{cases} \dot{\hat{q}}_1 = \frac{1}{\sqrt{m(q_1)}} \left( \sqrt{m(\hat{q}_1)} \dot{q}_2 + 1.95 \cdot \psi_1(e_1) \right) \\ \dot{\hat{q}}_2 = \frac{1}{\sqrt{m(\hat{q}_1)m(q_1)}} (u - g(q_1) + 1.87 \psi_2(q_1 - \hat{q}_1)) \\ \quad - \frac{1}{m(\hat{q}_1)} c(\hat{q}_1) \hat{q}_2^2 \end{cases}$$

where  $m(q_1) = 1 - \frac{3\varepsilon}{4} \cos^2(q_1)$ ,  $m(\hat{q}_1) = 1 - \frac{3\varepsilon}{4} \cos^2(\hat{q}_1)$ ,  $c(\hat{q}_1) = \frac{3\varepsilon}{8} \sin(2\hat{q}_1)$ ,  $\psi_1(e_1) = 1.1[e_1]^{\frac{1}{2}} + [e_1]$  and  $\psi_2(e_1) = \frac{(1.1)^2}{2} [e_1]^0 + \frac{3(1.1)}{2} [e_1]^{\frac{1}{2}} + [e_1]$ .

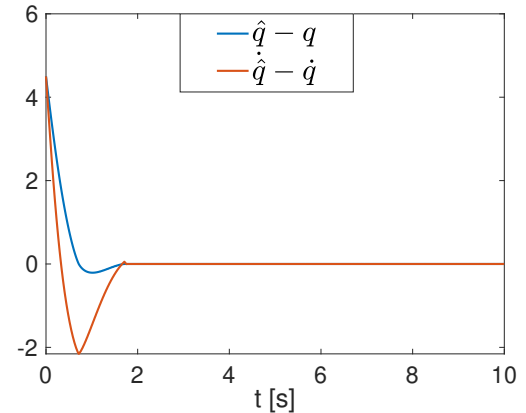
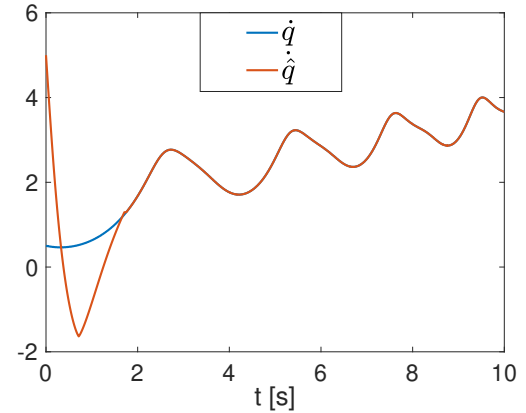
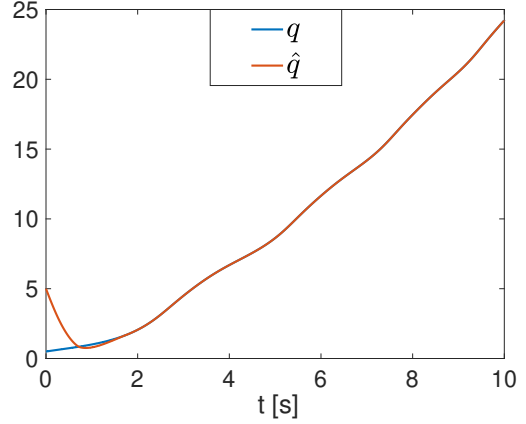


Fig. 2. Position, Velocity and Errors of system (22)

Figure 2. presents the simulation results for finite-time observer for initial conditions  $q(0) = 0.5$  and  $\hat{q}(0) = 5$ . Note

that the considered mechanical system is unstable, but the convergence of the observed state to the real state of the mechanical system is guaranteed in a finite-time (less than 2 seconds).

## VI. CONCLUSION

In this paper, we propose a global sliding mode observer for a class of mechanical systems with Coriolis and centrifugal forces and disturbances. The proposed observer does not require of a prior transformation and provides finite-time and fixed-time convergence to the position and velocity of the mechanical system. BIBS stability of mechanical system is not required for the observer design, and its gains can be obtained from a feasible matrix inequality.

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