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Uniform in time propagation of chaos for the 2D vortex model and other singular stochastic systems

Arnaud Guillin*, Pierre Le Bris† and Pierre Monmarché‡

Abstract

In this article, we adapt the work of Jabin and Wang in [JW18] to show the first result of uniform in time propagation of chaos for a class of singular interaction kernels. In particular, our models contain the Biot-Savart kernel on the torus and thus the 2D vortex model.

Keywords: propagation of chaos, relative entropy, logarithmic Sobolev inequality, 2D vortex equation, singular kernels.

2020 Mathematics Subject Classifications: 35Q30, 26D10, 60F17, 60H10, 47D07, 76R99.

1 Introduction

1.1 Framework

Our main subject is the convergence of the law of a stochastic particles system with mean field singular interactions towards its non linear limit. More precisely we will establish the first quantitative bounds in the number of particles uniformly in time. Let $K : \mathbb{T}^d \rightarrow \mathbb{R}^d$ be an *interaction kernel* on the d -dimensional ($d \geq 2$) 1-periodic torus \mathbb{T}^d (represented as $[-\frac{1}{2}, \frac{1}{2}]^d$), on which we will specify some assumptions later. In this paper, we consider the non linear stochastic differential equation of *McKean-Vlasov type*

$$\begin{cases} dX_t = \sqrt{2}dB_t + K * \bar{\rho}_t(X_t)dt \\ \bar{\rho}_t = \text{Density of Law}(X_t), \end{cases} \quad (1.1)$$

where $X_t \in \mathbb{T}^d$, $(B_t)_{t \geq 0}$ is a d -dimensional Brownian motion and $f * g(x) = \int_{\mathbb{T}^d} f(x-y)g(y)dy$ stands for the convolution operation on the torus. The density $\bar{\rho}_t$ satisfies

$$\partial_t \bar{\rho}_t = -\nabla \cdot (\bar{\rho}_t (K * \bar{\rho}_t)) + \Delta \bar{\rho}_t. \quad (1.2)$$

In the other words, the non-linear Equation (1.2) has the following natural probabilistic interpretation : the solution $\bar{\rho}_t$ is the density of the law at time t of the \mathbb{T}^d valued process $(X_t)_{t \geq 0}$ evolving according to (1.1). As we understand (1.1) to be the motion of a particle interacting with its own law, (1.2) thus describes the dynamic of a cloud of charged particles (where $(X_t)_{t \geq 0}$ would be one particle). In particular, it holds importance in plasma physics, see [Vla68]. We also consider the associated system of particles, describing the motion of N particles interacting with one another through the interaction kernel K .

$$dX_t^i = \sqrt{2}dB_t^i + \frac{1}{N} \sum_{j=1}^N K(X_t^i - X_t^j)dt, \quad (1.3)$$

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where $X_t^i \in \mathbb{T}^d$ is the position at time t of the i -th particle, and $(B_t^i, 1 \leq i \leq N)$ are independent Brownian motions in \mathbb{T}^d . We assume that $(X_0^i)_{i=1, \dots, N}$ are exchangeable, i.e. have a law which is invariant by permutation of the particles, so that this property is true for all times. We denote by ρ_N the density of the law of the system of particles, formally satisfying

$$\partial_t \rho_N = - \sum_{i=1}^N \nabla_{x_i} \cdot \left(\left(\frac{1}{N} \sum_{j=1}^N K(x_i - x_j) \right) \rho_N \right) + \sum_{i=1}^N \Delta_{x_i} \rho_N. \quad (1.4)$$

We define ρ_N^k the density of the law of the first k marginals of the N particles system

$$\rho_N^k(t, x_1, \dots, x_k) = \int_{\mathbb{T}^{(N-k)d}} \rho_N(t, x_1, \dots, x_N) dx_{k+1} \dots dx_N,$$

which is also, thanks to the exchangeability of particles, the density of the law of any k marginals. More precisely, in this work, we focus on the equation (1.4) and we will not address the question of the well-posedness of the stochastic equation (1.3).

Here, although we will consider general assumptions on K , the main example motivating our work is the singular interaction kernel known as *the Biot-Savart kernel*, defined in \mathbb{R}^2 by

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} = \frac{1}{2\pi} \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right). \quad (1.5)$$

Consider the 2D incompressible Navier-Stokes system on $x \in \mathbb{R}^2$

$$\begin{aligned} \partial_t u &= -u \cdot \nabla u - \nabla p + \Delta u \\ \nabla \cdot u &= 0, \end{aligned}$$

where p is the local pressure. Taking the curl of the equation above, we get that $\omega(t, x) = \nabla \times u(t, x)$ satisfies (1.2) with K given by (1.5) (see for instance Chapter 1 of [MP94]).

One can see equation (1.3) as an approximation of equation (1.1), where the law $\bar{\rho}_t$ is replaced by the empirical measure $\frac{1}{N} \sum_{i=1}^N \delta_{X_t^i}$. It is well known, at least in a setting where the interaction kernel K is Lipschitz continuous ([M96], [Szn91]), that, under some mild conditions on K , for all fixed $k \in \mathbb{N}$ and all $t \geq 0$, $\rho_N^k(t, \cdot)$ converges toward $\bar{\rho}_k(t, \cdot) = \bar{\rho}_t^{\otimes k}$ as N tends to infinity, where $\bar{\rho}_t$ is the density of the law of X_t solution of (1.1). Thus, provided the particles start independent, they will stay (more or less) independent, as the law of any k -uplet of particles converges toward a tensorized law. The expression *propagation of chaos* to describe this behavior was coined by Kac [Kac56]), and we refer to Sznitman [Szn91] for a landmark study of the phenomenon. Of course there is a huge literature on propagation of chaos however limited for uniform in time results, and always when the interaction potential is regular, see Malrieu [Mal01] for an example by a coupling approach under convexity conditions and the recent Durmus & al [DEGZ20] via reflection coupling allowing non convexity but where the interaction is considered small and acts mainly as a perturbation. For more recent results we refer to [Lac23] (and its uniform in time extension in [LLF23]) for a nice new approach for propagation of chaos furnishing better speed but strong assumptions on the interactions (regularity, integrability), including a nice survey of the existing results, and [DT21] using Lions derivatives for uniform in time results on the torus but also under regularity assumptions on the interaction kernel.

Hence, both these classical and recent results do not apply to the Biot-Savart kernel, which is singular at 0. For a convergence without rate, and specific to the vortex 2D equation, a first striking result appeared in [FHM14], relying on proving that close encounters of particles are rare and that the possible limits of the particles system are made of solutions of the nonlinear SDE. As a second step, in the recent work [JW18], Jabin and Wang have proven that propagation of chaos still holds in this case with a *quantitative* rate. The

goal of the present paper is to extend their works and show a *quantitative propagation of chaos uniform in time*. We refer to [FHM14, JW18, BJW19a, BJW19b, BJW20] for detailed discussions on the literature concerning propagation of chaos with singular kernels, which is still at its beginning for quantitative rates. Shortly after this work was submitted, an alternative approach to global in time estimates was developed in [RS23], see also the very recent preprint [CRS23].

Obtaining uniform in time estimates for propagation of chaos is an important challenge to tackle. One of its applications for instance concerns the use of particle system, which can easily be simulated numerically, to approximate the solution of a nonlinear physics motivated problem, such as here the vorticity equation arising from fluid mechanics. Likewise, it provides a framework for studying noisy gradient descent used in Machine Learning (see the recent [CRW22]) and thus attracts some attention.

The approach of Jabin and Wang [JW18] is to compute the time evolution of the relative entropy of ρ_N with respect to $\bar{\rho}_N$ and then to use an integration by parts to deal with the singularity of K thanks to the regularity of the probability density $\bar{\rho}_t$. In order to improve this argument to get uniform in time propagation of chaos, our main contribution is the proof of time-uniform bounds for $\bar{\rho}_t$, in Lemma 1, from which a time-uniform logarithmic Sobolev inequality is deduced. From the latter, in the spirit of the work of Malrieu [Mal01] in the smooth and convex case, the Fisher information appearing in the entropy dissipation yields a control on the relative entropy itself, inducing the time uniformity. However a major difficulty is that this quantities are expressed in terms of the solution of the nonlinear equation. We then have to prove a logarithmic Sobolev inequality, uniformly in time, for $\bar{\rho}_t$, and a sufficient decay of the derivatives of $\bar{\rho}_t$. To do so, it requires new estimates on regularity and a priori bounds of the solutions of non linear 2D vortex equation. Indeed, we prove that the bounds on the derivative of $\bar{\rho}_t$ decay sufficiently fast (see again Lemma 1) to ensure uniform in time convergence without smallness assumption on the interaction. Finally, the remaining error term in the entropy evolution due to the difference between (1.2) and (1.4) is tackled thanks to a law of large number already used in [JW18]. Compared to [FHM14] we thus obtain a quantitative and uniform in time result.

The organization of the article is as follows. For the remaining of this section, we state the main theorem as well as the various assumptions on both the initial condition and the interaction kernel K . In Section 2 we gather various tools that will be useful later on: we state the regularity of the solutions, the existence of uniform in time bounds on the density and its derivatives, and we prove a logarithmic Sobolev inequality. Finally, in Section 3, we prove the uniform in time propagation of chaos following the method described in [JW18].

1.2 Main results

First, let us describe the assumptions made on the initial condition. Unless otherwise specified, L^p and H^p respectively refer to the spaces $L^p(\mathbb{T}^d)$ and $H^p(\mathbb{T}^d)$. Given $\lambda > 1$, we denote by $\mathcal{C}_\lambda^\infty(\mathcal{X})$ the set of functions f in $\mathcal{C}^\infty(\mathcal{X})$ such that $0 < \frac{1}{\lambda} \leq f \leq \lambda < \infty$, and $\mathcal{C}_{>0}^\infty(\mathcal{X}) = \cup_{\lambda>1} \mathcal{C}_\lambda^\infty(\mathcal{X})$, which is simply the set of positive smooth functions when \mathcal{X} is compact.

Assumption 1. *We make the following assumptions on $\bar{\rho}_0$:*

- *There is $\lambda > 1$ such that $\bar{\rho}_0 \in \mathcal{C}_\lambda^\infty(\mathbb{T}^d)$*
- *For all $n \geq 1$, $C_n^0 := \|\nabla^n \bar{\rho}_0\|_{L^\infty} < \infty$*

Remark 1. *Let us discuss the smoothness assumption on the initial condition. Via Theorem 2 below, which follows from the result of [BA94], this will ensure the smoothness of $\bar{\rho}_t$. This fact (and the fact that we consider, as we will see later, a smooth solution ρ_N of (1.4)) allows us to justify all calculations in comfortable way. This could however be improved. First, as in [JW18], the calculations should hold for any entropy solution of (1.4). Second, it is also shown in [BA94], in the case of the vorticity equation, that an initial*

condition in L^1 yields existence and uniqueness of a solution of (1.2) which is smooth for positive times. One could thus think of using the non-uniform in time result of [JW18] on a small time interval $[0, \epsilon]$, and then complete the proof on $[\epsilon, \infty[$ with our result. We would then require some bounds on $\bar{\rho}_\epsilon$ and its derivatives of a sufficient order (depending on the Sobolev embedding, see the proof of Lemma 1 below) that we could propagate in time.

For the sake of clarity and conciseness however we choose not to insist in this direction.

Let us describe the assumptions on the interaction kernel K . Below, $\nabla \cdot$ stands for the divergence operator.

Assumption 2. We make the following assumptions on K :

- $\|K\|_{L^1} < \infty$.
- In the sense of distributions, $\nabla \cdot K = 0$,
- There is a matrix field $V \in L^\infty$ such that $K = \nabla \cdot V$, i.e for $1 \leq \alpha \leq d$, $K_\alpha = \sum_{\beta=1}^d \partial_\beta V_{\alpha,\beta}$.

The problem of finding a matrix field $V \in L^\infty(\mathbb{T}^d)$ such that $K = \nabla \cdot V$ for a given K is a complex mathematical question. We refer to [BB03] and [PT08] and the references therein for a more detailed discussion on the literature. As it was noted in Proposition 2 of [JW18], the existence of such a matrix V is true for any kernel $K \in L^d$ (using the results of [BB03]), or for any kernel K such that $\exists M > 0, \forall x \in \mathbb{T}^d, |K(x)| \leq M/|x|$ (using the results of [PT08]).

Remark 2. If a function a satisfies $\nabla \cdot a = 0$, then for $\psi : \mathbb{T}^d \mapsto \mathbb{R}$ we have $\nabla \cdot (a\psi) = (a \cdot \nabla)\psi$

Suppose \tilde{K} is an interaction kernel in \mathbb{R}^d (such as the Biot-Savart kernel). It is possible to periodize \tilde{K} on the torus as follows. For f a function on the torus (identified as a 1-periodic function on \mathbb{R}^d), writing $f *_{\mathcal{X}} g(x) = \int_{\mathcal{X}} f(x-y)g(y)dy$ the convolution operator on a space \mathcal{X} ,

$$\begin{aligned} \tilde{K} *_{\mathbb{R}^d} f(x) &= \int_{\mathbb{R}^d} \tilde{K}(x-y)f(y)dy = \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{T}^d} \tilde{K}(x-y+k)f(y-k)dy \\ &= \int_{\mathbb{T}^d} \left(\sum_{k \in \mathbb{Z}^d} \tilde{K}(x-y+k) \right) f(y)dy, \end{aligned}$$

and thus $\tilde{K} *_{\mathbb{R}^d} f(x) = K *_{\mathbb{T}^d} f(x)$, where $K(x) = \sum_{k \in \mathbb{Z}^d} \tilde{K}(x+k)$. In particular, the periodized Biot-Savart kernel obtained by taking \tilde{K} from (1.5) reads

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} + \frac{1}{2\pi} \sum_{k \in \mathbb{Z}^2, k \neq 0} \frac{(x-k)^\perp}{|x-k|^2} := \tilde{K}(x) + K_0(x). \quad (1.6)$$

It has been shown that the sum defining K_0 converges (in the sense that $K_0(x) = \lim_{N \rightarrow \infty} \sum_{|k|^2 \leq N, k \neq 0} \frac{(x-k)^\perp}{|x-k|^2}$) in \mathcal{C}^∞ (see for instance [Sch96]). It is straightforward to check that K is periodic, bounded in L^1 , and divergence free. Finally, Proposition 2 of [JW18] yields the existence of $V \in L^\infty$ such that $K = \nabla \cdot V$. As a consequence, Assumption 2 holds in the case of the periodized Biot-Savart kernel.

Remark 3. Notice that, for the Biot-Savart kernel on the whole space \mathbb{R}^2

$$\tilde{K}(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2},$$

the matrix field \tilde{V} such that $\tilde{K} = \nabla \cdot \tilde{V}$ can be chosen explicitly

$$V(x) = \frac{1}{2\pi} \begin{pmatrix} -\arctan\left(\frac{x_1}{x_2}\right) & 0 \\ 0 & \arctan\left(\frac{x_2}{x_1}\right) \end{pmatrix}.$$

One could also consider collision-like interactions, as mentioned in [JW18]. Let $\phi \in L^1$ be a function on the torus, M be a smooth antisymmetric matrix field and consider the kernel $K = \nabla \cdot (M \mathbb{1}_{\phi(x) \leq 0})$. By construction, K is the divergence of a L^∞ matrix field, and since M is antisymmetric K is divergence free.

Example 1. Consider in dimension 2 the function $\phi : x \mapsto |x|^2 - (2R)^2$ for a given radius $R > 0$ and the matrix

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which yield

$$K(x) = 2x^\perp \delta_{\phi(x)=0}.$$

This interaction kernel models particles, seen as balls of radius R , interacting via some form of collision.

The well-posedness of the equations (1.2) and (1.4) under Assumptions 1 and 2 will be discussed respectively in Sections 2.1 and 3.5. In particular we will see in Theorem 2 that $\bar{\rho}_t$ is in $\mathcal{C}_\lambda^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$.

The comparison between the law of the system of N interacting particles and the law of N independent particles satisfying the non-linear equation (1.1) is stated in terms of relative entropy.

Definition 1. Let μ and ν be two probability densities on \mathbb{T}^{dN} . We consider the rescaled relative entropy

$$\mathcal{H}_N(\nu, \mu) = \begin{cases} \frac{1}{N} \mathbb{E}_\mu \left(\frac{\nu}{\mu} \log \frac{\nu}{\mu} \right) & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases} \quad (1.7)$$

For the sake of conciseness, for all $k \in \mathbb{N}$ and $t \geq 0$, we denote $\rho_N(t) : \mathbf{x} \in \mathbb{T}^{dN} \mapsto \rho_N(t, \mathbf{x})$ and $\bar{\rho}_N(t) : \mathbf{x} \in \mathbb{T}^{dN} \mapsto \bar{\rho}_t^{\otimes N}(\mathbf{x})$. The main result is the following

Theorem 1. Under Assumptions 1 and 2, there are constants C_1, C_2 and C_3 such that for all $N \in \mathbb{N}$ and all exchangeable density probability $\rho_N(0) \in \mathcal{C}_{>0}^\infty(\mathbb{T}^{dN})$ there exists a weak solution ρ_N of (1.4) such that for all $t \geq 0$

$$\mathcal{H}_N(\rho_N(t), \bar{\rho}_N(t)) \leq C_1 e^{-C_2 t} \mathcal{H}_N(\rho_N(0), \bar{\rho}_N(0)) + \frac{C_3}{N} \quad (1.8)$$

In particular, if $\rho_N(0) = \bar{\rho}_N(0)$, the first term of the right-hand side vanishes, and this property has been called entropic propagation of chaos, see for example [HM14].

1.3 Strong propagation of chaos

We show that Theorem 1 yields strong propagation of chaos, uniform in time. For μ and ν two probability measures on \mathbb{T}^{dk} , denote by $\Pi(\mu, \nu)$ the set of couplings of μ and ν , i.e. the set of probability measures Γ on $\mathbb{T}^{dk} \times \mathbb{T}^{dk}$ with $\Gamma(A \times \mathbb{T}^{dk}) = \mu(A)$ and $\Gamma(\mathbb{T}^{dk} \times A) = \nu(A)$ for all Borel set A of \mathbb{T}^{dk} . Let us define the usual L^2 -Wasserstein distance by

$$\mathcal{W}_2(\mu, \nu) = \left(\inf_{\Gamma \in \Pi(\mu, \nu)} \int_{\mathbb{T}^{dk}} d_{\mathbb{T}^{dk}}(x, y)^2 \Gamma(dx dy) \right)^{1/2},$$

where $d_{\mathbb{T}^{dk}}$ is the usual distance on the torus. For $\mathbf{x} = (x_i)_{i \in [1, N]} \in \mathbb{T}^{dN}$, we write $\pi(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$ the associated empirical measure.

Corollary 1. *Under assumptions 1 and 2, assuming moreover that $\rho_N(0) = \bar{\rho}_N(0)$, there is a constant C such that for all $k \leq N \in \mathbb{N}$ and all $t \geq 0$,*

$$\|\rho_N^k(t) - \bar{\rho}_k(t)\|_{L^1} + \mathcal{W}_2\left(\rho_N^k(t), \bar{\rho}_k(t)\right) \leq C \left(\left\lfloor \frac{N}{k} \right\rfloor\right)^{-\frac{1}{2}}$$

and

$$\mathbb{E}_{\rho_N(t)}(\mathcal{W}_2(\pi(\mathbf{X}), \bar{\rho}_t)) \leq C\alpha(N)$$

where $\alpha(N) = N^{-1/2} \ln(1 + N)$ if $d = 2$ and $\alpha(N) = N^{-1/d}$ if $d > 2$.

As shown in [BGV07], the last result yields confidence interval in uniform norm when estimating $\bar{\rho}_t$ with $\pi(\mathbf{X}_t^N)$ convoluted to a smooth kernel.

We postpone the proof as it will rely on results shown later. It will however be a direct corollary of Theorem 1 and of the logarithmic Sobolev inequality proven in Corollary 2, which is a crucial ingredient in the proof of Theorem 1.

2 Preliminary work

2.1 First results on the non-linear PDE

We have the following result concerning the solution of (1.2).

Theorem 2. *Under Assumption 2, let $\mu_0 \in \mathcal{C}_\lambda^\infty(\mathbb{T}^d)$. Then the system*

$$\begin{cases} \partial_t \bar{\rho}_t = -\nabla \cdot ((K * \bar{\rho}_t) \bar{\rho}_t) + \Delta \bar{\rho}_t, \text{ in } \mathbb{R}^+ \times \mathbb{T}^d \\ \bar{\rho}_0 = \mu_0, \end{cases} \quad (2.1)$$

has, in the class of bounded solutions, a unique solution $\bar{\rho}(t, x) \in \mathcal{C}_\lambda^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$.

Proof. The first part of the theorem (existence, uniqueness and smoothness) can be proven by following closely the proof done by Ben-Artzi in [BA94]. For the sake of completeness, this is detailed in Appendix A. Note that a similar result has also been recently proven in [Wyn23], where the C^k regularity of $\bar{\rho}_t$ for any given k and any given t is shown. The proof relies heavily on the fact that the kernel K is divergence free, that the convolution operation tends to keep the regularity of the most regular term, and that the Fokker-Planck equation has a smoothing effect.

Let us now prove the second part of the result, namely the time uniform bounds on $\bar{\rho}_t$. Assume that $\mu_0 \in \mathcal{C}_\lambda^\infty(\mathbb{T}^d)$, which by definition implies $\frac{1}{\lambda} \leq \mu_0 \leq \lambda$, and consider $\bar{\rho}_t$ the unique solution of (2.1). We start by proving that $K * \bar{\rho}_t$ is in \mathcal{C}^∞ . By definition

$$K * \bar{\rho}_t(x) = \int_{\mathbb{T}^d} K(x - y) \bar{\rho}_t(y) dy = - \int_{\mathbb{T}^d} K(y) \bar{\rho}_t(x - y) dy.$$

Then

$$K * \bar{\rho}_t(x) = - \int_{\mathbb{T}^d} \nabla \cdot V(y) \bar{\rho}_t(x - y) dy = - \int_{\mathbb{T}^d} V(y) \nabla_y \bar{\rho}_t(x - y) dy.$$

Since $V \in L^\infty(\mathbb{T}^d)$ and $\bar{\rho} \in \mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$, we easily deduce that $K * \bar{\rho}$, as well as all its derivatives, are Lipschitz continuous on $[0, T] \times \mathbb{T}^d$ for all $T > 0$. Hence $K * \bar{\rho}$ is \mathcal{C}^∞ . Moreover, using that $\nabla \cdot K = 0$ (in the sense of distribution), we immediately get that $\nabla \cdot (K * \bar{\rho}_t) = 0$ for all $t \geq 0$.

For $t \geq 0$ and $x \in \mathbb{T}^d$, consider Z_s the strong solution of the following stochastic differential equation for $s \in [0, t]$

$$dZ_s = \sqrt{2} dB_s - K * \bar{\rho}_{t-s}(Z_s) ds, \quad Z_0 = x$$

which exists, is unique and non-explosive since $K * \bar{\rho}_{t-s}$ is smooth and bounded. Then

$$\bar{\rho}(t, x) = \mathbb{E}_x(\bar{\rho}_0(Z_t)).$$

The bounds on $\bar{\rho}_t$ follow. □

2.2 Higher order estimates

We have already established that $\bar{\rho}_t$ is bounded uniformly in time. In this section, we extend this result to all its derivatives.

Lemma 1. *For all $n \geq 1$ and $\alpha_1, \dots, \alpha_n \in \llbracket 1, d \rrbracket$, there exist $C_n^u, C_n^\infty > 0$ such that for all $t \geq 0$,*

$$\|\partial_{\alpha_1, \dots, \alpha_n} \bar{\rho}_t\|_{L^\infty} \leq C_n^u \quad \text{and} \quad \int_0^t \|\partial_{\alpha_1, \dots, \alpha_n} \bar{\rho}_s\|_{L^\infty}^2 ds \leq C_n^\infty$$

Proof. Thanks to Morrey's inequality and Sobolev embeddings, it is sufficient to prove such bounds in the Sobolev space H^m for all m , in other words it is sufficient to prove similar bounds for $\|\partial_{\alpha_1, \dots, \alpha_n} \bar{\rho}_s\|_{L^2}^2$ for all multi-indexes α . The proof is by induction on the order of the derivatives, we only detail the first iterations. We write $f = \nabla \cdot ((K * \bar{\rho}_t) \bar{\rho}_t) = (K * \bar{\rho}_t) \cdot \nabla \bar{\rho}_t$.

Integrated bound for $\|\nabla \bar{\rho}_t\|_{L^2}^2$. We have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} |\bar{\rho}_t|^2 = \int_{\mathbb{T}^d} \bar{\rho}_t \partial_t \bar{\rho}_t = \int_{\mathbb{T}^d} \bar{\rho}_t \Delta \bar{\rho}_t - \int_{\mathbb{T}^d} \bar{\rho}_t f.$$

On the one hand,

$$\int_{\mathbb{T}^d} \bar{\rho}_t \Delta \bar{\rho}_t = - \int_{\mathbb{T}^d} |\nabla \bar{\rho}_t|^2.$$

On the other hand,

$$\int_{\mathbb{T}^d} \bar{\rho}_t f = \int_{\mathbb{T}^d} \bar{\rho}_t \nabla \cdot ((K * \bar{\rho}_t) \bar{\rho}_t) = - \int_{\mathbb{T}^d} \nabla \bar{\rho}_t \cdot (K * \bar{\rho}_t) \bar{\rho}_t = - \int_{\mathbb{T}^d} \bar{\rho}_t f = 0.$$

Hence,

$$\frac{1}{2} \frac{d}{dt} \|\bar{\rho}_t\|_{L^2}^2 + \|\nabla \bar{\rho}_t\|_{L^2}^2 = 0.$$

By integrating the equality above, we get $\int_0^t \|\nabla \bar{\rho}_t\|_{L^2}^2 = \frac{\|\bar{\rho}_0\|_{L^2}^2 - \|\bar{\rho}_t\|_{L^2}^2}{2} \leq \frac{\lambda^2}{2} = C_1^\infty$.

Integrated bound for $\|\partial_{\alpha_1, \alpha_2} \bar{\rho}_t\|_{L^2}^2$ and uniform bound for $\|\nabla \bar{\rho}_t\|_{L^2}^2$. Similarly, we calculate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} |\partial_{\alpha_1} \bar{\rho}_t|^2 &= \int_{\mathbb{T}^d} \partial_{\alpha_1} \bar{\rho}_t \partial_{\alpha_1} (\partial_t \bar{\rho}_t) = \int_{\mathbb{T}^d} \partial_{\alpha_1} \bar{\rho}_t \partial_{\alpha_1} (\Delta \bar{\rho}_t - f) \\ &= - \sum_{\alpha_2} \int_{\mathbb{T}^d} |\partial_{\alpha_1, \alpha_2} \bar{\rho}_t|^2 + \int_{\mathbb{T}^d} \partial_{\alpha_1, \alpha_1} \bar{\rho}_t f. \end{aligned}$$

Bounding

$$\int_{\mathbb{T}^d} \partial_{\alpha_1, \alpha_1} \bar{\rho}_t f \leq \|\partial_{\alpha_1, \alpha_1} \bar{\rho}_t\|_{L^2} \|f\|_{L^2} \leq \frac{1}{2} \sum_{\alpha_2} \|\partial_{\alpha_1, \alpha_2} \bar{\rho}_t\|_{L^2}^2 + \frac{1}{2} \|f\|_{L^2}^2,$$

and

$$\|f\|_{L^2}^2 = \int_{\mathbb{T}^d} \left| \sum_{\gamma=1}^d (K_\gamma * \bar{\rho}_t) \partial_\gamma \bar{\rho}_t \right|^2 \leq \|K * \bar{\rho}_t\|_{L^\infty}^2 \|\nabla \bar{\rho}_t\|_{L^2}^2 \leq \|K\|_{L^1}^2 \|\bar{\rho}_t\|_{L^\infty}^2 \|\nabla \bar{\rho}_t\|_{L^2}^2,$$

where we used Young's convolution inequality, we get

$$\frac{1}{2} \frac{d}{dt} \|\partial_{\alpha_1} \bar{\rho}_t\|_{L^2}^2 + \frac{1}{2} \sum_{\alpha_2} \|\partial_{\alpha_1, \alpha_2} \bar{\rho}_t\|_{L^2}^2 \leq \frac{1}{2} \|K\|_{L^1}^2 \|\bar{\rho}_t\|_{L^\infty}^2 \|\nabla \bar{\rho}_t\|_{L^2}^2.$$

By integrating the equality above and using Theorem 2, we get

$$\begin{aligned} \frac{\|\partial_{\alpha_1} \bar{\rho}_t\|_{L^2}^2 - \|\partial_{\alpha_1} \bar{\rho}_0\|_{L^2}^2}{2} + \frac{1}{2} \int_0^t \sum_{\alpha_2} \|\partial_{\alpha_1, \alpha_2} \bar{\rho}_s\|_{L^2}^2 ds &\leq \frac{1}{2} \|K\|_{L^1}^2 \lambda^2 \int_0^t \|\nabla \bar{\rho}_s\|_{L^2}^2 ds \\ &\leq \frac{1}{2} \|K\|_{L^1}^2 \lambda^2 C_1^\infty. \end{aligned}$$

This provides both the existence of C_2^∞ such that for all $t \geq 0$, $\int_0^t \|\partial_{\alpha_1, \alpha_2} \bar{\rho}_s\|_{L^2}^2 ds \leq C_2^\infty$, and the existence of C_1^u such that for all $t \geq 0$, $\|\partial_{\alpha_1} \bar{\rho}_t\|_{L^2}^2 \leq C_1^u$.

Integrated bound bound for $\|\partial_{\alpha_1, \alpha_2, \alpha_3} \bar{\rho}_t\|_{L^2}^2$ and uniform bound for $\|\partial_{\alpha_1, \alpha_2} \bar{\rho}_t\|_{L^2}^2$. We have

$$\partial_\alpha f = \sum_{\gamma} (\partial_\alpha K_\gamma * \bar{\rho}_t) \partial_\gamma \bar{\rho}_t + \sum_{\gamma} (K_\gamma * \bar{\rho}_t) \partial_{\alpha, \gamma} \bar{\rho}_t,$$

and

$$\begin{aligned} \partial_\alpha K_\gamma * \bar{\rho}_t &= \int_{\mathbb{T}^d} \partial_\alpha K_\gamma(x-y) \bar{\rho}_t(y) dy = - \int_{\mathbb{T}^d} \partial_\alpha K_\gamma(y) \bar{\rho}_t(x-y) dy = - \int_{\mathbb{T}^d} K_\gamma(y) \partial_\alpha \bar{\rho}_t(x-y) dy \\ &= - \sum_{\beta} \int_{\mathbb{T}^d} V_{\gamma, \beta}(y) \partial_{\alpha, \beta} \bar{\rho}_t(x-y) dy = \sum_{\beta} V_{\gamma, \beta} * \partial_{\alpha, \beta} \bar{\rho}_t. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{\gamma} (\partial_\alpha K_\gamma * \bar{\rho}_t) \partial_\gamma \bar{\rho}_t &= \sum_{\gamma} \left(\sum_{\beta} V_{\gamma, \beta} * \partial_{\alpha, \beta} \bar{\rho}_t \right) \partial_\gamma \bar{\rho}_t \\ &= (V * \partial_\alpha \nabla \bar{\rho}_t) \nabla \bar{\rho}_t, \end{aligned}$$

and thus

$$\left\| \sum_{\gamma} (\partial_\alpha K_\gamma * \bar{\rho}_t) \partial_\gamma \bar{\rho}_t \right\|_{L^2} \leq \|V * \partial_\alpha \nabla \bar{\rho}_t\|_{L^\infty} \|\nabla \bar{\rho}_t\|_{L^2} \leq \|V\|_{L^\infty} \|\partial_\alpha \nabla \bar{\rho}_t\|_{L^1} \|\nabla \bar{\rho}_t\|_{L^2}.$$

Therefore

$$\|\partial_\alpha f\|_{L^2}^2 \leq 2 \|V\|_{L^\infty}^2 \|\partial_\alpha \nabla \bar{\rho}_t\|_{L^1}^2 \|\nabla \bar{\rho}_t\|_{L^2}^2 + 2 \|K\|_{L^1}^2 \|\bar{\rho}_t\|_{L^\infty}^2 \|\partial_\alpha \nabla \bar{\rho}_t\|_{L^2}^2.$$

Similarly to previous computations,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} |\partial_{\alpha_1, \alpha_2} \bar{\rho}_t|^2 &= \int_{\mathbb{T}^d} \partial_{\alpha_1, \alpha_2} \bar{\rho}_t \partial_{\alpha_1, \alpha_2} (\Delta \bar{\rho}_t - f) \\
&= - \sum_{\alpha_3} \int_{\mathbb{T}^d} |\partial_{\alpha_1, \alpha_2, \alpha_3} \bar{\rho}_t|^2 + \int_{\mathbb{T}^d} \partial_{\alpha_1, \alpha_2, \alpha_3} \bar{\rho}_t \partial_{\alpha_1} f \\
&\leq - \sum_{\alpha_3} \|\partial_{\alpha_1, \alpha_2, \alpha_3} \bar{\rho}_t\|_{L^2}^2 + \|\partial_{\alpha_1, \alpha_2, \alpha_3} \bar{\rho}_t\|_{L^2} \|\partial_{\alpha_1} f\|_{L^2} \\
&\leq - \sum_{\alpha_3} \|\partial_{\alpha_1, \alpha_2, \alpha_3} \bar{\rho}_t\|_{L^2}^2 + \frac{1}{2} \sum_{\alpha_3} \|\partial_{\alpha_1, \alpha_2, \alpha_3} \bar{\rho}_t\|_{L^2}^2 \\
&\quad + \|V\|_{L^\infty}^2 \|\partial_{\alpha_1} \nabla \bar{\rho}_t\|_{L^1}^2 \|\nabla \bar{\rho}_t\|_{L^2}^2 + \|K\|_{L^1}^2 \|\bar{\rho}_t\|_{L^\infty}^2 \|\partial_{\alpha_1} \nabla \bar{\rho}_t\|_{L^2}^2 \\
&\leq - \frac{1}{2} \sum_{\alpha_3} \|\partial_{\alpha_1, \alpha_2, \alpha_3} \bar{\rho}_t\|_{L^2}^2 + \|V\|_{L^\infty}^2 \|\nabla \bar{\rho}_t\|_{L^2}^2 \|\partial_{\alpha_1} \nabla \bar{\rho}_t\|_{L^2}^2 \\
&\quad + \|K\|_{L^1}^2 \|\bar{\rho}_t\|_{L^\infty}^2 \|\partial_{\alpha_1} \nabla \bar{\rho}_t\|_{L^2}^2,
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\partial_{\alpha_1, \alpha_2} \bar{\rho}_t\|_{L^2}^2 + \frac{1}{2} \sum_{\alpha_3} \|\partial_{\alpha_1, \alpha_2, \alpha_3} \bar{\rho}_t\|_{L^2}^2 &\leq \|V\|_{L^\infty}^2 \|\partial_{\alpha_1} \nabla \bar{\rho}_t\|_{L^2}^2 \|\nabla \bar{\rho}_t\|_{L^2}^2 \\
&\quad + \|K\|_{L^1}^2 \|\bar{\rho}_t\|_{L^\infty}^2 \|\partial_{\alpha_1} \nabla \bar{\rho}_t\|_{L^2}^2.
\end{aligned}$$

Integrating over time, and using Theorem 2

$$\begin{aligned}
\frac{\|\partial_{\alpha_1, \alpha_2} \bar{\rho}_t\|_{L^2}^2 - \|\partial_{\alpha_1, \alpha_2} \bar{\rho}_0\|_{L^2}^2}{2} + \frac{1}{2} \sum_{\alpha_3} \int_0^t \|\partial_{\alpha_1, \alpha_2, \alpha_3} \bar{\rho}_s\|_{L^2}^2 ds \\
\leq \|V\|_{L^\infty}^2 d C_1^u \int_0^t \|\partial_{\alpha_1} \nabla \bar{\rho}_s\|_{L^2}^2 ds + \|K\|_{L^1}^2 \lambda^2 \int_0^t \|\partial_{\alpha_1} \nabla \bar{\rho}_s\|_{L^2}^2 ds \\
\leq d (d \|V\|_{L^\infty}^2 C_1^u + \|K\|_{L^1}^2 \lambda^2) C_2^\infty
\end{aligned}$$

This provides both the existence of C_3^∞ such that for all $t \geq 0$, $\int_0^t \|\partial_{\alpha_1, \alpha_2, \alpha_3} \bar{\rho}_s\|_{L^2}^2 ds \leq C_3^\infty$, and the existence of C_2^u such that for all $t \geq 0$, $\|\partial_{\alpha_1, \alpha_2} \bar{\rho}_t\|_{L^2}^2 \leq C_2^u$.

The proof is then by induction on the order of derivative, iterating the same method. \square

2.3 Logarithmic Sobolev inequality

We now establish a logarithmic Sobolev inequality (LSI) for $\bar{\rho}_t$ solution of (1.2). To this end, we use the fact that the uniform distribution u on \mathbb{T}^d satisfies a LSI and that $\bar{\rho}_t$ is bounded (below and above) uniformly in time. Recall the following Holley-Stroock perturbation lemma, from [BGL14, Prop. 5.1.6].

Lemma 2. *Assume that ν is a probability measure on \mathbb{T}^d satisfying a logarithmic Sobolev inequality with constant C_ν^{LS} , i.e for all $f \in C_{>0}^\infty(\mathbb{T}^d)$,*

$$\text{Ent}_\nu(f) := \int_{\mathbb{T}^d} f \log f d\nu - \int_{\mathbb{T}^d} f d\nu \log \left(\int_{\mathbb{T}^d} f d\nu \right) \leq C_\nu^{LS} \int_{\mathbb{T}^d} \frac{|\nabla f|^2}{f} d\nu.$$

Let μ be a probability measure with density h with respect to ν such that, for some constant $\lambda > 0$, $\frac{1}{\lambda} \leq h \leq \lambda$. Then μ satisfies a logarithmic Sobolev inequality with constant $C_\mu^{LS} = \lambda^2 C_\nu^{LS}$, i.e for all $f \in C_{>0}^\infty(\mathbb{T}^d)$

$$\text{Ent}_\mu(f) \leq \lambda^2 C_\nu^{LS} \int_{\mathbb{T}^d} \frac{|\nabla f|^2}{f} d\mu.$$

We also know that the uniform distribution u (i.e the Lebesgue measure) on \mathbb{T}^d satisfies a LSI. See for instance Proposition 5.7.5 of [BGL14], or [Gen04] for a proof in dimension 1, the results in higher dimension being a consequence of tensorization properties.

Lemma 3. *Let u be the uniform distribution on \mathbb{T}^d . Then u satisfies a logarithmic Sobolev inequality : for all $f \in \mathcal{C}_{>0}^\infty(\mathbb{T}^d)$*

$$\text{Ent}_u(f) \leq \frac{1}{8\pi^2} \int_{\mathbb{T}^d} \frac{|\nabla f|^2}{f} du \quad (2.2)$$

A direct consequence of Lemma 2, Lemma 3 and the bounds on $\bar{\rho}_t$ given in Theorem 2 is the following theorem, as well as its corollary. It establishes a uniform in time logarithmic Sobolev inequality for $\bar{\rho}_t$, crucial for the uniform control of the Fisher information appearing in the study of the dissipation of the entropy between the law of the particles system and the nonlinear ones.

Theorem 3. *Under Assumptions 1 and 2, for all $t \geq 0$ and all function $f \in \mathcal{C}_{>0}^\infty(\mathbb{T}^d)$,*

$$\text{Ent}_{\bar{\rho}_t}(f) \leq \frac{\lambda^2}{8\pi^2} \int_{\mathbb{T}^d} \frac{|\nabla f|^2}{f} d\bar{\rho}_t$$

Corollary 2. *Under Assumptions 1 and 2, for all $N \in \mathbb{N}$, $t \geq 0$ and all probability density $\mu_N \in \mathcal{C}_{>0}^\infty(\mathbb{T}^{dN})$,*

$$\mathcal{H}_N(\mu_N, \bar{\rho}_N(t)) \leq \frac{\lambda^2}{8\pi^2} \frac{1}{N} \sum_{i=1}^N \int_{\mathbb{T}^d} \mu_N \left| \nabla_{x_i} \log \frac{\mu_N}{\bar{\rho}_N(t)} \right|^2$$

Proof. By tensorization of the logarithmic Sobolev inequality (see for instance Proposition 5.2.7 of [BGL14]), since $\bar{\rho}$ satisfies a LSI with constant $\frac{\lambda^2}{8\pi^2}$, so does $\bar{\rho}_N$. Using Theorem 3 for $f = \frac{\mu_N}{\bar{\rho}_N}$ we thus get

$$\mathcal{H}_N(\mu_N, \bar{\rho}_N(t)) = \frac{1}{N} \text{Ent}_{\bar{\rho}_N(t)} \left(\frac{\mu_N}{\bar{\rho}_N(t)} \right) \leq \frac{\lambda^2}{8\pi^2} \frac{1}{N} \mathbb{E}_{\bar{\rho}_N(t)} \left(\left| \nabla_x \frac{\mu_N}{\bar{\rho}_N(t)} \right|^2 \frac{\bar{\rho}_N(t)}{\mu_N} \right).$$

Hence the result. □

3 Proofs of the main results

From now on and up to Section 3.5 (excluded), in addition to Assumptions 1 and 2, we suppose that there exists a solution $\rho_N \in \mathcal{C}_{>0}^\infty(\mathbb{R}^+ \times \mathbb{T}^{dN})$ of (1.4). This justifies the validity of the various calculations conducted in this part of the proof. The question to lift this assumption (by taking a limit in a regularized problem) is addressed in Section 3.5

3.1 Time evolution of the relative entropy

We write

$$\mathcal{H}_N(t) = \mathcal{H}_N(\rho_N(t), \bar{\rho}_N(t)), \quad \mathcal{I}_N(t) = \frac{1}{N} \sum_i \int_{\mathbb{T}^{dN}} \rho_N(t) \left| \nabla_{x_i} \log \frac{\rho_N(t)}{\bar{\rho}_N(t)} \right|^2 dx.$$

as short hands for the relative entropy and relative Fisher information. We start by calculating the time evolution of the relative entropy.

Lemma 4. For all $t \geq 0$,

$$\frac{d}{dt} \mathcal{H}_N(t) \leq A_N(t) + \frac{1}{2} B_N(t) - \frac{1}{2} \mathcal{I}_N(t), \quad (3.1)$$

with

$$A_N(t) := \frac{1}{N^2} \sum_{i,j} \int_{\mathbb{T}^{dN}} \rho_N (V(x_i - x_j) - V * \bar{\rho}_t(x_i)) : \frac{\nabla_{x_i}^2 \bar{\rho}_N}{\bar{\rho}_N} d\mathbf{x}$$

$$B_N(t) := \frac{1}{N} \sum_i \int_{\mathbb{T}^{dN}} \rho_N \frac{|\nabla_{x_i} \bar{\rho}_N|^2}{\bar{\rho}_N^2} \left| \frac{1}{N} \sum_j V(x_i - x_j) - V * \bar{\rho}_t(x_i) \right|_f^2 d\mathbf{x}.$$

Here, $|\cdot|_f^2$ denotes the sum of the square of the coefficients of the matrix.

Proof. It has been shown, in [JW18], that

$$\frac{d}{dt} \mathcal{H}_N(t) \leq -\mathcal{I}_N(t) - \frac{1}{N^2} \sum_{i,j} \int_{\mathbb{T}^{dN}} \rho_N (K(x_i - x_j) - K * \bar{\rho}_t(x_i)) \cdot \nabla_{x_i} \log \bar{\rho}_N d\mathbf{x},$$

with

$$-\frac{1}{N^2} \sum_{i,j} \int_{\mathbb{T}^{dN}} \rho_N (K(x_i - x_j) - K * \bar{\rho}_t(x_i)) \cdot \nabla_{x_i} \log \bar{\rho}_N d\mathbf{x}$$

$$= \frac{1}{N^2} \sum_{i,j} \int_{\mathbb{T}^{dN}} \rho_N (V(x_i - x_j) - V * \bar{\rho}_t(x_i)) : \frac{\nabla_{x_i}^2 \bar{\rho}_N}{\bar{\rho}_N} d\mathbf{x}$$

$$+ \frac{1}{N^2} \sum_{i,j} \int_{\mathbb{T}^{dN}} (V(x_i - x_j) - V * \bar{\rho}_t(x_i)) : \nabla_{x_i} \bar{\rho}_N \otimes \nabla_{x_i} \frac{\rho_N}{\bar{\rho}_N} d\mathbf{x}.$$

Let us consider the latter term

$$\frac{1}{N^2} \sum_{i,j} \int_{\mathbb{T}^{dN}} (V(x_i - x_j) - V * \bar{\rho}_t(x_i)) : \nabla_{x_i} \bar{\rho}_N \otimes \nabla_{x_i} \frac{\rho_N}{\bar{\rho}_N} d\mathbf{x}$$

$$= \frac{1}{N} \sum_i \sum_{\alpha,\beta} \int_{\mathbb{T}^{dN}} \left(\frac{1}{N} \sum_j V(x_i - x_j) - V * \bar{\rho}_t(x_i) \right)_{\alpha,\beta} (\nabla_{x_i} \bar{\rho}_N)_\alpha \left(\nabla_{x_i} \frac{\rho_N}{\bar{\rho}_N} \right)_\beta d\mathbf{x}.$$

Let

$$y_\beta^i := \left(\nabla_{x_i} \frac{\rho_N}{\bar{\rho}_N} \right)_\beta \frac{\bar{\rho}_N}{\sqrt{\rho_N}}, \quad z_\alpha^i := (\nabla_{x_i} \bar{\rho}_N)_\alpha \frac{\sqrt{\rho_N}}{\bar{\rho}_N}, \quad \text{and} \quad x_{\alpha,\beta}^i := \left(\frac{1}{N} \sum_j V(x_i - x_j) - V * \bar{\rho}_t(x_i) \right)_{\alpha,\beta},$$

then, using $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$ for all $x, y \in \mathbb{R}$,

$$\sum_{\alpha,\beta} x_{\alpha,\beta}^i z_\alpha^i y_\beta^i = \sum_\beta y_\beta^i \left(\sum_\alpha x_{\alpha,\beta}^i z_\alpha^i \right) \leq \frac{1}{2} \sum_\beta (y_\beta^i)^2 + \frac{1}{2} \sum_\beta \left(\sum_\alpha x_{\alpha,\beta}^i z_\alpha^i \right)^2,$$

and thus, using the Cauchy-Schwarz inequality,

$$\sum_{\alpha,\beta} x_{\alpha,\beta}^i z_\alpha^i y_\beta^i \leq \frac{1}{2} \sum_\beta (y_\beta^i)^2 + \frac{1}{2} \sum_\beta \left(\sum_\alpha (x_{\alpha,\beta}^i)^2 \right) \left(\sum_\alpha (z_\alpha^i)^2 \right)$$

$$= \frac{1}{2} \sum_\beta (y_\beta^i)^2 + \frac{1}{2} \left(\sum_\alpha (z_\alpha^i)^2 \right) \left(\sum_{\alpha,\beta} (x_{\alpha,\beta}^i)^2 \right).$$

Hence

$$\begin{aligned}
& \frac{1}{N^2} \sum_{i,j} \int_{\mathbb{T}^{dN}} (V(x_i - x_j) - V * \bar{\rho}_t(x_i)) : \nabla_{x_i} \bar{\rho}_N \otimes \nabla_{x_i} \frac{\rho_N}{\bar{\rho}_N} d\mathbf{x} \\
& \leq \frac{1}{2N} \sum_i \int_{\mathbb{T}^{dN}} \frac{\bar{\rho}_N^2}{\rho_N} \left| \nabla_{x_i} \frac{\rho_N}{\bar{\rho}_N} \right|^2 d\mathbf{x} + \frac{1}{2N} \sum_i \int_{\mathbb{T}^{dN}} \rho_N \frac{|\nabla_{x_i} \bar{\rho}_N|^2}{\bar{\rho}_N^2} \left| \frac{1}{N} \sum_j V(x_i - x_j) - V * \bar{\rho}_t(x_i) \right|_f^2 d\mathbf{x} \\
& = \frac{1}{2} \mathcal{I}_N(t) + \frac{1}{2N} \sum_i \int_{\mathbb{T}^{dN}} \rho_N \frac{|\nabla_{x_i} \bar{\rho}_N|^2}{\bar{\rho}_N^2} \left| \frac{1}{N} \sum_j V(x_i - x_j) - V * \bar{\rho}_t(x_i) \right|_f^2 d\mathbf{x}.
\end{aligned}$$

This yields the desired result. \square

3.2 Change of reference measure and Law of Large Number

We now state three general results which will be useful in order to control the error terms A_N and B_N defined in Lemma 4. The first one will be used to perform a change of measure from ρ_N to $\bar{\rho}_N$.

Lemma 5. *Let $N \in \mathbb{N}$. For two probability densities μ and ν on \mathbb{T}^{dN} , and any $\Phi \in L^\infty(\mathbb{T}^{dN})$ and $\eta > 0$,*

$$\mathbb{E}^\mu \Phi \leq \eta \mathcal{H}_N(\mu, \nu) + \frac{\eta}{N} \log \mathbb{E}^\nu e^{N\Phi/\eta}.$$

Proof. Define

$$f = \frac{1}{\theta} e^{N\Phi/\eta} \nu, \quad \theta = \int_{\mathbb{T}^{dN}} e^{N\Phi/\eta} \nu d\mathbf{x}.$$

Notice f is a probability density. By convexity of the entropy

$$\frac{1}{N} \int_{\mathbb{T}^{dN}} \mu \log f d\mathbf{x} \leq \frac{1}{N} \int_{\mathbb{T}^{dN}} \mu \log \mu d\mathbf{x}.$$

On the other hand

$$\frac{1}{N} \int_{\mathbb{T}^{dN}} \mu \log f d\mathbf{x} = \frac{1}{\eta} \int_{\mathbb{T}^{dN}} \mu \Phi d\mathbf{x} + \frac{1}{N} \int_{\mathbb{T}^{dN}} \mu \log \nu d\mathbf{x} - \frac{\log \theta}{N}.$$

\square

The next two statements are crucial theorems of [JW18].

Theorem 4. *[Theorem 3 of [JW18]] Consider any probability measure μ on \mathbb{T}^d and a scalar function $\psi \in L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ with $\|\psi\|_{L^\infty} < \frac{1}{2e}$ and such that for all $z \in \mathbb{T}^d$, $\int_{\mathbb{T}^d} \psi(z, x) \mu(dx) = 0$. Then*

$$\int_{\mathbb{T}^{dN}} \exp\left(\frac{1}{N} \sum_{j_1, j_2=1}^N \psi(x_1, x_{j_1}) \psi(x_1, x_{j_2})\right) \mu^{\otimes N} d\mathbf{x} \leq C = 2 \left(1 + \frac{10\alpha}{(1-\alpha)^3} + \frac{\beta}{1-\beta}\right), \quad (3.2)$$

where

$$\alpha = (e\|\psi\|_{L^\infty})^4 < 1, \quad \beta = (\sqrt{2e}\|\psi\|_{L^\infty})^4 < 1.$$

The second one is a nice improvement of the usual level two large deviations bound for i.i.d. random variables.

Theorem 5. [Theorem 4 of [JW18]] Consider any probability measure μ on \mathbb{T}^d and $\phi \in L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ with

$$\gamma := (1600^2 + 36e^4) \left(\sup_{p \geq 1} \frac{\|\sup_z |\phi(\cdot, z)|\|_{L^p(\mu)}}{p} \right)^2 < 1. \quad (3.3)$$

Assume that ϕ satisfies the following cancellations

$$\forall z \in \mathbb{T}^d, \quad \int_{\mathbb{T}^d} \phi(x, z) \mu(dx) = 0 = \int_{\mathbb{T}^d} \phi(z, x) \mu(dx).$$

Then, for all $N \in \mathbb{N}$,

$$\int_{\mathbb{T}^{dN}} \exp\left(\frac{1}{N} \sum_{i,j=1}^N \phi(x_i, x_j)\right) \mu^{\otimes N} d\mathbf{x} \leq \frac{2}{1-\gamma} < \infty. \quad (3.4)$$

3.3 Bounding the error terms

Lemma 6. The terms A_N and B_N introduced in Lemma 4 are such that

$$A_N(t) + \frac{1}{2} B_N(t) \leq C \left(\mathcal{H}_N(t) + \frac{1}{N} \right)$$

with

$$C = \hat{C}_1 \lambda d \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \|V\|_{L^\infty} + \hat{C}_2 \lambda^2 d^2 \|V\|_{L^\infty}^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2$$

where \hat{C}_1, \hat{C}_2 are universal constants.

Proof. Recall from Theorem 2 that $\bar{\rho}_t \in C_\lambda^\infty(\mathbb{T}^d)$ for all $t \geq 0$. We first bound B_N . For $(X_t^i)_i$ given in (1.3), we have

$$\begin{aligned} B_N &= \frac{1}{N} \sum_i \int_{\mathbb{T}^{dN}} \rho_N \frac{|\nabla \bar{\rho}_t|^2}{\bar{\rho}_t^2}(x_i) \left| \frac{1}{N} \sum_j V(x_i - x_j) - V * \bar{\rho}_t(x_i) \right|_f^2 d\mathbf{x} \\ &= \frac{1}{N} \sum_i \mathbb{E} \left(\left| \frac{\nabla \bar{\rho}_t}{\bar{\rho}_t}(X_t^i) \right|^2 \left| \frac{1}{N} \sum_j V(X_t^i - X_t^j) - V * \bar{\rho}_t(X_t^i) \right|_f^2 \right) \\ &= \frac{1}{N} \sum_i \sum_{\alpha, \beta=1}^d \mathbb{E} \left(\left| \frac{\nabla \bar{\rho}_t}{\bar{\rho}_t}(X_t^i) \right|^2 \left(\frac{1}{N} \sum_j V_{\alpha, \beta}(X_t^i - X_t^j) - V_{\alpha, \beta} * \bar{\rho}_t(X_t^i) \right)^2 \right) \\ &\leq \frac{\lambda^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2}{N} \sum_i \sum_{\alpha, \beta=1}^d \mathbb{E} \left(\left(\frac{1}{N} \sum_j V_{\alpha, \beta}(X_t^i - X_t^j) - V_{\alpha, \beta} * \bar{\rho}_t(X_t^i) \right)^2 \right). \end{aligned}$$

We apply Lemma 5 to each

$$\Phi_{\alpha, \beta} = \left(\frac{1}{N} \sum_j V_{\alpha, \beta}(x_i - x_j) - V_{\alpha, \beta} * \bar{\rho}_t(x_i) \right)^2,$$

to get, for all $C_B > 0$,

$$\begin{aligned} & \mathbb{E} \left(\left(\frac{1}{N} \sum_j V_{\alpha,\beta}(X_t^i - X_t^j) - V_{\alpha,\beta} * \bar{\rho}_t(X_t^i) \right)^2 \right) \\ & \leq C_B \mathcal{H}_N(t) + \frac{C_B}{N} \log \mathbb{E} \left(\exp \left(\frac{1}{C_B} \left(\frac{1}{\sqrt{N}} \sum_j V_{\alpha,\beta}(\bar{X}_t^i - \bar{X}_t^j) - V_{\alpha,\beta} * \bar{\rho}_t(\bar{X}_t^i) \right)^2 \right) \right). \end{aligned}$$

This way,

$$\begin{aligned} B_N & \leq \frac{C_B \lambda^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2}{N^2} \sum_i \sum_{\alpha,\beta} \log \int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp \left(\frac{1}{C_B} \left(\frac{1}{\sqrt{N}} \sum_j V_{\alpha,\beta}(x_i - x_j) - V_{\alpha,\beta} * \bar{\rho}_t(x_i) \right)^2 \right) d\mathbf{x} \\ & \quad + C_B d^2 \lambda^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2 \mathcal{H}_N(t). \end{aligned}$$

In the following we choose $C_B = 64e^2 \|V\|_{L^\infty}^2$. Applying Theorem 4 to

$$\psi(z, x) = \frac{1}{8e \|V\|_{L^\infty}} (V(z - x) - V * \bar{\rho}_t(z)),$$

which satisfies $\|\psi\|_{L^\infty} \leq \frac{1}{4e}$ and is such that

$$\int_{\mathbb{T}^d} \psi(z, x) \bar{\rho}_t(x) dx = \frac{1}{8e \|V\|_{L^\infty}} \int_{\mathbb{T}^d} V(z - x) \bar{\rho}_t(x) dx - \frac{1}{8e \|V\|_{L^\infty}} \int_{\mathbb{T}^d} V * \bar{\rho}_t(z) \bar{\rho}_t(x) dx = 0,$$

we get

$$B_N \leq \hat{C}_B \|V\|_{L^\infty}^2 \lambda^2 d^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2 \left(\mathcal{H}_N(t) + \frac{\tilde{C}_B}{N} \right), \quad (3.5)$$

where \hat{C}_B and \tilde{C}_B are universal constants.

We now proceed with the bound on A_N . Applying Lemma 5 to

$$\Phi = \frac{1}{N^2} \sum_{i,j} (V(x_i - x_j) - V * \bar{\rho}_t(x_i)) : \frac{\nabla_{x_i}^2 \bar{\rho}_N}{\bar{\rho}_N},$$

we obtain, for all $C_A > 0$,

$$A_N \leq \frac{C_A}{N} \log \int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp \left(\frac{1}{C_A N} \sum_{i,j} (V(x_i - x_j) - V * \bar{\rho}_t(x_i)) : \frac{\nabla_{x_i}^2 \bar{\rho}_N}{\bar{\rho}_N} \right) d\mathbf{x} + C_A \mathcal{H}_N(t)$$

In the following we choose

$$C_A = 4\sqrt{1600^2 + 36e^4} \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \|V\|_{L^\infty} \lambda d := \hat{C}_A \lambda d \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \|V\|_{L^\infty}.$$

Then, we apply Theorem 5 to

$$\phi(z, x) = \frac{1}{C_A} \left((V(z - x) - V * \bar{\rho}_t(z)) : \frac{\nabla_{x_i}^2 \bar{\rho}_t}{\bar{\rho}_t}(z) \right),$$

which satisfies, thanks to Assumption 2

$$\begin{aligned}\int_{\mathbb{T}^d} \phi(z, x) \bar{\rho}_t(z) dz &= \frac{1}{C_A} \int_{\mathbb{T}^d} \left((V(z-x) - V * \bar{\rho}_t(z)) : \frac{\nabla^2 \bar{\rho}_t(z)}{\bar{\rho}_t(z)} \right) \bar{\rho}_t(z) dz \\ &= \frac{1}{C_A} \int_{\mathbb{T}^d} (\operatorname{div} K(z-x) - \operatorname{div} K * \bar{\rho}_t(z)) \bar{\rho}_t(z) dz = 0,\end{aligned}$$

and, thanks to $\int_{\mathbb{T}^d} (V(z-x) - V * \bar{\rho}_t(z)) \bar{\rho}_t(x) dx = 0$,

$$\int_{\mathbb{T}^d} \phi(z, x) \bar{\rho}_t(x) dx = 0.$$

Through our choice of C_A , (3.3) is verified, as $\gamma \leq (1600^2 + 36e^4) \left(\frac{2d\|V\|_{L^\infty} \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \lambda}{C_A} \right)^2 = \frac{1}{4} < 1$.
Hence

$$A_N \leq \hat{C}_A \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \|V\|_{L^\infty} \lambda d \left(\mathcal{H}_N(t) + \frac{\tilde{C}_A}{N} \right), \quad (3.6)$$

where \hat{C}_A and \tilde{C}_A are universal constants. The conclusion easily follows. \square

3.4 Proof of Theorem 1 in the smooth case

It only remains to gather the previous results. Equations (3.1), (3.5) and (3.6) yield

$$\begin{aligned}\frac{d}{dt} \mathcal{H}_N(t) &\leq \left(\hat{C}_A \lambda d \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \|V\|_{L^\infty} + \frac{\hat{C}_B \|V\|_{L^\infty}^2 \lambda^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2 d^2}{2} \right) \mathcal{H}_N(t) \\ &\quad + \frac{C_2}{N} - \frac{1}{2} \mathcal{I}_N(t),\end{aligned}$$

and using Corollary 2 and $\hat{C}_A \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \|V\|_{L^\infty} \lambda d \leq \frac{1}{2} \left(\frac{2\pi}{\lambda} \right)^2 + \frac{1}{2} \left(\frac{\lambda}{2\pi} \right)^2 \hat{C}_A^2 \|\nabla^2 \bar{\rho}_t\|_{L^\infty}^2 \|V\|_{L^\infty}^2 \lambda^2 d^2$

$$\begin{aligned}\frac{d}{dt} \mathcal{H}_N(t) &\leq - \left(\left(\frac{2\pi}{\lambda} \right)^2 - \hat{C}_A \lambda d \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \|V\|_{L^\infty} - \frac{\hat{C}_B \|V\|_{L^\infty}^2 \lambda^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2 d^2}{2} \right) \mathcal{H}_N(t) + \frac{C_2}{N} \\ &\leq - \frac{1}{2} \left(\left(\frac{2\pi}{\lambda} \right)^2 - \hat{C}_A^2 \frac{\lambda^4}{4\pi^2} d^2 \|\nabla^2 \bar{\rho}_t\|_{L^\infty}^2 \|V\|_{L^\infty}^2 - \hat{C}_B \|V\|_{L^\infty}^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2 \lambda^2 d^2 \right) \mathcal{H}_N(t) + \frac{C_2}{N}.\end{aligned}$$

In a more concise way, using Lemma 1, it means there are constants $C_1, C_2^\infty, C_3 > 0$ and a function $t \mapsto C_2(t) > 0$ with $\int_0^t C_2(s) ds \leq C_2^\infty$ for all $t \geq 0$ such that for all $t \geq 0$

$$\frac{d}{dt} \mathcal{H}_N(t) \leq -(C_1 - C_2(t)) \mathcal{H}_N(t) + \frac{C_3}{N}.$$

Multiplying by $\exp(C_1 t - \int_0^t C_2(s) ds)$ and integrating in time we get

$$\begin{aligned}\mathcal{H}_N(t) &\leq e^{-C_1 t + \int_0^t C_2(s) ds} \mathcal{H}_N(0) + \frac{C_3}{N} \int_0^t e^{C_1(s-t) + \int_s^t C_2(u) du} ds \\ &\leq e^{C_2^\infty - C_1 t} \mathcal{H}_N(0) + \frac{C_3}{C_1 N} e^{C_2^\infty},\end{aligned}$$

which concludes.

3.5 Dealing with the regularity of the density of the particle system

As mentioned at the beginning of Section 3, up to now we have proven the result under the additional assumption that there exists a smooth solution ρ_N to (1.4). Let us now remove this assumption. Consider $(\zeta_\epsilon)_{\epsilon \geq 0}$ a sequence of mollifiers such that $\|\zeta_\epsilon\|_{L^1} = 1$, whose compact support are assumed to be strictly contained within $[-\frac{1}{2}, \frac{1}{2}]^d$. Let us consider $K^\epsilon = K * \zeta_\epsilon$. We have $K^\epsilon \in \mathcal{C}^\infty(\mathbb{T}^d)$ and $\text{div}(K^\epsilon) = 0$.

Let ρ_N^ϵ be the unique smooth solution (see Lemma 8 below) of the parabolic equation with smooth coefficients

$$\partial_t \rho_N^\epsilon + \frac{1}{N} \sum_{i,j=1}^N K^\epsilon(x_i - x_j) \cdot \nabla_{x_i} \rho_N^\epsilon = \sum_{i=1}^N \Delta_{x_i} \rho_N^\epsilon, \quad (3.7)$$

with initial condition $\rho_N^\epsilon(0, \cdot) = \rho_N(0, \cdot)$.

We have the following bounds

Lemma 7. *Let $\gamma > 1$ be such that $\rho_N(0) \in \mathcal{C}_\gamma^\infty(\mathbb{T}^{dN})$. Then, for all $t \geq 0$ and all $\epsilon > 0$, $\rho_N^\epsilon(t) \in \mathcal{C}_\gamma^\infty(\mathbb{T}^{dN})$.*

Proof. Let $\mathbf{x} \in \mathbb{T}^{dN}$. Consider the particle system $dX_i^\epsilon(t) = -\frac{1}{N} \sum_{j=1}^N K^\epsilon(X_i^\epsilon(t) - X_j^\epsilon(t))dt + \sqrt{2}dB_t^i$ with initial condition $\mathbf{X}_0^\epsilon = \mathbf{x}$, where we denote $\mathbf{X}_t^\epsilon = (X_1^\epsilon(t), \dots, X_N^\epsilon(t))$. We have strong existence and uniqueness for this SDE. Then

$$\rho_N^\epsilon(t, \mathbf{x}) = \mathbb{E}(\rho_N^\epsilon(0, \mathbf{X}_t^\epsilon)).$$

The bounds on ρ_N^ϵ follow. □

Using Lemma 7, we get $(\rho_N^\epsilon)_\epsilon$ is a sequence of smooth functions uniformly bounded in $L^\infty(\mathbb{R}^+ \times \mathbb{T}^{Nd})$. This yields two results.

First, we can extract a weakly-* converging subsequence in $L^\infty(\mathbb{R}^+ \times \mathbb{T}^{Nd})$, i.e there exists $\rho_N \in L^\infty(\mathbb{R}^+ \times \mathbb{T}^{Nd})$ such that for all $f \in L^1(\mathbb{R}^+ \times \mathbb{T}^{Nd})$ we have

$$\int_{\mathbb{T}^{Nd}} \rho_N^\epsilon f \xrightarrow{\epsilon \rightarrow 0^+} \int_{\mathbb{T}^{Nd}} \rho_N f.$$

We finally check that ρ_N is indeed a weak solution of (1.4). For all $T \geq 0$ and for all f smooth test function on $[0, T] \times \mathbb{T}^{Nd}$

- We have, since $\partial_t f$ is smooth and therefore in $L^1([0, T] \times \mathbb{T}^{Nd})$

$$\int_{\mathbb{T}^{Nd}} \rho_N^\epsilon \partial_t f \rightarrow \int_{\mathbb{T}^{Nd}} \rho_N \partial_t f.$$

- Likewise, since $\Delta_{x_i} f$ is smooth and therefore in $L^1([0, T] \times \mathbb{T}^{Nd})$

$$\int_{\mathbb{T}^{Nd}} \rho_N^\epsilon \Delta_{x_i} f \rightarrow \int_{\mathbb{T}^{Nd}} \rho_N \Delta_{x_i} f.$$

- Finally

$$\begin{aligned} & \int_{\mathbb{T}^{Nd}} \rho_N^\epsilon K^\epsilon(x_i - x_j) \cdot \nabla_{x_i} f - \int_{\mathbb{T}^{Nd}} \rho_N K(x_i - x_j) \cdot \nabla_{x_i} f \\ &= \int_{\mathbb{T}^{Nd}} \rho_N^\epsilon (K^\epsilon(x_i - x_j) - K(x_i - x_j)) \cdot \nabla_{x_i} f + \int_{\mathbb{T}^{Nd}} (\rho_N^\epsilon - \rho_N) K(x_i - x_j) \cdot \nabla_{x_i} f \\ &\leq \|\rho_N^\epsilon\|_{L^\infty} \|\nabla_{x_i} f\|_{L^\infty} \|K^\epsilon - K\|_{L^1} + \int_{\mathbb{T}^{Nd}} (\rho_N^\epsilon - \rho_N) K(x_i - x_j) \cdot \nabla_{x_i} f \\ &\rightarrow 0, \end{aligned}$$

as $\|K^\epsilon - K\|_{L^1} \rightarrow 0$ and $K(x_i - x_j) \cdot \nabla_{x_i} f \in L^1([0, T] \times \mathbb{T}^{Nd})$.

We have thus proven that ρ_N is a weak solution of (1.4).

Likewise, we may consider $(\bar{\rho}^\epsilon)_\epsilon$, which weakly-* converges to a solution which, by uniqueness, is $\bar{\rho}$.

Second, ρ_N^ϵ satisfies the assumption made at the beginning of Section 3, i.e $\rho_N^\epsilon \in \mathcal{C}_{>0}^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$. Since by considering $V^\epsilon = V * \zeta_\epsilon$ we have $K^\epsilon = \text{div}(V^\epsilon)$, we get that K^ϵ satisfies Assumption 2 and that the calculations done in Section 3 are valid for this specific kernel, i.e

$$\mathcal{H}_N(\rho_N^\epsilon(t), \bar{\rho}_N^\epsilon(t)) \leq \mathcal{H}_N(\rho_N(0), \bar{\rho}_N(0))e^{-C_1^\epsilon t} e^{C^{\infty, \epsilon}} + \frac{C_3^\epsilon e^{C^{\infty, \epsilon}}}{C_1^\epsilon} \frac{1}{N}. \quad (3.8)$$

Notice how, in the proof of Lemma 1, the constants bounding the various derivatives of $\bar{\rho}$ only depend on the initial conditions, on $\|K\|_{L^1}$ and on $\|V\|_{L^\infty}$. Since $(\zeta_\epsilon)_{\epsilon \geq 0}$ is a sequence of mollifiers, we have $\|K^\epsilon\|_{L^1} \rightarrow \|K\|_{L^1}$ as ϵ tends to 0, and $\|V^\epsilon\|_{L^\infty} \leq \|V\|_{L^\infty}$. The righthand side of (3.8) can thus be chosen independent of ϵ .

We now use the fact that for $u \geq 0$ and $v \in \mathbb{R}$ we have $uv \leq u \log u - u + e^v$, to obtain the variational formulation of the entropy,

$$N\mathcal{H}_N(\rho_N^\epsilon(t), \bar{\rho}_N^\epsilon(t)) = \sup \left\{ \mathbb{E}_{\rho_N^\epsilon(t)}(g) - \mathbb{E}_{\bar{\rho}_N^\epsilon(t)}(e^g) + 1, g \in L^\infty \right\}, \quad (3.9)$$

the equality being attained for $g = \log\left(\frac{\rho_N^\epsilon}{\bar{\rho}_N^\epsilon}\right)$. We thus consider, for $g \in L^\infty$,

$$\frac{1}{N} \left(\mathbb{E}_{\rho_N^\epsilon(t)}(g) - \mathbb{E}_{\bar{\rho}_N^\epsilon(t)}(e^g) + 1 \right) \leq \mathcal{H}_N(\rho_N(0), \bar{\rho}_N(0))e^{-C_1 t} e^{C^\infty} + \frac{C_3 e^{C^\infty}}{1 + C_1} \frac{1}{N}.$$

By definition of the weak-* convergence in L^∞ (since both g and e^g are thus in L^1), we have the following convergence $\mathbb{E}_{\rho_N^\epsilon(t)}(g) \rightarrow \mathbb{E}_{\rho_N(t)}(g)$ and $\mathbb{E}_{\bar{\rho}_N^\epsilon(t)}(e^g) \rightarrow \mathbb{E}_{\bar{\rho}_N(t)}(e^g)$ as ϵ tends to 0. Therefore, for all $g \in L^\infty$,

$$\frac{1}{N} \left(\mathbb{E}_{\rho_N(t)}(g) - \mathbb{E}_{\bar{\rho}_N(t)}(e^g) + 1 \right) \leq \mathcal{H}_N(\rho_N(0), \bar{\rho}_N(0))e^{-C_1 t} e^{C^\infty} + \frac{C_3 e^{C^\infty}}{1 + C_1} \frac{1}{N},$$

which yields Theorem 1, using (3.9) for $\mathcal{H}_N(\rho_N(t), \bar{\rho}_N(t))$.

3.6 Proof of Corollary 1

Let $k \in \mathbb{N}$, and $N \geq k$. The sub-additivity of the entropy (see for instance Theorem 10.2.3 of [ABC⁺00]) implies that the (rescaled) relative entropy of the marginals is bounded by the total (rescaled) relative entropy

$$k \left\lfloor \frac{N}{k} \right\rfloor \mathcal{H}_k(\rho_N^k(t), \bar{\rho}_k(t)) \leq N\mathcal{H}_N(\rho_N(t), \bar{\rho}_N(t)).$$

The logarithmic Sobolev inequality established in Corollary 2 implies a Talagrand's transportation inequality (see [OV00]), so that the L^2 -Wasserstein distance is bounded by the relative entropy. Classically, this is also the case of the total variation thanks to Pinsker's inequality, and thus

$$\|\rho_N^k(t) - \bar{\rho}_k(t)\|_{L^1} + \mathcal{W}_2(\rho_N^k(t), \bar{\rho}_k(t)) \leq C \sqrt{k\mathcal{H}_k(\rho_N^k(t), \bar{\rho}_k(t))} \leq C \sqrt{\frac{N}{\lfloor \frac{N}{k} \rfloor} \mathcal{H}_N(\rho_N(t), \bar{\rho}_N(t))}.$$

With the additional assumption that $\mathcal{H}_N(\rho_N(0), \bar{\rho}_N(0)) = 0$, we thus get the result using Theorem 1. To obtain the result on the empirical measure, we recall for the sake of completeness the arguments of [JM22, Proposition 8]. Given $x, y \in \mathbb{T}^{dN}$, a coupling of $\pi(x)$ and $\pi(y)$ is obtained by considering (x_J, y_J) where J

is uniformly distributed over $\llbracket 1, N \rrbracket$. From this we get $\mathcal{W}_2(\pi(x), \pi(y)) \leq |x - y|/\sqrt{N}$. Considering (\mathbf{X}, \mathbf{Y}) an optimal coupling of $(\rho_N(t), \bar{\rho}_N(t))$, we bound

$$\begin{aligned} \mathbb{E}(\mathcal{W}_2(\pi(\mathbf{X}), \bar{\rho}_t)) &\leq \mathbb{E}(\mathcal{W}_2(\pi(\mathbf{X}), \pi(\mathbf{Y}))) + \mathbb{E}(\mathcal{W}_2(\pi(\mathbf{Y}), \bar{\rho}_t)) \\ &\leq \frac{1}{\sqrt{N}} \mathcal{W}_2(\rho_N(t), \bar{\rho}_N(t)) + \mathbb{E}(\mathcal{W}_2(\pi(\mathbf{Y}), \bar{\rho}_t)). \end{aligned}$$

The last term is tackled with the result for i.i.d. variables established in [FG15].

A Proof of Theorem 2

The proof is based on an iterative procedure, and relies heavily on the work of Ben-Artzi [BA94]. Let $\bar{\rho}^{(-1)} := 0$, and then for $k \in \mathbb{N}$ solve

$$\partial_t \bar{\rho}^{(k)} = - \left(u^{(k-1)} \cdot \nabla \right) \bar{\rho}^{(k)} + \Delta \bar{\rho}^{(k)}, \text{ in } \mathbb{R}^+ \times \mathbb{T}^d \quad (\text{A.1})$$

$$u^{(k)} = K * \bar{\rho}^{(k)} \quad (\text{A.2})$$

$$\bar{\rho}^{(k)}(0, \cdot) = \mu_0. \quad (\text{A.3})$$

Let us recall the following lemma concerning the regularity of the second order parabolic equation. We refer to Chapter 7 of [Eva10] for a proof on a bounded domain that can be extended to the torus.

Lemma 8. *Let $a(t, x)$ be a C^∞ function on $\mathbb{R}^+ \times \mathbb{T}^d$ and $\psi_0 \in C^\infty(\mathbb{T}^d)$. Then the problem*

$$\begin{aligned} \partial_t \psi &= -a \cdot \nabla \psi + \Delta \psi, \text{ in } \mathbb{R}^+ \times \mathbb{T}^d \\ \psi(0, \cdot) &= \psi_0, \end{aligned}$$

has a unique solution, which is C^∞ .

Lemma 9. *Suppose $\mu_0 \in C^\infty(\mathbb{T}^d)$. Then the system (A.1)-(A.3) defines successively a sequence of C^∞ solutions $\{\bar{\rho}^{(k)}, u^{(k)}\}_{k \in \mathbb{N}}$.*

Furthermore, for all $t \geq 0$ and all $k \in \mathbb{N}$, $\|\bar{\rho}^{(k)}(t, \cdot)\|_{L^\infty} \leq \|\mu_0\|_{L^\infty}$ and $\|u^{(k)}(t, \cdot)\|_{L^\infty} \leq \|K\|_{L^1} \|\mu_0\|_{L^\infty}$. Finally, given a final time $T \geq 0$, $\bar{\rho}^{(k)}$ (resp. $u^{(k)}$) and all its derivatives, both in time and in space, are bounded on $[0, T] \times \mathbb{T}^d$ uniformly in k

Proof. We use induction on k . The assertion is clear for $\bar{\rho}^{(0)}$ as the explicit solution to the heat equation. Suppose $\{\bar{\rho}^{(j)}, u^{(j-1)}\}_{j=0, \dots, k}$ have been shown to be C^∞ solutions bounded uniformly in time.

Regularity. By definition

$$u^{(k)}(t, x) = K * \bar{\rho}^{(k)}(t, x) = \int_{\mathbb{T}^d} K(x - y) \bar{\rho}^{(k)}(t, y) dy = - \int_{\mathbb{T}^d} K(y) \bar{\rho}^{(k)}(t, x - y) dy.$$

Then

$$u^{(k)}(t, x) = - \int_{\mathbb{T}^d} \text{div} V(y) \bar{\rho}^{(k)}(t, x - y) dy = - \int_{\mathbb{T}^d} V(y) \nabla_y \bar{\rho}^{(k)}(t, x - y) dy.$$

Since we are in the compact set \mathbb{T}^d , that $V \in L^\infty(\mathbb{T}^d)$, and that $\bar{\rho}^{(k)} \in C^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$ by induction hypothesis, we can easily show that $u^{(k)}$, as well as all its derivatives, are Lipschitz continuous. Hence $u^{(k)}$ is C^∞ . Using Lemma 8 in (A.1) with k replaced by $k + 1$ yields the desired result for $\bar{\rho}^{(k+1)}$.

Boundedness of $\bar{\rho}^{(k+1)}$ and $u^{(k)}$. Let us show that for all $T \geq 0$, $\bar{\rho}^{(k+1)}$ and $u^{(k)}$ are both bounded on $[0, T] \times \mathbb{T}^d$, with a bound independent of T . We have, using Young's convolution inequality and the induction hypothesis

$$\|u^{(k)}(t, \cdot)\|_{L^\infty} \leq \|K\|_{L^1} \|\bar{\rho}^{(k)}(t, \cdot)\|_{L^\infty} \leq \|K\|_{L^1} \|\mu_0\|_{L^\infty}.$$

Now $\bar{\rho}^{(k+1)}$ is the unique solution of

$$\begin{aligned} \partial_t \bar{\rho}^{(k+1)} &= - \left(u^{(k)} \cdot \nabla \right) \bar{\rho}^{(k+1)} + \Delta \bar{\rho}^{(k+1)} \\ \bar{\rho}^{(k+1)}(0, x) &= \mu_0(x). \end{aligned}$$

For $t \geq 0$, consider $Z_s^{(k+1)}$ the strong solution of the following stochastic differential equation for $s \in [0, t]$

$$dZ_s^{(k+1)} = \sqrt{2}dB_s - u^{(k)}(t-s, Z_s)ds,$$

which exists, is unique and non-explosive since $u^{(k)}$ is smooth, bounded and Lipschitz continuous. Then

$$\bar{\rho}^{(k+1)}(t, x) = \mathbb{E}_x \left(\mu_0(Z_t^{(k+1)}) \right).$$

We thus get

$$\|\bar{\rho}_t^{(k+1)}\|_{L^\infty} \leq \|\mu_0\|_{L^\infty}.$$

Notice that this is simply a probabilistic way of presenting the use of the maximum principle.

Boundedness of the derivatives of $\bar{\rho}^{(k+1)}$ and $u^{(k)}$. The boundedness of the derivatives of $u^{(k)}$ is a direct consequence of the boundedness of the derivatives of $\bar{\rho}^{(k)}$ thanks to Young's convolution inequality. Then, the proof for $\bar{\rho}^{(k+1)}$ similar to the proof of Lemma 1, using the boundedness of the derivatives of $u^{(k)}$. To show that the bounds are in fact independent of k , we follow the proof of Lemma 1, i.e by induction on the order of the derivative, and within each induction step we prove that both the integrated and uniform bounds are independent of k . This comes from the fact that the proof initially only relies on the bounds on $\|\bar{\rho}_t^{(k+1)}\|_{L^\infty}$ and $\|u_t^{(k)}\|_{L^\infty}$ -which, as we have shown, only depend on $\|\mu_0\|_{L^\infty}$ - and then for each induction step on the initial condition and on the bounds constructed at the previous step (therefore independent of k). The bounds concerning the derivatives involving time are then obtained thanks to the bounds on the space derivatives using (A.1). □

Proof of Theorem 2. It is sufficient to prove existence and uniqueness of the solution in $[0, T] \times \mathbb{T}^d$ for all $T \geq 0$, since then the solutions on $[0, T_1] \times \mathbb{T}^d$ and $[0, T_2] \times \mathbb{T}^d$, with $T_1 < T_2$, must coincide in $[0, T_1] \times \mathbb{T}^d$, leading to the existence and uniqueness of the global solution in $\mathbb{R}^+ \times \mathbb{T}^d$. Let us consider $T \geq 0$

Existence in $[0, T] \times \mathbb{T}^d$ for T small enough: Let us show the existence of the limit solution. We consider here T to be small enough (an explicit bound will be given later) Let $G(t, x) = \sum_{k \in \mathbb{Z}^d} \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp(-\frac{|x+k|^2}{4t})$ be the heat kernel on the d dimensional torus. We have

$$\bar{\rho}^{(k)}(t, x) = G(t, \cdot) * \mu_0(x) - \int_0^t \int_{\mathbb{T}^d} G(t-s, x-y) u^{(k-1)}(s, y) \cdot \nabla_y \bar{\rho}^{(k)}(s, y) dy ds.$$

Let us denote $N_k(t) = \sup_{0 \leq s \leq t} \|\bar{\rho}^{(k+1)}(s, \cdot) - \bar{\rho}^{(k)}(s, \cdot)\|_{L^\infty}$. We have, using $\nabla_y \cdot u^{(k)} = 0$

$$\begin{aligned} \bar{\rho}^{(k+1)}(t, x) - \bar{\rho}^{(k)}(t, x) &= - \int_0^t \int_{\mathbb{T}^d} \nabla_y G(t-s, x-y) \left(\bar{\rho}^{(k+1)}(s, y) - \bar{\rho}^{(k)}(s, y) \right) u^{(k)}(s, y) dy ds \\ &\quad - \int_0^t \int_{\mathbb{T}^d} \nabla_y G(t-s, x-y) \bar{\rho}^{(k)}(s, y) \left(u^{(k)}(s, y) - u^{(k-1)}(s, y) \right) dy ds. \end{aligned}$$

Remark (using the first moment of the chi distribution) that, for some constant $\beta > 0$

$$\int_{\mathbb{T}^d} |\nabla_x G(t, x)| dx \leq \beta t^{-\frac{1}{2}}.$$

We thus get

$$\begin{aligned} \|\bar{\rho}^{(k+1)}(t, \cdot) - \bar{\rho}^{(k)}(t, \cdot)\|_{L^\infty} &\leq \beta \|K\|_{L^1} \|\mu_0\|_{L^\infty} \int_0^t (t-s)^{-\frac{1}{2}} \|\bar{\rho}^{(k+1)}(s, \cdot) - \bar{\rho}^{(k)}(s, \cdot)\|_{L^\infty} ds \\ &\quad + \beta \|\mu_0\|_{L^\infty} \int_0^t (t-s)^{-\frac{1}{2}} \|u^{(k)}(s, \cdot) - u^{(k-1)}(s, \cdot)\|_{L^\infty} ds, \end{aligned}$$

and

$$\|u^{(k)}(s, \cdot) - u^{(k-1)}(s, \cdot)\|_{L^\infty} \leq \|K\|_{L^1} \|\bar{\rho}^{(k)}(s, \cdot) - \bar{\rho}^{(k-1)}(s, \cdot)\|_{L^\infty}.$$

Therefore

$$N_k(t) \leq \beta \|K\|_{L^1} \|\mu_0\|_{L^\infty} \int_0^t (t-s)^{-\frac{1}{2}} N_k(s) ds + \beta \|K\|_{L^1} \|\mu_0\|_{L^\infty} \int_0^t (t-s)^{-\frac{1}{2}} N_{k-1}(s) ds.$$

Denoting $C = \beta \|K\|_{L^1} \|\mu_0\|_{L^\infty}$ we get

$$N_k(t) \leq C \int_0^t (t-s)^{-\frac{1}{2}} (N_k(s) + N_{k-1}(s)) ds. \quad (\text{A.4})$$

Since N_k is continuous, there exists $R > 0$ such that for all $t \in [0, T]$ we have $N_k(t) \leq R$. We thus have, using this bound in (A.4) and assuming $2C\sqrt{T} \leq \frac{1}{2}$

$$\begin{aligned} N_k(t) &\leq RC \int_0^t (t-s)^{-\frac{1}{2}} ds + C \int_0^t (t-s)^{-\frac{1}{2}} N_{k-1}(s) ds \\ &\leq \frac{R}{2} + C \int_0^t (t-s)^{-\frac{1}{2}} N_{k-1}(s) ds. \end{aligned}$$

We use this bound in (A.4)

$$\begin{aligned} N_k(t) &\leq \frac{R}{2} C \int_0^t (t-s)^{-\frac{1}{2}} ds + C \int_0^t (t-s)^{-\frac{1}{2}} N_{k-1}(s) ds \\ &\quad + C^2 \int_0^t \int_0^s (t-s)^{-\frac{1}{2}} (s-u)^{-\frac{1}{2}} N_{k-1}(u) du ds. \end{aligned}$$

We deal with this last term

$$\begin{aligned} &C^2 \int_0^t \int_0^s (t-s)^{-1/2} (s-u)^{-\frac{1}{2}} N_{k-1}(u) du ds \\ &= C^2 \int_0^t N_{k-1}(u) \int_u^t (t-s)^{-\frac{1}{2}} (s-u)^{-\frac{1}{2}} ds du \\ &= C^2 \pi \int_0^t N_{k-1}(u) du. \end{aligned}$$

Let $\alpha = \sqrt{T}\pi C$ and choose T such that $\alpha \leq \frac{1}{2}$ (which in turns also yields the previous condition $2C\sqrt{T} \leq \frac{1}{2}$).

We have

$$\alpha C \int_0^t (t-s)^{-\frac{1}{2}} N_{k-1}(s) ds - C^2 \pi \int_0^t N_{k-1}(s) du = C \int_0^t N_{k-1}(s) \left(\alpha (t-s)^{-\frac{1}{2}} - \pi C \right) ds,$$

and since $\alpha = \sqrt{T}\pi C \geq \sqrt{t-s}\pi C$ for $0 \leq s \leq t \leq T$, we get

$$\alpha C \int_0^t (t-s)^{-\frac{1}{2}} N_{k-1}(s) ds \geq C^2 \pi \int_0^t N_{k-1}(s) ds,$$

and thus

$$N_k(t) \leq \frac{R}{4} + C(1+\alpha) \int_0^t (t-s)^{-\frac{1}{2}} N_{k-1}(s) ds.$$

Iterating this method, we obtain for all $n \in \mathbb{N}$

$$N_k(t) \leq 2^{-n} R + C(1+\alpha+\dots+\alpha^{n-1}) \int_0^t (t-s)^{-\frac{1}{2}} N_{k-1}(s) ds,$$

and thus

$$N_k(t) \leq 2C \int_0^t (t-s)^{-\frac{1}{2}} N_{k-1}(s) ds.$$

We now show that this implies that

$$N_k(t) \leq N_0(T) \left(2C\Gamma\left(\frac{1}{2}\right) \right)^k t^{k/2} \Gamma\left(\frac{k+2}{2}\right)^{-1}, \quad (\text{A.5})$$

where we denote $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$. We have that for $k=0$, (A.5) is satisfied and, by induction, we have

$$\int_0^t (t-s)^{-\frac{1}{2}} s^{\frac{k}{2}} ds = t^{\frac{k+1}{2}} \int_0^1 (1-u)^{-\frac{1}{2}} u^{\frac{k}{2}} du = t^{\frac{k+1}{2}} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{k+2}{2}\right)}{\Gamma\left(\frac{k+3}{2}\right)}$$

Using the fact that $\Gamma(k+1) = k!$ and $\Gamma\left(k+\frac{3}{2}\right) = k!\Gamma\left(\frac{1}{2}\right)$, we get that $\sum_{k=0}^\infty N_k(t)$ converges uniformly for $t \in [0, T]$ and the limits

$$\bar{\rho}(t, x) = \lim_{k \rightarrow \infty} \bar{\rho}^{(k)}(t, x) \quad \text{and} \quad u(t, x) = \lim_{k \rightarrow \infty} u^{(k)}(t, x)$$

exist in $\mathcal{C}([0, T] \times \mathbb{T}^d)$. Now, since for all $l, n \in \mathbb{N}$ and all $\alpha_1, \dots, \alpha_n$, $\|\partial_t^l \partial_{\alpha_1, \dots, \alpha_n} \bar{\rho}^{(k)}\|_{L^\infty}$ and $\|\partial_t^l \partial_{\alpha_1, \dots, \alpha_n} u^{(k)}\|_{L^\infty}$ are bounded uniformly in k , using Arzela-Ascoli theorem, we have uniform convergence, up to an extraction, of the derivatives. Hence the validity of the limits in $\mathcal{C}^\infty([0, T] \times \mathbb{T}^d)$, i.e there is convergence of the functions along with their derivatives of all order in $[0, T] \times \mathbb{T}^d$. This gives us the fact that the limit $\bar{\rho}$ satisfies (2.1).

Uniqueness in $[0, T] \times \mathbb{T}^d$. Suppose $\bar{\rho}^1$ and $\bar{\rho}^2$ are two bounded solutions of (2.1) on $[0, T] \times \mathbb{T}^d$. Then

$$\partial_t (\bar{\rho}^1 - \bar{\rho}^2) - \Delta (\bar{\rho}^1 - \bar{\rho}^2) = - (K * \bar{\rho}^1) \cdot \nabla (\bar{\rho}^1 - \bar{\rho}^2) - \nabla \cdot ((K * \bar{\rho}^1 - K * \bar{\rho}^2) \bar{\rho}^2),$$

so that

$$\begin{aligned} \bar{\rho}^1(t, x) - \bar{\rho}^2(t, x) &= - \int_0^t \int_{\mathbb{T}^d} \nabla_y G(t-s, x-y) \cdot (K *_{\bar{y}} \bar{\rho}^1(s, y)) (\bar{\rho}^1(s, y) - \bar{\rho}^2(s, y)) dy ds \\ &\quad - \int_0^t \int_{\mathbb{T}^d} \nabla_y G(x-y, t-s) \cdot (K *_{\bar{y}} \bar{\rho}^1(s, y) - K *_{\bar{y}} \bar{\rho}^2(s, y)) \bar{\rho}^2(s, y) dy ds. \end{aligned}$$

Let $N(t) := \sup_{0 \leq s \leq t} \|\bar{\rho}^1(s, \cdot) - \bar{\rho}^2(s, \cdot)\|_{L^\infty}$. Recall

$$\|K * \bar{\rho}^1(s, \cdot) - K * \bar{\rho}^2(s, \cdot)\|_{L^\infty} \leq \|K\|_{L^1} \|\bar{\rho}^1(s, \cdot) - \bar{\rho}^2(s, \cdot)\|_{L^\infty},$$

which implies, like previously, the existence of a constant C such that

$$N(t) \leq C \int_0^t (t-s)^{-1/2} N(s) ds.$$

We choose $L > 0$ such that $C \int_0^T s^{-\frac{1}{2}} e^{-Ls} ds \leq \frac{1}{2}$, and let $Q(t) = e^{-Lt} N(t)$, which satisfies for all t

$$Q(t) \leq C \int_0^t (t-s)^{-1/2} Q(s) e^{-L(t-s)} ds.$$

Let $R > 0$ be such that $Q(t) \leq R$.

Then

$$Q(t) \leq RC \int_0^t (t-s)^{-1/2} e^{-L(t-s)} ds \leq \frac{R}{2}.$$

By induction, we get $N(t) = 0$ for $t \in [0, T]$. This concludes the proof of uniqueness.

Existence in $\mathbb{R}^+ \times \mathbb{T}^d$. For T small enough, there exists a solution in $[0, T] \times \mathbb{T}^d$. Notice that T only depends on constants independent of time (it depends on the L^∞ bound of the initial condition, which we have shown propagates). It is therefore possible to construct the (unique) smooth solution on all intervals $[t_0, T + t_0] \times \mathbb{T}^d$. Uniqueness allows us to iteratively construct the (unique) smooth solution on $\mathbb{R}^+ \times \mathbb{T}^d$. This concludes the proof. □

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References

- [ABC⁺00] Cécile Ané, Sébastien Blachère, Djalil Chafaï, Pierre Fougères, Ivan Gentil, Florent Malrieu, Cyril Roberto, and Grégory Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2000. With a preface by Dominique Bakry and Michel Ledoux.
- [BA94] Matania Ben-Artzi. Global solutions of two-dimensional Navier-Stokes and Euler equations. *Arch. Rational Mech. Anal.*, 128(4):329–358, 1994.
- [BB03] Jean Bourgain and Haïm Brezis. On the equation $\operatorname{div} Y = f$ and application to control of phases. *J. Amer. Math. Soc.*, 16(2):393–426, 2003.
- [BGL14] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2014.

- [BGV07] François Bolley, Arnaud Guillin, and Cédric Villani. Quantitative concentration inequalities for empirical measures on non-compact spaces. *Probab. Theory Related Fields*, 137(3-4):541–593, 2007.
- [BJW19a] Didier Bresch, Pierre-Emmanuel Jabin, and Zhenfu Wang. Modulated free energy and mean field limit. *Séminaire Laurent Schwartz—EDP et applications*, pages 1–22, 2019.
- [BJW19b] Didier Bresch, Pierre-Emmanuel Jabin, and Zhenfu Wang. On mean-field limits and quantitative estimates with a large class of singular kernels: application to the Patlak-Keller-Segel model. *C. R. Math. Acad. Sci. Paris*, 357(9):708–720, 2019.
- [BJW20] Didier Bresch, Pierre-Emmanuel Jabin, and Zhenfu Wang. Mean-field limit and quantitative estimates with singular attractive kernels. *arXiv e-prints*, page arXiv:2011.08022, November 2020.
- [CRS23] Antonin Chodron de Courcel, Matthew Rosenzweig, and Sylvia Serfaty. Sharp uniform-in-time mean-field convergence for singular periodic Riesz flows. *arXiv e-prints*, page arXiv:2304.05315, April 2023.
- [CRW22] Fan Chen, Zhenjie Ren, and Songbo Wang. Uniform-in-Time Propagation of Chaos for Mean Field Langevin Dynamics. *arXiv e-prints*, page arXiv:2212.03050, December 2022.
- [DEGZ20] Alain Durmus, Andreas Eberle, Arnaud Guillin, and Raphael Zimmer. An elementary approach to uniform in time propagation of chaos. *Proc. Amer. Math. Soc.*, 148(12):5387–5398, 2020.
- [DT21] François Delarue and Alvin Tse. Uniform in time weak propagation of chaos on the torus. *arXiv e-prints*, page arXiv:2104.14973, April 2021.
- [Eva10] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.
- [FG15] Nicolas Fournier and Arnaud Guillin. On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields*, 162(3-4):707–738, 2015.
- [FHM14] Nicolas Fournier, Maxime Hauray, and Stéphane Mischler. Propagation of chaos for the 2D viscous vortex model. *J. Eur. Math. Soc. (JEMS)*, 16(7):1423–1466, 2014.
- [Gen04] Ivan Gentil. Inégalités de sobolev logarithmique et de poincaré pour la loi uniforme. *Unpublished note*, see <http://math.univ-lyon1.fr/homes-www/gentil/logsob-segment.pdf>, 2004.
- [HM14] Maxime Hauray and Stéphane Mischler. On Kac’s chaos and related problems. *J. Funct. Anal.*, 266(10):6055–6157, 2014.
- [JM22] Lucas Journal and Pierre Monmarché. Convergence of a particle approximation for the quasi-stationary distribution of a diffusion process: uniform estimates in a compact soft case. *ESAIM Probab. Stat.*, 26:1–25, 2022.
- [JW18] Pierre-Emmanuel Jabin and Zhenfu Wang. Quantitative estimates of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels. *Invent. Math.*, 214(1):523–591, 2018.
- [Kac56] Mark Kac. Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III*, pages 171–197. University of California Press, Berkeley-Los Angeles, Calif., 1956.

- [Lac23] Daniel Lacker. Hierarchies, entropy, and quantitative propagation of chaos for mean field diffusions. *Probab. Math. Phys.*, 4(2):377–432, 2023.
- [LLF23] Daniel Lacker and Luc Le Flem. Sharp uniform-in-time propagation of chaos. *Probab. Theory Related Fields*, 187(1-2):443–480, 2023.
- [M96] Sylvie Méléard. Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models. In *Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995)*, volume 1627 of *Lecture Notes in Math.*, pages 42–95. Springer, Berlin, 1996.
- [Mal01] Florent Malrieu. Logarithmic Sobolev inequalities for some nonlinear PDE’s. *Stochastic Process. Appl.*, 95(1):109–132, 2001.
- [MP94] Carlo Marchioro and Mario Pulvirenti. *Mathematical theory of incompressible nonviscous fluids*, volume 96 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.
- [OV00] F. Otto and C. Villani. Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.*, 173(2):361–400, 2000.
- [PT08] Nguyen Cong Phuc and Monica Torres. Characterizations of the existence and removable singularities of divergence-measure vector fields. *Indiana Univ. Math. J.*, 57(4):1573–1597, 2008.
- [RS23] Matthew Rosenzweig and Sylvia Serfaty. Global-in-time mean-field convergence for singular Riesz-type diffusive flows. *Ann. Appl. Probab.*, 33(2):754–798, 2023.
- [Sch96] Steven Schochet. The point-vortex method for periodic weak solutions of the 2-D Euler equations. *Comm. Pure Appl. Math.*, 49(9):911–965, 1996.
- [Szn91] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d’Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.
- [Vla68] A.A. Vlasov. The vibrational properties of an electron gas. *Soviet Physics Uspekhi*, 10(6):721–733, jun 1968.
- [Wyn23] Dominic Wynter. Quantitative propagation of chaos for the mixed-sign viscous vortex model on the torus. *Kinet. Relat. Models*, 16(2):294–310, 2023.