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Nondimensionalization of the Atmospheric Boundary-Layer System: Obukhov Length and Monin–Obukhov Similarity Theory

Jun–Ichi Yano¹ · Marta Waclawczyk²

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Abstract

The Obukhov length, although often adopted as a characteristic scale of the atmospheric boundary layer, has been introduced purely based on a dimensional argument without a deductive derivation from the governing equations. Here, its derivation is pursued by the nondimensionalization method in the same manner as for the Rossby deformation radius and the Ekman-layer depth. Physical implications of the Obukhov length are inferred by nondimensionalizing the turbulence-kinetic-energy equation for the horizontally homogeneous boundary layer. A nondimensionalization length scale for a full set of equations for boundary-layer flow formally reduces to the Obukhov length by dividing this scale by a rescaling factor. This rescaling factor increases with increasing stable stratification of the boundary layer, in which flows tend to be more horizontal and gentler; thus the Obukhov length increasingly loses its relevance. A heuristic, but deductive, derivation of Monin–Obukhov similarity theory is also outlined based on the obtained nondimensionalization results.

Keywords Nondimensionalization scale · Obukhov length · Stably stratified turbulence · Turbulence-kinetic-energy equation · Vertical scale

1 Introduction

Theories of atmospheric boundary-layer turbulence have been developed by heavily relying on so-called dimensional analysis (Barenblatt 1996). This methodology is alternatively called scaling in atmospheric boundary-layer studies, as reviewed by e.g., Holtslag and Nieuwstadt (1986), and Foken (2006). Some key variables controlling a given regime of turbulence are first identified, then various characteristic scales of the system (e.g., length, velocity, temperature) are determined from these key controlling variables by ensuring dimensional

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consistency. For example, the Obukhov length (Obukhov 1948) follows from dimensional analysis of the frictional velocity and the buoyancy flux. Resulting theories from these analyses are called “similarity theories”, because they remain similar regardless of specific cases by rescaling the relevant variables by characteristic scales.

In performing a dimensional analysis correctly, a certain ingenuity is required for choosing the proper controlling variables of a given system. No systematic methodology exists for choosing them, and the choice is solely based on physical intuition. A wrong choice of controlling variables can lead to totally meaningless results (cf., Batchelor 1954). With the absence of an analytical solution to turbulent flows as well as difficulties in observations and numerical modelling, the usefulness of those proposed scales is often hard to judge. As a result, the Obukhov length is hardly a unique choice. There are various efforts to introduce alternative scales, as further discussed in Sects. 2.4 and 6.1, but the theory based on gradient-based scales (Sorbjan 2006, 2010, 2016) is probably the most notable.

However, dimensional analysis is not a sole possibility of defining the characteristic scales of a system. In atmospheric large-scale dynamics, these scales are typically derived by nondimensionalizations. No doubt, this procedure is more straightforward and formal: characteristic scales are introduced for all the variables of a system for nondimensionalizing them. These scales cannot be arbitrary, because we expect that terms in an equation to balance each other, thus their orders of magnitudes must match. These conditions, in turn, constrain these characteristic scales in a natural manner. An advantage of the nondimensionalization procedure is that these scales are defined not only by dimensional consistencies, but also by requirements of balance by order of magnitude between the terms in a system. The Rossby radius of deformation is a classical example of a characteristic scale identified by nondimensionalization. This scale characterizes the quasi-geostrophic system (Sect. 3.12, Pedlosky 1987). The depth of the Ekman layer is another such example (Sect. 4.3, Pedlosky 1987). Yano and Tsujimura (1987), and Yano and Bonazzola (2009) systematically apply this methodology for their scale analysis.

In a certain sense, nondimensionalization is a brute-force approach without relying on any physical intuition, even observation or modelling. The procedure is totally formal and abstract. However, it can lead us to identify physically meaningful scales because it is applied to equations that govern a given system physically: see e.g., discussions concerning Eq. 5a, b in Sect. 2.3.

Note a subtle difference in the concept of the scale in nondimensionalization from that in similarity theories: in similarity theories, precise values are used to scale the variables in universal functions that define a solution of a system. On the other hand, for the nondimensionalization procedure, the scales are typically defined by orders of magnitudes, as just stated. This difference must clearly be kept in mind in the following analyses (see further discussions in Sect. 5). However, in principle, the characteristic scales adopted in boundary-layer similarity theories should be linked to the nondimensionalization scales identified by the nondimensionalization procedure, if the former have a physical basis for adoption. The most basic motivation of the present study is to investigate whether this is the case.

As a first step of such a systematic investigation, the present study focuses on the Obukhov length. This scale is a core of the celebrated Monin–Obukhov similarity theory (Monin and Obukhov 1954). In spite of its importance in describing boundary-layer turbulence, the basis of this Obukhov length is often questioned. In particular, previous studies suggest that this similarity theory breaks down in a strongly stratified limit (cf., King 1990; Howell and Sun 1999; Mahrt 1999). Various efforts for generalization of Monin–Obukhov theory already exist (e.g., Zilitinkevich and Calanca 2000; Zilitinkevich 2002; Zilitinkevich and Esau 2007). However, as far as the authors are aware, all these efforts are under the framework

of dimensional analysis. The present paper suggests a procedure beyond those efforts by analyzing the governing equations of the atmospheric boundary layer more directly.

The nondimensionalization procedure itself is hardly new in boundary-layer meteorology. For example, Mahrt (1982) adopts it for analyzing gravity-wave currents in the boundary layer. A full nondimensionalization performed by Nieuwstadt is closer to the spirit of the present study as it focuses on the stably stratified boundary layer. However, unlike the present study, the applicability of this nondimensionalization is rather limited by adopting a model already including a closure, thus a final nondimensionalization result also depends on this closure. In the present study, we consider the atmospheric boundary-layer governing equations without closure for this reason.

The present analysis is close to the spirit of George et al. (2000) in seeking to identify characteristic scales of a given system by directly examining a balance in the governing equations, although their study does not go through a path of nondimensionalization. This link further suggests possibilities of applying various methodologies of multiscale asymptotic expansions (cf., Yano and Tsujimura 1987) to atmospheric boundary-layer problems, though attempts are left for future studies.

Below, the basics of the nondimensionalization method are introduced in Sect. 2. First (Sect. 2.1), by taking a linear stability problem of a vertical-shear flow with stratification as a simple pedagogical example, it is demonstrated in an explicit manner how characteristic scales can be identified from the nondimensionalization procedure. The basic premises of the method are then discussed (Sect. 2.3), because they become crucial in its application to the atmospheric boundary layer in subsequent sections. The adopted example is a standard stability problem in fluid mechanics (cf., Miles 1961; Howard 1961). This example also serves the purpose of introducing the Richardson number, which plays a key role in Monin–Obukhov theory (cf., Łobocki 2013), as a nondimensional parameter of the problem. In standard analyses, the Richardson number is introduced in retrospect, only after solving the stability problem in dimensional form, with the exception of Sect. 8.1 of Townsend (1976), which outlines a nondimensionalization procedure of this problem.

The main analyses are presented in Sects. 3 and 4. The turbulence-kinetic-energy (TKE) equation is considered in Sect. 3 as a preliminary analysis. However, the TKE equation for atmospheric boundary-layer flow is hardly self-contained, thus its generalization is much desirable. For this reason, the linear analysis of Sect. 2 is generalized into a fully nonlinear case in Sect. 4. The analysis identifies a vertical scale for the nondimensionalization with an explicit dependence of its definition on the Richardson number. The identified nondimensionalization scale can directly be linked to the Obukhov length with a rescaling factor, which is defined in terms of the nondimensional eddy amplitude and the aspect ratio of the dominant turbulent eddies. Furthermore, the similarity theory is derived in Sect. 5 in a heuristic manner from a derived nondimensionalized system in Sect. 4. The paper concludes with final remarks in Sect. 6.

The paper adopts the Boussinesq approximation throughout the analysis for simplicity. Though an extension of the analysis under an anelastic approximation is straightforward, there is only a small advantage in making the analysis more involved. For the same reason, a two-dimensional system is considered in Sect. 2, with x and z taken as the horizontal and vertical coordinates, respectively. Also keep in mind that throughout, the vertical nondimensionalization scale is always that of the dominant eddy of a system. Thus, a certain caution is required to link it to the Obukhov length, because the latter is adopted to be in many studies a vertical characteristic scale for the vertical profiles of vertical heat and momentum fluxes. Obukhov (1948) originally introduced this scale to measure the height of the surface boundary layer (“sub-layer of dynamic turbulence”).

The overbar and prime signs are used throughout for designating the mean and the deviation from the former. The former is assumed to depend only on height, z . The latter corresponds to a turbulence fluctuation when a fully nonlinear problem is considered, as in Sects. 3 and 4. On the other hand, under a linear stability analysis in Sect. 2, the primed quantities designate the perturbation variables. Due to the linearization, the perturbation variables may grow to infinity in an unstable situation, whereas the turbulent fluctuations are bounded by full nonlinearity. Nevertheless, the formal definition of the prime sign itself does not change in these sections.

2 A Linear Perturbation Problem: Basic Premises of Nondimensionalization

2.1 Analysis

The purpose of this section is to introduce the basics of the nondimensionalization method that is systematically exploited in the following sections. As a simple concrete example for a demonstration, we consider a linear perturbation equation for a standard shear instability. Viscosity is neglected by following the standard formulation (cf., Miles 1961; Howard 1961). Thus, the governing equations of the problem are

$$\frac{\partial u'}{\partial t} + w' \frac{d\bar{u}}{dz} + \bar{u} \frac{\partial u'}{\partial x} = -\frac{\partial \phi'}{\partial x}, \tag{1a}$$

$$\frac{\partial w'}{\partial t} + \bar{u} \frac{\partial w'}{\partial x} = -\frac{\partial \phi'}{\partial z} + b', \tag{1b}$$

$$\frac{\partial b'}{\partial t} + \bar{u} \frac{\partial b'}{\partial x} + w' \frac{d\bar{b}}{dz} = 0, \tag{1c}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0. \tag{1d}$$

Here, u and w are the horizontal and the vertical components of the velocity, ϕ is a perturbation pressure divided by a reference density, b' is the perturbation buoyancy, which may be evaluated from the perturbation potential temperature, θ' , as $b' = g\theta'/\theta_0$, where g is the acceleration due to gravity, and θ_0 is a reference value of the potential temperature. The goal of a perturbation problem in this section is to identify the unstable eddy modes.

Nondimensionalizations are performed throughout on the variables by designating the nondimensional variables by the dagger, †. The nondimensionalization scales for given variables are designated by a superscript *. Note that the nondimensionalization scales are always defined to be positive definite, for example, regardless of whether the atmosphere is stably stratified or not, unlike the standard convention of boundary-layer meteorology. This definition is consistent with the basic nature of the nondimensionalization whereby only the balances by the order of magnitude are of concern, thus the signs of the terms are not issues.

Consequently,

$$u' = u^* u'^{\dagger}, \quad w' = u^* w'^{\dagger}, \quad \phi' = \phi^* \phi'^{\dagger}, \quad b' = b^* b'^{\dagger},$$

$$\frac{\partial}{\partial t} = \frac{1}{t^*} \frac{\partial}{\partial t'^{\dagger}}, \quad \frac{\partial}{\partial x} = \frac{1}{z^*} \frac{\partial}{\partial x'^{\dagger}}, \quad \frac{\partial}{\partial z} = \frac{1}{z^*} \frac{\partial}{\partial z'^{\dagger}},$$

and also

$$\bar{u} = \bar{u}^* \bar{u}^\dagger, \quad \frac{d\bar{u}}{dz} = \left(\frac{d\bar{u}}{dz}\right)^* \frac{d\bar{u}^\dagger}{dz^\dagger}, \quad \frac{d\bar{b}}{dz} = \left(\frac{d\bar{b}}{dz}\right)^* \frac{d\bar{b}^\dagger}{dz^\dagger}.$$

Here, the velocity and the spatial scales are not distinguished in the horizontal and vertical directions for simplicity, thus $x^* = z^*$ and $w^* = u^*$. Also note that the background-state gradients, $d\bar{u}/dz$ and $d\bar{b}/dz$, are nondimensionalized directly by using the scales for these gradients, rather than by setting \bar{u}/z^* and \bar{b}/z^* , respectively, for the purpose of deriving a standard definition of the Richardson number in the following. Furthermore, to simplify the notions, the prime sign, \prime , has been omitted from the nondimensionalization scales for the perturbations.

By substituting these expressions into Eq. 1, we obtain

$$\frac{u^*}{t^*} \frac{\partial u^\dagger}{\partial t^\dagger} + u^* \left(\frac{d\bar{u}}{dz}\right)^* w^\dagger \left(\frac{d\bar{u}}{dz}\right)^\dagger + \frac{\bar{u}^* u^*}{z^*} \bar{u}^\dagger \frac{\partial u^\dagger}{\partial x^\dagger} = -\frac{\phi^*}{z^*} \frac{\partial \phi^\dagger}{\partial x^\dagger}, \tag{2a}$$

$$\frac{u^*}{t^*} \frac{\partial w^\dagger}{\partial t^\dagger} + \frac{\bar{u}^* u^*}{z^*} \bar{u}^\dagger \frac{\partial w^\dagger}{\partial x^\dagger} = -\frac{\phi^*}{z^*} \frac{\partial \phi^\dagger}{\partial z^\dagger} + b^* b^\dagger, \tag{2b}$$

$$\frac{b^*}{t^*} \frac{\partial b^\dagger}{\partial t^\dagger} + \frac{b^* \bar{u}^*}{z^*} \bar{u}^\dagger \frac{\partial b^\dagger}{\partial x^\dagger} + u^* \left(\frac{d\bar{b}}{dz}\right)^* w^\dagger \left(\frac{d\bar{b}}{dz}\right)^\dagger = 0, \tag{2c}$$

$$\frac{u^*}{z^*} \frac{\partial u^\dagger}{\partial x^\dagger} + \frac{u^*}{z^*} \frac{\partial w^\dagger}{\partial z^\dagger} = 0. \tag{2d}$$

The basic procedure of defining the scales, u^* , ϕ^* , etc. under the nondimensionalization is to set the prefactors in front of all the terms in the equations to be unity as much as possible. Continuity (Eq. 2d) already satisfies this condition, because two prefactors are identical. A final nondimensional equation is obtained simply by dividing both sides by the common prefactor, u^*/z^* , thus a final prefactor in front of all the terms becomes unity. In the other equations, the same goal is achieved by setting those prefactors to be equal to each other as much as possible. These conditions define the characteristic scales of the system. For example, by setting the prefactors in front of the first and the third terms (temporal and advection tendencies) in Eq. 2a equal, we obtain

$$\frac{u^*}{t^*} = \frac{\bar{u}^* u^*}{z^*},$$

which leads to a definition of the time scale, t^* .

By proceeding in this manner, we identify the nondimensionalization scales as

$$t^* = \frac{z^*}{\bar{u}^*}, \tag{3a}$$

$$\phi^* = \bar{u}^* u^*, \tag{3b}$$

$$b^* = \frac{\bar{u}^* u^*}{z^*}. \tag{3c}$$

The resulting nondimensionalized set of equations is

$$\frac{\partial u^\dagger}{\partial t^\dagger} + \left[\frac{z^*}{\bar{u}^*} \left(\frac{d\bar{u}}{dz}\right)^* \right] \frac{d\bar{u}^\dagger}{dz^\dagger} w^\dagger + \bar{u}^\dagger \frac{\partial u^\dagger}{\partial x^\dagger} = -\frac{\partial \phi^\dagger}{\partial x^\dagger}, \tag{4a}$$

$$\frac{\partial w^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial w^\dagger}{\partial x^\dagger} = -\frac{\partial \phi^\dagger}{\partial z^\dagger} + b^\dagger, \tag{4b}$$

$$\frac{\partial b^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial b^\dagger}{\partial x^\dagger} + \left(\frac{z^*}{\bar{u}^\dagger}\right)^2 \left(\frac{d\bar{b}}{dz}\right)_* \frac{d\bar{b}^\dagger}{dz^\dagger} w^\dagger = 0, \tag{4c}$$

$$\frac{\partial u^\dagger}{\partial x^\dagger} + \frac{\partial w^\dagger}{\partial z^\dagger} = 0. \tag{4d}$$

As intended, there is no constant prefactor in front of almost all the terms in the equations, except for the two in Eq. 4a, c. Here, by choosing a length scale, z^* , in an appropriate manner, we can remove a constant prefactor from one of these two terms, but not from both of them.

As a result, in turn, there are only two options for defining the length scale, z^* , consistently under the nondimensionalization procedure: set either of those two undefined prefactors to be unity. This is a fundamental difference from dimensional analysis: with the latter, we can find a variety of different ways of defining a length scale solely based on a dimensional consistency by taking various available dimensional parameters. However, for the nondimensionalization procedure, we do not have such a liberty: the given equation set dictates in a more specific manner what the options are, and we have to follow them.

Thus, these two options for setting the length scale, z^* , are: (i) by setting a prefactor in front of the second term of Eq. 4a to be unity, or (ii) by setting a prefactor in front of the second term in Eq. 4c to be unity. Option (i) amounts to setting the spatial scale (shear scale) to be that of the background wind shear, and we obtain

$$z^* = z^{*u} \equiv \bar{u}^* / (d\bar{u}/dz)^*. \tag{5a}$$

Option (ii) leads to

$$z^* = z^{*b} \equiv \bar{u}^* / (d\bar{b}/dz)^{*1/2}. \tag{5b}$$

The latter may be called the buoyancy-gradient scale. Recall that all the nondimensionalization scales are defined to be positive definite, thus the definition of the buoyancy scale above is applicable to both the stable and the unstable situations.

Consequently, with option (i) the set of equations also reduces to

$$\frac{\partial u^\dagger}{\partial t^\dagger} + \frac{d\bar{u}^\dagger}{dz^\dagger} w^\dagger + \bar{u}^\dagger \frac{\partial u^\dagger}{\partial x^\dagger} = -\frac{\partial \phi^\dagger}{\partial x^\dagger}, \tag{6a}$$

$$\frac{\partial w^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial w^\dagger}{\partial x^\dagger} = -\frac{\partial \phi^\dagger}{\partial z^\dagger} + b^\dagger, \tag{6b}$$

$$\frac{\partial b^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial b^\dagger}{\partial x^\dagger} + Ri \left(\frac{d\bar{b}^\dagger}{dz^\dagger}\right) w^\dagger = 0, \tag{6c}$$

$$\frac{\partial u^\dagger}{\partial x^\dagger} + \frac{\partial w^\dagger}{\partial z^\dagger} = 0, \tag{6d}$$

and with option (ii) to

$$\frac{\partial u^\dagger}{\partial t^\dagger} + Ri^{-1/2} \frac{d\bar{u}^\dagger}{dz^\dagger} w^\dagger + \bar{u}^\dagger \frac{\partial u^\dagger}{\partial x^\dagger} = -\frac{\partial \phi^\dagger}{\partial x^\dagger}, \tag{7a}$$

$$\frac{\partial w^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial w^\dagger}{\partial x^\dagger} = -\frac{\partial \phi^\dagger}{\partial z^\dagger} + b^\dagger, \tag{7b}$$

$$\frac{\partial b^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial b^\dagger}{\partial x^\dagger} + \frac{db^\dagger}{dz^\dagger} w^\dagger = 0, \tag{7c}$$

$$\frac{\partial u^\dagger}{\partial x^\dagger} + \frac{\partial w^\dagger}{\partial z^\dagger} = 0. \tag{7d}$$

Here, Ri is the Richardson number defined by a ratio of the two characteristic scales

$$Ri = \left(\frac{z^{*u}}{z^{*b}} \right)^2 = \frac{(d\bar{b}/dz)^*}{(d\bar{u}/dz)^{*2}}. \tag{8}$$

Note again that the Richardson number is positive definite due to our definitions of the nondimensional scales to be positive definite.

We find that when shear is more dominant than the buoyancy gradient (stratification), i.e., $Ri < 1$, the nondimensionalization based on the shear scale, z^{*u} , (Eq. 5a) is relevant, and when stratification is more dominant than shear, i.e., $Ri > 1$, the nondimensionalization based on the buoyancy-gradient scale, z^{*b} , (Eq. 5b) becomes relevant.

2.2 Asymptotic Limit of Vanishing Richardson Number

The system under the asymptotic limit¹ of $Ri \rightarrow 0$, i.e., when the Richardson number is very small, warrants further discussions, because in this limit the leading order of the buoyancy equation (6c) reduces to the homogeneous solution

$$\frac{\partial b^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial b^\dagger}{\partial x^\dagger} = 0.$$

The evolution of buoyancy is simply described by advection by the background flow, \bar{u}^\dagger . The time-evolving buoyancy, in turn, acts as a forcing in Eq. 6b to generate a perturbation flow as a consequence.

However, this leading-order solution of the buoyancy never becomes unstable, being purely advective. Thus, if we decide to focus only on the unstable modes, the leading-order buoyancy can be neglected, and we must rescale it into

$$\tilde{b}^\dagger = Ri^{-1} b^\dagger, \tag{9}$$

so that the buoyancy equation is also rescaled into

$$\frac{\partial \tilde{b}^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial \tilde{b}^\dagger}{\partial x^\dagger} + \frac{d\tilde{b}^\dagger}{dz^\dagger} w^\dagger = 0.$$

After the rescaling, the third term with the background stratification, $d\tilde{b}^\dagger/dz^\dagger$, contributes to a weak generation of the perturbation buoyancy. In turn, however, the generated weak buoyancy no longer feeds back to the momentum equation (7b) to the leading order.

2.3 Discussion

We demonstrated in Sect. 2.1 how characteristic scales (not only the length scale) of a system can be determined by nondimensionalization naturally. Next we address whether the Obukhov length can be derived in a similar manner for atmospheric boundary-layer flows. In addressing this question, we also proceed with the identical basic strategy as in the present section: to constrain a given partial differential equation solely based on the nondimensionalization procedure. No observational measurement is taken into account.

From this demonstration, the following points may be noted:

¹ Refer to e.g., Olver (1974), Bender and Orszag (1978) for the concept of the asymptotic limit; see Sect. 5 of Yano (2015) for a pedagogical introduction. Not be confused with the *analytical* limit.

- (1) For maintaining generality, it is imperative to retain all the terms in the system of equations as much as possible, by maintaining prefactors the unity. In the example here, only a single term remains with a prefactor being non-unity (Ri or $Ri^{-1/2}$ depending on the size of Ri). There is no arbitrariness in the nondimensionalization: the result is unique.
- (2) The nondimensional set of equations obtained can solve the given shear-instability problem in a general manner, without any approximations, once a profile of the background flow, \bar{u} , is fixed, regardless of the actual scales (e.g., in a tank, over a planet) as well as the magnitude of a given flow, but only by specifying a value of a single nondimensional number, Ri . An asymptotic limit of $Ri \rightarrow 0$ or $Ri \rightarrow \infty$ may be taken to consider a limiting background state. In that case, one of the terms in the system can be dropped, but still a full set of solutions under these limiting states can be obtained.
- (3) If we drop any extra terms from this system, a resulting stability analysis loses its generality by limiting the possibility of solutions by further simplifications. That is why it is crucial to keep all the terms the same order of magnitude under nondimensionalization. Exactly the same principle is applied to boundary-layer problems in Sects. 3 and 4 for achieving the same goal.
- (4) Not all the terms may contribute equally, in practice, with some specific solutions arising from this system of equations. For example, as discussed in Sect. 2.2, under the asymptotic limit of $Ri \rightarrow 0$, buoyancy can be rescaled by Eq. 9. More generally, when steady solutions are sought, terms with time derivatives drop out. One can also consider a perturbation mode only slowly varying with time (e.g., a weakly unstable mode). We can focus on those situations by rescaling the time by $\partial/\partial t^\dagger = \hat{\epsilon} \partial/\partial \tau^\dagger$ with a small nondimensional parameter, $\hat{\epsilon}$, then τ^\dagger is a resulting nondimensional slow time scale. However, importantly, these cases are only subcategories of the general case retained by Eq. 4.

The definition of the time scale, t^* , by advection (Eq. 3a) may appear to be rather restrictive. However, this appearance is rather superfluous: by substituting the two possible spatial scales, given by Eq. 5a, b, we obtain $t^* = (d\bar{u}/dz)^{-1}$ and $t^* = (d\bar{b}/dz)^{-1/2}$, corresponding to the shear- and buoyancy-driven situations, respectively.

Keep in mind that in applying the same methodology to the atmospheric boundary layer in Sects. 3 and 4, the same level of generality as in the present section is also maintained. Due to this generality, no particular boundary-layer regime is specified in most of the analyses, simply because the analyses are performed in a general manner. Identification of the regimes only follow from there based on nondimensional parameters characterizing a given system.

2.4 Implications for Boundary-Layer Problems

The results obtained in this section already have implications for boundary-layer problems, because the identified characteristic scales, z^{u*} and z^{b*} , (Eq. 5a, b) are expected to characterize the typical size of eddies of given regimes, and thus, also characterize the resulting mixing lengths, l^* . In this respect, it may be worthwhile to note that, for example, Grisogono (2010) proposes the use of two different mixing lengths,

$$l^* = \frac{u^*}{(d\bar{u}/dz)^*}, \tag{10a}$$

$$l^* = \frac{u^*}{(d\bar{b}/dz)^{*/1/2}}, \tag{10b}$$

depending on the Richardson number, Ri . A similar scale (buoyancy scale),

$$l^* = \frac{w^*}{(\overline{db}/dz)^*1/2}, \tag{10c}$$

is introduced by Stull (1973), Zeman and Tennekes (1977), and Brost and Wyngaard (1978). Hunt et al. (1985), in turn, infer from field data that the buoyancy scale (Eq. 10c) characterizes both the vertical heat transport and the temperature-variance production in the stably stratified boundary layer. These definitions reduce to z^{*u} and z^{*b} , respectively, with small and large Richardson numbers by letting $u^* = \bar{u}^*$ and $w^* = \bar{w}^*$. This condition is expected to be satisfied when a system is fully turbulent.

3 Boundary-Layer System: Turbulence-Kinetic-Energy Equation

3.1 Obukhov Length

We turn to the nondimensionalization of boundary-layer systems, with the goal of deriving the Obukhov length as a natural consequence of nondimensionalization in a similar manner as in Sect. 2, and more specifically as discussed in Sect. 2.3.

The Obukhov length is defined by a ratio of fractional powers of the scales for the vertical momentum stress and the vertical buoyancy flux, $\overline{u'w'^*}$ and $\overline{w'b'^*}$

$$L = \frac{\overline{u'w'^*}^{3/2}}{\overline{w'b'^*}}. \tag{11}$$

The definition of the Obukhov length is often alternatively presented as

$$L = \frac{u_{*\tau}^3}{(\overline{w'b'})^*}$$

by introducing a friction velocity, $u_{*\tau}$, defined by

$$u_{*\tau}^2 = \overline{(u'w')^*}.$$

Note that here the Obukhov length (11) is introduced as one of the nondimensionalization scales, thus it differs from the standard definition in two major respects: i) scales of fluxes are used rather than flux values themselves; ii) these scales only represent orders of magnitudes rather than the actual measured values at the surface. Also keep in mind that the Obukhov length defined by Eq. 11 is positive definite with the flux scales chosen positive definite.

3.2 Turbulence-Kinetic-Energy Equation

Can we derive the Obukhov length by following a principle of the nondimensionalization? That is a question considered in the following two sections. As seen in Eq. 11, the Obukhov length is defined from the vertical buoyancy flux, $\overline{w'b'}$, and the vertical momentum stress, $\overline{u'w'}$. Thus, for the purpose of deriving this scale as directly as possible, we would need to take a system of equations that contains these two variables for nondimensionalization. We consider the TKE equation for this reason.

By following a standard formulation for atmospheric boundary-layer turbulence, only the vertical flux terms are retained assuming horizontal homogeneity. This is solely for simplifying the analysis and focusing on the goal of deriving the Obukhov length. Horizontal

heterogeneity is expected to be important for some stably stratified atmospheric boundary-layer flows, but this extension is left for a future study. Note that, as in Sect. 2, the following analyses are performed in a general manner without discriminating the sign of the stratification. However, as just suggested, our focus in applications is on stably stratified cases.

The TKE equation (e.g., Deardorff 1983) is given by

$$\frac{\partial}{\partial t} \frac{\overline{\mathbf{v}'^2}}{2} = \overline{w'b'} - \overline{u'w'} \frac{d\bar{u}}{dz} - \frac{\partial}{\partial z} \overline{w'(\frac{\mathbf{v}'^2}{2} + \phi')} - D, \tag{12}$$

where the overbar designates a horizontal average, \mathbf{v} is a velocity vector, and D the dissipation rate. We refer to Wyngaard and Coté (1971) for the basics of the boundary-layer TKE budget.

We retain the temporal tendency in the equations for the generality of the analyses. Although the original Monin–Obukhov similarity theory assumes quasi-stationarity, the question here is to what extent this theory may also be applicable to a transient system.

3.3 Nondimensionalization

In boundary-layer similarity theories, flux values are directly used as dimensional variables, corresponding to the nondimensionalization scales in the terminology here. By following this approach, fluxes are now nondimensionalized by their own nondimensionalization scales as

$$\begin{aligned} \overline{w'b'} &= (\overline{w'b'})^* \overline{w'b'}^\dagger, \\ \overline{u'w'} &= (\overline{u'w'})^* \overline{u'w'}^\dagger, \\ \overline{w'\mathbf{v}'^2} &= (\overline{w'\mathbf{v}'^2})^* \overline{w'\mathbf{v}'^2}^\dagger, \\ \overline{w'\phi'} &= (\overline{w'\phi'})^* \overline{w'\phi'}^\dagger, \\ \overline{w'^2} &= w^{*2} \overline{w'^2}^\dagger, \\ \overline{b'^2} &= b^{*2} \overline{b'^2}^\dagger. \end{aligned}$$

As in Sect. 2.1, the background-state gradients are also nondimensionalized directly by using the scales for these gradients. Also keep in mind that the nondimensionalization scales are assumed to be positive definite.

By nondimensionalizing Eq. 12 in this manner, we obtain

$$\begin{aligned} \frac{u^{*2}}{t^*} \frac{\partial}{\partial t^\dagger} \frac{\overline{\mathbf{v}'^2}}{2} &= (\overline{w'b'})^* \overline{w'b'}^\dagger - (\overline{u'w'})^* \left(\frac{d\bar{u}}{dz} \right)^* \overline{u'w'}^\dagger \left(\frac{d\bar{u}}{dz} \right)^\dagger \\ &\quad - \frac{(\overline{w'\mathbf{v}'^2})^*}{z^*} \left[\frac{\partial}{\partial z^\dagger} \overline{w'(\frac{\mathbf{v}'^2}{2} + \phi')}^\dagger \right] - D^* D^\dagger \end{aligned} \tag{13}$$

with $(\overline{w'\phi'})^* = (\overline{w'\mathbf{v}'^2})^*$.

The goal of the nondimensionalization is to set the constant prefactors in front of the terms equal as much as possible by following the principles outlined in Sect. 2.3. However, keep in mind that setting two prefactors equal does not mean that these two terms are well balanced with any boundary-layer regimes: this balance is only by an order of magnitude, and it does not say anything directly about the TKE budget. In the following two subsections, we examine possible balances in the equation with the goal of identifying the Obukhov length in mind.

3.4 The Balance between the Buoyancy and Shear Productions

The first balance to be considered by order of magnitude is between two terms that involve the vertical buoyancy flux, $\overline{w'b'}$ (buoyancy production), and the vertical momentum stress, $\overline{u'w'}$ (shear production), respectively. This leads to the constraint

$$(\overline{w'b'})^* = (\overline{u'w'})^* \left(\frac{d\bar{u}}{dz} \right)^*, \tag{14}$$

which can lead to the Obukhov length by introducing a vertical scale, z^* , by setting

$$\left(\frac{d\bar{u}}{dz} \right)^* = \frac{\bar{u}^*}{z^*}, \tag{15}$$

and substituting Eq. 15 into Eq. 14 to give

$$z^* = \frac{\bar{u}^* (\overline{u'w'})^*}{(\overline{w'b'})^*}.$$

This length scale reduces to the Obukhov length by further assuming

$$\bar{u}^* \overline{u'w'}^* \simeq \overline{u'w'}^{*3/2}. \tag{16}$$

3.5 The Balance between the Buoyancy Production and the Turbulent Transport

An alternative possibility to be considered is the order-of-magnitude balance between the buoyancy production (first) and the turbulent transport (third)

$$\overline{w'b'}^* = \frac{(\overline{w'u'^2})^*}{z^*},$$

which leads to a length scale

$$z^* = \frac{\overline{w'u'^2}^*}{(\overline{w'b'})^*}. \tag{17}$$

If one can further set

$$\overline{w'u'^2}^* \approx \overline{w'u'}^{*3/2}, \tag{18}$$

Eq. 17 reduces to the Obukhov length.

3.6 Discussion

In this section, we attempt to derive the Obukhov length directly by nondimensionalizing the TKE equation. The Obukhov length has been derived by considering two possible balances by order of magnitude in this equation, however, only with additional assumptions (Eqs. 16, 18). Making any further progress is difficult due to the fact that the TKE equation is not self-contained. Although adding more equations for turbulence statistics may help, they never close the system. Based on these considerations, in Sect. 4, we generalize a set of equations considered in Sect. 2 into a fully nonlinear three-dimensional version so that the TKE equation can be nondimensionalized in a self-consistent manner. However, a major drawback of this alternative approach is the length scale no longer defined by the ratio of fractional powers of the flux scales directly.

4 Equations for the Fully-Developed Eddies

4.1 Formulation

As decided at the end of Sect. 3, we now turn to a fully nonlinear set of equations, including diffusivity

$$\begin{aligned} \frac{\partial \mathbf{u}'}{\partial t} + w' \frac{d\bar{\mathbf{u}}}{dz} + \bar{\mathbf{u}} \cdot \nabla_H \mathbf{u}' + \nabla_H \cdot \mathbf{u}' \mathbf{u}' + \frac{\partial}{\partial z} [\mathbf{u}' w' - \overline{\mathbf{u}' w'}] \\ = -\nabla_H \phi' + \nu \nabla^2 \mathbf{u}', \end{aligned} \tag{19a}$$

$$\begin{aligned} \frac{\partial w'}{\partial t} + \bar{\mathbf{u}} \cdot \nabla_H w' + \nabla_H \cdot \mathbf{u}' w' + \frac{\partial}{\partial z} [w'^2 - \overline{w'^2}] \\ = -\frac{\partial \phi'}{\partial z} + b' + \nu \nabla^2 w', \end{aligned} \tag{19b}$$

$$\begin{aligned} \frac{\partial b'}{\partial t} + \bar{\mathbf{u}} \cdot \nabla_H b' + w' \frac{d\bar{b}}{dz} + \nabla_H \cdot \mathbf{u}' b' + \frac{\partial}{\partial z} [w' b' - \overline{w' b'}] \\ = \kappa \nabla^2 b', \end{aligned} \tag{19c}$$

$$\nabla_H \cdot \mathbf{u}' + \frac{\partial w'}{\partial z} = 0. \tag{19d}$$

Here, \mathbf{u} and w are the horizontal and the vertical components of the velocity, and ν and κ are coefficients for the diffusion of momentum and heat.

4.2 Nondimensionalization Scales

The above system can be nondimensionalized by setting the variables as

$$\begin{aligned} \mathbf{u}' = u^* \mathbf{u}^\dagger, \quad w' = w^* w^\dagger, \quad \phi' = \phi^* \phi^\dagger, \quad b' = b^* b^\dagger, \\ \frac{\partial}{\partial t} = \frac{1}{t^*} \frac{\partial}{\partial t^\dagger}, \quad \nabla_H = \frac{1}{x^*} \nabla_H^\dagger, \quad \frac{\partial}{\partial z} = \frac{1}{z^*} \frac{\partial}{\partial z^\dagger}, \end{aligned}$$

and also

$$\bar{\mathbf{u}} = \bar{u}^* \bar{\mathbf{u}}^\dagger, \quad \frac{d\bar{\mathbf{u}}}{dz} = \left(\frac{d\bar{\mathbf{u}}}{dz}\right)^* \frac{d\bar{\mathbf{u}}^\dagger}{dz^\dagger}, \quad \frac{d\bar{b}}{dz} = \left(\frac{d\bar{b}}{dz}\right)^* \frac{d\bar{b}^\dagger}{dz^\dagger}.$$

By substituting these expressions into Eq. 19, we obtain

$$\begin{aligned} \frac{u^*}{t^*} \frac{\partial \mathbf{u}^\dagger}{\partial t^\dagger} + \frac{\bar{u}^* u^*}{x^*} \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger \mathbf{u}^\dagger + w^* \left(\frac{d\bar{\mathbf{u}}}{dz}\right)^* w^\dagger \frac{d\bar{\mathbf{u}}^\dagger}{dz^\dagger} + \frac{u^{*2}}{x^*} \nabla_H^\dagger \mathbf{u}^{\dagger 2} + \frac{u^* w^*}{z^*} \frac{\partial}{\partial z^\dagger} [\mathbf{u}^\dagger w^\dagger - \overline{\mathbf{u}^\dagger w^\dagger}] \\ = -\frac{\phi^*}{x^*} \nabla_H^\dagger \phi^\dagger + \nu u^* \left(\frac{1}{x^{*2}} \nabla_H^{\dagger 2} + \frac{1}{z^{*2}} \frac{\partial^2}{\partial z^{\dagger 2}}\right) \mathbf{u}^\dagger, \end{aligned} \tag{20a}$$

$$\begin{aligned} \frac{w^*}{t^*} \frac{\partial w^\dagger}{\partial t^\dagger} + \frac{\bar{u}^* w^*}{x^*} \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger w^\dagger + \frac{u^* w^*}{x^*} \nabla_H^\dagger \cdot \mathbf{u}^\dagger w^\dagger + \frac{w^{*2}}{z^*} \frac{\partial}{\partial z^\dagger} [w^{\dagger 2} - \overline{w^{\dagger 2}}] \\ = -\frac{\phi^*}{z^*} \frac{\partial \phi^\dagger}{\partial z^\dagger} + b^* b^\dagger + \nu w^* \left(\frac{1}{x^{*2}} \nabla_H^{\dagger 2} + \frac{1}{z^{*2}} \frac{\partial^2}{\partial z^{\dagger 2}}\right) w^\dagger, \end{aligned} \tag{20b}$$

$$\frac{b^*}{t^*} \frac{\partial b^\dagger}{\partial t^\dagger} + \frac{\bar{u}^* b^*}{x^*} \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger b^\dagger + w^* \left(\frac{d\bar{b}}{dz}\right)^* w^\dagger \left(\frac{d\bar{b}}{dz}\right)^\dagger + \frac{u^* b^*}{x^*} \nabla_H^\dagger \cdot \mathbf{u}^\dagger b^\dagger$$

$$+ \frac{w^* b^*}{z^*} \frac{\partial}{\partial z^\dagger} [w^\dagger b^\dagger - \overline{w^\dagger b^\dagger}] = \kappa b^* \left(\frac{1}{x^{*2}} \nabla_H^{\dagger 2} + \frac{1}{z^{*2}} \frac{\partial^2}{\partial z^{\dagger 2}} \right) b^\dagger, \tag{20c}$$

$$\frac{u^*}{x^*} \nabla_H^\dagger \cdot \mathbf{u}^\dagger + \frac{w^*}{z^*} \frac{\partial w^\dagger}{\partial z^\dagger} = 0. \tag{20d}$$

First, we note from the continuity equation (20d) that

$$\frac{u^*}{x^*} = \frac{w^*}{z^*}.$$

To take into account this condition, we introduce the aspect ratio, α , defined by

$$\alpha = \frac{w^*}{u^*} = \frac{z^*}{x^*}. \tag{21a}$$

Second, we introduce an amplitude, $\hat{\epsilon}$, for the eddies defined by

$$\hat{\epsilon} = \frac{u^*}{\bar{u}^*}. \tag{21b}$$

These two unspecified parameters, α and $\hat{\epsilon}$, become necessary for closing the nondimensionalized system in the following.

By setting the factors for the first two terms in Eq. 20a, b, c equal, i.e., the temporal and advection tendencies, we define the time scale as

$$t^* = \frac{x^*}{\bar{u}^*} = \left(\frac{1}{\alpha} \right) \frac{z^*}{\bar{u}^*}, \tag{21c}$$

i.e., the advective time scale. Furthermore, by equating the factors for the second terms on the left-hand and right-hand sides, i.e., the advection tendency and the pressure-gradient force, respectively, in Eq. 20a, the pressure scale is defined by

$$\phi^* = \hat{\epsilon} \bar{u}^{*2}, \tag{21d}$$

which means that the magnitude of the pressure force is constrained by the advection term. By equating the factors for the first two terms on the right-hand side of Eq. 20b, i.e., the pressure gradient and buoyancy forces, then

$$b^* = \hat{\epsilon} \frac{\bar{u}^{*2}}{z^*}. \tag{21e}$$

Thus, the magnitude of buoyancy is constrained by the vertical pressure gradient, as expected from the hydrostatic balance.

The resulting nondimensionalized set of equations is

$$\begin{aligned} & \frac{\partial u^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger \mathbf{u}^\dagger + \left[\frac{z^*}{\bar{u}^*} \left(\frac{d\bar{u}}{dz} \right)^* \right] w^\dagger \frac{d\bar{\mathbf{u}}^\dagger}{dz^\dagger} + \hat{\epsilon} \left[\nabla_H^\dagger \mathbf{u}^{\dagger 2} + \frac{\partial(\mathbf{u}^\dagger w^\dagger - \overline{\mathbf{u}^\dagger w^\dagger})}{\partial z^\dagger} \right] \\ & = -\nabla_H^\dagger \phi^\dagger + \frac{1}{Re} \left(\frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) \mathbf{u}^\dagger, \end{aligned} \tag{22a}$$

$$\begin{aligned} & \frac{\partial w^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger w^\dagger + \hat{\epsilon} \left[\nabla_H^\dagger \cdot \mathbf{u}^\dagger w^\dagger + \frac{\partial(w^{\dagger 2} - \overline{w^{\dagger 2}})}{\partial z^\dagger} \right] \\ & = \frac{1}{\alpha^2} \left[-\frac{\partial \phi^\dagger}{\partial z^\dagger} + b^\dagger \right] + \frac{1}{Re} \left(\frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) w^\dagger, \end{aligned} \tag{22b}$$

$$\begin{aligned} \frac{\partial b^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger b^\dagger + \left(\frac{z^*}{\bar{u}^*}\right)^2 \left(\frac{d\bar{b}}{dz}\right)^* w^\dagger \frac{db^\dagger}{dz^\dagger} + \hat{\epsilon} \left[\nabla_H^\dagger \cdot \mathbf{u}^\dagger b^\dagger + \frac{\partial(w^\dagger b^\dagger - \overline{w^\dagger b^\dagger})}{\partial z^\dagger} \right] \\ = \frac{1}{Pr Re} \left(\frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) b^\dagger, \end{aligned} \tag{22c}$$

$$\nabla_H^\dagger \cdot \mathbf{u}^\dagger + \frac{\partial w^\dagger}{\partial z^\dagger} = 0, \tag{22d}$$

where

$$Re = \alpha \frac{z^* \bar{u}^*}{\nu}, \tag{23a}$$

$$Pr = \frac{\nu}{\kappa}, \tag{23b}$$

are the Reynolds number and the Prandtl number, respectively.

The nondimensionalized version of the TKE equation (Eq. 12) can be obtained directly from Eq. 22a, b, and we find

$$\frac{\partial \overline{\mathbf{v}^{\dagger 2}}}{\partial t^\dagger} = \overline{w^\dagger b^\dagger} - \left[\frac{z^*}{\bar{u}^*} \left(\frac{d\bar{u}}{dz}\right)^* \right] \overline{u^\dagger w^\dagger} \frac{d\bar{u}^\dagger}{dz^\dagger} - \hat{\epsilon} \frac{\partial}{\partial z^\dagger} w^\dagger \left(\frac{\mathbf{v}^{\dagger 2}}{2} + \phi^\dagger \right) - \frac{1}{Re} D^\dagger, \tag{24a}$$

where

$$D^\dagger = \left(\frac{\partial \mathbf{u}}{\partial z}\right)^2 + \alpha^2 \left[\left(\frac{\partial \mathbf{u}}{\partial x}\right)^2 + \left(\frac{\partial \mathbf{u}}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right] + \alpha^4 \overline{(\nabla_H w)^2}. \tag{24b}$$

4.3 Reynolds Number and the Dissipation Term

Note that the kinematic viscosity of air is $\nu \sim 10^{-5} \text{ m}^2 \text{ s}^{-1}$. Also assuming $z^* \sim 10^2 \text{ m}$ and $\bar{u}^* \sim 1 \text{ m s}^{-1}$, we obtain $Re \sim 10^7$ as an estimate of the Reynolds number in the atmospheric boundary layer. Further note that the Prandtl number of air is $Pr \sim 1$. These values suggest that both kinetic and thermal diffusions are negligible in a typical atmospheric boundary layer: the orders of magnitudes of those terms, i.e., 10^{-7} , are simply too small to treat properly, even numerically. It is known that those molecular-diffusion terms play a leading role over a very thin molecular-dissipative layer of only a few centimetres above the surface (cf., Eq. 30 below). For this reason, in standard large-eddy simulations of the atmospheric boundary layer, these molecular diffusions are effectively replaced by the so-called eddy diffusivities.

However, this conclusion immediately contradicts with another known fact that the dissipation term, D , in the TKE equation balances well with the shear-production term (cf., Wyngaard and Coté 1971; Lenschow et al. 1988), i.e.,

$$\overline{\mathbf{u}'w'} \cdot \frac{d\bar{\mathbf{u}}}{dz} \sim D \tag{25}$$

in typical stably stratified boundary layers.

More generally, under a standard “cascade” picture of turbulence (cf., Pope 2000), the energy-cascade rate in wavenumber space balances with the molecular energy dissipation rate. In the TKE equation, this picture is translated as the local turbulent-transport term balances with the local energy dissipation

$$D \sim \nabla \cdot \mathbf{v}^{\prime 2} \mathbf{v}'.$$

Note that in this balance statement, no spatial average is applied: a different conclusion follows after a horizontal average. Thus, we expect the magnitude of the energy dissipation to be constrained by

$$D^* = \frac{u^{*3}}{x^*}. \tag{26}$$

This relation is often referred to as Taylor’s dissipation law (Taylor 1935). However this phenomenological conclusion obviously contradicts with the order of magnitude of the dissipation term, which is estimated to be $Re^{-1} \sim 10^{-7}$ by nondimensionalization in Eq. 24a. The tendency that the dissipation term does not asymptotically vanish in the high-Reynolds-number limit is sometimes called anomalous dissipation (e.g., Salewski et al. 2012).

To resolve this ostensive contradiction, we have to first note that in deriving Eq. 24a, we have assumed that the nondimensionalization scale for the energy dissipation is

$$D^* = \hat{\epsilon}^2 \frac{v\bar{u}^{*2}}{z^{*2}} = \left(\frac{1}{\hat{\epsilon}\alpha Re} \right) \frac{u^{*3}}{x^*}. \tag{27}$$

An obvious conclusion is that for the nondimensionalization scale to be consistent with the phenomenologically-identified scale (26), the nondimensionalization scale must be rescaled into

$$\tilde{D}^* = \hat{\epsilon}\alpha Re D^* \tag{28}$$

so that the new nondimensionalization scale, \tilde{D}^* becomes identical to the phenomenologically identified scale (26).

Thus, the question reduces to why such a rescaling is necessary, although it appears to be not necessary for the momentum equation. Realize that in performing the nondimensionalization, we have assumed that the system is dominated by a single pair of spatial scales, x^* and z^* . This reasoning works to a good extent for flow simulations. However, when the concern is of closing the horizontally-averaged energy budget, different issues arise.

The problem may be better understood in wavenumber space with $k_0 \sim x^{*-1}$. The higher wavenumbers, $k_1 (\gg k_0)$, do not contribute to the motions of the scale, k_0 , in any significant manner, because any possible nonlinear interactions arising with those higher wavenumbers, $k_1 \pm k_1, k_1 \pm k_0$, do not project to the scale, k_0 , of the interest significantly. On the other hand, when the horizontally averaged energy budget is considered, the higher wavenumbers can project to the horizontal average by a nonlinear interaction, $k_1 - k_1$, in association with the second-order spatial derivative in the dissipation term, which amplifies its contribution by the factor of $(k_1/k_0)^2$.

As a result, the nondimensionalization scale of the dissipation must be rescaled by taking into account the contributions from those higher wavenumbers by

$$D^\dagger \sim \left(\frac{z^*}{\delta^*} \right)^2, \tag{29}$$

where δ^* is a dissipation scale, which constitutes the rescaling factor for D^* . Therefore, by comparing it with the factor in Eq. 28, we find that the dissipation scale δ^* is defined by

$$\delta^*/x^* = (\hat{\epsilon}\alpha Re)^{-1/2} \sim 10^{-7/2}, \tag{30}$$

which is much smaller than the characteristic scale, x^* , of dominant boundary-layer flows. A similar conclusion also follows from a typical boundary-layer balance (25).

4.4 Two Regimes

As in Sect. 2, the two regimes are identified depending on the magnitude of the Richardson number, Ri . In respective cases, the equation set reduces to:

(i) $Ri \ll 1, z^* = \bar{u}^* \left(\frac{d\bar{u}}{dz} \right)^{-1}$:

$$\begin{aligned} \frac{\partial \mathbf{u}^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger \mathbf{u}^\dagger + \frac{d\bar{\mathbf{u}}^\dagger}{dz^\dagger} w^\dagger + \hat{\epsilon} \left[\nabla_H^\dagger \mathbf{u}^{\dagger 2} + \frac{\partial(\mathbf{u}^\dagger w^\dagger - \overline{\mathbf{u}^\dagger w^\dagger})}{\partial z^\dagger} \right] \\ = -\nabla_H^\dagger \phi^\dagger + \frac{1}{Re} \left(\frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) \mathbf{u}^\dagger, \end{aligned} \tag{31a}$$

$$\begin{aligned} \frac{\partial b^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger b^\dagger + Ri w^\dagger \frac{d\bar{b}^\dagger}{dz^\dagger} + \hat{\epsilon} \left[\nabla_H^\dagger \cdot \mathbf{u}^\dagger b^\dagger + \frac{\partial(w^\dagger b^\dagger - \overline{w^\dagger b^\dagger})}{\partial z^\dagger} \right] \\ = \frac{1}{Pr Re} \left(\frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) b^\dagger, \end{aligned} \tag{31b}$$

$$\frac{\partial \overline{\mathbf{v}^{\dagger 2}}}{\partial t^\dagger} = \overline{w^\dagger b^\dagger} - \overline{u^\dagger w^\dagger} \frac{d\bar{u}^\dagger}{dz^\dagger} - \hat{\epsilon} \frac{\partial}{\partial z^\dagger} \overline{w^\dagger \left(\frac{\mathbf{v}^{\dagger 2}}{2} + \phi^\dagger \right)} - \frac{1}{Re} D^\dagger. \tag{31c}$$

(ii) $Ri \gg 1, z^* = \bar{u}^* \left(\frac{d\bar{b}}{dz} \right)^{-\frac{1}{2}}$:

$$\begin{aligned} \frac{\partial \mathbf{u}^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger \mathbf{u}^\dagger + Ri^{-\frac{1}{2}} w^\dagger \frac{d\bar{\mathbf{u}}^\dagger}{dz^\dagger} + \hat{\epsilon} \left[\nabla_H^\dagger \mathbf{u}^{\dagger 2} + \frac{\partial(\mathbf{u}^\dagger w^\dagger - \overline{\mathbf{u}^\dagger w^\dagger})}{\partial z^\dagger} \right] \\ = -\nabla_H^\dagger \phi^\dagger + \frac{1}{Re} \left(\frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) \mathbf{u}^\dagger, \end{aligned} \tag{32a}$$

$$\begin{aligned} \frac{\partial b^\dagger}{\partial t^\dagger} + \bar{\mathbf{u}}^\dagger \cdot \nabla_H^\dagger b^\dagger + w^\dagger \frac{d\bar{b}^\dagger}{dz^\dagger} + \hat{\epsilon} \left[\nabla_H^\dagger \cdot \mathbf{u}^\dagger b^\dagger + \frac{\partial(w^\dagger b^\dagger - \overline{w^\dagger b^\dagger})}{\partial z^\dagger} \right] \\ = \frac{1}{Pr Re} \left(\frac{\partial^2}{\partial z^{\dagger 2}} + \alpha^2 \nabla_H^{\dagger 2} \right) b^\dagger, \end{aligned} \tag{32b}$$

$$\frac{\partial \overline{\mathbf{v}^{\dagger 2}}}{\partial t^\dagger} = \overline{w^\dagger b^\dagger} - Ri^{-\frac{1}{2}} \overline{u^\dagger w^\dagger} \frac{d\bar{u}^\dagger}{dz^\dagger} - \hat{\epsilon} \frac{\partial}{\partial z^\dagger} \overline{w^\dagger \left(\frac{\mathbf{v}^{\dagger 2}}{2} + \phi^\dagger \right)} - \frac{1}{Re} D^\dagger. \tag{32c}$$

As already found in Sect. 2, the stratification term with $d\bar{b}^\dagger/dz^\dagger$ in the buoyancy equation (31b) is scaled by Ri in the regime with weak stratification, $Ri \ll 1$, whereas the shear-driven terms with $d\bar{u}^\dagger/dz^\dagger$ in the momentum equation (32a) and the TKE budget (32c) are scaled by $Ri^{-1/2}$ in the regimes with strong stratification, $Ri \gg 1$. Thus, those respective terms become less significant in these respective limits. Note that the nonlinear terms solely due to the eddies are scaled by the nondimensional eddy amplitude, $\hat{\epsilon}$. As already discussed in Sect. 4.3, due to a very large value of Re , diffusion terms proportional to $1/Re$ both in the momentum and heat equations practically drop off. On other hand, in the TKE budget, the energy-dissipation term may be rescaled by resetting $D^\dagger/Re = \tilde{D}^\dagger$, as also suggested in Sect. 4.3.

4.5 Asymptotic Limit of Vanishing Richardson Number

The system under the asymptotic limit of $Ri \rightarrow 0$ warrants further discussion, as in the linear case (Sect. 2.2). Under this limit, the leading order of the fully nonlinear buoyancy equation (31b) reduces to

$$\frac{\partial b^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial b^\dagger}{\partial x^\dagger} + \hat{\epsilon} \left[\nabla_H^\dagger \cdot \mathbf{u}^\dagger b^\dagger + \frac{\partial (w^\dagger b^\dagger - \overline{w^\dagger b^\dagger})}{\partial z^\dagger} \right] = 0.$$

With the absence of background stratification, the buoyancy is simply advective. The time-evolving buoyancy, in turn, drives the flow in the vertical momentum equation (22b).

However, in this regime, the buoyancy is never generated to the leading order. Thus, when a system is initialized with no buoyancy, only a weak buoyancy is generated from the term with the background-buoyancy stratification, $d\bar{b}/dz$, which remains $O(Ri)$. Thus, in this case, the buoyancy must be rescaled into

$$\tilde{b}^\dagger = Ri^{-1} b^\dagger, \tag{33}$$

so that the buoyancy equation is also rescaled into

$$\frac{\partial \tilde{b}^\dagger}{\partial t^\dagger} + \bar{u}^\dagger \frac{\partial \tilde{b}^\dagger}{\partial x^\dagger} + w^\dagger \frac{d\tilde{b}^\dagger}{dz^\dagger} + \hat{\epsilon} \left[\nabla_H^\dagger \cdot \mathbf{u}^\dagger \tilde{b}^\dagger + \frac{\partial (w^\dagger \tilde{b}^\dagger - \overline{w^\dagger \tilde{b}^\dagger})}{\partial z^\dagger} \right] = 0$$

to the leading order.

After this rescaling, in turn, the buoyancy no longer contributes to the vertical momentum equation (22b) to the leading order. In other words, such flows are no longer driven by buoyancy. Here, we face a minor difficulty with this regime, because the pressure gradient $\partial\phi^\dagger/\partial z$ no longer has any term to balance it, to the leading order, when $\alpha \ll 1$. It simply suggests that the only types of flows consistent with weak buoyancy, $b^\dagger = O(Ri)$, are isotropic with $\alpha = 1$.

When the buoyancy is rescaled by Eq. 33, the buoyancy flux, $\overline{w^\dagger b^\dagger}$, in the TKE equation must also be rescaled accordingly, and we obtain

$$\frac{\partial}{\partial t^\dagger} \frac{\overline{\mathbf{v}^{\dagger 2}}}{2} = Ri \overline{w^\dagger \tilde{b}^\dagger} - \overline{u^\dagger w^\dagger} \frac{d\bar{u}^\dagger}{dz^\dagger} - \hat{\epsilon} \frac{\partial}{\partial z^\dagger} \overline{w^\dagger \left(\frac{\mathbf{v}^{\dagger 2}}{2} + \phi^\dagger \right)} - \frac{1}{Re} D^\dagger. \tag{34}$$

This result would be consistent with typically observed stably stratified boundary-layer states (cf., Wyngaard and Coté 1971; Lenschow et al. 1988), if the conditions $1 \gg Ri \gg \hat{\epsilon}$ are satisfied in an asymptotic sense.

We may further speculate that the buoyancy flux must be rescaled with $Ri \ll 1$ even when buoyancy is found to be of the leading order, because the buoyancy would simply act like a random force in the momentum equation (22b) to which the vertical velocity component responds off phase, without spatial coherency. Thus, vertical motions generated by a finite buoyancy perturbation do not correlate with the buoyancy. In this case, we need to more directly set

$$\overline{w^\dagger \tilde{b}^\dagger} = Ri^{-1} \overline{w^\dagger b^\dagger}. \tag{35}$$

Equation 35 also leads to the rescaled TKE equation (34).

4.6 Obukhov Length: Derivation

The buoyancy scale defined by Eq. 21e can alternatively be interpreted as a definition of the vertical scale,

$$z^* = \hat{\epsilon} \frac{\bar{u}^{*2}}{b^*}. \tag{36}$$

This scale can also be interpreted, effectively, to be equivalent to the Obukhov length (Eq. 11) as seen by noting that

$$\begin{aligned} \overline{u'w'}^* &= u^* w^* = \alpha u^{*2} = \hat{\epsilon}^2 \alpha \bar{u}^{*2}, \\ \overline{w'b'}^* &= w^* b^* = \hat{\epsilon} \alpha b^* \bar{u}^*, \end{aligned}$$

in which Eq. 21a, b are invoked in deriving final expressions. Substituting these two expressions into (11), we obtain

$$L = \hat{\epsilon}^2 \alpha^{1/2} z^*. \tag{37}$$

Furthermore, when $Ri \ll 1$, there is a possibility that the buoyancy must be further rescaled by Eq. 33, and as a result, the buoyancy-flux scale must be revised into

$$\overline{w'b'}^* = Ri w^* b^* = \hat{\epsilon} Ri b^* \bar{u}^*,$$

also using the constraint, $\alpha = 1$, under the rescaling. As a result, the Obukhov length becomes.

$$L = \frac{\hat{\epsilon}^2}{Ri} z^*. \tag{38}$$

4.7 Obukhov Length: Discussions

The obtained main result is Eq. 37, which shows that the Obukhov length, L , underestimates the characteristic vertical scale, z^* , of the system by the factors determined by the nondimensional eddy amplitude, $\hat{\epsilon}$, and the aspect ratio, α , of the flow: the discrepancy between the Obukhov length, L , and the characteristic scale, z^* , increases for the smaller $\hat{\epsilon}$ and α . Thus, the observationally known increasing discrepancy in the limit of strong stable stratification can easily be explained by the dramatic decrease of both factors, $\hat{\epsilon}$ and α . On the other hand, with a strongly unstably stratified regime (i.e., $Ri \gg 1$ with a negative stratification), we rather expect $\alpha \sim 1$ and $\hat{\epsilon}$ to increase with the increasing Ri . In contrast, an overall relevance of the Obukhov length, L , identified in observations with weak stratification may partially be explained by a rescaling by the Richardson number, Ri , in Eq. 38, which compensates for the tendency of making the Obukhov length smaller than the actual length scale by the factor, $\hat{\epsilon}^2$.

Keep in mind that z^* measures a typical vertical scale of turbulent eddies in a given boundary layer, and not necessarily the actual vertical scale of the boundary layer itself. Although it may be intuitive enough to expect that the most dominant sale of the eddies in a boundary layer is comparable to the boundary-layer depth, this claim is still to be substantiated.

5 Heuristic Derivation of Monin–Obukhov Similarity Theory

From the nondimensionalization considered so far, the essence of Monin–Obukhov similarity theory can be derived in a heuristic manner (cf., Calder 1966, 1968). If the nondimensionaliza-

tion has been properly accomplished for a given system of equations describing a boundary layer, in principle, steady solutions for all the dependent variables in the equation can be expressed as functions of $z^\dagger = z/z^*$, where z^* is the vertical scale defined by Eq. 36, after nondimensionalization. In other words, steady solutions for these nondimensional variables are given in terms of nondimensional functions, say, $f^\dagger(z^\dagger)$, $g^\dagger(z^\dagger)$, $q^\dagger(z^\dagger)$, by

$$\overline{w'b'}^\dagger = f^\dagger(z^\dagger), \tag{39a}$$

$$\overline{u'w'}^\dagger = g^\dagger(z^\dagger), \tag{39b}$$

$$\frac{d\bar{u}^\dagger}{dz^\dagger} = q^\dagger(z^\dagger), \tag{39c}$$

etc. We can consider these functions to be universal in the sense that we obtain the same solutions whenever the same boundary conditions are imposed on the system, regardless of its dimension, because the system reduces to an identical form by rescaling by z^* . After dimensionalizations, the above equations read as

$$\overline{w'b'} = w^*b^* f^\dagger\left(\frac{z}{z^*}\right), \tag{40a}$$

$$\overline{u'w'} = u^*w^* g^\dagger\left(\frac{z}{z^*}\right), \tag{40b}$$

$$\frac{d\bar{u}}{dz} = \left(\frac{d\bar{u}}{dz}\right)^* q^\dagger\left(\frac{z}{z^*}\right). \tag{40c}$$

Furthermore, let us suppose that the only necessary boundary conditions required for determining these variables are their surface values. In that case, we may set $w^*b^* = (\overline{w'b'})_0$, $u^*w^* = (\overline{u'w'})_0$, and $(d\bar{u}/dz)^* = (d\bar{u}/dz)_0$ with the subscript, 0, designating the surface values. Here, the surface values, more precisely, refer to those at the top of the viscous boundary layers for the fluxes. Defining the wind shear close to the surface is even trickier because, over the surface layer, the wind speed typically increases logarithmically with height. For this reason, the Monin–Obukhov theory further replaces $(d\bar{u}/dz)_0$ by \bar{u}^*/z .

As a result, general solutions to the system are given by

$$\overline{w'b'} = (\overline{w'b'})_0 f^\dagger\left(\frac{z}{z^*}\right), \tag{41a}$$

$$\overline{u'w'} = (\overline{u'w'})_0 g^\dagger\left(\frac{z}{z^*}\right), \tag{41b}$$

$$\frac{d\bar{u}}{dz} = \frac{\bar{u}^*}{z} q^\dagger\left(\frac{z}{z^*}\right). \tag{41c}$$

Based on the arguments so far, we may conclude that the nondimensional functions, $f^\dagger(z/z^*)$, $g^\dagger(z/z^*)$, $q^\dagger(z/z^*)$, are universal only depending on the nondimensionalization scale, z^* . In this manner, Monin–Obukhov similarity theory (cf., Sorbjan 1989) is essentially derived simply as steady solutions of the system under nondimensionalization.

Realize, however, that there are various caveats in this derivation. The first is the replacement of the nondimensionalization scales on the right-hand side of Eq. (40) by the surface values in Eq. 41. As already emphasized in the introduction, the nondimensionalization scales only refer to the typical physical values by orders of magnitudes. Conversely, the surface values may not necessarily represent typical values of the whole boundary-layer depth. However,

this merely reduces to a matter of rescaling the nondimensional functions, f^\dagger , g^\dagger , q^\dagger with no further consequence.

A more serious problem is in replacing the nondimensionalization length scale, z^* , defined by Eq. 36 with the Obukhov length. The nondimensionalization length scale, z^* , is related to the Obukhov length by Eq. 37. Even after these rescalings, these functions remain universal so long as these rescaling factors are constants for a given dynamical regime as defined, say, by the Richardson number, thus a rescaling of Eq. 37 into Eq. 38 is not an issue by adding an extra Richardson-number dependence. The most serious constraint of the above derivation of Monin–Obukhov similarity theory from a nondimensionalized equation is an assumption that the system is solely controlled by surface values. It may not be the case with all types of boundary-layer turbulence.

6 Summary and Further Remarks

6.1 Summary and Link to Previous Studies

An important aim behind our study has been to demonstrate how characteristic scales of atmospheric boundary-layer systems can be identified directly by nondimensionalization of partial differential equations describing the system. For a demonstrative purpose, the analysis has begun with a linear-stability problem (Sect. 2), then we turned to the TKE equation for the atmospheric boundary layer (Sect. 3). Finally, a full set of nonlinear equations for the boundary layer has been examined (Sect. 4). All the analyses have been performed under the Boussinesq approximation, and also assuming horizontal homogeneity for simplicity.

The present study has focused on identifying the Obukhov length. Nondimensionalization of the full nonlinear equation set in Sect. 4 shows that the characteristic vertical scale is related to the Obukhov length by Eq. 37, or alternatively by Eq. 38 in the limit of weak stratification (i.e., $Ri \rightarrow 0$). Note that equivalence of these two scales can be established only by including a rescaling factor, which increases with the increasing relative amplitude of the eddies with respect to the background flow, as well as the increasing aspect ratio of the flow. The value of these factors can be substantial, and as a result, the Obukhov length substantially underestimates the characteristic vertical scale of the boundary layer, especially with increasing stratification.

Under weak stratification, the shear-production term significantly contributes in defining the characteristic vertical scale (cf., Eq. 5a). This is consistent with a preliminary stand-alone analysis with the TKE equation in Sect. 3.4, which suggests that the Obukhov length may essentially be derived from a condition of an order-of-magnitude balance between buoyancy- and shear-production.

On the other hand, with increasing stratification with the Richardson number much larger than unity (i.e., $Ri \rightarrow \infty$), shear production is expected to become insignificant, being scaled by $Ri^{-1/2}$. In this limit, a more relevant vertical scale becomes the buoyancy-gradient scale, z^{*b} , defined by Eq. 5b. This scale (5b) is somehow akin to the external static-stability scale

$$L_N = \frac{\overline{(u'w')^*}^{1/2}}{(\overline{db/dz})^*1/2}$$

introduced by Kitaigorodskii (1988), and considered, especially, by Zilitinkevich and Esau (2005). Moreover, Zilitinkevich and Calanca (2000) introduce a nondimensional parameter, L/L_N , to define a transition from a regime dominated by the Obukhov length, L , to L_N .

Zilitinkevich and Esau (2005) argue that the scale, L_N , becomes relevant when the vertical turbulent heat flux is small. Note that the static-stability scale, L_N , is further linked to the buoyancy scale (Eq. 10b, c) introduced by Stull (1973), Zeman and Tennekes (1977), and Brost and Wyngaard (1978). Sorbjan (2006, 2010, 2016), in turn, develops his gradient-based similarity theory based on the buoyancy scale.

By performing nondimensionalization of a full set of boundary-layer equations, we have also arrived at the order-of-magnitude estimate of each term in the TKE equation separately for two regimes with $Ri \ll 1$ and $Ri \gg 1$ in Eqs. 31c and 32c, respectively. In the case with $Ri \ll 1$, Eq. 31c may further be rescaled into Eq. 34, which appears to be consistent with the observed budget of weakly stratified regimes (cf., Wyngaard and Coté 1971; Lenschow et al. 1988).

It is important to keep in mind that those order-of-magnitude estimates of the TKE budget terms have been classified solely in terms of the absolute value of the Richardson number, without discriminating between the stable and unstable stratification. Thus, rather unintuitively, if this analysis is self-consistent, the characteristics of the TKE budget, say, under strong stratification must somehow be akin to that of the convective boundary layer with the absence of wind shear. Certainly, the signs of the budget terms would change with the change in sign of the stratification. However, importance of the buoyancy-generation term compared to shear generation must be valid in both regimes, although buoyancy generation may not be positive under strong stratification. Also keep in mind that two additional nondimensional parameters, α and $\hat{\epsilon}$, are expected to depend on Ri in different manners depending on the sign of the stratification.

Finally, a heuristic derivation of Monin–Obukhov similarity theory from a steady nondimensionalized solution has been outlined in Sect. 5.

6.2 Further Remarks

The present study should be considered only a first step for more extensive nondimensionalization analyses to the atmospheric boundary-layer system, especially for the stably stratified regimes. Nevertheless, the present preliminary analysis already suggests the fruitfulness of such an investigation. The present study has suggested that different atmospheric boundary-layer regimes can be identified by changing orders of magnitudes of the Richardson number. Such an analysis is expected to provide a more solid theoretical basis for interpreting the various different regimes phenomenologically identified for the stably stratified boundary layer (cf., Holtslag and Nieuwstadt 1986; Mahrt 1999). Yano and Tsujimura (1987) demonstrate how such a systematic analysis is possible by the nondimensionalization method for a different system.

Various further generalizations are equally feasible. The present analysis has focused on quasi-steady states. However, some of the turbulence regimes under stable stratification may be fundamentally transient (cf., Caughey et al. 1979). The role of horizontal heterogeneity is another aspect to be investigated, especially in the context of stably stratified boundary layers. For example, under certain situations, the horizontal heat transport, a term that is often neglected in theoretical studies, becomes a key process in the heat budget (e.g., Wittch 1991). The role of anisotropy of the flow with $\alpha \ll 1$ is still to be fully examined as well.

Another aspect, not discussed herein, is the role of boundary conditions in solving the turbulence problems. While our focus is on a layer close enough to the surface (e.g., surface layer), a contribution from the top of the boundary layer may be neglected, as a basic premise of Monin–Obukhov theory as well as in subsequent generalizations. However, when a prob-

lem concerns a whole depth of the boundary layer, this depth becomes another parameter to be considered. As pointed out by, e.g., Holtslag and Nieuwstadt (1986), the problem must be solved by explicitly taking into account the condition at the top of the boundary layer.

In all these respects, the major weakness of the present study is to take the Richardson number as a sole parameter for characterizing the boundary-layer regimes. Although the Richardson number can be interpreted as a measure of the stratification of the system, its correspondence to the Obukhov length, which is traditionally adopted for this measure, is not quite obvious. It is most likely that we still need to identify another controlling parameter of the system. However, the identification of this parameter is left for a future study.

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