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THE POROUS MEDIUM EQUATION AS A SINGULAR LIMIT OF THE THIN FILM MUSKAT PROBLEM

PHILIPPE LAURENÇOT AND BOGDAN-VASILE MATIOC

ABSTRACT. The singular limit of the thin film Muskat problem is performed when the density (and possibly the viscosity) of the lighter fluid vanishes and the porous medium equation is identified as the limit problem. In particular, the height of the denser fluid is shown to converge towards the solution to the porous medium equation and an explicit rate for this convergence is provided in space dimension $d \leq 4$. Moreover, the limit of the height of the lighter fluid is determined in a certain regime and is given by the corresponding initial condition.

1. INTRODUCTION AND MAIN RESULTS

The thin film Muskat problem is the strongly coupled degenerate parabolic system

$$\begin{cases} \partial_t f = \operatorname{div}[f \nabla((1+R)f + Rg)], \\ \partial_t g = \mu R \operatorname{div}[g \nabla(f + g)], \end{cases} \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \quad d \geq 1, \quad (1.1)$$

which describes the motion of two thin fluid layers on an impermeable horizontal bottom, identified with the plane $\{x_{d+1} = 0\}$, under the influence of gravity. Here $f(t, x) \geq 0$ is the thickness of the layer which has density ρ_- and viscosity μ_- and $g(t, x) \geq 0$ is the thickness of the layer with density ρ_+ and viscosity μ_+ . In particular, (1.1) is a model for the spreading of two fluid blobs of different fluids on an impermeable surface. We assume that $\rho_- > \rho_+ > 0$, the positive constants in (1.1) being given by the relations

$$R := \frac{\rho_+}{\rho_- - \rho_+} \quad \text{and} \quad \mu := \frac{\mu_-}{\mu_+}.$$

The system (1.1) is derived in [10, 11, 23] in a one-dimensional setting and most of the analysis devoted so far to this problem, which we review now, is restricted to the one-dimensional case $d = 1$. The well-posedness of (1.1) in the setting of classical solutions is addressed in [10], while non-negative global weak solutions are constructed, by different approaches, in [3, 8, 9, 13, 14]. The rich dynamics described by the thin film Muskat problem (1.1) is very well illustrated by the laboratory experiments described in [23], but also by the numerical simulations reported in [1, 2, 15]. Besides, from a mathematical point of view, many of these experimental and numerical observations are rigorously established. In particular, non-negative weak solutions to (1.1) possess finite speed of propagation and solutions emanating from certain initial data feature the waiting time phenomenon, see [14]. However, the finite speed of propagation property for a single fluid phase, that is, when only $f(0)$ (or $g(0)$) is compactly supported, is still an open problem. When the system (1.1) is posed on \mathbb{R} , the large time behavior of weak solutions is investigated in [15]. More precisely, it is shown that

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solutions starting from even initial data converge towards the (unique) even self-similar solution, a property which is in perfect agreement with the laboratory experiments reported in [23]. It is worthwhile to mention, as a special feature of the one-dimensional thin film Muskat problem (1.1), that, depending on the values of the parameters R and μ , there may exist a continuum of self-similar solutions which are not symmetric, see [15], and every weak solution to (1.1) converges to one of these special solutions for large times. Determining which self-similar solutions are attained in the large time limit is still an open problem, but numerical simulations performed in [15] seem to indicate that some of these non-symmetric self-similar solutions attract certain non-negative weak solutions. The dynamics is much simpler in dimension $d = 2$, as there exists only one self-similar solution to (1.1) which is radially symmetric and a global attractor, see [2].

Though the thin film Muskat problem is formally derived as the singular limit of the Muskat problem when letting the thickness of the fluid layers vanish [10], it is rather immediate to notice that (1.1) can be viewed as a two-phase generalization of the well-known porous medium equation

$$\partial_t f = \operatorname{div}(f \nabla f), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \quad (1.2)$$

see [6, 22]. Indeed, when $g = 0$, f solves, up to a multiplicative factor which can be scaled out, the equation (1.2). The goal of this paper is to establish the latter observation rigorously by performing the singular limit

$$\rho_+ \rightarrow 0,$$

with μ kept constant or even letting $\mu \rightarrow \infty$, to recover the porous medium equation (1.2) in the limit, see Theorem 1.2-Theorem 1.5 below. In order to present our results, we first quantify how ρ_+ (and possibly also μ_+) vanishes by setting

$$R = \varepsilon \quad \text{and} \quad \mu = \mu(\varepsilon) = \frac{\bar{\mu}}{\varepsilon^\alpha}, \quad (1.3)$$

with $\varepsilon \in (0, 1)$, $\alpha \in [0, \infty)$, and a positive constant $\bar{\mu}$ (we will let $\varepsilon \rightarrow 0$). For this choice of the parameters R and μ , the system (1.1) becomes

$$\begin{cases} \partial_t f_\varepsilon = \operatorname{div}[f_\varepsilon \nabla((1 + \varepsilon)f_\varepsilon + \varepsilon g_\varepsilon)], \\ \partial_t g_\varepsilon = \mu \varepsilon \operatorname{div}[g_\varepsilon \nabla(f_\varepsilon + g_\varepsilon)], \end{cases} \quad (t, x) \in (0, \infty) \times \mathbb{R}^d. \quad (1.4)$$

We supplement (1.4) with initial data

$$(f_\varepsilon(0), g_\varepsilon(0)) = (f_0, g_0) \in \mathcal{K}^2, \quad (1.5)$$

where

$$\mathcal{K} := \left\{ h \in L_1(\mathbb{R}^d, (1 + |x|^2)dx) \cap L_2(\mathbb{R}^d) : h \geq 0 \text{ a.e. and } \int_{\mathbb{R}^d} h(x) dx = 1 \right\}.$$

We next recall that the system (1.1) is a gradient flow for the energy functional

$$\mathcal{E}_\varepsilon(f, g) := \frac{1}{2} \int_{\mathbb{R}^d} [f^2 + \varepsilon(f + g)^2] dx \quad (1.6)$$

with respect to the 2-Wasserstein distance, see [13], a similar property being available for the porous medium equation (1.2) as observed earlier in [18, 19]. Additionally, the entropy functional

$$\mathcal{H}_\varepsilon(f, g) := \int_{\mathbb{R}^d} \left[f \ln f + \frac{1}{\mu(\varepsilon)} g \ln g \right] dx \quad (1.7)$$

is also non-increasing along solutions to (1.1). The gradient flow structure and the time monotonicity of (1.7) are used in [13] to construct non-negative global weak solutions to (1.1). The next theorem

just recalls the statement of [13, Theorem 1.1] and provides also the corresponding results in the case $d \geq 2$, as the strategy used in [13] can be easily adapted to establish Theorem 1.1 in arbitrary space dimension $d \geq 1$.

Theorem 1.1. *Let $\varepsilon \in (0, 1)$, $(f_0, g_0) \in \mathcal{K}^2$, and assume (1.3). Then, there exists a pair*

$$(f_\varepsilon, g_\varepsilon) : [0, \infty) \rightarrow \mathcal{K}^2$$

such that

$$(i) \quad (f_\varepsilon, g_\varepsilon) \in L_\infty(0, \infty; L_2(\mathbb{R}^d; \mathbb{R}^2)), \quad (f_\varepsilon, g_\varepsilon) \in L_2(0, t; H^1(\mathbb{R}^d; \mathbb{R}^2)),$$

$$(ii) \quad (f_\varepsilon, g_\varepsilon) \in C([0, \infty), (W_4^1(\mathbb{R}^d; \mathbb{R}^2))') \text{ with } (f_\varepsilon, g_\varepsilon)(0) = (f_0, g_0),$$

and $(f_\varepsilon, g_\varepsilon)$ is a weak solution to (1.4)-(1.5) in the sense that

$$\int_{\mathbb{R}^d} f_\varepsilon(t)\xi \, dx - \int_{\mathbb{R}^d} f_0\xi \, dx + \int_0^t \int_{\mathbb{R}^d} f_\varepsilon \nabla((1 + \varepsilon)f_\varepsilon + \varepsilon g_\varepsilon) \cdot \nabla \xi \, dx \, ds = 0, \quad (1.8a)$$

$$\int_{\mathbb{R}^d} g_\varepsilon(t)\xi \, dx - \int_{\mathbb{R}^d} g_0\xi \, dx + \mu\varepsilon \int_0^t \int_{\mathbb{R}^d} g_\varepsilon \nabla(f_\varepsilon + g_\varepsilon) \cdot \nabla \xi \, dx \, ds = 0, \quad (1.8b)$$

for all $\xi \in C_c^\infty(\mathbb{R}^d)$ and $t \geq 0$. In addition, $(f_\varepsilon, g_\varepsilon)$ satisfy the following estimates:

$$(a) \quad \mathcal{H}_\varepsilon(f_\varepsilon(t), g_\varepsilon(t)) + \int_0^t \int_{\mathbb{R}^d} [|\nabla f_\varepsilon|^2 + \varepsilon|\nabla(f_\varepsilon + g_\varepsilon)|^2] \, dx \, ds \leq \mathcal{H}_\varepsilon(f_0, g_0),$$

$$(b) \quad \mathcal{E}_\varepsilon(f_\varepsilon(t), g_\varepsilon(t)) + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} [f_\varepsilon |\nabla((1 + \varepsilon)f_\varepsilon + \varepsilon g_\varepsilon)|^2 + \mu\varepsilon^2 g_\varepsilon |\nabla(f_\varepsilon + g_\varepsilon)|^2] \, dx \, ds \leq \mathcal{E}_\varepsilon(f_0, g_0)$$

for almost all $t \in (0, \infty)$.

From now on f_0 and g_0 are fixed in \mathcal{K} and $(f_\varepsilon, g_\varepsilon)$, $\varepsilon \in (0, 1)$, denotes the solution to the evolution problem (1.4)-(1.5) provided by Theorem 1.1. In our first main result, see Theorem 1.2 below, we establish the convergence of the family $(f_\varepsilon)_{\varepsilon \in (0, 1)}$ found in Theorem 1.1 towards a weak solution to the porous medium equation (1.2) as $\varepsilon \rightarrow 0$ along a suitable sequence.

Theorem 1.2. *Let $d \geq 1$, $\alpha \in [0, \infty)$, and assume (1.3). There exists a sequence $(\varepsilon_k)_{k \geq 1} \subset (0, 1)$ with $\varepsilon_k \rightarrow 0$ and a function $f : [0, \infty) \rightarrow \mathcal{K}$ with*

$$\begin{aligned} f &\in L_\infty(0, \infty; L_2(\mathbb{R}^d)) \cap C([0, \infty), (W_4^1(\mathbb{R}^d))'), \\ f &\in L_2(0, t; H^1(\mathbb{R}^d)) \cap L_\infty(0, t; L_1(\mathbb{R}^d, |x|^2 dx)), \\ \sqrt{f} \nabla f &\in L_2((0, t) \times \mathbb{R}^d), \end{aligned} \quad (1.9)$$

such that $f_{\varepsilon_k} \rightarrow f$ in $L_2((0, t) \times \mathbb{R}^d)$ for all $t > 0$ as $k \rightarrow \infty$. Moreover, f is a weak solution to the porous medium equation (1.2) determined by the initial condition f_0 in the sense that it satisfies (1.9) and

$$\int_{\mathbb{R}^d} f(t)\xi \, dx - \int_{\mathbb{R}^d} f_0\xi \, dx + \int_0^t \int_{\mathbb{R}^d} f \nabla f \cdot \nabla \xi \, dx \, ds = 0$$

for all $\xi \in C_c^\infty(\mathbb{R}^d)$ and $t \geq 0$.

Even though there are several uniqueness results available for the porous medium equation (1.2) in the literature, see [4, 5, 7, 17, 20, 22] and the references therein, uniqueness of a weak solution to (1.2) in the sense of Theorem 1.2 does not seem to be dealt with and is thus reported below when the space dimension satisfies $d \leq 4$, see Theorem 1.3. In fact, Theorem 1.3 improves Theorem 1.2

in space dimension $d \leq 4$ by providing rates for the convergence of the whole family $(f_\varepsilon)_{\varepsilon \in (0,1)}$ as $\varepsilon \rightarrow 0$ towards the solution f to the porous medium equation.

Theorem 1.3. *Let $d \leq 4$, $\alpha \in [0, \infty)$, and assume (1.3). Then, the porous medium equation (1.2) with initial data $f(0) = f_0$ has a unique solution f in the sense of Theorem 1.2 and there exists a positive constant $C = C(f_0, g_0, \bar{\mu})$ such that*

$$\|f_\varepsilon(t) - f(t)\|_{H^{-1}} \leq C e^{Ct} \varepsilon^{\frac{6d+36}{11d+36}} \quad \text{for all } t \geq 0 \text{ and } \varepsilon \in (0, 1). \quad (1.10)$$

The dimension-dependent exponent of ε featured in (1.10) is connected to the low regularity assumed on the solutions to (1.2) and (1.4). Under the additional assumption that an ε -independent L_∞ -bound is available for the solutions to (1.2) and (1.4), the outcome of Theorem 1.3 can be improved as follows.

Corollary 1.4. *Let $d \geq 1$, $\alpha \in [0, \infty)$, and assume (1.3). If $(f_0, g_0) \in L_\infty(\mathbb{R}^d; \mathbb{R}^2)$ and if there exists a positive constant κ such that*

$$\|f_\varepsilon(t)\|_\infty + \|f(t)\|_\infty \leq \kappa, \quad (t, \varepsilon) \in [0, \infty) \times (0, 1), \quad (1.11)$$

then there exists a positive constant $C = C(f_0, g_0, \bar{\mu}, \kappa)$ such that

$$\|f_\varepsilon(t) - f(t)\|_{H^{-1}} \leq C e^{Ct} \varepsilon \quad \text{for all } t \geq 0 \text{ and } \varepsilon \in (0, 1).$$

The boundedness (1.11) is well-known for the solution to the porous medium equation (1.2), as the comparison principle ensures that $\|f(t)\|_\infty \leq \|f_0\|_\infty$ for $t \geq 0$. Such a bound is far from being obvious for solutions to (1.4) and we refer to the forthcoming paper [12] for results in that direction.

Finally, in Theorem 1.5 we establish the convergence of the family $(g_\varepsilon)_{\varepsilon \in (0,1)}$ towards the initial condition g_0 in the regime where $\alpha \in [0, 1/(d+2))$.

Theorem 1.5. *Let $d \geq 1$, $\alpha \in [0, 1/(d+2))$, and assume (1.3). Then, there exists a positive constant $C = C(f_0, g_0, \bar{\mu})$ such that*

$$\|g_\varepsilon(t) - g_0\|_{H^{-1-d}} \leq C(1+t) \varepsilon^{\frac{1}{d+2}-\alpha} \quad \text{for all } t \geq 0 \text{ and } \varepsilon \in (0, 1).$$

The outline of the paper is as follows. In Section 2 we deduce from Theorem 1.1 a handful of estimates for the solutions $(f_\varepsilon, g_\varepsilon)$ to (1.4)-(1.5) which form the basis of the proof of the convergence result stated in Theorem 1.2. Section 3 is next devoted to the proofs of Theorem 1.3 and Corollary 1.4, which use the estimates from Section 2, a duality technique, and Gronwall's lemma. Finally, in Section 4, we establish Theorem 1.5, using once more the estimates established in Section 2.

2. ε -INDEPENDENT ESTIMATES AND PROOF OF THEOREM 1.2

To begin with, we derive from Theorem 1.1 estimates for the solutions $(f_\varepsilon, g_\varepsilon)$ to (1.4)-(1.5), see Lemma 2.1. These estimates, together with Lemma 2.2 and a classical compactness result [21, Corollary 4], enable us to establish the convergence of $(f_\varepsilon)_{\varepsilon \in (0,1)}$ along a sequence $\varepsilon_k \rightarrow 0$ towards the solution to the porous medium equation (1.2), see Lemma 2.3. We conclude the section with the proof of Theorem 1.2. In the following we use the shorthand notation

$$h_\varepsilon := f_\varepsilon + g_\varepsilon, \quad \varepsilon \in (0, 1).$$

Lemma 2.1. *Let $d \geq 1$, $\alpha \in [0, \infty)$, and assume (1.3). Then:*

- (i) $(f_\varepsilon)_{\varepsilon \in (0,1)}$ and $(\sqrt{\varepsilon}h_\varepsilon)_{\varepsilon \in (0,1)}$ are bounded in $L_\infty(0, \infty; L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d))$;
- (ii) $(\sqrt{f_\varepsilon}\nabla(f_\varepsilon + \varepsilon h_\varepsilon))_{\varepsilon \in (0,1)}$ is bounded in $L_2((0, \infty) \times \mathbb{R}^d)$;
- (iii) $(\nabla f_\varepsilon)_{\varepsilon \in (0,1)}$ and $(\sqrt{\varepsilon}\nabla h_\varepsilon)_{\varepsilon \in (0,1)}$ are bounded in $L_2((0, t) \times \mathbb{R}^d)$ for all $t \geq 0$;
- (iv) $(\partial_t f_\varepsilon)_{\varepsilon \in (0,1)}$ is bounded in $L_2(0, \infty; (W_4^1(\mathbb{R}^d))')$;
- (v) $(f_\varepsilon)_{\varepsilon \in (0,1)}$ is bounded in $L_\infty(0, t; L_1(\mathbb{R}^d, |x|^2 dx))$ for all $t \geq 0$.

Proof. Since $\varepsilon \in (0, 1)$ and

$$\mathcal{E}_\varepsilon(f_0, g_0) \leq \frac{\|f_0\|_2^2 + \|f_0 + g_0\|_2^2}{2},$$

the estimates (i)-(ii) directly follow from the definition of \mathcal{K} and the energy inequality, see Theorem 1.1 (b).

In order to prove (iii), let us consider, for each $n \in \mathbb{N}$, the function $\xi_n : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$\xi_n(x) := \begin{cases} |x|^2 & , \quad |x| \leq n, \\ 4n|x| - |x|^2 - 2n^2 & , \quad n \leq |x| \leq 2n, \\ 2n^2 & , \quad |x| \geq 2n. \end{cases}$$

We point out that $\xi_n - 2n^2$ is continuously differentiable and has compact support, hence it can be approximated in the W_∞^1 -norm by functions in $C_c^\infty(\mathbb{R})$. Moreover, the properties of the solutions to (1.4)-(1.5) listed in Theorem 1.1 enable us to show that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_\varepsilon(t, x) \xi_n(x) dx &= \int_{\mathbb{R}^d} f_\varepsilon(t, x) |x|^2 dx, \\ \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} g_\varepsilon(t, x) \xi_n(x) dx &= \int_{\mathbb{R}^d} g_\varepsilon(t, x) |x|^2 dx, \\ \lim_{n \rightarrow \infty} \int_0^t \int_{\mathbb{R}^d} f_\varepsilon \nabla(f_\varepsilon + \varepsilon h_\varepsilon) \cdot \nabla \xi_n dx ds &= 2 \int_0^t \int_{\mathbb{R}^d} f_\varepsilon \nabla(f_\varepsilon + \varepsilon h_\varepsilon) \cdot x dx, \\ \lim_{n \rightarrow \infty} \int_0^t \int_{\mathbb{R}^d} g_\varepsilon \nabla h_\varepsilon \cdot \nabla \xi_n dx ds &= 2 \int_0^t \int_{\mathbb{R}^d} g_\varepsilon \nabla h_\varepsilon \cdot x dx ds, \quad t \geq 0. \end{aligned}$$

Hence, using $\xi_n - 2n^2$, $n \in \mathbb{N}$, as test functions in (1.8), these convergences yield in the limit $n \rightarrow \infty$ that

$$\int_{\mathbb{R}^d} \left(f_\varepsilon(t) + \frac{1}{\mu} g_\varepsilon(t) \right) |x|^2 dx - d \int_0^t \int_{\mathbb{R}^d} (f_\varepsilon^2 + \varepsilon h_\varepsilon^2) dx ds = \int_{\mathbb{R}^d} \left(f_0 + \frac{1}{\mu} g_0 \right) |x|^2 dx, \quad t \geq 0.$$

In particular, it follows from (1.3) and Theorem 1.1 (b) that there exists a constant $C = C(f_0, g_0, \bar{\mu}) > 0$ such that

$$\int_{\mathbb{R}^d} \left(f_\varepsilon(t) + \frac{1}{\mu} g_\varepsilon(t) \right) |x|^2 dx \leq C(1+t), \quad t \geq 0. \quad (2.1)$$

Taking advantage of [13, Lemma A.1], we find a positive universal constant C_E such that

$$-\mathcal{H}_\varepsilon(f_\varepsilon(t), g_\varepsilon(t)) \leq C_E \left(1 + \frac{1}{\mu} \right) + \int_{\mathbb{R}^d} \left(f_\varepsilon(t) + \frac{1}{\mu} g_\varepsilon(t) \right) |x|^2 dx, \quad t \geq 0,$$

which, together with (2.1) and the entropy estimate in Theorem 1.1 (a), shows that

$$\int_0^t \int_{\mathbb{R}^d} [|\nabla f_\varepsilon|^2 + \varepsilon |\nabla h_\varepsilon|^2] dx ds \leq \mathcal{H}_\varepsilon(f_0, g_0) + C(1+t), \quad t \geq 0.$$

Since $(f_0, g_0) \in \mathcal{K}^2$ and $r \ln r \leq r + r^2$ for $r \geq 0$, we conclude, together with (1.3), that there exists a constant $C = C(f_0, g_0, \bar{\mu}) > 0$ with the property that

$$\int_0^t \int_{\mathbb{R}^d} [|\nabla f_\varepsilon|^2 + \varepsilon |\nabla h_\varepsilon|^2] dx ds \leq C(1+t), \quad t \geq 0, \quad (2.2)$$

and (iii) follows from the above inequality.

Next, a classical consequence of Theorem 1.1 (see Lemma 3.1 below for a related result) ensures that $(f_\varepsilon, g_\varepsilon)$ solves (1.4) in distributional sense; that is,

$$\partial_t f_\varepsilon = \operatorname{div} J_{f_\varepsilon} \quad \text{and} \quad \partial_t g_\varepsilon = \operatorname{div} J_{g_\varepsilon} \quad \text{in } \mathcal{D}'((0, \infty) \times \mathbb{R}^d), \quad (2.3)$$

where the fluxes J_{f_ε} and J_{g_ε} are given by

$$J_{f_\varepsilon} := f_\varepsilon \nabla (f_\varepsilon + \varepsilon h_\varepsilon) \quad \text{and} \quad J_{g_\varepsilon} := \mu \varepsilon g_\varepsilon \nabla h_\varepsilon.$$

The estimates (i)-(ii) from Theorem 1.1, along with Hölder's inequality, lead us to

$$\begin{aligned} \|J_{f_\varepsilon}\|_{L_2(0, \infty; L_{4/3}(\mathbb{R}^d))}^2 &= \int_0^\infty \left(\int_{\mathbb{R}^d} |f_\varepsilon \nabla (f_\varepsilon + \varepsilon h_\varepsilon)|^{4/3} dx \right)^{3/2} dt \\ &= \int_0^\infty \left(\int_{\mathbb{R}^d} |f_\varepsilon^2|^{1/3} |\sqrt{f_\varepsilon} \nabla (f_\varepsilon + \varepsilon h_\varepsilon)|^{4/3} dx \right)^{3/2} dt \\ &\leq \int_0^\infty \|f_\varepsilon\|_2 \left\| \sqrt{f_\varepsilon} \nabla (f_\varepsilon + \varepsilon h_\varepsilon) \right\|_2^2 dt \\ &\leq \|f_\varepsilon\|_{L_\infty(0, \infty; L_2(\mathbb{R}^d))} \left\| \sqrt{f_\varepsilon} \nabla (f_\varepsilon + \varepsilon h_\varepsilon) \right\|_{L_2((0, \infty) \times \mathbb{R}^d)}^2, \end{aligned}$$

which shows that $(J_{f_\varepsilon})_{\varepsilon \in (0, 1)}$ is bounded in $L_2(0, \infty; L_{4/3}(\mathbb{R}^d))$. This property immediately implies (iv) by a duality argument.

Finally, the bound (v) is a straightforward consequence of (2.1). \square

The next step is the continuity and compactness of some embeddings involving weighted L_p -spaces, which will serve when establishing the convergence of the family $(f_\varepsilon)_{\varepsilon \in (0, 1)}$ (along a suitable sequence $\varepsilon_k \rightarrow 0$).

Lemma 2.2.

- (i) Given $p \in (2, \infty]$, the embedding $L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d, |x|^2 dx) \hookrightarrow L_2(\mathbb{R}^d, |x|^{\frac{2(p-2)}{p-1}} dx)$ is continuous.
- (ii) The embedding $H^1(\mathbb{R}^d) \cap L_1(\mathbb{R}^d, |x|^2 dx) \hookrightarrow L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ is compact.

Proof. The claim (i) with $p = \infty$ is obvious. For $p \in (2, \infty)$, Hölder's inequality leads us to

$$\int_{\mathbb{R}^d} |f(x)|^2 |x|^{\frac{2(p-2)}{p-1}} dx \leq \left(\int_{\mathbb{R}^d} |f(x)| |x|^2 dx \right)^{\frac{p-2}{p-1}} \|f\|_p^{\frac{p}{p-1}} \quad (2.4)$$

and (i) follows.

With respect to (ii), let $(f_n)_{n \geq 1}$ be a bounded sequence in $H^1(\mathbb{R}^d) \cap L_1(\mathbb{R}^d, |x|^2 dx)$ and set

$$M := \sup_{n \geq 1} \{ \|f_n\|_{H^1} + \|f_n\|_{L_1(\mathbb{R}^d, |x|^2 dx)} \}.$$

Owing to the compactness of the embedding of $H^1(B_R(0))$ in $L_2(B_R(0))$ for any $R > 0$, where we set $B_R(0) := \{x \in \mathbb{R}^d : |x| < R\}$, it follows from a standard Cantor diagonal procedure that there exist a function $f \in H^1(\mathbb{R}^d) \cap L_1(\mathbb{R}^d, |x|^2 dx)$ and a subsequence of $(f_n)_{n \geq 1}$ (not relabeled) such that $f_n \rightarrow f$ in $L_2(B_R(0))$ for all $R > 0$ and

$$\|f\|_{H^1} + \|f\|_{L_1(\mathbb{R}^d, |x|^2 dx)} \leq M.$$

Choosing $p \in (2, \infty)$ such that $p < 2d/(d-2)$ when $d \geq 3$, Sobolev's embedding and (2.4) then lead us to

$$\begin{aligned} \|f_n - f\|_2^2 &\leq \int_{B_R(0)} |f_n(x) - f(x)|^2 dx + R^{-\frac{2(p-2)}{p-1}} \int_{\{|x| > R\}} |f_n(x) - f(x)|^2 |x|^{\frac{2(p-2)}{p-1}} dx \\ &\leq \|f_n - f\|_{L_2(B_R(0))}^2 + R^{-\frac{2(p-2)}{p-1}} \|f_n - f\|_{L_1(\mathbb{R}^d, |x|^2 dx)}^{\frac{p-2}{p-1}} \|f_n - f\|_p^{\frac{p}{p-1}} \\ &\leq \|f_n - f\|_{L_2(B_R(0))}^2 + CR^{-\frac{2(p-2)}{p-1}} (2M)^{\frac{p-2}{p-1}} \|f_n - f\|_{H^1}^{\frac{p}{p-1}} \\ &\leq \|f_n - f\|_{L_2(B_R(0))}^2 + CM^2 R^{-\frac{2(p-2)}{p-1}} \end{aligned}$$

for all $n \in \mathbb{N}$ and $R > 1$. Letting first $n \rightarrow \infty$ and then $R \rightarrow \infty$, we deduce that $f_n \rightarrow f$ in $L_2(\mathbb{R}^d)$. Finally, for $R > 1$,

$$\begin{aligned} \|f_n - f\|_1 &\leq \|f_n - f\|_{L_1(B_R(0))} + R^{-2} \|f_n - f\|_{L_1(\mathbb{R}^d, |x|^2 dx)} \\ &\leq \sqrt{|B_R(0)|} \|f_n - f\|_{L_2(B_R(0))} + 2MR^{-2}. \end{aligned}$$

Arguing as before completes the proof of (ii). \square

We now use the estimates derived in Lemma 2.1 and the previous result to deduce the following convergences.

Lemma 2.3. *Let $d \geq 1$, $\alpha \in [0, \infty)$, and assume (1.3). There exists a sequence $(\varepsilon_k)_{k \geq 1} \subset (0, 1)$ with $\varepsilon_k \rightarrow 0$ and a function $f \in L_2(0, t; H^1(\mathbb{R}^d)) \cap L_\infty(0, \infty; L_2(\mathbb{R}^d))$ for all $t > 0$ such that*

- (i) $f_{\varepsilon_k} \rightarrow f$ in $L_2((0, t) \times \mathbb{R}^d)$ and in $C([0, t], (W_4^1(\mathbb{R}^d))')$ for all $t > 0$;
- (ii) $\nabla f_{\varepsilon_k} \rightarrow \nabla f$ in $L_2((0, t) \times \mathbb{R}^d)$ for all $t > 0$;
- (iii) $\partial_t f_{\varepsilon_k} \rightarrow \partial_t f$ in $L_2(0, \infty; (W_4^1(\mathbb{R}^d))')$;
- (iv) $\sqrt{f_{\varepsilon_k}} \nabla(f_{\varepsilon_k} + \varepsilon_k h_{\varepsilon_k}) \rightarrow \sqrt{f} \nabla f$ in $L_2((0, t) \times \mathbb{R}^d)$ for all $t > 0$.

Proof. Let $t > 0$ be fixed. In view of Lemma 2.1 (i), (iii), and (v), we obtain the boundedness of $(f_\varepsilon)_{\varepsilon \in (0, 1)}$ in $L_2(0, t; H^1(\mathbb{R}^d))$ and in $L_\infty(0, t; L_1(\mathbb{R}^d, |x|^2 dx))$, as well as that of $(\partial_t f_\varepsilon)_{\varepsilon \in (0, 1)}$ in $L_2(0, t; (W_4^1(\mathbb{R}^d))')$. Hence, the family $(f_\varepsilon)_{\varepsilon \in (0, 1)}$ is bounded in $L_2(0, t; H^1(\mathbb{R}^d) \cap L_1(\mathbb{R}^d, |x|^2 dx))$. Since

$$H^1(\mathbb{R}^d) \cap L_1(\mathbb{R}^d, |x|^2 dx) \hookrightarrow L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d) \hookrightarrow (W_4^1(\mathbb{R}^d))'$$

and the first embedding is compact according to Lemma 2.2 (ii), we infer from a classical compactness result, see [21, Corollary 4], and a Cantor diagonal argument that there exist a sequence $\varepsilon_k \rightarrow 0$ and a function f such that $f_{\varepsilon_k} \rightarrow f$ in $L_2((0, t) \times \mathbb{R}^d)$ and in $C([0, t], (W_4^1(\mathbb{R}^d))')$ for all $t > 0$. Note that Lemma 2.1 (i) now immediately implies that $f \in L_\infty(0, \infty; L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d))$.

Next, the convergence (ii) is a straightforward consequence of Lemma 2.1 (iii) and the just established Lemma 2.3 (i), from which we also deduce that $f \in L_2(0, t; H^1(\mathbb{R}^d))$ for all $t > 0$.

Recalling Lemma 2.1 (iv), the convergence $\partial_t f_{\varepsilon_k} \rightharpoonup \partial_t f$ in $L_2(0, t; (W_4^1(\mathbb{R}^d))')$ (after possibly extracting a further subsequence) follows from Lemma 2.3 (i) and the reflexivity of $L_2(0, t; (W_4^1(\mathbb{R}^d))')$.

With respect to (iv), due to Lemma 2.1 (ii), we may assume that there is $\mathbf{F} \in L_2((0, \infty) \times \mathbb{R}^d)$ such that

$$\sqrt{f_{\varepsilon_k}} \nabla(f_{\varepsilon_k} + \varepsilon_k h_{\varepsilon_k}) \rightharpoonup \mathbf{F} \quad \text{in } L_2((0, \infty) \times \mathbb{R}^d). \quad (2.5)$$

Moreover, the bounds in Lemma 2.1 (iii), along with Lemma 2.3 (ii), imply that

$$\nabla(f_{\varepsilon_k} + \varepsilon_k h_{\varepsilon_k}) \rightharpoonup \nabla f \quad \text{in } L_2((0, t) \times \mathbb{R}^d),$$

while Lemma 2.3 (i) leads us to

$$\sqrt{f_{\varepsilon_k}} \rightarrow \sqrt{f} \quad \text{in } L_4((0, t) \times \mathbb{R}^d). \quad (2.6)$$

Combining the last two convergences we find

$$\sqrt{f_{\varepsilon_k}} \nabla(f_{\varepsilon_k} + \varepsilon_k h_{\varepsilon_k}) \rightharpoonup \sqrt{f} \nabla f \quad \text{in } L_{4/3}((0, t) \times \mathbb{R}^d).$$

Recalling (2.5), we conclude that $\mathbf{F} = \sqrt{f} \nabla f$ and therewith establish (iv). \square

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $(\varepsilon_k)_k \subset (0, 1)$ and f be as found in Lemma 2.3. As a direct consequence of Lemma 2.1 (v) and Lemma 2.3 (i), which in particular implies that $f_{\varepsilon_k}(t) \rightarrow f(t)$ in $L_2(\mathbb{R}^d)$ for almost all $t > 0$, we deduce that $f \in L_\infty(0, t; L_1(\mathbb{R}^d, (1 + |x|^2)dx))$ for all $t > 0$. Taking also into account that $f \in C([0, \infty), (W_4^1(\mathbb{R}^d))')$, it follows that $f(t) \geq 0$ a.e. in \mathbb{R}^d for all $t \geq 0$. Moreover, in view of Lemma 2.3 (i) and (iv), it is straightforward to pass to the limit $k \rightarrow \infty$ in (1.8a) with $\varepsilon = \varepsilon_k$ and obtain

$$\int_{\mathbb{R}^d} f(t) \xi \, dx - \int_{\mathbb{R}^d} f_0 \xi \, dx + \int_0^t \int_{\mathbb{R}^d} f \nabla f \cdot \nabla \xi \, dx \, ds = 0$$

for all $\xi \in C_c^\infty(\mathbb{R}^d)$ and all $t \geq 0$, thereby establishing (1.2).

Finally, choosing $\zeta_n : \mathbb{R}^d \rightarrow \mathbb{R}$ with

$$\zeta_n(x) := \begin{cases} 1 & , \quad |x| \leq n, \\ n+1-|x| & , \quad n \leq |x| \leq n+1, \\ 0 & , \quad |x| \geq n+1, \end{cases} \quad n \in \mathbb{N},$$

as test function in (1.2), we find that $\|f(t)\|_1 = \|f_0\|_1 = 1$ for all $t \geq 0$. Therefore $f(t) \in \mathcal{K}$ for all $t \geq 0$. \square

3. ESTIMATING THE ERROR $\|f_\varepsilon(t) - f(t)\|_{H^{-1}}$

In this section we restrict our arguments to the case when the space dimension satisfies $d \leq 4$. The main goal is to provide an estimate for the error $\|f_\varepsilon(t) - f(t)\|_{H^{-1}}$ with $t \geq 0$, cf. Theorem 1.3. In particular, we also prove that the porous medium equation (1.2) with initial data $f_0 \in \mathcal{K}$ has a unique solution in the sense of Theorem 1.2. Hence, this improves Theorem 1.2 in the sense that now the whole family $(f_\varepsilon)_{\varepsilon \in (0, 1)}$ converges for $\varepsilon \rightarrow 0$ towards the corresponding solution to the porous medium equation. In order to prepare the proof of Theorem 1.3, which we postpone to the

end of the section, we first introduce some notation. We recall that $1 - \Delta : W_p^2(\mathbb{R}^d) \rightarrow L_p(\mathbb{R}^d)$ is an isomorphism for all $p \in (1, \infty)$ and

$$\|f\|_{H^{-1}} = \|(1 - \Delta)^{-1}[f]\|_{H^1} \quad \text{for all } f \in L_2(\mathbb{R}^d).$$

Theorem 1.3 then amounts to estimate the norm $\|F_\varepsilon(t) - F(t)\|_{H^1}$, where

$$F_\varepsilon := (1 - \Delta)^{-1}[f_\varepsilon] \quad \text{and} \quad F := (1 - \Delta)^{-1}[f].$$

Testing the equations (1.8a) and (1.2) by $(1 - \Delta)^{-1}[\xi]$, with $\xi \in C_c^\infty(\mathbb{R}^d)$ and using the self-adjointness of $(1 - \Delta)^{-1}$, we arrive at

$$\begin{aligned} & \int_{\mathbb{R}^d} F_\varepsilon(t)\xi \, dx - \int_{\mathbb{R}^d} F_0\xi \, dx \\ & \quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \left[f_\varepsilon^2 \xi - \xi(1 - \Delta)^{-1}[f_\varepsilon^2] + 2\varepsilon(1 - \Delta)^{-1}[f_\varepsilon \nabla h_\varepsilon] \cdot \nabla \xi \right] dx \, ds = 0, \\ & \int_{\mathbb{R}^d} F(t)\xi \, dx - \int_{\mathbb{R}^d} F_0\xi \, dx + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \left[f^2 \xi - \xi(1 - \Delta)^{-1}[f^2] \right] dx \, ds = 0, \end{aligned}$$

where $F_0 := (1 - \Delta)^{-1}[f_0]$. Hence, letting

$$d_\varepsilon = f_\varepsilon - f \quad \text{and} \quad D_\varepsilon := F_\varepsilon - F = (1 - \Delta)^{-1}[d_\varepsilon],$$

we have $d_\varepsilon(0) = D_\varepsilon(0) = 0$ and, subtracting the above identities, we deduce that

$$\begin{aligned} & \int_{\mathbb{R}^d} D_\varepsilon(t)\xi \, dx + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \left[d_\varepsilon(f_\varepsilon + f)\xi - \xi(1 - \Delta)^{-1}[d_\varepsilon(f_\varepsilon + f)] \right] dx \, ds \\ & \quad + \varepsilon \int_0^t \int_{\mathbb{R}^d} (1 - \Delta)^{-1}[f_\varepsilon \nabla h_\varepsilon] \cdot \nabla \xi \, dx \, ds = 0. \end{aligned} \tag{3.1}$$

As a consequence of (3.1), Theorem 1.1, and Theorem 1.2, we obtain the following result.

Lemma 3.1. *Let $t > 0$ and $\xi \in C_c^\infty([0, t] \times \mathbb{R}^d)$. Then*

$$\begin{aligned} & \int_{\mathbb{R}^d} D_\varepsilon(t)\xi(t) \, dx - \int_0^t \int_{\mathbb{R}^d} D_\varepsilon \partial_t \xi \, dx \, ds \\ & = -\frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \left[d_\varepsilon(f_\varepsilon + f)\xi - \xi(1 - \Delta)^{-1}[d_\varepsilon(f_\varepsilon + f)] + 2\varepsilon(1 - \Delta)^{-1}[f_\varepsilon \nabla h_\varepsilon] \cdot \nabla \xi \right] dx \, ds \end{aligned} \tag{3.2}$$

and

$$\int_{\mathbb{R}^d} d_\varepsilon(t)\xi(t) \, dx - \int_0^t \int_{\mathbb{R}^d} d_\varepsilon \partial_t \xi \, dx \, ds = \int_0^t \int_{\mathbb{R}^d} \left[f \nabla f - f_\varepsilon \nabla(f_\varepsilon + \varepsilon h_\varepsilon) \right] \cdot \nabla \xi \, dx \, ds \tag{3.3}$$

for all $\varepsilon \in (0, 1)$.

Proof. This is a classical result and therefore we omit its proof. \square

The next lemma provides an integral identity, cf. (3.4), which is obtained when formally testing with D_ε in (3.3). This identity is the starting point in the proof of Theorem 1.3.

Lemma 3.2. *For all $t \geq 0$ and $\varepsilon \in (0, 1)$,*

$$\|d_\varepsilon(t)\|_{H^{-1}}^2 = \|D_\varepsilon(t)\|_{H^1}^2 = \int_0^t \int_{\mathbb{R}^d} \left[-d_\varepsilon^2(f_\varepsilon + f) + D_\varepsilon d_\varepsilon(f_\varepsilon + f) + 2\varepsilon f_\varepsilon \nabla h_\varepsilon \cdot \nabla D_\varepsilon \right] dx \, ds. \tag{3.4}$$

Proof. Since $D_\varepsilon = (1 - \Delta)^{-1}[d_\varepsilon]$ with $d_\varepsilon \in L_\infty(0, \infty; L_2(\mathbb{R}^d))$, we have $D_\varepsilon \in L_\infty(0, \infty; H^2(\mathbb{R}^d))$. Let $\tilde{D}_\varepsilon \in L_\infty(\mathbb{R}; H^2(\mathbb{R}^d))$ denote the even reflection of D_ε with respect to the boundary $\{t = 0\}$ of $(0, \infty) \times \mathbb{R}^d$ and choose $\varphi \in C_c^\infty(\mathbb{R}^d, [0, 1])$. We then define $\xi_\delta \in C_c^\infty(\mathbb{R}^{d+1})$ as the convolution $\xi_\delta := \rho_\delta * (\varphi \tilde{D}_\varepsilon)$, where $(\rho_\delta)_{\delta \in (0, 1)}$ is a standard mollifier on \mathbb{R}^{d+1} . We first note that there exists a positive integer $N \in \mathbb{N}$ with the property that $\text{supp } \xi_\delta(t) \subset [-N, N]^d$ for all $\delta \in (0, 1)$ and $t \geq 0$. Using ξ_δ as a test function in (3.3), we get

$$\begin{aligned} \int_{\mathbb{R}^d} d_\varepsilon(t) \xi_\delta(t) dx - \int_0^t \int_{\mathbb{R}^d} d_\varepsilon \partial_t \xi_\delta dx ds \\ = \int_0^t \int_{\mathbb{R}^d} \left[f \nabla f - f_\varepsilon \nabla (f_\varepsilon + \varepsilon h_\varepsilon) \right] \cdot \nabla \xi_\delta dx ds \quad \text{for all } t > 0. \end{aligned} \quad (3.5)$$

To pass to the limit $\delta \rightarrow 0$ in the term on the right-hand side of (3.5) we recall that

$$f \nabla f - f_\varepsilon \nabla (f_\varepsilon + \varepsilon h_\varepsilon) \in L_{4/3}((0, t) \times \mathbb{R}^d)$$

by Lemma 2.1 (i)-(ii), Theorem 1.3, and Hölder's inequality. Moreover, the regularity of \tilde{D}_ε , the restriction to space dimension $d \leq 4$, and Sobolev's embedding ensure that $\varphi \tilde{D}_\varepsilon \in L_\infty(\mathbb{R}; W_4^1(\mathbb{R}^d))$, from which we deduce that $\nabla \xi_\delta \rightarrow \nabla(\varphi D_\varepsilon)$ in $L_4((0, t) \times \mathbb{R}^d)$ for all $t > 0$ when letting $\delta \rightarrow 0$. We therewith get

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int_0^t \int_{\mathbb{R}^d} \left[f \nabla f - f_\varepsilon \nabla (f_\varepsilon + \varepsilon h_\varepsilon) \right] \cdot \nabla \xi_\delta dx ds \\ = \int_0^t \int_{\mathbb{R}^d} \left[f \nabla f - f_\varepsilon \nabla (f_\varepsilon + \varepsilon h_\varepsilon) \right] \cdot \nabla(\varphi D_\varepsilon) dx ds. \end{aligned}$$

To deal with the first term on the left-hand side of the equality (3.5), we note that $d_\varepsilon \xi_\delta \rightarrow \varphi d_\varepsilon D_\varepsilon$ in $L_1((0, t) \times \mathbb{R}^d)$ for $\delta \rightarrow 0$ and all $t > 0$, and therefore we may assume that

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}^d} d_\varepsilon(t) \xi_\delta(t) dx = \int_{\mathbb{R}^d} \varphi d_\varepsilon(t) D_\varepsilon(t) dx \quad \text{for almost all } t > 0.$$

To pass to the limit $\delta \rightarrow 0$ in the second term on the left-hand side of (3.5), we note that Lemma 2.3 (iii) guarantees that $\partial_t d_\varepsilon \in L_2(0, \infty, (W_4^1(\mathbb{R}^d))')$. Since $\partial_t D_\varepsilon = (1 - \Delta)^{-1}[\partial_t d_\varepsilon]$, we have $\partial_t D_\varepsilon \in L_2(0, \infty, W_{4/3}^1(\mathbb{R}^d))$ and this implies that $\partial_t \tilde{D}_\varepsilon$ lies in $L_2(\mathbb{R}; L_{4/3}(\mathbb{R}^d))$. It follows that $\partial_t \xi_\delta = \rho_\delta * (\varphi \partial_t \tilde{D}_\varepsilon)$ in \mathbb{R}^{d+1} and therefore $\partial_t \xi_\delta \rightarrow \varphi \partial_t D_\varepsilon$ in $L_2(0, t; L_{4/3}(\mathbb{R}^d))$ as $\delta \rightarrow 0$. Taking also into account that $d_\varepsilon \in L_2(0, t; L_4(\mathbb{R}^d))$, due to Lemma 2.3 (i)-(ii) and the continuous embedding of $H^1(\mathbb{R}^d)$ in $L_4(\mathbb{R}^d)$, we may now pass to the limit $\delta \rightarrow 0$ in (3.5) to conclude that the identity

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi d_\varepsilon(t) D_\varepsilon(t) dx - \int_0^t \int_{\mathbb{R}^d} \varphi d_\varepsilon \partial_t D_\varepsilon dx ds \\ = \int_0^t \int_{\mathbb{R}^d} \left[f \nabla f - f_\varepsilon \nabla (f_\varepsilon + \varepsilon h_\varepsilon) \right] \cdot \nabla(\varphi D_\varepsilon) dx ds \end{aligned}$$

is satisfied for almost all $t \in (0, \infty)$. Now, owing to (3.2),

$$\partial_t D_\varepsilon = -\frac{d_\varepsilon(f_\varepsilon + f)}{2} + (1 - \Delta)^{-1} \left[\frac{d_\varepsilon(f_\varepsilon + f)}{2} \right] + \varepsilon \text{div} \left((1 - \Delta)^{-1} [f_\varepsilon \nabla h_\varepsilon] \right)$$

in $\mathcal{D}'((0, \infty) \times \mathbb{R}^d)$. Combining the above two identities, we end up with

$$\begin{aligned} & \int_{\mathbb{R}^d} \varphi d_\varepsilon(t) D_\varepsilon(t) dx \\ &= \int_0^t \int_{\mathbb{R}^d} \varphi d_\varepsilon \left(-\frac{d_\varepsilon(f_\varepsilon + f)}{2} + (1 - \Delta)^{-1} \left[\frac{d_\varepsilon(f_\varepsilon + f)}{2} \right] + \varepsilon \nabla \cdot (1 - \Delta)^{-1} [f_\varepsilon \nabla h_\varepsilon] \right) dx ds \\ & \quad - \int_0^t \int_{\mathbb{R}^d} \left[\nabla \left(\frac{d_\varepsilon(f_\varepsilon + f)}{2} \right) + \varepsilon f_\varepsilon \nabla h_\varepsilon \right] \cdot \nabla (\varphi D_\varepsilon) dx ds \end{aligned}$$

for almost every $t \in (0, \infty)$. Choosing a suitable approximating sequence $(\varphi_n)_n \subset C_c^\infty(\mathbb{R}^d, [0, 1])$ for the constant function 1, we infer from the above identity, after passing to the limit $n \rightarrow \infty$ and using the relation $D_\varepsilon = (1 - \Delta)^{-1} [d_\varepsilon]$, that (3.4) holds true (for all $t \geq 0$ due to the fact that $D_\varepsilon \in C([0, \infty); W_{4/3}^1(\mathbb{R}^d))$). \square

We are now in a position to establish our second main result, see Theorem 1.3.

Proof of Theorem 1.3. Given $t \geq 0$, it follows from (3.4) that

$$\|D_\varepsilon(t)\|_{H^1}^2 + \int_0^t \int_{\mathbb{R}^d} d_\varepsilon^2(f_\varepsilon + f) dx ds \leq T_1 + T_2, \quad (3.6)$$

where

$$T_1 := \int_0^t \int_{\mathbb{R}^d} D_\varepsilon d_\varepsilon(f_\varepsilon + f) dx ds \quad \text{and} \quad T_2 := 2\varepsilon \int_0^t \int_{\mathbb{R}^d} f_\varepsilon \nabla h_\varepsilon \cdot \nabla D_\varepsilon dx ds.$$

Below we estimate the terms T_1 and T_2 separately in order to obtain an integral inequality to which we may apply Gronwall's inequality and conclude in this way our claim (1.10).

The term T_1 . Using Hölder's and Young's inequalities, we find

$$\begin{aligned} T_1 &\leq \left(\int_0^t \int_{\mathbb{R}^d} d_\varepsilon^2(f_\varepsilon + f) dx ds \right)^{1/2} \left(\int_0^t \int_{\mathbb{R}^d} D_\varepsilon^2(f_\varepsilon + f) dx ds \right)^{1/2} \\ &\leq \frac{1}{4} \int_0^t \int_{\mathbb{R}^d} d_\varepsilon^2(f_\varepsilon + f) dx ds + \int_0^t \int_{\mathbb{R}^d} D_\varepsilon^2(f_\varepsilon + f) dx ds. \end{aligned} \quad (3.7)$$

The first term on the right-hand of (3.7) is clearly controlled by the left-hand side of (3.6) and we are thus left with estimating the second term. In view of the continuous embedding $H^1(\mathbb{R}^d) \hookrightarrow L_4(\mathbb{R}^d)$ (recall that $d \leq 4$), we infer from Hölder's inequality that

$$\int_0^t \int_{\mathbb{R}^d} D_\varepsilon^2(f_\varepsilon + f) dx ds \leq \int_0^t \|D_\varepsilon\|_4^2 \|f_\varepsilon + f\|_2 ds \leq C \int_0^t \|f_\varepsilon + f\|_2 \|D_\varepsilon\|_{H^1}^2 ds.$$

Hence, in view of Theorem 1.2 and Lemma 2.1 (i), we have

$$\int_0^t \int_{\mathbb{R}^d} D_\varepsilon^2(f_\varepsilon + f) dx ds \leq C \int_0^t \|D_\varepsilon\|_{H^1}^2 ds. \quad (3.8)$$

The term T_2 . In order to estimate T_2 , we first observe that $W_3^1(\mathbb{R}^d)$ embeds continuously in $L_{12}(\mathbb{R}^d)$ due to $d \leq 4$. We then infer from Gagliardo-Nirenberg's inequality [16] that

$$\|\nabla D_\varepsilon\|_{12} \leq C \|D_\varepsilon\|_{W_3^2}^a \|\nabla D_\varepsilon\|_2^{1-a} \leq C \|d_\varepsilon\|_3^a \|\nabla D_\varepsilon\|_2^{1-a}, \quad (3.9)$$

where

$$a := \frac{5d}{2(d+6)} \in (0, 1].$$

Hölder's inequality and (3.9) now imply that

$$\begin{aligned} \|f_\varepsilon \nabla h_\varepsilon \cdot \nabla D_\varepsilon\|_1 &\leq \|f_\varepsilon\|_{12/5} \|\nabla h_\varepsilon\|_2 \|\nabla D_\varepsilon\|_{12} \\ &\leq C \|f_\varepsilon\|_{12/5} \|\nabla h_\varepsilon\|_2 \|d_\varepsilon\|_3^a \|\nabla D_\varepsilon\|_2^{1-a}. \end{aligned}$$

Taking advantage of Young's inequality, we then get

$$\begin{aligned} 2\varepsilon \|f_\varepsilon \nabla h_\varepsilon \cdot \nabla D_\varepsilon\|_1 &\leq \frac{1}{4} \|d_\varepsilon\|_3^3 + C \left(\varepsilon \|f_\varepsilon\|_{12/5} \|\nabla h_\varepsilon\|_2 \|\nabla D_\varepsilon\|_2^{1-a} \right)^{\frac{3}{3-a}} \\ &= \frac{1}{4} \|d_\varepsilon\|_3^3 + C \left(\varepsilon^{1/2} \|\nabla h_\varepsilon\|_2 \|\nabla D_\varepsilon\|_2 \right)^{\frac{3(1-a)}{3-a}} \left(\varepsilon^{1/2} \|\nabla h_\varepsilon\|_2 \right)^{\frac{3a}{3-a}} \left(\varepsilon^{1/2} \|f_\varepsilon\|_{12/5} \right)^{\frac{3}{3-a}} \\ &\leq \frac{1}{4} \|d_\varepsilon\|_3^3 + \varepsilon \|\nabla h_\varepsilon\|_2^2 \|\nabla D_\varepsilon\|_2^2 + C \left(\varepsilon^{1/2} \|\nabla h_\varepsilon\|_2 \right)^{\frac{6a}{3+a}} \left(\varepsilon^{1/2} \|f_\varepsilon\|_{12/5} \right)^{\frac{6}{3+a}}. \end{aligned}$$

Consequently, since $d_\varepsilon \leq f_\varepsilon + f$,

$$\begin{aligned} T_2 &\leq \frac{1}{4} \int_0^t \int_{\mathbb{R}^d} (f_\varepsilon + f) d_\varepsilon^2 dx ds + \varepsilon \int_0^t \|\nabla h_\varepsilon\|_2^2 \|D_\varepsilon\|_{H^1}^2 ds \\ &\quad + C \varepsilon^{\frac{3}{3+a}} \int_0^t \left(\varepsilon^{1/2} \|\nabla h_\varepsilon\|_2 \right)^{\frac{6a}{3+a}} \|f_\varepsilon\|_{12/5}^{\frac{6}{3+a}} ds. \end{aligned}$$

Using Hölder's inequality and (2.2), we further have

$$\begin{aligned} \int_0^t \left(\varepsilon^{1/2} \|\nabla h_\varepsilon\|_2 \right)^{\frac{6a}{3+a}} \|f_\varepsilon\|_{12/5}^{\frac{6}{3+a}} ds &\leq \left(\int_0^t \varepsilon \|\nabla h_\varepsilon\|_2^2 ds \right)^{\frac{3a}{3+a}} \left(\int_0^t \|f_\varepsilon\|_{12/5}^{\frac{6}{3-2a}} ds \right)^{\frac{3-2a}{3+a}} \\ &\leq C(1+t)^{\frac{3a}{3+a}} \left(\int_0^t \|f_\varepsilon\|_{12/5}^{\frac{6}{3-2a}} ds \right)^{\frac{3-2a}{3+a}}. \end{aligned}$$

Furthermore, by Gagliardo-Nirenberg's inequality and Lemma 2.1 (i),

$$\|f_\varepsilon\|_{12/5} \leq C \|\nabla f_\varepsilon\|_2^b \|f_\varepsilon\|_2^{1-b} \leq C \|\nabla f_\varepsilon\|_2^b \quad \text{with } b = \frac{d}{12} \in (0, 1),$$

which implies, together with (2.2) and the property $d/(2(3-2a)) \leq 2$, that

$$\left(\int_0^t \|f_\varepsilon\|_{12/5}^{\frac{6}{3-2a}} ds \right)^{\frac{3-2a}{3+a}} \leq C \left(\int_0^t \|\nabla f_\varepsilon\|_2^{\frac{d}{2(3-2a)}} ds \right)^{\frac{3-2a}{3+a}} \leq C(1+t)^{\frac{3-2a}{3+a}}.$$

Recalling the definition of a , we conclude that

$$T_2 \leq \frac{1}{4} \int_0^t \int_{\mathbb{R}^d} (f_\varepsilon + f) d_\varepsilon^2 dx ds + \varepsilon \int_0^t \|\nabla h_\varepsilon\|_2^2 \|D_\varepsilon\|_{H^1}^2 ds + C(1+t) \varepsilon^{\frac{6d+36}{11d+36}} \quad (3.10)$$

for all $t > 0$.

Applying Gronwall's lemma. Gathering (3.6), (3.7), (3.8), and (3.10), we arrive at

$$\|D_\varepsilon(t)\|_{H^1}^2 \leq \int_0^t (C + \varepsilon \|\nabla h_\varepsilon\|_2^2) \|D_\varepsilon\|_{H^1}^2 ds + C(1+t)\varepsilon^{\frac{6d+36}{11d+36}}, \quad t \geq 0. \quad (3.11)$$

Recalling (2.2), a direct application of Gronwall's inequality now leads us to the desired estimate (1.10).

Uniqueness of the solution to (1.2). If f_1 and f_2 are two solutions to (1.2) corresponding to the same initial data f_0 , then we can perform similar computations as those leading to (3.11) to obtain that

$$\|(1 - \Delta)^{-1}[(f_1 - f_2)(t)]\|_{H^1}^2 \leq C \int_0^t \|(1 - \Delta)^{-1}[(f_1 - f_2)]\|_{H^1}^2 ds, \quad t \geq 0,$$

and Gronwall's lemma then implies $f_1 = f_2$. \square

We complete this section with the proof of Corollary 1.4.

Proof of Corollary 1.4. Keeping the notation introduced in the proof of Theorem 1.3, it readily follows from (1.11) and (3.7) that

$$T_1 \leq \frac{1}{4} \int_0^t \int_{\mathbb{R}^d} d_\varepsilon^2(f_\varepsilon + f) dx ds + \kappa \int_0^t \|D_\varepsilon\|_2^2 ds. \quad (3.12)$$

Using again (1.11) along with (2.2) and Hölder's and Young's inequality, we find

$$T_2 \leq 2\kappa\varepsilon \int_0^t \|\nabla h_\varepsilon\|_2 \|D_\varepsilon\|_2 ds \leq \int_0^t \|D_\varepsilon\|_{H^1}^2 ds + C\varepsilon(1+t). \quad (3.13)$$

Combining (3.6), (3.12), and (3.13) gives

$$\|D_\varepsilon(t)\|_{H^1}^2 \leq (1 + \kappa) \int_0^t \|D_\varepsilon\|_{H^1}^2 ds + C\varepsilon(1+t), \quad t \geq 0,$$

and applying Gronwall's lemma completes the proof. \square

4. THE LIMITING BEHAVIOR OF g_ε

In this section we establish our last main result stated in Theorem 1.5. Before going on, we point out that all the estimates for the family $(g_\varepsilon)_{\varepsilon \in (0,1)}$ provided by Theorem 1.1 involve the function g_ε multiplied by a positive power of ε when $\alpha > 0$, see Lemma 2.1, except the conservation of mass $\|g_\varepsilon(t)\|_1 = 1$, which stems from $g_\varepsilon(t) \in \mathcal{K}$, $t \geq 0$. Nevertheless, exploiting this property, we establish below the convergence of $(g_\varepsilon)_{\varepsilon \in (0,1)}$ towards the initial condition g_0 for $0 \leq \alpha < 1/(d+2)$, without any restriction on the space dimension $d \geq 1$.

Proof of Theorem 1.5. To start, we use (1.8b) and the continuous embedding $H^{1+d}(\mathbb{R}^d) \hookrightarrow W_\infty^1(\mathbb{R}^d)$ to deduce that

$$\begin{aligned} \left| \int_{\mathbb{R}^d} (g_\varepsilon(t) - g_0)\xi dx \right| &\leq \int_0^t \int_{\mathbb{R}^d} |J_{g_\varepsilon} \cdot \nabla \xi| dx dt \leq \|J_{g_\varepsilon}\|_{L_1((0,t) \times \mathbb{R}^d)} \|\nabla \xi\|_\infty \\ &\leq C \|J_{g_\varepsilon}\|_{L_1((0,t) \times \mathbb{R}^d)} \|\xi\|_{H^{1+d}}, \end{aligned}$$

for all $t \geq 0$ and $\xi \in C_c^\infty(\mathbb{R}^d)$, where we recall that $J_{g_\varepsilon} = \mu\varepsilon g_\varepsilon \nabla h_\varepsilon$. Consequently we have

$$\|g_\varepsilon(t) - g_0\|_{H^{-1-d}} \leq C \|J_{g_\varepsilon}\|_{L_1((0,t) \times \mathbb{R}^d)} \quad \text{for } t \geq 0 \text{ and } \varepsilon > 0,$$

and it remains to estimate the norm $\|J_{g_\varepsilon}\|_{L_1((0,t)\times\mathbb{R}^d)}$.

Recalling that $\|g_\varepsilon(t)\|_1 = 1$, $t \geq 0$, Gagliardo-Nirenberg's inequality yields

$$\begin{aligned} \|g_\varepsilon(t)\|_2 &\leq C \|\nabla g_\varepsilon(t)\|_2^{\frac{d}{d+2}} \|g_\varepsilon(t)\|_1^{\frac{2}{2+d}} \\ &\leq C \|\nabla g_\varepsilon(t)\|_2^{\frac{d}{d+2}} \\ &\leq C (\|\nabla f_\varepsilon(t)\|_2 + \|\nabla h_\varepsilon(t)\|_2)^{\frac{d}{d+2}} \\ &\leq C \varepsilon^{-\frac{d}{2(d+2)}} (\|\nabla f_\varepsilon(t)\|_2 + \sqrt{\varepsilon} \|\nabla h_\varepsilon(t)\|_2)^{\frac{d}{d+2}}, \quad t \geq 0. \end{aligned}$$

Taking now advantage of (2.2), we get

$$\varepsilon^{\frac{d}{2(d+2)}} \|g_\varepsilon\|_{L_2(d+2)/d(0,t;L_2(\mathbb{R}^d))} \leq C(1+t)^{\frac{d}{2(d+2)}}, \quad t \geq 0,$$

and Hölder's inequality leads us to

$$\varepsilon^{\frac{d}{2(d+2)}} \|g_\varepsilon\|_{L_2((0,t)\times\mathbb{R}^d)} \leq C(1+t)^{\frac{1}{2}}, \quad t \geq 0.$$

Recalling from (2.2) that

$$\sqrt{\varepsilon} \|\nabla h_\varepsilon\|_{L_2((0,t)\times\mathbb{R}^d)} \leq C(1+t)^{\frac{1}{2}}, \quad t \geq 0,$$

we obtain, by using once more Hölder's inequality and the definition of J_{g_ε} , that

$$\|J_{g_\varepsilon}\|_{L_1((0,t)\times\mathbb{R}^d)} \leq C(1+t)\varepsilon^{\frac{1}{d+2}-\alpha}, \quad t \geq 0,$$

which proves the claim. \square

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