Discrete planes: comparison of geometric and topological representations

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Abstract

There are three different representations for discrete planes in a 3-dimensional space which are often used for geometric modeling, image analysis and image recognition: discrete analytical representation based on analytical geometry, discrete morphological representation based on general topology, and discrete combinatorial representation based on combinatorial and algebraic topology. In this paper, we present the equivalence relations of the representations and derive some new aspects of properties for discrete planes from the equivalence.

1 Introduction

For analysis, recognition and modeling of two- or three-dimensional objects for computer imagery, several shape representations have been introduced and used in different approaches and tasks. Each representation has its own characteristics with consideration on topology or geometry of an object and we sometimes need some of these characteristics of several different representations simultaneously. For example, we may need topological properties which give us the guarantee such that our object of interest is a 2-manifold and geometric properties which enable us to calculate the curvatures for our shape representation. In this paper, we focus on planes which is one of simple geometric objects in a 3-dimensional space and discuss the differences and relations between several geometric and topological representations for such planes.

Let $\mathcal{Z}^3$ be the 3-dimensional discrete space where $\mathcal{Z}$ is the set of all integers. We consider the following three discrete representations in $\mathcal{Z}^3$ for a plane in the 3-dimensional Euclidean space $\mathcal{R}^3$.

1. **Discrete analytical planes** (**DAP**): **DAP**s were first introduced by Reveillès [12]. For each Euclidean plane, a **DAP** is obtained such as a set of points in $\mathcal{Z}^3$ which are between two parallel

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Euclidean planes given by two inequations. Depending on the width between the two Euclidean planes, we have two different types of **DAP**s: standard planes (**SP**) [5] and naive planes (**NP**) [12]. The geometric and topological properties of **DAP**s were discussed in [4, 12, 13] and [2, 3, 5, 6], respectively.

2. **Discrete morphological planes** (**DMP**): **DMP**s are based on the definition of border points in general topology and point set theory in $\mathcal{R}^3$. Any Euclidean plane separates $\mathcal{Z}^3$ into two regions. For one of the regions, we define a set of border points by using one of the 6-, 18- and 26-neighborhood systems, such that the neighborhood of a border point contains at least one point of the complement of the region. As a set of border points, we obtain a $\mathbf{DMP}_m$ for each $m = 6, 18, 26$. The definitions of border points of objects in $\mathcal{Z}^3$ and the related discussions are found in [10, 14, 16].

3. **Discrete combinatorial planes** (**DCP**): Similarly to $\mathbf{DMP}_m$, we construct a discrete combinatorial plane $\mathbf{DCP}_m$ for each $m = 6, 18, 26$ as the combinatorial boundary of a polyhedral complex for one of the discrete regions which are separated by a Euclidean plane. In [9], we give the definition and the construction algorithm for combinatorial boundaries of objects in $\mathcal{Z}^3$. Such a combinatorial boundary is considered to be a triangulation of border points. Thus, $\mathbf{DCP}_m$s have combinatorial topological structures.

The **DAP** is the most similar representation to the analytical form of an original Euclidean plane among the above representations. Thus, once we have a **DAP** in $\mathcal{Z}^3$, we can easily obtain the geometric features such as the normal vector of a **DAP**. The recognition problem of a plane for a given set of points in $\mathcal{Z}^3$ by using a **DAP** is studied in [4, 6]. However, it is obvious that we cannot explicitly obtain topological properties of a **DAP**. Although some topological study for **DAP**s has been presented in [2, 3, 5, 6, 12], the additional topological tools such as the notions of digital topology or combinatorial topology are necessary.

The definition of **DMP**s is one of the simplest constructive definitions for border points of objects in $\mathcal{Z}^3$. It is well known and often used in the field of medical image analysis. However, a $\mathbf{DMP}_m$ contain no combinatorial topological structure; the connectivity of $\mathbf{DMP}_m$ is studied in [11] and it is shown that $\mathbf{DMP}_m$ is m'-connected for $(m, m') = (6, 18), (18, 6), (26, 6)$. In order to have the combinatorial topological structures, we need to apply an additional triangulation algorithm after obtaining a $\mathbf{DMP}_m$.

The definition of **DCP**s is also constructive, so that **DCP**s are obtained by tracking border points of objects in $\mathcal{Z}^3$ with combinatorial topological structures. In our recent work [9], we derived the inclusion relations between $\mathbf{DMP}_m$ and the set of vertices of $\mathbf{DCP}_{m'}$ which is called the skeleton $Sk(\mathbf{DCP}_{m'})$ for $(m, m') = (6, 18), (6, 26), (18, 6), (26, 6)$. There are two type of **DAP**s: naive planes **NP** and standard planes **SP**. In [8], we have shown the equivalence relations between **NP** and either $Sk(\mathbf{DCP}_{18})$ or $Sk(\mathbf{DCP}_{26})$.

In this paper, first we give the definitions of **DAP** (or **SP** and **NP**), **DMP** and **DCP** and present

their properties which have been already obtained by several authors in Sections 2, 3 and 4, respectively. We then show the relations of **DAP**, **DMP** and **DCP** such as the equivalence relations between **SP** and $\mathbf{DMP}_{26}$, **NP** and $\mathbf{DMP}_6$, $\mathbf{DMP}_{26}$ and $Sk(\mathbf{DCP}_6)$, and $\mathbf{DMP}_6$ and $Sk(\mathbf{DCP}_{26})$ in Section 5. From the equivalence relations we succeed to derive some new aspects of properties for discrete planes. In Section 6, for example, we show that any discrete plane is conceivable to be a 2-dimensional combinatorial manifold.

2 Discrete Analytical Planes

2.1 Definitions

A plane **P** in the 3-dimensional Euclidean space $\mathcal{R}^3$ is given by an analytical form such as

$$\mathbf{P} = \{(x, y, z) \in \mathcal{R}^3 \ : \ ax + by + cz + d = 0\} \tag{1}$$

where a, b, c, d are real numbers. The analytical form of discrete planes in $\mathcal{Z}^3$ was introduced and studied by Reveillès [12]. The definition of discrete analytical plane with respect to **P** is given by

$$\mathbf{DAP} = \{(x, y, z) \in \mathcal{Z}^3 \ : \ 0 \leq ax + by + cz + d < w\} \tag{2}$$

where w is the width of **DAP**. If $w = |a| + |b| + |c|$, **DAP** is called a standard plane **SP** [5], and if $w = max\{|a|, |b|, |c|\}$, **DAP** is called a naive plane **NP** [12].

2.2 Properties

2.2.1 Connectivities

Some digital topological properties such as connectivities of **DAP**s were considered by several authors. In $\mathcal{Z}^n$ for $n = 2, 3$ the m-neighborhoods are defined by

$$\mathbf{N}_m(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathcal{Z}^n : \|\boldsymbol{x} - \boldsymbol{y}\|^2 \leq t\} \tag{3}$$

setting $t = 1, 2$ for each $m = 4, 8$ if $n = 2$, and $t = 1, 2, 3$ for each $m = 6, 18, 26$ if $n = 3$. A subset $\mathbf{A} \subset \mathcal{Z}^n$ is said to be m-connected if any pair of elements $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{A}$ has a path $\boldsymbol{a}_1 = \boldsymbol{a}, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_p = \boldsymbol{b}$ such that $\boldsymbol{a}_{i+1} \in \mathbf{N}_m(\boldsymbol{a}_i)$ and $\boldsymbol{a}_i \in \mathbf{A}$ for every $i = 1, 2, \ldots, p-1$. In two dimensions, discrete analytical lines

$$\mathbf{DAL} = \{(x, y) \in \mathcal{Z}^2 \ : \ 0 \leq ax + by + c < w\}$$

are given similarly to **DAP** [12] and it was shown that **DAL** is 4-connected if $w = |a| + |b|$ and 8-connected if $w = max\{|a|, |b|\}$ [12]. For **DAP**s in three dimensions, Andrès derived the property such that any **NP** is 18-connected without 6-connected holes [2]. The following more general properties are also given in [3], with the definitions of k-tunnel and k-separating[1] If the complement $\overline{\mathbf{DAP}}$ of **DAP** in $\mathbf{Z}^3$ is not

[1] In [3], k is set to be $0, 1, 2$. In this paper we set $k = 26, 18, 6$ instead to avoid the confusion.

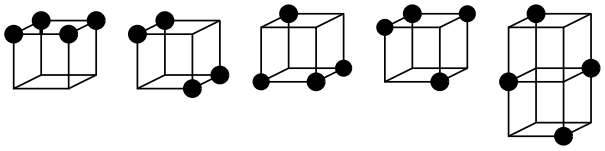


Figure 1: Five bicubes for naive planes.

k-connected, **DAP** is said to be k-separating for $k = 6, 18, 26$. Considering the two regions such as

$$\begin{aligned} \mathbf{A} &= \{(x, y, z) \in \mathcal{Z}^3 : ax + by + cz + d < 0\}, \\ \mathbf{B} &= \{(x, y, z) \in \mathcal{Z}^3 : ax + by + cz + d \geq w\}, \end{aligned}$$

if there are two k-neighboring points $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a} \in \mathbf{A}$, $\boldsymbol{b} \in \mathbf{B}$, **DAP** is said to have a k-tunnel for $k = 6, 18, 26$. The connectivities of **SP**s are also studied in [5].

Property 1 *A standard plane* **SP** *is tunnel free [3], and* 6*-connected [3, 5].*

Property 2 *[3] A naive plane* **NP** *may have* 18*-tunnel but no* 6*-tunnel, and is* 6*-separating, i.e.* **NP** *is* 18*-connected but not* 6*-connected.*

2.2.2 Bicubes and Tricubes for Naive Planes

The geometric properties such as local configurations of lattice points are also investigated in [4, 5, 6, 13], but the properties were obtained only for naive planes **NP**. Let us consider the configurations of lattice points at local regions which project on the coordinate plane $z = 0$ as a rectangle whose sizes are $\lambda \times \mu$. If $\lambda = \mu = 2$, such local configurations are called bicubes and if $\lambda = \mu = 3$, they are called tricubes.

Property 3 *[6] There exist five different bicubes in* **NP***s with respect to* **P***s such that* $0 \leq a \leq b \leq c$, $c > 0$ *as shown in Figure 1.*

Property 4 *[6, 13] At most four different bicubes are contained in an* **NP***.*

Property 5 *[4] There exist forty different tricubes in* **NP***s with respect to* **P***s such that* $0 \leq a \leq b \leq c$, $c > 0$ *as shown in Figure 2.*

Property 6 *[13] At most nine different tricubes are contained in an* **NP***.*

Note that bicubes and tricubes for other **NP**s with any real numbers a, b, c, d, which may not satisfy $0 \leq a \leq b \leq c$, $c > 0$, can be obtained by rotations of the five bicubes and forty tricubes in Figures 1 and 2. In Section 6, we succeed to obtain bicubes and tricubes for standard planes **SP**s.

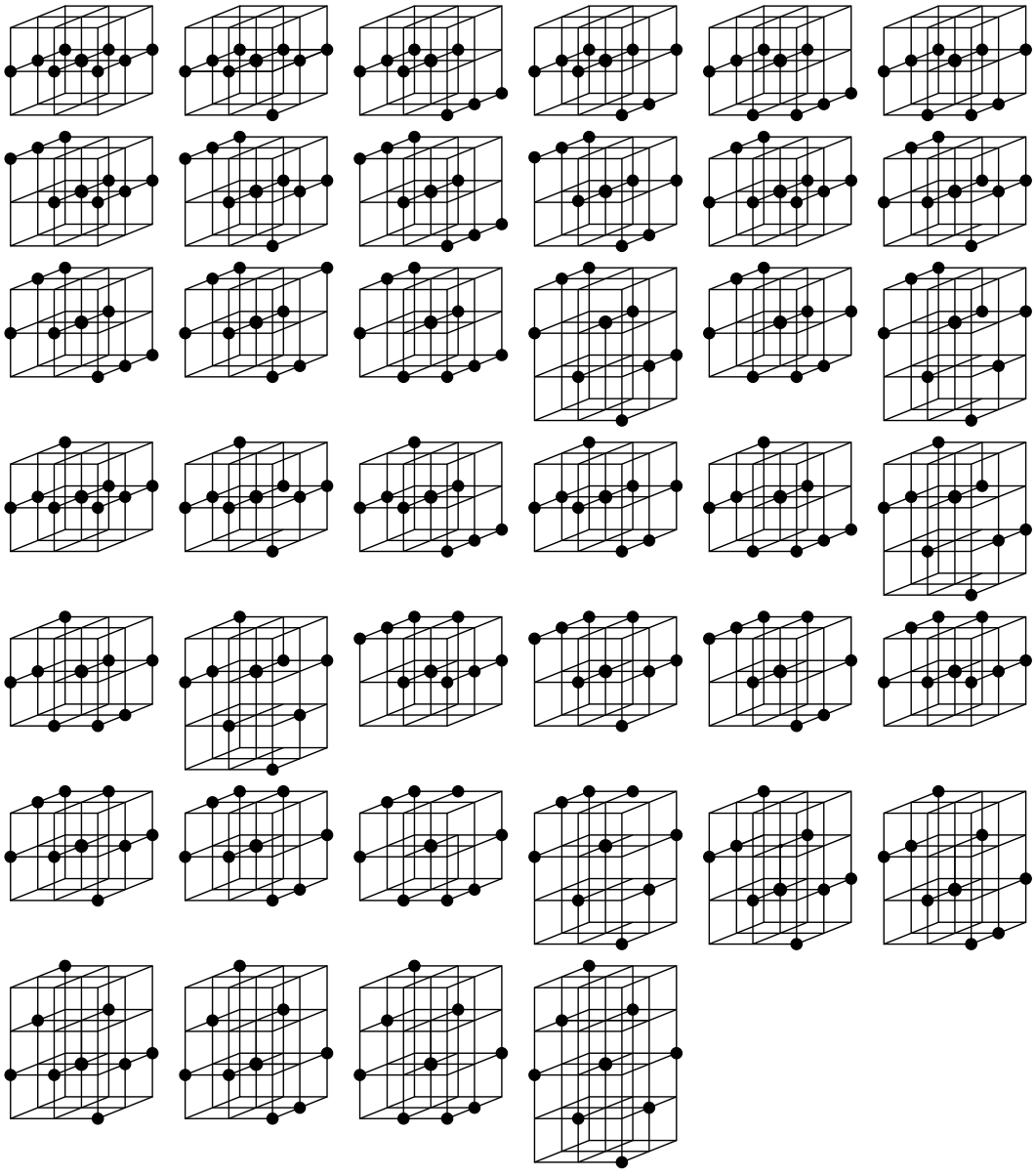


Figure 2: Forty tricubes for naive planes.

2.2.3 Combinatorial Topological Structures

The combinatorial topological properties are also investigated for both **SP** and **NP**.

Property 7 *[5] An* **SP** *has the structure of a* 2*-dimensional combinatorial manifold without boundary, whose faces are squares and vertices are points in* **SP**.

Property 8 *[6] An* **NP** *has the structure of a* 2*-dimensional combinatorial manifold without boundary, whose faces are bicubes and whose umbrella are tricubes.*

In Section 6, we also derive the similar properties such that **SP**s and **NP**s are 2-dimensional combinatorial manifolds in a different way from those in [5, 6].

3 Discrete Morphological Planes

3.1 Definitions

A plane **P** of (1) defines two half spaces such as

$$\begin{aligned} \mathbf{H}^- &= \{(x,y,z) \in \mathcal{R}^3 : ax + by + cz + d \leq 0\}, \\ \mathbf{H}^+ &= \{(x,y,z) \in \mathcal{R}^3 : ax + by + cz + d \geq 0\} \end{aligned}$$

and the digitization of $\mathbf{H}^-$ and $\mathbf{H}^+$ are obtained by

$$\begin{aligned} \mathbf{I}^- &= \{(x,y,z) \in \mathcal{Z}^3 : ax + by + cz + d \leq 0\}, \\ \mathbf{I}^+ &= \{(x,y,z) \in \mathcal{Z}^3 : ax + by + cz + d \geq 0\}. \end{aligned} \tag{4}$$

For each of $\mathbf{I}^-$ and $\mathbf{I}^+$, we can define the morphological border by using one of the 6-, 18- and 26-neighborhood systems of (3). In this paper, we focus on $\mathbf{I}^+$ and consider the m-neighborhood border of $\mathbf{I}^+$ such as

$$\mathbf{DMP}_m = \{\boldsymbol{x} \in \mathbf{I}^+ : \mathbf{N}_m(\boldsymbol{x}) \cap \overline{\mathbf{I}^+} \neq \emptyset\} \tag{5}$$

for each $m = 6, 18, 26$ where $\overline{\mathbf{I}^+}$ is the complement of $\mathbf{I}^+$ in $\mathcal{Z}^3$. This representation is based on the definition of border points in general topology and point set theory in $\mathcal{R}^3$. It has been introduced as one of the classical definitions for the set of border points of an object in $\mathcal{Z}^3$. The definitions and discussions of border points for general objects which are not only planar-form but free-form can be found in [10, 14, 16].

3.2 Properties

The digital topological properties such as connectivities of $\mathbf{DMP}_m$ can be derived in the following. In [11], it has been shown that the m-neighborhood border of any subset $\mathbf{V} \subset \mathcal{Z}^3$ is m'-connected if $\mathbf{V}$ and $\overline{\mathbf{V}}$

are m'- and m''-connected, respectively, where $(m, m') = (6, 18), (18, 6), (26, 6)$[2] and any $m'' = 6, 18, 26$. Because $\mathbf{I}^+$ and $\overline{\mathbf{I}^+}$ are considered to be m'- and m''-connected for any $m', m'' = 6, 18, 26$, we obtain the following property.

Property 9 *A* $\mathbf{DMP}_m$ *of (5) is* m'*-connected for the pairs* $(m, m') = (6, 18), (18, 6), (26, 6)$.

4 Discrete Combinatorial Planes

Similarly to **DMP**s of (5), we obtain discrete combinatorial planes **DCP** to apply the combinatorial boundary tracking algorithm introduced in [9] to $\mathbf{I}^+$ of (4). The difference between **DMP** and **DCP** is that **DCP** is considered to be a triangulation of **DMP**, or a polygonal decomposition of **DMP**. We first give the definitions of discrete convex polyhedra and discrete polyhedral complexes such that all vertices are points in $\mathcal{Z}^3$ and the adjacent vertices are m-neighboring each other for $m = 6, 18, 26$ by following the notions of combinatorial topology [1, 15, 17]. Note that our notions are based on not simplexes [1, 15] but convex polyhedra [17]. We then present the algorithm for construction of discrete combinatorial planes **DCP** which are 2-dimensional discrete polyhedral complexes modifying the algorithm in [9]. The details of discrete convex polyhedra, discrete polyhedral complexes and the algorithm for border tracking of general objects in $\mathcal{Z}^3$ are seen in [9].

4.1 Discrete Convex Polyhedra and Discrete Polyhedral Complexes

In $\mathcal{R}^n$, a convex polyhedron σ is the convex hull of a finite set of points in some $\mathcal{R}^d$ where $d \leq n$. The dimension of σ is the dimension of its affine hull. An n-dimensional convex polyhedron σ is abbreviated to an n-polyhedron. A linear inequality $\boldsymbol{a} \cdot \boldsymbol{x} \leq z$ is valid for σ if it is satisfied for all points $\boldsymbol{x} \in \sigma$. A face of σ is then defined by any set of the form

$$\delta = \sigma \cap \{\boldsymbol{x} \in \mathcal{R}^d : \boldsymbol{a} \cdot \boldsymbol{x} \leq z\}$$

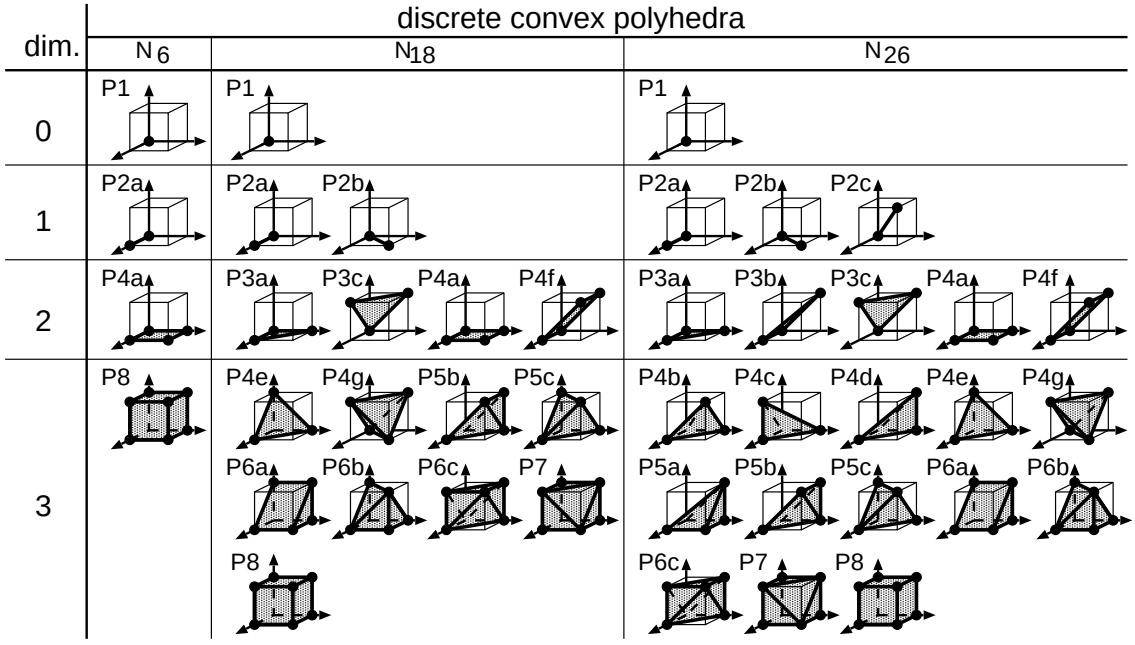
where $\boldsymbol{a} \cdot \boldsymbol{x} \leq z$ is valid for σ. For example, a point is a 0-polyhedron, a line segment is 1-polyhedron, a triangle is a 2-polyhedron, and a tetrahedron is a 3-polyhedron. The point of a 0-polyhedron, the endpoints of a 1-polyhedron and the vertices of 2- and 3-polyhedra are called the vertices of convex polyhedra.

Definition 1 *A polyhedral complex* **K** *is a finite collection of convex polyhedra such that*

1. *the empty polyhedron is in* **K**,
2. *if* $\sigma \in \mathbf{K}$, *then all faces of* σ *are also in* **K**,

[2] Precisely, the possible pairs are given such as $(m, m') = (6, 18), (6, 26), (18, 6), (18, 18), (18, 26), (26, 6), (26, 18), (26, 26)$ in [11] due to the inclusion relations of the 6-, 18- and 26-neighborhoods.

Table 1: All discrete n-polyhedra for $n = 0, 1, 2, 3$ such that the vertices are all lattice points in $\mathcal{Z}^3$ and the adjacent vertices are m-neighboring for $m = 6, 18, 26$.



3. the intersection $\sigma \cap \tau$ of two convex polyhedra $\sigma, \tau \in \mathbf{K}$ is a face both of σ and of τ.

The dimension of $\mathbf{K}$ is the largest dimension of a convex polyhedron in $\mathbf{K}$.

In $\mathcal{Z}^3$, we consider all convex polyhedra such that the vertices are all lattice points and any adjacent vertices are m-neighboring each other for $m = 6, 18, 26$. We call such convex polyhedra discrete convex polyhedra. Such convex polyhedra are obtained in the following. Let us consider all possible convex polyhedra in a unit cubic region such that the vertices of each convex polyhedron are vertices of a unit cube. A unit cube has eight lattice points for the vertices. Each lattice point is assigned a value of either 1 or 0 and called a 1- or 0-point. There are 256 configurations of 1- and 0-points for the eight lattice points in a unit cubic region which can be reduced to 23 with considering the congruent configurations by rotations. For each configuration, we obtain a convex polyhedron such that the vertices of the polyhedron are 1-points. We then classify each convex polyhedron into a set of discrete convex polyhedra with the dimension of $n = 0, 1, 2, 3$ and with the m-neighborhood relations between the adjacent vertices for $m = 6, 18, 26$ as shown in Table 1. From Table 1, we see that there are a finite number of discrete convex polyhedra for each neighborhood system and for each dimension from 0 to 3. For the abbreviation, we call the n-dimensional discrete convex polyhedra in Table 1 discrete n-polyhedra hereafter.

In Table 1, we see that every n'-dimensional face of any discrete n-polyhedron for $n' < n$ is also a discrete n'-polyhedron for each m-neighborhood system, $m = 6, 18, 26$. This is important because it enables us to construct a discrete polyhedral complex which is a finite collection of discrete convex polyhedra satisfying the three conditions in Definition 1 for each m-neighborhood system. We abbreviate

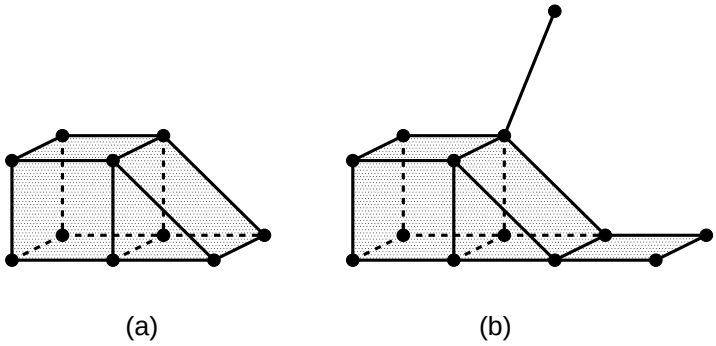


Figure 3: Examples of (a) a pure discrete 3-complex and (b) a non-pure discrete 3-complex for the 26-neighborhood system.

n-dimensional discrete polyhedral complexes discrete n-complexes hereafter.

4.2 Combinatorial Boundaries of Discrete Polyhedral Complexes

We give some topological notions for discrete polyhedral complexes [1]. A discrete n-complex $\mathbf{K}$ is said to be pure if every discrete n'-polyhedron of $\mathbf{K}$ where $n' < n$ is a face of some discrete n-polyhedron. Figure 3 illustrates examples of pure and non-pure discrete 3-complexes for the 26-neighborhood system. If $\mathbf{K}_0$ is any subcomplex of $\mathbf{K}$, the complex consisting of all the elements of $\mathbf{K}_0$ and of all the elements of $\mathbf{K}$ each of which is a face of at least one element of $\mathbf{K}_0$ is called the combinatorial closure $Cl(\mathbf{K}_0)$ of $\mathbf{K}_0$ in $\mathbf{K}$.

We consider a discrete polyhedral complex $\mathbf{C}$ as a topological representation of any subset $\mathbf{V} \subset \mathcal{Z}^3$, i.e. a topological space by topologizing $\mathbf{V}$. Because we would like to have combinatorial surface structures in our boundary representation, we consider a pure discrete 3-subcomplex $\mathbf{O} \subseteq \mathbf{C}$ and define the combinatorial boundary $\partial\mathbf{O}$ of $\mathbf{O}$. The notion of such combinatorial boundary $\partial\mathbf{O}$ is based on algebraic topology [15].

Definition 2 *Let* $\mathbf{O}$ *be a pure discrete* 3*-complex and* $\mathbf{G}$ *be the set of all discrete* 2*-polyhedra in* $\mathbf{O}$ *each of which is a face of exactly one discrete* 3*-polyhedron in* $\mathbf{O}$*. The combinatorial boundary of* $\mathbf{O}$ *is defined such that*

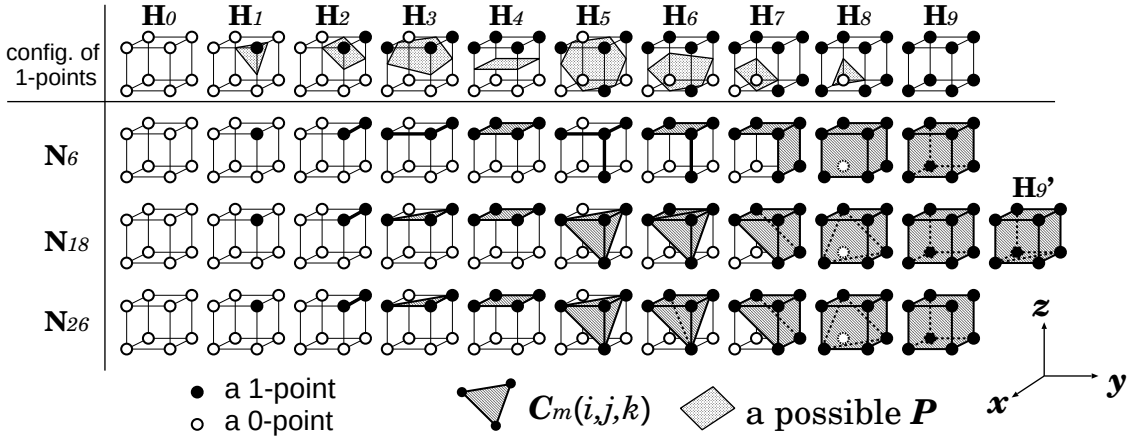
$$\partial\mathbf{O} = Cl(\mathbf{G}).$$

From Definition 2, we obtain the following proposition.

Proposition 1 *The boundary* $\partial\mathbf{O}$ *of a pure discrete* 3*-complex* $\mathbf{O}$ *is a pure discrete* 2*-subcomplex of* $\mathbf{O}$*.*

Because discrete convex polyhedra are defined for each m-neighborhood system where $m = 6, 18, 26$, a discrete polyhedral complex $\mathbf{C}$, a discrete pure 3-polyhedron $\mathbf{O}$ and the combinatorial boundary $\partial\mathbf{O}$

Table 2: A 2-dimensional polyhedral complex $\mathbf{C}_m(i,j,k)$ for each of ten possible configurations $\mathbf{H}(i,j,k)$ of 1- and 0-points for $m = 6, 18, 26$. In the table, we consider $\mathbf{P}$ such that $0 \leq a \leq b \leq c$, $c > 0$.



are also defined for each m-neighborhood system. When we insist a m-neighborhood system considering for them, they are denoted by $\mathbf{C}_m$, $\mathbf{O}_m$ and $\partial\mathbf{O}_m$ instead.

4.3 Construction of Discrete Combinatorial Planes by Combinatorial Boundary Tracking

In [9], we show how to construct a pure discrete 2-complex $\partial\mathbf{O}_m$ from any given subset $\mathbf{V} \subset \mathcal{Z}^3$ for $m = 6, 18, 26$. By applying the same algorithm in [9] to $\mathbf{I}^+$ of (4) as a subset of $\mathcal{Z}^3$, we can obtain a discrete combinatorial plane $\mathbf{DCP}_m$ which is a pure discrete 2-complex $\partial\mathbf{O}_m$ with respect to a Euclidean plane $\mathbf{P}$ of (1). The details and discussion for general boundary tracking are found in [9].

We assign a value of 1 (resp. 0) to every point in $\mathbf{I}^+$ (resp. the complement $\overline{\mathbf{I}^+} = \mathcal{Z}^3 \setminus \mathbf{I}^+$), and it is called a 1-point (resp. a 0-point). Considering a set of 1-points at a unit cubic region of $(i,j,k) \in \mathcal{Z}^3$ such as

$$\mathbf{H}(i,j,k) = \mathbf{I}^+ \cap \{(x,y,z) \in \mathcal{Z}^3 : i \leq x \leq i+1, j \leq y \leq j+1, k \leq z \leq k+1\},$$

the number of all possible 1-point configurations of $\mathbf{H}(i,j,k)$ is ten as shown in the first line of Table 2 if we consider Euclidean planes $\mathbf{P}$s of (1) such that $0 \leq a \leq b \leq c$, $c > 0$. In Table 2, we also show a discrete polyhedral complex $\mathbf{C}_m(i,j,k)$ with respect to each $\mathbf{H}(i,j,k)$ such that we put as many and large discrete convex polyhedra chosen from Table 1 as possible into a unit cube and all the vertices of discrete convex polyhedra are points of $\mathbf{H}(i,j,k)$. Note that the configuration H0 appears for $\mathbf{H}(i,j,k)$ which is empty, and thus all points in the unit cube are in $\overline{\mathbf{I}^+}$. The configuration H9 appears in a unit cube whose eight points are all in $\mathbf{I}^+$.

Combining $\mathbf{C}_m(i,j,k)$ for all $(i,j,k) \in \mathcal{Z}^3$ such that

$$\mathbf{C}_m = \bigcup_{(i,j,k)\in\mathcal{Z}^3} \mathbf{C}_m(i,j,k),$$

now we have to prove that $\mathbf{C}_m$ becomes a discrete polyhedral complex, satisfying the conditions in Definition 1. The proof is obtained in a similar way to the proof for more general cases in [9]. In fact this case is much simpler than the case in [9] because the number of 1-point configurations is much smaller than that in [9]; we have ten in Table 2 while 23 in [9]. If we consider any pair of adjacent $\mathbf{C}_m(\boldsymbol{x})$ and $\mathbf{C}_m(\boldsymbol{y})$ such that $\boldsymbol{x} \in \mathbf{N}_6(\boldsymbol{y})$, then we see that $\mathbf{C}_m(\boldsymbol{x}) \cup \mathbf{C}_m(\boldsymbol{y})$ becomes a discrete polyhedral complex except for an adjacent pair of $\mathbf{C}_{18}(i,j,k)$ of H6 and $\mathbf{C}_{18}(i,j,k+1)$ of H9. In this case, we have to consider $\mathbf{C}_{18}(i,j,k+1)$ of H9$'$ instead of H9 as shown in Table 2.

Following the procedure in [9], after obtaining a discrete 3-complex $\mathbf{C}_m$, we make a pure discrete 3-complex $\mathbf{O}_m$ from $\mathbf{C}_m$. In this paper, we show the following lemma which does not hold generally.

Lemma 1 *If we obtain a discrete* 3*-complex* $\mathbf{C}_m$ *for each* $m = 6, 18, 26$ *by referring to Table 2 with respect to* $\mathbf{I}^+$*, then the relation*

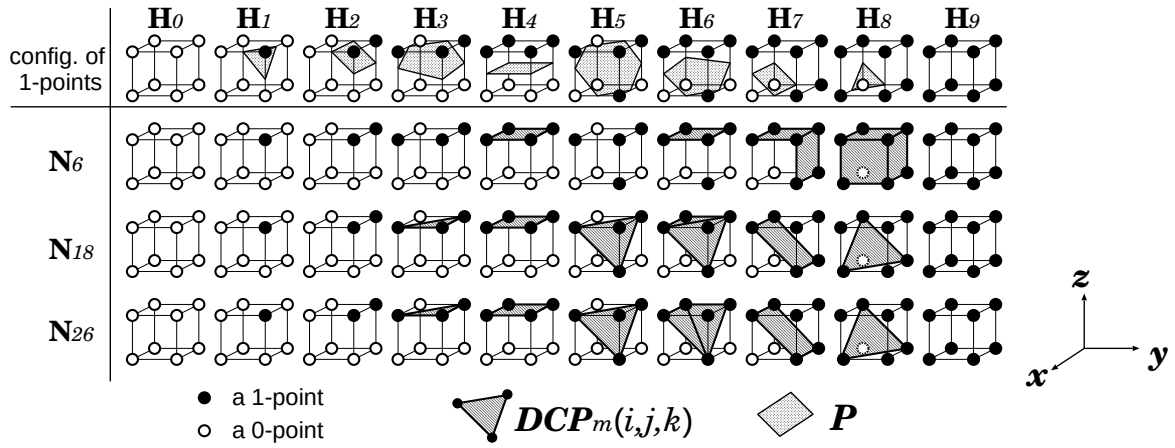
$$\mathbf{C}_m = \mathbf{O}_m$$

always holds.

(Proof) In order to prove this, we have to show that any discrete n-polyhedron in $\mathbf{C}_m$ $(n < 3)$ is a face of some discrete 3-polyhedron of $\mathbf{C}_m$. We first consider the case of $m = 6$. Let us consider a discrete 2-polyhedron σ_2 in Table 2, for example, H4. From the 1-point configuration $\mathbf{H}(i,j,k)$ of H4, we see that $\mathbf{H}(i,j,k+1)$ can be only H9 which has a 3-polyhedron σ_3. Thus σ_2 is a face of σ_3; the faces of σ_2 are also faces of σ_3. Similarly, we can show that other discrete 2-polyhedra of H6, H7 and H8 are also faces of some discrete 3-polyhedra of H9 if we consider the possible 1-point configurations of the adjacent cubes. Let us consider discrete 1-polyhedra which are not faces of discrete 2-polyhedra in Table 2, for example, a discrete 1-polyhedron σ_1 of H2 From the 1-point configuration $\mathbf{H}(i,j,k)$ of H2, we see that $\mathbf{H}(i,j,k+1)$ can be only H7 and σ_1 is a face of the right-side discrete 2-polyhedron σ_2. Such σ_2 is a face of some discrete 3-polyhedron of H9 as we have already shown in the above. Similarly, we can show that other discrete 1-polyhedra of H3, H5 and H6 are also faces of some discrete 2-polyhedra which are faces of some discrete 3-polyhedra of H9. Finally, let us consider discrete 0-polyhedra which are not faces of any discrete 1-polyhedra in Table 2, such as a discrete 0-polyhedron σ_0 of H1. From the 1-point configuration $\mathbf{H}(i,j,k)$ of H1, we see that $\mathbf{H}(i,j,k+1)$ can be H5 or H6 which has a discrete 1-polyhedron σ_1 such that σ_0 is a face of σ_1 and σ_1 is a face of some discrete 3-polyhedron of H9.

Let us consider the cases of $m = 18, 26$. In this case, we need to check only discrete 0-, 1- and 2-polyhedra of H1, H2, H3 and H4 for $m = 18, 26$, and H6 only for $m = 18$. Similarly to the case of $m = 6$, we find possible configurations $\mathbf{H}(i,j,k+1)$ adjacent to $\mathbf{H}(i,j,k)$ of H1, H2, H3 and H4: $\mathbf{H}(i,j,k+1)$ can be only H5 or H6 for $\mathbf{H}(i,j,k)$ of H1, H7 for H2, H8 for H3, and H6 or H9 for H4. For $\mathbf{H}(i,j,k)$ of H6 with $m = 18$, we have the special configuration $\mathbf{H}(i,j,k+1)$ of H9$'$. Therefore, all discrete 0-, 1- and 2-polyhedra are faces of some discrete 3-polyhedra. *(Q.E.D.)*

Table 3: A pure discrete 2-complex $\mathbf{DCP}_m(i,j,k)$ for each of ten possible configurations $\mathbf{H}(i,j,k)$ of 1- and 0-points for $m = 6, 18, 26$. In the table, we consider $\mathbf{P}$ such that $0 \leq a \leq b \leq c$, $c > 0$.



We obtain the combinatorial boundary $\partial\mathbf{O}_m$ of a pure discrete 3-complex $\mathbf{O}_m$ following Definition 2. From Lemma 1, we can obtain $\partial\mathbf{O}_m$ more easily and directly from $\mathbf{I}^+$ without considering $\mathbf{O}_m$ or $\mathbf{C}_m$. As shown in [9], we prepare Table 3 instead of Table 2 for referring a pure discrete 2-complex $\mathbf{DCP}_m(i,j,k)$ which constitutes a part of $\mathbf{DCP}_m = \partial\mathbf{O}_m$ at each unit cube of $\mathbf{H}(i,j,k)$.

Algorithm 1

input: $\mathbf{I}^+$.
output: $\mathbf{DCP}_m$ *for each* $m = 6, 18, 26$.

begin

1. *for each* $(i,j,k) \in \mathcal{Z}^3$, *refer a pure discrete* 2*-complex* $\mathbf{DCP}_m(i,j,k)$ *to Table 3;*
2. *obtain*

$$\mathbf{DCP}_m = \bigcup_{(i,j,k)\in\mathcal{Z}^3} \mathbf{DCP}_m(i,j,k) \ .$$

end

4.4 Properties

From Lemma 1, we obtain the next property.

Property 10 *A* $\mathbf{DCP}_m$ *for each* $m = 6, 18, 26$ *is uniquely obtained from* $\mathbf{I}^+$ *by Algorithm 1 and is a pure discrete* 2*-complex.*

Let $Sk(\mathbf{DCP}_m)$ be the set of all vertices of discrete convex polyhedra in $\mathbf{DCP}_m$. By comparing $Sk(\mathbf{DCP}_m(i,j,k))$ for all $m = 6, 18, 26$ at each $(i,j,k) \in \mathcal{Z}^3$ in Table 3, we obtain the following inclusion relations; the complete proof is found in [8].

Property 11 *For any plane* $\mathbf{P}$*, we have the inclusion and equality relation*

$$Sk(\mathbf{DCM}_6) \supseteq Sk(\mathbf{DCM}_{18}) = Sk(\mathbf{DCM}_{26}).$$

5 Equivalence Relations of Discrete Planes

The aim of this section is to prove the following relations of **DAP**, namely **SP** and **NP**, $\mathbf{DMP}_m$ for $m = 6, 18, 26$ and $\mathbf{DCP}_{m'}$ for $m' = 6, 18, 26$.

Theorem 1 *Let us consider* **SP**, **NP**, $\mathbf{DMP}_m$ *for* $m = 6, 18, 26$ *and* $\mathbf{DCP}_{m'}$ *for* $m' = 6, 18, 26$ *with respect to any* $\mathbf{P}$*. Then we have*

$$\mathbf{SP} = \mathbf{DMP}_{26} = Sk(\mathbf{DCP}_6), \tag{6}$$

$$\mathbf{NP} = \mathbf{DMP}_6 = Sk(\mathbf{DCP}_{18}) = Sk(\mathbf{DCP}_{26}). \tag{7}$$

A part of the above equivalence relations has been already proved in [8]: the equivalence relation between **NP** and $Sk(\mathbf{DCP}_{26})$ (or $Sk(\mathbf{DCP}_{18})$). In this paper, we give a different proof in the following steps; first, we show the equivalence relations between $\mathbf{DMP}_{26}$ and $Sk(\mathbf{DCP}_6)$ and between $\mathbf{DMP}_6$ and $Sk(\mathbf{DCP}_{26})$ or $Sk(\mathbf{DCP}_{18})$, and then show the equivalence relations between **SP** and $\mathbf{DMP}_{26}$, and between **NP** and $\mathbf{DMP}_6$.

5.1 Relations between $\mathbf{DMP}_m$ and $\mathbf{DCP}_{m'}$

From Theorem 1 in [9], we have

$$\begin{aligned} \mathbf{DMP}_6 &= Sk(\mathbf{DCP}_{26}) \cup (Sk(\mathbf{C}_{26}) \setminus Sk(\mathbf{O}_{26})), \\ \mathbf{DMP}_{26} &= Sk(\mathbf{DCP}_6) \cup (Sk(\mathbf{C}_6) \setminus Sk(\mathbf{O}_6)). \end{aligned}$$

From Lemma 1 and Property 11, we obtain the following relations.

Lemma 2 *For any plane* $\mathbf{P}$*, we have*

$$\begin{aligned} \mathbf{DMP}_6 &= Sk(\mathbf{DCP}_{18}) = Sk(\mathbf{DCP}_{26}), \\ \mathbf{DMP}_{26} &= Sk(\mathbf{DCP}_6). \end{aligned}$$

5.2 Relations between DAP and $\mathbf{DMP}_m$

Lemma 3 *For any plane* $\mathbf{P}$*, we have*

$$\mathbf{SP} = \mathbf{DMP}_{26}. \tag{8}$$

In order to prove this lemma, we need the following lemma.

Lemma 4 *For any plane* $\mathbf{P}$ *such that* $0 \leq a \leq b \leq c$, $c > 0$, *if a point* $(u-1, v-1, w-1) \in \mathbf{I}^+$, *then*

$$\mathbf{N}_{26}(u, v, w) \subset \mathbf{I}^+.$$

(Proof) Because $(u-1, v-1, w-1) \in \mathbf{I}^+$,

$$a(u-1) + b(v-1) + c(w-1) + d \geq 0$$

from (4). Setting $(u', v', w') \in \mathbf{N}_{26}(u, v, w)$, we have $u - 1 \leq u'$, $v - 1 \leq v'$, $w - 1 \leq w'$, thus

$$au' + bv' + cw' + d \geq a(u-1) + b(v-1) + c(w-1) + d \geq 0$$

because a, b, c are not negative. *(Q.E.D.)*

(Proof of Lemma 3) For simplification, we set $w = a + b + c$ for $\mathbf{SP}$ of (2) and give the proof for $\mathbf{P}$ such that $0 \leq a \leq b \leq c$, $c > 0$. The similar proof is easily derived for other $\mathbf{P}$s. Let us consider two Euclidean planes $\mathbf{P}$ of (1) and

$$\mathbf{P}' = \{(x, y, z) \in \mathcal{R}^3 \ : \ ax + by + cz + d = a + b + c\}. \tag{9}$$

We see that $\mathbf{SP}$ is a set of lattice points between $\mathbf{P}$ and $\mathbf{P}'$. From (1) and (9), it is obvious that a point $(p, q, r) \in \mathcal{R}^3$ is on $\mathbf{P}'$ if $(p-1, q-1, r-1) \in \mathcal{R}^3$ is on $\mathbf{P}$. Geometrically, this means that there is a unit cube between $\mathbf{P}$ and $\mathbf{P}'$ such that the two vertices (p, q, r) and $(p-1, q-1, r-1)$ of the unit cube are on $\mathbf{P}$ and $\mathbf{P}'$, respectively.

Taking a point $(u, v, w) \in \mathbf{SP}$, i.e. a point (u, v, w) between $\mathbf{P}$ and $\mathbf{P}'$ (can be on $\mathbf{P}$ but not be on $\mathbf{P}'$), we have

$$0 \leq au + bv + cw + d < a + b + c$$

from (2). Thus,

$$-(a + b + c) \leq a(u-1) + b(v-1) + c(w-1) + d < 0,$$

so that $(u-1, v-1, w-1) \in \overline{\mathbf{I}^+}$. Because $(u-1, v-1, w-1) \in \mathbf{N}_{26}(u, v, w)$, we have

$$\mathbf{N}_{26}(u, v, w) \cap \overline{\mathbf{I}^+} \neq \emptyset \tag{10}$$

for any $(u, v, w) \in \mathbf{SP}$.

Taking a point $(u, v, w) \in \mathbf{I}^+ \setminus \mathbf{SP}$, we have

$$au + bw + cw + d \geq a + b + c,$$

thus

$$a(u-1)+b(v-1)+c(w-1)+d \geq 0.$$

Therefore we say $(u-1, v-1, w-1) \in \mathbf{I}^+$ and obtain

$$\mathbf{N}_{26}(u,v,w) \cap \overline{\mathbf{I}^+} = \emptyset \tag{11}$$

for any $(u,v,w) \in \mathbf{I}^+ \setminus \mathbf{SP}$ from Lemma 4.

From (10) and (11), we have (8). *(Q.E.D.)*

Similarly to the above, we show the following lemma for naive planes.

Lemma 5 *For any plane* $\mathbf{P}$*, we have*

$$\mathbf{NP} = \mathbf{DMP}_6. \tag{12}$$

In order to prove this lemma, we need the following lemma.

Lemma 6 *For any plane* $\mathbf{P}$ *such that* $0 \leq a \leq b \leq c$*,* $c > 0$*, if a point* $(u,v,w-1) \in \mathbf{I}^+$*, then*

$$\mathbf{N}_6(u,v,w) \subset \mathbf{I}^+.$$

(Proof) Because $(u,v,w-1) \in \mathbf{I}^+$,

$$au+bv+c(w-1)+d \geq 0$$

from (4), thus

$$a(u-1)+bv+cw+d \geq au+b(v-1)+cw+d \geq au+bv+c(w-1)+d \geq 0$$

because $0 \leq a \leq b \leq c$. Setting $(u',v',w') \in \mathbf{N}_6(u,v,w)$, so that $u-1 \leq u'$, $v-1 \leq v'$, $w-1 \leq w'$, we then obtain

$$au'+bv'+cw'+d \geq au+bv+c(w-1)+d \geq 0.$$

(Q.E.D.)

(Proof of Lemma 5) For simplification, we set $w=c$ for $\mathbf{NP}$ of (2) and give the proof for $\mathbf{P}$ such that $0 \leq a \leq b \leq c$, $c > 0$. The similar proof is easily derived for other $\mathbf{P}$s. Let us consider two Euclidean planes $\mathbf{P}$ of (1) and

$$\mathbf{P}'' = \{(x,y,z) \in \mathcal{R}^3 \ : \ ax+by+cz+d = c\}. \tag{13}$$

We see that $\mathbf{NP}$ is a set of lattice points between $\mathbf{P}$ and $\mathbf{P}''$. From (1) and (13), it is obvious that a point $(p,q,r) \in \mathcal{R}^3$ is on $\mathbf{P}''$ if $(p,q,r-1) \in \mathcal{R}^3$ is on $\mathbf{P}$.

Taking a point $(u, v, w) \in \mathbf{NP}$, i.e. a point (u, v, w) between $\mathbf{P}$ and $\mathbf{P}''$ (can be on $\mathbf{P}$ but not be on $\mathbf{P}''$), we have

$$0 \leq au + bv + cw + d < c$$

from (2), thus,

$$-c \leq au + bv + c(w-1) + d < 0,$$

so that $(u, v, w-1) \in \overline{\mathbf{I}^+}$. Because $(u, v, w-1) \in \mathbf{N}_6(u, v, w)$, we have

$$\mathbf{N}_6(u, v, w) \cap \overline{\mathbf{I}^+} \neq \emptyset \tag{14}$$

for any $(u, v, w) \in \mathbf{NP}$.

Taking a point $(u, v, w) \in \mathbf{I}^+ \setminus \mathbf{NP}$, we have

$$au + bw + cw + d \geq c,$$

thus

$$au + bv + c(w-1) + d \geq 0.$$

Therefore we say $(u-1, v-1, w-1) \in \mathbf{I}^+$ and then obtain

$$\mathbf{N}_6(u, v, w) \cap \overline{\mathbf{I}^+} = \emptyset \tag{15}$$

for any $(u, v, w) \in \mathbf{I}^+ \setminus \mathbf{NP}$ from Lemma 6.

From (14) and (15), we have (12). *(Q.E.D.)*

6 New Aspects of Properties for Discrete Planes

From Theorem 1, all properties which are presented for each of **NP**, **SP**, $\mathbf{DMP}_m$ and $\mathbf{DCP}_{m'}$ for $m, m' = 6, 18, 26$ are considered for any discrete planes. Bicubes and tricubes of Properties 3 and 5 are, for example, presented for **NP**. Because $\mathbf{NP} = Sk(\mathbf{DCP}_{18}) = Sk(\mathbf{DCP}_{26})$ from Theorem 1, we can obtain discrete combinatorial planes $\mathbf{DCP}_{18}$ and $\mathbf{DCP}_{26}$ for bicubes and tricubes as shown in Figure 4 (a) and (b), and Figures 5 and 6. From Table 3, we also know a $\mathbf{DCP}_6$ which corresponds to each $\mathbf{DCP}_{18}$ or $\mathbf{DCP}_{26}$ for a 1-point configuration at a unit cubic region. Therefore, we can also obtain bicubes and tricubes for $\mathbf{DCP}_6$, namely **SP** in Figures 4 (c) and 7.

Let $\mathbf{K}$ be a polyhedral complex. For each vertex $\boldsymbol{v} \in Sk(\mathbf{K})$, the subcomplex consisting of all convex polyhedra σ of $\mathbf{K}$ which contain $\boldsymbol{v}$ such that $\boldsymbol{v} \in Sk(\{\sigma\})$ is called the star $St(\boldsymbol{v})$ of $\boldsymbol{v}$ in $\mathbf{K}$ [1]. For each tricube in Figures 5, 6 and 7, we also draw the star of a white vertex which is the central point of the tricube. We then obtain the following property which is similar to Properties 7 and 8.

Property 12 *Any* $\mathbf{DCP}_m$ *for* $m = 6, 18, 26$ *is a 2-dimensional combinatorial manifold.*

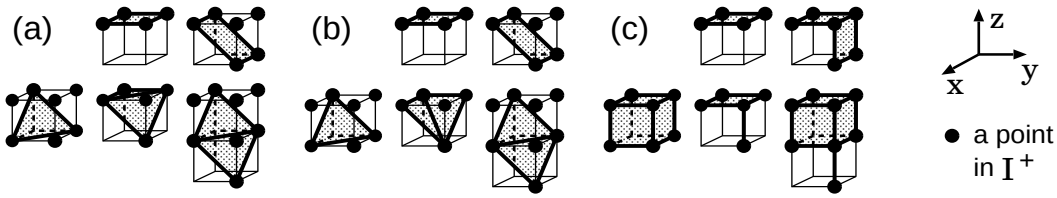


Figure 4: Discrete combinatorial planes $\mathbf{DCP}_{18}$ (a), $\mathbf{DCP}_{26}$ (b) and $\mathbf{DCP}_{6}$ (c) for all five bicubes in Figure 1.

The total number of different configurations of stars is less than the number of tricubes, i.e. 40, because the same configurations of stars clearly appear in Figures 5, 6 and 7. Precisely, there are 4, 29 and 34 different configurations of stars for $m = 6, 18, 26$ respectively. The same 4 configurations of stars for $m = 6$, i.e. stars for **SP**, are also presented in [5].

7 Conclusions

In this paper, we consider the three different representations for discrete planes in $\mathcal{Z}^3$, **DAP**, or **SP** and **NP**, $\mathbf{DMP}_m$ for $m = 6, 18, 26$ and $\mathbf{DCP}_{m'}$ for $m' = 6, 18, 26$ and show their equivalence relations in Theorem 1. Thus, afterwards, we succeeded to view all properties which had been obtained for each representation of discrete planes as the properties for the other equivalent representations. For example, bicubes and tricubes had been presented only for **NP**s [4, 6] and in this paper we extended bicubes and tricubes for **SP**s. In addition, we also succeeded to show that discrete planes are 2-dimensional combinatorial manifolds in Property 12, because $\mathbf{DCP}_{m'}$s enable us to investigate combinatorial topological structures of discrete planes.

From the equivalence relations, we can also transform the representation of a discrete plane from one to another. Therefore, we can choose one or more than one representations whose properties are suitable for our task. For example, given a subset $\mathbf{V} \subset \mathcal{Z}^3$, let us consider global polyhedrization of $\mathbf{V}$ [7], i.e. looking for a collection of pieces of discrete planes whose half spaces contain $\mathbf{V}$ such that a set of lattice points of discrete plane pieces is equivalent to the set of border points of $\mathbf{V}$. The goal of this problem is to obtain a set of **DAP**s from $\mathbf{V}$. The procedure will be in the following; first we extract the border points of $\mathbf{V}$ as the initial set which is considered as a set of $\mathbf{DMP}_6$s (resp. $\mathbf{DMP}_{26}$s), and then tackle the recognition problem which corresponds to the transformation from $\mathbf{DMP}_6$s (resp. $\mathbf{DMP}_{26}$s) to **NP**s (resp. **SP**s). The equivalence relations guarantee the correspondence between different representations of discrete planes during such transformation.

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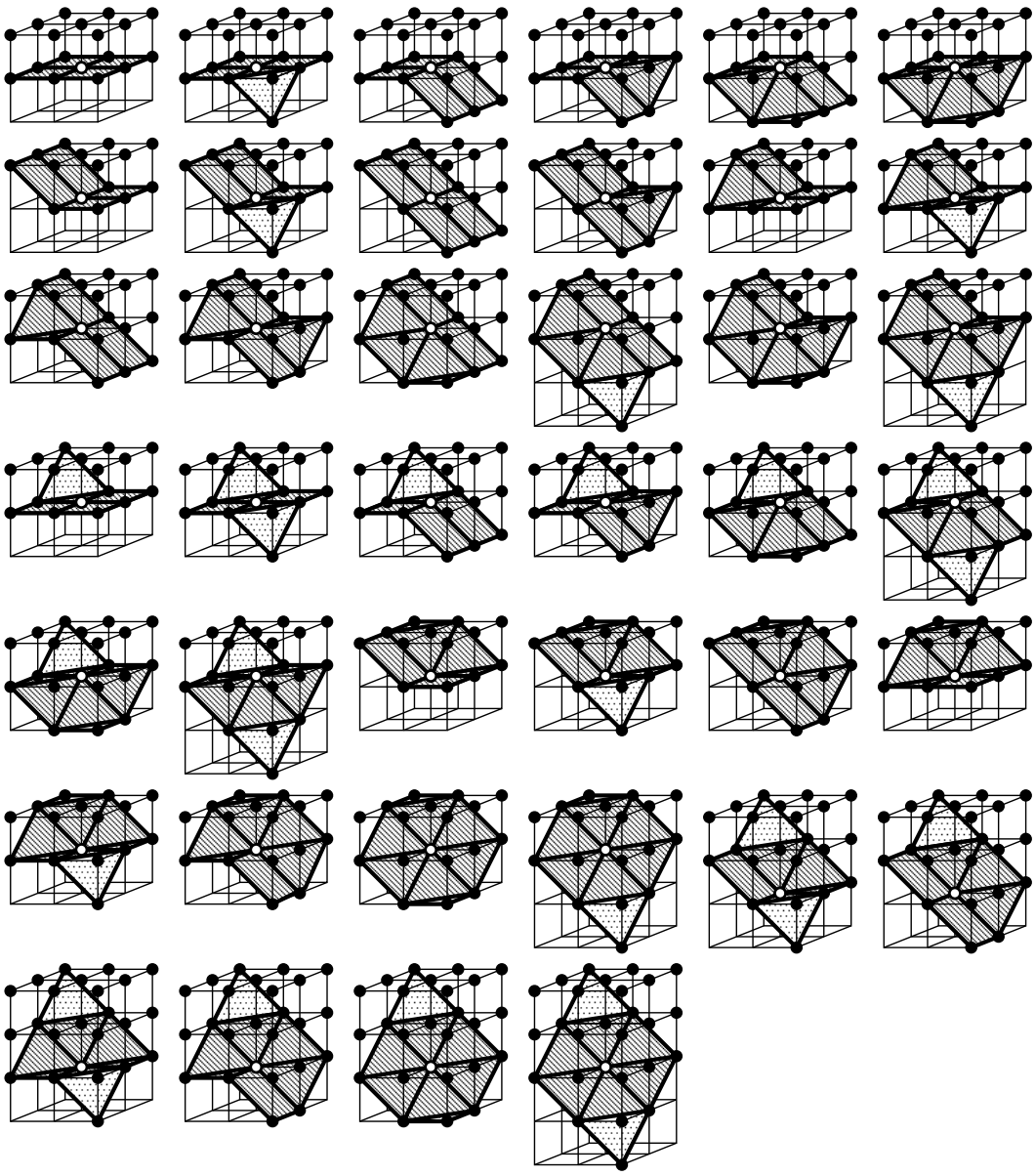

Figure 5: Discrete combinatorial planes $\mathbf{DCP}_{18}$ for all 40 tricubes in Figure 2. The star of each white point is also shown a part with diagonal lines in the figures.

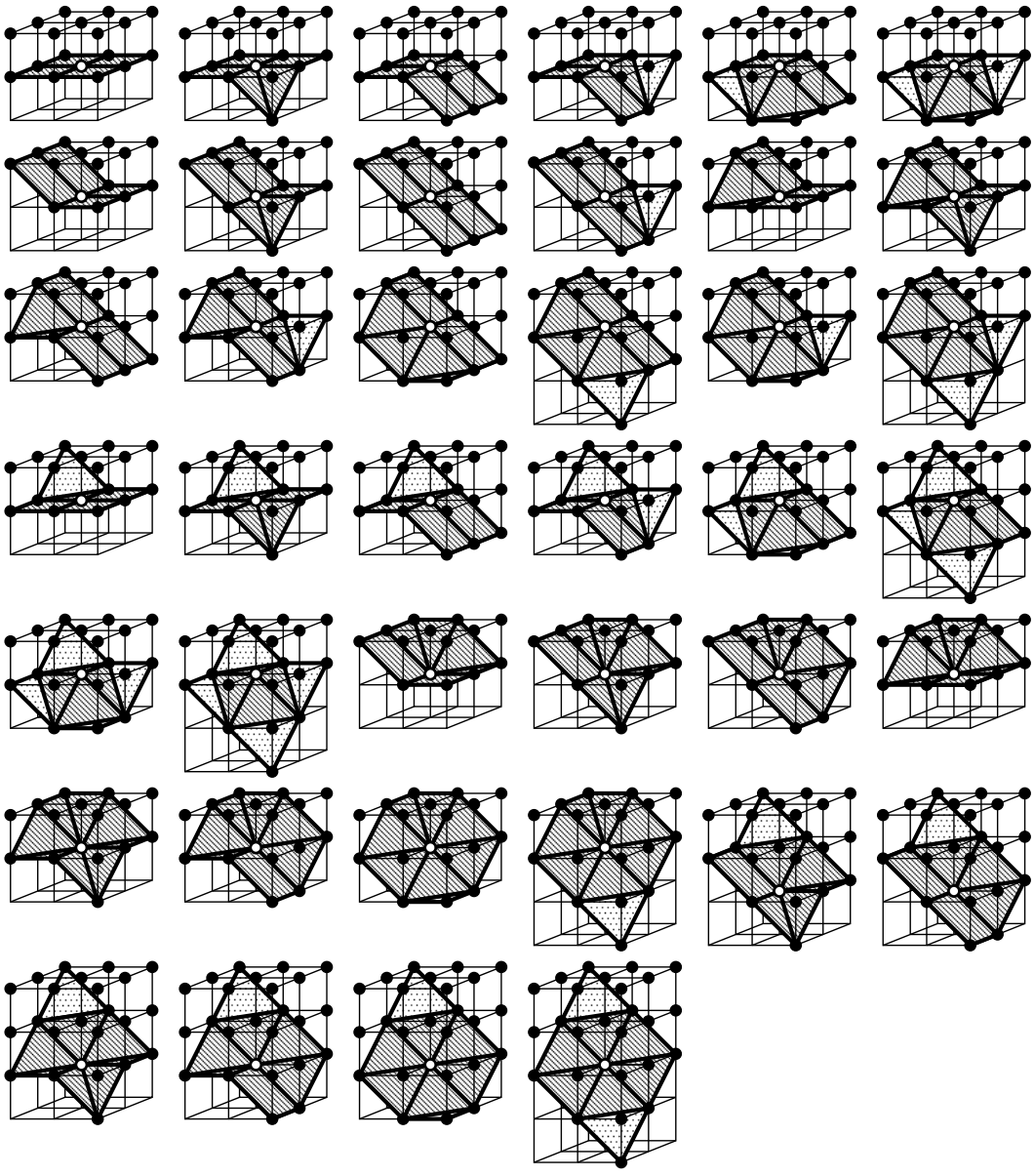

Figure 6: Discrete combinatorial planes $\mathbf{DCP}_{26}$ for all 40 tricubes in Figure 2. The star of each white point is also shown a part with diagonal lines in the figures.

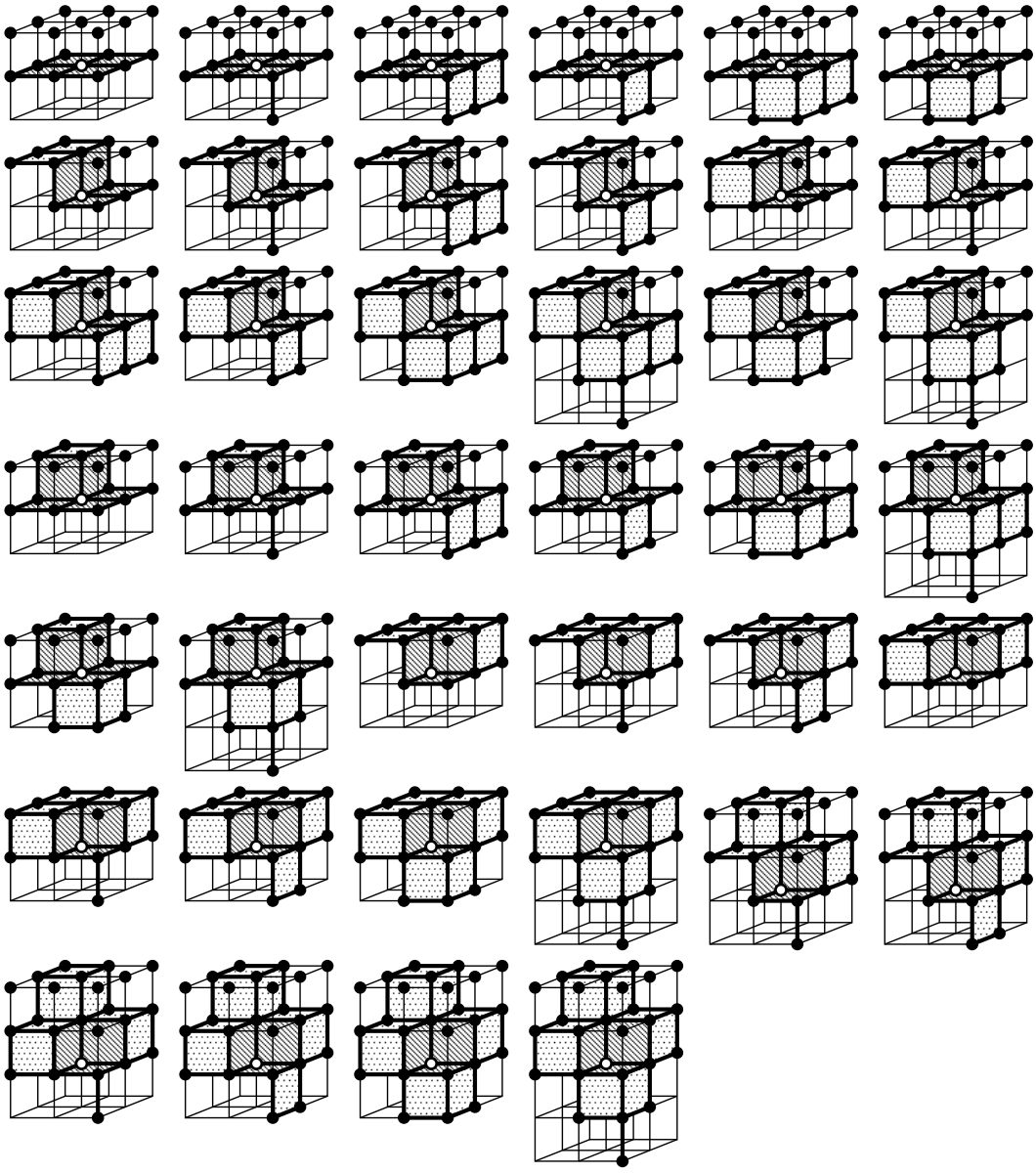

Figure 7: Discrete combinatorial planes $\mathbf{DCP}_6$ for all 40 tricubes in Figure 2. The star of each white point is also shown a part with diagonal lines in the figures.

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