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# VOLUME FUNCTIONS ON BLOW-UPS AND SESHADRI CONSTANTS 

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#### Abstract

We exhibit some relations between the Seshadri constant of an ample divisor along a closed subscheme and the behaviour of the volume function on the corresponding blow-up. As an application, we give an equivalent formulation of Nagata's conjecture in terms of the differentiability of a real valued function.


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## 1. Introduction

Let $X$ be a projective variety of dimension $d \geq 1$ over an algebraically closed field $k$. The volume of a Cartier divisor $D$ on $X$ measures the asymptotic growth of the linear systems $|n D|, n \in \mathbb{N}$ :

$$
\operatorname{vol}(D)=\underset{n \rightarrow \infty}{\limsup } \frac{h^{0}(X, n D)}{n^{d} / d!}
$$

This invariant depends only on the numerical equivalence class of $D$, and can be extended to define a continuous function

$$
\operatorname{vol}: N^{1}(X)_{\mathbb{R}} \rightarrow \mathbb{R}
$$

on the real Néron-Severi group $N^{1}(X)_{\mathbb{R}}=N^{1}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ of $X$ (see [12, section 2.2.C] and [5, section 2.4] for details). The $\operatorname{big}$ cone $\operatorname{Big}(X) \subseteq N^{1}(X)_{\mathbb{R}}$ is the convex cone of classes $\alpha \in N^{1}(X)_{\mathbb{R}}$ with $\operatorname{vol}(\alpha)>0$. The study of the volume function is an interesting topic of research, as emphasized in the survey [9]. For example, it is believed that vol is real analytic on some "large" open subset of the big cone [9, Conjecture 2.18]. The differentiability of vol on $\operatorname{Big}(X)$ was established by Boucksom, Favre and Jonsson [3] in characteristic zero and by Cutkosky [5] in general.

In this paper we are interested in the behaviour of the volume function on blowups. More precisely, let $D$ be an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$ and let $Z \mp X$ be a closed proper subscheme, with defining ideal $I_{Z}$. Let $\pi_{Z}: \widetilde{X} \rightarrow X$ be the blow up along $I_{Z}$ and let $E=\pi_{Z}^{-1}(Z)$ be the exceptional divisor, with the scheme structure defined by the ideal $I_{Z} \mathcal{O}_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}(1)$. For $t \in \mathbb{R}$, we let $D_{t}=\pi_{Z}^{*} D-t E$. The purpose of this short article is to observe that the behaviour of the function

$$
\varphi_{D, Z}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}, t \mapsto \operatorname{vol}\left(D_{t}\right)
$$

is closely related to the Seshadri constant of $D$ along $Z$, defined to be

$$
\epsilon_{Z}(D)=\sup \left\{t \in \mathbb{R} \mid D_{t} \text { is ample }\right\}
$$

Seshadri constants were originally introduced by Demailly [7] in the case where $Z=\{x\}$ is a closed point. They have been widely studied since then, at least when $\operatorname{dim} Z=0$, and have become an important tool to understand the geometry of projective varieties. We refer the reader to [2] and [12, Chapter 5] for background and motivation about this invariant. We shall prove the following theorem and study some of its consequences.

Theorem 1.1. Assume that the line bundle $\mathcal{O}_{E}(-E)=\mathcal{O}_{\tilde{X}}(1)_{\mid E}$ is nef on $E$. Then

$$
\epsilon_{Z}(D)=\sup \left\{t \geq 0 \mid \varphi_{D, Z} \text { is polynomial on }[0, t]\right\} .
$$

We remark that the assumption on $\mathcal{O}_{E}(-E)$ is automatically satisfied when $Z=\left\{x_{1}, \ldots, x_{r}\right\}$ is a collection of $r \geq 1$ distinct closed points in $X$, since in that case the line bundle $\mathcal{O}_{\tilde{X}}(1)$ is relatively ample. When the inclusion $Z \nsubseteq X$ is a regular embedding (e.g. when $X$ and $Z$ are smooth), $\mathcal{O}_{E}(-E)$ is nef if and only if the anti-conormal bundle $I_{Z} / I_{Z}^{2}$ is nef. We shall actually prove a generalization of Theorem 1.1, valid without any positivity assumption on $\mathcal{O}_{E}(-E)$ (see Theorem 3.1). The proof of Theorem 1.1 builds on ideas of McKinnon and Roth [13], and is based on a criterion for amplitude in terms of higher cohomological functions due to de Fernex, Küronya and Lazarsfeld [6] and to Murayama [15] (see Theorem 2.1 below). We use the assumption on $\mathcal{O}_{E}(-E)$ to guarantee the vanishing of the higher cohomological functions $\widehat{h}^{i}\left(\widetilde{X}, D_{t}\right)$ when $i \geq 2$ and $t \geq 0$ (see Lemma 3.3), in order to combine the main result of $[6]$ with the asymptotic Riemann-Roch theorem. This leads to the identities

$$
\epsilon_{Z}(D)=\sup \left\{t \geq 0 \mid \operatorname{vol}\left(D_{t}\right)=D_{t}^{\operatorname{dim} X}\right\}=\sup \left\{t \geq 0 \mid \widehat{h}^{1}\left(\widetilde{X}, D_{t}\right)=0\right\}
$$

from which we easily deduce Theorem 1.1.
Let $\gamma_{Z}(D)$ be the supremum of the real numbers $t$ such that $\varphi_{D, Z}(t)>0$. We consider the function

$$
\phi_{D, Z}:\left[0, \gamma_{Z}(D)\right] \rightarrow \mathbb{R}, t \mapsto\left\langle D_{t}^{\operatorname{dim} X-1}\right\rangle \cdot E
$$

where $\left\langle D_{t}^{\operatorname{dim} X-1}\right\rangle$ denotes the positive intersection product introduced in [3, section 2] (see also [5, section 4]). By [5, Theorem 5.6], the function $\varphi_{D, Z}$ is differentiable on $\left(0, \gamma_{Z}(D)\right)$ and its derivative is given by $\varphi_{D, Z}^{\prime}=-\operatorname{dim}(X) \phi_{D, Z}$. In particular, Theorem 1.1 remains true if $\varphi_{D, Z}$ is replaced by $\phi_{D, Z}$.

Corollary 1.2. If $\mathcal{O}_{E}(-E)$ is nef, then

$$
\epsilon_{Z}(D)=\sup \left\{t \geq 0 \mid \phi_{D, Z} \text { is polynomial on }[0, t]\right\} .
$$

When $\tilde{X}$ is a smooth surface (for example when $X$ and $Z$ are smooth and $\operatorname{dim} X=2), \varphi_{D, Z}$ is a piecewise polynomial function of degree at most 2 by a theorem of Bauer, Küronya and Szemberg [1, Theorem]. It follows that $\varphi_{D, Z}$ and $\phi_{D, Z}$ are polynomial on an interval $[0, t]$ if and only if $\phi_{D, Z}$ is differentiable on $(0, t)$.

Corollary 1.3. Assume that $\widetilde{X}$ is a smooth surface and that $\mathcal{O}_{E}(-E)$ is nef. Then

$$
\epsilon_{Z}(D)=\sup \left\{t \geq 0 \mid \phi_{D, Z} \text { is differentiable on }(0, t)\right\} .
$$

It is worth to note that when $\tilde{X}$ is a smooth surface, the function $\phi_{D, Z}$ is simply given by $\phi_{D, Z}(t)=P_{t} \cdot E$ for all $t \in\left[0, \gamma_{Z}(D)\right)$, where $P_{t}$ is the positive part in the Zariski decomposition of $D_{t}=\pi_{Z}^{*} D-t E$. This follows from the definition of the positive intersection product, as explained in [3, section 3.4]. Let us also mention that when $X$ is a smooth surface and $Z \varsubsetneqq X$ is a smooth irreducible curve, Theorem 1.1 and Corollary 1.3 remain valid even if $\mathcal{O}_{E}(-E)$ is not nef (see Corollary 3.5).

Let us now focus on zero-dimensional subschemes, that is $Z=\left\{x_{1}, \ldots, x_{r}\right\}$ is a collection of closed points in $X$. When $k$ is uncountable and the points $x_{1}, \ldots, x_{r}$
are in very general position, we shall see that the function $\varphi_{D, Z}$ depends only on $D$ and $r$ (see Proposition 4.2 for a precise statement, and Proposition 4.1 for a more general result valid for higher dimensional subschemes). We denote it by $\varphi_{D, r}$. It is differentiable on some non-empty interval $\left[0, \gamma_{r}(D)\right)$, and we let $\phi_{D, r}=$ $-\varphi_{D, r}^{\prime} / \operatorname{dim} X$. In particular, we recover from Theorem 1.1 the well-known fact that Seshadri constants at $r$ very general points all take the same value $\epsilon_{r}(D)$. Indeed, we have

$$
\epsilon_{r}(D)=\sup \left\{t \geq 0 \mid \varphi_{D, r} \text { is polynomial on }[0, t]\right\} .
$$

Of course we also have analogues of Corollaries 1.2 and 1.3: for example, if $X$ is a smooth surface then

$$
\begin{equation*}
\epsilon_{r}(D)=\sup \left\{t \geq 0 \mid \phi_{D, r} \text { is differentiable on }(0, t)\right\} . \tag{1}
\end{equation*}
$$

A celebrated conjecture of Nagata predicts that for $r \geq 9$ very general points $x_{1}, \ldots, x_{r}$ in $\mathbb{P}_{\mathbb{C}}^{2}$ and for any integral curve $C \subset \mathbb{P}_{\mathbb{C}}^{2}$, we have

$$
\operatorname{deg}(C) \geq \frac{1}{\sqrt{r}} \sum_{i=1}^{r} \operatorname{mult}_{x_{i}} C
$$

This conjecture was settled by Nagata when $r$ is a perfect square, but the general case remains open despite many attempts in the past decades. It can be formulated in terms of Seshadri constants as follows: if $D$ is a line in $\mathbb{P}_{\mathbb{C}}^{2}$, then $\epsilon_{r}(D) \geq 1 / \sqrt{r}$ whenever $r \geq 9$ ([12, Remark 5.1.14]). In view of (1), we have the following equivalent formulation of
Nagata's conjecture. Let $X=\mathbb{P}_{\mathbb{C}}^{2}$ and $D$ be a line. Then for any $r \geq 9$, the function $\phi_{D, r}$ is differentiable on $(0,1 / \sqrt{r})$.

We mention that similar reformulations of Nagata's conjecture can also be derived from the computation of Newton-Okounkov bodies of line bundles on blow-ups of $\mathbb{P}_{\mathbb{C}}^{2}$ obtained by Eckl ([8, Theorems 3.4 and 3.5]).
Organization of the paper. We recall some preliminary results on higher cohomological functions in section 2. We then prove Theorem 1.1 in section 3. In section 4 we study the behaviour of $\varphi_{D, Z}$ when $Z$ varies in a flat family of subschemes (Proposition 4.1). In the case $\operatorname{dim} Z=0$, we prove that the function $\varphi_{D, Z}=\varphi_{D, r}$ depends only on $D$ and $r=\operatorname{card}(Z)$ when $Z$ consists of very general points (Proposition 4.2).

## 2. Conventions and background

Throughout this paper we work over an algebraically closed field $k$. We denote by $\operatorname{Div}(X)$ the group of Cartier divisors on a projective scheme $X$. A $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$ is an element of $\operatorname{Div}(X)_{\mathbb{R}}:=\operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. We let $N^{1}(X)_{\mathbb{R}}$ be the real Néron-Severi space of $X$ ([12, section 1.3.B]). A projective variety is a reduced and irreducible projective scheme.
2.1. Higher cohomological functions. Let $X$ be a projective variety of dimension $d \geq 1$. For any integer $0 \leq i \leq d$, we denote by $\widehat{h}^{i}: \operatorname{Div}(X) \rightarrow \mathbb{R}$ the higher cohomological function introduced by Küronya in [11]: for any $D \in \operatorname{Div}(X)$ we have

$$
\widehat{h}^{i}(X, D)=\limsup _{n \rightarrow+\infty} \frac{h^{i}(X, n D)}{n^{d} / d!}
$$

where $h^{i}(X, n D)=h^{i}\left(X, \mathcal{O}_{X}(n D)\right)$ is the dimension of $H^{i}\left(X, \mathcal{O}_{X}(n D)\right)$ as a $k$ vector space. Note that $\widehat{h}^{0}(X, D)=\operatorname{vol}(D)$ coincides with the volume of $D$. Moreover, the functions $\widehat{h}^{i}$ are homogeneous of degree $d$ and induce well defined functions

$$
\widehat{h}^{i}: \operatorname{Div}(X)_{\mathbb{R}} \rightarrow \mathbb{R}
$$

for $0 \leq i \leq d$, that are continuous on every finite-dimensional $\mathbb{R}$-linear subspace with respect to any norm (see [11, Corollary 5.3] and [4, Proposition 3.4.8]). We will use the following theorem characterizing ampleness in terms of vanishing of higher cohomological functions. It was proved by de Fernex, Küronya and Lazarsfeld [6, Theorem A] over the complex numbers, and generalized in arbitrary characteristic by Murayama [15].

Theorem 2.1 ([15], Theorem B). Let $D, A \in \operatorname{Div}(X)_{\mathbb{R}}$ be two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on $X$, with $A$ ample. Then $D$ is ample if and only if there exists a real $\gamma>0$ such that

$$
\widehat{h}^{i}(X, D-t A)=0
$$

for all $i \in\{1, \ldots, d\}$ and $t \in(0, \gamma)$.
2.2. Cohomology of nef divisors. We need the following technical lemma for the proof of Theorem 1.1.

Lemma 2.2. Let $A_{1}, \ldots, A_{\ell} \in \operatorname{Div}(X)$ be nef Cartier divisors on a projective scheme $X$ of dimension $d \geq 0$. Then for any coherent sheaf $\mathcal{F}$ on $X$, there exists a real number $C$ such that

$$
h^{i}\left(X, \mathcal{O}_{X}\left(n_{1} A_{1}+\ldots+n_{\ell} A_{\ell}\right) \otimes_{\mathcal{O}_{X}} \mathcal{F}\right) \leq C \max _{1 \leq j \leq \ell} n_{j}^{d-i}
$$

for any integers $i, n_{1}, \ldots, n_{\ell} \geq 0$.
Proof. We adapt the arguments of [12, Theorem 1.4.40]. We may assume by induction that the statement is true for any projective scheme of dimension $\leq d-1$. For any $\mathbf{n}=\left(n_{1}, \ldots, n_{\ell}\right) \in \mathbb{N}^{\ell}$, we let $\mathbf{n} A \in \operatorname{Div}(X)$ be the nef Cartier divisor given by $\mathbf{n} A=\sum_{j=1}^{\ell} n_{j} A_{j}$. By Fujita's vanishing theorem [12, Theorem 1.4.35 and Remark 1.4.36], there exists a very ample divisor $H \in \operatorname{Div}(X)$ which does not contain any subvariety of $X$ defined by the associated primes of $\mathcal{F}$ and such that

$$
H^{i}\left(X, \mathcal{O}_{X}(H+\mathbf{n} A) \otimes_{\mathcal{O}_{X}} \mathcal{F}\right)=0
$$

for any $i \geq 1$ and $\mathbf{n} \in \mathbb{N}^{\ell}$. We have an exact sequence of sheaves

$$
0 \rightarrow \mathcal{O}_{X}(\mathbf{n} A) \otimes_{\mathcal{O}_{X}} \mathcal{F} \rightarrow \mathcal{O}_{X}(H+\mathbf{n} A) \otimes_{\mathcal{O}_{X}} \mathcal{F} \rightarrow \mathcal{O}_{H}(H+\mathbf{n} A) \otimes_{\mathcal{O}_{H}} \mathcal{F}_{\mid H} \rightarrow 0
$$

from which we obtain

$$
h^{i}\left(X, \mathcal{O}_{X}(\mathbf{n} A) \otimes_{\mathcal{O}_{X}} \mathcal{F}\right) \leq h^{i-1}\left(H, \mathcal{O}_{H}(H+\mathbf{n} A) \otimes_{\mathcal{O}_{H}} \mathcal{F}_{\mid H}\right)=O\left(\max _{1 \leq j \leq \ell} n_{j}^{d-i}\right)
$$

for any integer $i \geq 1$ by induction. Since the Euler characteristic

$$
\chi\left(\mathcal{O}_{X}(\mathbf{n} A) \otimes_{\mathcal{O}_{X}} \mathcal{F}\right)=\sum_{i=0}^{d}(-1)^{i} h^{i}\left(X, \mathcal{O}_{X}(\mathbf{n} A) \otimes_{\mathcal{O}_{X}} \mathcal{F}\right)
$$

is a polynomial of total degree at most $d$ in $n_{1}, \ldots, n_{\ell}$ [10, I.1], we deduce that $h^{0}\left(X, \mathcal{O}_{X}(\mathbf{n} A) \otimes_{\mathcal{O}_{X}} \mathcal{F}\right)=O\left(\max _{1 \leq j \leq \ell} n_{j}^{d}\right)$ and the lemma is proved.

## 3. Proof of Theorem 1.1

Let $X$ be a projective variety of dimension $d \geq 1$ and let $D \in \operatorname{Div}(X)_{\mathbb{R}}$ be an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. We fix a closed proper subscheme $Z \nsubseteq X$, and we denote by $\pi_{Z}: \widetilde{X} \rightarrow X$ the blow-up of $X$ along the ideal sheaf $I_{Z}$ defining $Z$ in $X$. We let $E$ be the exceptional divisor and for any $t \in \mathbb{R}$, we consider the $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D_{t}=\pi_{Z}^{*} D-t E$ on $\widetilde{X}$. Recall that the Seshadri constant of $D$ along $Z$ is

$$
\epsilon_{Z}(D)=\sup \left\{t \in \mathbb{R}_{\geq 0} \mid D_{t} \text { is ample }\right\}
$$

Since $\mathcal{O}_{\tilde{X}}(-E)=\mathcal{O}_{\tilde{X}}(1)$ is relatively ample with respect to $\pi_{Z}, \epsilon_{Z}(D)>0$ is positive by [12, Proposition 1.7.10]. We also consider the invariant $\omega_{Z}(D) \in[0, \infty]$ defined by

$$
\omega_{Z}(D)=\sup \left\{t \in \mathbb{R}_{\geq 0} \mid D_{t \mid E} \text { is ample }\right\}
$$

where by a slight abuse of notation we denoted by $D_{t \mid E}$ the image in $N^{1}(E)_{\mathbb{R}}$ of the class of $D_{t}$ in $N^{1}(X)_{\mathbb{R}}$. Note that $\omega_{Z}(D) \geq \epsilon_{Z}(D)$. The goal of this section is to prove the following result, which generalizes Theorem 1.1 in the introduction.

Theorem 3.1. We have

$$
\epsilon_{Z}(D)=\sup \left\{t \in\left[0, \omega_{Z}(D)\right) \mid \varphi_{D, Z} \text { is polynomial on }[0, t]\right\} .
$$

Note that when $\mathcal{O}_{E}(-E)$ is nef, then $\omega_{Z}(D)=\infty$. Hence Theorem 3.1 indeed implies Theorem 1.1. We shall derive it from the following proposition.

Proposition 3.2. We have

$$
\begin{aligned}
\epsilon_{Z}(D) & =\sup \left\{t \in\left[0, \omega_{Z}(D)\right) \mid \operatorname{vol}\left(D_{t}\right)=D_{t}^{d}\right\} \\
& =\sup \left\{t \in\left[0, \omega_{Z}(D)\right) \mid \widehat{h}^{1}\left(\widetilde{X}, D_{t}\right)=0\right\}
\end{aligned}
$$

When $Z=\{x\}$ is a point, this statement is implicitly proved by McKinnon and Roth in [13, Proof of Theorem 9.1]. We shall prove Proposition 3.2 by combining the arguments of [13] with Lemma 2.2. We need the following lemma, which improves and generalizes [13, Lemma 4.1].

Lemma 3.3. For any integer $i \in[2, d]$ and any real number $t \in\left[0, \omega_{Z}(D)\right]$, we have $\widehat{h}^{i}\left(\widetilde{X}, D_{t}\right)=0$. Moreover,

$$
\operatorname{vol}\left(D_{t}\right)=D_{t}^{d}+\widehat{h}^{1}\left(\widetilde{X}, D_{t}\right) .
$$

Proof. We first assume that $D \in \operatorname{Div}(X)$ is a Cartier divisor. Let $i \in\{2, \ldots, d\}$ and $t \in\left[0, \omega_{Z}(D)\right]$. In order to prove that $\widehat{h}^{i}\left(\widetilde{X}, D_{t}\right)=0$, we may assume that $t \in\left(0, \omega_{Z}(D)\right) \cap \mathbb{Q}$ by continuity of $\widehat{h}^{i}$. By homogeneity of $\widehat{h}^{i}$, we may even assume that $t$ is a positive integer. For any integer $n \geq 1$, we have an exact sequence

$$
\begin{equation*}
0 \rightarrow \mathcal{O}_{\tilde{X}}\left(n D_{t}\right) \rightarrow \mathcal{O}_{\tilde{X}}\left(n D_{0}\right) \rightarrow \mathcal{O}_{\tilde{X}}\left(n D_{0}\right) \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{n t E} \rightarrow 0 \tag{2}
\end{equation*}
$$

where $n t E$ denotes the subscheme of $\widetilde{X}$ defined by the $n t$-th power of the ideal sheaf $I_{E}$ of the Cartier divisor $E$. Since $D$ is ample, we have $H^{j}\left(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\left(n D_{0}\right)\right)=0$ for all $n \gg 1$ and $j>0$. It follows from (2) that

$$
\begin{equation*}
H^{i}\left(\widetilde{X}, n D_{t}\right)=H^{i-1}\left(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\left(n D_{0}\right) \otimes \mathcal{O}_{n t E}\right), \tag{3}
\end{equation*}
$$

for $n$ large enough. For any integer $\ell \geq 1$ we have $\mathcal{O}_{E}(-\ell E)=I_{E}^{\ell} / I_{E}^{\ell+1}$. This yields an exact sequence of sheaves on $\widetilde{X}$
$0 \rightarrow \mathcal{O}_{\widetilde{X}}\left(n D_{0}\right) \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{E}(-\ell E) \rightarrow \mathcal{O}_{\widetilde{X}}\left(n D_{0}\right) \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{(\ell+1) E} \rightarrow \mathcal{O}_{\widetilde{X}}\left(n D_{0}\right) \otimes \mathcal{O}_{\widetilde{X}} \mathcal{O}_{\ell E} \rightarrow 0$, from which we obtain
$h^{i-1}\left(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\left(n D_{0}\right) \otimes \mathcal{O}_{(\ell+1) E}\right) \leq h^{i-1}\left(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\left(n D_{0}\right) \otimes \mathcal{O}_{\ell E}\right)+h^{i-1}\left(E, \mathcal{O}_{E}\left(n D_{0}-\ell E\right)\right)$.
Combining this inequality with (3), we get

$$
\begin{equation*}
h^{i}\left(\widetilde{X}, n D_{t}\right)=h^{i-1}\left(\widetilde{X}, \mathcal{O}_{\tilde{X}}\left(n D_{0}\right) \otimes \mathcal{O}_{n t E}\right) \leq \sum_{\ell=0}^{n t-1} h^{i-1}\left(E, \mathcal{O}_{E}\left(n D_{0}-\ell E\right)\right) \tag{4}
\end{equation*}
$$

Let $0 \leq \ell \leq n t-1$ be an integer and let $k=\lceil\ell / t\rceil$ be the least integer with $k \geq \ell / t$. We have

$$
\mathcal{O}_{E}\left(n D_{0}-\ell E\right)=\mathcal{O}_{E}\left((n-k) D_{0}+k D_{t}+j E\right)
$$

where $j=t k-\ell \in\{0, \cdots, t\}$. Since $\mathcal{O}_{E}\left(D_{0}\right)$ and $\mathcal{O}_{E}\left(D_{t}\right)$ are nef, Lemma 2.2 applied with $\mathcal{F}=\mathcal{O}_{E}(j E)$ for all possible values of $j$ gives a real number $C>0$ such that

$$
h^{i-1}\left(E, \mathcal{O}_{E}\left(n D_{0}-\ell E\right)\right) \leq C n^{d-i}
$$

for all $n \in \mathbb{N}$ and $\ell \in\{0, \ldots, n t-1\}$. By (4), it follows that

$$
\widehat{h}^{i}\left(\widetilde{X}, D_{t}\right)=\lim _{n \rightarrow+\infty} \frac{h^{i}\left(X, n D_{t}\right)}{n^{d} / d!}=0
$$

By the asymptotic Riemann-Roch theorem, we have $\operatorname{vol}\left(D_{t}\right)-\widehat{h}^{1}\left(\widetilde{X}, D_{t}\right)=D_{t}^{d}$, and the lemma is proved in the case where $D \in \operatorname{Div}(X)$.

In general, there exist Cartier divisors $D_{1}, \ldots, D_{\ell} \in \operatorname{Div}(X)$ such that $D \in V:=$ $\operatorname{Span}_{\mathbb{R}}\left(D_{1}, \ldots, D_{\ell}\right)$, and there exists a sequence $\left(D_{n}\right)_{n \in \mathbb{N}}$ in $\operatorname{Span}_{\mathbb{Q}}\left(D_{1}, \ldots, D_{\ell}\right) \subseteq V$ converging to $D$ in $V$. Let $t \in\left(0, \omega_{Z}(D)\right)$ be a real number. Then $D_{t \mid E}$ is ample, and therefore $D_{n, t \mid E}$ is ample for $n$ sufficiently large by [12, Example 1.3.14]. In particular, $t \in\left(0, \omega_{Z}\left(D_{n}\right)\right]$ for $n \gg 1$. By the above and by homogeneity of the $\widehat{h}^{i}$, we have

$$
\begin{equation*}
\operatorname{vol}\left(D_{n, t}\right)-\widehat{h}^{1}\left(\widetilde{X}, D_{n, t}\right)=D_{n, t}^{d} \text { and } \widehat{h}^{i}\left(D_{n, t}\right)=0 \forall i \in\{2, \ldots, d\} \tag{5}
\end{equation*}
$$

for $n \gg 1$. By continuity, (5) also holds for $D$ and the lemma is proved.
Proposition 3.2 is a consequence of Theorem 2.1 and Lemma 3.3.
Proof of Proposition 3.2. The second equality is implied by the first one by using Lemma 3.3. For any $t \in\left[0, \epsilon_{Z}(D)\right], D_{t}$ is nef and therefore $\operatorname{vol}\left(D_{t}\right)=D_{t}^{d}$ by $[12$, Corollary 1.4.41]. Since $t \leq \omega_{Z}(D)$, this gives the inequality

$$
\epsilon_{Z}(D) \leq \sup \left\{t \in\left[0, \omega_{Z}(D)\right] \mid \operatorname{vol}\left(D_{t}\right)=D_{t}^{d}\right\} .
$$

To prove the converse, let $\alpha$ be the supremum on the right hand side. By Lemma 3.3, we have $\widehat{h}^{1}\left(D_{t}\right)=0$ for all $t \in[0, \alpha)$. Let $s \in\left(0, \epsilon_{Z}(D)\right)$ and $t \in(0, \alpha)$ be real numbers. By definition of $\epsilon_{Z}(D), D_{s}$ is ample. For any $\lambda>0$ small enough we have $0 \leq \frac{t-\lambda s}{1-\lambda}<\alpha$, and by homogeneity of $\widehat{h}^{1}$ we deduce that

$$
\widehat{h}^{1}\left(\widetilde{X}, D_{t}-\lambda D_{s}\right)=(1-\lambda)^{d} \widehat{h}^{1}\left(\widetilde{X}, D_{\frac{t-\lambda s}{1-\lambda}}\right)=0 .
$$

By homogeneity and Lemma 3.3, we also have $\widehat{h}^{i}\left(\tilde{X}, D_{t}-\lambda D_{s}\right)=0$ for all $i \geq 2$. By Theorem 2.1 $D_{t}$ is ample, hence $t \leq \epsilon_{Z}(D)$. By letting $t$ tend to $\alpha$, we obtain

$$
\sup \left\{t \in\left[0, \omega_{Z}(D)\right] \mid \operatorname{vol}\left(D_{t}\right)=D_{t}^{d}\right\}=\alpha \leq \epsilon_{Z}(D)
$$

We can easily derive Theorem 3.1 from Proposition 3.2 as follows. Recall that $\varphi_{D, Z}(t)=\operatorname{vol}\left(D_{t}\right)$ for all $t \in \mathbb{R}_{\geq 0}$.

Proof of Theorem 3.1. By Proposition 3.2, it suffices to show that the following implication holds for any $\gamma>0$ :

$$
\varphi_{D, Z} \text { is polynomial on }[0, \gamma] \Longrightarrow \forall t \in[0, \gamma], \varphi_{D, Z}(t)=D_{t}^{d} .
$$

Let $\gamma>0$ be a real number such that $\varphi_{D, Z_{\mid[0, \gamma]}}$ is a polynomial. Then

$$
P: t \mapsto \varphi_{D, Z}(t)-D_{t}^{d}
$$

is a polynomial function on $[0, \gamma]$. On the other hand, $P(t)=0$ for all $t \in\left[0, \epsilon_{Z}(D)\right]$. Since $\epsilon_{Z}(D)>0$, it follows that $P=0$ on $[0, \gamma]$.

In the end of this section, we study in more detail the case where $Z \nsubseteq X$ is a smooth irreducible curve. The following example gives an alternative description of the invariant $\omega_{Z}(D)$.

Example 3.4. (1) Assume that $X$ is a smooth surface and that $Z=C \varsubsetneqq X$ is a smooth irreducible curve. In that case, $\widetilde{X}=X$ and $E=C$. In particular, $\mathcal{O}_{E}(-E)$ is nef if and only if $C^{2} \leq 0$. On the other hand, $D_{t \mid E}$ is ample if and only if $D_{t} \cdot E=D \cdot C-t C^{2}>0$. It follows that $\omega_{C}(D)=\infty$ if $C^{2} \leq 0$, and $\omega_{C}(D)=(D \cdot C) / C^{2}$ otherwise.
(2) Assume that $X$ is smooth and that $Z=C \varsubsetneqq X$ is a smooth irreducible curve. In that case, the normal sheaf $\mathcal{N}_{C}:=\left(I_{C} / I_{C}^{2}\right)^{\vee}$ is a vector bundle of rank $d-1$ on $C$ and we have $E=\operatorname{Proj}\left(\mathcal{N}_{C}^{\vee}\right)$. We denote by $\mu_{\max }\left(\mathcal{N}_{C}\right)$ the maximal slope of $\mathcal{N}_{C}$, defined as

$$
\mu_{\max }\left(\mathcal{N}_{C}\right)=\max _{0 \neq \mathcal{F} \subseteq \mathcal{N}_{C}} \frac{\operatorname{deg}(\mathcal{F})}{\operatorname{rk}(\mathcal{F})},
$$

where $\mathcal{F}$ runs over the non-zero sub-bundles of $\mathcal{N}_{C}$. We also define the strong maximal slope of $\mathcal{N}_{C}$ as $\bar{\mu}_{\max }\left(\mathcal{N}_{C}\right)=\sup _{\tau: C^{\prime} \rightarrow C} \mu_{\max }\left(\tau^{*} \mathcal{N}_{C}\right)$, where the supremum is over the finite $k$-morphisms $\tau: C^{\prime} \rightarrow C$ of smooth projective integral curves (when $k$ has characteristic zero, we have $\bar{\mu}_{\max }\left(\mathcal{N}_{C}\right)=$ $\mu_{\max }\left(\mathcal{N}_{C}\right)$ by [14, Proposition $\left.\left.7.1(3)\right]\right)$. Let $\xi \in N^{1}(E)_{\mathbb{R}}$ be the class of $\mathcal{O}_{E}(1)$ and let $f \in N^{1}(E)_{\mathbb{R}}$ be the class of a fiber of $E \rightarrow C$. For any $t \in \mathbb{R}_{\geq 0}$, the class of $D_{t \mid E}$ in $N^{1}(E)_{\mathbb{R}}$ is $(D \cdot C) f+t \xi$ for any $t \geq 0$. By [14, Proposition $7.1(3)], D_{t \mid E}$ is ample if and only if $(D \cdot C)>t \bar{\mu}_{\max }\left(\mathcal{N}_{C}\right)$, and therefore

$$
\omega_{C}(D)= \begin{cases}(D \cdot C) / \bar{\mu}_{\max }\left(\mathcal{N}_{C}\right) & \text { if } \bar{\mu}_{\max }\left(\mathcal{N}_{C}\right)>0 \\ \infty & \text { if } \bar{\mu}_{\max }\left(\mathcal{N}_{C}\right) \leq 0 .\end{cases}
$$

We refer the reader to [16, section 3] for more explicit computations of the invariant $\omega_{C}(D)$ in the case where $X=\mathbb{P}_{k}^{3}$.

In view of Example 3.4 (1), we see that the assumption of Theorem 1.1 is rather restrictive when $\operatorname{dim} Z>0$. Nevertheless, the following corollary shows that when $Z=C$ is a smooth irreducible curve in a smooth surface, Theorem 1.1 and Corollary 1.3 remain valid even if $\mathcal{O}_{E}(-E)$ is not nef. Recall that $\phi_{D, C}(t)=-\varphi_{D, C}^{\prime}(t) / d$ for every $t \geq 0$ such that $\varphi_{D, C}(t)>0$.

Corollary 3.5. Assume that $X$ is a smooth surface and that $Z=C \varsubsetneqq X$ is a smooth irreducible curve. Then

$$
\begin{aligned}
\epsilon_{C}(D) & =\sup \left\{t \geq 0 \mid \varphi_{D, C} \text { is polynomial on }[0, t]\right\} \\
& =\sup \left\{t \geq 0 \mid \phi_{D, C} \text { is differentiable on }(0, t)\right\} .
\end{aligned}
$$

Proof. If $C^{2} \leq 0$, then $\mathcal{O}_{E}(-E)$ is nef and the result is given by Theorem 1.1 and Corollary 1.3. In the following, we assume that $C^{2}>0$. In particular, $\omega_{C}(D)=$ $(D \cdot C) / C^{2}$ by Example 3.4 (1). Let

$$
\beta_{C}(D):=\sup \left\{t \geq 0 \mid \varphi_{D, C} \text { is polynomial on }[0, t]\right\},
$$

and let $\gamma \in\left(0, \beta_{C}(D)\right)$ be a real number. Then $\varphi_{D, C}$ is a decreasing polynomial function on $[0, \gamma]$. As in the proof of Theorem 3.1, we have

$$
\varphi_{D, C}(t)=D_{t}^{2}=D^{2}-2 t D \cdot C+t^{2} C^{2}
$$

for every $t \in[0, \gamma]$. Since $\varphi_{D, C}(\gamma)=\operatorname{vol}\left(D_{\gamma}\right) \geq 0$, studying the variation of the polynomial $\varphi_{D, C_{\mid[0, \gamma]}}$ leads to the inequalities

$$
\gamma \leq \frac{D \cdot C-\sqrt{(D \cdot C)^{2}-D^{2} C^{2}}}{C^{2}} \leq \frac{D \cdot C}{C^{2}}=\omega_{C}(D)
$$

(note that $(D \cdot C)^{2}-D^{2} C^{2} \geq 0$ by the Hodge index Theorem). Letting $\gamma$ tend to $\beta_{C}(D)$, we obtain $\beta_{C}(D) \leq \omega_{C}(D)$. On the other hand we have $\epsilon_{C}(D)=$
$\min \left\{\beta_{C}(D), \omega_{C}(D)\right\}$ by Theorem 3.1, and therefore

$$
\epsilon_{C}(D)=\beta_{C}(D)=\sup \left\{t \geq 0 \mid \varphi_{D, C} \text { is polynomial on }[0, t]\right\} .
$$

Combining this equality with [1, Theorem] as in the introduction, we also have

$$
\epsilon_{C}(D)=\sup \left\{t \geq 0 \mid \phi_{D, C} \text { is differentiable on }(0, t)\right\} .
$$

Remark 3.6. Assume that $X$ is smooth (of arbitrary dimension $d \geq 1$ ) and that $Z \varsubsetneqq X$ is a smooth divisor with Picard $\operatorname{rank} \rho(Z)=1$. In that case, we also have

$$
\epsilon_{Z}(D)=\beta_{Z}(D):=\sup \left\{t \geq 0 \mid \varphi_{D, Z} \text { is polynomial on }[0, t]\right\} .
$$

Indeed, if $t>\omega_{Z}(D)$ then $D_{t \mid Z}$ is not ample, hence it is not big since $\rho(Z)=1$. Arguing as in the proof of Lemma 3.3, it follows that $\varphi_{D, Z}(t)=\varphi_{D, Z}\left(t^{\prime}\right)$ for all $t^{\prime}>t$. This implies that $\omega_{Z}(D) \geq \beta_{Z}(D)$, and therefore $\epsilon_{Z}(D)=\beta_{Z}(D)$ by Theorem 3.1.

## 4. Variation of volume functions in families

We retain the notation of section 3 , and we assume that the field $k$ is uncountable. Our goal in this section is to study the behaviour of $\varphi_{D, Z}$ when the subscheme $Z \varsubsetneqq X$ varies in a flat family. The following proposition implies in particular that the functions $\varphi_{D, Z}$ all coincide for sufficiently general zero-dimensional subschemes of a given cardinality, as claimed in the introduction (see Proposition 4.2 below).

Proposition 4.1. Let $S$ be a Noetherian scheme locally of finite type over $k$, and let $\mathcal{Z} \varsubsetneqq X \times_{k} S$ be a closed subscheme, flat over $S$. For any $s \in S(k)$, let $\mathcal{Z}_{s}$ be the fiber of $\mathcal{Z}$ above $s$. Then there exists a countable union $V=\cup_{n \in \mathbb{N}} V_{n} \nsubseteq S$ of proper subvarieties of $S$ such that

$$
\varphi_{D, \mathcal{Z}_{s}}(t)=\varphi_{D, \mathcal{Z}_{s^{\prime}}}(t)
$$

for all $t \in \mathbb{R}_{\geq 0}$ and $s, s^{\prime} \in S(k) \backslash V$.
Proof. We assume that $S$ is irreducible without loss of generality. We first consider the case where $D \in \operatorname{Div}(X)$ is a Cartier divisor. Let $I_{\mathcal{Z}} \subset \mathcal{O}_{X \times_{k} S}$ be the ideal sheaf defining $\mathcal{Z}$ in $X \times_{k} S$, and let $s \in S(k)$. We identify the fiber $X \times_{k}\{s\}$ with $X$ and we denote by $j_{s}: X \hookrightarrow X \times_{k} S$ the corresponding closed immersion. Let $\pi_{s}: \widetilde{X}_{s} \rightarrow X$ be the blow-up of $X$ along the ideal $I_{\mathcal{Z}_{s}}$ defining $\mathcal{Z}_{s}$ in $X$, and let $E_{s}$ be the exceptional divisor. By flatness, $j_{s}^{*} I_{\mathcal{Z}}=I_{\mathcal{Z}_{s}}$ is the ideal defining $\mathcal{Z}_{s}$ in $X$. By the semi-continuity theorem applied to $X \times_{k} S \rightarrow S$, the function

$$
s \in S(k) \mapsto h^{0}\left(X, \mathcal{O}_{X}(p D) \otimes_{\mathcal{O}_{X}} I_{\mathcal{Z}_{s}}^{q}\right)
$$

is upper semicontinuous for any integers $p, q \geq 0$. On the other hand, we have $\pi_{s *} \mathcal{O}_{\tilde{X}_{s}}(-q E)=I_{\mathcal{Z}_{s}}^{q}$ for any sufficiently large integer $q$, and therefore the projection formula gives

$$
h^{0}\left(\widetilde{X}_{s}, p \pi_{s}^{*} D-q E_{s}\right)=h^{0}\left(X, \mathcal{O}_{X}(p D) \otimes I_{\mathcal{Z}_{s}}^{q}\right)
$$

for any integers $p, q \geq 0$ with $q$ large enough. It follows that

$$
\left\{s \in S(k) \mid \varphi_{D, \mathcal{Z}_{s}}(t)<\alpha\right\}
$$

is a countable intersection of open subsets in $S(k)$ for any $t \in \mathbb{Q} \geq 0$ and $\alpha \in \mathbb{R}$. This implies the existence of a countable union of proper subvarieties $V=\cup_{n \in \mathbb{N}} V_{n} \varsubsetneqq S$ such that for all $s \in S(k) \backslash V$, we have

$$
\begin{equation*}
\forall t \in \mathbb{Q}_{\geq 0}, \quad \varphi_{D, \mathcal{Z}_{s}}(t)=\operatorname{vol}\left(\pi_{s}^{*} D-t E_{s}\right)=\inf _{s^{\prime} \in S(k)} \varphi_{D, \mathcal{Z}_{s^{\prime}}}(t) \tag{6}
\end{equation*}
$$

By continuity of vol, (6) actually holds for all $t \in \mathbb{R}_{\geq 0}$. Therefore the function $\varphi_{D, \mathcal{Z}_{s}}$ is independent of the choice of $s \in S(k) \backslash V$, and the proposition is proved in the case where $D \in \operatorname{Div}(X)$ is a Cartier divisor.

In general, there exists a sequence $\left(D_{n}\right)_{n \in \mathbb{N}}$ of elements of $\operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\left(D_{n}\right)_{n \in \mathbb{N}}$ converges to $D$ in $N^{1}(X)_{\mathbb{R}}$. By the above and by homogeneity of vol, for every $n \in \mathbb{N}$ there exists a countable union of proper subvarieties $W_{n} \varsubsetneqq S$ such that $\varphi_{D_{n}, \mathcal{Z}_{s}}(t)=\varphi_{D_{n}, \mathcal{Z}_{s^{\prime}}}(t)$ for all $t \in \mathbb{R}_{\geq 0}$ and all $s, s^{\prime} \in S(k) \backslash W_{n}$. Let $V=\cup_{n \in \mathbb{N}} W_{n}$ and let $s, s^{\prime} \in S(k) \backslash V$. Then for every $t \in \mathbb{R}_{\geq 0}$, we have

$$
\varphi_{D, \mathcal{Z}_{s}}(t)=\lim _{n \rightarrow \infty} \varphi_{D_{n}, \mathcal{Z}_{s}}(t)=\lim _{n \rightarrow \infty} \varphi_{D_{n}, \mathcal{Z}_{s^{\prime}}}(t)=\varphi_{D, \mathcal{Z}_{s^{\prime}}}(t)
$$

by continuity of vol.
Proposition 4.1 applies in particular when $S$ is a Hilbert scheme as follows. Given a very ample line bundle $\mathcal{O}_{X}(1)$ on $X$ and a polynomial $\Psi \in \mathbb{Q}[T]$, we denote by Hilb ${ }_{X / k}^{\Psi}$ the Hilbert scheme of closed subschemes of $X$ whose Hilbert polynomial computed with respect to $\mathcal{O}_{X}(1)$ is $\Psi$. By Proposition 4.1 applied to $S=\operatorname{Hilb}_{X / k}^{\Psi}$, there exist a countable union of proper subvarieties $V_{\Psi} \nsubseteq \operatorname{Hilb}_{X / k}^{\Psi}$ and a function $\varphi_{D, \Psi}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ such that

$$
\varphi_{D, Z}=\varphi_{D, \Psi}
$$

for every closed subscheme $Z \in \operatorname{Hilb}_{X / k}^{\Psi}(k) \backslash V_{\Psi}$. When we restrict our attention to zero-dimensional subschemes, we obtain the following result. Let $r \geq 1$ be an integer and let $X^{r}=X \times_{k} \cdots \times_{k} X$ be the fiber product of $r$ copies of $\bar{X}$.

Proposition 4.2. There exist a countable union of proper subvarieties $V \nsubseteq X^{r}$ and a function $\varphi_{D, r}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ such that $\varphi_{D,\left\{x_{1}, \ldots, x_{r}\right\}}=\varphi_{D, r}$ for all $\left(x_{1}, \ldots, x_{r}\right) \in$ $X^{r}(k) \backslash V$.

Proof. This follows from the above discussion by taking $\Psi=r$. Alternatively, this result follows directly from Proposition 4.1 applied to the schemes $S$ and $\mathcal{Z} \subset X \times{ }_{k} S$ defined by

$$
S(k)=\left\{\left(x_{1}, \cdots, x_{r}\right) \in X_{\mathrm{sm}}^{r}(k) \mid x_{i} \neq x_{j} \forall i \neq j\right\}
$$

and

$$
\mathcal{Z}(k)=\left\{\left(x,\left(x_{1}, \ldots, x_{r}\right)\right) \in X(k) \times S(k) \mid x \in\left\{x_{1}, \ldots, x_{r}\right\}\right\},
$$

where $X_{\mathrm{sm}}$ denotes the smooth locus of $X$.
Remark 4.3. As mentioned in the introduction, Theorem 1.1 and Proposition 4.2 imply that the function

$$
\left(x_{1}, \ldots, x_{r}\right) \in X^{r}(k) \mapsto \epsilon_{\left\{x_{1}, \ldots, x_{r}\right\}}(D) \in \mathbb{R}
$$

is constant outside a countable union of proper subvarieties $V \nsubseteq X^{r}$. This result is well-known and can also be derived directly from the definition of $\epsilon_{\left\{x_{1}, \ldots, x_{r}\right\}}(D)$, by using that ampleness is an open condition in a proper family of line bundles (see [12, Theorem 1.2.17 and Example 5.1.11]).

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