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Multiple T -values with one parameter

Frédéric Chapoton

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Introduction

Multiple zeta values can be defined as the iterated integrals of the differential forms dt/t and $dt/(1-t)$ on the real interval $[0, 1]$. They are very important and central objects, in relation with many fields, including knot theory, quantum groups and perturbative quantum field theory [9, 4]. They form an algebra \mathcal{A}_{MZV} over \mathbb{Q} , which has been studied a lot, and is conjecturally graded by the weight.

Many variants of the multiple zeta values have been considered in the literature. Some of these variants are also numbers, for example when allowing various roots of unity as poles of the differential forms. Other variants are functions, with one or more arguments, such as multiple polylogarithms [5] or hyperlogarithms [8].

Among all these variations on a theme, a specific one, recently introduced by Kaneko and Tsumura in [7, §5] and further studied in [6], deals with the iterated integrals of the forms dt/t and $2dt/(1-t^2)$. The resulting numbers, called multiple T -values, also form an algebra \mathcal{A}_{MTV} over \mathbb{Q} under the shuffle product. This new algebra has been less studied than the algebra of multiple zeta values. Conjecturally, there is an inclusion $\mathcal{A}_{\text{MZV}} \subset \mathcal{A}_{\text{MTV}}$.

The present article introduces a new algebra $\mathcal{A}_{\text{MTV},c}$ of iterated integrals, whose elements are functions of a parameter c . This algebra can be seen as a common deformation of both the algebras \mathcal{A}_{MZV} and \mathcal{A}_{MTV} , in the following manner.

For every admissible index (k_1, \dots, k_r) , there is a function $Z_c(k_1, \dots, k_r)$ in $\mathcal{A}_{\text{MTV},c}$. These functions span the algebra $\mathcal{A}_{\text{MTV},c}$, and their product is given by the usual shuffle rule, exactly as for multiple zeta values. One can evaluate the function $Z_c(k_1, \dots, k_r)$ when the parameter c is a real number with $c < 1$. When $c = 0$, one recovers the multiple zeta value $\zeta(k_1, \dots, k_r)$ and when $c = -1$, one recovers the multiple T -value $T(k_1, \dots, k_r)$. It follows that both \mathcal{A}_{MZV} and \mathcal{A}_{MTV} are quotient algebras of $\mathcal{A}_{\text{MTV},c}$.

We start here the study of $\mathcal{A}_{\text{MTV},c}$. Our original motivation was to generalize both multiple zeta values and multiple T -values while keeping the duality relations. We therefore show that for every c , there is an involution of \mathbb{P}^1 that implies the duality relations for the functions Z_c . We extend a result of Kaneko and Tsumura relating a generating function of some Z_c to the hypergeometric function ${}_2F_1$. Using computer experiments, we determine the first few dimensions of the graded pieces of the algebra $\mathcal{A}_{\text{MTV},c}$, assuming that it is graded by the weight. We also propose a guess for the generating series of the graded dimensions of the algebra \mathcal{A}_{MTV} .

It seems that the algebra $\mathcal{A}_{\text{MTV},c}$ is a new object, although it is difficult to be sure, given the very large number of articles related to multiple zeta values and their many variants. Let us also note that a related study can be seen in §4.2 of [3], where our main differential form appears in formula (4.47).

1 Definition and first properties

The letter c will denote a parameter, either complex or real, not equal to 1. In most of the article, we will assume that c is real and $c < 1$.

Consider the two differential forms:

$$\omega_0(t) = \frac{dt}{t} \quad \text{and} \quad \omega_1(t) = \frac{(1-c)dt}{(1-t)(1-ct)}. \quad (1)$$

Note that one can also write

$$\omega_1(t) = \frac{dt}{1-t} - \frac{cdt}{1-ct}. \quad (2)$$

Because of the assumption $c < 1$, the only singularity of the differential form ω_1 on the interval $[0, 1]$ is therefore the simple pole at 1.

When $c = 0$, these two differential forms become the two differential forms dt/t and $dt/(1-t)$ whose iterated integrals are the classical multiple zeta values (MZV). When $c = -1$, they become the two differential forms $\Omega_0 = dt/t$ and $\Omega_1 = 2dt/(1-t^2)$ whose iterated integrals are Kaneko-Tsumura's multiple T-values (MTV) [6].

For general c , one can consider the iterated integrals of ω_0 and ω_1 as functions of the parameter c . We will use the definition

$$I(\varepsilon_1, \dots, \varepsilon_k) = \int_{0 < t_1 < \dots < t_k < 1} \dots \int \omega_{\varepsilon_1}(t_1) \dots \omega_{\varepsilon_k}(t_k), \quad (3)$$

where each ε_i is either 0 or 1, with $\varepsilon_1 = 1$ and $\varepsilon_k = 0$ to ensure convergence.

Using the standard conversion of indices, let us introduce the functions defined by

$$Z_c(k_1, \dots, k_r) = I(1, 0^{k_1-1}, 1, 0^{k_2-1}, \dots, 1, 0^{k_r-1}) \quad (4)$$

for $r \geq 1$ with $k_i \geq 1$ and $k_r \geq 2$. Here powers of 0 stand for repeated zeroes.

We have used above the same conventions as Kaneko and Tsumura in [6], so that the comparison with their results would be simple.

By their definition as iterated integrals, the functions Z_c satisfy the same shuffle product rule as multiple zeta values and multiple T-values. This is most easily described as a sum over the shuffle product of indices in the notation (3). For example,

$$Z_c(2)Z_c(3) = 6Z_c(1, 4) + 3Z_c(2, 3) + Z_c(3, 2). \quad (5)$$

The vector space over \mathbb{Q} spanned by all the functions Z_c is therefore a commutative algebra, denoted by $\mathcal{A}_{\text{MTV},c}$. Moreover, for any fixed c , the vector space over \mathbb{Q} spanned by all the values of Z_c is also a commutative algebra.

As a side remark, one could wonder what happens to the relationship of multiple zeta values with the Drinfeld associator. One could naively replace the KZ equation by

$$\frac{dF}{dz} = \left(\frac{e_0}{z} + \frac{(1-c)e_1}{(1-z)(1-cz)} \right) F \quad (6)$$

and ask about properties of the solution.

1.1 Duality

The functions Z_c also satisfy the duality property, using the change of variable

$$s = \frac{t-1}{ct-1}, \quad (7)$$

which makes sense as soon as $c \neq 1$. This is an involution of \mathbb{P}^1 that exchanges 0 and 1, ∞ with $1/c$ and maps the interval $[0, 1]$ to itself. It also exchanges ω_0 and ω_1 up to sign:

$$-\frac{ds}{s} = \frac{dt}{1-t} - \frac{cdt}{1-ct}, \quad (8)$$

which is ω_1 by (2).

This implies, with the usual proof by change of variables, that the standard duality relations known for multiple zeta values and multiple T -values also hold for the functions Z_c . In the notation of (3), the sequences $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ and $(1 - \varepsilon_k, \dots, 1 - \varepsilon_2, 1 - \varepsilon_1)$ give the same iterated integral. This translates via (4) into equalities between two functions Z_c , including for example

$$Z_c(1, 2) = Z_c(3). \quad (9)$$

The unique fixed point in $[0, 1]$ of the involution (7) is

$$-\frac{\sqrt{-c+1}-1}{c}. \quad (10)$$

This can be used for the purpose of numerical computations, as a convenient cut-point where to apply the composition-of-paths formula for iterated integrals.

Theorem 1. *There is an equality between generating series:*

$$1 - \sum_{m,n \geq 1} Z_c(\underbrace{1, \dots, 1}_{n-1}, m+1) X^m Y^n = (1-c) \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)} {}_2F_1(1-X, 1-Y; 1-X-Y; c). \quad (11)$$

Proof. The proof is essentially the proof of Theorem 3.6 in Kaneko and Tsumura [6]. Let us only sketch the main steps. Define an auxiliary function

$$\mathcal{L}(t) = \int_0^t \omega_1.$$

Because the integrand of ω_1 is positive on $[0, 1]$, the function \mathcal{L} maps as a diffeomorphism the open interval $(0, 1)$ to the positive real line $\mathbb{R}_{>0}$. From the

iterated integral (3) and the conversion rule (4), one gets by symmetrization that

$$Z_c(\underbrace{1, \dots, 1}_{n-1}, m+1) = \frac{1}{(n-1)!m!} \int_0^1 \mathcal{L}(t)^{n-1} \log(1/t)^m \omega_1(t).$$

Therefore

$$\sum_{m, n \geq 1} Z_c(\underbrace{1, \dots, 1}_{n-1}, m+1) X^m Y^n = \int_0^1 e^{\mathcal{L}(t)Y} (t^{-X} - 1) \omega_1(t).$$

Let us write this integral as the sum of I_1 and $-I_2$, where

$$I_1 = \int_0^1 e^{\mathcal{L}(t)Y} t^{-X} \omega_1(t) = (1-c) \int_0^1 \left(\frac{1-t}{1-ct} \right)^{-Y} t^{-X} \frac{dt}{(1-t)(1-ct)}$$

and

$$I_2 = \int_0^1 e^{\mathcal{L}(t)Y} \omega_1(t) = \int_0^\infty e^{wY} dw = -1/Y.$$

In I_1 , one uses the involution (7), which implies (8) and $-\log(s) = \mathcal{L}(t)$. In I_2 , one uses the change of variables $w = \mathcal{L}(t)$. One concludes by evaluating I_1 using the classical Euler integral expression for the hypergeometric function:

$$\frac{\Gamma(A)\Gamma(C-A)}{\Gamma(C)} {}_2F_1(A, B; C; z) = \int_0^1 t^{A-1} (1-t)^{C-A-1} (1-zt)^{-B} dt.$$

□

As for the MTV, one can expand the iterated integrals for Z_c into iterated sums using (2), but this will involve not only one but a linear combination of iterated sums. For example,

$$Z_c(2) = \int_{0 < s < t < 1} \omega_1(s) \frac{dt}{t} = \sum_{n \geq 1} \frac{1-c^n}{n^2} = \text{Li}_2(1) - \text{Li}_2(c).$$

The same argument shows that more generally

$$Z_c(m) = \text{Li}_m(1) - \text{Li}_m(c) \tag{12}$$

for all integers $m \geq 2$, where Li_m is the m -th polylogarithm.

2 Dimensions

Let us declare that the function $Z_c(k_1, \dots, k_r)$ has weight $k_1 + \dots + k_r$. This is also the number of integration signs in the iterated integral (3).

Note that it is not clear at all that only weight-homogeneous linear relations can exist between the functions Z_c . This is probably expected from the motivic philosophy, as in the classical case of MZV [2] and also for MTV.

Assuming that, one can then ask the following question: what are the graded dimensions (with respect to the weight) of the vector space $\mathcal{A}_{\text{MTV},c}$ spanned by the functions $Z_c(k_1, \dots, k_r)$ over \mathbb{Q} ? One can also ask the same question for any fixed value of c .

When $c = 0$, this question is about the algebra \mathcal{A}_{MZV} of MZV and has been studied a lot. The conjecture by Zagier states that the dimensions are the Padovan numbers. This has been proved by Brown in the setting of motivic multiple zeta values [1].

When $c = -1$, the question has been considered in [6], where the authors have performed large-scale computations to find the expected dimensions in low degrees. Based on this data, one could try to find an analogue of Zagier's conjecture. A proposal is made below in §2.2.

Here is a little table for \mathcal{A}_{MZV} and \mathcal{A}_{MTV} .

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
MZV	1	0	1	1	1	2	2	3	4	5	7	9	12	16
MTV	1	0	1	1	2	2	4	5	9	10	19	23	42	49

It would also be interesting to consider other special values for c , such as roots of unity or the inverse golden ratio, as some related variants of multiple zeta values have already been studied. Besides 0 and -1 , what are the exceptional values for c where the dimensions are smaller than those of $\mathcal{A}_{\text{MTV},c}$?

2.1 Constraints of duality

As the functions Z_c satisfy the duality relations and form an algebra $\mathcal{A}_{\text{MTV},c}$ under the shuffle product, one can try to get purely algebraic upper bounds on the graded dimensions of this algebra.

For this, consider the shuffle algebra \mathcal{A} on all words in letters 0 and 1. The subspace \mathcal{A}' of \mathcal{A} spanned by all words starting with 1 and ending with 0 is a subalgebra. One can define in \mathcal{A}' formal elements $Z(k_1, \dots, k_r)$ using the conversion rule (4).

Let \mathcal{B} be the quotient of \mathcal{A}' by the ideal generated by elements $Z(\alpha) - Z(\alpha^*)$ for all indices α , where $*$ is the duality of indices. This ideal contains more linear relations, the first one being

$$Z(3)(Z(3) - Z(1, 2)) = 0. \tag{13}$$

Using a computer, one can find the first few dimensions of \mathcal{B} in small degrees:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
\mathcal{B}_n	1	0	1	1	3	4	9	15	31	55	109	203	397	754

It would be good to have some kind of formula for these dimensions.

2.2 The case of \mathcal{A}_{MTV}

In this section, we propose a guess for the dimensions of the algebra \mathcal{A}_{MTV} of Kaneko-Tsumura. It would be rather bold to call this a conjecture.

Let us start from the data given in Kaneko and Tsumura article:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A	1	0	1	1	2	2	4	5	9	10	19	23	42	49	91	110

which gives in line A the conjectural dimensions obtained from numerical experiments.

Let us add some lines to these table. First, add one line B by computing the sum of two consecutive terms in A , suitably aligned. In the next line, compute the difference $B - A$ between the lines B and A . Next, do something rather strange, namely define $A\sharp B$ as the column-per-column product of A and B :

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A	1	0	1	1	2	2	4	5	9	10	19	23	42	49	91	110
B			1	1	2	3	4	6	9	14	19	29	42	65	91	140
$B - A$			0	0	0	1	0	1	0	4	0	6	0	16	0	30
$A\sharp B$			1	1	4	6	16	30	81	140	361	667				

Now comes the key observation: the lines $B - A$ and $A\sharp B$ seem to be the same, up to insertion of one 0 between any two consecutive terms in $A\sharp B$.

Assuming that this equality continues to hold at every order, one can compute as many terms as one wants in the sequence A , using this presumed identity between $B - A$ and $A\sharp B$. This gives the sequence

1, 0, 1, 1, 2, 2, 4, 5, 9, 10, 19, 23, 42, 49, 91, 110, 201, 230, 431, 521, 952, 1112, 2064, 2509, 4573, 5318, 9891, 12024, 21915, 25658, 47573, 57831, 105404, 122834, ...

This is of course a rather strange procedure, which lacks an interpretation in terms of the structure of the algebra of MTV.

In term of generating series, this can be summarized as

$$\begin{aligned} A &= 1 + O(t^2), \\ B &= (t + t^2)A - t, \\ B &= A - 1 + t \operatorname{Diag}(A, B), \end{aligned}$$

where $\operatorname{Diag}(A, B)$ keeps only the diagonal terms in the product AB .

Another conjecture for the dimensions of \mathcal{A}_{MTV} has been proposed in Remark 2.2 of [6].

2.3 Dimensions with parameter c

Let us now consider the dimensions for the algebra spanned by all functions Z_c .

This section is the result of some experimental work, based on an implementation of these functions using Pari. Linear relations were searched in the intersection of the space of relations for MTV and MZV, and only beyond the obvious relations deduced from duality.

Up to weight 6, there seems to be no relation beyond the relations that can be deduced from the duality relations. For example, in weight 4, there does not seem to be any relation over \mathbb{Q} between $Z_c(1, 3)$, $Z_c(2, 2)$ and $Z_c(4)$. This implies that the graded dimensions of $\mathcal{A}_{\text{MTV},c}$ should be strictly larger than those of \mathcal{A}_{MTV} .

In weight 7, one finds 2 linearly independant relations, not in the span of relations implied by duality. So the expected dimension is $13 = 15 - 2$. In extenso, these relations are

$$\begin{aligned} Z_c(1, 2, 4) - 2Z_c(1, 3, 3) - 4Z_c(2, 1, 1, 3) + 3Z_c(2, 1, 4) \\ + Z_c(2, 2, 3) - Z_c(2, 3, 2) + 2Z_c(3, 1, 3) = 0 \quad (14) \end{aligned}$$

and

$$\begin{aligned}
& 18Z_c(1, 1, 5) + 26Z_c(1, 3, 3) - 30Z_c(1, 6) + 45Z_c(2, 1, 1, 3) - 27Z_c(2, 1, 4) \\
& - 8Z_c(2, 2, 3) + 12Z_c(2, 3, 2) - 15Z_c(2, 5) - 19Z_c(3, 1, 3) + Z_c(3, 2, 2) \\
& - 4Z_c(3, 4) + Z_c(4, 1, 2) = 0 \quad (15)
\end{aligned}$$

One can check that these 2 relations hold exactly for MZV and numerically for MTV.

In weight 8, one finds 3 linearly independant relations, not in the span of relations implied by duality. So the expected dimension is $28 = 31 - 3$. Here are these three relations as lists of coefficients:

$$\begin{aligned}
& [0, 0, 0, 0, 0, -1, 0, 0, -1, -3, 0, 2, 5, 10, -1, -5, 21, -2, \\
& \quad -10, -18, 5, 0, 13, 27, -6, 52, -13, -48, 90, -45, -72] \\
& [0, 0, 0, -2, 0, 0, 1, -8, 0, -2, -20, 10, 6, -10, 0, -2, -16, \\
& \quad 1, 4, 6, 20, -40, 14, -8, -4, -16, 7, 24, 0, 19, 16] \\
& [0, 0, -5, 9, -5, 0, 0, 48, 4, 29, 135, -45, -56, -50, 4, 38, -144, \\
& \quad 7, 54, 117, -100, 270, -154, -227, 60, -479, 82, 345, -975, 366, 672]
\end{aligned}$$

between the 31 elements

$$\begin{aligned}
& Z_c(8), Z_c(6, 2), Z_c(5, 1, 2), Z_c(4, 4), Z_c(4, 2, 2), Z_c(4, 1, 3), Z_c(4, 1, 1, 2), Z_c(3, 5), \\
& Z_c(3, 1, 2, 2), Z_c(3, 1, 1, 3), Z_c(2, 6), Z_c(2, 4, 2), Z_c(2, 3, 3), Z_c(2, 2, 4), Z_c(2, 2, 2, 2), \\
& Z_c(2, 2, 1, 3), Z_c(2, 1, 5), Z_c(2, 1, 3, 2), Z_c(2, 1, 2, 3), Z_c(2, 1, 1, 4), Z_c(2, 1, 1, 1, 3), \\
& Z_c(1, 7), Z_c(1, 4, 3), Z_c(1, 3, 4), Z_c(1, 3, 1, 3), Z_c(1, 2, 5), Z_c(1, 2, 2, 3), Z_c(1, 2, 1, 4), \\
& Z_c(1, 1, 6), Z_c(1, 1, 2, 4), Z_c(1, 1, 1, 5)
\end{aligned}$$

that span the space modulo the relations induced by duality. These 3 relations also hold exactly for MZV and numerically for MTV.

In weight 9, a similar search found 15 relations beyond duality. So the expected dimension would be $40 = 55 - 15$. This is slightly surprising, as one may have expected to get $41 = 13 + 28$ from the idea of summing the two previous terms when the weight is odd. This idea seems to work for even weights in the case of \mathcal{A}_{MTV} . Either our experimental work has a flaw, or this idea must be abandoned for $\mathcal{A}_{\text{MTV},c}$.

Here is a table summarizing the experimental results.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
MZV	1	0	1	1	1	2	2	3	4	5	7	9	12	16
MTV	1	0	1	1	2	2	4	5	9	10	19	23	42	49
$\mathcal{A}_{\text{MTV},c}$	1	0	1	1	3	4	9	13	28	40				
\mathcal{B}	1	0	1	1	3	4	9	15	31	55	109	203	397	754

The line labelled \mathcal{B} is the upper bound assuming only the relations implied by the duality relations, as explained in §2.1.

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