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Tabulated EOS and solution of the Riemann problem for diphasic water.

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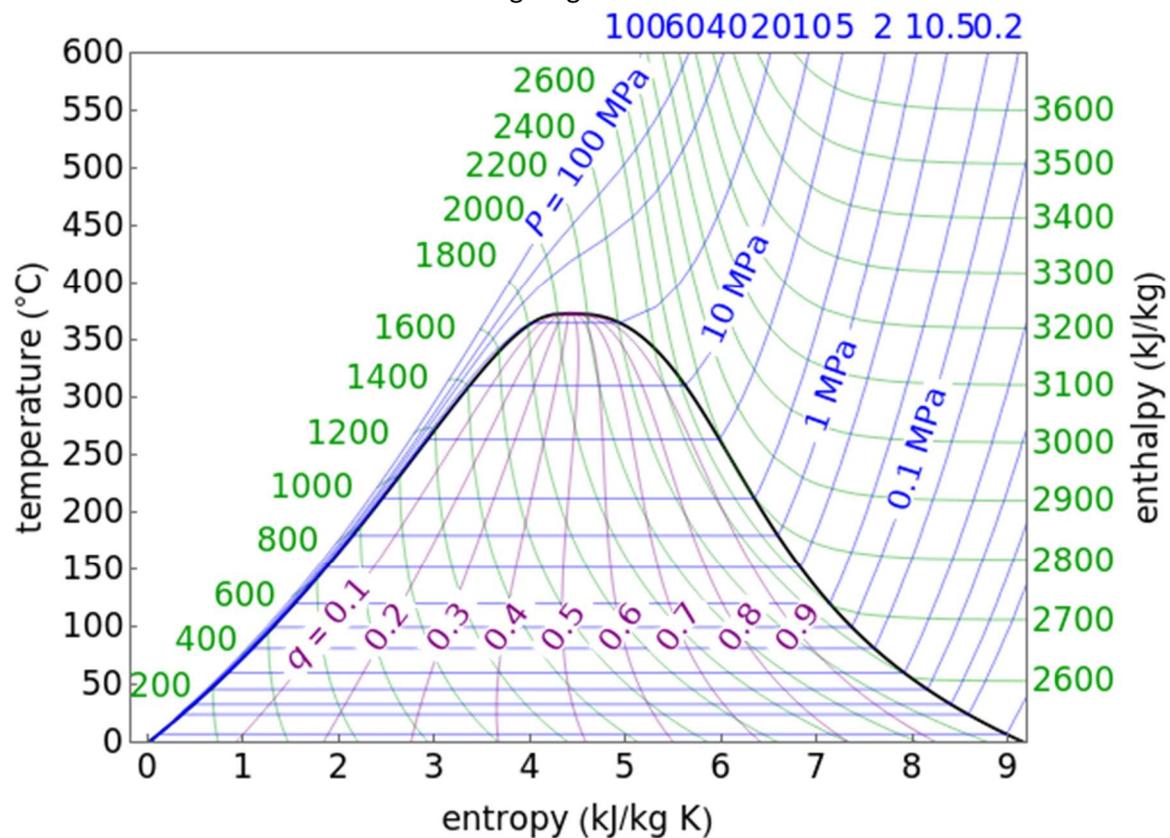
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Introduction

For real materials, the Riemann Problem has been considered in the pioneering work by R. Menikoff & B. Blohr [1]. In particular, they show that when the isentropes in the (p, τ) plane are convex, then the Riemann problem has a unique solution.

As we shall see, for water, we get convex isentropes.

The main reason comes from the following diagram



<https://demonstrations.wolfram.com/TemperatureEntropyDiagramForWater/>

This diagram shows that when one follows a given isentrope, the saturation line can be crossed only once : from the liquid to the diphasic domain or from the steam to the diphasic domain.

Since the sound speed is higher in the liquid domain than in the diphasic domain and also higher in the steam domain than in the diphasic domain, we shall deduce that for water, isentropes are convex in the (p, τ) domain.

The Van Der Waals model is the simplest EOS taking phase transition into account.

However, people working far from the critical point don't use it since it is not sufficiently accurate. Of course they could use the IAPSW97 EOS for water [11]. This one is very accurate, but very costly as regards computing time. See more details in [10].

People rather use stiffened gas EOS for pure liquid water and another one for pure steam water.

Then they apply thermodynamic laws to obtain an EOS in the diphasic domain, as explained in [5, 9].

This induces a computational cost, and this is the reason why people build look-up tables as in [10] or [12].

In the present paper, we use a stiffened gas EOS for pure liquid water, a perfect gas for pure steam water and, for the diphasic domain, a table provided by Faccononi [6].

This table gives

$$T, p(T), \tau_f(T), \tau_v(T), \varepsilon_f(T), \varepsilon_v(T), s_f(T), s_v(T)$$

where $p(T)$ is the saturation pressure, $\tau_f(T)$, (*resp.* $\tau_v(T)$) is the specific volume at saturation in the liquid (*resp.* steam) phase, $\varepsilon_f(T)$, $\varepsilon_v(T)$ (*resp.* $s_f(T)$, $s_v(T)$) are similarly the specific energy (*resp.* entropy) at saturation.

In §1 we show how to use our table to derive an EOS in the diphasic domain.

In §2, we show how to combine our diphasic EOS with a stiffened gas EOS in the pure liquid domain.

In §3, we show how to combine our diphasic EOS with a perfect gas EOS in the steam domain.

In §4 we address the solution of the Riemann problem with our combined equations of state.

We show that the isentropes we obtain in the (p, τ) plane are convex, which, according to [1], proves that the Riemann problem has a unique solution.

Like in [1] we use a graphical method for solving the Riemann problem.

Finally we give some specific examples in connection with depressurization.

1 Equation of state for diphasic water.

Formalism

In what follows, the subscript f (*resp.* v) stands for liquid (*resp.* steam)

We want to define an equation of state for an equilibrium diphasic mixture (steam + liquid).

Such a mixture is at the saturation temperature T .

Let y denote the liquid mass fraction.

We know that the saturation pressure p and the Gibbs potential g depend on T only, not on y , provided that $0 < y < 1$.

We have the following result Faccononi et al [8] or Corot [4 ,p.158] :

Result : Let τ and ε be given :

1. Assume there exists $(\tau_f^*, \tau_v^*, \varepsilon_f^*, \varepsilon_v^*, y^*)$ with $0 < y^* < 1$ and solving the following system

$$p_f(\tau_f, \varepsilon_f) = p_v(\tau_v, \varepsilon_v)$$

$$T_f(\tau_f, \varepsilon_f) = T_v(\tau_v, \varepsilon_v)$$

$$g_f(\tau_f, \varepsilon_f) = g_v(\tau_v, \varepsilon_v)$$

$$\tau = y \tau_f + (1 - y) \tau_v$$

$$\varepsilon = y \varepsilon_f + (1 - y) \varepsilon_v$$

Then the equilibrium state is a diphasic mixture and the associated entropy is

$$s = y^* s_f(\tau_f^*, \varepsilon_f^*) + (1 - y^*) s_v(\tau_v^*, \varepsilon_v^*)$$

2. Otherwise (τ, ε) defines a monophasic state. ■

If we have an EOS for each phase, this result is used to evaluate the specific volume, the internal energy and the entropy of each phase at saturation.

However if we know them, this result can also be used to find p, T and s , when τ and ε are given.

We use numerical values given by Faccononi [6].

There are 99 lines in her table. For $1 \leq i \leq 99$, the table gives a value T_i for the saturation temperature and the 7 values $p(T_i)$, $\tau_f(T_i)$, $\tau_v(T_i)$, $\varepsilon_f(T_i)$, $\varepsilon_v(T_i)$, $s_f(T_i)$, $s_v(T_i)$.

When $T = (1 - \theta)T_{i+1} + \theta T_i$ we interpolate these values linearly so that e.g.

$$\tau_f(T) = (1 - \theta)\tau_f(T_{i+1}) + \theta\tau_f(T_i)$$

Method A : to compute p, T and s , when τ and ε are given :

Let

$$y_\tau(T) = (\tau - \tau_v(T)) / (\tau_f(T) - \tau_v(T))$$

$$y_\varepsilon(T) = (\varepsilon - \varepsilon_v(T)) / (\varepsilon_f(T) - \varepsilon_v(T))$$

To compute T we just have to solve the equation $y_\tau(T) = y_\varepsilon(T)$.

This is a non linear equation with one unknown T which can be easily solved by

- finding i such that $y_\tau(T_i) > y_\varepsilon(T_i)$ and $y_\tau(T_{i+1}) < y_\varepsilon(T_{i+1})$
- solving a second degree equation to find θ such that

$$(\tau - \tau_v(T)) (\varepsilon_f(T) - \varepsilon_v(T)) = (\varepsilon - \varepsilon_v(T)) (\tau_f(T) - \tau_v(T))$$

(indeed, see above, the functions where $\tau_f(T) =, \tau_v(T), \varepsilon_f(T)$ and $\varepsilon_v(T)$ are all linear in θ)

- From the value of θ , compute T (and similarly p) by $T = (1 - \theta)T_{i+1} + \theta T_i$
- Let y^* denote the common value of $y_\tau(T)$ and $y_\varepsilon(T)$ we let
 $s = y^* s_f(T) + (1 - y^*) s_v(T)$ ■

Method B : to compute p, T and ε , when τ and s are given :

In the same way, we solve $y_\tau(T) = y_s(T) = y^*$ where

$$y_\tau(T) = (\tau - \tau_v(T)) / (\tau_f(T) - \tau_v(T))$$

$$y_s(T) = (s - s_v(T)) / (s_f(T) - s_v(T))$$

The details are left to the reader. ■

Method C : to compute ε , when τ and p are given :

This is still easier : p being given, first we evaluate T and then compute

$$y^* = (\tau - \tau_v(T)) / (\tau_f(T) - \tau_v(T))$$

So that $\varepsilon = y^* \varepsilon_f(T) + (1 - y^*) \varepsilon_v(T)$. ■

Test of our equation of state.

We let $\varepsilon = f(\tau, s)$: a well-known result in thermodynamics (see e.g. [4]) is that we should have

$$(1) \quad \frac{\partial f}{\partial \tau} = -p$$

To check that this is the case, we have selected $s = 4.4957$ and $5.81494 \leq \tau \leq 10.46689$.

We compute the derivative of ε w.r.t τ both by forward and backward difference. The curve “depsdtau1” is obtained by forward difference. The other one by backward difference.

The results given in Fig.1 show a rather good agreement that make us confident with the validity of our equation of state.

Sound speed

We notice on Fig.1 that, if we write $p = f(\tau, s)$, for fixed s , $f(\tau, s)$ is a decreasing function of τ , so that the sound speed exists :

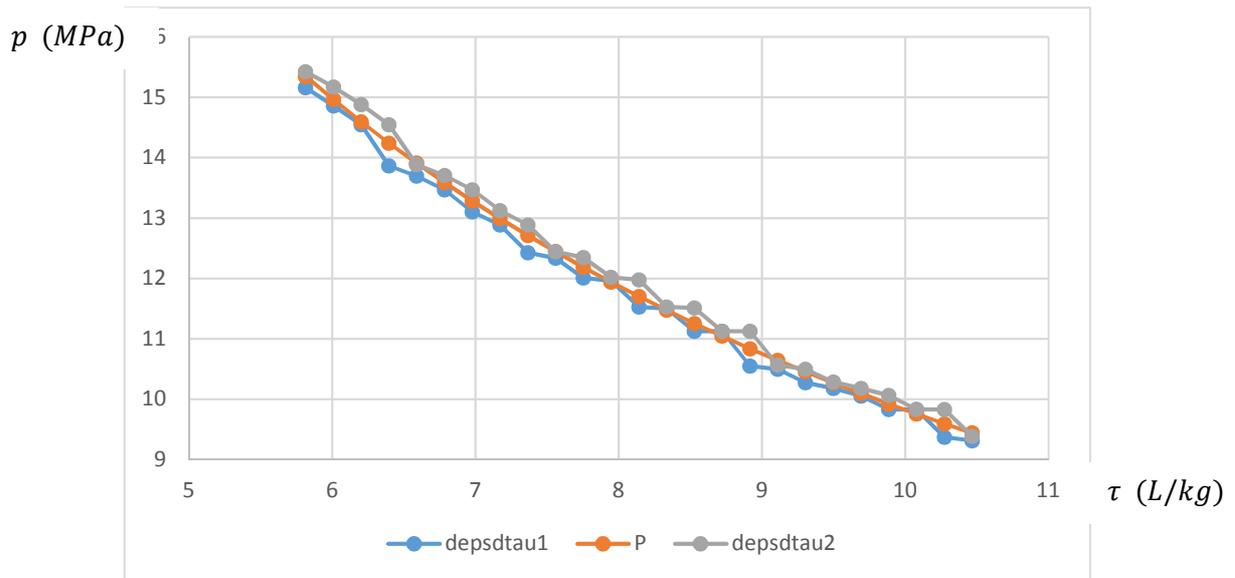


Fig.1 pressure p vs $-\partial\varepsilon/\partial\tau$

We have

$$(2) \quad c = \tau \sqrt{-\partial p / \partial \tau}$$

provided we use international units for each variable.

When we select τ, ε as the primitive thermodynamic variables, we use that

$$(3) \quad c = \tau \sqrt{p \cdot \partial p / \partial \varepsilon - \partial p / \partial \tau}$$

also in international units.

In the following test we replace partial derivatives by finite differences, and we get the results given in Fig 2 for $p = 12.9 \text{ MPa}$.

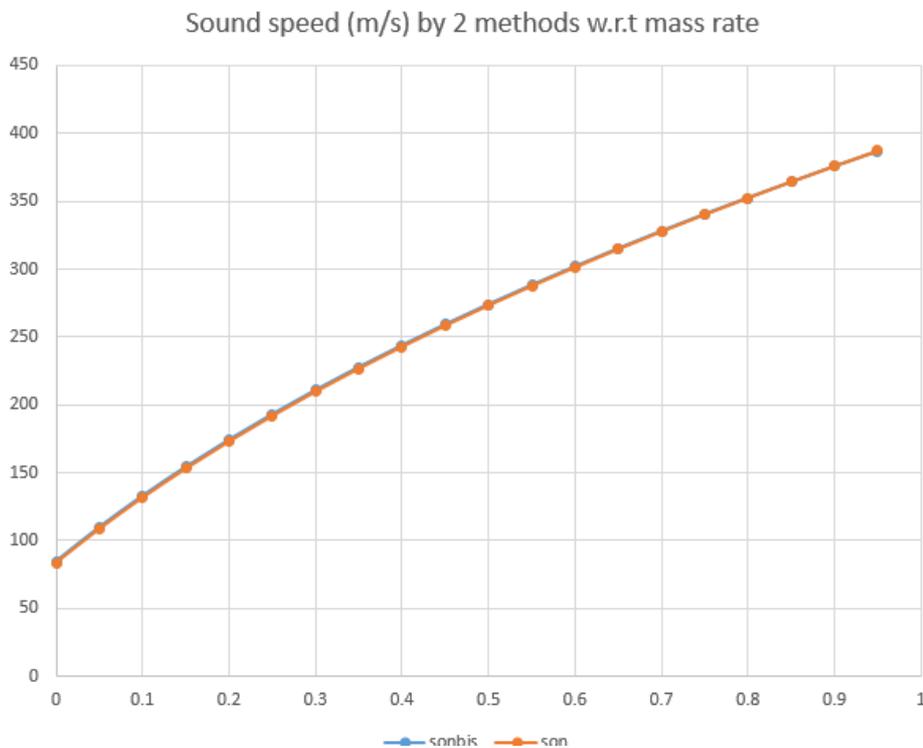


Fig 2 Sound speed evaluated either with (2) or (3) as a function of the steam mass fraction x (there are 2 superposed curves)

We notice that the sound speed in a diphasic mixture is much lower than in the liquid phase, where it is of the order of 800 to 1200 m/s. This result is well known.

2 Equation of state for the liquid phase

For the pure liquid phase, we shall use a stiffened gas EOS.[6]

$$(4) \quad p = -\gamma p_\infty + (\gamma - 1)(\varepsilon - q)/\tau$$

From(3), we have

$$c = \tau \sqrt{p \cdot \partial p / \partial \varepsilon - \partial p / \partial \tau}$$

From (4) we get

$$\partial p / \partial \varepsilon = (\gamma - 1) / \tau$$

$$\partial p / \partial \tau = -(\gamma - 1)(\varepsilon - q) / \tau^2$$

$$c^2 = \tau^2 (p \cdot \partial p / \partial \varepsilon - \partial p / \partial \tau) = \tau^2 (-\gamma(\gamma - 1) p_\infty / \tau + \gamma(\gamma - 1)(\varepsilon - q) / \tau^2)$$

$$= \gamma \tau (-\gamma p_\infty + p_\infty + (\gamma - 1)(\varepsilon - q) / \tau) = \gamma (p + p_\infty) \tau$$

So that we get

$$c = \sqrt{\gamma (p + p_\infty) \tau}$$

Remark :

In [6] Faccononi shows that, when τ and s are selected as the primitive variables, the SG equation of state can be written

$$p = -p_\infty + (\gamma - 1) \tau^{-\gamma} \exp((s - m) / C_v)$$

Along the isentrope, we have $p + p_\infty = (p_0 + p_\infty) \left(\frac{\tau_0}{\tau}\right)^\gamma$ so that

$$\frac{\partial}{\partial \tau} p = \frac{\partial}{\partial \tau} (p_0 + p_\infty) \left(\frac{\tau_0}{\tau}\right)^\gamma = (-\gamma) \tau^{-1} (p_0 + p_\infty) (\tau_0)^\gamma \tau^{-\gamma} = (-\gamma) \tau^{-1} (p + p_\infty)$$

Another way to compute the sound speed is to use (2) which gives

$$c = \tau \sqrt{-\partial p / \partial \tau} = \sqrt{\gamma (p + p_\infty) \tau}. \blacksquare$$

In what follows, we shall select $p_0 = 5.664 \text{ MPa}$ and $\tau_0 = 1.3083 \text{ L/kg}$ which correspond to saturated liquid water at $T = 545 \text{ K}$, and $s = 2.9935 \text{ kJ/kg/K}$.

To define our equation of state we just have to select p_∞ and γ .

We have selected $p_\infty = 186 \text{ MPa}$ and $\gamma = 2.79$, but other choices are possible (see [10]).

Our sub-saturated fluid will be initially at specific volume $\tau_1 = 1.30098 \text{ L/kg}$.

We complement our isentrope in the two-phase mixture domain by using the second method described in §1. We get the result shown on Fig. 3.

Obviously the isentrope is continuous but there is a strong slope discontinuity between both parts.

This corresponds to a strong discontinuity of the sound speed c . Note that such an isentrope is

convex. It has a slope discontinuity on the saturation line. But since the slope depends on c^2 and since c decreases, the isentrope is globally convex.

3 Equation of state for the steam phase

For the pure steam phase, we have selected a perfect gas with $\gamma = 1.21$ ([10] chooses $\gamma = 1.15$).

We have checked that the sound speed is higher in the pure steam domain than in the diphasic domain. For example at $\tau = 34.535 \text{ L/kg}$, $p = 5.664 \text{ MPa}$ the isentrope is shown on Fig.4

Note that for $p > 5.664 \text{ MPa}$ we are in the pure steam domain and for $p < 5.664 \text{ MPa}$ in the diphasic domain.

At the slope discontinuity we easily compute by (2) that $c = 486 \text{ m/s}$ on the pure steam side and $c = 441 \text{ m/s}$ on the diphasic side.

This proves that the isentrope is also convex as can be seen on Fig.4

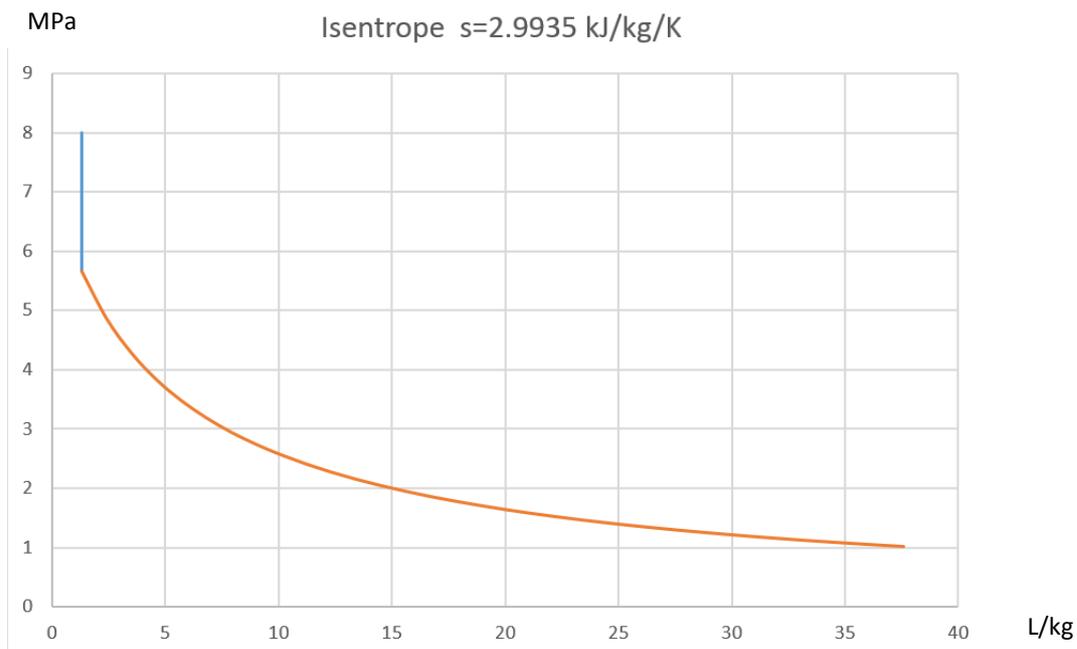


Fig 3 Isentrope in a τ, p diagram. The SG part is shown in blue. The diphasic part in red.

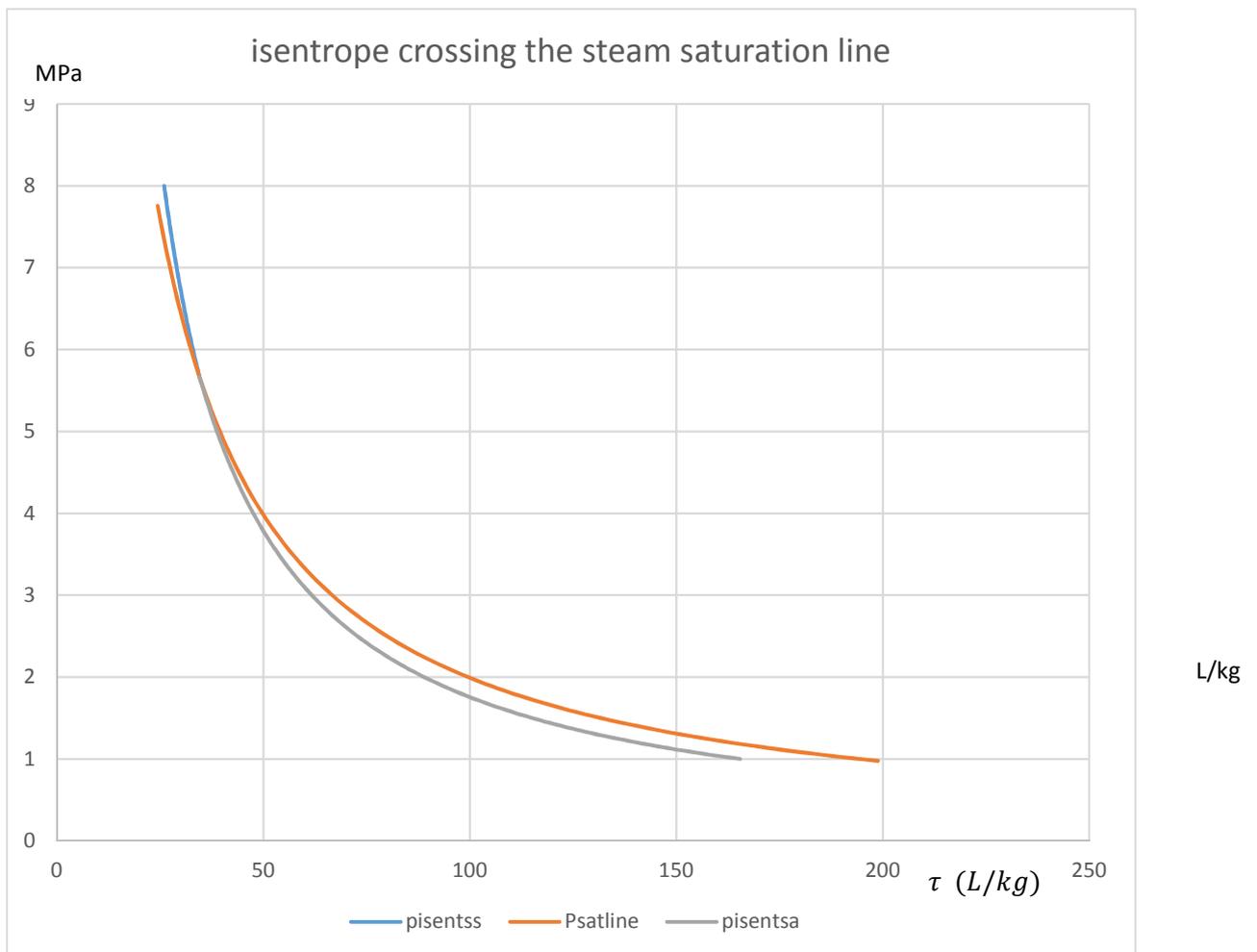


Fig 4 Isentrope crossing the steam saturation line at $\tau = 34.535 \text{ L/kg}$, $p = 5.664 \text{ MPa}$

4 : Solution of the Riemann problem with our equation of state

We shall first consider the case where we have the same diphasic fluid with two different states separated by a diaphragm which is to be removed at time $t=0$.

u_L, p_L, τ_L	u_R, p_R, τ_R
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We then have $u_R = u_L = 0$ and we shall assume that $p_R > p_L$.

We anticipate that we shall have a 1-shock (propagating to the left) and a 3-rarefaction wave propagating to the right.

For $t > 0$ we shall have an intermediate constant state u_*, p_* , itself subdivided in 2 parts separated by a contact discontinuity. On the left (resp. on the right) of the contact discontinuity, we shall have $\tau = \tau_1$ (resp. $\tau = \tau_2$).

We have 4 unknowns u_*, p_*, τ_2, τ_1 , and we need 4 scalar equations.

First we shall use the fact that the following Riemann invariant is constant along a 3-rarefaction wave. We remind the reader that in Eulerian coordinates

$$\frac{\partial}{\partial t} \begin{pmatrix} \tau \\ u \\ \varepsilon \end{pmatrix} + \begin{pmatrix} u & -\tau & 0 \\ \tau p_\tau & u & \tau p_\varepsilon \\ 0 & p\tau & u \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \tau \\ u \\ \varepsilon \end{pmatrix} = 0$$

Let us call λ_1, λ_2 and λ_3 the 3 eigenvalues of the matrix of this hyperbolic system they satisfy

$$(u - \lambda)^3 + \tau^2 \cdot p_\tau \cdot (u - \lambda) - \tau^2 \cdot p \cdot p_\varepsilon \cdot (u - \lambda) = 0$$

So that

$$(u - \lambda)[(u - \lambda)^2 - \tau^2(p \cdot p_\varepsilon - p_\tau)] = (u - \lambda)[(u - \lambda)^2 - c^2]$$

And we get the well-known result that $\lambda_1 = u - c, \lambda_2 = u, \lambda_3 = u + c$.

Riemann invariants :

We check that $r_3 = \begin{pmatrix} -\tau \\ c \\ p\tau \end{pmatrix}$ is the eigenvector associated to λ_3 indeed

$$\begin{pmatrix} -c & -\tau & 0 \\ \tau p_\tau & -c & \tau p_\varepsilon \\ 0 & p\tau & -c \end{pmatrix} \begin{pmatrix} -\tau \\ c \\ p\tau \end{pmatrix} = \begin{pmatrix} -\tau^2 p_\tau - c^2 + p p_\varepsilon \tau^2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

A function $R = R(\tau, u, \varepsilon)$ is a 3-Riemann invariant iff $\nabla R \cdot r_3 = 0$ i-e

$$-\tau R_\tau + c R_u + p\tau R_\varepsilon = 0$$

Then $R = u - g(\tau)$ is a 3-Riemann invariant iff $c = \tau g'(\tau)$ or

$$g'(\tau) = c/\tau$$

As a second Riemann invariant we can choose the entropy s which is constant in a rarefaction wave.

Let s_R denote the entropy of the right state, we let

$$c_R(\tau) = c(\tau, s_R)$$

We can choose

$$g(\tau) = \int_{\tau_0}^{\tau} c_R(\sigma)/\sigma d\sigma$$

We now get our first two equations :

$$(5) \quad u_* - g(\tau_2) - (u_R - g(\tau_R)) = 0$$

$$(6) \quad p_* - f(\tau_2) = 0$$

Remark :

1°/ By using Method B introduced in §1, we can tabulate the isentrope associated to s_R . More precisely, we compute a 5-column table such that we find τ, p, y, c and g in the 5 columns.

So that we have tabulated values for $g(\tau)$, but also for $p = f(\tau)$

By assuming linear interpolation, we can also evaluate $g'(\tau)$ and $f'(\tau)$.

2°/ Knowing the velocity u_R , for all values of τ , we can compute $u = u(\tau)$ by using

$$u(\tau) - g(\tau) - (u_R - g(\tau_R)) = 0.$$

We still call "isentrope" the so obtained curve $\tau \rightarrow \{u(\tau), p(\tau)\}$. ■

Hugoniot curves.

Now what happens along the 1-shock ?

We have the Rankine-Hugoniot relations. Let σ denote the speed of the shock, we should have

$$(7) \quad \rho_1 u_1 - \rho_L u_L = \sigma(\rho_1 - \rho_L)$$

$$(8) \quad (\rho_1 u_1^2 + p_*) - (\rho_L u_L^2 + p_L) = \sigma(\rho_1 u_1 - \rho_L u_L)$$

$$(9) \quad (\rho_1 E_1 + p_*)u^* - (\rho_L E_L + p_L)u_L = \sigma(\rho_1 E_1 - \rho_L E_L)$$

where (noting $u_1 = u^*$)

$$E_1 = \varepsilon_1 + \frac{1}{2}u_1^2 \quad E_L = \varepsilon_L + \frac{1}{2}u_L^2$$

Proceeding as DESPRÉS B. [5, p.155], we introduce :

$$j = \rho_L(\sigma - u_L) = \rho_1(\sigma - u_1)$$

So that

$$\begin{aligned} u_L &= \sigma - j\tau_L & \text{and} & & u_1 &= \sigma - j\tau_1 \\ u_1 - u_L &= j(\tau_L - \tau_1) & \text{or} & & j &= -[u]/[\tau] \end{aligned}$$

Then from (8) we get

$$\begin{aligned} (\rho_1 u_1^2 + p_1) - (\rho_L u_L^2 + p_L) &= \rho_1 \sigma u_1 - \rho_L \sigma u_L = (\rho_1 u_1 + j) u_1 - (\rho_L u_L + j) u_L \\ &= \rho_1 u_1^2 + j u_1 - \rho_L u_L^2 - j u_L \end{aligned}$$

So that we get :

$$p_1 - p_L = j(u_1 - u_L) \quad \text{or} \quad j = [p]/[u]$$

We also have

$$(10) \quad (p_1 - p_L) \frac{u_1 + u_L}{2} = j \frac{u_1^2 - u_L^2}{2}$$

Finally from (9) we get

$$\begin{aligned} (\rho_1 E_1 + p_1)u_1 - (\rho_L E_L + p_L)u_L &= \sigma(\rho_1 E_1 - \rho_L E_L) = (\rho_1 u_1 + j)E_1 - (\rho_L u_L + j)E_L \\ p_1 u_1 - p_L u_L &= j(E_1 - E_L) \end{aligned}$$

Finally

$$E_1 - E_L = \varepsilon_1 - \varepsilon_L + \frac{1}{2}(u_1^2 - u_L^2)$$

So that

$$\begin{aligned} p_1 u_1 - p_L u_L &= j(E_1 - E_L) = j(\varepsilon_1 - \varepsilon_L) + j \frac{u_1^2 - u_L^2}{2} = j(\varepsilon_1 - \varepsilon_L) + (p_1 - p_L) \frac{u_1 + u_L}{2} \\ j(\varepsilon_1 - \varepsilon_L) &= p_1 u_1 - p_L u_L - (p_1 - p_L) \frac{u_1 + u_L}{2} = \frac{1}{2}[p_1 u_1 - p_L u_L - p_1 u_L + p_L u_1] \\ &= \frac{1}{2}(u_1 - u_L)(p_1 + p_L) = \frac{j}{2}(\tau_L - \tau_1)(p_1 + p_L) \end{aligned}$$

Hence

$$(11) \quad (\varepsilon_1 - \varepsilon_L) + \frac{1}{2}(p_1 + p_L)(\tau_1 - \tau_L) = 0$$

Since $\varepsilon_1 = f(\tau_1, p_1)$ equation (11) defines a (so called Hugoniot) curve in the plane (τ, p) .

We denote by

$$p_1 = p_{HO}(\tau_1)$$

the relation so obtained between p_1 et τ_1 .

On Fig. 5 we compare the isentrope passing at $\tau_L = 313.7083 \text{ L/kg}$; $P_L = 0.15 \text{ MPa}$; $u_L = 0$ and the Hugoniot curve starting at the same point. (P_1 in MPa and τ_1 in L/kg). We notice that both curves are very close to each other around the point $\{\tau_L, P_L\}$, but this is a well known result.

Note that on Fig 5 the isentrope starts from a saturated liquid state $\{1.3083 \text{ L/kg}, 5.664 \text{ MPa}\}$

Remark : From relations $j = -[u]/[\tau]$ and $j = [p]/[u]$ we get that $[p][\tau] + [u]^2 = 0$.

The parameter j is called the Lagrangian velocity of the shock.

If $j < 0$ we have a 1-shock, whereas with $j > 0$ a 3-shock (and for $j = 0$ we have a contact discontinuity).

We have $j^2 = -[p]/[\tau]$ which proves that the Hugoniot curve is decreasing. ■

Remark : Lax's entropy condition for a 1-shock requires that

$$\sigma < u_L - c_L \text{ and } u_1 - c_1 < \sigma < u_1$$

For a 3- shock, we have

$$u_1 + c_1 < \sigma \text{ and } u_L < \sigma < u_L + c_L. \blacksquare$$

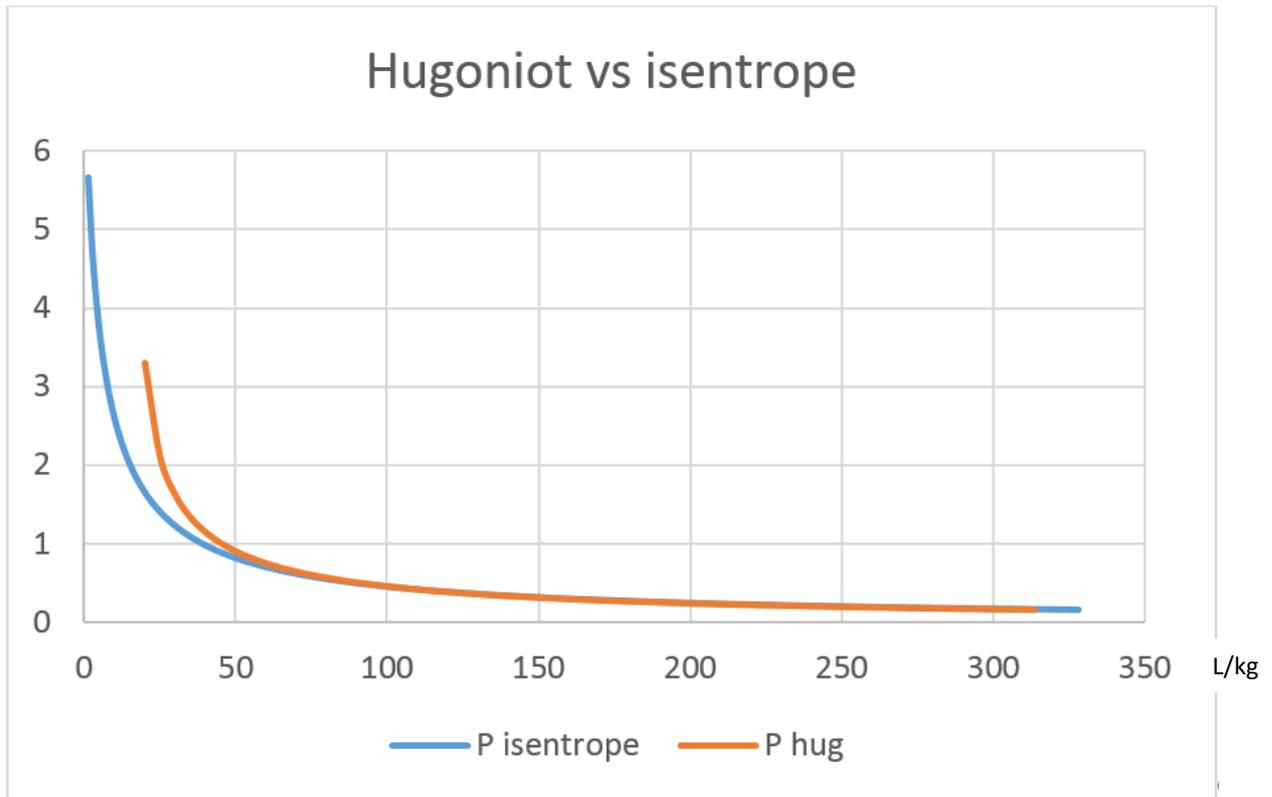


Fig.5 Hugoniot curve and Isentrope starting from the same point $\{\tau_L, p_L\}$.

Convexity of the Hugoniot curves

An example is shown in [1] where the Hugoniot curves both in the $\{\tau, p\}$ plane and the $\{u, p\}$ plane are not convex. This is not what we find here.

We shall consider 2 examples :

- The first one is a case where the Hugoniot curve is crossing the saturation curve on the steam side
- The second one is a case where the Hugoniot curve is crossing the saturation curve on the liquid side

Example 1 :

On Fig. 6, we represent the Hugoniot curve crossing the saturation line in $\{34.53 \text{ L/kg}, 5.664 \text{ MPa}\}$.

The steam saturation line is in grey.

The Hugoniot curve is made of a blue part (in the diphasic domain) and a red part (in the pure steam domain). Both are almost tangent. In any case the curve is convex. We have assumed that steam is a perfect gas with $\gamma = 1.21$ but it does not change significantly if we take $\gamma = 1.4$.

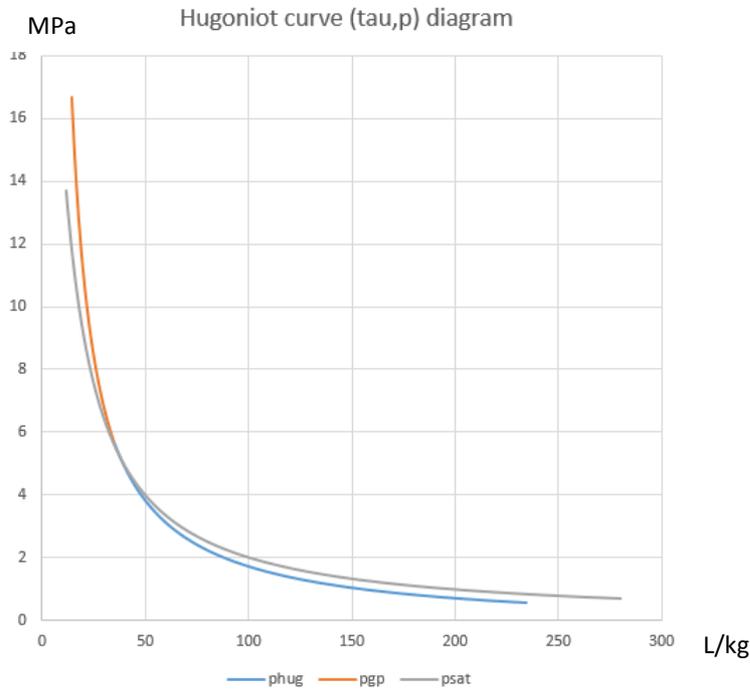


Fig 6 Hugoniot curve crossing the steam saturation line in $\{ 34.53 \text{ L/kg}, 5.664 \text{ MPa} \}$ in $\{\tau, p\}$ axes.

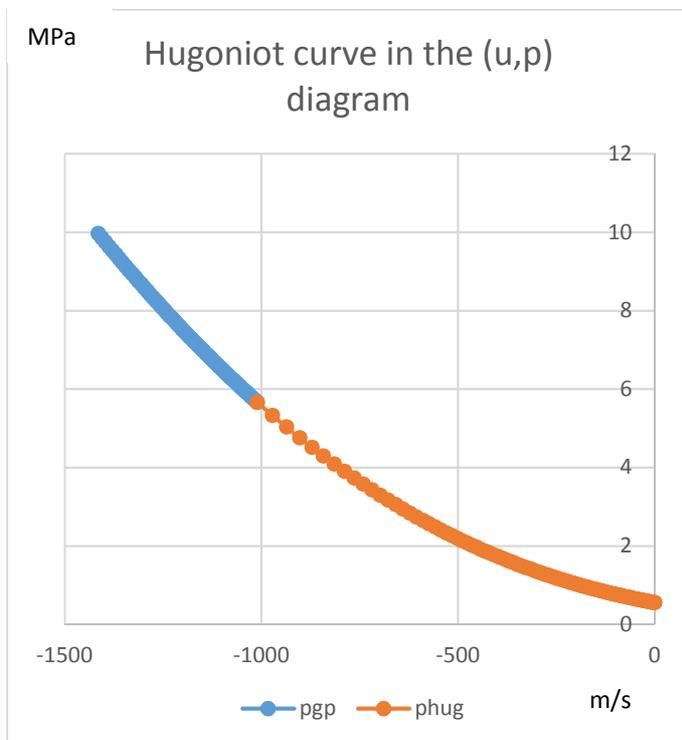


Fig 7 Hugoniot curve in $\{u, p\}$ axes for example 1.

Example 2 :

On Fig. 8, we represent the Hugoniot curve crossing the saturation line in

$\{1,3083 \text{ L/kg}, 5.664 \text{ MPa}\}$.

For the pure liquid phase we have selected $\gamma = 2$ and $p_\infty = 215,863 \text{ MPa}$.

The Hugoniot curve in the diphasic (resp. liquid) domain is in blue (resp. grey). The saturation curve is in red.

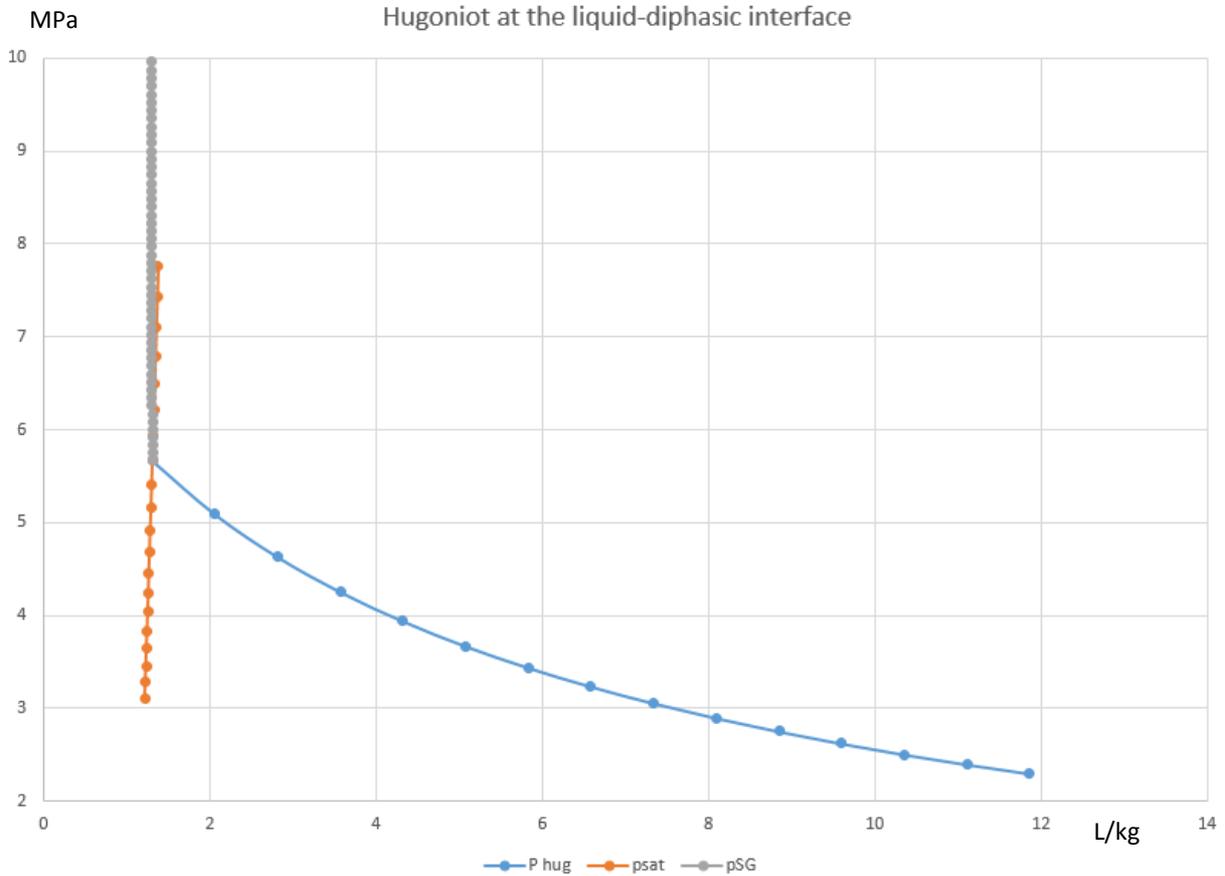


Fig 8 Hugoniot curve crossing the liquid saturation line in $\{1.3083 \text{ L/kg}, 5.664 \text{ MPa}\}$ in $\{\tau, p\}$ axes.

We see that the Hugoniot curve is continuous, but it has a slope discontinuity when it crosses the saturation curve. However it is convex.

Graphical solution to the Riemann problem

From the so obtained Hugoniot curve $\tau \rightarrow p_{HO}(\tau)$ we can deduce another Hugoniot curve in the plane $\{u, p\}$ by $\tau \rightarrow \{u_{HO}(\tau), p_{HO}(\tau)\}$

Where

$$u_{HO}(\tau) = u_L - \sqrt{(\tau_L - \tau)(p_{HO}(\tau) - P_L)}$$

To graphically solve the Riemann problem, we just have to find the intersection in the plane $\{u, p\}$ of the "isentropes" starting from the state $\{\tau_R, p_R, u_R\}$ and the "Hugoniot" starting from the state $\{\tau_L, p_L, u_L\}$.

Here is an example :

We start from $\{1.4746 \text{ L/kg}, 15 \text{ MPa}, 0 \text{ m/s}\}$ on the right and $\{600.6 \text{ L/kg}, 0.226 \text{ MPa}, 0 \text{ m/s}\}$ on the left. Here is what we get on Fig 9.

The intersection is obtained for $p^* \cong 1.07 \text{ MPa}$ and $u_* \cong -601 \text{ m/s}$.

This corresponds to $\tau_2 \cong 95 \text{ L/kg}$ on the isentrope and $\tau_1 \cong 172.6 \text{ L/kg}$ on the Hugoniot.

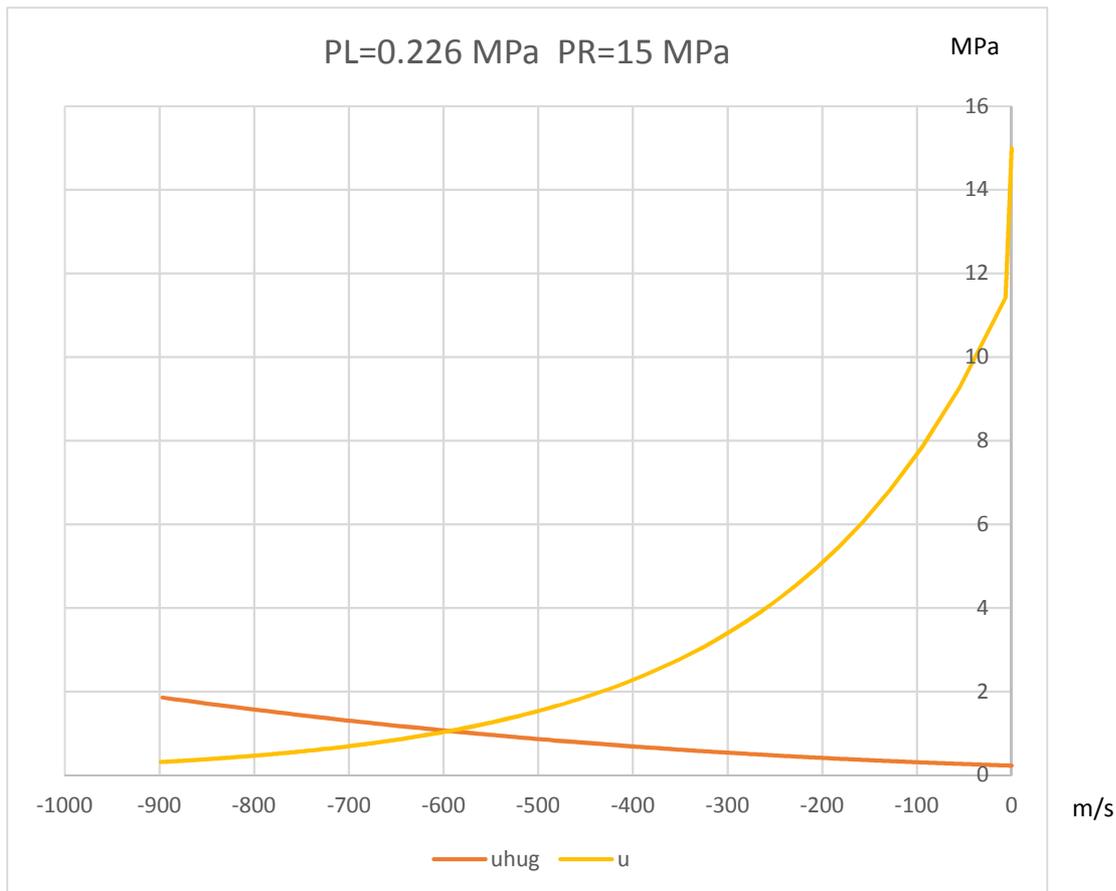


Fig 9 : Graphical solution to the Riemann Problem in a diagram $\{u, p\}$

Computer solution of the Riemann Problem.

We proceed as follows :

1. We build the isentrope starting from $\tau_R; P_R; u_R$
2. We build the Hugoniot curve starting from $\tau_L; P_L; u_L$
3. We define a function $\tau_2 \rightarrow f(\tau_2)$ such that $f(\tau_2) = u_G - u_D$ where
 - a. $\{\tau_2, P_*\}$ is on the same isentrope as $\{\tau_R, P_R\}$
 - b. $u_D = g(\tau_2) + (u_R - g(\tau_R)) = 0$
 - c. $\{\tau_1, P_*\}$ is on the same Hugoniot curve as $\{\tau_L, P_L\}$
 - d. $u_G = u_L - \sqrt{(\tau_L - \tau_1)(P_* - P_L)}$
4. We use the dichotomy method to solve $f(\tau_2) = 0$

Example 3:

With

$$\tau_R = 1.3083 \text{ L/kg}; P_R = 5.664 \text{ MPa}; u_R = 0$$

$$\tau_L = 313.7083 \text{ L/kg}; P_L = 0.15 \text{ MPa}; u_L = 0$$

we get $|f(\tau_2)| < 10^{-10}$ in 40 steps.

We get :

$$\tau_2 = 49.5734 \text{ L/kg}$$

$$\tau_1 = 54.9847 \text{ L/kg}$$

$$P_* = 0.80977 \text{ MPa}$$

$$u_* = u_G = u_D = -413.137 \text{ m/s}$$

We give below a plot of the solution of this Riemann problem at $t = 2.5 \text{ ms}$.

(Note that we need first to use (7) to find $\sigma = -500.938$ m/s)

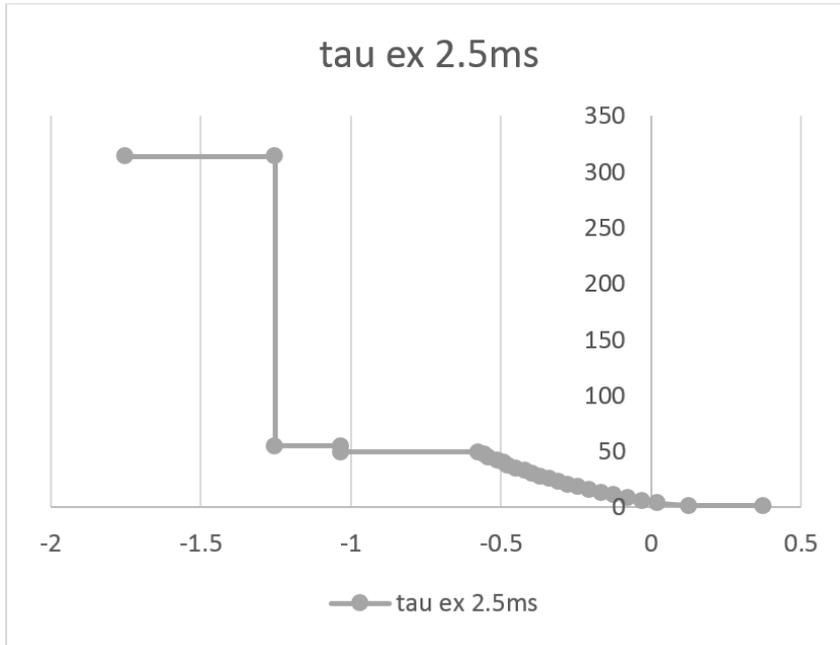


Fig 10 Solution to the Riemann Problem at $t=2.5$ ms. Specific volume L/kg wrt x (m).

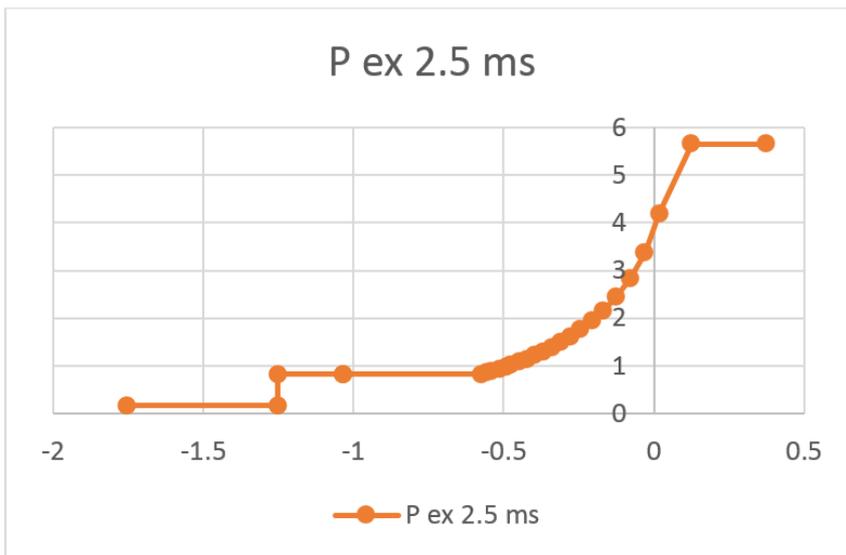


Fig 11 Solution to the Riemann Problem at $t=2.5$ ms. Pressure in MPa wrt x (m)..

We note that the rarefaction wave propagates relatively slowly (~ 50 m/s) to the right.

We also note that, on this specific case, τ has a weak jump at the contact discontinuity.

This is due to the fact that we have selected the right state and the left state with the same entropy.

If we start from

$$\tau_R = 1.3083 \text{ L/kg} ; P_R = 5.664 \text{ MPa} ; u_R = 0$$

$$\tau_L = 1000. ; P_L = 0.15 \text{ MPa} ; u_L = 0$$

We get $\tau_1 \cong 334$ L/kg and $\tau_2 \cong 81$ L/kg

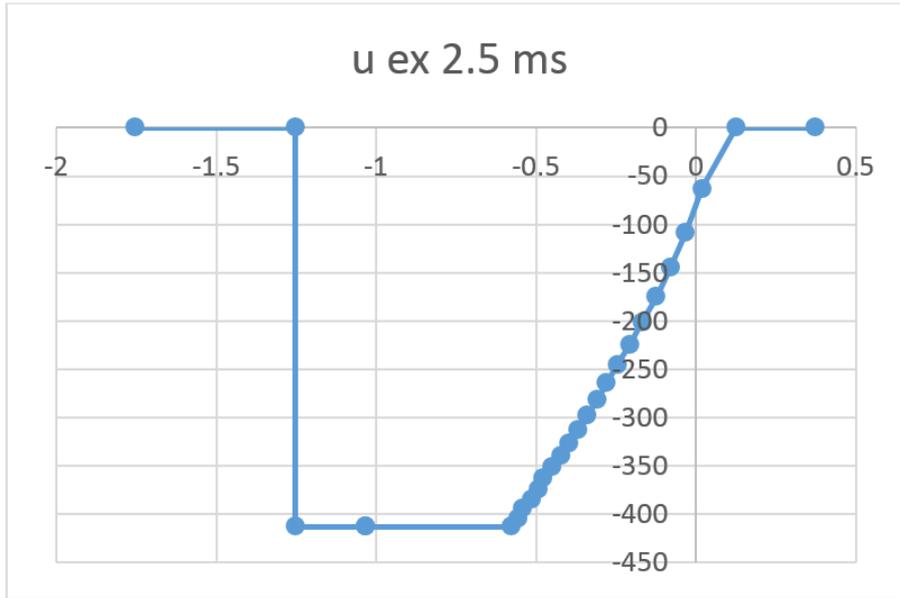


Fig 12 Solution to the Riemann Problem at $t=2.5$ ms. Velocity in m/s wrt x (m)..

Another Case

Case where we have air on the left and subsaturated water on the right.

For air we choose a perfect gas equation of state.

In such a case we use $(p, \tau) = \frac{p\tau}{\gamma-1}$, with $\gamma=1.4$ to evaluate

$$\varepsilon_1 = f(\tau_1, p_1)$$

in (11).

Here are the results when we start from a depressurization case found in [7]

$\tau_R = 1.4745$ L/kg ; $P_R = 15$ MPa ; $u_R = 0$ (pressurized subsaturated liquid water)

$\tau_L = 773.395$ L/kg ; $P_L = 0.1$ MPa ; $u_L = 0$ (air at the atmospheric pressure)

We use either

(a) choice BM2 $\gamma = 2$ and $p_\infty = 215.86$ MPa

(b) choice AF $\gamma = 2.2045$ and $p_\infty = 194.45$ MPa

(both choices give the same sound speed 800 m/s in the pure liquid phase).

The results are given on Fig 13, 14 and 15.

We can see that the rarefaction wave is made of 2 parts :

- a fast wave propagating at about 800 m/s in the pure liquid phase which reduces its pressure from 15 MPa to the saturation pressure (11.4 MPa) on the same isentrope.
- a slow wave propagating at about 50 m/s in the high pressure domain.

This phenomenon explains some facts about the Chernobyl accident (see [13]).

Conclusion :

We have shown that, with a simple tabulated EOS for water in the diphasic domain, a stiffened gas EOS for pure liquid water and a perfect gas EOS for pure steam, we obtain both convex isentropes and convex Hugoniot curves so that the Riemann problem has a unique solution and can be solved easily.

We have given examples which are useful to understand the depressurization process in a tube.

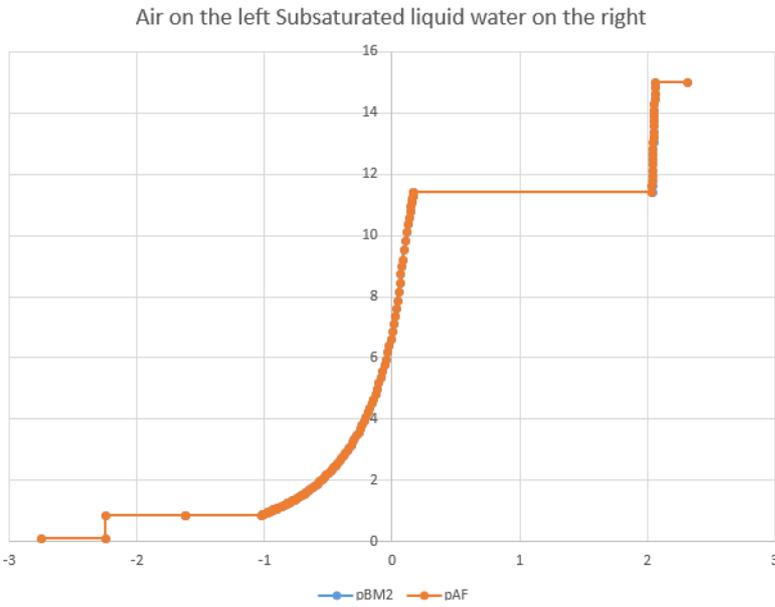


Fig 13 Solution to the Riemann Problem with air on the left at $t=2.5$ ms. Pressure in MPa wrt x (m)..

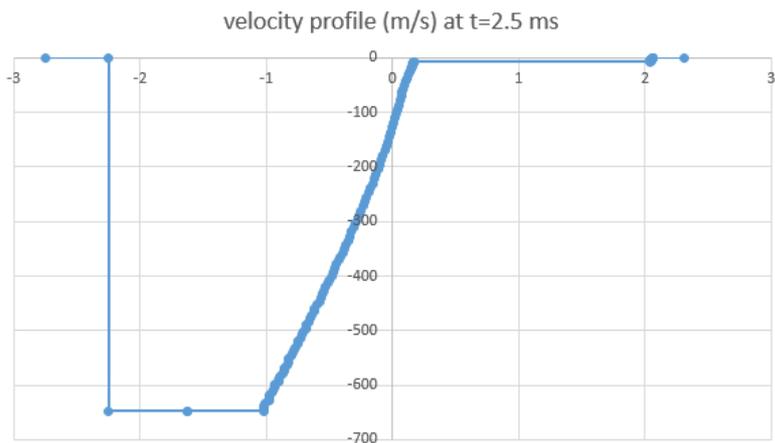


Fig 14 Solution to the Riemann Problem with air on the left at $t=2.5$ ms. Velocity in m/s wrt x (m).

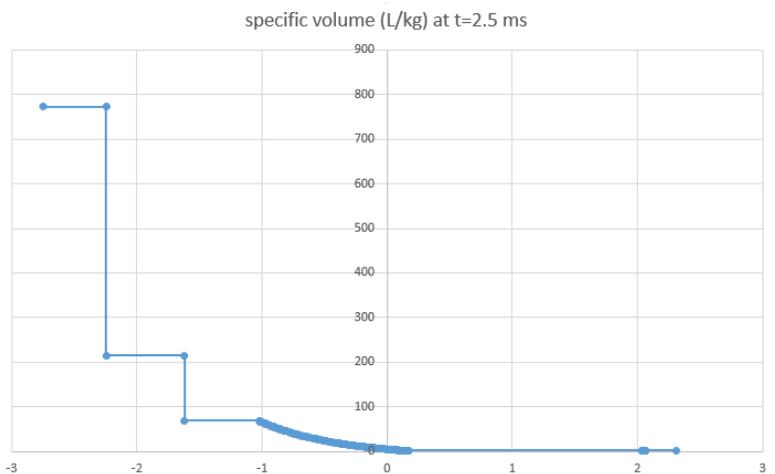


Fig 15 Solution to the Riemann Problem with air on the left at $t=2.5$ ms. Specific volume L/kg wrt x (m).

References

- [1] MENIKOFF R., PLOHR B., Riemann Problem for fluid flow of real materials, *Rev. Mod. Phys.* **61** (1989) 75-130.
- [2] MENIKOFF R., Applications of non-reactive compressible fluids (2001) LANL technical report, 221 p.
- [3] COROT T., MERCIER B., A new nodal solver for the two dimensional Lagrangian hydrodynamics, *J. Computational Physics* **353** (2018)1-25.
- [4] COROT T. Numerical simulation of shock waves in a bi-fluid flow: application to steam explosion. PhD thesis, Conservatoire national des arts et métiers - CNAM, 2017.
- [5] DESPRÉS B. Numerical Methods for Eulerian and Lagrangian Conservation Laws Springer International Publishing, 2017
- [6] G. FACCANONI. Étude d'un modèle fin de changement de phase liquide-vapeur. PhD thesis, École Polytechnique, 2009
- [7] T. BARBERON, Ph. HELLUY, Finite volume simulation of cavitating flows, *Computers & Fluids* **34** (2005) 832–858
- [8] Gloria Faccanoni, Samuel Kokh, Grégoire Allaire. “Numerical Simulation with Finite Volume of Dynamic Liquid-Vapor Phase Transition.” FVCA5, Jun 2008, Aussois, France. pp.391-398. HAL-00976927
- [9] P. HELLUY. “Simulation numérique des écoulements multiphasiques: de la théorie aux applications”. Habilitation à diriger des recherches. Université du Sud Toulon Var, 2005. URL: <https://tel.archives-ouvertes.fr/tel-00657839>.
- [10] L. Quibel, Simulation of water-vapor two-phase flows with non-condensable gas., PhD thesis, 2020, <https://tel.archives-ouvertes.fr/tel-02941486v3>
- [11] W. WAGNER and H.-J. KRETZSCHMAR. *International Steam Tables: Properties of Water and Steam Based on the Industrial Formulation IAPWS-IF97.* Springer-Verlag Berlin Heidelberg, 2008. ISBN: 9783540742340. DOI: <http://dx.doi.org/10.1007/978-3-540-74234-0>.
- [12] Saurel, R., Cocchi, J. P. & Butler, P. B. A numerical study of cavitation in the wake of a hypervelocity underwater projectile. *J. Propulsion Power* **15**,(1999) 513–522.
- [13] B. Mercier, D. Yang, Z. Zhuang, J. Liang, A simplified analysis of the Chernobyl accident, *EPJ Nuclear Sci. Technol.* **7**, 1 (2021)