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Existence of an unbounded nodal hypersurface for smooth Gaussian fields in dimension $d \geq 3$

Hugo Duminil-Copin[*], Alejandro Rivera[†],
Pierre-François Rodriguez[‡], Hugo Vanneuville[§]

Abstract

For the Bargmann–Fock field on $\mathbb{R}^d$ with $d \geq 3$, we prove that the critical level $\ell_c(d)$ of the percolation model formed by the excursion sets $\{f \geq \ell\}$ is strictly positive. This implies that for every ℓ sufficiently close to 0 (in particular for the nodal hypersurfaces corresponding to the case $\ell = 0$), $\{f = \ell\}$ contains an unbounded connected component that visits "most" of the ambient space. Our findings actually hold for a more general class of positively correlated smooth Gaussian fields with rapid decay of correlations. The results of this paper show that the behaviour of nodal hypersurfaces of these Gaussian fields in $\mathbb{R}^d$ for $d \geq 3$ is very different from the behaviour of nodal lines of their two-dimensional analogues.

Contents

[*]Institut des Hautes Études Scientifiques and Université de Genève
[†]École Polytechnique Fédérale de Lausanne
[‡]Imperial College London
[§]CNRS and Université Grenoble Alpes

1 Introduction

Let f be a stationary, isotropic, centered and smooth Gaussian field on $\mathbb{R}^d$, and let $\ell \in \mathbb{R}$. Percolation properties of level sets $\{f = \ell\}$ and of excursion sets $\{f \geq \ell\}$ have been extensively investigated in recent years in the case $d = 2$, starting with the work of Beffara and Gayet [BG17] on box-crossing properties. In particular, under very mild conditions, it has been proved that the critical level is 0 if $d = 2$, in the sense that if $\ell > 0$ then a.s. there is no unbounded component in $\{f \geq \ell\}$ while such an unbounded component exists a.s. if $\ell < 0$ [MRVK20]. When f is positively correlated and if $d = 2$, it is also known that a.s. there is no unbounded component in $\{f \geq 0\}$ [Ale96].

In the present paper, we study these questions in the case $d \geq 3$. Our main result is that for a class of positively correlated Gaussian fields with fast decay of correlations – including the Bargmann–Fock field introduced below – the critical level is strictly larger than 0. In particular, contrary to the case $d = 2$, a.s. there exists an unbounded component in $\{f = 0\}$. Let us first state our main results in the specific case of the Bargmann–Fock field.

1.1 Existence of an unbounded nodal hypersurface for the Bargmann–Fock field in dimension $d \geq 3$

Let $d \geq 2$. The Bargmann–Fock field in $\mathbb{R}^d$ is the analytic centered Gaussian field f defined by the following covariance kernel:

$$\forall x, y \in \mathbb{R}^d, \qquad \mathbb{E}[f(x)f(y)] = \exp(-\tfrac{1}{2}|x-y|^2).$$

This field can be realized as the following entire series, where $a_{i_1,\dots,i_d}$ are i.i.d. standard Gaussian variables:

$$f(x) = \exp(-\tfrac{1}{2}|x|^2) \sum_{i_1,\dots,i_d \in \mathbb{N}} a_{i_1,\dots,i_d} \frac{x_1^{i_1} \cdots x_d^{i_d}}{\sqrt{i_1! \dots i_d!}}.$$

One can note that if $d' \leq d$ then $f_{|\mathbb{R}^{d'}}$ is the Bargmann–Fock field in $\mathbb{R}^{d'}$. As explained for instance in the introduction of [BG17], this field arises naturally from real algebraic geometry when considering suitably rescaled random homogenous polynomials of degree n in $d+1$ variables in the limit when $n \to \infty$. In the present paper, given some level $\ell \in \mathbb{R}$, we study the connectivity properties of the random sets

$$\{f = \ell\} := \{x \in \mathbb{R}^d : f(x) = \ell\} \quad \text{and} \quad \{f \geq \ell\} := \{x \in \mathbb{R}^d : f(x) \geq \ell\}.$$

By ergodicity (see e.g. [Adl10, Theorem 6.5.4]), for every $\ell \in \mathbb{R}$, the event that there is an unbounded component in $\{f \geq \ell\}$ (resp. $\{f = \ell\}$) has probability either 0 or 1. The critical level is defined as follows:

$$\ell_c(d) := \sup\big\{\ell \in \mathbb{R} : \mathbb{P}\big[\exists \text{ an unbounded component in } \{f \geq \ell\}\big] = 1\big\}. \tag{1}$$

It is known that $\ell_c(2) = 0$ [RV20]. Moreover, still if $d = 2$, it is known that there is no unbounded connected component in $\{f \geq 0\}$ [Ale96]. In the present paper, we prove that the critical level strictly increases between dimensions 2 and 3.

Theorem 1.1. *If $d \geq 3$ then $\ell_c(d) > 0$.*

We refer to [CR85, GM90, AG91] and [DPR18a] for analogous results in the context of Bernoulli percolation and the discrete Gaussian free field respectively. Note that Theorem 1.1 is a pure existence result. For these other examples, a (strong) uniqueness result also holds, cf. [CCN87, GM90, DPR18b]. By analogy, we expect that the unbounded component of the Bargmann–Fock field is unique (see Section 1.2 for a general conjecture).

From Theorem 1.1, we deduce the following result.

Corollary 1.2. *There exists $\delta > 0$ such that the following holds for any $\ell \in [-\delta, \delta]$. If $d \geq 3$ then a.s. there exists an unbounded component in $\{f = \ell\}$.*

We actually prove more precise results, namely that

- Such an unbounded connected component still exists if we restrict the field to a thick slab;
- There is an unbounded component that visits "most of the space";

see Theorem 1.7 for a formal (and more general) statement, which also directly implies Theorem 1.1 and Corollary 1.2.

Note that the level $\ell = 0$ plays a special role since it is the planar critical level. To illustrate this, it seems worth restating here the following conjecture. For $t \in \mathbb{R}$, let $\mathcal{P}_t$ denote the plane $\{x \in \mathbb{R}^3 : x_3 = t\}$.

Conjecture 1.3 ([GV20])**.** *Assume that $d = 3$ and let $t < t'$. A.s. there exists $s \in [t, t']$ such that $\{f = 0\} \cap \mathcal{P}_s$ contains an unbounded component. Moreover, the Hausdorff dimension of the set $\{t \in \mathbb{R} : \{f = 0\} \cap \mathcal{P}_t$ contains an unbounded component$\}$ equals $2/3$ a.s.*

1.2 Extension to a family of Gaussian fields

The previous results extend to a class of positively correlated smooth Gaussian fields with fast decay of correlations. We now state the precise assumptions. Let $d \geq 2$ and f be a continuous modification of

$$q \star W, \tag{2}$$

where $\star$ denotes the convolution, W is a d-dimensional L^2-white noise and $q : \mathbb{R}^d \to \mathbb{R}$ satisfies Assumption 1.4 below.

Assumption 1.4. Let $\beta > d$. We say that q *satisfies Assumption 1.4 for β* if

- (regularity) q is C^{10} and there exists $\varepsilon > 0$ such that $\partial^\alpha q(x) = O(|x|^{-(d/2+\varepsilon)})$ for every α with $|\alpha| \leq 10$;

- (decay of correlations) $\partial^\alpha q(x) = O(|x|^{-\beta})$ for every multi-index α with $|\alpha| \leq 1$;
- (isotropy) q is radial;
- (positivity) $q \geq 0$;
- (non-triviality) q is not identically equal to 0.

If q satisfies Assumption 1.4 (for some $\beta > d$) then f defined by (2) is a stationary, isotropic and centered Gaussian process with covariance

$$\mathbb{E}[f(x)f(y)] = (q \star q)(x - y).$$

Moreover, $q \star q$ is C^{10} so f is a.s. C^4 (see for instance Sections A.3 and A.9 of [NS16]) and one can check that

$$(q \star q)(x) = O\big(|x|^{-\beta}\big).$$

Remark 1.5. In several lemmas, conjectures etc., one could also assume that $\beta \in (d/2, d]$ and ask some conditions on the spectral measure (as e.g. in [MV20]), but we have chosen to assume that $\beta > d$ to simplify the statements. In particular, this implies that the spectral measure of $f_{|\mathbb{R}^{d'}}$ is continuous and strictly positive at 0 for all $d' \in [1, d]$.

Remark 1.6. If $q(x) := (2/\pi)^{d/4} e^{-|x|^2}$ (which satisfies Assumption 1.4 for every β) then f is the Bargmann–Fock field.

Our main result is the following. Here and elsewhere, if $1 \leq d' \leq d$ and $D \subset \mathbb{R}^{d'}$, we identify D with $D \times \{0\}^{d-d'}$.

Theorem 1.7. *Let $d \geq 3$. There exists $\beta_0 > 0$ such that the following holds. Let $q : \mathbb{R}^d \to \mathbb{R}$ satisfy Assumption 1.4 for some $\beta > \beta_0$. Then, there exist $L, N, \delta > 0$ such that, for every $\ell \in [-\delta, \delta]$, a.s.*

i) there exists an unbounded component in $\{f = \ell\} \cap (\mathbb{R}^2 \times [0, L])$, and

ii) for every $d' \in [3, d]$ there exist an unbounded component $\mathcal{C}$ of $\{f = \ell\} \cap \mathbb{R}^{d'}$ and some (random) $R_0 > 0$ such that, for every $R \geq R_0$, $\mathcal{C}$ intersects all the (Euclidean, closed) balls of radius $(\log R)^N$ that are included in $[-R, R]^{d'}$.

It is instructive to compare with the planar case, for which we have the following result. Let $d \geq 2$ and $q : \mathbb{R}^d \to \mathbb{R}$ satisfy Assumption 1.4. A result from [Ale96] implies that for $\ell \geq 0$ there is no unbounded component in $\{f \geq \ell\} \cap \mathbb{R}^2$ a.s. In particular, for all $\ell \in \mathbb{R}$ there is no unbounded component in $\{f = \ell\} \cap \mathbb{R}^2$ a.s. Conversely, [MRVK20] (see also [RV20, MV20, Riv21b, GV20] for previous yet less general results) implies that for $\ell < 0$, there exists a unique unbounded component in $\{f \geq \ell\} \cap \mathbb{R}^2$ a.s. The results at $\ell \neq 0$ are also known without any positivity assumption, see [MRVK20].

We conjecture that the unbounded component of $\{f = \ell\}$ (or $\{f \geq \ell\}$) is a.s. unique when it exists. We state this conjecture as we believe it to be of importance to understand further the properties of the nodal hypersurfaces of Gaussian fields.

Conjecture 1.8. *Let $d \geq 3$ and q satisfy Assumption 1.4 for some $\beta > d$. Then, the unbounded component of $\{f = \ell\}$ is a.s. unique when it exists (and similarly for $\{f \geq \ell\}$).*

Remark 1.9. In fact, much more is likely true. In particular, we expect the following local uniqueness statement to hold: the probability that

$$\begin{array}{l}\{f \geq \ell\} \cap [-R,R]^d \text{ contains two or more macroscopic connected components} \\ \text{(of diameter at least } R/100\text{, say) which are not connected inside } [-R,R]^d\end{array} \tag{3}$$

decays very rapidly in R for all $\ell < \ell_c$ (in particular for $\ell = 0$). For Bernoulli or Gaussian free field percolation in $d \geq 3$, such a result holds [GM90, DPR18b]. Item ii) in Theorem 1.7 can be regarded as evidence towards its validity for f. Another piece of evidence is the recent sharpness result proved by Severo, see Theorem 1.2 of [Sev21]: it is shown therein that, for any f satisfying Assumption 1.4, the connection probabilities decay exponentially fast in the subcritical phase while the probability that $[-R,R]^d$ is not connected to infinity is less than $e^{-cR^{d-1}}$ for some $c > 0$ in the supercritical phase.

We further expect the results of the present paper to hold more generally, notably when f denotes the monochromatic random wave, which is the centered stationary smooth Gaussian field whose covariance function is the Fourier transform of the uniform measure on the $(d-1)$-dimensional sphere. The monochromatic random wave is not positively correlated and its covariance decays as $|x|^{-(d-1)/2}$. The following conjecture was stated in [Sar17]. We refer to video simulations[1] by A. Barnett for supporting numerical evidence.

Conjecture 1.10 (Sarnak)**.** *Let $d \geq 3$ and let f be the monochromatic random wave in $\mathbb{R}^d$. There exists $\delta > 0$ such that, for any $\ell \in [-\delta, \delta]$, a.s. there exists an unbounded component in $\{f = \ell\} \cap \mathbb{R}^3$.*

In a forthcoming companion paper, the second author derives this conjecture for large values of d by using the main intermediate result of the present paper and a comparison argument with the Bargmann-Fock model.

Theorem 1.11 ([Riv21a])**.** *There exist $d_0 \geq 3$ and $\delta > 0$ such that the following holds for each $d \geq d_0$. Let f be the monochromatic random wave in $\mathbb{R}^d$. Then, for any $\ell \in [-\delta, \delta]$, a.s. there exists an unbounded component in $\{f = \ell\} \cap \mathbb{R}^3$.*

1.3 Strategy of the proof

In this section, we explain the general strategy of the proof that there exists an unbounded component in $\{f \geq 0\} \cap (\mathbb{R}^2 \times [0, L])$ if L is sufficiently large. Let us fix some small $a > 0$ and some large $\beta = \beta(a)$, and let us assume that q satisfies Assumption 1.4 for such a β. Below (and in all the paper), for any $d' \leq d$, we routinely view $D \subset \mathbb{R}^{d'}$ as the subset $D \times \{0\}^{d-d'} \subset \mathbb{R}^d$.

[1] available at https://math.ethz.ch/fim/activities/conferences/past-conferences/2017/random-geometries-topologies/talks/videos-barnett.html

A quasi-planar uniqueness property. The first important aspect of our proof is an analysis of so-called crossing probabilities. While it is known from Russo–Seymour–Welsh type arguments (see Section 2.3) that in *planar* rectangles crossing probabilities for $\{f \geq 0\}$ are bounded away from 0 and 1, it is not difficult to show that these crossing probabilities tend to 1 as soon as one works in a "thick" rectangle. More precisely, for instance by approximating f by an R^ε-dependent field (for some suitable $\varepsilon = \varepsilon(\beta)$) and by looking at the 2D slices $[0,R]^2 \times \{kR^\varepsilon\}$ for all $k \in \{0, \dots, R^{a-\varepsilon}\}$, one can deduce that

$$\lim_{R\to\infty} \mathbb{P}\big[\{f \geq 0\} \text{ contains a path from left to right in } [0,R]^2 \times [0,R^a]\big] = 1.$$

While this may be interpreted as a sign of supercriticality (dimension 3 is expected to be below the upper critical dimension, so that crossing probabilities should remain bounded away from 1 at criticality, and tend to 0 exponentially fast below criticality) there is still some work to be done to construct the unbounded connected component of $\{f \geq 0\}$ "by hand".

One difficulty comes from the fact that crossings in thick rectangles are not straightforward to combine into longer paths as one can often do in the planar case. In order to circumvent this difficulty, we will prove (a variant of) the following uniqueness property for crossings of such thick rectangles:

$$\begin{gathered}\text{With high probability, for large } R, \text{ any two components of} \\ \text{of } \{f \geq R^{-3/2}\} \cap [0,R]^2 \text{ with diameter} \geq R/100 \\ \text{are connected by a path in } \{f \geq 0\} \cap ([0,R]^2 \times [0,R^a]).\end{gathered} \tag{4}$$

Let us first note that we expect that (4) holds with $\{f \geq 0\}$ instead of $\{f \geq R^{-3/2}\}$ (similarly, recall the conjecture (3)). We refer to the beginning of Section 3 for a discussion about the reason why we need this $R^{-3/2}$-sprinkling.

Once (4) is proved, one can launch a renormalisation procedure to construct an unbounded connected component in $\{f \geq 0\} \cap (\mathbb{R}^2 \times [0,R^a])$ for some R sufficiently large (the reader may note that for this procedure to work, we need to show that there exists a component in $\{f \geq R^{-3/2}\}$ of diameter at least $R/100$ with high probability, which is the case since $R^{-3/2}$ is small enough to apply some approximation tools; see Section 5 for more details).

Setting of the proof of (4). The previous discussion highlights the fact that the proof of (4) is the heart of our paper. We now describe it in some detail. First of all it is a good place to mention that we will work with a finite-range approximation f_r (see the definition in Section 2.1) of f instead of f itself, where f_r has range of dependence r. Our goal is essentially to show that the following holds with very high probability for some well-chosen $\gamma \in (0,a)$:

$$\begin{gathered}\text{Let us condition on } f_{R^\gamma} \text{ restricted to } \mathbb{R}^2 = \mathbb{R}^2 \times \{0\}^{d-2} \text{ and let } \mathcal{C}, \mathcal{C}' \\ \text{be two components of } \{f_{R^\gamma} \geq R^{-3/2}\} \cap [0,R]^2 \text{ of diameter at least } R/100. \\ \text{Then, the (conditional) probability that } \mathcal{C} \text{ and } \mathcal{C}' \text{ are connected by} \\ \text{a path in } \{f_{R^\gamma} \geq 0\} \cap ([0,R]^2 \times [0,R^a]) \text{ is very close to 1.}\end{gathered} \tag{5}$$

The proof for Bernoulli percolation. To explain the proof of (5), let us make a brief detour to present an analogous strategy in the context of Bernoulli percolation at $p = 1/2$ on $\mathbb{Z}^3$, for which the argument is maybe more transparent. For this model, we can show the result obtained from (4) by replacing $R^{-3/2}$ by 0 and R^a by a large (but independent of R) number L. Let us explain this proof. We recall that in this model each edge is open with probability $1/2$ and closed otherwise, and that the states of the edges are independent of each other. In order to connect two large connected components $\mathcal{C}$ and $\mathcal{C}'$ of the square $[0,R]^2$ in $[0,R^2] \times [0,L]$ for some well-chosen L, one may proceed as follows. First, recall that by the Russo–Seymour–Welsh theory, two macroscopic connected sets in some planar domain are connected with probability uniformly bounded away from 0 and 1. By looking at Bernoulli percolation in the vertical translates $[0,R]^2 \times \{k\}$ of the square (for $k \in \{1,\dots,L\}$) and by using the independence structure of Bernoulli percolation, one obtains that if L is large then with high probability there exists some height k and a cluster $\mathcal{C}''$ in $[0,R]^2 \times \{k\}$ whose projection intersects both $\mathcal{C}$ and $\mathcal{C}'$. One can actually further prove that with high probability, the projection of $\mathcal{C}''$ intersects $\mathcal{C}$ and $\mathcal{C}'$ in *many* places. Then, one can try to connect $\mathcal{C}$ and $\mathcal{C}'$ to $\mathcal{C}''$ by using the vertical paths made of k edges at each of these many places, and one obtains that $\mathcal{C}$ and $\mathcal{C}'$ are connected in $[0,R]^2 \times [0,L]$ with high probability.

Adapting the strategy to smooth Gaussian fields. Adapting this strategy to smooth Gaussian fields raises substantial challenges. First, the field f itself is not finite range and this is the reason for working with an approximation f_{R^γ} of it. Second (and more importantly), a new difficulty emerges from the lack of a fundamental property of Bernoulli percolation and various other dependent percolation models: the *finite-energy property*, by which we mean that conditioned on everything outside a finite set A, the configuration in A takes any possible value with probability bounded from below uniformly in what happens outside. This seemingly harmless property is in fact extremely powerful. For instance when conditioning on $\mathcal{C}, \mathcal{C}', \mathcal{C}''$, one could use it to create vertical paths connecting the connected components with reasonable probability. Not being able to invoke it substantially complicates our argument. In order to overcome this issue, we will rely on one of the main innovations of this paper and prove that large clusters in $\{f_{R^\gamma} \geq R^{-3/2}\}$ typically belong to clusters in $\{f_{R^\gamma} \geq 0\}$ which are not confined to thin slabs. More precisely, we show the following result, which we raise to the level of a proposition to highlight its importance in our argument.

For $R \geq 0$, let $D(R) := \{x \in \mathbb{R}^2 : |x| \leq R\}$, $\mathcal{P}_t := \{x \in \mathbb{R}^3 : x_3 = t\}$ and recall that if $d' \leq d$ and $D \subset \mathbb{R}^{d'}$ then we identify D with $D \times \{0\}^{d-d'}$. We also refer to the definition of f_r (and r_q) in the next section.

Proposition 1.12. *Let $d \geq 3$ and let q satisfy Assumption 1.4 for some $\beta > d$. For any $\theta_0 > 0$ there exist $a, \gamma, c, R_0 > 0$ such that for every $R \geq R_0$, $r \in [r_q, R^\gamma]$ and $\ell \in \mathbb{R}$,*

$$\mathbb{P}\left[\begin{array}{c}\text{Every continuous path in } \{f_r \geq \ell + R^{-2+\theta_0}\} \cap D(2R) \\ \text{from } D(R) \text{ to } \partial D(2R) \text{ belongs to a connected component} \\ \text{of } \{f_r \geq \ell\} \cap (D(2R) \times [0,R^a]) \text{ that intersects } \mathcal{P}_{R^a}\end{array}\right] \geq 1 - \exp(-R^c).$$

Remark 1.13. Let us make two observations:

- We will actually only use Proposition 1.12 for $\theta_0 = 1/2$, and any fixed $\theta_0 < 1$ would have sufficed. However, we state and prove this stronger result as we believe it could be of interest in other contexts.
- If we could prove this proposition with ℓ instead of $\ell + R^{-2+\theta_0}$ then we would be able to replace $R^{-3/2}$ by 0 in (4) (for f_{R^γ} instead of f).

To the risk of repeating ourselves, this proposition will be one of the main novelties of this paper. It uses in a central fashion the fact that f_r is defined on a continuous space, which makes the argument one of the few instances where continuous fields are easier to handle than their lattice analogues. The argument, inspired by the Mermin–Wagner theorem [MW66] – see in particular the arguments developped by Pfister [Pfi81] – may have other applications of the same kind in the future.

End of the proof of (4) (for f_{R^γ}). With Proposition 1.12 at hand, the end of the proof that (5) holds with high probability consists in

- observing, by Proposition 1.12, with high probability, one can condition on the fact that several random points u_i at a vertical distance R^a from $\mathcal{C}$ (resp. $\mathcal{C}'$) are likely to be connected to $\mathcal{C}$ (resp. $\mathcal{C}'$);
- then trying to connect the u_i's together.

While the model is very biased in the neighbourhood of $\mathcal{C}$ and $\mathcal{C}'$, we have much more freedom in the neighborhood of the u_i's. This will be crucial for us. Using this property and the previous observation that Russo–Seymour–Welsh type arguments enable to connect such areas with relatively good probabilities will conclude the proof of (5) and as a result of (4) also (we are omitting a fair amount of details here and postpone the discussion to the corresponding section).

Organization of the paper. In Section 2, we give some notations and recall some preliminary results. Section 3 contains the proof of Proposition 1.12. Section 4 is devoted to the proof of a Russo–Seymour–Welsh type result with many contact points. At this point, we will have all the tools in hand to prove (an analogue of) (4). This is done in Section 5. In Section 6, we state and prove a general renormalization scheme that will be used together with (the analogue of) (4) in the two last sections in order to conclude the proof of our main result Theorem 1.7. The first and second parts i) and ii) of this theorem are proved in Sections 7 and 8 respectively.

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2 Notations and preliminary properties

In this section, we state some notation used in all the paper as well as two preliminary properties, namely the FKG inequalities and the RSW theorem. Let us note that these two properties are not used in Section 3, which contains the proof of Proposition 1.12.

2.1 Truncation and other notations

Let $q : \mathbb{R}^d \to \mathbb{R}^d$ satisfy Assumption 1.4 for some $\beta > d$. Most of our intermediate results deal with a finite-range approximation of f defined by truncating q. Below, $|\cdot|$ denotes the Euclidean norm. For $r \geq 1$, let $\chi_r : \mathbb{R}^d \to [0,1]$ be a smooth isotropic function satisfying

$$\chi_r(x) = \begin{cases} 1 & \text{if } |x| \leq r/2 - 1/4, \\ 0 & \text{if } |x| \geq r/2, \end{cases}$$

and whose k^{th} derivatives, for all $k \geq 1$, are uniformly bounded in $x \in \mathbb{R}^d$ and $r \geq 1$. We let $q_r = q\chi_r$ and define (f, f_r) as a continuous modification of the pair

$$(q \star W, q_r \star W)\,.$$

The field f_r is r-dependent if the sense that $\mathbb{E}[f_r(x)f_r(y)] = 0$ if $|x-y| \geq r$. In all the paper, we let

$$r_q = 1 + \sup\{r \geq 1 : q_r \text{ is identically equal to } 0\}$$

(with $\sup \emptyset := 1$). One can note that q_r satisfies Assumption 1.4 for every $r \geq r_q$ and that the O estimates from this assumption are uniform in r. To see that f and f_r are close to each other if r is large, we will use some classical approximation techniques (Cameron–Martin, Kolmogorov and BTIS lemmas), see Section B.1.

We will use the following notation/conventions in all the paper.

- We let $B(R)$ denote the Euclidean (closed) ball of radius R centered at 0 and we let $B(x,R) := x + B(R)$.
- If $1 \leq d' \leq d$, if we work in $\mathbb{R}^d$ and if $D \subset \mathbb{R}^{d'}$, then we identify D with $D \times \{0\}^{d-d'}$.
- We let $D(R) := \{x \in \mathbb{R}^2 : |x| \leq R\}$ and $D(x,R) := x + D(R) = x + (D(R) \times \{0\}^{d-2})$ for any $x \in \mathbb{R}^d$.
- We let $\mathcal{P}_t := \mathbb{R}^2 \times \{t\}$.
- If $U \subset \mathbb{R}^d$, let $\mathcal{F}_U$ be the σ-algebra on the set of continuous functions $C(\mathbb{R}^d)$ generated by the projections $u \mapsto u(x)$ for $x \in U$. We say that $\phi : C(\mathbb{R}^d) \to \mathbb{R}$ is measurable if it is $\mathcal{F}_{\mathbb{R}^d}$-measurable. Let us note that $\mathcal{F}_{\mathbb{R}^d}$ is the Borel σ-algebra for the topology of uniform convergence on every compact subset.

2.2 The Gaussian FKG inequalities

Let us first state the Fortuin–Kasteleyn–Ginibre (or FKG) inequality for continuous Gaussian fields.

Lemma 2.1 (Continuous Gaussian FKG inequality)**.** *Let $\phi, \psi : C(\mathbb{R}^d) \to \mathbb{R}$ be two non-decreasing bounded measurable functions and let f be a centered continuous Gaussian field on $\mathbb{R}^d$. Assume that $\mathbb{E}[f(x)f(y)] \geq 0$ for every $x, y \in \mathbb{R}^d$. Then,*

$$\mathbb{E}\left[\phi(f)\psi(f)\right] \geq \mathbb{E}\left[\phi(f)\right]\mathbb{E}\left[\psi(f)\right].$$

This lemma is proven in Section A.1 by relying on the analogous result for Gaussian vectors proven by Pitt [Pit82]. In the present paper, we also need a generalization of this lemma to "locally monotone" functions. Lemma A.6 is such a generalization. In this section, we state Corollary 2.2 which is a consequence of Lemma A.6 and which is sufficient for us (to see that Corollary 2.2 is indeed a consequence of Lemma A.6, the reader can note that if $U \subset \mathbb{R}^d$ and $\phi : C(\mathbb{R}^d) \to \mathbb{R}$ is $\mathcal{F}_U$-measurable, then ϕ does not depend on U^c in the sense that $\phi(u) = \phi(v)$ for every u, v that agree in U).

We first need some vocabulary/notation. Given some $U \subset \mathbb{R}^d$, let U^r denote the r-neighborhood of U, i.e. $U^r := \{x \in \mathbb{R}^d : \exists y \in U, |x - y| < r\}$. If $\phi : C(\mathbb{R}^d) \to \mathbb{R}$, say that ϕ is non-decreasing in U if $\phi(u) \geq \phi(v)$ for every u, v that agree outside of U and satisfy $u_{|U} \geq v_{|U}$.

Corollary 2.2 (Local FKG inequality)**.** *Let $\phi, \psi : C(\mathbb{R}^d) \to \mathbb{R}$ be two bounded measurable functions and let f be a centered continuous Gaussian field on $\mathbb{R}^d$. Also, let $r \in (0, +\infty]$ such that f is r-dependent (i.e. $\mathbb{E}[f(x)f(y)] = 0$ for every $x, y \in \mathbb{R}^d$ satisfying $|x - y| \geq r$). Finally, let $\delta > 0$ and $U, V \subset \mathbb{R}^d$ and assume that*

- *ϕ is non-decreasing in $V^{r+\delta}$ and $\mathcal{F}_U$-measurable,*
- *ψ is non-decreasing and $\mathcal{F}_V$-measurable,*
- *for every $x, y \in U^{r+\delta}$, $\mathbb{E}[f(x)f(y)] \geq 0$.*

Then,

$$\mathbb{E}\left[\phi(f)\psi(f)\right] \geq \mathbb{E}\left[\phi(f)\right]\mathbb{E}\left[\psi(f)\right].$$

2.3 Russo–Seymour–Welsh theory

Box-crossing properties for excursion and level sets of smooth positively correlated Gaussian fields with sufficiently fast decay of correlations have been proven in [BG17, BM18, BMW17, RV19, MV20] by relying on [Tas16]. Recently, Köhler-Schindler and Tassion [KT20] have developed a strategy that enables to remove the assumption on the speed of decay of the correlations in the case of the excursion sets. Moreover, the constants obtained in [KT20] do not depend on the model. In the context of the present paper, this gives Theorem 2.4 below. Note that the theorem also applies to f_r for any $r \geq r_q$. Before stating it, we note that it relies on the following simple lemma, which guarantees planar duality for crossings.

Lemma 2.3. *Assume that q satisfies Assumption 1.4 for some $\beta > d$. Let $g = f_{|\mathbb{R}^2}$, $\ell \in \mathbb{R}$ and let $L \subset \mathbb{R}^2$ be a line. The following holds a.s.:*

- *the sets $\{g \geq \ell\}$ and $\{g \leq \ell\}$ are two $\mathcal{C}^1$-smooth 2-dimensional manifolds with boundary,*
- *$\partial\{g \geq \ell\} = \partial\{g \leq \ell\} = \{g = \ell\}$,*
- *$\{g = \ell\}$ intersects L transversally.*

Proof. It is for instance a direct consequence of Lemma A.9 of [RV19] (in this lemma, one needs the additional hypothesis that the field is not degenerate, which is actually a consequence of the fact that the support of the Fourier transform of q contains an open set – which holds since q is non-negative, not identically equal to 0 and L^1 – see e.g. Theorem 6.8 of [Wen04]). □

Given a continuous field f in $\mathbb{R}^d$ and $\rho_1, \rho_2 > 0$, we let $\mathrm{Cross}(\rho_1, \rho_2)$ be the event that there is a path included in $\{f \geq 0\} \cap ([0, \rho_1] \times [0, \rho_2])$ from the left side of $[0, \rho_1] \times [0, \rho_2]$ to its right side.

Theorem 2.4 ([KT20])**.** *Assume that q satisfies Assumption 1.4 for some $\beta > d$. Then, for every $\rho > 0$, there exists a constant $c > 0$ that depends only on ρ such that, for every $R > 0$,*

$$c < \mathbb{P}\left[\mathrm{Cross}(\rho R, R)\right] < 1 - c.$$

It is important to note that c does not depend on q.

Proof. Let $a > 0$. In [KT20], Köhler-Schindler and Tassion prove a box-crossing property (with constants that only depend on a) for any FKG bond percolation model on $\mathbb{Z}^2$ that satisfies the following two properties: the model is invariant under the symmetries of $\mathbb{Z}^2$ and the probability of the left-right crossing of $[0, n]^2$ is at least a for every $n \geq 1$.

The proof directly extends to positively correlated, stationary, isotropic, centered Gaussian fields as soon as one can prove some smoothness properties about the nodal lines. More precisely, by using for instance Lemma 2.3 (with $\ell = 0$) one obtains that $\mathbb{P}[\mathrm{Cross}(R, R)] = 1/2$ for every $R > 0$ and that all the geometric constructions from [KT20] hold a.s., and one thus obtains Theorem 2.4. □

For future reference, let us note that Lemma 2.3 also holds in higher dimensions (the proof is the same):

Lemma 2.5. *Assume that q satisfies Assumption 1.4 for some $\beta > d$. Let $d' \leq d$, $\ell \in \mathbb{R}$ and let $g = f_{|\mathbb{R}^{d'}}$. Then, the following holds a.s.: the sets $\{g \geq \ell\}$ and $\{g \leq \ell\}$ are two $\mathcal{C}^1$-smooth d'-dimensional manifolds with boundary. Moreover, $\partial\{g \geq \ell\} = \partial\{g \leq \ell\} = \{g = \ell\}$ a.s.*

Using Theorem 2.4 in conjunction with Lemma 2.1, one can easily construct various gluing patterns by considering crossings of suitable rectangles. These gluing constructions are classical in planar percolation theory and will be used freely throughout this article. We briefly list a few simple geometric facts related to these constructions and refer the reader to [Wer07] for more details on this matter.

i) For each $\delta > 0$, there exist $c, N > 0$ such that for any $R > 0$ there exists a collection of less than N rectangles of size $3cR \times cR$ such that any continuous path in $[-R, R]^2$ of diameter at least δR must cross at least one of these rectangles.

ii) There exists an integer k and rectangles $Q_1, \dots, Q_k$ such that if γ_i is a continuous path crossing Q_i in the long direction for $i = 1, \dots, k$, then, $\cup_i \gamma_i$ contains a circuit in $D(2)$ surrounding $D(1)$.

iii) There exists an integer k' and $x_1, \dots, x_{k'} \in \mathbb{R}^2$ such that if η_i is a circuit around $D(x_i, 1)$ in $D(x_i, 2)$ for $i = 1, \dots, k'$, then $\cup_i \eta_i$ contains a crossing in the long direction of $[0, 20] \times [0, 10]$.

iv) If $0 < r_1 \leq r_2 \leq r_3$ with $r_1 \leq r_2/2$ and $r_2 \leq r_3/2$, if there exist three continuous paths $\gamma_1, \gamma_2, \gamma_3$ that cross the annuli $D(r_2) \setminus D(r_1)$, $D(2r_2) \setminus D(r_2/2)$ and $D(r_3) \setminus D(r_2)$ respectively, and if there exist two circuits γ_4, γ_5 in the annuli $D(2r_2) \setminus D(r_2)$ and $D(r_2) \setminus D(r_2/2)$ respectively, then $\cup_i \gamma_i$ contains a path that crosses the annulus $D(r_3) \setminus D(r_1)$ from inside to outside. The same also holds in a half-plane for instance.

v) Item ii) together with Theorem 2.4 and Lemma 2.1 imply that there exists $\eta > 0$ such that for each $R \geq 1$ and $r \geq r_q$ the probability that $\{f_r \geq 0\}$ contains a path connecting $D(R)$ to $\partial D(2R)$ is at most η. If furthermore $R \geq r$, then by considering a well-chosen family of $O(\log(R/r))$ concentric annuli at mutual distance at least r, one deduces that the probability that $D(r)$ is connected to $\partial D(R)$ in $\{f_r \geq 0\}$ is at most $C(r/R)^c$ for some universal $c, C > 0$.

vi) By considering the endpoint of a square crossing, one deduces that there exists $c > 0$ such that for every $R > 0$ and $r \geq r_q$, the probability that 0 is connected to $\partial D(R)$ by a path in $\{f_r \geq 0\} \cap \{x \in \mathbb{R}^2 : x_2 \geq 0\}$ is at least cR^{-1}. In fact, this bound is not optimal at all but it will be sufficient for our purposes.

3 Clusters are not confined in slabs

In this section, we prove Proposition 1.12.

3.1 Setting of the proof

In this section, we fix some $d \geq 3$ and a function $q : \mathbb{R}^d \to \mathbb{R}$ that satisfies Assumption 1.4 for some $\beta > d$. Also, we work with three numbers $0 < \gamma < a < b < 1$ as well as a parameter $\theta_0 > 0$. Throughout the section, $R > 0$ and $r \geq r_q$ will denote two scale parameters and ℓ some level. Moreover, we let $\ell \in \mathbb{R}$ and we write $\varepsilon_0 := (1/2)R^{-2+\theta_0}$. Unless ortherwise stated, the constants below may depend on $q, \gamma, a, b, \theta_0$, but are uniform in R, r and ℓ. Recall that for any $d' \in [1, d]$ and any set $D \subset \mathbb{R}^{d'}$, we identify D with $D \times \{0\}^{d-d'}$.

We start with a definition.

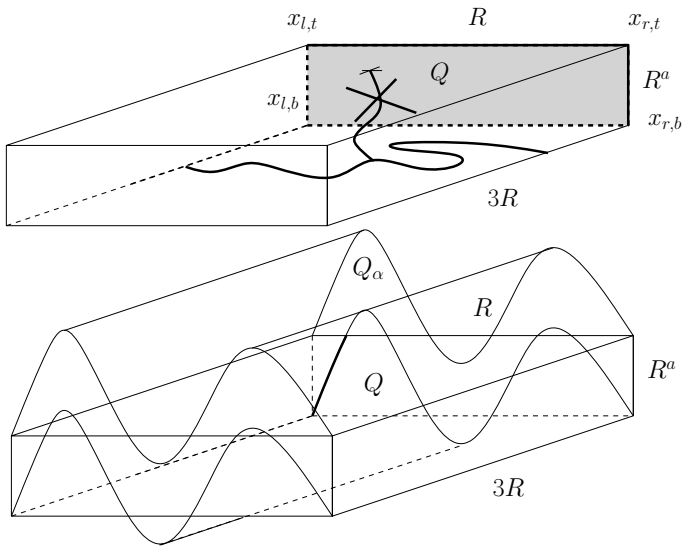

Figure 1: i) The quad Q (in grey) and the event $E_\ell(R)$ in $[0,3R]\times Q$; ii) a quad Q_α and the set $[0,3R]\times Q_\alpha$. A segment (in bold) of the bottom side of Q_α connects the left-bottom corner of Q to its top side.

Definition 3.1. Let $R>0$ and $\ell,\ell'\in\mathbb{R}$. We define the event $E_{\ell,\ell'}(R)$ as follows: a function $u\in C(\mathbb{R}^d)$ belongs to $E_{\ell,\ell'}(R)$ if there exists a connected component $\mathcal{C}$ of $\{u\geq\ell'\}\cap([0,3R]\times[0,R])$ such that

i) $\mathcal{C}$ contains a crossing from top to bottom in $[0,3R]\times[0,R]$ (i.e. from $[0,3R]\times\{0\}$ to $[0,3R]\times\{R\}$) and
ii) $\mathcal{C}$ is *not* connected to $[0,3R]\times[0,R]\times\{R^a\}$ by a continuous path in $\{u\geq\ell\}\cap([0,3R]\times[0,R]\times[0,R^a])$.

Moreover, we let $E_\ell(R)=E_{\ell,\ell}(R)$.

As we will see at the end of this subsection, the proof of Proposition 1.12 essentially boils down to bounding the probability of $\mathbb{P}[f_r\in E_{\ell,\ell+2\varepsilon_0}(R)]$. We will do that by introducing a family of events $(E^\alpha_{\ell+\varepsilon_0}(R):\alpha\in A)$ for some very large set A that satisfy

$$\mathbb{P}[f_r\in E_{\ell,\ell+2\varepsilon_0}(R)]\leq\mathbb{P}[f_r\in E^\alpha_{\ell+\varepsilon_0}(R)]+\exp(-R^c)$$

for every α and some constant $c>0$ (recall that $\varepsilon_0=(1/2)R^{-2+\theta_0}$). If the events $E^\alpha_{\ell+\varepsilon_0}(R)$ were mutually *disjoint*, this would imply that $\min_\alpha\mathbb{P}[E^\alpha_{\ell+\varepsilon_0}(R)]$ is small and as a result that $\mathbb{P}[E_{\ell,\ell+2\varepsilon_0}(R)]$ is also small. What we will actually do is relate the $E^\alpha_{\ell+\varepsilon_0}(R)$'s to some events $A^\alpha_{\ell+\varepsilon_0}(x,R)$'s which are mutually disjoint and deduce the desired bound. The $E^\alpha_{\ell+\varepsilon_0}(R)$ events will be constructed from $E_{\ell+\varepsilon_0}(R)$ by *slightly deforming* the ambient

space and proving that the law of the field is not altered too much by this deformation. This is a feature that is specific to this percolation model in the continuum and we are unaware of similar arguments for percolation models on lattices. The deformation amounts to a slight sprinkling in the percolation model, which is why we can only bound the probability of the event $E_{\ell,\ell+2\varepsilon_0}(R)$ instead of that of $E_\ell(R)$.

Let us note that similar disjoint events have been used in [Wer95] in order to study Brownian disconnection exponents (see in particular Section 4 therein).

The events $(E_\ell^\alpha(R) : \alpha \in A)$ are defined as

$$\{u \in E_\ell^\alpha(R)\} := \{u \circ (Id_1 \times F_\alpha \times Id_{d-3}) \in E_\ell(R)\},$$

where the maps $(F_\alpha, \alpha \in A)$ are introduced in Lemma 3.2 below and, by definition, for $x \in \mathbb{R}^d$,

$$u \circ (Id_1 \times F_\alpha \times Id_{d-3})(x) = u(x_1, F_\alpha(x_2, x_3), x_4, \dots, x_d)\,.$$

In a first stage, the reader may skip the details of this lemma and just keep track in their mind of the "nicknames" of the three items, and come back to it in the next subsection.

Lemma 3.2 (Existence of a large family of rippling quads)**.** *Recall that we work with some numbers $0 < a < b < 1$. There exist $c_1, \dots, c_4, R_0 > 0$ such that for every $R \geq R_0$, there exists a family $(F_\alpha)_{\alpha \in A}$ of embeddings (i.e. C^2 diffeomorphisms onto their images)*

$$F_\alpha \,:\, \mathbb{R} \times [-R^a, 2R^a] \to \mathbb{R}^2$$

mapping 0 to 0 with the following three properties:

1. *(A is large) $|A| \geq \exp(c_1 R^{1-b})$.*
2. *(the F_α's are sufficiently different) For each distinct $\alpha, \alpha' \in A$, the following or its analogue obtained by interchanging α and α' holds: there exists a continuous path included in $F_\alpha([0,R]\times\{0\}) \cap F_{\alpha'}([0,R]\times[0,R^a])$ going from 0 to $F_{\alpha'}([0,R]\times\{R^a\})$.*
3. *(the F_α's are almost affine isometries) The F_α's are C^2 and for each $\alpha \in A$ and $x \in \mathbb{R} \times [-R^a, 2R^a]$ we have the following:*
   - *i) (F_α is quasi-affine) $|d_x^2 F_\alpha| \leq c_2 R^{2(a-b)}$;*
   - *ii) (F_α is a quasi-isometry) there exists a rotation $J_x^\alpha \in O(2)$ such that*
   $$|d_x F_\alpha - J_x^\alpha| \leq c_3 R^{2(a-b)};$$
   - *iii) (F_α is bi-Lipschitz) for each $y \in \mathbb{R} \times [-R^a, 2R^a]$,*
   $$|x-y|/c_4 \leq |F_\alpha(x) - F_\alpha(y)| \leq c_4 |x-y|.$$

The third property above implies that F_α is sufficiently close to an affine isometry. Since χ_r and q are radial, composition by $Id_1 \times F_\alpha \times Id_{d-3}$ will almost preserve the law of f_r. In the following lemma, we will use this to compare the probability of the events $E_{\ell,\ell+2\varepsilon_0}(R)$ and $E^\alpha_{\ell+\varepsilon_0}(R)$.

Lemma 3.3. *Assume that* $\gamma(2+d/2)+3(a+1-b)<\theta_0$. *There exist* $c, R_0>0$ *such that for every* $\ell\in\mathbb{R}$, $R\geq R_0$, $r\in[r_q,R^\gamma]$ *and every* $\alpha\in A$,

$$\mathbb{P}\left[f_r\in E_{\ell,\ell+2\varepsilon_0}(R)\right]\leq\mathbb{P}\left[f_r\in E^\alpha_{\ell+\varepsilon_0}(R)\right]+\exp(-R^c).$$

In Section 3.4, we will relate the events $E^\alpha_\ell(R)$ to certain events $A^\alpha_\ell(R)$ which we will show are mutually disjoint for different $\alpha\in A$ using the second property in Lemma 3.2. With some work, this will imply the following upper bound on the probability of $E^\alpha_\ell(R)$.

Lemma 3.4. *There exist* $C, R_0>0$ *such that for every* $\ell\in\mathbb{R}$, $R\geq R_0$ *and* $r\geq r_q$,

$$\min_{\alpha\in A}\mathbb{P}\left[f_r\in E^\alpha_\ell(R)\right]\leq\frac{CR}{|A|}.$$

The proofs of the three lemmata are dispatched in the following three subsections. Before diving into these proofs, let us explain how Proposition 1.12 is derived.

Proof of Proposition 1.12. Take $0<\gamma<a<b<1$ such that $\gamma(2+d/2)+3(a+1-b)<\theta_0$. Then, Lemma 3.3 and Lemma 3.4 (applied to level $\ell+\varepsilon_0$) together imply that there exist $c, R_0>0$ such that, for every $R\geq R_0$ and $r\in[r_q,R^\gamma]$,

$$\mathbb{P}\left[f_r\in E_{\ell,\ell+2\varepsilon_0}(R)\right]\leq\exp(-R^c).$$

By definition of $E_{\ell,\ell+2\varepsilon_0}(R)$, Proposition 1.12 follows by tiling $D(2R)$ with a suitable collection of $O(1)$ rectangles of size $3R\times R$ and considering the union of the copies of the events $E_{\ell,\ell+2\varepsilon_0}(R)$ for each of these rectangles (see for instance the discussion at the end of Section 2.3). □

3.2 Proof of Lemma 3.2

Let $R>0$ and introduce the notation $Q:=[0,R]\times[0,R^a]$ and $S:=\mathbb{R}\times[-R^a,2R^a]$.

Construction of the family $(F_\alpha)_{\alpha\in A}$**.** Fix $N=\lfloor R^{1-b}/3\rfloor$ and $A=\{-1,1\}^N$. Fix $e_1:[0,3]\to[0,1]$ a smooth function which vanishes on $[0,1]$ and is equal to 1 on $[2,3]$ and write $e_{-1}=-e_1$.

Given $\alpha=(\alpha_1,\dots,\alpha_N)\in A$ for some $N\geq 1$, define a smooth function $e_\alpha:\mathbb{R}\to\mathbb{R}$ recursively (note that e_α is piecewise constant outside of $[0,3N]$) as follows:

(a) $e_\alpha(x)=0$ for $x\leq 0$,
(b) for each $0\leq k<N$ and $3k\leq x\leq 3(k+1)$, $e_\alpha(x):=e_{\alpha_{k+1}}(x-3k)+e_\alpha(3k)$,
(c) $e_\alpha(x)=e_\alpha(3N)$ for $x\geq 3N$.

We then define

$$\begin{aligned}\gamma_\alpha:\quad&\mathbb{R}\to\mathbb{R}^2,\\&t\mapsto(t,R^a e_\alpha(t/R^b)).\end{aligned}$$

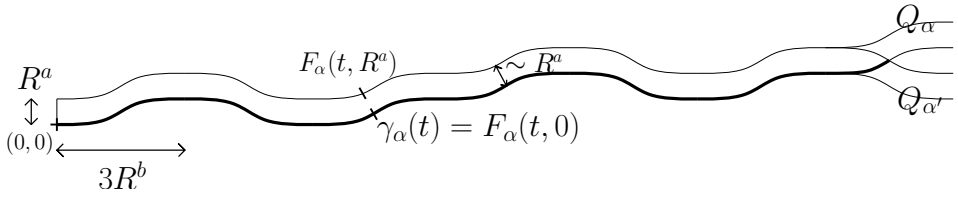

Figure 2: Two quads Q_α and $Q_{\alpha'}$ until they differ and (in bold) the path included in the bottom side of Q_α that connects 0 to the top side of $Q_{\alpha'}$.

In addition, define $\nu_\alpha(t)$ as the $\pi/2$-counterclockwise rotation of $\gamma'_\alpha(t)$ and finally F_α as follows:

$$\begin{aligned} F_\alpha : \quad S \quad &\to F_\alpha(S) \subset \mathbb{R}^2, \\ (t, s) &\mapsto \gamma_\alpha(t) + s\nu_\alpha(t). \end{aligned}$$

For future reference, let $Q_\alpha = F_\alpha(Q)$ (see Figure 2).

Checking that the construction satisfies the required properties. We first note that the first point is obvious as

$$|A| = 2^N \geq \exp(c_1 R^{1-b})$$

for some constant $c_1 > 0$ independent of everything else. Next, the second point of the lemma is easily checked to hold for every two distinct $\alpha, \alpha' \in A$ by considering the first letter where the two words differ (see Figure 2). Let us also note that the third point will imply that the F_α's are embeddings (if R is sufficiently large). Indeed, by 3.iii) they are bijections and by 3.ii) they are local C^1-diffeomorphisms.

All that remains is to show the third point of the lemma. To show it, we need the following claims whose proofs follow by direct computation and are left for the reader. In these claims, the constants only depend on the choice of the function e_1.

Claim 3.5. *There exists $C_0 > 0$ such that the following holds for each $\alpha \in A$ and $t \in \mathbb{R}$,*

- $1 - |\gamma'_\alpha(t)| = 1 - |\nu_\alpha(t)| \leq C_0 R^{2(a-b)}$*;*
- *the absolute value of the second component of $\gamma'_\alpha(t)$ is no greater than $C_0 R^{a-b}$.*

Claim 3.6. *There exists $C_1 > 0$ such that for each $\alpha \in A$ and each $t \in \mathbb{R}$,*

$$|\gamma''_\alpha(t)| + |\gamma'''_\alpha(t)| \leq C_1 R^{a-2b} \mathbf{1}_{0 \leq t \leq R}.$$

Let us now conclude the proof of the third point. Fix $\alpha \in A$. To begin with,

$$\begin{aligned} dF_\alpha &= (\gamma'_\alpha(t) + s\nu'_\alpha(t))dt + \nu_\alpha(t)ds, \\ d^2F_\alpha &= (\gamma''_\alpha(t) + s\nu''_\alpha(t))dt^2 + 2\nu'_\alpha(t)dsdt. \end{aligned}$$

By Claim 3.6, we get

$$dF_\alpha = \gamma'_\alpha(t)dt + \nu_\alpha(t)ds + O(R^{2(a-b)}),$$
$$d^2F_\alpha = O(R^{2(a-b)}).$$

The second estimate establishes the quasi-affine property i). Moreover, by the first estimate of Claim 3.5 (and since $\nu_\alpha(t)$ and $\gamma'_\alpha(t)$ are orthogonal), this also establishes the quasi-isometry property ii).

All that remains is to prove iii). Let $(s_1,t_1),(s_2,t_2) \in S$ and let us note that

$$F_\alpha((s_1,t_1)) - F_\alpha((s_2,t_2)) = \gamma_\alpha(t_1) - \gamma_\alpha(t_2) + (s_1-s_2)\nu_\alpha(t_1) + s_2(\nu_\alpha(t_1) - \nu_\alpha(t_2)).$$

By Claim 3.6,

$$\|s_2(\nu_\alpha(t_1) - \nu_\alpha(t_2))\| = O(|t_1-t_2|R^{2(a-b)}) = o(|t_1-t_2|).$$

The first component of $\gamma_\alpha(t_1) - \gamma_\alpha(t_2)$ equals $t_1 - t_2$ and by Claim 3.5 its second component is $O(|t_1-t_2|R^{a-b}) = o(|t_1-t_2|)$. Moreover, still by Claim 3.5, the first component of $(s_1-s_2)\nu_\alpha(t_1)$ is $O(|s_1-s_2|R^{a-b}) = o(|s_1-s_2|)$ and the absolute value of its second component is $\Theta(|s_1-s_2|)$. These estimates imply the desired bi-Lipschitz property (for instance by distinguishing between the cases $|t_1-t_2| \geq |s_1-s_2|$ and $|t_1-t_2| \leq |s_1-s_2|$).

3.3 Proof of Lemma 3.3

We keep the notation $Q = [0,R] \times [0,R^a]$ and $S = \mathbb{R} \times [-R^a, 2R^a]$ from the previous section. Fix $\alpha \in A$, recall that $q_r = q\chi_r$ and let $G_\alpha := Id_1 \times F_\alpha \times Id_{d-3}$.

We first note that $f_r \in E^\alpha_{\ell+\varepsilon_0}(R)$ if and only if $g^\alpha_r \in E_{\ell+\varepsilon_0}(R)$, where g^α_r is the Gaussian field defined for $x \in \mathbb{R} \times Q \times \mathbb{R}^{d-3}$ by

$$g^\alpha_r(x) := \int q_r(G_\alpha(x) - y)dW_y = \int_{G_\alpha(\mathbb{R}\times S\times\mathbb{R}^{d-3})} q_r(G_\alpha(x) - y)dW_y, \tag{6}$$

where the second equality is justified for R sufficiently large since q_r is supported in B_r, $r \leq R^\gamma$, and by Item 3.iii) of Lemma 3.2. We wish to prove that g^α_r behaves like f_r. To this purpose, consider the following intermediary field defined for $x \in \mathbb{R} \times S \times \mathbb{R}^{d-3}$ by

$$h^\alpha_r(x) := \int_{\mathbb{R}\times S\times\mathbb{R}^{d-3}} q_r\big(G_\alpha(x) - G_\alpha(y)\big)\,|\det(\mathrm{Jac}_y G_\alpha)|^{1/2}dW_y.$$

By the second formulation of $g^\alpha_r(x)$ from (6), the field $(h^\alpha_r)_{|\mathbb{R}\times Q\times\mathbb{R}^{d-3}}$ has the same law as (g^α_r) (if R is sufficiently large). Hence, in order to prove the lemma, it is sufficient to prove that there exist $c, R_0 > 0$ such that for any $\alpha \in A$, $R \geq R_0$ and $r \in [r_q, R^\gamma]$,

$$\mathbb{P}[f_r \in E_{\ell,\ell+2\varepsilon_0}(R)] \leq \mathbb{P}[h^\alpha_r \in E_{\ell+\varepsilon_0}(R)] + \exp(-R^c). \tag{7}$$

To this purpose, we are going to prove that the infinite norm of $f_r - h^\alpha_r$ on $[0,3R] \times Q \times [0,1]^{d-3}$ is typically small. We start with a claim.

Claim 3.7. *There exists $C > 0$ such that for every vector $u \in \mathbb{R}^d$ of Euclidean norm 1 and $x \in \mathbb{R} \times Q \times \mathbb{R}^{d-3}$,*

$$\mathrm{Var}(f_r(x) - h_r^\alpha(x)) + \mathrm{Var}(d_x(f_r - h_r^\alpha)(u)) \le C \frac{r^{4+d}}{R^{4(b-a)}}.$$

Before proving the claim, let us conclude the proof of the lemma by showing (7). Let $\mathcal{K} = K \times [0,1]^{d-3} \subseteq \mathbb{R} \times Q \times [0,1]^{d-3}$ with K a unit cube. Claim 3.7 and Kolmogorov's theorem (see for instance [NS16, Appendix A.9]) imply that for all $r \in [r_q, R^\gamma]$,

$$\mathbb{E}\left[\|f_r - h_r^\alpha\|_{\infty,\mathcal{K}}\right] = O\big(R^{-2+\theta_1}\big),$$

where $\theta_1 := \gamma(2 + d/2) + 2(a + 1 - b) < \gamma(2 + d/2) + 3(a + 1 - b) =: \theta < \theta_0$. The BTIS inequality (see for instance [AW09, Theorem 2.9]) gives that for R large enough and some constant $c_0 > 0$,

$$\mathbb{P}\big[\|f_r - h_r^\alpha\|_{\infty,\mathcal{K}} \ge R^{-2+\theta}\big] \le \exp(-R^{c_0}).$$

Taking a union bound on unit cubes $\mathcal{K}$ covering $[0, 3R] \times Q \times [0,1]^{d-3}$ gives that for R large enough,

$$\mathbb{P}\big[\|f_r - h_r^\alpha\|_{\infty,[0,3R]\times Q\times[0,1]^{d-3}} \ge R^{-2+\theta}\big] \le C_1 R^{2+a} \exp(-R^{c_0}) \le \exp(-R^{c_1}), \quad (8)$$

for some $C_1, c_1 > 0$. Now, note that since we have assumed that $\theta < \theta_0$ and since by definition, $\varepsilon_0 = (1/2)R^{-2+\theta_0}$, there exists $R_0 > 0$ such that $R \ge R_0$ implies

$$\{f_r \in E_{\ell,\ell+2\varepsilon_0}(R)\} \cap \big\{\|f_r - h_r^\alpha\|_{\infty,[0,3R]\times Q\times[0,1]^{d-3}} < R^{-2+\theta}\big\} \subset \{h_r^\alpha \in E_{\ell+\varepsilon_0}(R)\}. \quad (9)$$

Hence, (8) implies (7). This leaves the proof of Claim 3.7 to conclude the proof of the lemma.

Remark 3.8. Rather than using an inclusion such as (9), the reader could be tempted to apply the approximation result Lemma B.2 to $E_\ell(R)$. However, Lemma B.2 only applies to events which are in the Boolean algebra of a not too large family of monotonic events, which does not seem to be case of $E_\ell(R)$ (and $f_r - h_r^\alpha$ is too large for the other approximation results that we thought about – see e.g [RV19, BMR20] – to apply). If one could apply Lemma B.2 to $E_\ell(R)$ then one would obtain that $\mathbb{P}[E_\ell(R)] \simeq \mathbb{P}[E_\ell^\alpha(R)]$, which would imply that Proposition 1.12 holds with ℓ instead of $\ell + R^{-2+\theta_0}$.

Proof of Claim 3.7. Fix $x \in \mathbb{R} \times Q \times \mathbb{R}^{d-3}$. We start by bounding $\mathrm{Var}(f_r(x) - h_r^\alpha(x))$. First, we have

$$f_r(x) - h_r^\alpha(x) = \int_{\mathbb{R}\times S\times\mathbb{R}^{d-3}} \big[q_r(x-y) - q_r(G_\alpha(x) - G_\alpha(y))|\det(\mathrm{Jac}_y G_\alpha)|^{1/2}\big] dW_y$$

and therefore

$$\mathrm{Var}(f_r(x) - h_r^\alpha(x)) = \int_{\mathbb{R}\times S\times\mathbb{R}^{d-3}} \big[q_r(x-y) - q_r(G_\alpha(x) - G_\alpha(y))|\det(\mathrm{Jac}_y G_\alpha)|^{1/2}\big]^2 dy.$$

We let $L_x^\alpha := Id_1 \times J_x^\alpha \times Id_{d-3}$, where $J_x^\alpha \in O(2)$ is the rotation from Lemma 3.2. Below, we will use several times without mentioning it that the k^{th} derivatives of q_r with $k \leq 2$ are bounded. By Items 3.i) and 3.ii) of Lemma 3.2, we have

$$|\det(\mathrm{Jac}_y G_\alpha)| = 1 + O(R^{2(a-b)}) = 1 + O(R^{2(a-b)}) \tag{10}$$

and

$$\begin{aligned} q_r(G_\alpha(x) - G_\alpha(y)) &= q_r\big(d_x G^\alpha(x-y) + |x-y|^2 O(R^{2(a-b)})\big) \\ &= q_r\big(L_x^\alpha(x-y) + |x-y| O(R^{2(a-b)}) + |x-y|^2 O(R^{2(a-b)})\big) \\ &= q_r(L_x^\alpha(x-y)) + (|x-y| + |x-y|^2) O(R^{2(a-b)}). \end{aligned} \tag{11}$$

Since q_r is radial, $q_r(L_x^\alpha(x-y)) = q_r(x-y)$ and therefore (10) and (11) imply that

$$\begin{aligned} &q_r(x-y) - q_r\big(G_\alpha(x) - G_\alpha(y)\big)|\det(\mathrm{Jac}_y G_\alpha)|^{1/2} \\ &\quad = q_r(x-y) - \big[q_r(x-y) + (|x-y| + |x-y|^2) O(R^{2(a-b)})\big](1 + O(R^{2(a-b)})) \\ &\quad = (1 + |x-y| + |x-y|^2) O(R^{2(a-b)}). \end{aligned} \tag{12}$$

(The constant 1 comes from the fact that q_r is bounded.) By Item 3.iii) of Lemma 3.2, $|G_\alpha(x) - G_\alpha(y)| \asymp |x-y|$ so there exists a constant $C > 0$ such that for every $|x-y| \geq Cr$,

$$q_r(x-y) = q_r(G_\alpha(x) - G_\alpha(y)) = 0. \tag{13}$$

By combining (12) and (13) (more precisely, by using (12) on a ball of radius of order r around x and (13) on the complement of this ball) we obtain that

$$\mathrm{Var}(f_r(x) - h_r^\alpha(x)) = O(R^{4(a-b)}) \int_0^{O(r)} (1 + s + s^2)^2 s^{d-1} ds = O\Big(\frac{r^{4+d}}{R^{4(b-a)}}\Big),$$

which is the desired result.

Let us now estimate $\mathrm{Var}(d_x(f_r - h_r^\alpha)(u))$. We have

$$d_x(f_r - h_r^\alpha)(u) = \int_{\mathbb{R}\times S\times\mathbb{R}^{d-3}} d_{x-y} q_r(u) - d_{G_\alpha(x)-G_\alpha(y)} q_r \circ d_x G_\alpha(u) |\det(\mathrm{Jac}_y G_\alpha)|^{1/2} dW_y.$$

A computation similar to (11) gives

$$\begin{aligned} d_{G_\alpha(x)-G_\alpha(y)} q_r \circ d_x G_\alpha(u) &= d_{G_\alpha(x)-G_\alpha(y)} q_r(L_x^\alpha(u)) + (|x-y| + |x-y|^2) O(R^{2(a-b)}) \\ &= d_{L_x^\alpha(x-y)} q_r(L_x^\alpha(u)) + (|x-y| + |x-y|^2) O(R^{2(a-b)}). \end{aligned} \tag{14}$$

By (10) and (14), we have

$$\begin{aligned} &d_{x-y} q_r(u) - d_{G_\alpha(x)-G_\alpha(y)} q_r \circ d_x G_\alpha(u) |\det(\mathrm{Jac}_y G_\alpha)|^{1/2} \\ &= d_{x-y} q_r(u) - \big[d_{L_x^\alpha(x-y)} q_r(L_x^\alpha(u)) + (|x-y| + |x-y|^2) O(R^{2(a-b)})\big](1 + O(R^{2(a-b)})) \\ &= d_{x-y} q_r(u) - d_{L_x^\alpha(x-y)} q_r(L_x^\alpha(u)) + (1 + |x-y| + |x-y|^2) O(R^{2(a-b)}) \\ &= (1 + |x-y| + |x-y|^2) O(R^{2(a-b)}). \end{aligned}$$

(In the last line we use that q_r is radial). From there, the proof is the same as for $\mathrm{Var}(f_r(x) - h_r^\alpha(x))$. □

3.4 Proof of Lemma 3.4

Given $x \in [0,3R] (= [0,3R] \times \{0\}^{d-1})$, we define the events $A_\ell^\alpha(x,R)$ as

$$\{u \in A_\ell^\alpha(x,R)\} := \{u \circ (Id_1 \times F_\alpha \times Id_{d-3}) \in A_\ell(x,R)\},$$

where $A_\ell(x,R) \subset E_\ell(R)$ is as follows: a function $u \in C(\mathbb{R}^d)$ belongs to $A_\ell(x,R)$ if there exists a connected component $\mathcal{C}$ of $\{u \geq \ell\} \cap ([0,3R] \times [0,R] \times [0,R^a])$ such that

i) $\mathcal{C}$ contains a crossing from top to bottom in $[0,3R] \times [0,R]$;
ii) $\mathcal{C}$ does *not* intersect $[0,3R] \times [0,R] \times \{R^a\}$;
iii) $x \in \mathcal{C}$.

The reader can think about these "A-events" as the same as the "E-events" except that they are arm-type events (starting from x) rather than crossing-type events. We also let

$$B_\ell^\alpha(x,R) := A_\ell^\alpha(x,R) \setminus \bigcup_{y \in [0,3R] \times \{0\}^{d-1} : y_1 < x_1} A_\ell^\alpha(y,R),$$

which corresponds to the event that x is the "lowest point" for which $A_\ell^\alpha(x,R)$ occurs. In particular, we have $E_\ell^\alpha(R) = \sqcup_{x \in [0,3R]} B_\ell^\alpha(x,R)$. The following claim translates this decomposition into a tractable formula.

Claim 3.9. *There exists $R_0 > 0$ such that for every $\ell \in \mathbb{R}$, $R \geq R_0$ and $r \geq r_q$,*

$$\mathbb{P}[f_r \in E_\ell^\alpha(R)] = \mathbb{P}[f_r \in A_\ell^\alpha(0,R)] + \int_0^{3R} \mathbb{E}[|\partial_{x_1} f_r(x)| 1_{B_\ell^\alpha(x,R)} \mid f_r(x) = \ell] \gamma_{f_r(x)}(\ell) dx_1,$$

where $\gamma_{f_r(x)}$ is the density function of $f_r(x)$, and $x := (x_1, 0, \dots, 0)$.

The proof of Claim 3.9 is given in Section 3.5. Let us now conclude the proof of the lemma. We have

$$\begin{aligned} |A| \min_{\alpha \in A} \mathbb{P}[f_r \in E_\ell^\alpha(R)] &\leq \sum_{\alpha \in A} \mathbb{P}[f_r \in E_\ell^\alpha(R)] \\ &= \sum_{\alpha \in A} \mathbb{P}[f_r \in A_\ell^\alpha(0,R)] + \sum_{\alpha \in A} \int_0^{3R} \mathbb{E}[|\partial_{x_1} f_r(x)| 1_{B_\ell^\alpha(x,R)} \mid f_r(x) = \ell] \gamma_{f_r(x)}(\ell) dx. \end{aligned}$$

The crucial point is that Item 2 of Lemma 3.2 implies that for every fixed x, the $(A_\ell^\alpha(x,R) : \alpha \in A)$ are disjoint. By first using this with $x = 0$, we obtain that the first sum is bounded by one. Let us deal with the second sum. Since $B_\ell^\alpha(x,R) \subset A_\ell^\alpha(x,R)$, the $B_\ell^\alpha(x,R)$ are also disjoint and $\sum_{\alpha \in A} 1_{B_\ell^\alpha(x,R)} \leq 1$ so the second sum is at most

$$\int_0^{3R} \mathbb{E}\left[|\partial_{x_1} f_r(x)| \mid f_r(x) = \ell\right] \gamma_{f_r(x)}(\ell) dx = 3R \mathbb{E}[|\partial_{x_1} f_r(0)|] \gamma_{f_r(0)}(\ell)$$

by stationarity. Since the right-hand side is $O(R)$, all in all, there is some constant $C > 0$ such that

$$|A| \min_{\alpha \in A} \mathbb{P}[f_r \in E_\ell^\alpha(R)] \leq 1 + CR.$$

(The reader can note that we have used that $r \geq r_q$ in order to obtain that C does not depend on r.) This implies the desired result.

3.5 Proof of Claim 3.9

In this section, we prove Claim 3.9. Since in Bernoulli percolation, which is our guiding model, Claim 3.9 would simply be replaced by a decomposition of a crossing event into arm events, from the point of view of percolation, this section is just a technical obstacle and should be skipped upon first reading. However, we believe that the proof is interesting from the point of view of the geometry of Gaussian fields. It is inspired by the proof of the Kac–Rice formula given in Appendix C of [Let16]. With this tool in mind, the formula is easily established for fields taking values in finite dimensional function spaces. We must then take the formula to the limit.

Proof of Claim 3.9. We prove the claim in a more general setting. Moreover, for brevity, we prove it for $E_\ell(R)$ instead of $E_\ell^\alpha(R)$. The proof in the general case is similar although one must consider curved rectangles instead of flat ones. Throughout the proof, $R > 0$ will stay fixed and we consider a compact subset $\mathcal{K} \subset \mathbb{R}^d$ with smooth boundary that is a neighbourhood of $[0, 3R] \times Q = [0, 3R] \times [0, R] \times [0, R^a]$. Let g be an a.s. C^2 centered Gaussian field on $\mathcal{K}$ with covariance $(x, y) \mapsto K(x, y) = \mathbb{E}[g(x)g(y)]$ in $C^{2,2}(\mathcal{K}, \mathcal{K})$ (i.e., $\partial_x^\alpha \partial_y^\beta K \in C(\mathcal{K} \times \mathcal{K})$ for $|\alpha|, |\beta| \leq 2$) such that

$$\forall x, y \in \mathcal{K} \text{ with } x \neq y,\ (g(x), g(y), d_y g) \text{ is a non-degenerate Gaussian vector.} \tag{15}$$

Our goal is to prove that

$$\mathbb{P}[g \in E_\ell(R) \setminus A_\ell(0, R)] = \int_0^{3R} \mathbb{E}\left[|\partial_{x_1} g(x)| 1_{B_\ell(x,R)} \mid g(x) = \ell\right] \gamma_{g(x)}(\ell) dx_1. \tag{16}$$

Before proving (16), let us use it in order to conclude the proof of the claim.

Proof of Claim 3.9. In order to prove the claim, we only need to show that f_r satisfies (15). If this were not the case, then there would exist some $x \neq y$ and some real numbers $a, b, c_1, \dots, c_d$ not all equal to 0 such that $a f_r(x) + b f_r(y) + \sum_{i=1}^d c_i \partial_{x_i} f_r(y) = 0$. As a result, there would exist some vector $v \in \mathbb{R}^d \setminus \{0\}$ and three real numbers a, b, c not all equal to 0 such that

$$a f_r(x) + b f_r(y) + c \partial_v f_r(y) = 0. \tag{17}$$

To conclude, we need the following lemma.

Lemma 3.10 (Lemma A.2 of [BMM20])**.** *Let f be a centered, stationary and C^3 Gaussian field on $\mathbb{R}^2$. If the support of the spectral measure of f contains an open set, then for every distinct $x, y \in \mathbb{R}^2$, $(f(x), f(y), d_x f, d_y f)$ is non-degenerate.*

Let $\mathcal{P} \subset \mathbb{R}^d$ denote a plane such that $x, y, x + v \in \mathcal{P}$. The spectral measure of $(f_r)_{|\mathcal{P}}$ is continuous and not identically equal to 0, so Lemma 3.10 implies that (17) cannot be true. □

Let us now prove (16). The proof will follow from the co-area formula applied to a well chosen functional defined and differentiable on the complement of some exceptional

subset $W \subset C^1(\mathcal{K})$. Before moving forward, let us define W and prove a few facts about this set (and (15)). Let $\mathcal{F}$ be the set whose elements are the corners, open sides, open 2-dimensional faces and the interior of $[0,R] \times Q$. We define W as the set of $u \in C^1(\mathcal{K})$ for which there exists $F \in \mathcal{F}$ for which $u_{|F}$ has a critical point at height ℓ.

Claim 3.11. *1. The set of covariance functions $K \in C^{1,1}(\mathcal{K},\mathcal{K})$ such that the a.s. C^1 field g with covariance K satisfies (15) is open in the $C^{1,1}$ topology;*
2. For every $x \in (0,3R)$, $\mathbb{P}[g \in W] = \mathbb{P}[g \in W | g(x) = \ell] = 1$;
3. The boundary of $U := (E_\ell(R) \setminus A_\ell(0,R)) \setminus W$ in $C^1(\mathcal{K})$ is included in W;
4. U is open in $C^1(\mathcal{K})$;
5. For every $x \in (0,3R)$, the boundary of $B_\ell(x,R) \setminus W$ in $\{u \in C^1(\mathcal{K}) : u(x) = \ell\}$ is included in W.

Proof. We prove the points one by one.

For Item 1, proceed as follows. The negation of property (15) at $x \in \mathcal{K}$ is equivalent to a polynomial equation in $K(\cdot,\cdot)$ and its derivatives of order up to one in each variable, at (x,x) and (x,y). Since $\mathcal{K}$ is compact, this condition is open in $C^{1,1}(\mathcal{K},\mathcal{K})$.

We turn to Item 2. Since $y \mapsto (g(y), d_y g)$ is a.s. C^1 and we have assumed that for each $y \in \mathcal{K}$, $(g(y), d_y g)$ is non-degenerate, then, by [AT07, Lemma 11.2.10], $g \notin W$ a.s. Moreover, if we condition on $\{g(x) = \ell\}$ then $y \mapsto (g(y), d_y g)$ still satisfies this property on $\mathcal{K} \setminus \{x\}$ (indeed, if a Gaussian vector $(X_1, \dots, X_{k+1})$ is non-degenerate, then the law of $(X_1, \dots, X_k)$ is still non-degenerate when we condition on the value of X_{k+1})[2]. As a result, still by [AT07, Lemma 11.2.10], we also have $g \notin W$ a.s. under this conditioning.

In order to prove Item 3, let us consider some $u \notin W$. Then, the set $\{u = \ell\}$ is the intersection of $[0,3R] \times Q$ and of a C^1-manifold that intersects every $F \in \mathcal{F}$ transversally. As a result, for every $v \in C^1(\mathcal{K},\mathcal{K})$ with $||v||_{C^1(\mathcal{K})}$ sufficiently small, $\{u + v \geq \ell\}$ homotopy retracts to $\{u \geq \ell\}$ in a way that preserves every $F \in \mathcal{F}$. In particular, $u + v \in E_\ell(R) \setminus A_\ell(0,R)$ if and only if $u \in E_\ell(R) \setminus A_\ell(0,R)$.

We now show Item 4. By Item 3, if $u \in U$ then there is a $C^1(\mathcal{K})$-neighbourhood of u that is included in $E_\ell(R) \setminus A_\ell(0,R)$. Since W is closed in $C^1(\mathcal{K})$ (see e.g. the short proof of Lemma C.1 of [BMR20]), this implies the desired result.

For Item 5, let us consider some $u \notin W$. Then, the set $\{u = \ell\}$ is the intersection of $[0,3R] \times Q$ and of a C^1-manifold that intersects every $F \in \mathcal{F}$ transversally (in particular at x). As a result, for every $v \in C^1(\mathcal{K},\mathcal{K})$ with $v(x) = 0$ and $\|v\|_{C^1(\mathcal{K})}$ sufficiently small, $\{u + v \geq \ell\}$ homotopy retracts to $\{u \geq \ell\}$ in a way that preserves both each $F \in \mathcal{F}$ and each x. In particular, $u + v \in B_\ell(x,R)$ if and only if $u \in B_\ell(x,R)$. □

A. Proof in the finite-dimensional case. We are now ready to prove (16). We start by assuming that g belongs to a finite-dimensional space $V \subset C^2(\mathcal{K})$. Thus, there exists a scalar product $\langle \cdot, \cdot \rangle$ on V for which g has the law of the standard Gaussian distribution $\gamma_V(u)du$ on $(V, \langle \cdot, \cdot \rangle)$.

[2] To see this one can use that the variance of $(X_1, \dots, X_k)$ conditioned on the value of X_{k+1} does not depend on this value and that its mean is linear with respect to this value.

For every $u \in E_\ell(R)$, we let $X(u) \in [0,3R] = [0,3R] \times \{0\}^{d-1}$ be the (unique) point such that $u \in B_\ell(X(u), R)$. Recall that by Item 4 of Claim 3.11, $U \cap V$ is open in V. The map

$$\Phi : U \cap V \to (0, 3R) \tag{18}$$

sending u to the first coordinate of $X(u)$ is well defined and differentiable with differential

$$d_u\Phi(v) = \frac{-v(X(u))}{(\partial_{x_1} u)(X(u))}.$$

Thus, the norm of the differential at u is $|\partial_{x_1} u(X(u))|^{-1}$ times the norm of the evaluation map at $X(u)$. By construction of V, the norm of the evaluation map is[3] $\sqrt{K(X(u), X(u))}$. Then, the co-area formula (see e.g. [BZ13, Corollary 13.4.6]) applied to Φ yields

$$\begin{aligned}\mathbb{P}[g \in E_\ell(R) \setminus A_\ell(0,R)] = \int_U \gamma_V(u)du &= \int_0^{3R} \int_{U_x} \frac{\gamma_V(u)}{\|d_u\Phi\|} \overline{du} dx_1 \\ &= \int_0^{3R} \int_{U_x} \frac{|\partial_{x_1} u(x)| \gamma_V(u)}{\sqrt{K(x,x)}} \overline{du} dx_1,\end{aligned}$$

where the first equality comes from Item 2 of Claim 3.11 and

- $\gamma_V : U \to \mathbb{R}_+$ is the density function of g;
- $x = (x_1, 0, \dots, 0)$ and $U_x := \Phi^{-1}(x_1) = B_\ell(x, R) \cap (V \setminus W) \subset V_x := \{u \in V : u(x) = \ell\}$;
- $\overline{du}$ is the Lebesgue density on V_x.

As a result, we have

$$\mathbb{P}[g \in E_\ell(R) \setminus A_\ell(0,R)] = \int_0^{3R} \mathbb{E}[|\partial_{x_1} g(x)| 1_{g \in B_\ell(x,R)} \mid g(x) = \ell] \frac{\int_{V_x} \gamma_V(u) \overline{du}}{\sqrt{K(x,x)}} dx_1.$$

Next, we observe that $\int_{V_x} \gamma_V(u)\overline{du}$ is the density of $\mathcal{N}(0,1)$ at $\ell/\sqrt{K(x,x)}$ (because, by the observations from Footnote 3, V_x is a hyperplane of V translated by a vector in $V_x^\perp$ of norm $\ell/\sqrt{K(x,x)}$). This implies that $\int_{V_x} \gamma_V(u)\overline{du} = \sqrt{K(x,x)}\gamma_{g(x)}(\ell)$ and ends the proof of the claim in the finite dimensional case.

[3] Indeed, let $(\Psi_1, \dots, \Psi_{k-1})$ be an orthonormal basis of the kernel of the non-trivial linear form $L \in V^*$ defined by $L(v) = v(x)$ for some $x \in \mathcal{K}$ and let $\Psi_k \in \mathrm{Ker}(L)^\perp$ be a unit vector. Then,

$$\|L\| = |L(\Psi_k)| = \sqrt{\sum_{j=1}^k L(\Psi_j)^2} = \sqrt{\sum_{j=1}^k \Psi_j(x)^2} = \sqrt{K(x,x)}.$$

In the last equality, we have used that $K(x,y) = \sum_{j=1}^k \Psi_j(x)\Psi_j(y)$ for any $x, y \in \mathcal{K}$. (A more general form of this result follows from (A.8) and (C.6) from [Let16].)

B. Proof in the infinite-dimensional case. To cover the general (infinite dimensional) case, we start by arguing that (16) is stable by approximation. To this end, we first show the following claim:

Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of C^2 and centered Gaussian fields that converges a.s. in $C^2(\mathcal{K})$ to g and satisfies (16) for each n. Then, g also satisfies (16).

To show this claim, we use the following three properties:

i) The covariance kernels $K_n \in C^{2,2}(\mathcal{K},\mathcal{K})$ converge in $C^{2,2}(\mathcal{K},\mathcal{K})$ to K;
ii) The conditioning on $g(x) = \ell$ makes g a Gaussian field with covariance $(y,z) \mapsto K(y,z) - K(y,x)K(x,z)$ and mean $y \mapsto K(x,y)\ell/K(x,x)$;
iii) For every $x \in \mathcal{K}$, the law of g_n conditioned on $g_n(x) = \ell$ converges in distribution on the Borel-σ-algebra of the C^1-topology to the law of g conditioned on $g(x) = \ell$.

(To prove iii), one can use i), ii) and that the sequence is tight due to Kolmogorov's theorem – see e.g. Appendix A.9 of [NS16].) The points i), ii) and iii) above, Claim 3.11 and dominated convergence imply the claim.

So let us exhibit such a sequence $(g_n)_n$. (We follow the analogous arguments in the proof of [MRVK20, Lemma 3.1].) Let us first assume furthermore that g is a.s. C^∞ and let $H^{2d}(\mathcal{K})$ be the L^2-Sobolev space of order $2d$ on $\mathcal{K}$ (here $2d$ is arbitrary but we need it to be larger than $d/2+3$). Then, $H^{2d}(\mathcal{K}) \subset C^2(\mathcal{K})$ and the injection is continuous. Moreover, $g \in H^{2d}(\mathcal{K})$ since $g \in C^\infty(\mathcal{K})$. Let $(e_k)_k$ be an orthonormal basis for $H^{2d}(\mathcal{K})$ and let g_n denote the projection of g on the subspace generated by $e_1,\dots,e_n$. Then, g_n converges to g a.s. in $H^{2d}(\mathcal{K})$, and so converges to g a.s. in $C^2(\mathcal{K})$. This ends the proof in the case where g is C^∞.

To finish the argument, we remove the C^∞-smoothness assumption. To this purpose, we use that g is defined on a neighborhood $\mathcal{K}$ of $[0,3R]\times Q$ rather than on $[0,3R]\times Q$ only (on the contrary, all the above arguments also work with $\mathcal{K}$ replaced by $[0,3R]\times Q$). Let $\varepsilon > 0$ denote the distance between $[0,3R]\times Q$ and $\mathcal{K}$ and define g_n on $[0,3R]\times Q$ as the convolution between g and a Dirac approximation which is compactly supported in $B(\varepsilon/2)$. Then, g_n is smooth and converges to g a.s. in C^2, which is the desired property except that $\mathcal{K}$ is replaced by $[0,3R]\times Q$. But, as we have already observed, all what has been done above also works with $\mathcal{K}$ replaced by $[0,3R]\times Q$, so this ends the proof. □

4 A box-crossing property with polynomially many contact points

Let $d \geq 2$ and let q satisfy Assumption 1.4 for some $\beta > d$. In this section, we only consider $(f_r)_{|\mathbb{R}^2}$ and prove Proposition 4.1 below. All the constants in this section are universal since they are functions of the RSW constants from Theorem 2.4. We will often use the gluing constructions described in Section 2.3 in conjunction with the FKG inequality and the RSW theorem without mentioning them explicitely. Recall that we identify all subsets $D \subset \mathbb{R}^2$ with $D \times \{0\}^{d-2} \subset \mathbb{R}^d$ and that $D(R)$ and $D(x,R)$ are Euclidean discs (see Section 2.1).

Proposition 4.1. *There exists some universal $\eta > 0$ such that the following holds. For every $r_q \leq r \leq \rho \leq R$ and every continuous path $\mathcal{C} \subset \mathbb{R}^2$ that intersects both $D(R)$ and $\partial D(2R)$, there exists a (deterministic) countable family of points $(y_i)_{i\in I} \subset D(2R)$ at mutual distances at least 20ρ and at a distance at most ρ from $\mathcal{C}$ such that*

$$\mathbb{P}\left[\begin{array}{c}\#\{i \in I : D(y_i, \rho/100) \text{ is connected to } \partial D(3R) \\ \text{in } (D(3R) \cap \{f_r \geq 0\}) \setminus (\cup_{j\neq i} D(y_j, 2\rho))\} \geq \eta(R/\rho)^\eta\end{array}\right] \geq \eta.$$

Results of a similar flavor have appeared e.g. in the context of planar random cluster models, see Proposition 7.4 in [DMT21], cf. also the proof of Proposition 1.5 therein.

In order to prove Proposition 4.1, we introduce the following notion of R-good pair.

Definition 4.2 (Good pairs)**.** Let $x, z \in \mathbb{R}^2$, $R > 0$, and $\mathcal{C} \subset \mathbb{R}^2$ be a continuous path. Then (x, z) is called an *R-good pair* for $\mathcal{C}$ if it has the following properties:

1. $\mathcal{C}$ intersects both $D(x, 2R)$ and $\partial D(x, 10R)$;
2. $z \in \partial D(x, 10R)$ and is at a distance at least $11R$ from $\mathcal{C}$.

Observe for later purposes that with $\mathcal{C}$ as in the statement of Proposition 4.1 and $R' \leq R/100$, one can always find an R'-good pair (x, z) for $\mathcal{C} \cap D(1.9R)$ with $x \in \partial D(1.9R + 2R')$ and $z \in \partial D(1.9R + 12R')$.

We start with a first lemma.

Lemma 4.3 (First moment estimate)**.** *There exists a universal constant $\eta_0 > 0$ such that, for all $K \geq 100$, $r_q \leq r \leq R$, every continuous path $\mathcal{C}$ and every (KR)-good pair (x, z) for $\mathcal{C}$, there exists a countable collection $(y_i, z_i)_{i\in I}$ of R-good pairs for $\mathcal{C}$ such that*

- *For every $i \in I$, $y_i \in D(x, 5KR)$;*
- *The y_i's are at mutual distances at least $20R$;*
- *Let H_i be the event that there exists a circuit in $\{f_r \geq 0\} \cap D(z_i, 3R)$ around $D(z_i, 2R)$ that is connected to $D(z, 2KR)$ in $D(x, 9KR)$ by a path in $\{f_r \geq 0\}$ staying at a distance at least $12R$ from $\mathcal{C}$ and at a distance at least $11R$ from the y_j's, $j \neq i$. Then*

$$\mathbb{E}\left[\#\{i \in I : H_i \text{ occurs}\}\right] \geq \eta_0 K^{\eta_0}.$$

Remark 4.4. We first note, for future reference, that the following analogous (but much more direct) result holds by the RSW theorem. For all $R > 0$, $r \geq r_q$, every continuous path $\mathcal{C}$ and every R-good pair (x, z) for $\mathcal{C}$, there exists a point $y \in D(x, 5R)$ at distance at most R from $\mathcal{C}$ such that the following holds. Let $\widetilde{H}$ denote the event that there exists a path in $\{f_r \geq 0\} \cap D(x, 9R)$ from $D(y, R/100)$ to $D(z, 2R)$. Then, there exists a universal constant $\eta_1 > 0$ such that $\mathbb{P}[\widetilde{H}] \geq \eta_1$.

Proof of Lemma 4.3. Let Δ' be the boundary in $D(x, 5KR)$ of the union Γ of the balls $D(y, 11R)$ where y ranges over the set of points in $D(x, 5KR) \cap \frac{1}{100}R\mathbb{Z}^2$ at a distance at most $2R$ from $\mathcal{C}$. To avoid having to deal with what happens close to $\partial D(x, 5KR)$, we

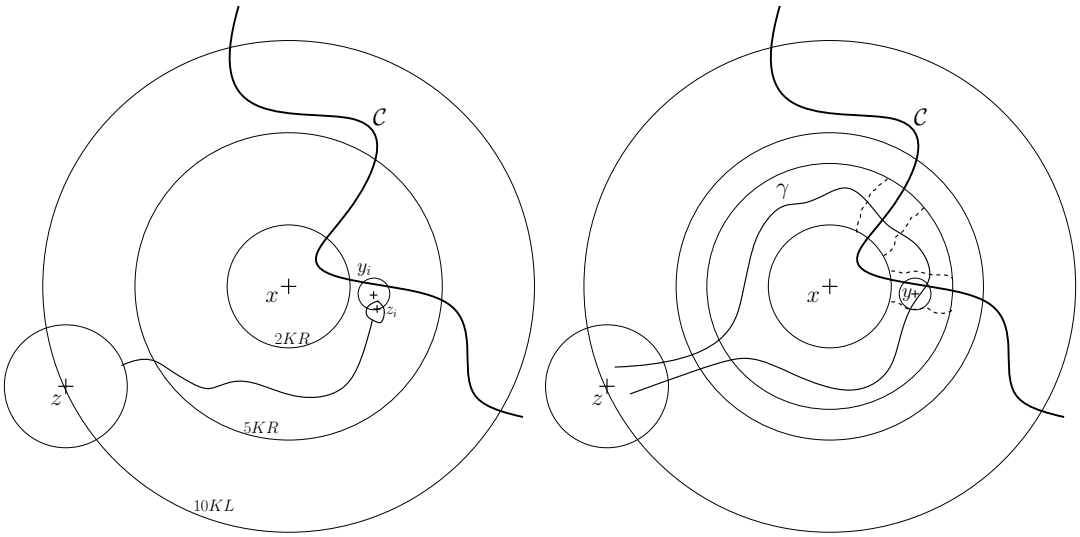

Figure 3: (a) The event H_i. (b) The path γ and (in dashed lines) the set η.

also consider $\Delta := \Delta' \cap D(x, 4KR)$. Furthermore, let Y be the set of such y for which $\partial D(y, 11R) \cap \Delta \neq \emptyset$.

For each $y \in Y$, let $G(y)$ (resp. $G^\star(y)$) denote the event that $D(y, 11R)$ is connected to $D(z, 2KR)$ in $D(x, 9KR)$ by a path in $\{f_r \geq 0\}$ (resp. $\{f_r \leq 0\}$) staying at a distance at least $12R$ from $\mathcal{C}$ and not intersecting the interior of Γ.

Suitable applications of the RSW property Theorem 2.4 (together with Lemma 2.3 and the fact that f_r is r-dependent), imply the existence of $c_1 > 0$ such that with probability at least c_1, there exists a path γ in $\{f_r = 0\} \cap D(x, 9KR)$ with endpoints in $D(z, 2KR)$ such that $D(z, 2KR) \cup \gamma$ contains a circuit in the annulus $D(x, 10KR) \setminus D(x, 2KR)$ separating its two boundary components. But $\mathcal{C}$ crosses this annulus and does not intersect $D(z, 2KR)$ so it must intersect γ (see Figure 3). Considering the first time γ intersects $\cup_{y \in Y} D(y, 11R)$ gives

$$\sum_{y \in Y} \mathbb{P}[G(y) \cap G^\star(y)] \geq c_1.$$

By the FKG inequality (see Lemma 2.1),

$$\mathbb{P}[G(y) \cap G^\star(y)] \leq \mathbb{P}[G(y)]\mathbb{P}[G^\star(y)]$$

and by spatial independence (using that $R \geq r$) and the RSW property, as mentioned at the end of Section 2.3 one may infer that $\mathbb{P}[G^\star(y)] \leq c_2 K^{-c_3}$, so we deduce overall that

$$\sum_{y \in Y} \mathbb{P}[G(y)] \geq c_4 K^{c_3},$$

for some $c_2, c_3, c_4 > 0$. We then extract a family $(y_i)_{i \in I} \subset Y$ of points at a mutual

distance at least $20R$ such that

$$\sum_{i\in I}\mathbb{P}[G(y_i)] \geq \frac{c_4}{10^4}K^{c_3}\,.$$

Applying the RSW property again near the starting point of the path described by $G(y_i)$ implies that there exists $c_5 > 0$ and a family $(z_i)_{i\in I}$ of points such that for each $i \in I$, z_i belongs to $\partial D(y_i, 10R)$ and is at a distance at least $11R$ from $\mathcal{C}$ and

$$\mathbb{P}[H_i] \geq c_5\mathbb{P}[G(y_i)],$$

where H_i is as in the statement of the lemma. The result follows readily. □

We are now in a position to prove the proposition. We will prove Proposition 4.1 by constructing a tree whose nodes are the good pairs obtained thanks to Lemma 4.3 and whose edges are open if the corresponding events H_i occur. Then, the existence of many open paths from the root to the leaves will imply the event under consideration in the proposition.

Proof of Proposition 4.1. Consider η_0 given by Lemma 4.3 and η_1 given by Remark 4.4 and fix $K \geq 100$ such that $\eta_0 K^{\eta_0} > 1$. For each integer $j \geq 0$, let

$$R_j = K^j\rho, \tag{19}$$

so in particular $R_0 = \rho$. Define J to be the supremum over $j \geq 0$ such that $R_j \leq R/100$. If $J = 0$ then the result is a direct consequence of the RSW theorem so we assume that $J \geq 1$.

We introduce a tree $\mathcal{T}$ of depth J as follows. All the vertices will be pairs $(x, z) \in \mathbb{R}^2$ except leaves which will be points $y \in \mathbb{R}^2$.

- The root of $\mathcal{T}$ is a R_J-good pair $(x_{\text{root}}, z_{\text{root}})$ for $\mathcal{C}\cap D(1.9R)$ with $x_{\text{root}} \in \partial D(1.9R+2R_J)$ and $z_{\text{root}} \in \partial D(1.9R + 12R_J)$ (cf. the observation following Definition 4.2 regarding its existence).

- The vertices of $\mathcal{T} \setminus \{(x_{\text{root}}, z_{\text{root}})\}$ which are not leaves are defined recursively as follows. For some $0 \leq j < J$, suppose that we consider a vertex $(x, z) \in \mathcal{T}$ which is a R_{J-j}-good pair for $\mathcal{C} \cap D(1.9R)$. Then, we add to $\mathcal{T}$ the vertices $(y_i, z_i)_{i\in I}$ given by Lemma 4.3 when applied with (x, z), K and $R = R_{J-j}$, and join each of them to (x, z) by an edge. We call the (y_i, z_i)'s the *descendants* of (x, z). Finally, we say that the edge between (x, z) and (y_i, z_i) (where (y_i, z_i) is a descendant of (x, z)) is *open* if H_i occurs, with H_i as appearing in the statement of Lemma 4.3.

- The leaves are defined as follows. Let us consider a vertex $(x, z) \in \mathcal{T}$ which is a ρ-good pair for $\mathcal{C}\cap D(1.9R)$ (note that $\rho = R_j$ with $j = 0$). Then, we add to $\mathcal{T}$ the vertex y given by Remark 4.4 when applied to (x, z), $R = \rho$, and join y to (x, z) by an edge. We say that this edge is *open* if the event $\widetilde{H}$ appearing in Remark 4.4 occurs.

For every $0 \le j < J$, let $\mathbf{V}_j$ be the set of vertices in $\mathcal{T}$ that are R_j-good pairs and are connected by an open path to $(x_{\text{root}}, z_{\text{root}})$. Also, let $\mathbf{V}_J = \{(x_{\text{root}}, z_{\text{root}})\}$ and let $\mathbf{V}_{-1}$ denote the set of leaves which are connected by an open path to $(x_{\text{root}}, z_{\text{root}})$.

Due to the nested structure of the events H_i and the notion of open edge in $\mathcal{T}$, a path of open edges between the root and a leaf y guarantees the existence of a path from $D(y, \rho)$ to $D(z_{\text{root}}, 2R_J)$ in $\{f_r \ge 0\} \cap D(x_{\text{root}}, 10R_J)$. Therefore, upon setting

$$\mathbf{N} = \#\Big\{ i \in I : D(y_i, \rho) \leftrightarrow \partial D(3R) \text{ in } \{f_r \ge 0\} \cap \big(D(3R) \setminus \cup_{j \ne i} D(y_j, 2\rho)\big) \Big\},$$

where $\{y_i : i \in I\}$ refers to the set of leaves of $\mathcal{T}$, the RSW theorem and the FKG inequality imply that for every $\eta > 0$,

$$\mathbb{P}[\mathbf{N} \ge \eta(R/\rho)^\eta] \ge c_0 \mathbb{P}[|\mathbf{V}_{-1}| \ge \eta(R/\rho)^\eta],$$

for some constant $c_0 > 0$.

The previous observation implies that it suffices to show the existence of some $\eta > 0$ such that

$$\mathbb{P}[|\mathbf{V}_{-1}| \ge \eta(R/\rho)^\eta] \ge \eta. \tag{20}$$

This claim is a direct consequence of repeated applications of the inequality

$$\mathbb{P}\left[|\mathbf{V}_{j-1}| \ge \kappa |\mathbf{V}_j| \,\middle|\, \mathcal{F}_j\right] \ge 1 - e^{-c|\mathbf{V}_j|}, \tag{21}$$

for all $0 \le j \le J+1$ and some $c > 0$ and where

- $\mathcal{F}_j$ is the σ-algebra generated by the events H_i for the (y_i, z_i)'s that are R_i-good pairs with $i \ge j$ (here and below, we use the convention that $\mathcal{F}_{J+1}$ is the trivial σ-algebra);
- $\kappa := (1 + \eta_0 K^{\eta_0})/2$ if $1 \le j \le J$ and $\kappa := \eta_1/2$ if $j = 0$. Here $\eta_0 > 0$ is as in Lemma 4.3 and η_1 is some positive constant.

We therefore focus on proving (21). We write the proof for $1 \le j \le J$ since the proof in the case $j = 0$ is essentially the same (but easier). For each (x, z) in the tree at a graph distance at least 2 from the leaves, let $\mathbf{N}_{(x,z)}$ be the number of descendants (x', z') of (x, z) such that the edge of $\mathcal{T}$ between the two vertices is open. We claim that, for any $A \in \mathcal{F}_{j+1}$, any possible outcome V for $\mathbf{V}_j$, and any $M > 0$, the event

$$\Big\{ \sum_{(x,z) \in V} \mathbf{N}_{(x,z)} > M \Big\} \tag{22}$$

is positively correlated with $\{\mathbf{V}_j = V\} \cap A$. Before proving (22), let us conclude the proof. By (22), we have

$$\mathbb{P}\left[|\mathbf{V}_{j-1}| < \kappa |V| \,\middle|\, \mathbf{V}_j = V, \mathcal{F}_{j+1}\right] \le \mathbb{P}\Big[\sum_{(x,z) \in V} \mathbf{N}_{(x,z)} < \kappa |V| \Big] \text{ a.s.}$$

for any possible outcome V for $\mathbf{V}_j$. By definition of the tree, there exists a constant $C > 0$ such that for each $(x, z) \in V$, $0 \le \mathbf{N}_{(x,z)} \le CK^2$. Moreover, the variables $\mathbf{N}_{(x,z)}$ are independent from each other since the $(x, z) \in V$ are at a distance at least $20R_j$ from each other and f_r is r-dependent. Moreover, by Lemma 4.3, their expectation is at least $\eta_0 K^{\eta_0} > \kappa$. By a standard concentration inequality for sums of i.i.d. random variables (Hoeffding's inequality) applied to the family of variables $(\kappa - \mathbf{N}_{(x,z)})_{(x,z)\in V}$, we have

$$\mathbb{P}\Big[\sum_{(x,z)\in V} \mathbf{N}_{(x,z)} < \kappa|V|\Big] \le \exp\Big(-\frac{2(\eta_0 K^{\eta_0} - \kappa)^2}{(CK^2 + \kappa)^2}|V|\Big).$$

Since this is true for any V and since $\mathcal{F}_j$ is the smallest σ-algebra containing $\mathcal{F}_{j+1}$ and $\mathbf{V}_j$, this implies (21).

It only remains to show (22). To this purpose, we observe that $\{\mathbf{V}_j = V\} \cap A$ is increasing in the $11R_j$-neighborhood of $\cup_{(x,z)\in V}\{x\}$ and that $\{\sum_{(x,z)\in V} \mathbf{N}_{(x,z)} > M\}$ is increasing and measurable with respect to $\cup_{(x,z)\in V} D(x, 9R_j)$. So the result holds by applying the local FKG inequality Corollary 2.2 to $U = \mathbb{R}^2$, $V = \cup_{(x,z)\in V} D(x, 9R_j)$, $\delta = 1$, $\phi = 1_{\{\mathbf{V}_j = V\}\cap A}$ and $\psi = 1_{\{\sum_{(x,z)\in V} \mathbf{N}_{(x,z)} > M\}}$. □

5 A two-arms estimate in a slab

Let $q : \mathbb{R}^d \to \mathbb{R}$ satisfying Assumption 1.4 for some $\beta > d$. In this section, we consider two parameters $0 < \gamma < a^2 < 1$ and study connectivity properties in $[-4R, 4R]^2 \times [0, R^a]$. Our main goal is to prove the following uniqueness quasi-planar result that plays the role of (4) from the sketch of proof. Recall that for $d' \le d$, we routinely view $D \subset \mathbb{R}^{d'}$ as the subset $D \times \{0\}^{d-d'}$ and that we let $\mathcal{P}_t := \{x \in \mathbb{R}^3 : x_3 = t\}$.

Proposition 5.1. *There exists $b > 0$ that depends only on the dimension d such that the following holds if $0 < \gamma < a^2 < b$. For all $\delta > 0$ there exist $R_0, c > 0$ such that for every $R \ge R_0$,*

$$\mathbb{P}\Big[\begin{array}{c}\text{all the c.c. of } \{f_{R^\gamma} \ge 2R^{-3/2}\} \cap [-2R, 2R]^2 \text{ of diameter at least } \delta R \\ \text{belong to the same c.c. of } \{f_{R^\gamma} \ge R^{-3/2}\} \cap [-4R, 4R]^2 \times [0, R^a]\end{array}\Big] \ge 1 - \exp(-R^c).$$

Remark 5.2. The same proof gives Proposition 5.1 with $2R^{-3/2}$ (resp. $R^{-3/2}$) replaced by $R^{-3/2}$ (resp. 0).

Before proving Proposition 5.1, let us state the following elementary planar existence result which, in combination with Proposition 5.1, will help us create large components in $\{f_{R^\gamma} \ge R^{-3/2}\}$ with high probability.

Lemma 5.3 (Existence of a macroscopic planar component)**.** *For every $\delta > 0$ there exists $R_0 > 0$ such that for every $R \ge R_0$,*

$$\mathbb{P}\big[\{f_{R^\gamma} \ge 2R^{-3/2}\} \cap [-R, R]^2 \text{ contains a c.c. of diameter} \ge \delta R\big] \ge 1 - \Big(\frac{1}{2}\Big)^{\lfloor 1/(2\delta)\rfloor^2 - 1}.$$

Proof. For every $\delta > 0$, there exist at least $\lfloor 1/(2\delta) \rfloor^2$ squares of side length δR which are included in $[-R, R]^2$ and are at a mutual distance larger than δR. An easy application of Lemma 2.3 and the duality between $\{f_{R^\gamma} > 0\}$ and $\{f_{R^\gamma} < 0\}$ implies that each square is crossed by $\{f_{R^\gamma} \geq 0\}$ with probability $1/2$. Choose R_0 such that $R^\gamma \leq \delta R$ for all $R \geq R_0$. Then, for all such R,

$$\mathbb{P}\big[\{f_{R^\gamma} \geq 0\} \cap [-R, R]^2 \text{ contains a c.c. of diameter at least } \delta R\big] \geq 1 - \left(\frac{1}{2}\right)^{\lfloor 1/(2\delta) \rfloor^2}.$$

We conclude by applying Lemma B.2 to the event on the left-hand side with $r = R^\gamma$ and $t = -2R^{-3/2}$. □

Before proving Proposition 5.1, let us write two remarks that are central in the proof. The first remark is mainly about the use of the local FKG inequality in the proof (which is more subtle than one may expect). The second remark is about the fact that we can use RSW results at level $R^{-3/2}$ instead of 0.

Remark 5.4. In the proof of Proposition 5.1, we will work in $\mathbb{R}^3 \subset \mathbb{R}^d$ and frequently condition the field f_{R^γ} with respect to its values on certain subsets. Note that, since the field is a.s. analytic, it is determined by its values on a countable number of points. In particular, for any subset $U \subseteq \mathbb{R}^3$, the law of f_{R^γ} conditioned on $(f_{R^\gamma})_{|U}$ is still Gaussian.

Moreover, if $V \subseteq \mathbb{R}^3$ is such that for each $x \in U$ and $y \in V$, $\mathbb{E}[f_{R^\gamma}(x) f_{R^\gamma}(y)] = 0$, then the law of $(f_{R^\gamma})_{|V}$ is unaffected under the conditioning on $(f_{R^\gamma})_{|U}$. Furthermore, by the local FKG inequality, see Corollary 2.2, the following property holds, which is tailored to the purposes of the present section:

Fix $W, U_A, U_B \subseteq \mathbb{R}^3$ and let $A \in \mathcal{F}_{U_A}$ and $B \in \mathcal{F}_{U_B}$ (recall the definition of these σ-algebras from Section 2.1). Assume that B is increasing, that A is increasing on the set of points in $\mathbb{R}^3$ at a distance at most $R^\gamma + 1$ from U_B and that $\operatorname{dist}(U_A, W) \geq 2R^\gamma + 1$. Then, applying Corollary 2.2 to the conditional (Gaussian) measure $\mathbb{P}[\cdot \,|\, (f_{R^\gamma})_{|W}]$ with the choices $U = U_A$, $V = U_B$, $\phi = 1_A$, $\psi = 1_B$ and $\delta = 1$, one obtains that

$$\mathbb{P}[f_{R^\gamma} \in A \cap B \mid (f_{R^\gamma})_{|W}] \geq \mathbb{P}[f_{R^\gamma} \in A \mid (f_{R^\gamma})_{|W}] \times \mathbb{P}[f_{R^\gamma} \in B \mid (f_{R^\gamma})_{|W}]. \tag{23}$$

Remark 5.5. As one can notice in Proposition 5.1, our goal in this section is to connect planar components by using paths in $\{f_{R^\gamma} \geq R^{-3/2}\}$. As a result, it will be very useful to have at our disposal RSW results at level $R^{-3/2}$ instead of 0. Let us first recall that since the RSW theorem (Theorem 2.4) holds with universal constants, then it holds with f_{R^γ} instead of f. Next, by applying Lemma B.2 (to $r = R^\gamma$ and $t = -R^{-3/2}$), we obtain that there exits $R_0 > 0$ such that the following holds:

$$\begin{gathered}\text{If } R > R_0 \text{ then Theorem 2.4 holds with } \{f \geq 0\} \text{ replaced by } \{f_{R^\gamma} \geq R^{-3/2}\} \\ \text{in the definition of } \operatorname{Cross}(\rho R, R).\end{gathered} \tag{24}$$

The same argument implies that there exists $R_0 > 0$ such that we have the following:

$$\begin{gathered}\text{If } R > R_0 \text{ then Proposition 4.1 holds for any } \rho \in [R^\gamma, R] \\ \text{and with } \{f_r \geq 0\} \text{ replaced by } \{f_{R^\gamma} \geq R^{-3/2}\}.\end{gathered} \tag{25}$$

Let us now prove Proposition 5.1.

Proof of Proposition 5.1. Recall that we consider two parameters $0 < \gamma < a^2 < 1$ which are both assumed to be small. Throughout the proof, we will frequently condition on $(f_{R^\gamma})_{|\mathcal{P}_0}$. We will denote by $\widetilde{\mathbb{P}}$ the probability law with this conditioning (viewed as a regular conditional probability measure). Consider the event

$$\mathrm{Sprouts}(R) := \left\{ \begin{array}{c} \text{every continuous path included in } \{f_{R^\gamma} \geq 2R^{-3/2}\} \cap [-2R, 2R]^2 \text{ of} \\ \text{diameter at least } R^a \text{ is connected to } \mathcal{P}_{R^{a^2}} \text{ by a path included in} \\ \{f_{R^\gamma} \geq R^{-3/2}\} \cap ([-2R, 2R]^2 \times [0, R^{a^2}]) \text{ of diameter} \leq 3R^a \end{array} \right\}. \tag{26}$$

At this point, we assume that a and γ/a are small enough for Proposition 1.12 to apply at scale R^a, with truncation exponent γ/a instead of γ and with $\theta_0 = 1/2$. By Proposition 1.12 (at $\ell = R^{-3/2}$) followed by a union bound, there exist $R_0, c_0 > 0$ such that if $R \geq R_0$,

$$\mathbb{P}[\mathrm{Sprouts}(R)] \geq 1 - e^{-R^{c_0}}. \tag{27}$$

Let $\widetilde{\mathrm{Sprouts}}(R)$ be the $(f_{R^\gamma})_{|\mathcal{P}_0}$-measurable event

$$\widetilde{\mathrm{Sprouts}}(R) := \left\{ \widetilde{\mathbb{P}}[\mathrm{Sprouts}(R)] \geq 1 - e^{-R^{c_0}/2} \right\}. \tag{28}$$

Then, by (27) and Markov's inequality applied to $\widetilde{\mathrm{Sprouts}}(R)^c$, one finds for $R \geq R_0$,

$$\mathbb{P}[\widetilde{\mathrm{Sprouts}}(R)] \geq 1 - e^{-R^{c_0}/2}. \tag{29}$$

Let $\delta > 0$ and $x \in [-2R, 2R]^2$ and consider a deterministic path $\mathcal{C}_0 \subset D(x, 5\delta R)$ of diameter δR. By Proposition 4.1 (that we can apply since $\gamma < a$) and (25), there exist $R_0 > 0$, a universal $\eta > 0$, and a family $(y_i)_i$ of points at mutual distances at least $20R^a$ and at a distance at most R^a from $\mathcal{C}_0$ such that if $R \geq R_0$ and if we let

$$\begin{aligned} \mathrm{Conn}_i(R) &:= \left\{ \begin{array}{c} D(y_i, R^a) \text{ is connected to } \partial D(x, 20\delta R) \\ \text{in } (\{f_{R^\gamma} \geq R^{-3/2}\} \cap D(x, 20\delta R)) \setminus (\cup_{j \neq i} D(y_j, 10R^a)) \end{array} \right\}, \\ \mathrm{Poly}_{\mathcal{C}_0}(R) &:= \{\#\{i : \mathrm{Conn}_i(R) \text{ holds}\} \geq (\delta R^{(1-a)})^\eta\}, \end{aligned} \tag{30}$$

then

$$\mathbb{P}[\mathrm{Poly}_{\mathcal{C}_0}(R)] \geq \eta. \tag{31}$$

The following lemma is the core of the proof of Proposition 5.1. Conditionally on $(f_{R^\gamma})_{|\mathcal{P}_0}$ and on the (very likely) event $\widetilde{\mathrm{Sprouts}}$, one may connect two given large connected components of $\{f_{R^\gamma} \geq 2R^{-3/2}\} \cap [-2R, 2R]^2$ by a path in $\{f_{R^\gamma} \geq R^{-3/2}\} \cap [-4R, 4R]^2 \times [0, R^a]$ with very good probability. To conclude the proof, we will essentially apply this lemma to all pairs of connected components of this set.

Lemma 5.6. *Suppose that a satisfies $5a/(1-a) < \eta$. There exist $c, R_0 > 0$ such that the following holds. Assume that $\widetilde{\text{Sprouts}}(R)$ is satisfied and let $\mathcal{C}$ and $\mathcal{C}'$ be two continuous paths included in $\{f_{R^\gamma} \geq 2R^{-3/2}\} \cap [-2R, 2R]^2$ of diameter δR and at a mutual distance at least $100\delta R$. If $R \geq R_0$ then*

$$\widetilde{\mathbb{P}}\left[\mathcal{C} \leftrightarrow \mathcal{C}' \text{ in } ([-4R, 4R]^2 \times [0, R^a]) \cap \{f_{R^\gamma} \geq R^{-3/2}\}\right] \geq 1 - e^{-R^c}.$$

Before proving the lemma, we briefly conclude the proof of Proposition 5.1. Recall that we have already assumed that a and γ/a are small enough. In order to apply Lemma 5.6, we assume in addition that $5a/(1-a) < \eta$. This also holds if a is small enough.

Now let $\delta_0 := \delta/1000$ and observe that if $\mathcal{D}$ and $\mathcal{D}'$ are two connected components of $\{f_{R^\gamma} \geq 2R^{-3/2}\} \cap [-2R, 2R]^2$ of diameter larger than or equal to δR then there exist two continuous paths $\mathcal{C}$ and $\mathcal{C}'$ of diameter $\delta_0 R$, included in $\mathcal{D}, \mathcal{D}'$ respectively and satisfying $\text{dist}(\mathcal{C}, \mathcal{C}') \geq 100\delta_0 R$. As a result, Proposition 5.1 is a consequence of Lemma 5.6 and a union bound over all pairs of connected components of $\{f_{R^\gamma} \geq 2R^{-3/2}\} \cap [-2R, 2R]^2$ of diameter $\geq \delta R$, the number of which can be controlled except on an event of suitably small probability as follows.

Let $c > 0$ as in Lemma 5.6. The probability that there are more than $e^{R^c/2}$ connected components in $\{f_{R^\gamma} \geq 2R^{-3/2}\} \cap [-2R, 2R]^2$ is less than $e^{-R^c/4}$ if R is sufficiently large. This is a direct consequence of Markov's inequality and the fact that the expectation of the number of connected components of $\{f_{R^\gamma} \geq 2R^{-3/2}\} \cap [-2R, 2R]^2$ is less than CR^2 for some $C > 0$ that depends only on q, as soon as R is sufficiently large, see Lemma B.3. This completes the proof of Proposition 5.1, subject to the validity of Lemma 5.6. □

We now proceed with the proof of Lemma 5.6.

Proof of Lemma 5.6. Throughout the proof, crucially, the constants do not depend on $\mathcal{C}$ and $\mathcal{C}'$. The proof is split into two steps.

Step 1. In this step, we consider copies of the event $\text{Poly}_{\mathcal{C}}(R)$ at different heights and show, using (31), that with very good probability, at some height between $R^a/2$ and R^a there exist polynomially many paths that, when projected onto $\mathcal{P}_0$, connect $\mathcal{C}$ and $\mathcal{C}'$ up to distance $O(R^a)$.

Let $x, x' \in [-2R, 2R]^2$ be such that $\mathcal{C} \subset D(x, 5\delta R)$ and $\mathcal{C}' \subset D(x', 5\delta R)$. Let $(y_i)_i$ (resp. $(y'_j)_j$) be a family of points associated to $\mathcal{C}$ (resp. $\mathcal{C}'$) in the same way as the points associated to $\mathcal{C}_0$ in the definition of $\text{Poly}_{\mathcal{C}_0}(R)$ in (30) above. For every i (resp. j), let $\mathcal{C}_i$ (resp. $\mathcal{C}'_j$) be a path included in $\mathcal{C}$ (resp. $\mathcal{C}'$) of diameter R^a and at a distance at most R^a from y_i (resp. y'_j). (These sub-paths exist and are disjoint as i and j vary if R is sufficiently large.)

Let $\text{Circ}_{\mathcal{C},\mathcal{C}'}(R)$ denote the event that

i) there is a circuit in $\{f_{R^\gamma} \geq R^{-3/2}\} \cap (D(x, 20\delta R) \setminus D(x, 10\delta R))$ which surrounds the inner disc $D(x, 10\delta R)$,
ii) the analogous event holds with x' instead of x, and
iii) $D(x, 10\delta R)$ is connected to $D(x', 10\delta R)$ by a path included in $\{f_{R^\gamma} \geq R^{-3/2}\} \cap [-4R, 4R]^2$.

Introduce

$$\mathrm{Poly}_{\mathcal{C},\mathcal{C}'}(R) := \mathrm{Poly}_{\mathcal{C}}(R) \cap \mathrm{Poly}_{\mathcal{C}'}(R) \cap \mathrm{Circ}_{\mathcal{C},\mathcal{C}'}(R).$$

For $0 \leq k \leq \bar{k}$ with $\bar{k} := \lfloor R^{a-\gamma}/2 \rfloor$, let $\mathrm{Poly}^k_{\mathcal{C},\mathcal{C}'}(R)$ (resp. $\mathrm{Conn}^k_i(R)$) denote the event $\mathrm{Poly}_{\mathcal{C},\mathcal{C}'}(R)$ (resp. $\mathrm{Conn}_i(R)$) translated by $h_k \mathbf{e}_3$, where $h_k := \frac{R^a}{2} + kR^\gamma$.

Note that $\mathcal{P}_{h_0} = \mathcal{P}_{R^a/2}$ and that the planes $\mathcal{P}_{h_k}$ are at mutual distances R^γ and are included in $\mathbb{R}^2 \times [R^a/2, R^a]$. Since f_{R^γ} is R^γ-dependent, (31), the RSW theorem (see (24)) and standard gluing constructions (as described in Section 2.3) imply the existence of $c > 0$ that depends only on δ such that for all R sufficiently large

$$\widetilde{\mathbb{P}}\Big[\bigcup_{0\leq k\leq \bar{k}} \mathrm{Poly}^k_{\mathcal{C},\mathcal{C}'}(R)\Big] = \mathbb{P}\Big[\bigcup_{0\leq k\leq \bar{k}} \mathrm{Poly}^k_{\mathcal{C},\mathcal{C}'}(R)\Big] \geq 1 - e^{-cR^{a-\gamma}}. \tag{32}$$

If there exists $k \in \{0, \dots, \bar{k}\}$ such that $\mathrm{Poly}^k_{\mathcal{C},\mathcal{C}'}(R)$ holds, let $k_{\max}$ be the largest such k and define $\bar{I}, \bar{I}'$ by $i \in \bar{I}$ if and only if $\mathrm{Conn}^{k_{\max}}_i(R)$ occurs (and similarly for $\bar{I}'$). For every $k_0 \in \{0, \dots, \bar{k}\}$ and every $\bar{I}_0, \bar{I}'_0$, let

$$A_{k_0,\bar{I}_0,\bar{I}'_0} = \Big(\bigcup_k \mathrm{Poly}^k_{\mathcal{C},\mathcal{C}'}(R)\Big) \cap \{k_{\max} = k_0, \bar{I} = \bar{I}_0, \bar{I}' = \bar{I}'_0\}.$$

The event $A_{k_0,\bar{I}_0,\bar{I}'_0}$ is measurable with respect to the set

$$\{x_3 \geq h_{k_0}\} \setminus \Big(\bigcup_i D(y_i, R^a) \cup \bigcup_j D(y'_j, R^a)\Big) \tag{33}$$

and increasing in the set

$$\Big(\bigcup_{i\in\bar{I}_0} D(y_i, 10R^a) \cup \bigcup_{j\in\bar{I}'_0} D(y'_j, 10R^a)\Big) \times [0, h_{k_0} + R^\gamma/2]. \tag{34}$$

Step 2. In this step we show that, given $k_0 \in \{1, \dots, \bar{k}\}$ and I_0 a family of indices, it is very likely that there exists some $i \in I_0$ such that $\mathcal{C}_i$ is connected to a path in $D(y_i, 5R^a) \times \{h_{k_0}\}$ of diameter at least R^a. Here the difficulty is that we want to do this under $\widetilde{\mathbb{P}}$, i.e., conditionally on $(f_{R^\gamma})_{|\mathcal{P}_0}$. In doing so we lose the independence of f_{R^γ} restricted to the tubes $D(y_i, 5R^a) \times [0, h_{k_0}]$. This lack of independence is replaced with Claim 5.7 below.

Fix $k_0 \in \{1, \dots, \bar{k}\}$ and I_0 a family of indices such that

$$|I_0| \geq (\delta R^{(1-a)})^\eta. \tag{35}$$

Denote the elements of I_0 by $i_1, \dots, i_{|I_0|}$. For each $l \in \{1, \dots, |I_0|\}$, let H_l be the event that there exists an index $1 \leq l' \leq l$ such that $\mathcal{C}_{i_{l'}}$ is connected by a path in $(D(y_{i_{l'}}, 5R^a) \times [0, h_{k_0}]) \cap \{f_{R^\gamma} \geq R^{-3/2}\}$ to a circuit surrounding $D(y_{i_{l'}}, R^a) \times \{h_{k_0}\}$ in $(D(y_{i_{l'}}, 5R^a) \times \{h_{k_0}\}) \cap \{f_{R^\gamma} \geq R^{-3/2}\}$.

Claim 5.7. *Let c_0 as in (27). There exist $c, R_0 > 0$ such that the following holds if $R \geq R_0$. Let $l \in \{1, \dots, |I_0| - 1\}$ and assume that*

$$\widetilde{\mathbb{P}}\left[H_l\right] \leq 1 - e^{-R^{c_0}/4}.$$

Then, on $\widetilde{\text{Sprouts}}(R)$,

$$\widetilde{\mathbb{P}}\left[H_{l+1} \,|\, H_l^c\right] \geq cR^{-5a}.$$

Moreover, on $\widetilde{\text{Sprouts}}(R)$, *we have* $\widetilde{\mathbb{P}}[H_1] \geq cR^{-5a}$.

Before proving Claim 5.7, let us complete the proof of Lemma 5.6. By Claim 5.7, we deduce that on $\widetilde{\text{Sprouts}}(R)$, the probability of the event $H_{|I_0|}$ that there exists $i \in I_0$ for which $\mathcal{C}_i$ is connected by a path in $(D(y_i, 5R^a) \times [0, h_{k_0}]) \cap \{f_{R^\gamma} \geq R^{-3/2}\}$ to a circuit surrounding $D(y_i, R^a) \times \{h_{k_0}\}$ in $(D(y_i, 5R^a) \times \{h_{k_0}\}) \cap \{f_{R^\gamma} \geq R^{-3/2}\}$ satisfies

$$\widetilde{\mathbb{P}}[H_{|I_0|}] \geq 1 - (1 - cR^{-5a})^{|I_0|} \geq 1 - e^{-R^{c'}}$$

for R large enough and some $c' > 0$ since $|I_0| \geq (\delta R^{(1-a)})^\eta$ by (35) and since we have assumed that $5a < \eta(1-a)$. Clearly, the same holds for the analogous construction with $\mathcal{C}'$ instead of $\mathcal{C}$. In addition to k_0 and I_0, let I_0' be a set of indices j of the family $(y_j')_j$ satisfying $|I_0'| \geq (\delta R^{(1-a)})^\eta$. Let $B_{k_0, I_0, I_0'}$ be the event that $H_{|I_0|}$ holds and that the analogous event for $\mathcal{C}'$ holds as well. By union bound, we deduce that on the event $\widetilde{\text{Sprouts}}(R)$, for R large enough,

$$\widetilde{\mathbb{P}}[B_{k_0, I_0, I_0'}] \geq 1 - 2e^{-R^{c'}}. \tag{36}$$

Now this event is increasing and measurable with respect to the set

$$\Big(\bigcup_{i \in I_0} D(y_i, 5R^a) \cup \bigcup_{j \in I_0} D(y_j', 5R^a) \Big) \times [0, h_{k_0}].$$

Since, $A_{k_0, I_0, I_0'}$ is measurable with respect to the set (33) and increasing on the set (34) we deduce by applying (23) that on the event $\widetilde{\text{Sprouts}}(R)$, for R large,

$$\widetilde{\mathbb{P}}[A_{k_0, I_0, I_0'} \cap B_{k_0, I_0, I_0'}] \geq \widetilde{\mathbb{P}}[B_{k_0, I_0, I_0'}] \widetilde{\mathbb{P}}[A_{k_0, I_0, I_0'}] \geq (1 - 2e^{-R^{c'}}) \widetilde{\mathbb{P}}[A_{k_0, I_0, I_0'}].$$

Recall now that the disjoint union of the $A_{k_0, I_0, I_0'}$ over all the choices of k_0, I_0 and I_0' is $\bigcup_{k=1}^{\overline{k}} \text{Poly}^k_{\mathcal{C}, \mathcal{C}'}(R)$. Consequently, summing over all these possible choices, we deduce

that, on the event $\widetilde{\text{Sprouts}}(R)$, for R large enough,

$$\widetilde{\mathbb{P}}\Big[\bigsqcup_{k_0,I_0,I_0'} A_{k_0,I_0,I_0'} \cap B_{k_0,I_0,I_0'}\Big] \geq (1-2e^{-R^{c'}})\widetilde{\mathbb{P}}[\cup_{k=1}^{\overline{k}} \text{Poly}^k_{\mathcal{C},\mathcal{C}'}(R)]$$
$$\overset{(32)}{\geq} (1-2e^{-R^{c'}})(1-e^{-c_2 R^{a-\gamma}}). \tag{37}$$

But notice that the event $\bigsqcup_{k_0,I_0,I_0'}(A_{k_0,I_0,I_0'} \cap B_{k_0,I_0,I_0'})$ implies that $\mathcal{C}$ and $\mathcal{C}'$ are connected by a continuous path in $([-4R,4R]^2 \times [0,R^a]) \cap \{f \geq R^{-3/2}\}$. Therefore, (37) completes the proof of Lemma 5.6 (assuming Claim 5.7 holds true). □

It remains to give the proof of Claim 5.7.

Proof of Claim 5.7. Recall the definition (28) of $\widetilde{\text{Sprouts}}(R)$. Assume that

$$\widetilde{\mathbb{P}}[H_l] \leq 1 - e^{-R^{c_0}/4}.$$

Let Up_{l+1} be the event that $\mathcal{C}_{i_{l+1}}$ is connected to $D(y_{i_{l+1}}, 5R^a) \times \{R^{a^2}\}$ by a continuous path in $\{f_{R^\gamma} \geq R^{3/2}\} \cap (D(y_{i_{l+1}}, 5R^a) \times [0, R^{a^2}])$. Then, $\text{Sprouts}(R) \subset \text{Up}_{l+1}$ so that, on $\widetilde{\text{Sprouts}}(R)$, for R large enough,

$$\widetilde{\mathbb{P}}[\text{Up}_{l+1} \mid H_l^c] \geq \widetilde{\mathbb{P}}[\text{Sprouts}(R) \mid H_l^c] \geq 1 - \frac{\widetilde{\mathbb{P}}[\text{Sprouts}(R)^c]}{\widetilde{\mathbb{P}}[H_l^c]} \geq 1 - e^{-R^{c_0}/4}.$$

In particular, by union bound and FKG, we deduce that there exist a constant $c_1 > 0$ and a (random $(f_{R^\gamma})_{|\mathcal{P}_0}$-measurable) $u_{l+1} \in D(y_{i_{l+1}}, 5R^a) \times \{R^{a^2}\}$ such that the event $\text{Up}^\star_{l+1}$ that $\mathcal{C}_{i_{l+1}}$ is connected to u_{l+1} by a continuous path in $\{f_{R^\gamma} \geq R^{-3/2}\} \cap D(y_{i_{l+1}}, 5R^a) \times [0, R^{a^2}]$ satisfies, for R large enough,

$$\widetilde{\mathbb{P}}[\text{Up}^\star_{l+1} \,|\, H_l^c] \geq c_1 R^{-2a}. \tag{38}$$

Let

$$\text{Tube}_{l+1} := \left\{ \begin{array}{c} \exists \text{ a c.c. of } \{f_{R^\gamma} \geq R^{-3/2}\} \cap (D(y_{i_{l+1}}, 5R^a) \times [R^{a^2}, h_{k_0}]) \text{ that contains both} \\ u_{l+1} \text{ and a circuit in } \{f_{R^\gamma} \geq R^{-3/2}\} \cap (D(y_{i_{l+1}}, 5R^a) \times \{h_{k_0}\}) \\ \text{that surrounds } D(y_{i_{l+1}}, R^a) \times \{h_{k_0}\} \end{array} \right\}.$$

Note that $\text{Up}^\star_{l+1} \cap \text{Tube}_{l+1} \subset H_{l+1}$ so our goal will be to show that, for some constant $c > 0$ and R large enough,

$$\widetilde{\mathbb{P}}[\text{Up}^\star_{l+1} \cap \text{Tube}_{l+1} \,|\, H_l^c] \geq cR^{-5a}. \tag{39}$$

Let $\mathcal{E}_{l+1} = \mathbb{R}^3 \setminus (D(y_{i_{l+1}}, 10R^a) \times [0, h_{k_0}])$. Note that H_l is measurable with respect to $(f_{R^\gamma})_{|\mathcal{E}_{l+1}}$ while Tube_{l+1} is measurable with respect to f_{R^γ} restricted to $D(y_{i_{l+1}}, 5R^a) \times [0, h_{k_0}]$. On the other hand, by the RSW theorem (see (24)) and by using standard gluing constructions from Section 2.3 (more precisely, by using three times Item vi) of

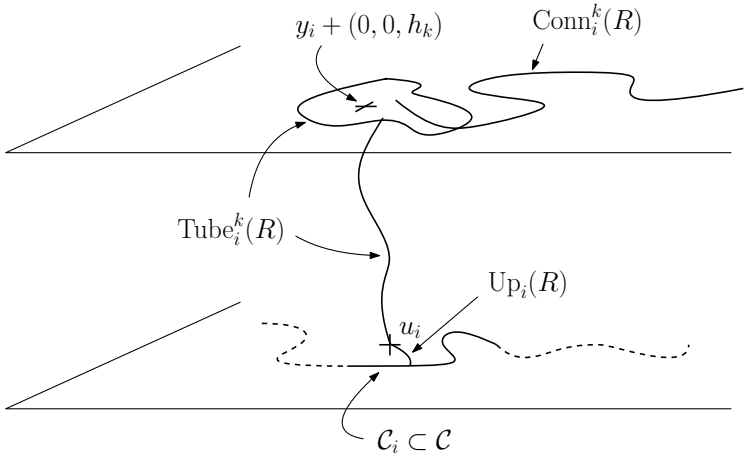

Figure 4: The events $\mathrm{Up}_i(R)$, $\mathrm{Tube}_i^k(R)$ and $\mathrm{Conn}_i^k(R)$.

this section at scale R^a: once in $\mathcal{P}_{h_k}$ and twice in a vertical plane in order to connect u_i to a well-chosen point of $\mathcal{P}_{h_k}$), one can show that Tube_{l+1} occurs with probability $\Omega(R^{-3a})$ so that, for some constant $c_2 > 0$,

$$\widetilde{\mathbb{P}}[\mathrm{Tube}_{l+1} \,|\, (f_{R^\gamma})_{|\mathcal{E}_{l+1}}] = \mathbb{P}[\mathrm{Tube}_{l+1}] \geq c_2 R^{-3a}. \tag{40}$$

Now, note that the events $\mathrm{Up}_{l+1}^\star$ and Tube_{l+1} are both increasing and recall that $\gamma < a^2$. Hence, for R large enough,

$$\begin{aligned}\widetilde{\mathbb{P}}[\mathrm{Up}_{l+1}^\star \cap \mathrm{Tube}_{l+1} \,|\, (f_{R^\gamma})_{|\mathcal{E}_{l+1}}] &\overset{(23)}{\geq} \widetilde{\mathbb{P}}[\mathrm{Up}_{l+1}^\star \,|\, (f_{R^\gamma})_{|\mathcal{E}_{l+1}}]\widetilde{\mathbb{P}}[\mathrm{Tube}_{l+1} \,|\, (f_{R^\gamma})_{|\mathcal{E}_{l+1}}] \\ &\overset{(40)}{\geq} c_2 R^{-3a}\widetilde{\mathbb{P}}[\mathrm{Up}_{l+1}^\star \,|\, (f_{R^\gamma})_{|\mathcal{E}_{l+1}}]\end{aligned}$$

so that

$$\begin{aligned}\widetilde{\mathbb{P}}[\mathrm{Up}_{l+1}^\star \cap \mathrm{Tube}_{l+1} \,|\, H_l^c] &= \frac{1}{\widetilde{\mathbb{P}}[H_l^c]}\widetilde{\mathbb{E}}[\widetilde{\mathbb{P}}[\mathrm{Up}_{l+1}^\star \cap \mathrm{Tube}_{l+1} \,|\, (f_{R^\gamma})_{|\mathcal{E}_{l+1}}]\mathbf{1}_{H_l^c}] \\ &\geq c_2 R^{-3a}\frac{1}{\widetilde{\mathbb{P}}[H_l^c]}\widetilde{\mathbb{E}}[\widetilde{\mathbb{P}}[\mathrm{Up}_{l+1}^\star \,|\, (f_{R^\gamma})_{|\mathcal{E}_{l+1}}]\mathbf{1}_{H_l^c}] \\ &\geq c_2 R^{-3a}\widetilde{\mathbb{P}}[\mathrm{Up}_{l+1}^\star \,|\, H_l^c] \\ &\overset{(38)}{\geq} c_1 c_2 R^{-5a},\end{aligned}$$

which yields (39) as required.

The remaining statement, i.e. the proof that on $\widetilde{\mathrm{Sprouts}}(R)$, $\widetilde{\mathbb{P}}[H_1] \geq cR^{-5a}$, follows from the same construction. □

We conclude this section with a result analogous to Proposition 5.1. Below, given some $d' \in [3, d]$, $F_R^1, \ldots, F_R^{N_{d'}}$ denote all the 2-dimensional faces of the cube $[-R/2, R/2]^{d'}$,

i.e. the F_R^i's are the sets obtained from the sets $[-R/2, R/2]^2 \times \{\pm R/2\}^{d'-2}$ by permuting the coordinates. Moreover, for every $i \in \{1, \dots, N_{d'}\}$ we let $\tilde{F}_R^i \subset F_R^i$ be the concentric square with side length equal to $R/2$.

Proposition 5.8. *There exists $b > 0$ that depends only on the dimension d such that the following holds if $0 < \gamma < b$. Let $d' \in [3, d]$. For all $\delta > 0$ there exist $R_0, c > 0$ such that for every $R \geq R_0$,*

$$\mathbb{P}\left[\bigcap_{i,j}\left\{\begin{array}{c}\text{all the c.c. of } \{f_{R^\gamma} \geq 2R^{-3/2}\} \cap \tilde{F}_R^i \text{ of diameter at least } \delta R \\ \text{and all the c.c. of } \{f_{R^\gamma} \geq 2R^{-3/2}\} \cap \tilde{F}_R^j \text{ of diameter at least} \\ \delta R \text{ belong to the same c.c. of } \{f_{R^\gamma} \geq R^{-3/2}\} \cap [-R/2, R/2]^{d'}\end{array}\right\}\right] \geq 1 - e^{-R^c},$$

where the intersection in the probability is taken over the pairs of indices $1 \leq i, j \leq N_{d'}$ for which F_R^i and F_R^j share a side.

Proof. As in Proposition 5.1, we also consider some $a \in (0, 1)$ such that $0 < \gamma < a^2 < b$ and let $h_k := \frac{R^a}{2} + kR^\gamma$. The proof is essentially the same as Proposition 5.1 but we need to replace the construction in $[-4R, 4R]^2 \times \{h_k\}$ by a construction in the 2-dimensional faces of $[-(R/2 - h_k), R/2 - h_k]^{d'}$ that correspond to F_R^i and F_R^j.

The only new technicality (which appears only if $i \neq j$, so let assume this) is about how to adapt point iii) in the definition of $\mathrm{Circ}_{\mathcal{C},\mathcal{C}'}(R)$ (see the beginning of the proof of Lemma 5.6 for the definition of this event). In order to connect two macroscopic paths which belong to two adjacent 2-dimensional faces – which is what will replace $\mathrm{Circ}_{\mathcal{C},\mathcal{C}'}(R)$, one would probably like to use a box-crossing property for f_{R^γ} retricted to the union of two orthogonal half-spaces: $(\mathbb{R} \times \{0\} \times [0, +\infty)) \cup (\mathbb{R} \times [0, +\infty) \times \{0\})$. Such a property is probably tractable (at least for γ sufficiently small) but it is sufficient for us to use the following weaker result (together with Lemma B.2 as in Remark 5.5).

Lemma 5.9. *Given some $r \geq 1$, consider the following union $\mathcal{S}_r := ([0, r] \times \{0\} \times [0, r]) \cup ([0, r] \times [0, r] \times \{0\})$ of two orthogonal squares. There exists $c > 0$ such that, if $R \geq 1$ satisfies $r_q \leq R^\gamma \leq r$, then*

$$\mathbb{P}\left[\begin{array}{c}\exists \text{ a c. path in } \mathcal{S}_r \cap \{f_{R^\gamma} \geq 0\} \text{ that connects the} \\ \text{two sides } [0, r] \times \{0\} \times \{r\} \text{ and } [0, r] \times \{r\} \times \{0\}\end{array}\right] \geq \frac{c}{\log(r) + R^\gamma}.$$

The proof of Lemma 5.9 is a variation of a standard percolation argument. We present it in Appendix C below. Using Lemma 5.9 with r of the order of δR, one can follow the proof of Proposition 5.1 in order to prove Proposition 5.8. The only difference is that we need to replace point iii) in the definition of $\mathrm{Circ}_{\mathcal{C},\mathcal{C}'}(R)$ by an event of probability at least c/R^γ. As a result, we need to replace the lower bound from (32) by $1 - e^{-c_2 R^{a-2\gamma}}$. Taking a small enough followed by γ small enough, we obtain that the analogue of Lemma 5.6 holds with $\mathcal{C} \subset \tilde{F}_R^i$ and $\mathcal{C}' \subset \tilde{F}_R^j$ where F_R^i and F_R^j share a side. The rest of the proof is exactly the same as Proposition 5.1 so we omit the details. □

6 Renormalization

We now supply a suitable renormalization scheme, with two purposes in mind. First, an application of this scheme and of the 2-arms estimate Proposition 5.1 yields the proof of existence of an unbounded nodal component, in a thick (i.e. Item i) of Theorem 1.7) see Section 7. Later, the scheme will also be used in Section 8 in the course of proving Item ii) Theorem 1.7 to generate an ambient cluster with good properties.

Definition 6.1 (Renormalization scheme)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a measure preserving action of $\mathbb{Z}^d$ which we denote by $(\tau_x)_{x\in\mathbb{Z}^d}$. Call *renormalization scheme* the following data:

- A *relative scale parameter* $\lambda \in \mathbb{N}$ such that $\lambda \geq 2$. For each $n \in \mathbb{N}$, we let $L_n := \lambda^n$;
- An initial event $G_{0,0} \in \mathcal{F}$;
- For each $n \in \mathbb{N} \setminus \{0\}$, an event $H_n \in \mathcal{F}$;
- A *range parameter* $\rho \in \mathbb{N} \setminus \{0\}$;
- A *separation parameter* $\sigma \in \mathbb{N} \setminus \{0\}$.

Given this data, define a family of events $(G_{n,x})$ indexed by $n \in \mathbb{N}$ and $x \in L_n\mathbb{Z}^d$ in the following way (the following approach was pioneered in [Szn10, Szn12]). The event $G_{0,0}$ is the initial event; for each $n \in \mathbb{N}$ and $x \in L_n\mathbb{Z}^d$,

$$G_{n,x} := \tau_x^{-1}(G_{n,0})$$

and for each $n \in \mathbb{N} \setminus \{0\}$,

$$G_{n,0} := H_n \cap \bigcap_{x_1,x_2} (G_{n-1,x_1} \cup G_{n-1,x_2}) \tag{41}$$

where the intersection runs over the pairs $x_1, x_2 \in L_{n-1}\mathbb{Z}^d \cap [-4\rho L_n, 4\rho L_n]^d$ such that

$$\mathrm{dist}(x_1, x_2) \geq \sigma L_{n-1}.$$

Remark 6.2. In absence of H_n, the events $G_{n,0}$ in (41) are called *cascading* in the context of [Szn12, Section 3]; for the benefits of adding events H_n at renormalized scales, see [DGRS20, Section 2]. The events $G_{n,0}$ and H_n will be typical in applications, see (42), (43), (44) below. One central aspect of the definition of the events $G_{n,x}$ is that if $x_1 \in L_{n-1}\mathbb{Z}^d \cap [-4\rho L_n, 4\rho L_n]^d$ and if $G_{n,0} \setminus G_{n-1,x_1}$ holds, then G_{n-1,x_2} holds for all $x_2 \in L_{n-1}\mathbb{Z}^d \cap [-4\rho L_n, 4\rho L_n]^d$ which are sufficiently far from x_1 in the sense that $\mathrm{dist}(x_1, x_2) \geq \sigma L_{n-1}$. This property propagates down to level $n = 0$. As to the events H_n, if one removes from $[-4\rho L_n, 4\rho L_n]^d$ the bad regions (where $G^c_{k,y}$ occurs for some $y \in [-4\rho L_n, 4\rho L_n]^d$ and $k < n$), then any point z in the remaining region "sits" in a tower of renormalized events (suitable shifts of H_k for $k \leq n$), which all occur. This observation motivates the notion of a black vertex below.

Proposition 6.3 (Stretched-exponential decay of probabilities)**.** *Fix a renormalization scheme with data λ, $G_{0,0}$, $(H_n)_n$, ρ and σ. Assume that for any $n \in \mathbb{N}$ and $x_1, x_2 \in L_n\mathbb{Z}^d$ such that $\mathrm{dist}(x_1, x_2) \geq \sigma L_n$, the events G_{n,x_1} and G_{n,x_2} are independent. For each $n \in \mathbb{N}$, set $q_n := \mathbb{P}[G_{n,0}^c]$ and assume that*

$$q_0 \leq \overline{q}_0 := \frac{1}{4(3\rho\lambda)^{2d}} \,. \tag{42}$$

Also, assume that for all $n \geq 1$,

$$\mathbb{P}[H_n^{\mathsf{c}}] \leq \overline{q}_0 2^{-2^n} \,. \tag{43}$$

Then, for all $n \geq 0$,

$$q_n \leq (2\overline{q}_0) 2^{-2^n} \,. \tag{44}$$

Proof. We prove the result by induction on n. For each $n \in \mathbb{N}$, apply a union bound over the pairs (x_1, x_2) defining $G_{n,0}$ to get

$$q_{n+1} = \mathbb{P}[G_{n+1,0}^c] \leq \mathbb{P}[H_{n+1}^c] + (3\rho\lambda)^{2d} q_n^2 \,.$$

By induction (or (42) for $n = 0$) and (43), $q_n \leq (2\overline{q}_0)2^{-2^n}$ and $\mathbb{P}[H_{n+1}^c] \leq \overline{q}_0 2^{-2^{n+1}}$. Plugging these estimates in the previous displayed equation implies that

$$q_{n+1} \leq \overline{q}_0 2^{-2^{n+1}} + 4(3\rho\lambda)^{2d} \overline{q}_0^2 \times 2^{-2^{n+1}} \,.$$

By (42) we obtain the desired bound for q_{n+1}. □

We now introduce the notion of black vertex.

Definition 6.4 (Black vertex)**.** This notion depends on a scale parameter $n \in \mathbb{N}$. We call a vertex $x \in \mathbb{Z}^d$ *black* (at scale n) if $G_{0,x}$ is satisfied and if, for each $m \in \{1, \dots, n\}$, there exists $y \in L_m\mathbb{Z}^d$ such that $\tau_y^{-1}(H_m)$ holds and $x \in y + [-4\rho L_m, 4\rho L_m]^d$.

Below, we consider "connected subsets" and "paths" for the usual hypercubic lattice with vertex-set $\mathbb{Z}^d$.

The following (deterministic) result is an adaptation of [DPR18b, Lemma 8.6].

Proposition 6.5 (Geometric properties of the renormalization scheme)**.** *Fix a renormalization scheme with data λ, $G_{0,0}$, $(H_n)_n$, ρ and σ. Assume that*

$$\lambda\rho \geq 100\sigma \text{ and } \rho \geq 2. \tag{45}$$

Let $n \geq 1$ and assume $G_{n,0}$ is satisfied. Then, for any connected sets $S_1, S_2 \subset \mathbb{Z}^d \cap [-\rho L_n, \rho L_n]^d$ such that $\mathrm{diam}(S_1), \mathrm{diam}(S_2) \geq 10\sigma L_{n-1}$, there exists a path of black vertices in $\mathbb{Z}^d \cap [-4\rho L_n, 4\rho L_n]^d$ whose endpoints belong to S_1 and S_2 respectively.

Remark 6.6. The proof of the previous proposition is a little easier in the particular case $d = 2$ (which is the only case used in the proof that $\ell_c < 0$), so the reader only interested in the existence of the unbounded nodal component can use planarity to simplify the proof below. Let us also stress that these methods are rather robust and do not in fact rely on the specific symmetries of $\mathbb{R}^d$ (essentially, as long as balls have polynomial volume growth and the base geometry does not exhibit "large bottlenecks", these techniques are likely to apply, cf. [DPR18b]).

Proof. We proceed by induction on n and prove the property for $G_{n,x}$ and $S_1, S_2 \subset x + [-\rho L_n, \rho L_n]^d$, with $x \in L_n\mathbb{Z}^d$ instead of $x = 0$.

Let us start with the case $n = 1$. For this case, we use neither (45) nor the fact that S_1, S_2 are connected. Let $S_1, S_2 \subset \mathbb{Z}^d \cap [-\rho L_1, \rho L_1]^d$ be two sets of diameter at least $10\sigma L_0 = 10\sigma$. There exist two paths of vertices in $\mathbb{Z}^d \cap [-2\rho L_1, 2\rho L_1]^d$, at a mutual distance at least σ, whose endpoints belong to S_1 and S_2 respectively[4]. In particular, if $G_{1,0}$ holds then one of these two paths must be entirely black. This proves the property for $n = 1$ (and with $[-2\rho L_n, 2\rho L_n]^d$ instead of $[-4\rho L_n, 4\rho L_n]^d$). The proof for translates of $G_{1,0}$ is identical.

Let $n \geq 2$, assume that the proposition holds for $n - 1$ and let $S_1, S_2 \subset \mathbb{Z}^d \cap [-\rho L_n, \rho L_n]^d$ be two connected sets of diameter at least $10\sigma L_{n-1}$. By the same reasoning as above, we get that there exist two sequences of vertices $(y_j)_j$ in $L_{n-1}\mathbb{Z}^d \cap [-2\rho L_n, 2\rho L_n]^d$ such that for each j,

$$\sum_{i=1}^{d} |(y_j)_i - (y_{j+1})_i| = L_{n-1},$$

at mutual distances at least σL_{n-1} and whose endpoints are at a distance at most L_{n-1} from S_1 and S_2 respectively. By definition of $G_{n,0}$, at least one of these, say $(y_j)_{1 \leq j \leq N}$, must be such that G_{n-1,y_j} is satisfied for each j. Without loss of generality, we assume that y_1 (resp. y_N) is at a distance at most L_{n-1} from S_1 (resp. S_2).

Let us note that since $G_{n,0} \subset H_n$, black vertices at scale $n - 1$ in $[-4\rho L_n, 4\rho L_n]$ are also black at scale n.

Let us first deal with the case where the path is of length 1, i.e. that $N = 1$. Since S_1 and S_2 are connected, and since $\lambda\rho \geq 100\sigma$, $\rho \geq 2$, and y_1 is at a distance smaller than or equal to L_{n-1} from S_1 and S_2, there exist $S_1' \subset S_1$ and $S_2' \subset S_2$ which are connected, of diameter larger than or equal to $10\sigma L_{n-2}$ and included in $y_1 + [-\rho L_{n-1}, \rho L_{n-1}]^d$. By the induction hypothesis, there exists a path of black vertices in $y_1 + [-4\rho L_{n-1}, 4\rho L_{n-1}]^d$ that connects S_1' and S_2'. This ends the proof in the case $N = 1$ since these paths are necessarily included in $[-4\rho L_n, 4\rho L_n]^d$.

[4] One can prove this as follows: Let $x_i, y_i \in S_i$ such that $\mathrm{dist}(x_i, y_i) \geq 10\sigma$, $i \in \{1, 2\}$ and choose the indexation of the points so that $\mathrm{dist}(x_1, y_2), \mathrm{dist}(x_2, y_1) \geq \mathrm{dist}(x_1, x_2)$. Then, one can let one of the two paths be an approximation of the segment $[x_1, x_2]$, let U be the 5σ-neighborhood of this path and use that

i) since $\mathrm{dist}(x_1, y_1), \mathrm{dist}(x_2, y_2) \geq 10\sigma$ and $\mathrm{dist}(x_1, y_2), \mathrm{dist}(x_2, y_1) \geq \mathrm{dist}(x_1, x_2)$, we have $y_1, y_2 \notin U$;
ii) $(\mathbb{Z}^d \cap [-2\rho L_1, 2\rho L_1]) \setminus U$ is connected.

Let us now consider the general case of an arbitrary path, i.e. let us assume that $N \geq 2$. Since $\lambda\rho \geq 100\sigma$ and by the induction hypothesis, for every $j \in \{1, \dots, N-1\}$ there exists a connected set

$$T_j \subset \mathbb{Z}^d \cap (y_{j+1} + [-\rho L_{n-1}, \rho L_{n-1}]^d) \cap (y_j + [-\rho L_{n-1}, \rho L_{n-1}]^d)$$

made of black vertices and of diameter larger than or equal to $10\sigma L_{n-2}$. By the induction hypothesis again, for every $j \in \{1, \dots, N-2\}$, T_j and T_{j+1} are connected by a black path included in $y_{j+1} + [-4\rho L_{n-1}, 4\rho L_{n-1}]^d$. Moreover, by reasoning as in the case $N = 1$, we obtain that T_1 (resp. T_{N-1}) is connected to S_1 (resp. S_2) by a path of black vertices included in $y_j + [-4\rho L_{n-1}, 4\rho L_{n-1}]^d$ with $j = 1$ (resp. $j = N$). This concludes the proof. □

7 Existence of an unbounded component in a slab

We now investigate the existence of unbounded components in $\{f \geq \ell\}$ intersected with a thick slab, for some (small) $\ell > 0$. The main result of this section is the following:

Theorem 7.1. *Let $d \geq 3$. There exists $\beta_0 > 0$ such that the following holds. Let q satisfying Assumption 1.4 for some $\beta > \beta_0$. Then, there exist $\ell, L > 0$ such that a.s. the set*

$$\{f \geq \ell\} \cap (\mathbb{R}^2 \times [0, L])$$

contains an unbounded connected component.

Before proving Theorem 7.1, let us deduce Item i) of Theorem 1.7 from it.

Proof of Item i) of Theorem 1.7. By Theorem 7.1 (together with translation invariance and equality in law of f and $-f$), there exist $\delta, L > 0$ such that, for every $\ell \in [-\delta, \delta]$, almost surely there is an unbounded component in both $\{f \geq \ell\} \cap (\mathbb{R}^2 \times [1, L-1])$ and $\{f \leq \ell\} \cap (\mathbb{R}^2 \times [1, L-1])$. Now, the result is a direct consequence of Lemma 2.5 (applied to $d' = 3$) and Lemma D.1 (applied to $E = \mathbb{R}^2 \times (0, L)$ and $\Sigma = \{f = \ell\} \cap E$). □

7.1 Definition of (very) good points

Let $0 < \gamma < a^2 < 1$ such that the hypotheses of Proposition 5.1 hold.

Definition 7.2 (good point)**.** Let $\delta, R > 0$. A point $x \in \mathbb{R}^2$ is called (δ-)*good* at scale R if the following three properties occur:

i) there exists a connected component of $\{f_{R^\gamma} \geq 2R^{-3/2}\} \cap (x + [-R, R]^2)$ of diameter larger than or equal to δR;
ii) all the connected components of $\{f_{R^\gamma} \geq 2R^{-3/2}\} \cap (x + [-2R, 2R]^2)$ of diameter larger than or equal to δR belong to the same connected component of $\{f_{R^\gamma} \geq R^{-3/2}\} \cap (x + [-4R, 4R]^2 \times [0, R^a])$.
iii) $\|f - f_{R^\gamma}\|_{\infty, B} \leq \frac{1}{2} R^{-3/2}$, where $B = x + [-4R, 4R]^2 \times [0, R^a]$.

7.2 The renormalization scheme

Let γ and a as above, assume that

$$\beta > \frac{5}{2\gamma} + \frac{d}{2}$$

and define a renormalization scheme (see Definition 6.1) depending on two parameters $\delta > 0$ and $R \geq 1$ as follows.

- Consider the probability space used in the rest of the paper with the $\mathbb{Z}^2$ action in which $x \in \mathbb{Z}^2$ acts by translating the field by Rx.
- Fix $\rho = 2$, $\sigma = 1000$ and $\lambda = 10^{10}$.
- Set $G_{0,0}$ to be the event that the point 0 satisfies Items i) and ii) from Definition 7.2 (recall that these items depend on δ and R). Since these two items are defined in terms of f_{R^γ} on $[-4R, 4R]^2 \times \mathbb{R}^{d-2}$, the event $G_{0,0}$ is measurable with respect to the white noise W restricted to $[-5R, 5R]^2 \times \mathbb{R}^{d-2}$ (if R is large enough). By Lemma 5.3 and Proposition 5.1, for every $h > 0$ there exist δ, R_0 such that for all $R \geq R_0$,
$$\mathbb{P}[G_{0,0}^c] \leq h. \tag{46}$$
- For each $n \geq 0$, write $f_{(n)} := f_{(RL_n)^\gamma}$ and if $n \geq 1$,
$$H_n := \{\|f_{(n)} - f_{(n-1)}\|_{\infty, B_n} < \varepsilon_n\},$$
where
$$B_n := [-8\rho RL_n+, 8\rho RL_n]^2 \times [0, R^a] \quad \text{and} \quad \varepsilon_n := (RL_{n-1})^{-\gamma(\beta - \frac{d}{2})+1}.$$
- Define the events $G_{n,x}$ for $n \in \mathbb{N}$ and $x \in L_n\mathbb{Z}^2$ as in Definition 6.1.

We start with the following lemma.

Lemma 7.3. *Consider the renormalization scheme defined above. For every $n \in \mathbb{N}$, the event $G_{n,0}$ is measurable with respect to the white noise W restricted to*

$$[-18\rho L_n, 18\rho L_n]^2 \times \mathbb{R}^{d-2}.$$

By our choice of ρ and σ, we have $\sigma > 36\sqrt{2}\rho$. As a result, G_{n,x_1} and G_{n,x_2} are independent for all $x_1, x_2 \in L_n\mathbb{Z}^2$ such that $\operatorname{dist}(x_1, x_2) \geq \sigma L_n$.

Proof. We construct by induction a sequence of positive real numbers $(r_n)_n$ such that for each $n \in \mathbb{N}$, $G_{n,0}$ is measurable with respect to W restricted to

$$[-r_nR, r_nR]^2 \times \mathbb{R}^{d-2}.$$

For $n = 0$, as explained after the definition of $G_{0,0}$ we may take $r_0 = 5$. Assume that we have constructed r_n. Then, $G_{n+1,0}$ is measurable with respect to H_{n+1} and all the events $G_{n,x}$ for $x \in L_n\mathbb{Z}^2 \cap [-4\rho L_{n+1}, 4\rho L_{n+1}]^2$. The event $G_{n,x}$ is the translate of $G_{n,0}$ by Rx so it is measurable with respect to the white noise restricted to

$$[-(4\rho L_{n+1} + r_n)R, (4\rho L_{n+1} + r_n)R]^2 \times \mathbb{R}^{d-2}.$$

The event H_n is measurable with respect to the white noise W on

$$[-9\rho L_n R, 9\rho L_n R]^2 \times \mathbb{R}^{d-2}$$

if R is larger than some constant. Altogether, we may take

$$r_{n+1} := 9\rho L_{n+1} + r_n.$$

Since $r_0 \le 9\rho$, the sequence $(r_n)_n$ satisfies

$$r_n \le \sum_{k=0}^{n} 9\rho\lambda^k \le \frac{9\rho\lambda}{\lambda - 1} L_n \le 18\rho L_n. \qquad \square$$

Corollary 7.4. *There exist $\delta > 0$ and $R \ge 1$ such that*

i) the renormalization scheme satisfies the conditions of Propositions 6.3 and 6.5;
ii) $\sum_{n\ge1} \varepsilon_n \le \frac{R^{-3/2}}{2}$.

Proof. We begin with i). By Lemma B.1, there exist $c, C > 0$ such that if $RL_{n-1} \ge C \log(RL_n)$ then

$$\mathbb{P}[H_n^c] \le e^{-c(RL_{n-1})^2}. \tag{47}$$

As a result, if R is sufficiently large then

$$\mathbb{P}[H_n^c] \le \frac{1}{4(3\rho\lambda)^4} 2^{-2^n}$$

for every $n \ge 1$. The rest of the assumptions of Propositions 6.3 and 6.5 follow from our choices for λ, ρ, σ, Lemma 7.3 and (46).

As for ii), $\beta > \frac{5}{2\gamma} + \frac{d}{2}$ implies that

$$\sum_{n\ge1} \varepsilon_n \le \frac{R^{-\gamma(\beta - \frac{d}{2})+1}}{1 - \lambda^{-\gamma(\beta-\frac{d}{2})+1}} = o(R^{-3/2}). \qquad \square$$

7.3 Proof of the theorem

We start with a lemma.

Lemma 7.5. *Let $\delta > 0$ and $R \ge 1$. If there is an infinite path $\pi_1, \pi_2, \dots$ of the lattice $\mathbb{Z}^2$ such that for every i, $R\pi_i$ is δ-very good, then there is an unbounded component in $\{f \ge \frac{1}{2}R^{-3/2}\} \cap (\mathbb{R}^2 \times [0, R^a])$.*

Proof. Let $\pi_1, \pi_2, \dots$ be such a path. By i) and ii) from Definition 7.2, for every j there exists a connected component of $\{f_{R^\gamma} \geq 2R^{-3/2}\}$ of diameter larger than or equal to δR in both $R\pi_j + [-R, R]^2$ and in $R\pi_{j+1} + [-R, R]^2$, and any two such components are connected in $\{f_{R^\gamma} \geq R^{-3/2}\} \cap (R\pi_j + [-4R, 4R]^2 \times [0, R^a])$. We obtain that there exists an unbounded component in

$$\{f_{R^\gamma} \geq R^{-3/2}\} \cap (\cup_{j\geq 1} \pi_j + [-4R, 4R]^2 \times [0, R^a]).$$

By iii) from Definition 7.2 there exists an unbounded component in

$$\{f \geq \tfrac{1}{2}R^{-3/2}\} \cap (\cup_{j\geq 1} \pi_j + [-4R, 4R]^2 \times [0, R^a]). \qquad \square$$

We now have all the tools to prove Theorem 7.1.

Proof of Theorem 7.1. Consider the renormalization scheme defined above and R, δ as in Corollary 7.4. We prove the theorem with $\ell = \frac{1}{2}R^{-3/2}$ and $L = R^a$. Let m_n be the largest integer such that $L_{m_n} \leq 2^n$, let $\Lambda_n := L_{m_n}\mathbb{Z}^2 \cap [-2^n, 2^n]^2$, and consider the following event:

$$A_n := \bigcap_{x\in\Lambda_n} \Big(G_{m_n,x} \cap \bigcap_{k\geq m_n+1} \tau_x^{-1}(H_k) \Big).$$

We first note that by Corollary 7.4, Proposition 6.3 applies and

$$\mathbb{P}[A_n^c] \leq C\Big(2^{1-2^{m_n}} + \sum_{k\geq m_n+1} 2^{-2^k}\Big) = O(2^{-2^{-m_n}})$$

for some $C > 0$ that depends only on d, λ, ρ. As a result, a.s. there exists n_0 such that A_n holds for all $n \geq n_0$.

A first claim: "black points imply very good points". For some $n \geq 1$, assume that A_n is satisfied and let $x \in \cup_{y\in\Lambda_n}(y + [-4\rho L_{m_n}, 4\rho L_{m_n}]^2)$ be a black vertex at scale m_n (recall Definition 6.4). We claim that Rx is a very good point. To show this claim, we note that $G_{0,x}$ is satisfied and that for each $m \geq 1$, there exists $y \in L_m\mathbb{Z}^2$ such that $x \in y + [-4\rho L_m, 4\rho L_m]^2$ and $\tau_y^{-1}(H_m)$ is satisfied. In particular,

$$\|f - f_{R^\gamma}\|_{\infty,B} < \sum_{n\geq 0} \varepsilon_n \leq \tfrac{1}{2}R^{-3/2},$$

where

$$B = Rx + ([-4R, 4R]^2 \times [0, R^a]).$$

The event $G_{0,x}$ implies that Rx satisfies Items i) and ii) of Definition 7.2 and the above implies that it satisfies Item iiii) of Definition 7.2. Altogether, Rx is a very good point.

Conclusion via a second claim. Consider some $n_0 \geq 1$ and let us assume that A_n holds for all $n \geq n_0$. Lemma 7.5 and the following claim enable us to conclude: There is an infinite path $\pi_1, \pi_2, \cdots \in \mathbb{Z}^2$ from $\partial[-2^{n_0-1}, 2^{n_0-1}]^2$ to infinity such that for every i, $R\pi_i$ is very good.

To derive this second claim, for all $n \geq n_0$ we construct a path $\theta_n \subset \mathbb{Z}^2 \cap [-2^n, 2^n]^2$ made of black vertices at level m_n and such that

- θ_{n_0} connects $\partial[-2^{n_0-1}, 2^{n_0-1}]^2$ to $\partial[-2^{n_0}, 2^{n_0}]^2$;
- if $n \geq n_0$, then θ_{n+1} connects θ_n to $\partial[-2^{n+1}, 2^{n+1}]^2$.

The construction of the infinite path $\pi_1, \pi_2, \ldots$ then follows from the existence of the paths θ_n and from the previous claim.

We only explain how to construct the path θ_{n_0} since the construction for other n's follows the same lines. To this purpose, first consider a path $x_1, \ldots, x_k \in \Lambda_{n_0}$ in the lattice $L_{m_{n_0}}\mathbb{Z}^d$ such that

- $x_1 + [-L_{m_{n_0}}, L_{m_{n_0}}]^2$ intersects $\partial[-2^{n_0-1}, 2^{n_0-1}]^2$ and
- $x_k + [-L_{m_{n_0}}, L_{m_{n_0}}]^2$ intersects $\partial[-2^{n_0}, 2^{n_0}]^2$.

By Proposition 6.5, for every $i \in \{1, \ldots, k-1\}$ there exists a connected set

$$T_i \subset \mathbb{Z}^2 \cap (x_{i+1} + [-\rho L_{m_{n_0}}, \rho L_{m_{n_0}}]^2) \cap (x_i + [-\rho L_{m_{n_0}}, \rho L_{m_{n_0}}]^2)$$

made of black vertices and of diameter larger than or equal to $10\sigma L_{m_{n_0}-1}$ (recall that we set $\rho = 2, \sigma = 1000, \lambda = 10^{10}$). By Proposition 6.5 once more, for every $i \in \{1, \ldots, k-2\}$, T_i and T_{i+1} are connected by a black path included in $x_{i+1} + [-4\rho L_{m_{n_0}}, 4\rho L_{m_{n_0}}]^2$. Still by Proposition 6.5, T_1 (resp. T_{k-1}) is connected to $\partial[-2^{n_0-1}, 2^{n_0-1}]^2$ (resp. $\partial[-2^{n_0}, 2^{n_0}]^2$) by a path of black vertices included in $x_i + [-4\rho L_{m_{n_0}}, 4\rho L_{m_{n_0}}]^2$ with $i = 1$ (resp. $i = k$). This ends the construction of θ_{n_0} and the proof of the theorem. □

8 Existence of a giant component "almost everywhere"

In this section, we prove the following result.

Theorem 8.1. *Let $d \geq 3$ and $d' \in [3, d]$. There exists $\beta_0 > 0$ such that the following holds. Let q satisfy Assumption 1.4 for some $\beta > \beta_0$. There exist $\ell, N > 0$ such that a.s. there is an unbounded component component $\mathcal{C}$ of $\{f \geq \ell\} \cap \mathbb{R}^{d'}$ and some (random) R_0 such that for every $R \geq R_0$, $\mathcal{C}$ instersects all the (Euclidean, closed) balls of radius $(\log R)^N$ that are included in $[-R, R]^{d'}$.*

Exactly like in Section 7, Item ii) of Theorem 1.7 is a direct consequence of Theorem 8.1. We omit this short argument and refer to Section 7. Let us now prove Theorem 8.1 (the proof is very similar to the proofs of the previous section). We write the proof in the case $d' = d$ to simplify the notations and since the proof in the general case is exactly the same.

8.1 Definition of (very) good points

We start by defining a slightly different notion of good and very good points. The essential difference is that we ask connected components to exist on each "face" of the boxes.

Below, for every $x \in \mathbb{R}^d$, $F_R^1(x), \dots, F_R^{N_d}(x)$ are the 2-dimensional faces of the hypercube $x + [-R/2, R/2]^d$ and for every $i \in \{1, \dots, N_d\}$, $\tilde{F}_R^i(x) \subset F_R^i(x)$ is the concentric square with side length equal to $R/2$. Let γ be such that Proposition 5.8 holds.

Definition 8.2 (Good point)**.** Let $\delta, R > 0$. A point $x \in \mathbb{R}^d$ is called (δ-)*good* at scale R if the following properties occur:

i) for every $i \in \{1, \dots, N_d\}$, there exists a connected component of $\{f_{R^\gamma} \geq 2R^{-3/2}\} \cap \tilde{F}_R^i(x)$ of diameter larger than or equal to δR;
ii) all the above connected components belong to the same connected component of $\{f_{R^\gamma} \geq R^{-3/2}\} \cap (x + [-R/2, R/2]^d)$;
iv) $\|f - f_{R^\gamma}\|_{\infty, x+[-R/2,R/2]^d} \leq \frac{1}{2} R^{-3/2}$.

8.2 The renormalization scheme

Concerning the renormalization scheme, it accommodates the fact that we now work with a d-dimensional renormalization instead of a planar one. In particular, the renormalization space is $\mathbb{Z}^d$ instead of $\mathbb{Z}^2$, B_n is defined differently, and $G_{0,0}$ depends on the white noise in $[-2R, 2R]^d$. Finally, the fact that $G_{0,0}^c$ has small probability will be a consequence of Proposition 5.8.

Let γ as above, assume that

$$\beta > \frac{5}{2\gamma} + \frac{d}{2},$$

and define a renormalization scheme depending on two parameters $\delta > 0$ and $R \geq 1$ as follows.

- Consider the probability space used in the rest of the paper with the $\mathbb{Z}^d$ action given by $x \mapsto Rx$.
- Fix $\rho = 2$, $\sigma = 1000d$ and $\lambda = 10^{10}d$.
- Set $G_{0,0}$ to be the event that the point 0 satisfies Items i) and ii) from Definition 8.2. The event $G_{0,0}$ is measurable with respect to the white noise W restricted to $[-2R, 2R]^d$ (if R is large enough). By Lemma 5.3 and Proposition 5.8, for every $h > 0$ there exist δ, R_0 such that for all $R \geq R_0$,

$$\mathbb{P}[G_{0,0}^c] \leq h. \tag{48}$$

- For each $n \geq 0$, write $f_{(n)} := f_{(RL_n)^\gamma}$ and if $n \geq 1$,

$$H_n := \{\|f_{(n)} - f_{(n-1)}\|_{C^1(B_n)} < \varepsilon_n\}$$

where

$$B_n := [-8\rho R L_n, 8\rho R L_n]^d \quad \text{and} \quad \varepsilon_n := (RL_{n-1})^{-\gamma(\beta-\frac{d}{2})+1}.$$

- Define the events $G_{n,x}$ for $n \in \mathbb{N}$ and $x \in L_n\mathbb{Z}^d$ as in Definition 6.1.

The proofs of the following results are exactly the same as those of Lemma 7.3 and Corollary 7.4 (the statements are also very similar, except the appearance of the dimension d instead of 2). We omit the proofs.

Lemma 8.3. *Consider the renormalization scheme defined above. For $n \in \mathbb{N}$, the event $G_{n,0}$ is measureable with respect to the white noise W restricted to $[-18\rho L_n, 18\rho L_n]^d$. By our choice of ρ, σ, we have $\sigma > 36\sqrt{d}\rho$. As a result, G_{n,x_1} and G_{n,x_2} are independent for all $x_1, x_2 \in L_n\mathbb{Z}^2$ such that $\operatorname{dist}(x_1, x_2) \geq \sigma L_n$.*

Corollary 8.4. *We can choose $\delta > 0$, $R \geq 1$ such that*

i) the renormalization scheme satisfies the conditions of Propositions 6.3 and 6.5;
ii) $\sum_{n\geq 1} \varepsilon_n \leq \frac{1}{2}R^{-3/2}$.

8.3 Proof of the theorem

We start with a lemma.

Lemma 8.5. *Let $\delta > 0$, $R \geq 1$ and let $\mathcal{C}$ be a connected subset of $\mathbb{Z}^d$ such that, for every $x \in \mathcal{C}$, Rx is δ-very good. Then, there is a connected component of $\{f \geq \frac{R^{-3/2}}{2}\} \cap (\cup_{x\in\mathcal{C}} Rx + [-R/2, R/2]^d)$ that intersects $Rx + [-R/2, R/2]^d$ for every $x \in \mathcal{C}$.*

The proof is essentially the same as the proof of Lemma 7.5 and is omitted here. We now give the proof of the theorem.

Proof of Theorem 8.1. Recall that we write the proof in the case $d' = d$ since the proof in the general case is exactly the same. Consider the renormalization scheme defined above and R, δ as in Corollary 8.4. We prove the theorem with $\ell = \frac{1}{2}R^{-3/2}$. The proof is essentially the same as that of Theorem 7.1, so we leave the details to the reader and only mention the two main differences:

- We let m_n be the largest integer such that

$$L_{m_n} \leq n^N,$$

where $N > 0$ is any number such that

$$(2^n/n^N)^d(2^{1-2^{m_n}} + \sum_{k\geq m_n+1} 2^{-2^k})$$

is summable. Moreover, we let $\Lambda_n = L_{m_n}\mathbb{Z}^d \cap [-2^n, 2^n]^d$.

- Given some n_0, we construct some paths $\theta_n \subset \mathbb{Z}^d \cap [-2^n, 2^n]^d$, $n \geq n_0$, made of black vertices at level m_n and such that

– θ_{n_0} connects $\partial[-2^{n_0-1}, 2^{n_0-1}]^d$ to $\partial[-2^{n_0}, 2^{n_0}]^d$;
– if $n \geq n_0$, then θ_{n+1} connects θ_n to $\partial[-2^{n+1}, 2^{n+1}]^d$;
– for every $n \geq n_0$, θ_n intersects $x + [-\rho L_{m_n}, \rho L_{m_n}]^d$ for *every* $x \in \Lambda_n$.

This completes the proof. □

A The Gaussian FKG inequalities

This section is devoted to the proofs of the FKG inequalities Lemma 2.1 and Corollary 2.2 (as well as generalizations and finite dimensional analogues). The proofs have been provided to us by Matthis Lehmkühler and are included here with his permission. Below and in all the appendix, we equip $C(\mathbb{R}^d)$ with the topology of uniform convergence on every compact subset.

A.1 Proof of the continuous FKG inequality

In this section, we prove Lemma 2.1. The proof is based on the following result by Pitt.

Lemma A.1 (Finite dimensional Gaussian FKG inequality, [Pit82])**.** *Let $\phi, \psi : \mathbb{R}^n \to \mathbb{R}$ be two non-decreasing bounded measurable functions and let X be a centered Gaussian vector such that for every $i, j \in \{1, \dots, n\}$, $\mathbb{E}[X_i X_j] \geq 0$. Then,*

$$\mathbb{E}\left[\phi(X)\psi(X)\right] \geq \mathbb{E}\left[\phi(X)\right]\mathbb{E}\left[\psi(X)\right].$$

Proof of Lemma 2.1. This is a direct consequence of Lemma A.1 and of the two following claims, where $\phi : C(\mathbb{R}^d) \to \mathbb{R}$ and f are as in the statement of Lemma 2.1.

Claim A.2. *Assume furthermore that ϕ is continuous. Then, there exists a sequence of non-decreasing functions $\phi_n : C(\mathbb{R}^d) \to \mathbb{R}$ bounded by $\|\phi\|_\infty$ that depend on only finitely many points*[5] *and such that for every $u \in C(\mathbb{R}^d)$, we have $\phi_n(u) \to \phi(u)$.*

Claim A.3. *There exists a sequence of non-decreasing continuous functions $\phi_n : C(\mathbb{R}^d) \to \mathbb{R}$ bounded by $\|\phi\|_\infty$ such that a.s. we have $\phi_n(f) \to \phi(f)$.*

Proof of Claim A.2. Let χ_ε be a smooth approximation of the identity with compact support. Also, let θ_R be a smooth function which is compactly supported and equal to 1 in $[-R, R]^d$. For every n, R, ε and for every continuous function $u : \mathbb{R}^d \to \mathbb{R}$ we let

$$u_{R,n,\varepsilon}(x) := \theta_R(x)\frac{1}{n^d}\sum_{v \in \frac{1}{n}\mathbb{Z}^d} u(v)\chi_\varepsilon(x - v) \quad \text{and} \quad \phi_{n,R,\varepsilon}(u) := \phi(u_{R,n,\varepsilon}).$$

Then, $u_{R,n,\varepsilon}$ converges to $\theta_R \times (u \star \chi_\varepsilon)$ as n goes to infinity, $\theta_R \times (u \star \chi_\varepsilon)$ converges to $u \star \chi_\varepsilon$ as R goes to infinity, and $u \star \chi_\varepsilon$ converges to u as ε goes to 0. Moreover, $\phi_{n,R,\varepsilon}$ is non-decreasing in u since $u \mapsto u_{R,n,\varepsilon}$ is non-decreasing. □

[5] I.e. for all n there exist $x_1, \dots, x_{m_n} \in \mathbb{R}^d$ and a measurable function $\widetilde{\phi}_n : \mathbb{R}^{m_n} \to \mathbb{R}$ such that $\phi_n(u) = \widetilde{\phi}_n(u(x_1), \dots, u(x_{m_n}))$.

Proof of Claim A.3. We follow the proof of the lemma from Section 3 of [Pit82] (which is the analogous result for functions on $\mathbb{R}^n$ rather than $C(\mathbb{R}^d)$). First, exactly as in [Pit82], we note that we can assume that ϕ is the indicator of some event A and we work in this case.

By regularity of the law of f (because $C(\mathbb{R}^d)$ is a Polish space), there exist two sequences of compact sets $C_n, K_n \subset C(\mathbb{R}^d)$ and a sequence of open sets $O_n \subset C(\mathbb{R}^d)$ with

$$K_n \subset C_n \cap A; \quad A \subset O_n; \quad \mathbb{P}[f \notin C_n] < 1/n^2; \quad \mathbb{P}[f \in O_n \setminus K_n] < 1/n^2. \tag{49}$$

For every $\varepsilon, M > 0$ and $u \in C(\mathbb{R}^d)$, we define the following non-decreasing neighborhood of u:

$$R^M(u, \varepsilon) := \{v \in C(\mathbb{R}^d) : \forall x \in [-M, M]^d, v(x) > u(x) - \varepsilon\}.$$

As in [Pit82], we fix

i) two sequences ε_n, M_n such that, for every $u \in K_n$, $C_n \cap R^{M_n}(u, 2\varepsilon_n) \subset O_n$,
ii) a finite cover $\{R^{M_n}(u^i, \varepsilon_n)\}_{i=1}^{N_n}$ of K_n,

and we let

$$L_n := \bigcup_{i=1}^{N_n} R^{M_n}(u^i, \varepsilon_n).$$

Then, we define the following continuous and non-decreasing functions:

$$\phi_n : u \mapsto \left(1 - \varepsilon_n^{-1} \inf_{v \in L_n} \|v - u\|_{\infty, [-M_n, M_n]^d}\right)_+.$$

It remains to show that $\phi_n(f)$ converges to $\phi(f)$ a.s. To prove this, we use (49) and we observe that

$$\begin{aligned}\mathbf{1}_{K_n}(u) \le \mathbf{1}_{L_n}(u) \le \phi_n(u) \le \mathbf{1}\{\exists i, \inf_{v \in R^{M_n}(u^i, \varepsilon_n)} \|v - u\|_{\infty, [-M_n, M_n]^d} < \varepsilon_n\} \\ \le \mathbf{1}\{\exists i, u \in R^{M_n}(u^i, 2\varepsilon_n)\} \le \mathbf{1}_{O_n}(u) + \mathbf{1}_{C_n^c}(u).\end{aligned}$$

□

This concludes the proof of Lemma 2.1. □

A.2 The local FKG inequalities

In this section, we prove discrete and continuous local FKG inequalities (Lemmas A.4 and A.6). Corollary 2.2 is a direct consequence of Lemma A.6.

Let us start with some notations. For every $\phi : \mathbb{R}^n \to \mathbb{R}$ and for every $i \in \{1, \dots, n\}$ such that ϕ is monotonic in i, we let $\epsilon_i^\phi = 0$ if ϕ does not depend on x_i (in the sense that if x and y agree outside of the i^{th} coordinate then $\phi(x) = \phi(y)$); $\epsilon_i^\phi = 1$ if ϕ depends on x_i and is non-decreasing in x_i; $\epsilon_i^\phi = -1$ if ϕ depends on x_i and is non-increasing in x_i.

Lemma A.4 (Finite dimensional local Gaussian FKG inequality)**.** *Let $\phi, \psi : \mathbb{R}^n \to \mathbb{R}$ be two bounded measurable functions and let X be a centered Gaussian vector. Assume that for every $(i,j) \in \{1,\dots,n\}^2$ one of the following properties hold:*

- $\mathbb{E}[X_i X_j] = 0$,
- *ϕ does not depend on x_i,*
- *ψ does not depend on x_j,*
- *ϕ is monotonic in x_i, ψ is monotonic in x_j and $\mathbb{E}[X_i X_j]\epsilon_i^{\phi}\epsilon_j^{\psi} \geq 0$.*

Then,

$$\mathbb{E}\left[\phi(X)\psi(X)\right] \geq \mathbb{E}\left[\phi(X)\right]\mathbb{E}\left[\psi(X)\right].$$

Proof. The proof is essentially the same as Lemma A.1 (proven in [Pit82]). Pitt proves Lemma A.1 by first showing it with the further hypothesis that ϕ and ψ are continuous and then shows an approximation result. The proof of Lemma A.4 with the further hypothesis that ϕ and ψ are continuous is exactly the same as in [Pit82]. Then, we can conclude by using the following claim, where ϕ and X are as in the statement of Lemma A.4.

Claim A.5. *Let I (resp. J) denote the set of coordinates on which ϕ is non-decreasing (resp. non-increasing).*[6] *There exists a sequence of continuous functions $\phi_n : \mathbb{R}^n \to \mathbb{R}$ bounded by $\|\phi\|_\infty$ that are non-decreasing (resp. non-increasing) in I (resp. J) and such that a.s. $\phi_n(X) \to \phi(X)$.*

Proof. The proof is essentially the same as that of Claim A.3. As previously, we can assume that ϕ is the indicator of some event A and we consider $C_n, K_n \subset \mathbb{R}^n$ two sequences of compact sets and $O_n \subset \mathbb{R}^n$ a sequence of open sets that satisfy (49). We then define the following set for every $x \in \mathbb{R}^d$ and $\varepsilon > 0$:

$$R(x,\varepsilon) := \left\{ y \in \mathbb{R}^k : \begin{array}{l} \forall i \in I \setminus J, y_i \geq x_i - \varepsilon, \\ \forall i \in J \setminus I, y_i \leq x_i + \varepsilon, \\ \forall i \notin I \cup J, |x_i - y_i| \leq \varepsilon. \end{array} \right\}.$$

As previously, we fix

- a sequence $\varepsilon_n > 0$ such that for every $x \in K_n$, $C_n \cap R(x,\varepsilon_n) \subset O_n$,
- a finite cover $\{R(x^i,\varepsilon_n)\}_{i=1}^{N_n}$ of K_n,

and we let

$$L_n := \bigcup_{i=1}^{N_n} R(x^i,\varepsilon_n) \quad \text{and} \quad \phi_n(x) := \left(1 - \varepsilon_n^{-1} \inf_{y \in L_n} \|y - x\|_\infty\right)_+.$$

As in the proof of Claim A.3, the functions ϕ_n satisfy the desired properties. □

This concludes the proof of Lemma A.4. □

[6] Note that $I \cap J$ is the set of coordinates on which ϕ does not depend.

We end the section by showing a general local FKG inequality in the continuum. As previously, we need some notation. If $x \in \mathbb{R}^d$ and $\delta > 0$, we let $Q_\delta(x) := x + [-\delta, \delta]^d$. Monoticity properties for functions on $C(\mathbb{R}^d)$ are defined above Corollary 2.2. Moreover, we say that a function $\phi : C(\mathbb{R}^d) \to \mathbb{R}$ does not depend on some set $U \subset \mathbb{R}^d$ if $\phi(u) = \phi(v)$ for every u, v that agree on U^c. Let $\phi : C(R^d) \to \mathbb{R}$ and $U \subset \mathbb{R}^d$. If ϕ is monotonic in U, we let $\epsilon_U^\phi = 0$ if ϕ does not depend on U (in the sense that $\phi(u) = \phi(v)$ for every u, v that agree outside of U); $\epsilon_U^\phi = 1$ if ϕ depends on U and is non-decreasing in U; $\epsilon_U^\phi = -1$ if ϕ depends on U and is non-increasing in U.

Lemma A.6 (Local continuous Gaussian FKG inequality)**.** *Let $\phi, \psi : C(\mathbb{R}^d) \to \mathbb{R}$ be two bounded measurable functions and let f be a centered continuous Gaussian field on $\mathbb{R}^d$. Assume that there exists $\delta > 0$ such that for every $(x, y) \in (\mathbb{R}^d)^2$ one of the following properties holds:*

- $\forall (s, t) \in Q_\delta(x) \times Q_\delta(y), \mathbb{E}[f(s)f(t)] = 0$,
- *ϕ does not depend on $Q_\delta(x)$,*
- *ψ does not depend on $Q_\delta(y)$,*
- *ϕ is monotonic in $Q_\delta(x)$, ψ is monotonic in $Q_\delta(y)$, the sign of $\mathbb{E}[f(s)f(t)]$ is constant in $Q_\delta(x) \times Q_\delta(y)$, and*

$$\mathbb{E}[f(x)f(y)]\epsilon^\phi_{Q_\delta(x)}\epsilon^\psi_{Q_\delta(y)} \geq 0.$$

Then,

$$\mathbb{E}\left[\phi(f)\psi(f)\right] \geq \mathbb{E}\left[\phi(f)\right]\mathbb{E}\left[\psi(f)\right].$$

Proof. Let $\eta > 0$, let $\rho : C(\mathbb{R}^d) \to \mathbb{R}$ be a bounded measurable function and let $U, V \subset \mathbb{R}^d$ defined by $x \in U$ (resp. V) if ϕ is non-decreasing (resp. non-increasing) in $Q_\eta(x)$. Lemma A.6 is a direct consequence of Lemma A.4 and of the two following claims.

Claim A.7. *Assume furthermore that ρ is continuous. Then, there exists a sequence of functions $\rho_n : C(\mathbb{R}^d) \to \mathbb{R}$ bounded by $\|\rho\|_\infty$ that depend on only finitely many points and such that:*

- *for every $u \in C(\mathbb{R}^d)$, we have $\rho_n(u) \to \rho(u)$,*
- *for every $x \in U$ (resp. V), ρ_n is non-decreasing (resp. non-increasing) in $Q_{\eta/2}(x)$.*

Claim A.8. *There exists a sequence of continuous functions $\rho_n : C(\mathbb{R}^d) \to \mathbb{R}$ bounded by $\|\rho\|_\infty$ such that*

- *a.s. we have $\rho_n(f) \to \rho(f)$,*
- *for every $x \in U$ (resp. V), ρ_n is non-decreasing (resp. non-increasing) in $Q_{\eta/2}(x)$.*

Proof of Claim A.7. The proof is the same as Claim A.2 with the further hypothesis that the support of χ_ε is included in $[-\eta/2, \eta/2]^d$. □

Proof of Claim A.8. The proof is essentially a combination of the proofs of Claims A.3 and A.5. More precisely, we define $I, J \subset (\eta/4)\mathbb{Z}^d$ by $x \in I$ (resp. J) if ρ is non-decreasing (resp. non-decreasing) in $Q_{\eta/4}(x)$ and we follow the proof of Claim A.8 but replacing the sets $R^M(u,\varepsilon)$ by

$$\left\{ v \in C(\mathbb{R}^d) : \begin{array}{l} \forall x \in (I \setminus J) \cap [-M,M]^d,\ \forall y \in x + [0,\eta/4)^d,\ v(y) \geq v(x) - \varepsilon, \\ \forall x \in (J \setminus I) \cap [-M,M]^d,\ \forall y \in x + [0,\eta/4)^d,\ v(y) \leq u(y) + \varepsilon, \\ \forall x \in (I \cup J)^c \cap [-M,M]^d,\ \forall y \in x + [0,\eta/4)^d,\ |v(y) - u(y)| \leq \varepsilon. \end{array} \right\}.$$

The rest of the proof is the same so we omit the details. □

This concludes the proof of Lemma A.6. □

B Other properties of Gaussian fields

B.1 Approximation by truncation and sprinkling

In this section, we present two approximation results that essentially come from [MV20]. The first one is based on Kolmogorov and BTIS lemmas and shows that the field does not vary too much under truncation. The second one is based on a Cameron–Martin type argument and shows that the probability of monotone events does not change much under a small sprinkling.

Lemma B.1. *Assume that q satisfies Assumption 1.4 for some $\beta > d$. Then, there exist $c, C > 0$ such that, for all $r, R \geq 1$ and $t \geq C \log R$,*

$$\mathbb{P}\big[\|f - f_r\|_{\infty, B(R)} \geq t r^{-(\beta - \frac{d}{2})}\big] \leq e^{-ct^2}. \tag{50}$$

Proof. The proof follows the lines of that of Proposition 3.11 in [MV20], noting that Assumption 1.4 implies that

$$\sup_x \sup_{|\alpha| \leq 1} \mathbb{E}\big[(\partial^\alpha (f - f_r)(x))^2\big] = O(r^{d-2\beta}),$$

which, upon applying the Kolmogorov and BTIS lemmas, yields the desired bound. □

Lemma B.2 (Proposition 3.6 in [MV20])**.** *Let q satisfying Assumption 1.4 for some $\beta > d$. There exist $C, r_0, R_0 > 0$ such that for every $r \in [r_0, \infty]$, $R \geq R_0$, $t \in \mathbb{R}$ and every monotonic event $A \in \mathcal{F}_{D(0,R)\times[0,1]^{d-2}}$, we have*

$$|\mathbb{P}[f_r \in A] - \mathbb{P}[f_r + t \in A]| \leq C|t|R.$$

It is important to note that C does not depend on $r \geq r_0$.

Proof. The proof is the same as the one of the analogous result Proposition 3.6 in [MV20]. Indeed, the conditions q non-identically equal to 0 and $q \geq 0$ as well as the decay condition of Assuption 1.4 imply that $\mathcal{F}(q_r \star q_r)$ is well-defined and larger than

some constant c that does not depend on r in some neighbourhood of 0 (as soon as r is sufficiently large). The only difference in the proof is that since f_r is defined in $\mathbb{R}^d$, the function h that one should choose is rather

$$\mathcal{F}[R^{2+a}c^{-d}\mathbf{1}_{[0,cR]^2\times\times[0,c]^{d-2}}]. \qquad \square$$

B.2 A bound on the number of connected components of the excursion sets

Lemma B.3. *Let q satisfying Assumption 1.4 for some $\beta > d$. There exists $C > 0$ such that for every $r \geq r_q$ and $\ell \in \mathbb{R}$, the expectation of the number of connected components of $\{f_r \geq \ell\} \cap \mathbb{R}^2$ that intersect $D(0,1)$ is less than C.*

Proof. Let $N_2(r)$ (resp. $N_1(r)$) denote the number of critical points of the restriction of f_r to $D(0,1)$ (resp. $\partial D(0,1)$). Then, the number considered in the statement is less than or equal to $N_1(r) + N_2(r)$. Moreover, by Kac–Rice formula (see e.g. Theorem 6.2 and Proposition 6.5 of [AW09]) and stationarity, we have

$$\begin{aligned}\mathbb{E}[N_1(r)] &= \int_0^{2\pi} \frac{1}{\sqrt{2\pi|\partial_\theta^2\kappa_r(0)|}}\mathbb{E}\big[|\partial_\theta^2 f_r(e^{i\theta})| \mid \partial_\theta f_r(e^{i\theta}) = 0\big]d\theta \\ &= \frac{2\pi}{\sqrt{2\pi|\partial_\theta^2\kappa_r(0)|}}\mathbb{E}[|\partial_\theta^2 f_r(0)|],\end{aligned}$$

where $\kappa_r = q_r \star q_r$ is the covariance function of f_r. Similarly, setting $g_r = (f_r)_{|\mathbb{R}^2}$ and using stationarity, we have

$$\begin{aligned}\mathbb{E}[N_2(r)] &= \int_{D(0,1)} \frac{1}{\sqrt{(2\pi)^2\det(\mathrm{Cov}(\nabla g_r(0)))}}\mathbb{E}\big[|\nabla^2 g_r(x)| \mid \nabla g_r(x) = 0\big]dx \\ &= \frac{\pi}{\sqrt{(2\pi)^2\det(\mathrm{Cov}(\nabla g_r(0)))}}\mathbb{E}[|\nabla^2 g_r(0)|].\end{aligned}$$

The above quantities are bounded uniformly in $r \geq r_q$. $\square$

C A percolation estimate: proof of Lemma 5.9

In this section we provide a proof of Lemma 5.9 based on a classical method from planar percolation. We expect that a stronger result could be reached using a deformation estimate as in the proof of Lemma 3.3 to compare the probability of crossing a flat rectangle to the probability of crossing the same rectangle bent into the shape of a quarter of a cylinder. However, this argument relies on full rotational symmetry and on Gaussianity so it does not invite generalization as much as the argument presented here.

Proof of Lemma 5.9. The proof is via a second moment method involving arm events in half-planes. Recall that $B(R)$ is the Euclidean ball centered at 0.

An arm from scale r_1 to scale r_2 (where $0 < r_1 \leq r_2$) is a path in $\{f_{R^\gamma} \geq 0\}$ from $\partial B(r_1)$ to $\partial B(r_2)$. Let α_{r_1,r_2} denote the probability that there exist two arms from scale r_1 to scale r_2 that are included in the half planes $\mathbb{R} \times \{0\} \times \mathbb{R}_+$ and $\mathbb{R} \times \mathbb{R}_+ \times \{0\}$ respectively. We first note that by the FKG inequality (Lemma 2.1), the RSW theorem (Theorem 2.4) and standard gluing properties (see Item iv) at the end of Section 2.3), for all $r_3 \geq r_2 \geq r_1 > 0$,

$$\alpha_{r_1,r_2}\alpha_{r_2,r_3} \leq C\alpha_{r_1,r_3} \tag{51}$$

for some universal $C > 0$. Similarly, if $0 < r_1 \leq 4r_2$, then $\alpha_{r_1/2,2r_2}$ is of the same order as α_{r_1,r_2}. We will use the latter below without mentioning it.

Claim C.1. *There exists a universal constant $c > 0$ such that if $r_2 \geq R^\gamma$ and $1 \leq r_1 \leq r_2$ then*

$$\alpha_{r_1,r_2} \geq cr_1/r_2. \tag{52}$$

Proof. By RSW and Lemma 2.3 and since f_{R^γ} is R^γ-dependent, for any $r \geq 2R^\gamma$ the probability that an $r \times r$ square is crossed by a path in $\{f_{R^\gamma} = 0\}$ is larger than some universal positive constant. Any ball centered at the extremity of this path touching a given side is then connected to the opposite side by a path in $\{f_{R^\gamma} \geq 0\}$ and a path in $\{f_{R^\gamma} \leq 0\}$. These two events are negatively correlated by FKG. By using this observation and the union bound, we deduce that the probability that there exists an arm from scale r_1 to scale $r_2 \geq R^\gamma$ in $\mathbb{R} \times \mathbb{R}_+$ is at least $c(r_2/r_1)^{-1/2}$ for some universal constant $c > 0$. Applying FKG once again we obtain, as required,

$$\alpha_{r_1,r_2} \geq (c(r_2/r_1)^{-1/2})^2 = c^2(r_1/r_2). \qquad \square$$

Now, let $\mathcal{S}_r^1 := [0,r] \times \{0\} \times [0,r]$ and $\mathcal{S}_r^2 := [0,r] \times [0,r] \times \{0\}$ (so that $\mathcal{S}_r = \mathcal{S}_r^1 \cup \mathcal{S}_r^2$), let I be the set of points of integer coordinates that belong to $[r/4, 3r/4] \times \{0\} \times \{0\}$ (so in particular I is included in the common side of $\mathcal{S}_r^1$ and $\mathcal{S}_r^2$) and for every $i \in I$ let A_i (resp. B_i) denote the event that there is a path in $\{f_{R^\gamma} \geq 0\} \cap \mathcal{S}_r^1$ (resp. $\{f_{R^\gamma} \geq 0\} \cap \mathcal{S}_r^2$) from i to the opposite side of $\mathcal{S}_r^1$ (resp. $\mathcal{S}_r^2$). Finally, let X denote the number of points $i \in I$ such that both A_i and B_i hold. Note that if $X > 0$ then the event from the lemma holds. We now apply the second moment method to X.

We first note that by FKG and RSW, $\mathbb{E}[X] \geq c'r\alpha_{1,r}$ for some $c' > 0$. In order to estimate $\mathbb{E}[X^2] = \sum_{i,j} \mathbf{1}_{A_i \cap B_i}\mathbf{1}_{A_j \cap B_j}$ we need to consider separately the cases $|i-j| \geq 2R^\gamma$ and $|i-j| < 2R^\gamma$. When $|i-j| \geq 2R^\gamma$ (which corresponds to the first sum below), three independent events happen: two events between scale 1 and $|i-j|/2$ and an event between scales $|i-j|$ and r), so we obtain that

$$\mathbb{E}[X^2] \leq C'r \sum_{R^\gamma \leq k \leq r} \alpha_{1,k}^2 \alpha_{k,r} + C'r \sum_{1 \leq k \leq R^\gamma} \alpha_{1,r},$$

for some $C' > 0$. By the quasi-multiplicativity property (51), the above is at most a positive constant times

$$r\alpha_{1,r}^2 \sum_{R^\gamma \leq k \leq r} 1/\alpha_{k,r} + r\alpha_{1,r}R^\gamma.$$

Moreover, since by (52), $\alpha_{k,r} \geq ck/r$ (so in particular $r\alpha_{1,r} \geq c$), we obtain that the above is at most a positive constant times

$$r^2\alpha_{1,r}^2(\log(r) + R^\gamma).$$

Finally, by the Cauchy–Schwarz inequality,

$$\mathbb{P}[X > 0] \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]} \geq \frac{c''}{\log(r) + R^\gamma}$$

for some $c'' > 0$, which is the desired result. □

D A topological lemma

Lemma D.1. *Let $E = \mathbb{R}^d$ or $\mathbb{R}^d \times (L, L')$ for some $d \geq 1$ and $L < L'$, let $\mathcal{U}$ be a collection of simply connected open subsets of E and let Σ be a C^1-smooth embedded hypersurface which is a closed (though not necessarily compact) subset of E. Assume that there exist two distinct connected components of $E \setminus \Sigma$ which intersect each $U \in \mathcal{U}$. Then, Σ has a connected component which also intersects each $U \in \mathcal{U}$.*

Proof. Without loss of generality we assume that $E \in \mathcal{U}$. If $U \in \mathcal{U}$ and Σ' is a connected component of $U \cap \Sigma$, then we claim that $U \setminus \Sigma'$ has exactly two connected components U', U'' and $\partial U' = \partial U'' = \Sigma'$. Indeed, using the fact that Σ' is connected, it is easy to show that it has a neighbourhood $V \subset U$ such that $V \setminus \Sigma'$ has exactly two connected components. But since U is connected, this is also the case of $U \setminus \Sigma'$. Next, by considering a local chart of Σ' one can find two points $x, y \in U \setminus \Sigma'$ which are connected by a smooth path in U intersecting Σ' transversally and exactly once. Since U is simply connected, if x, y belonged to the same connected component of $U \setminus \Sigma'$, this path would be homotopic to a path which does not intersect Σ'. But this would violate the homotopy invariance of intersection numbers (See e.g. the proposition on Page 108 of [GP10]:"Homotopic maps always have the same intersection numbers." The reader can note that this proposition is stated for C^∞-smooth manifolds and that we are working with C^1-smooth manifolds, but one can easily conclude by approximating f by a C^∞ function in a neighborhood of the circuit.).

Let $\mathcal{T}_U = (V_U, E_U)$ be the graph whose vertices are connected components of $U \setminus \Sigma$ and whose edges are the set of connected components of $\Sigma \cap U$. An edge connects two vertices if it is contained in both of their boundaries in U. The previous observation implies that this graph is actually a tree (indeed, this graph is connected and if one erases any edge in this graph then one obtains two connected components). To each pair of elements $U, U' \in \mathcal{U}$ such that $U \subset U'$ we associate a morphism of graphs

$$\Psi_{U,U'} : \mathcal{T}_U \to \mathcal{T}_{U'}$$

which maps each vertex of $\mathcal{T}_U$ to the unique vertex of $\mathcal{T}_{U'}$ containing it. The set of vertices in the image of $\Psi_{U,U'}$ is the set of connected components of $\Sigma \cap U'$ which intersect U.

Note that a path between two vertices of $\mathcal{T}_U$ is mapped by $\Psi_{U,U'}$ to a path between their images in $\mathcal{T}_{U'}$. In particular, the image path contains the geodesic between the two image vertices (indeed, since $\mathcal{T}_{U'}$ is a tree, the geodesic is the unique injective path between the two vertices).

Now let $A, B \subset E \setminus \Sigma$ be two connected components which intersect each element of $\mathcal{U}$. Then, for each $U \in \mathcal{U}$, there exist $v_A, v_B \in V_U$ such that $\Psi_{U,E}(v_A) = A$ and $\Psi_{U,E}(v_B) = B$. In particular, the image of E_U by $\Psi_{U,E}$ contains the geodesic between A and B in $\mathcal{T}_E$, which implies that for each edge Σ' in this geodesic and each $U \in \mathcal{U}$, $\Sigma' \cap U \neq \emptyset$. Hence any such Σ' satisfies the required property. □

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