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# Fredholm transformation on Laplacian and rapid stabilization for the heat equations

Ludovick Gagnon\*, Amaury Hayat†, Shengquan Xiang‡, Christophe Zhang§

August 13, 2021

## Abstract

We revisit the rapid stabilization of the heat equation on the 1-dimensional torus using the backstepping method with a Fredholm transformation. We prove that, under some assumption on the control operator, two scalar controls are necessary and sufficient to get controllability and rapid stabilization. This classical framework allows us to present the backstepping method with the Fredholm transformation upon Laplace operators in a sharp functional setting, which is the major objective of this work, from the Riesz basis properties and the operator equality to the stabilizing spaces. Finally, we prove that the same Fredholm transformation also leads to the local rapid stability of the viscous Burgers equation.

KEYWORDS: Fredholm transformation, backstepping, rapid stabilization, controllability

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\*Université de Lorraine, CNRS, Inria équipe SPHINX, F-54000 Nancy, France. E-mail: ludovick.gagnon@inria.fr.

†CERMICS, École des Ponts ParisTech, 6 - 8, Avenue Blaise Pascal, Cité Descartes—Champs sur Marne, 77455 Marne la Vallée, France. E-mail: amaury.hayat@enpc.fr.

‡Bâtiment des Mathématiques, EPFL, Station 8, CH-1015 Lausanne, Switzerland. E-mail: shengquan.xiang@epfl.ch.

§Université de Lorraine, CNRS, Inria équipe SPHINX, F-54000 Nancy, France. E-mail: Christophe.zhang@polytechnique.org.

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# 1 Introduction

We consider the following heat equation with two internal control

$$\begin{cases} \partial_t u - \Delta u = v_1(t)\phi_1 + v_2(t)\phi_2, & (t, x) \in (0, T) \times \mathbb{T}, \\ u|_{t=0} = u_0(x), & x \in \mathbb{T}, \end{cases} \quad (1.1)$$

with  $\mathbb{T}$  denoting the one-dimensional torus  $\mathbb{R}/2\pi\mathbb{Z}$ ,  $(v_1, v_2) \in L^2((0, T); \mathbb{R})$  are real-valued scalar controls to be defined and  $(\phi_1, \phi_2) \in H^\sigma(\mathbb{T}; \mathbb{R})$  with  $\sigma \in \mathbb{R}$  are real-valued potentials. Our goal is to design a suitable feedback law to stabilize the system (1.1) exponentially quickly with a decay rate arbitrary large. A first natural question could be to wonder why not considering the simpler system with only one scalar control

$$\begin{cases} \partial_t u - \Delta u = v(t)\phi, & (t, x) \in (0, T) \times \mathbb{T}, \\ u|_{t=0} = u_0(x), & x \in \mathbb{T}, \end{cases} \quad (1.2)$$

with  $v \in L^2((0, T); \mathbb{R})$  a real-valued control to be defined and  $\phi \in H^\sigma(\mathbb{T}; \mathbb{R})$ . As it turns out, as simple as it is, this system is not controllable, due to the degeneracy of the eigenvalues of the

Laplacian operator defined on  $H^s(\mathbb{T}; \mathbb{R})$  (see Section 2). In this paper we show that at least two internal controls are required and that, in fact, two controls are enough provided that  $\phi_1$  and  $\phi_2$  satisfy some good conditions. In order to obtain the rapid stabilization result we propose a double backstepping method, detailed in Section 4.

## 1.1 Main result

Our goal is to show the following result:

**THEOREM 1.1.** *Let  $m \in \mathbb{R}_+$  and  $\phi_1, \phi_2 \in H^{m-1/2-}$ , such that*

$$\phi_1 = \sum_{n \in \mathbb{N}^*} a_n^1 \sin(nx), \quad \phi_2 = \sum_{n \in \mathbb{N}} a_n^2 \cos(nx) \quad (1.3)$$

with,

$$a_0^2 \neq 0 \quad \text{and} \quad cn^{-m} < |a_n^k| < Cn^{-m}, \quad \text{for } k \in \{1, 2\} \text{ and } n \in \mathbb{N}^*.$$

For any  $\lambda > 0$ , there exist  $K_1$  and  $K_2$  bounded feedback functionals on  $H^{m+1/2+}$  such that for any  $y_0 \in H^{m+r}$  with  $r \in (-1/2, 1/2)$ , the equation

$$\begin{cases} \partial_t y - \Delta y = \phi_1 K_1(y) + \phi_2 K_2(y), & (t, x) \in (0, T) \times \mathbb{T}, \\ y(0, \cdot) = y_0, & x \in \mathbb{T}, \end{cases} \quad (1.4)$$

has a unique solution  $y$  that is satisfied in  $L_{loc}^2(0, +\infty; H^{m+r-1})$  sense, and

$$y \in C^0([0, +\infty); H^{m+r}(\mathbb{T})) \cap L_{loc}^2((0, +\infty); H^{m+r+1}(\mathbb{T})) \cap H_{loc}^1((0, +\infty); H^{m+r-1}(\mathbb{T})). \quad (1.5)$$

Moreover, we have the following exponential stability estimate

$$\|y(t, \cdot)\|_{H^{m+r}} \leq Ce^{-\lambda t} \|y_0\|_{H^{m+r}}, \quad \forall t \in [0, +\infty), \quad (1.6)$$

where  $C = C_r(\lambda, m)$  is a constant independent of  $y_0$ .

**REMARK 1.2.** *We can make an interesting remark: the preceding feedback law given by  $K_1$  and  $K_2$  actually stabilizes the system in  $H^{m+r}$  space with  $r \in (-1/2, 1/2)$ , while the feedback law and the transformation does not depend on  $r \in (-1/2, 1/2)$ .*

**REMARK 1.3.** *The assumption on  $a_0^2 \neq 0$  is necessary, as it is related to the eigenfunction 1 of 0 eigenvalue. Otherwise, one can easily check that the ‘‘mass’’,  $\int_{\mathbb{T}} y(t, x) dx$ , is conserved. In this case, instead of converging to the zero state, the solution of the closed-loop system converge exponentially to the final equilibrium state  $\tilde{y}(x) := \int_{\mathbb{T}} y_0(x) dx$ .*

The same feedback also stabilizes several related nonlinear systems such as the viscous Burgers equation and the nonlinear heat equations. More precisely, simply as an example, we have the following theorem which corresponds to the case that  $m = 0$  and  $r = 0$ .

**THEOREM 1.4.** *Let  $\phi_1, \phi_2 \in H^{-1/2-}$ , such that*

$$\phi_1 = \sum_{n \in \mathbb{N}^*} a_n^1 \sin(nx), \quad \phi_2 = \sum_{n \in \mathbb{N}} a_n^2 \cos(nx) \quad (1.7)$$

with

$$a_0^2 \neq 0 \quad \text{and} \quad cn^{-m} < |a_n^k| < Cn^{-m}, \quad \text{for } k \in \{1, 2\} \text{ and } n \in \mathbb{N}^*.$$

For any  $\lambda > 0$ , there exists  $K_1$  and  $K_2$  bounded feedback functionals on  $H^{1/2+}$  and  $\delta > 0$  such that, for any  $y_0 \in L^2$ , the equation

$$\begin{cases} \partial_t y - \Delta y + \partial_x(y^2/2) = \phi_1 K_1(y) + \phi_2 K_2(y), \\ y(0) = y_0, \end{cases} \quad (1.8)$$

has a unique solution  $y$  that is satisfied in  $L^2_{loc}(0, +\infty; H^{-1}(\mathbb{T}))$  sense, and

$$y \in C^0([0, +\infty); L^2(\mathbb{T})) \cap L^2_{loc}((0, +\infty); H^1(\mathbb{T})) \cap H^1_{loc}((0, +\infty); H^{-1}(\mathbb{T})). \quad (1.9)$$

Moreover, for any  $\|y_0\|_{L^2} < \delta$ , we have the following exponential stability estimate

$$\|y(t, \cdot)\|_{L^2} \leq C e^{-\lambda t} \|y_0\|_{L^2}, \forall t \in [0, +\infty), \quad (1.10)$$

where  $C = C(\lambda)$  is a constant independent of  $y_0$ .

## 1.2 Related results

There exists various way to design feedback laws for infinite dimensional systems: Riccati equations (see for instance [28, 35] and reference therein), Gramian approach [25, 36], through Lyapunov functionals (see for instance [3, 19, 23, 24, 38]) or with pole-shifting techniques [30, 31], and among others. The backstepping method is also among these methods, and one traces back its origin for infinite dimensional systems to Russell [32] and to Balogh and Krstic [2] (we refer to [9, 26, 33] for an introduction to the finite dimensional backstepping method). Seen as a limit of a finite dimensional system, the backstepping method relied at first on a Volterra transformation of the second kind mapping the solution to stabilize to the solution of a stable target system. The Volterra transformation having the advantage of always being invertible, only the existence is needed to be proven, which is equivalent to solving a PDE of the kernel on a triangular domain. These PDEs usually do not enter in the classical Cauchy problem framework, but different techniques are now known to solve the kernel equation: successive approximations [27], explicit representations [27] or method of characteristics [17]. There exists now a vast literature on the backstepping with the Volterra transformation, from which only cite a few for the heat/parabolic equation [2, 7, 18], hyperbolic systems [4] and the viscous Burgers equation [15]. We refer to [27] to a general overview of the backstepping method with the Volterra transformation.

As of late, the Fredholm transformation was introduced for the backstepping method as an alternative for certain limitations of the Volterra transformation. The idea to prove the exponential stability remains the same, but the existence and invertibility of the transformation is different and oftentimes more involved. We distinguish two main procedure to prove the existence of the transformation, either by direct methods [16, 17] or, more commonly, by proving the existence of a Riesz basis. For the latter, we again distinguish two cases: either the Riesz basis is deduced directly by an isomorphism applied on an eigenbasis [12, 41, 40] or the existence of a Riesz basis follows by controllability assumptions and sufficient growth of the eigenvalues of the spatial operator allowing in particular to prove that the family is quadratically close to the eigenfunctions [11, 13, 14, 22] (see Section 2.2 and Section 5 for a definition).

## 1.3 Contribution of this paper

We believe that one of the major contribution of this paper is on the thorough study of the Fredholm transformation on Laplace operator, that is for given operator  $(A, B)$  and controlled

systems

$$\dot{x} = Ax + Bu, \tag{1.11}$$

to find suitable invertible operator  $T$  and  $K$  such that

$$\begin{cases} TA + BK = AT - \lambda T, \\ TB = B. \end{cases} \tag{1.12}$$

The preceding equality is called the operator equality. Actually, exponential stabilization is a direct consequence of this equality. At least formally, suppose that

$$\dot{x}_1 = Ax_1 + Bu_1, \tag{1.13}$$

by defining  $x_2 := Tx_1$  and  $u_1 := Kx_1$ , and by adapting  $T$  on the preceding equation we get

$$T\dot{x}_1 = TAx_1 + TBKx_1 = TAx_1 + BKx_1 = (AT - \lambda T)x_1. \tag{1.14}$$

Therefore

$$\dot{x}_2 = (A - \lambda)x_2, \tag{1.15}$$

which implies that  $x_2$  decays exponentially with rate  $\lambda$  and, since  $T$  is an invertible operator, so does  $x_1$ .

A central problem on the study of the operator equality, therefore on the rapid stabilization via backstepping using Fredholm transformation, is on the existence of a unique solution  $(T, K)$ :

**Open Problem 1.** *Let given  $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times l}$ . What is the necessary and sufficient condition on  $(A, B)$  to guarantee the existence and the uniqueness of the solution  $(T, K) \in GL_n(\mathbb{R}) \times \mathbb{R}^{l \times n}$  to the operator equality (1.12)?*

*What about the infinite dimensional cases that are related to partial differential equations, say  $A$  and  $B$  are (unbounded) operators?*

This problem was proved in the finite dimensional case and when the degree of  $B$  is 1 (namely  $l = 1$ ) in [10], where the condition is given by  $(A, B)$  verifying Kalman condition for the exact controllability. Then, somewhat it indicates that controllability is a sufficient condition, which has been further verified to several important PDE models, as transport equation [41], KdV equation [13], the linearized Schrödinger equation [11], the linearized Saint-Venant equation [12], etc. We emphasize that one common credit of these results is that the dimension of the operator  $B$  is one, or more essentially the eigenvalues of the operator  $A$  are simple and isolated. We also notice that in [8] concerning KdV-KdV equation the authors used backstepping of degree two as dealing with coupled systems, which somehow indicates  $\dim B = 2$  but the eigenvalues are still simple and isolated. For more broadly kinds of PDE models, for example when eigenvalues with higher multiplicity appear, this problem is still widely open. In this paper, for the first time, we treat the case that  $\dim B = 2$  while only one control is not able to stabilize the system. This phenomenon appears quite often when we study on compact Riemannian manifolds, for example Schrödinger equations on torus.

Another important contribution of our paper is to present a sharp functional setting, with respect to the state space and control space, for the application of the backstepping method with a Fredholm transform in the case of the Laplacian with periodic boundary conditions. In particular, we deduce the sharp spaces  $H^s(\mathbb{T})$  for which the Riesz basis exists, which is crucial for the application of the backstepping method with a Fredholm transformation. We hope

this precise framework could extend our knowledge on backstepping method using Fredholm type transformation, for example on the use of nonlinear systems, and on other important models. For instance, Proposition 4.1 Corollary 4.2 and Lemma 5.1 can be similarly proved for Schrödinger equations on  $\mathbb{T}$ , which somehow extends the analysis of [11]. It is interesting to further investigate whether these new observation could be applied to the bi-linear Schrödinger equations. This analysis could also be applied to evolution equations with fractional Laplacian  $(-\Delta)^\alpha$ , at least for  $\alpha$  strictly larger than  $3/4$ , where similar results (at least partially, depending on the value of  $\alpha$ ) to Lemma 5.1 can be proved.

Indeed, it seems that the critical growth of the eigenvalues for the existence of a quadratically close Riesz basis is  $|\lambda_n| \simeq n^{3/2+\epsilon}$  (i.e.  $\alpha > 3/4$  is the case of the fractional Laplacian), meaning that for a growth of order  $|\lambda_n| \simeq n^s, 1 \leq s \leq 3/2$  does not seem enough to prove that the family is quadratically close to the eigenbasis. An interesting open problem is therefore to apply the backstepping method with a Fredholm type transformation for spatial operator with eigenvalues with growth  $|\lambda_n| \simeq n^s, 1 < s \leq 3/2$ , the first-order equation being excluded due to the positive answer [12, 41] and its link to other transformation such as the Hilbert transform. However the case  $1 < s \leq 3/2$  remains and this analysis may give some inspiration on the study of fractional Laplacian with the value of  $\alpha$  lower than this threshold.

Moreover, we also highlight that the framework investigated here is closely related to the one found for the linear Schrödinger equation in [11]. A major distinction between [11] and the present article is that the well-posedness for the closed-loop system here relies on the dissipation properties of the heat equation. Hence, there is no need to satisfy the operator equality  $T(A + BK) = (A - \lambda I)T$  for functions in  $D(A + BK)$  (see Remark 4.3).

Finally, as we can see from Theorem 1.1 (more precisely, from Section 7), it is totally a new observation that the same feedback law stabilizes the system in  $H^r$  sense with some cost  $C_r(\lambda, m)$  depending on  $r \in (-1/2, 1/2)$  and  $\lambda \notin \mathcal{N}$ . It is an important but challenging problem to get any quantitative description of this constant  $C_r(\lambda, m)$ , as it is linked with the cost of the stabilization procedure from the application point of interest, moreover, it also may linked with finite time stabilization problems, namely  $Ce^{C\lambda^\beta}$  type estimates. So far such estimates has been achieved via different methods, rely on direct energy estimates [18], on the use of Bessel functions [20], with the help of induction on successive iteration [37], and benefiting spectral inequality [38, 39]. However, so far such important property has not yet been discovered for Fredholm type backstepping methods. Armed with the precise description introduced in this paper, we believe that we are more close to an answer.

## 1.4 Organization of the paper

This paper is organized as follows: in Section 1.2 we discuss some related results and previous work, in Section 2 we define the functional setting, in Section 3 we show that the system (1.2) with a single control is not controllable, in Section 4 we present the double backstepping approach, in Section 5 and 6 we prove the main propositions and lemma and finally in Section 7 we prove Theorem 1.1–1.4.

## 2 Functional setting

### 2.1 Function spaces

We start by recalling some results on the eigenvectors and eigenvalues of the Laplacian on the torus. Observe that the classical Fourier series  $\{e^{inx}\}_{n \in \mathbb{Z}}$  in  $\mathbb{T}$  are eigenfunctions of the Laplacian operator  $\Delta$  associated to the eigenvalues  $\lambda_n := -n^2$ , and form an orthonormal basis of  $L^2(\mathbb{T})$ . Note that thanks to the fact that we are working on  $\mathbb{T}$  without boundary, the Sobolev space  $H^s$  coincide with the span of  $\{n^s e^{inx}\}$ . Note also that

$$\lambda_n = \lambda_{-n} = -n^2,$$

and therefore the eigenvalues are degenerated. Except in Section 3 where using the Fourier series  $\{e^{inx}\}_{k \in \mathbb{Z}}$  is convenient, we will use in the following basis of eigenfunctions of the Laplacian

$$\begin{aligned} f_n^1 &:= \sin nx, \quad f_n^2 := \cos nx, \quad \text{associated to } \lambda_n := -n^2, \quad \forall n \in \mathbb{N}^*, \\ f_0^2 &:= 1 \quad \text{associated to } \lambda_0 := 0. \end{aligned} \tag{2.1}$$

We further define

$$L_1^2 := \text{span}\{\sin nx\}_{n \in \mathbb{N}^*}, \quad \text{describing the odd functions,} \tag{2.2}$$

$$L_2^2 := \text{span}\{\cos nx\}_{n \in \mathbb{N}}, \quad \text{describing the even functions,} \tag{2.3}$$

$$L_i^2 \text{ is a subspace of } L^2 \text{ that is endowed with the same norm,} \tag{2.4}$$

$$L^2(\mathbb{T}) = L_1^2 \oplus L_2^2. \tag{2.5}$$

Similarly, concerning Sobolev space  $s$  in  $\mathbb{T}$ , recall that one has

$$H^m(\mathbb{T}) = \left\{ f = \sum_{n \in \mathbb{N}^*} a_n^1 f_n^1 + \sum_{n \in \mathbb{N}} a_n^2 f_n^2 \mid \sum_{n \in \mathbb{N}^*} n^{2m} ((a_n^1)^2 + (a_n^2)^2) < +\infty \right\}, \tag{2.6}$$

with the (inhomogeneous) Sobolev norm

$$\|f\|_{H^m}^2 := (a_0^2)^2 + \sum_{n \in \mathbb{N}^*} n^{2m} ((a_n^1)^2 + (a_n^2)^2). \tag{2.7}$$

In this paper we say that the function  $y$  belongs to  $H^{m-}$ , and that the functional  $\mathcal{L} : H^{m+} \rightarrow \mathbb{R}$  is bounded, when

$$y \in H^{m-\varepsilon} \quad \text{for all } \varepsilon > 0, \tag{2.8}$$

$$\mathcal{L} : H^{m+\varepsilon} \rightarrow \mathbb{R} \text{ is bounded for all } \varepsilon > 0. \tag{2.9}$$

We also define the sub-spaces  $H_i^m$ ,  $i \in \{1, 2\}$  as follows, for  $m \in \mathbb{R}$

$$H_1^m := \left\{ a \in H^m(\mathbb{T}) \mid a = \sum_{n \in \mathbb{N}^*} a_n f_n^1 \right\}, \tag{2.10}$$

$$H_2^m := \left\{ a \in H^m(\mathbb{T}) \mid a = \sum_{n \in \mathbb{N}} a_n f_n^2 \right\}, \tag{2.11}$$

$$H^m(\mathbb{T}) = H_1^m \oplus H_2^m. \tag{2.12}$$



Notice that

$$\Delta : H_k^m \rightarrow H_k^{m-2}. \quad (2.13)$$

We remark here that for  $f \in H_1^{m+1}, g \in H_1^{m-1}$ , the inner product  $\langle \cdot, \cdot \rangle_{H_1^m}$  is well-defined and is given by

$$\langle f, g \rangle_{H_1^m} = \sum_{n \in \mathbb{N}^*} (n^m f_n)(n^m g_n) = \sum_{n \in \mathbb{N}^*} (n^{m+1} f_n)(n^{m-1} g_n),$$

which, inspired by the last term of the preceding formula, can be also denoted as  $\langle \cdot, \cdot \rangle_{H_1^{m+1}, H_1^{m-1}}$ .

In order to describe the precise definition domain of the operator  $T$ , we recall that for the Schwartz type space  $\mathcal{S}(\mathbb{T})$  satisfying fast decay at high frequency, one has

$$\mathcal{S}(\mathbb{T}) = \{f = \sum_{n \in \mathbb{N}^*} a_n^1 f_n^1 + \sum_{n \in \mathbb{N}} a_n^2 f_n^2 \mid \forall m \in \mathbb{N}, \forall \varepsilon > 0, \exists M \text{ such that} \quad (2.14)$$

$$n^{2m} ((a_n^1)^2 + (a_n^2)^2) < \varepsilon, \forall n > M\}. \quad (2.15)$$

We also define the decomposition of in odd and even function as follows

$$\mathcal{S}_1 := \{a \in \mathcal{S} \mid \langle a, f_n^2 \rangle = 0, \forall n \in \mathbb{N}\}, \quad (2.16)$$

$$\mathcal{S}_2 := \{a \in \mathcal{S} \mid \langle a, f_n^1 \rangle = 0, \forall n \in \mathbb{N}^*\}. \quad (2.17)$$

While, by denoting the space  $\mathcal{S}'$  as the dual of  $\mathcal{S}$ , we also define

$$\mathcal{S}'_1 := \{a \in \mathcal{S}' \mid \langle a, f_n^2 \rangle = 0, \forall n \in \mathbb{N}\}, \quad (2.18)$$

$$\mathcal{S}'_2 := \{a \in \mathcal{S}' \mid \langle a, f_n^1 \rangle = 0, \forall n \in \mathbb{N}^*\} \quad (2.19)$$

which strictly speaking is not the dual of  $\mathcal{S}_k$ , but as the quotient space  $\mathcal{S}'/\mathcal{S}_{k+1}$ . We easily observe that

$$\mathcal{S} \subset H^s \subset \mathcal{S}', \forall s \in \mathbb{R}, \quad (2.20)$$

$$\mathcal{S}_k \subset H_k^s \subset \mathcal{S}'_k, \forall s \in \mathbb{R}, \forall k \in \{1, 2\}. \quad (2.21)$$

## 2.2 Riesz basis

Finally we recall here the definition of a Riesz basis, see for instance a monograph on the moment theory [1] (or related papers [11]).

**DEFINITION 2.1** (Vector family). *Let  $X$  be a Hilbert space. A family of vectors  $\{\xi_n\}_{n \in \mathcal{I}}$ , where  $\mathcal{I} = \mathbb{Z}, \mathbb{N}$ , or  $\mathbb{N}^*$  is said to be*

(1) **Minimal** in  $X$ , if for every  $k \in \mathcal{I}$ ,  $\xi_k \notin \overline{\text{span}\{\xi_i; i \in \mathcal{I} - \{k\}\}}$ .

(2) **Dense** in  $X$ , if  $\overline{\text{span}\{\xi_i; i \in \mathcal{I}\}} = X$ .

(3)  **$\omega$ -independent** in  $X$ , if

$$\sum_{k \in \mathcal{I}} c_k \xi_k = 0 \text{ in } X \text{ with } \{c_n\}_{n \in \mathcal{I}} \in l^2(\mathcal{I}) \implies c_n = 0, \forall n \in \mathcal{I}. \quad (2.22)$$

(4) **Quadratically close** to a family of vector  $\{e_n\}_{n \in \mathcal{I}}$ , if

$$\sum_{k \in \mathcal{I}} \|\xi_k - e_k\|_X^2 < +\infty. \quad (2.23)$$

(5) **Riesz basis** of  $X$ , if it is the image of an isomorphism (on  $X$ ) of some orthonormal basis.

(5)' **Riesz basis** of  $X$  (an equivalent definition of (5)), if it is dense in  $X$  and if there exist  $C_1, C_2 > 0$  such that for any  $\{a_n\}_{n \in \mathcal{I}} \in l^2(\mathcal{I})$  we have

$$C_1 \sum_{k \in \mathcal{I}} |a_k|^2 \leq \left\| \sum_{k \in \mathcal{I}} a_k \xi_k \right\|_X^2 \leq C_2 \sum_{k \in \mathcal{I}} |a_k|^2. \quad (2.24)$$

These definitions allow us to give the following criteria for a Riesz basis which will be used later on in Section 5.2.

**LEMMA 2.2.** *Let  $\{\xi_n\}_{n \in \mathcal{I}}$  be quadratically close to an orthonormal basis  $\{e_n\}_{n \in \mathcal{I}}$ . Suppose that  $\{\xi_n\}_{n \in \mathcal{I}}$  is either dense in  $X$  or  $\omega$ -independent in  $X$ , then  $\{\xi_n\}_{n \in \mathcal{I}}$  is a Riesz basis of  $X$ .*

It is noteworthy that under the assumption that  $\{\xi_n\}_{n \in \mathcal{I}}$  is quadratically close to some orthonormal basis  $\{e_n\}_{n \in \mathcal{I}}$ , if for some coefficients  $\{c_n\}_{n \in \mathcal{I}} \in \mathbb{R}^{\mathcal{I}}$  (or  $\mathbb{C}^{\mathcal{I}}$ ) the following series converge in the Cauchy sense

$$\sum_{k \in \mathcal{I}} c_k \xi_k \text{ converges in } X, \quad (2.25)$$

which in particular contains the case that it converges to 0, then automatically we know that the coefficients  $\{c_n\}_{n \in \mathcal{I}}$  belong to  $l^2(\mathcal{I})$ . Indeed, we know from the fact that the series converges in  $X$  that, for every  $N$ ,

$$\sum_{k > |N|, k \in \mathcal{I}} c_k \xi_k \text{ converges in } X, \quad (2.26)$$

which means that the  $X$  norm of the preceding series is finite:

$$\left\| \sum_{k > |N|, k \in \mathcal{I}} c_k \xi_k \right\|_X < +\infty. \quad (2.27)$$

Since  $\{\xi_n\}_{n \in \mathcal{I}}$  is quadratically close to  $\{e_n\}_{n \in \mathcal{I}}$ , for some  $N$  sufficiently large we have

$$\sum_{k > |N|, k \in \mathcal{I}} \|(\xi_k - e_k)\|_X^2 < \frac{1}{4}, \quad (2.28)$$

which, to be combined with the fact that  $\{e_n\}_{n \in \mathcal{I}}$  is an orthonormal basis of  $X$ , yield

$$\begin{aligned} \left\| \sum_{k > |N|, k \in \mathcal{I}} c_k \xi_k \right\|_X &= \left\| \sum_{k > |N|, k \in \mathcal{I}} c_k e_k + \sum_{k > |N|, k \in \mathcal{I}} c_k (\xi_k - e_k) \right\|_X, \\ &\geq \left\| \sum_{k > |N|, k \in \mathcal{I}} c_k e_k \right\|_X - \left\| \sum_{k > |N|, k \in \mathcal{I}} c_k (\xi_k - e_k) \right\|_X \\ &\geq \left( \sum_{k > |N|, k \in \mathcal{I}} (c_k)^2 \right)^{1/2} - \left( \frac{1}{4} \sum_{k > |N|, k \in \mathcal{I}} (c_k)^2 \right)^{1/2} \end{aligned}$$

$$= \frac{1}{2} \left( \sum_{k>|N|, k \in \mathcal{I}} (c_k)^2 \right)^{1/2}.$$

Therefore,  $\{c_n\}_{n \in \mathcal{I}} \in l^2(\mathcal{I})$ .

We also have the following Lemma that will be useful in the Section 5.2.

**LEMMA 2.3.** *Let  $X, Y$  be Hilbert spaces. Let  $T : X \rightarrow Y$  be an isomorphism. Suppose that  $\{\xi_n\}_{n \in \mathcal{I}}$  is a Riesz basis of  $X$ , then with  $\zeta_n := T\xi_n$ , the family  $\{\zeta_n\}_{n \in \mathcal{I}}$  is a Riesz basis of  $Y$ .*

*Proof.* If  $X = Y$  we can directly use Definition 2.1 (5) to show that it is a Riesz basis as the image of an orthonormal basis by an isomorphism on  $X$ . Otherwise, a quite straightforward proof is given according to Definition 2.1 (5)'.

We first show the inequality (2.24). Thanks to the fact that  $T$  is an isomorphism and that  $\{\xi_n\}_{n \in \mathcal{I}}$  is a Riesz basis of  $X$ , there exist constants  $C, C_1$  and  $C_2$  such that

$$\left\| \sum_{k \in \mathcal{I}} a_k \zeta_k \right\|_Y^2 = \left\| T \sum_{k \in \mathcal{I}} a_k \xi_k \right\|_Y^2 \leq C \left\| \sum_{k \in \mathcal{I}} a_k \xi_k \right\|_X^2 \leq CC_2 \sum_{k \in \mathcal{I}} |a_k|^2,$$

and

$$\left\| \sum_{k \in \mathcal{I}} a_k \zeta_k \right\|_Y^2 = \left\| T \sum_{k \in \mathcal{I}} a_k \xi_k \right\|_Y^2 \geq C^{-1} \left\| \sum_{k \in \mathcal{I}} a_k \xi_k \right\|_X^2 \geq C^{-1} C_1 \sum_{k \in \mathcal{I}} |a_k|^2.$$

Next, we show that  $\{\zeta_n\}_{n \in \mathcal{I}}$  is dense in  $Y$ . For any  $\zeta \in Y$ , since  $T^{-1}\zeta =: \xi \in X$ , for any  $\varepsilon > 0$  there exist a finite combination such that

$$\left\| \sum_{|k| \leq N} a_k \xi_k - \xi \right\|_X < \varepsilon. \quad (2.29)$$

Thus

$$\begin{aligned} \left\| \zeta - \sum_{|k| \leq N} a_k \zeta_k \right\|_Y &= \left\| T\xi - T \sum_{|k| \leq N} a_k \xi_k \right\|_Y \\ &\leq C \left\| \zeta - \sum_{|k| \leq N} a_k \xi_k \right\|_X \\ &\leq C\varepsilon, \end{aligned}$$

which concludes the proof of the lemma.  $\square$

**LEMMA 2.4** (Proposition 19 of [6]). *Let  $X$  be a Hilbert space. Suppose that  $\{\xi_n\}_{n \in \mathcal{I}}$  is a Riesz basis of  $X$ , then its bi-orthogonal sequence  $\{\xi'_n\}_{n \in \mathcal{I}}$  is also a Riesz basis of  $X$ , where bi-orthogonal means,*

$$\langle \xi_n, \xi'_m \rangle_X = \delta_{n,m}, \quad \forall n, m \in \mathcal{I}. \quad (2.30)$$

For any  $f \in X$ , there exists a unique sequence  $\{a_k\}_{k \in \mathcal{I}} \in l^2(\mathcal{I})$  such that

$$f = \sum_{k \in \mathcal{I}} a_k \xi_k \text{ in } X, \quad (2.31)$$

where the series converges in  $X$  under Cauchy sequence sense. Moreover,

$$a_k := \langle f, \xi'_k \rangle_X, \quad (2.32)$$

$$C_1 \sum_{k \in \mathcal{I}} |a_k|^2 \leq \|f\|_X^2 \leq C_2 \sum_{k \in \mathcal{I}} |a_k|^2. \quad (2.33)$$

### 3 Non-Stabilizability and non-controllability with a single internal control

#### 3.1 Non-controllable with one scalar control

In this section we show that the system (1.2) is not controllable. Let  $T > 0$ . Recall that  $\{e^{inx}\}_{n \in \mathbb{Z}}$  is a basis of eigenfunctions of  $A$  associated to the eigenvalue  $\lambda_n = -n^2$ . For ease of the presentation and for symmetry considerations, in this section we choose to work with the orthonormal basis  $\{e^{inx}\}_{n \in \mathbb{Z}}$  instead of  $\{\sin nx, \cos nx\}$ . Therefore, either

$$\langle \phi, e^{inx} \rangle \neq 0, \quad \forall n \in \mathbb{Z}, \quad (3.1)$$

or the system is non-controllable as there exists  $n_0$  such that  $\langle \phi, e^{in_0x} \rangle = 0$  and therefore the control has no effect on the associated one dimensional vector space:  $e^{-n_0^2t + in_0x}$  is a solution of (1.2) with initial condition  $e^{in_0x}$ , whatever the control is. Assume now that (3.1) holds. Motivated by the moment method (see for instance [21]) and the fact that  $\lambda_n = \lambda_{-n}$  we have for a solution  $u$  to (1.2) with  $u(0) = 0$ ,

$$\frac{\langle u(T, \cdot), e^{inx} \rangle}{\langle \phi, e^{inx} \rangle} = \frac{\langle u(T, \cdot), e^{-inx} \rangle}{\langle \phi, e^{-inx} \rangle}, \quad (3.2)$$

thus let us denote

$$d_n := \frac{\langle \phi, e^{inx} \rangle}{\langle \phi, e^{-inx} \rangle}, \quad (3.3)$$

one has

$$\langle u(T, \cdot), e^{inx} \rangle = d_n \langle u(T, \cdot), e^{-inx} \rangle. \quad (3.4)$$

Hence the states  $u(T, \cdot)$  that are reachable at time  $T$  satisfy the following

$$u(T, x) = d_0 + \sum_{n \geq 1} k_n (d_n e^{inx} + e^{-inx}), \quad (3.5)$$

which means that the projection of the reachable space on the two dimensional space  $\text{Span}\{e^{inx}, e^{-inx}\}$  is always of one dimension, hence the system is not controllable.

This non-controllability prevents any stabilization result. For instance, we can simply consider the space  $\text{Span}\{\sin nx, \cos nx\}$ , as we are only allowed to change the direction of  $(a_n e^{inx} + e^{-inx})$ , the projection of the solution on its co-direction  $(e^{inx} - a_n e^{-inx})$  does not change, thus the solution is not asymptotically stable whatever the feedback control.

This is different from the paper by [29] for controllability and [38] for finite time stabilization (the special case for  $\mathbb{T}$ , as these papers deal with general compact Riemannian manifolds), where the controllability and the stabilizability is obtained under the assumption that the controlled domain is  $\omega \subset \mathbb{T}$ , for which the control has infinite dimension of degree and not a one dimensional scalar control.

#### 3.2 Controllable with two scalar controls

According to the preceding section, two controls are required for the controllability of the heat equation on  $\mathbb{T}$ :  $\phi_1 v_1(t) + \phi_2 v_2(t)$ , which corresponds to the system 1.1. In fact, two controls

are eventually sufficient. In the following we prove the controllability in  $L^2(\mathbb{T})$  space, while the other spaces can be treated similarly.

Due to the smoothing effect of the heat equation we only consider the so called null controllability, *i.e.* for any  $u_0 \in L^2(\mathbb{T})$  there exist  $v_1, v_2 \in L^2(0, T)$  (this space is the natural space according to Lions' Hilbert Uniqueness Method, though this is not the optimal candidate) such that the final state becomes 0. In order to simplify the presentation, we always assume the projections of  $\phi_1, \phi_2, u_0$  on the direction corresponding to the eigenfunction  $e^{i0x}$  to be 0. Assuming that

$$u_0 = \sum_{n \in \mathbb{N}^*} b_n^1 e^{inx} + b_n^2 e^{-inx} \in L^2(\mathbb{T}),$$

Direct calculation yields,

$$\begin{aligned} u(T) &= \int_0^T e^{A(T-s)} (\phi_1 v_1(s) + \phi_2 v_2(s)) ds + \int_0^T e^{A(T-s)} u_0 ds \\ &= \sum_{n \in \mathbb{N}^*} \left( \int_0^T e^{\lambda_n(T-s)} (\langle \phi_1, e^{inx} \rangle v_1(s) + \langle \phi_2, e^{inx} \rangle v_2(s)) ds \right) e^{inx} \\ &\quad + \sum_{n \in \mathbb{N}^*} \left( \int_0^T e^{\lambda_n(T-s)} (\langle \phi_1, e^{-inx} \rangle v_1(s) + \langle \phi_2, e^{-inx} \rangle v_2(s)) ds \right) e^{-inx}, \\ &\quad + \sum_{n \in \mathbb{N}^*} \left( \frac{b_n^1}{e^{n^2 T}} e^{inx} + \frac{b_n^2}{e^{n^2 T}} e^{-inx} \right). \end{aligned}$$

The preceding formula indicates that the null controllability requires that

$$\left( \int e^{\lambda_n(T-s)} v_1(s), \int e^{\lambda_n(T-s)} v_2(s) \right) \begin{pmatrix} \langle \phi_1, e^{inx} \rangle & \langle \phi_1, e^{-inx} \rangle \\ \langle \phi_2, e^{inx} \rangle & \langle \phi_2, e^{-inx} \rangle \end{pmatrix} = - \left( \frac{b_n^1}{e^{n^2 T}}, \frac{b_n^2}{e^{n^2 T}} \right).$$

Consequently, for any  $n \in \mathbb{N}^*$  provided that the following matrix is invertible, we can control the two dimensional space  $\text{Span}\{e^{inx}, e^{-inx}\} = \text{Span}\{\cos nx, \sin nx\}$ ,

$$\begin{pmatrix} \langle \phi_1, e^{inx} \rangle & \langle \phi_1, e^{-inx} \rangle \\ \langle \phi_2, e^{inx} \rangle & \langle \phi_2, e^{-inx} \rangle \end{pmatrix}, \quad \forall n \in \mathbb{N}^*.$$

It is a quite general assumption to achieve, a simple example can be

$$\phi_1(x) = \sum_{n \in \mathbb{N}^*} c_n^1 e^{inx}, \quad \phi_2(x) = \sum_{n \in \mathbb{N}} c_n^2 e^{-inx},$$

with  $c_n^1 c_n^2 \neq 0$  for any  $n \in \mathbb{N}^*$ . Note that, as mentioned earlier, we excluded the direction  $1 = e^{i0x}$  corresponding to the case  $n = 0$  to simplify the presentation but it could be included as well. Heuristically, this case already provides the exact controllability in projections on finite dimensional subspaces, for example,  $\text{span}\{e^{inx} : -N < n < N\}$ . Moreover, if further  $c_n^1, c_n^2$  verify some suitable growth assumption, the system is even exact null controllable in  $L^2(\mathbb{T})$ .

**PROPOSITION 3.1.** *If there exist  $-\infty < \alpha \leq \beta < 1/2$  and  $c, C > 0$  such that*

$$cn^\alpha \leq |c_n^1|, |c_n^2| \leq Cn^\beta, \quad \forall n \in \mathbb{N}^* \tag{3.6}$$

*then the system (1.1) is  $L^2(\mathbb{T})$  exact null controllable.*

**REMARK 3.2.** The assumption on  $\beta < 1/2$  is here to guarantee that the functions  $\phi_i \in H^{-1}$ . Thus for any given  $v(t) \in L^2(0, T)$  the inhomogeneous term  $v(t)\phi_i(x)$  belongs to  $L^2(0, T; H^{-1})$ , which indicates that the open-loop system is well-posed in  $C^0([0, T]; L^2) \cap L^2(0, T; H^1)$ . On the other hand, the assumption  $-\infty < \alpha$  is used for the null controllability property that will be proved in the following. The lower bound on  $c_n^i$  proposed here is not the sharp condition. As we can see from the following proof, to get the null controllability in  $L^2(\mathbb{T})$  space, it suffices to find some  $0 < T_0 < T$  and  $C > 0$  such that

$$Ce^{-T_0 n^2} \leq |c_n^1|, |c_n^2|, \forall n \in \mathbb{N}^*, \quad (3.7)$$

which is of course weaker than the condition proposed in Proposition 3.1.

*Proof of Proposition 3.1.* In order to solve the controllability problem, it suffices to treat the following moment problem: show that for any  $\{b_n^1\}_{n \in \mathbb{N}^*}, \{b_n^2\}_{n \in \mathbb{N}^*} \in l_{\mathbb{N}^*}^2$ , there exist  $v_1(t), v_2(t) \in L^2(0, T)$  such that

$$\int_0^T e^{-n^2(T-s)} v_1(s) ds = \frac{b_n^1}{e^{n^2 T} c_n^1}, \quad \forall n \in \mathbb{N}^*, \quad (3.8)$$

$$\int_0^T e^{-n^2(T-s)} v_2(s) ds = \frac{b_n^2}{e^{n^2 T} c_n^2}, \quad \forall n \in \mathbb{N}^*. \quad (3.9)$$

Solving this problem is, in fact, a direct consequence of the following moment theory.

**LEMMA 3.3** ([21], Section 3, Equation (3.25)). *For any  $T > 0$ . The sequence  $\{e^{-n^2(T-s)}|_{s \in (0, T)}\}_{n \in \mathbb{N}^*}$  is minimal in  $L^2(0, T)$ , thus admits a bi-orthogonal sequence  $\{\Psi_n\}_{n \in \mathbb{N}^*}$  satisfying*

$$\int_0^T e^{-n^2(T-s)} \Psi_m(s) ds = \delta_{n,m}, \quad \forall n, m \in \mathbb{N}^*. \quad (3.10)$$

Moreover, there exists  $C > 0$  such that

$$\|\Psi_n\|_{L^2(0, T)} \leq Ce^{Cn}. \quad (3.11)$$

By adapting Lemma 3.3, and assuming (3.6), we know that the moment problem (3.8) can be solved by setting

$$v_1(s) := \sum_{n \in \mathbb{N}^*} \frac{b_n^1}{e^{n^2 T} c_n^1} \Psi_n, \quad (3.12)$$

satisfying, from (3.6),

$$\|v_1\|_{L^2(0, T)} \leq \sum_{n \in \mathbb{N}^*} C b_n^1 n^{-\alpha} e^{Cn - Tn^2} \leq C_2 \sum_{n \in \mathbb{N}^*} b_n^1 e^{-Tn^2/2} < +\infty, \quad (3.13)$$

where  $C_2 > 0$  is a constant independent of  $n$ . Then a similar procedure leads to  $v_2(t)$  as the solution of the moment problem (3.9).  $\square$

## 4 Double backstepping: strategy and outline

Inspired by the fact that  $(\sin(nx), \cos(nx))$  form an orthonormal basis of the two-dimensional eigenspaces corresponding to the eigenvalue  $\lambda_n$ , we directly consider the special form of  $\phi_k$ :

$$\phi_1 := \sum_{n \in \mathbb{N}^*} a_n^1 \sin nx \in L_1^2, \quad (4.1)$$

$$\phi_2 := \sum_{n \in \mathbb{N}} a_n^2 \cos nx \in L_2^2. \quad (4.2)$$

We can similarly separate the function  $y(t)$  in

$$y(t) =: y_1(t) + y_2(t) \text{ with } y_k \in L_k^2. \quad (4.3)$$

Therefore,

$$\partial_t y_k - \partial_x^2 y_k = \phi_k u_k(t), \quad \forall k \in \{1, 2\}. \quad (4.4)$$

The logic behind is to deal with the odd functions using the first control and with the even functions using the other one. What we are going to show is that each of the systems (4.4) can be rapidly stabilized for  $k \in \{1, 2\}$ .

To do so, we would like to show that for any  $k \in \{1, 2\}$  and for any  $\lambda > 0$ , under some conditions on  $\phi_k$ , there exists an isomorphism  $T_k(\lambda) : L_k^2 \rightarrow L_k^2$  as well as a feedback  $u_k(t) := K_k(\lambda)y_k(t, x)$  such that the solution of

$$\begin{cases} \partial_t y_k - \partial_x^2 y_k = \phi_k K_k y_k, \\ y_k(0) \in L_k^2 \end{cases}$$

satisfies that,  $z_k := T_k(\lambda)y_k$  verifies the following equation

$$\partial_t z_k - \partial_x^2 z_k - \lambda z_k = 0, \quad z_k \in L_k^2. \quad (4.5)$$

With this property, the stabilization result would follow simply by a decomposition of  $y$  in odd and even parts provided that the system is well-posed.

This existence of an isomorphism  $T_k$  and a feedback law  $K_k$  is given by the following key proposition.

**PROPOSITION 4.1.** *Let the countable set*

$$\mathcal{N} := \{i^2 - j^2 : i, j \in \mathbb{N}\}, \quad (4.6)$$

let  $m \in \mathbb{R}$  and  $k \in \{1, 2\}$ . Assume that the sequence  $\{a_n^k\}_n$  satisfies

$$cn^{-m} < |a_n^k| < Cn^{-m}, \quad \text{for } k \in \{1, 2\}, \quad \text{for } n \in \mathbb{N}^*, \quad (4.7)$$

$$a_0^2 \neq 0. \quad (4.8)$$

Then for any  $\lambda \notin \mathcal{N}$ , there exists a sequence  $\{K_n^k\}_n$  satisfying

$$K_0^2 \neq 0, \quad (4.9)$$

$$cn^m < |K_n^k| < Cn^m, \quad \text{for } k \in \{1, 2\}, \quad \text{for } n \in \mathbb{N}^*, \quad (4.10)$$

$$\{(\lambda + a_n^k K_n^k) n^r\}_n \in l^2, \quad \forall r \in [0, 1/2), \quad (4.11)$$

$$K_k \text{ is a bounded functional on } H_k^{m+1/2+}, \quad (4.12)$$

such that the linear operator  $T_k$  defined as follows

$$T_k : \mathcal{S} \rightarrow \mathcal{S}'_k, \quad (4.13)$$

$$f_n^k \mapsto -K_n^k \sum_p \frac{a_p f_p^k}{p^2 + \lambda - n^2}, \quad (4.14)$$

$$f_n^{3-k} \mapsto 0, \quad (4.15)$$

can be linearly extended on  $H^{m-3/2+}$  satisfying

$$T_k \text{ is an isomorphism on } H_k^{m+s} \text{ for any } s \in (-3/2, 3/2), \quad (4.16)$$

$$T_k \phi_k = \phi_k \text{ in } H_k^{m-1/2-}, \quad (4.17)$$

and moreover, for any  $r \in (-1/2, 1/2)$ , for any  $\varphi \in H_k^{m+r+1}$  we have that

$$(T_k A + T_k \phi_k K_k) \varphi = (A T_k - \lambda T_k) \varphi \text{ in } H_k^{m+r-1}. \quad (4.18)$$

This proposition gives exactly what we want. Indeed, if we denote by

$$B = (\phi_1, \phi_2), \quad K = (K_1, K_2)^T$$

and the linear operator  $T$  by

$$Tf := T_1 f + T_2 f, \quad \forall f \in \mathcal{S}, \quad (4.19)$$

then immediately we get the following.

**COROLLARY 4.2.** *Under the assumption of Proposition 4.1, the transformation  $T$  can be linearly extended on  $H^{m-3/2+}$ , moreover we have that*

$$T \text{ is an isomorphism on } H^{m+s} \text{ for any } s \in (-3/2, 3/2),$$

and that for any  $r \in (-1/2, 1/2)$ , for any  $\varphi \in H^{m+r+1}$ ,

$$(TA + TBK) \varphi = (AT - \lambda T) \varphi \text{ in } H^{m+r-1}, \quad (4.20)$$

$$TB = B \text{ in } H^{m-1/2-}. \quad (4.21)$$

The proof of Proposition 4.1 will be shown in Section 5. What remains to be done is to ensure that the system is well-posed. This is the object of the next subsection.

**REMARK 4.3.** *The functional setting of (4.20) is optimal without assuming additional compatibility conditions. Indeed, as shown in [11], the operator equality (4.20) is satisfied for  $r = 1$  if  $\phi \in H^{m+r+1}$  satisfies additional compatibility conditions, for  $\varphi$  satisfying some regularity requirement, namely  $D(A + BK)$ . In [11], this can be seen from the fact that  $r \geq 1/2$  is the precise space for which the trace of  $\phi$  make sense and for which the compatibility conditions can be ensured.*



## 4.1 On the well-posedness of the closed-loop systems

In this section we show that the closed-loop systems provided in Theorem 1.1 and Theorem 1.4 are actually well-posed.

We mainly focus on the well-posedness of the heat equation, while similar proof yields the well-posedness of the viscous Burgers equation.

It suffices to consider the case of  $m = 0$ , while the other cases can be proved similarly. Furthermore, for the ease of notations, here we only prove the following special case corresponding to  $r = 0$ , while the other cases where  $r \in (-1/2, 1/2)$  can be proved similarly. Inspired by the decomposition (4.4) and the fact that  $H^m = H_1^m \oplus H_2^m$  it suffices to consider the well-posedness in  $H_1^m$  and  $H_2^m$  separately. The well-posedness of the even and odd part which is given by the following lemma, whose proof is shown in Section 6.

**LEMMA 4.4** (Well-posedness of the odd and even part:  $m = 0, r = 0$ ). *Let  $k \in \{1, 2\}$ . Let  $y_0 \in L_k^2$ . Let  $\phi \in H_k^{-1}$ . Let  $K_k : H_k^{3/4} \rightarrow \mathbb{R}$  be bounded. The equation*

$$\begin{cases} \partial_t y - \Delta y = \phi_k K_k(y), \\ y(0) = y_0, \end{cases} \quad (4.22)$$

*has a unique solution that is satisfied in  $L^2(0, T; H_k^{-1})$  sense, and*

$$y(t) \in C^0([0, +\infty); L_k^2) \cap L_{loc}^2(0, +\infty; H_k^1) \cap H_{loc}^1(0, +\infty; H_k^{-1}). \quad (4.23)$$

**COROLLARY 4.5** (Case  $m = 0, r \in (-1/2, 1/2)$ ). *Let  $k \in \{1, 2\}$ . Let  $r \in (-1/2, 1/2)$ . Let  $y_0 \in H_k^r$ . Let  $\phi \in H_k^{-1/2-}$ . Let  $K_k : H_k^{1/2+} \rightarrow \mathbb{R}$  be bounded. The equation (4.22) has a unique solution in the  $L^2(0, T; H_k^{r-1})$  sense, and*

$$y(t) \in C^0([0, +\infty); H_k^r) \cap L_{loc}^2(0, +\infty; H_k^{r+1}) \cap H_{loc}^1(0, +\infty; H_k^{r-1}). \quad (4.24)$$

**REMARK 4.6** (Cases  $m \neq 0, r \in (-1/2, 1/2)$ ). *Let  $k \in \{1, 2\}$ . For the other cases where  $m \neq 0$ , we are dealing with  $y_0 \in H_k^{m+r}, \phi_k \in H_k^{m-1/2-}, K_k : H_k^{m+1/2+} \rightarrow \mathbb{R}$ . Either we can perform the same proof with respect to the pivot space  $H_k^m$  but, this time, or we can consider the isomorphism  $D^m : H_k^m \rightarrow L_k^2$ :*

$$D^m : n^{-m} f_n^k \mapsto f_n^k, \quad (4.25)$$

*with convention that  $D^m(f_0^2) = f_0^2$ , where we recall that  $f_n^k$  is an eigenfunction of  $A$  given by (2.1). Observe that  $D^m$  commute with Laplacian, thus the equation (4.22) is equivalent to*

$$\begin{cases} \partial_t w - \Delta w = (D^m \phi_k) K_k D^{-m}(w), \\ w(0) = D^{-m} y_0 \in H_k^r, \end{cases} \quad (4.26)$$

*with  $y = D^m w$ , which goes back to the case of Corollary 4.5.*

By combining Lemma 4.4, Corollary 4.5, Remark 4.6 and the fact that  $H^m = H_1^m \oplus H_2^m$ , we immediately get the well-posedness of the equation (1.4).

**COROLLARY 4.7** (Well-posedness of the heat equation (1.4)). *Let  $m \in \mathbb{R}$ . Let  $r \in (-1/2, 1/2)$ . Let  $y_0 \in H^{m+r}$ . Let  $\phi_k \in H_k^{m-1/2-}$  for every  $k \in \{1, 2\}$ . Let  $K_k : H^{m+1/2+} \rightarrow \mathbb{R}$  be bounded satisfying  $K_k : H_{3-k}^{m+1/2+} \rightarrow 0$  for every  $k \in \{1, 2\}$ . The equation (1.4) has a unique solution in the  $L^2(0, T; H^{m+r-1})$  sense, and*

$$y(t) \in C^0([0, +\infty); H^{m+r}) \cap L_{loc}^2(0, +\infty; H^{m+r+1}) \cap H_{loc}^1(0, +\infty; H^{m+r-1}). \quad (4.27)$$

Moreover, similar to Lemma 4.4 we also get the following well-posedness of the closed-loop viscous Burgers system (1.8).

**LEMMA 4.8** (Well-posedness of the viscous Burgers equation (1.8)). *Let  $y_0 \in L^2$ . Let  $\phi_1, \phi_2 \in H^{-1}$ . Let  $K_1, K_2 : H^{3/4} \rightarrow \mathbb{R}$  be bounded. The equation*

$$\begin{cases} \partial_t y - \Delta y + \partial_x(y^2/2) = \phi_1 K_1(y) + \phi_2 K_2(y), \\ y(0) = y_0, \end{cases} \quad (4.28)$$

has a unique solution that is satisfied in  $L^2(0, T; H^{-1})$  sense, and

$$y(t) \in C^0([0, +\infty); L^2) \cap L^2_{loc}(0, +\infty; H^1) \cap H^1_{loc}(0, +\infty; H^{-1}). \quad (4.29)$$

Moreover,

$$\|y(t, \cdot)\|_{L^2} \leq e^{Ct} \|y_0\|_{L^2}, \quad \forall t \in [0, +\infty). \quad (4.30)$$

## 5 Proof of Proposition 4.1

In this section we only work on the odd functions which correspond to the  $H_1^m$  spaces. We assume  $k = 1$  in the following. Similar results hold for even functions which correspond to the  $H_2^m$  spaces. To simplify the notations, in this section we ignore the index  $k$  and we also denote  $T := T_k, K := K_k$  and  $\phi := \phi_k$ . This Section will be divided in several parts: first we reformulate the problem by projecting the equation on the eigenfunctions of  $A$  and we define a candidate  $T$  to satisfies the operator equality (4.18); second we prove some Riesz basis properties; then we construct a candidate  $K$  that satisfies the condition  $T\phi = \phi$  weakly; and finally we show that this candidate allows  $T$  to be an isomorphism that satisfies the operator equality (4.18).

### 5.1 Setting up for the backstepping transformation

We want to map the solution of

$$y_t - y_{xx} = \phi K y, \quad y \in L_1^2, \quad (5.1)$$

via transformation  $T$ , to the solution of

$$z_t - z_{xx} - \lambda z = 0, \quad z \in L_1^2. \quad (5.2)$$

To achieve this aim we would like  $T$  to satisfies formally the backstepping conditions

$$TA + \phi K = AT - \lambda T \quad (5.3)$$

$$T\phi = \phi \quad (5.4)$$

for a suitable feedback law  $K$ :

$$K : f_n \mapsto \langle f_n, K \rangle =: K_n \in \mathbb{R}. \quad (5.5)$$

Projected on the eigenvectors  $f_n := \sin nx$  with eigenvalues  $\lambda_n = -n^2$ , the formal relation (5.3) becomes

$$T(\Delta f_n) + \langle f_n, K \rangle \phi = \Delta(T f_n) - \lambda(T f_n), \quad (5.6)$$

$$\langle T \phi, f_n \rangle = \langle \phi, f_n \rangle. \quad (5.7)$$

Defining

$$h_n := T f_n, \quad (5.8)$$

the first condition becomes

$$\lambda_n h_n + \langle f_n, K \rangle \phi = \Delta h_n - \lambda h_n. \quad (5.9)$$

Projecting the preceding equation now on  $f_p$ , defining

$$a_p := \langle \phi, f_p \rangle$$

and using the fact that  $\Delta$  is self-adjoint we get

$$\lambda_n \langle h_n, f_p \rangle + \langle f_n, K \rangle a_p = (\lambda_p - \lambda) \langle h_n, f_p \rangle, \quad (5.10)$$

Hence, for any  $n, p \in \mathbb{N}^*$ ,

$$\langle h_n, f_p \rangle = \frac{-K_n a_p}{\lambda_n - \lambda_p + \lambda} \quad (5.11)$$

Therefore

$$q_n := -\frac{h_n}{K_n} = \sum_{p \in \mathbb{N}^*} \frac{a_p f_p}{\lambda_n - \lambda_p + \lambda}, \forall n \in \mathbb{N}^*. \quad (5.12)$$

Inspired by the preceding formula, the number  $\lambda$  should be selected in such a way that

$$\lambda_n - \lambda_p + \lambda \neq 0, \forall p, n \in \mathbb{N}^*, \quad (5.13)$$

which is rather easy to achieve, for example to choose from  $\mathbb{N}^* + 1/2$ . More precisely, it suffices to choose

$$\lambda \notin \mathcal{N} := \{i^2 - j^2 : i, j \in \mathbb{N}\}. \quad (5.14)$$

## 5.2 Riesz basis properties

Recall that for any  $n \in \mathbb{N}^*$ ,

$$f_n = \sin nx, \quad \lambda_n = -n^2, \quad (5.15)$$

$$A := \Delta, \quad \phi = \sum_{n \in \mathbb{N}^*} a_n f_n, \quad (5.16)$$

$$h_n := T f_n, \quad -K_n q_n := h_n, \quad (5.17)$$

$$q_n = \sum_{p \in \mathbb{N}^*} \frac{a_p f_p}{\lambda_n - \lambda_p + \lambda}, \quad (5.18)$$

$$g_n := \sum_{p \in \mathbb{N}^*} \frac{f_p}{\lambda_n - \lambda_p + \lambda} \in H^{3/2-}. \quad (5.19)$$

This last claim on the regularity of the  $g_n$  comes from the growth of the eigenvalues  $\lambda_p$ , i.e., for any  $n \in \mathbb{N}^*$  we have

$$\|g_n\|_{H^s}^2 = \sum_{p \in \mathbb{N}^*} \frac{p^{2s}}{(p^2 - n^2 + \lambda)^2} < +\infty, \forall s \in (-\infty, 3/2), \quad (5.20)$$

$$\|g_n\|_{H^s}^2 = \sum_{p \in \mathbb{N}^*} \frac{p^{2s}}{(p^2 - n^2 + \lambda)^2} = +\infty, \text{ for } s = 3/2. \quad (5.21)$$

Notice that  $\{a_n\}_{n \in \mathbb{N}^*}$  is uniquely determined by the value of the function  $\phi$ , while the sequences  $\{g_n\}_{n \in \mathbb{N}^*}$  and  $\{q_n\}_{n \in \mathbb{N}^*}$  are independent of the choice of  $\{K_n\}_{n \in \mathbb{N}^*}$ . Hence any sequence  $\{K_n\}_{n \in \mathbb{N}^*}$  determines the value of  $\{h_n\}_{n \in \mathbb{N}^*}$ , thus the operator  $T$ , and such operator  $T$  (at least formally) satisfies the equation (5.3).

The following lemma is devoted to the properties of  $\{g_n\}_{n \in \mathbb{N}^*}$ , to the properties of  $\{q_n\}_{n \in \mathbb{N}^*}$  provided some suitable assumption on  $\{a_n\}_{n \in \mathbb{N}^*}$ , and to the properties of the transformation  $T$  provided some assumption on both  $\{a_n\}_{n \in \mathbb{N}^*}$  and  $\{K_n\}_{n \in \mathbb{N}^*}$ .

**LEMMA 5.1.** *Let  $m \geq 0$ . Let  $a_n \neq 0$  behaves like  $cn^{-m} < |a_n| < Cn^{-m}$ . Let  $\lambda \notin \mathcal{N}$ . Successively we are able to prove the following properties. Moreover, all the choices of  $s$  and  $r$  in the following are sharp.*

(1)  $\{g_n\}_{n \in \mathbb{N}^*}$  is a Riesz basis of  $L_1^2$ .

(2) Let  $s \in (-3/2, 3/2)$ .  $\{n^{-s}g_n\}_{n \in \mathbb{N}^*}$  is a Riesz basis of  $H_1^s$ .

(3) Let  $s \in (-3/2, 3/2)$ .  $\{n^{-s}q_n\}_{n \in \mathbb{N}^*}$  is a Riesz basis of  $H_1^{m+s}$ .

(4) Let  $m \geq 0$ . Let  $s \in (-3/2, 3/2)$ . If  $K_n := \langle f_n, K \rangle$  is chosen in such a way that  $|K_n| < Cn^m$ , then the transformation  $T$  is bounded from  $H_1^{m+s}$  to itself. Moreover, if  $cn^m < |K_n| < Cn^m$ , then the transformation  $T$  is an isomorphism from  $H_1^{m+s}$  to itself.

(5) Let  $r \in [0, 1/2)$ . There is the following smoothing effect,

$$\sum_{n \in \mathbb{N}^*} \|q_n - a_n f_n / \lambda\|_{H_1^{m+r}}^2 < +\infty. \quad (5.22)$$

(6) Let  $r \in [0, 1/2)$ . Similar smoothing effect also holds in the space  $H^{-1+m}$ ,

$$\sum_{n \in \mathbb{N}^*} \|n(q_n - a_n f_n / \lambda)\|_{H_1^{-1+m+r}}^2 < +\infty. \quad (5.23)$$

(7) Let  $m = 0$ . Let  $r \in [0, 1/2)$ .

$$\sum_{n \in \mathbb{N}^*} (q_n - \frac{a_n f_n}{\lambda}) \in H_1^r. \quad (5.24)$$

*Proof of Lemma 5.1.* For ease of notations, in this paper we fix

$$\lambda = N = 4M + 2,$$

which guarantees the fact that  $p^2 + \lambda - n^2 \neq 0$ . However, all the results hold with similar calculation for any  $\lambda$  outside the special subset  $\mathcal{N}$ .

**(1)  $\{g_n\}_{n \in \mathbb{N}^*}$  is a Riesz basis of  $L_1^2$ .** We proceed in two steps. We start by showing that  $\{g_n\}_{n \in \mathbb{N}^*}$  is quadratically close to a Riesz basis in  $L_1^2$ . Then we show that it is  $\omega$ -independent or dense in  $L_1^2$ , which, together with the quadratically close behavior, ensures that it is a Riesz basis of  $L_1^2$ .

$$\left\{ \sum_{p \in \mathbb{N}^*} \frac{f_p}{\lambda_n - \lambda_p + \lambda} \right\}_{n \in \mathbb{N}^*} \text{ is quadratically close to } \left\{ \frac{f_p}{\lambda} \right\}_{n \in \mathbb{N}^*} \text{ in } L_1^2. \quad (5.25)$$

It suffices to show that

$$\sum_{n \in \mathbb{N}^*} \sum_{p \neq n} \left( \frac{1}{p^2 + \lambda - n^2} \right)^2 < +\infty. \quad (5.26)$$

Thus it further suffices to prove

$$\sum_{n > N} \left( \sum_{p > n} + \sum_{p < n} \right) \left( \frac{1}{p^2 + \lambda - n^2} \right)^2 < +\infty. \quad (5.27)$$

as well as

$$I := \sum_{n \leq N} \left( \sum_{p > n} + \sum_{p < n} \right) \left( \frac{1}{p^2 + \lambda - n^2} \right)^2 < +\infty : \quad (5.28)$$

We can express  $I$  in the following fashion,

$$I = \sum_{j=p^2+\lambda-n^2, n < N} \frac{1}{j^2}, \text{ counting multiplicity of } j, \quad (5.29)$$

for any possible  $j$  the multiplicity count at most as  $N$ , thus

$$I < N \sum \frac{1}{j^2} < +\infty. \quad (5.30)$$

For the first part of (5.27) as  $p > n$ , we have

$$\begin{aligned} \sum_{n > N} \sum_{p > n} \left( \frac{1}{p^2 + \lambda - n^2} \right)^2 &\leq \sum_{n=N} \sum_{k=1} \left( \frac{1}{k^2 + 2kn} \right)^2, \\ &\leq \sum_{n=1} \sum_{k=1} \left( \frac{1}{kn} \right)^2 < +\infty. \end{aligned}$$

For the second part as  $p < n$ , there exists  $j \in \{1, \dots, n-1\}$  such that  $p = (n-j)$  and we know that for any such  $j$

$$n^2 - (n-j)^2 - \lambda \geq (n^2 - (n-j)^2)/2. \quad (5.31)$$

Indeed, using that  $\lambda = N$ .

$$\begin{aligned} n^2 - (n-j)^2 - \lambda - (n^2 - (n-j)^2)/2 &= n^2/2 - (n-j)^2/2 - N, \\ &\geq jn - j^2/2 - N, \end{aligned} \quad (5.32)$$

and the right-hand side is a second order polynomial whose minimum is achieved either for  $j = 1$  or for  $j = n - 1$ . As  $n \geq N + 1$ ,

$$\begin{aligned} n^2 - (n-j)^2 - \lambda - (n^2 - (n-j)^2)/2 &\geq 1/2 \\ &\geq 0. \end{aligned} \quad (5.33)$$

**REMARK 5.2.** For the general case that  $\lambda \notin \mathcal{N}$ , there exists  $C(\lambda) > 0$  such that

$$|p^2 + \lambda - n^2| \geq C(\lambda)|p^2 - n^2|, \quad \forall p, n \in \mathbb{N}^*. \quad (5.34)$$

Indeed, as for  $p \geq n$  we always have  $|p^2 + \lambda - n^2| \geq |p^2 - n^2|$ , it suffices to consider the case that  $p \leq n - 1$ . Thus, it is equivalent to show that for  $p \leq n - 1$ ,

$$\frac{|p^2 + \lambda - n^2|}{n^2 - p^2} > C(\lambda). \quad (5.35)$$

For  $n \geq \lambda + 1$  and  $p \leq n - 1$ , we know that

$$\frac{|p^2 + \lambda - n^2|}{n^2 - p^2} = \frac{n^2 - p^2 - \lambda}{n^2 - p^2} > \frac{n^2 - p^2 - n}{n^2 - p^2} > \frac{1}{2}. \quad (5.36)$$

For  $n \leq \lambda$  and  $p \leq n - 1$  containing finitely many pairs, thanks to the definition of  $\lambda$ , it is clear that such  $C(\lambda)$  exists.

Thus

$$\begin{aligned} \sum_{n>N} \sum_{p<n} \left( \frac{1}{p^2 + \lambda - n^2} \right)^2 &= \sum_{n>N} \sum_{k=1}^{n-1} \left( \frac{1}{k^2 + \lambda - n^2} \right)^2, \\ &\leq 4 \sum_{n>N} \sum_{k=1}^{n-1} \left( \frac{1}{n^2 - k^2} \right)^2, \\ &\leq 4 \sum_{n>N} \sum_{k=1}^{n-1} \left( \frac{1}{n(n-k)} \right)^2, \\ &\leq 4 \sum_{n>N} \sum_{j=1}^{n-1} \left( \frac{1}{nj} \right)^2 < +\infty. \end{aligned}$$

Let us denote

$$g_n := \sum_{p \in \mathbb{N}^*} \frac{f_p}{\lambda_n - \lambda_p + \lambda} = \sum_{p \in \mathbb{N}^*} \frac{f_p}{-n^2 + p^2 + \lambda} \in H_1^{3/2-}. \quad (5.37)$$

Then

$$(-\Delta + \lambda - n^2)g_n = \sum_{p \in \mathbb{N}^*} f_p = \frac{1}{2} \cot \frac{x}{2} =: h, \quad \text{in } H_1^{-1}. \quad (5.38)$$

The following proof to show that  $g_n$  is either  $\omega$ -independent or dense in  $L_1^2$  is inspired by [13], though even the transformation type that is adapted here is slightly different from the one given in [13].

Recalling that  $\mathcal{A} := -A = -\Delta$  and defining  $k_n := (n^2 - \lambda)^{-1}$ , from (5.38) we notice that

$$\mathcal{A}^{-1}g_n = k_n g_n - k_n \mathcal{A}^{-1}h. \quad (5.39)$$

If  $\{g_n\}_{n \in \mathbb{N}^*}$  is  $\omega$ -independent then we conclude the proof. Suppose that  $\{g_n\}_{n \in \mathbb{N}^*}$  is not  $\omega$ -independent, thus by the definition there exists a nontrivial sequence  $\{c_n\}_{n \in \mathbb{N}^*}$  belongs to  $l^2(\mathbb{N}^*)$  such that

$$\sum_{n \in \mathbb{N}^*} c_n g_n = 0, \text{ in } L_1^2. \quad (5.40)$$

The preceding formula is well-defined, in fact, thanks to (5.25) that we just proved,

$$\sum_{n \in \mathbb{N}^*} c_n g_n = \sum_{n \in \mathbb{N}^*} c_n \frac{f_n}{\lambda} + \sum_{n \in \mathbb{N}^*} c_n (g_n - \frac{f_n}{\lambda}) \quad (5.41)$$

converges in  $L_1^2$  sense.

Next, by applying  $\mathcal{A}^{-1}$  to this equality we conclude

$$\sum_{n \in \mathbb{N}^*} c_n k_n g_n = \sum_{n \in \mathbb{N}^*} c_n k_n \mathcal{A}^{-1}h, \text{ in } L_1^2, \quad (5.42)$$

where we have used the fact  $\sum_n c_n k_n$  converges.

Then, by applying again  $\mathcal{A}^{-1}$  to this equality we get that in  $L_1^2$  space,

$$\sum_{n \in \mathbb{N}^*} c_n k_n^2 g_n = \sum_{n \in \mathbb{N}^*} c_n k_n^2 \mathcal{A}^{-1}h + c_n k_n \mathcal{A}^{-2}h, \quad (5.43)$$

and applying it again we have still in  $L_1^2$  space,

$$\sum_{n \in \mathbb{N}^*} c_n k_n^3 g_n = \sum_{n \in \mathbb{N}^*} c_n k_n^3 \mathcal{A}^{-1}h + c_n k_n^2 \mathcal{A}^{-2}h + c_n k_n \mathcal{A}^{-3}h. \quad (5.44)$$

By induction we easily arrive at, for any  $m \in \mathbb{N}^*$ ,

$$\sum_{n \in \mathbb{N}^*} c_n k_n^m g_n = \sum_{i=1}^m \left( \sum_{n \in \mathbb{N}^*} c_n k_n^{m+1-i} \mathcal{A}^{-i}h \right) = \sum_{i=1}^m C_{m+1-i} \mathcal{A}^{-i}h, \quad (5.45)$$

where

$$C_l := \sum_{n \in \mathbb{N}^*} c_n k_n^l < +\infty. \quad (5.46)$$

Let us now proceed by cases:

- **First case:**  $C_1 \neq 0$ . Then we conclude from the preceding equation that for all  $m \in \mathbb{N}^*$  we have that  $\mathcal{A}^{-m}h \in \text{span}\{g_n\}_{n \in \mathbb{N}^*}$ . Suppose that  $\text{span}\{g_n\}_{n \in \mathbb{N}^*}$  is not dense in  $L_1^2$ , then we can find  $d = \sum d_n f_n \in L_1^2$  (thus  $\{d_n\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*)$ ) such that  $d \neq 0$  and

$$\langle g, d \rangle_{L_1^2} = 0, \forall g \in \text{span}\{g_n\}_{n \in \mathbb{N}^*}, \quad (5.47)$$

which in particular yields,

$$\langle \mathcal{A}^{-m}h, d \rangle_{L_1^2} = 0, \forall m \in \mathbb{N}^*. \quad (5.48)$$

Recalling that  $h = \sum f_n \in H_1^{-1}$ , we get that

$$\sum_n d_n \frac{1}{n^{2m}} = 0, \forall m \in \mathbb{N}^*. \quad (5.49)$$

By defining the complex function

$$G(z) := \sum_{n \in \mathbb{N}^*} d_n n^{-2} e^{n^{-2}z}, \forall z \in \mathbb{C}. \quad (5.50)$$

By checking that the series expansion of the right-hand side is absolutely convergent, we deduce that this function is holomorphic. For example, for any  $z \in \mathbb{C}$ , the following series is absolutely convergent,

$$\sum_{n \in \mathbb{N}^*} d_n n^{-2} e^{n^{-2}z} = \sum_{n \in \mathbb{N}^*} d_n n^{-2} \sum_{j \geq 0} \frac{n^{-2j}}{j!} z^j = \sum_{j \geq 0} \frac{1}{j!} \sum_{n \in \mathbb{N}^*} (d_n n^{-2}) (n^{-2j} z^j).$$

Similar calculation yields the absolute convergence of  $G^{(m)}(z)$ , for any  $m \in \mathbb{N}$ .

From (5.49) we know that  $G^{(m)}(0) = 0, m \in \mathbb{N}$ . Thus  $G = 0$ , and further  $d_n = 0$ , which leads to a contradiction. Therefore

$$\text{span}\{g_n\}_{n \in \mathbb{N}^*} \text{ is dense in } L_1^2. \quad (5.51)$$

- **Second case: there exists  $m > 1$  such that  $C_m \neq 0$** , without loss of generality we can assume that  $m$  is the first integer such that  $C_m \neq 0$ . We can also conclude that  $A^{-m}h \in \text{span}\{g_n\}_{n \in \mathbb{N}^*}$ . The same reasoning as above proves again (5.51).

- **Third case: for all  $l \in \mathbb{N}^*$  we have  $C_l = 0$** . Then we set complex function

$$\tilde{G}(z) := \sum_{n \in \mathbb{N}^*} c_n k_n e^{k_n z}. \quad (5.52)$$

This function is holomorphic and satisfies that  $\tilde{G}^{(m)}(0) = 0$  for any  $m \in \mathbb{N}$ , thus as previously  $\tilde{G} = 0$  and therefore  $c_n = 0$  for all  $n \in \mathbb{N}^*$ , which is in contradiction with the choice of  $\{c_n\}_{n \in \mathbb{N}^*}$ .

Consequently,

$$\{g_n\} \text{ is either } \omega \text{ independent in } L_1^2 \text{ or dense in } L_1^2. \quad (5.53)$$

Properties (5.25) and (5.53), together with Lemma 2.2 (which is also [11, Theorem 3.2 and Theorem 3.3]), lead to the proof of Lemma 5.1 (1).

**(2)  $\{n^{-s}g_n\}_{n \in \mathbb{N}^*}$  is a Riesz basis of  $H_1^s$ .**

As in (1), we first show that  $\{n^{-s}g_n\}_{n \in \mathbb{N}^*}$  is quadratically close to  $\{\frac{n^{-s}f_n}{\lambda}\}_{n \in \mathbb{N}^*}$  in  $H_1^s$ , and then prove that  $\{n^{-s}g_n\}_{n \in \mathbb{N}^*}$  is  $\omega$ -independent in  $H_1^s$  or dense in  $H_1^s$  to conclude the proof.

★ **Quadratically close.**

We notice that  $s = 0$  is exactly the case of (1).

i) If  $s = -1$  then

$$\sum_n \left\| ng_n - \frac{nf_n}{\lambda} \right\|_{H^{-1}}^2 = \sum_n \left\| \sum_{p \neq n} \frac{nf_p}{p^2 + \lambda - n^2} \right\|_{H^{-1}}^2,$$



$$\begin{aligned}
&= \sum_n \sum_{p \neq n} \frac{n^2}{p^2(p^2 + \lambda - n^2)^2}, \\
&= \sum_n \left( \sum_{p < n} + \sum_{p > n} \right) \frac{n^2}{p^2(p^2 + \lambda - n^2)^2}, \\
&\leq C + \sum_n \sum_{p < n} \frac{n^2}{p^2(p^2 + \lambda - n^2)^2}, \\
&\leq C + \sum_{n \geq N} \sum_{p < n} \frac{n^2}{p^2(n^2 - p^2 - \lambda)^2}, \\
&\leq C + 4 \sum_{n \geq N} \sum_{p < n} \frac{n^2}{p^2(n^2 - p^2)^2}, \\
&\leq C + C \sum_{n \geq N} \sum_{p < n} \frac{n^2}{p^2 n^2 (n-p)^2}, \\
&\leq C + C \sum_{n \geq N} \sum_{p < n} \frac{1}{n^2} \left( \frac{1}{p} + \frac{1}{n-p} \right)^2, \\
&\leq C + C \sum_{n \geq N} \sum_{p < n} \frac{1}{n^2} \left( \frac{1}{p^2} + \frac{1}{(n-p)^2} \right), \\
&\leq C + C \sum_{n \geq N} \sum_{p < n} \frac{1}{p^2 n^2} < +\infty,
\end{aligned}$$

where  $C$  is a constant that can change between lines and where used (5.26), (5.31), the fact that  $\lambda = N$ , and

$$\frac{1}{(n-p)^2 p^2} = \left( \frac{1}{n} \left( \frac{1}{n-p} + \frac{1}{p} \right) \right)^2 \leq \frac{2}{n^2} \left( \frac{1}{(n-p)^2} + \frac{1}{p^2} \right). \quad (5.54)$$

ii) From the preceding proof we easily conclude the case  $s \in (-1, 0)$  by using that  $(n/p)^{-s} \leq (n/p)$  for  $0 < p < n$ .

iii) If  $s \in (-3/2, -1)$ , it suffices to show that

$$\sum_{n \geq N} \sum_{p < n} \frac{1}{(n-p)^2 n^2} \frac{n^{-2s}}{p^{-2s}} < +\infty.$$

since the case  $p > n$  follows directly from (5.26) and  $(n/p)^{-2s} \leq 1$ . For  $p < n/2$ , we have

$$\begin{aligned}
\sum_{n \geq N} \sum_{p < n/2} \frac{1}{(n-p)^2 n^2} \frac{n^{-2s}}{p^{-2s}} &\leq C \sum_{n \geq N} \sum_{p < n/2} \frac{1}{n^4} \frac{n^{-2s}}{p^{-2s}} \\
&\leq C \sum_{n \geq N} \frac{1}{n^{4+2s}} \sum_{p < n/2} \frac{1}{p^{-2s}} \\
&\leq C \left( \sum_{n \geq N} \frac{1}{n^{4+2s}} \right) \left( \sum_{p \in \mathbb{N}^*} \frac{1}{p^{-2s}} \right)
\end{aligned}$$

$< +\infty$ .

For  $n/2 \leq p < n$ , we have

$$\sum_{n \geq N} \sum_{n/2 \leq p < n} \frac{1}{(n-p)^2 n^2} \frac{n^{-2s}}{p^{-2s}} \leq C \sum_{n \geq N} \frac{1}{n^2} \sum_{n/2 \leq p < n} \frac{1}{(n-p)^2} < +\infty. \quad (5.55)$$

iii) If  $s = \frac{3}{2} - \varepsilon$  with  $\varepsilon \in (0, \frac{3}{2})$ , then

$$\begin{aligned} \sum_n \left\| n^{-s} g_n - \frac{n^{-s} f_n}{\lambda} \right\|_{H^s}^2 &= \sum_n \left\| \sum_{p \neq n} \frac{n^{-s} f_p}{p^2 + \lambda - n^2} \right\|_{H^s}^2, \\ &= \sum_n \sum_{p \neq n} \frac{n^{-2s} p^{2s}}{(p^2 + \lambda - n^2)^2}, \\ &= \sum_n \left( \sum_{p < n} + \sum_{n < p < 2n} + \sum_{p \geq 2n} \right) \frac{1}{(p^2 + \lambda - n^2)^2} \frac{p^{2s}}{n^{2s}}, \\ &\leq C + \sum_n \sum_{p \geq 2n} \frac{1}{(p^2 + \lambda - n^2)^2} \frac{p^{2s}}{n^{2s}}, \\ &\leq C + \sum_n \sum_{p \geq 2n} \frac{1}{(p^2 - n^2)^2} \frac{p^{2s}}{n^{2s}}, \\ &\leq C + C \sum_n \sum_{p \geq 2n} \frac{1}{(p^2)^2} \frac{p^{2s}}{n^{2s}}, \\ &\leq C + C \sum_n \frac{1}{n^{2s}} \sum_{p \geq 2n} \frac{1}{p^{4-2s}}. \end{aligned}$$

Because  $s \in (0, 3/2)$ , we know that

$$\sum_{p \geq 2n} \frac{1}{p^{4-2s}} \leq C \frac{1}{n^{3-2s}}$$

and therefore  $\left( \frac{1}{n^{2s}} \sum_{p \geq 2n} \frac{1}{p^{4-2s}} \right)_{n \in \mathbb{N}^*}$  is absolutely convergent.

This concludes the proof of the quadratically close behavior.

★ **Let  $s \in (-3/2, 3/2)$ .  $\{n^{-s} g_n\}_{n \in \mathbb{N}^*}$  is  $\omega$ -independent in  $H_1^s$  or dense in  $H_1^s$ .**

Similar to the case that  $s = 0$ . Suppose that there exists  $\{c_n\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*)$  such that

$$\sum_{n \in \mathbb{N}^*} \frac{c_n}{n^s} g_n = 0, \text{ in } H_1^s, \quad (5.56)$$

which is well defined as

$$\sum_{n \in \mathbb{N}^*} c_n n^{-s} g_n = \sum_{n \in \mathbb{N}^*} c_n n^{-s} \frac{f_n}{\lambda} + \sum_{n \in \mathbb{N}^*} c_n n^{-s} \left( g_n - \frac{f_n}{\lambda} \right) \quad (5.57)$$

converges in  $H_1^s$  space thanks to the fact that  $s \in (-3/2, 3/2)$ , and the quadratically close result in  $H_1^s$  that we just proved. By the same reasoning as the case  $s = 1$ , we get

$$\sum_{n \in \mathbb{N}^*} \frac{c_n}{n^s} k_n g_n = \sum_{n \in \mathbb{N}^*} \frac{c_n}{n^s} k_n \mathcal{A}^{-1} h, \text{ in } H_1^s, \quad (5.58)$$

where we have used the fact  $\sum_n \frac{c_n}{n^s} k_n$  converges for  $s > -3/2$ .

Then, we can further consider

$$C_m := \sum_{n \in \mathbb{N}^*} \frac{c_n}{n^s} k_n^m < +\infty. \quad (5.59)$$

Exactly the same as in the case  $s = 0$ , it suffices to consider two cases,  $C_1 \neq 0$ , or  $C_m = 0$  for all  $m \in \mathbb{N}^*$ .

In the first case, we find that  $\mathcal{A}^{-m} h \in \text{span}\{n^{-s} g_n\}_{n \in \mathbb{N}^*}$  in  $H_1^s$ . Suppose that  $\text{span}\{n^{-s} g_n\}_{n \in \mathbb{N}^*}$  is not dense in  $H_1^s$ , then we can find  $d = \sum_n \frac{d_n}{n^s} f_n \in H_1^s$  (thus  $\{d_n\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*)$ ) such that

$$\langle \mathcal{A}^{-m} h, d \rangle_{H_1^s} = 0, \quad \forall m \in \mathbb{N}^*, \quad (5.60)$$

which is well-defined as  $h \in H_1^{-1/2-}$  and  $\mathcal{A}^{-1} h \in H_1^{3/2-} \subset H_1^s$ . Recalling the exact definition of  $h = \sum_n f_n \in H_1^{-1/2-}$ , we get that

$$\sum_n d_n \frac{n^s}{n^{2m}} = 0, \quad \forall m \in \mathbb{N}^*, \quad (5.61)$$

which further implies that  $d_n = 0$  using the holomorphic function technique.

In the second case where  $C_m = 0$  for all  $m \in \mathbb{N}^*$ , we can also prove that  $c_n = 0$ , similarly as in the case  $s = 0$ , which is in contradiction with the choice of  $\{c_n\}_{n \in \mathbb{N}^*}$ .

**(3)  $\{n^{-s} q_n\}_{n \in \mathbb{N}^*}$  is a Riesz basis of  $H_1^{m+s}$ .**

Let  $s \in (-3/2, 3/2)$ . We introduce  $\tau : H_1^s \rightarrow H_1^{m+s}$  defined by,

$$\tau : n^{-s} f_n \mapsto n^{-s} a_n f_n, \quad (5.62)$$

which is an isomorphism thanks to the fact that  $cn^{-m} < |a_n| < Cn^{-m}$ . We immediately notice that

$$\tau : n^{-s} g_n = n^{-s} \sum_{p \in \mathbb{N}^*} \frac{f_p}{\lambda_n - \lambda_p + \lambda} \mapsto n^{-s} \sum_{p \in \mathbb{N}^*} \frac{a_p f_p}{\lambda_n - \lambda_p + \lambda} = n^{-s} q_n. \quad (5.63)$$

Consequently, thanks to the fact that  $\{n^{-s} g_n\}_{n \in \mathbb{N}^*}$  is a Riesz basis of  $H_1^s$ , from Lemma 2.3 we know that  $\{n^{-s} q_n\}_{n \in \mathbb{N}^*}$  is a Riesz basis of  $H_1^{m+s}$ .

**(4) Boundedness of the transformation  $T$  and isomorphism**

If  $K_n \neq 0$  is chosen in such a way that  $cn^m < |K_n| < n^m$ , then from the definition of  $T$  we know that

$$T : \frac{n^{-s} f_n}{n^m} \mapsto \frac{-K_n}{n^m} (n^{-s} q_n). \quad (5.64)$$

Since  $\{n^{-s} f_n\}_{n \in \mathbb{N}^*}$  is an orthonormal basis of  $H^s$ , we know that  $\{\frac{n^{-s} f_n}{n^m}\}_{n \in \mathbb{N}^*}$  is an orthonormal basis of  $H_1^{m+s}$ . Moreover, because  $\{n^{-s} q_n\}_{n \in \mathbb{N}^*}$  is a Riesz basis of  $H_1^{m+s}$ , from Lemma 2.3 we know that  $\{\frac{-K_n}{n^m} (n^{-s} q_n)\}_{n \in \mathbb{N}^*}$  is also a Riesz basis of  $H_1^{m+s}$ .

Therefore, the transformation  $T$  is an isomorphism from  $H_1^{m+s}$  to  $H_1^{m+s}$ . When  $m = 2$ , we remark, however, here that  $T$  is not a isomorphism from  $L_1^2$  to itself.

Moreover, if we only assume that  $|K_n| \leq Cn^m$ , then  $T$  is a bounded operator from  $H_1^{m+s}$  to  $H_1^{m+s}$ . However, in this case  $T$  might not be an isomorphism.

**(5) Smoothing effect in  $H_1^{m+r}$ , for  $r \in [0, 1/2)$**

$$\begin{aligned} \sum_n \|q_n - a_n f_n / \lambda\|_{H_1^{m+r}}^2 &= \sum_n \left\| \sum_{p \neq n} \frac{a_p f_p}{p^2 + \lambda - n^2} \right\|_{H_1^{m+r}}^2, \\ &\leq C \sum_n \sum_{p \neq n} \frac{p^{2r}}{(p^2 + \lambda - n^2)^2}, \\ &= C \sum_p p^{2r} \sum_{n \neq p} \frac{1}{(p^2 + \lambda - n^2)^2}, \\ &\leq C + C \sum_{p \geq N} p^{2r} \sum_{n \neq p} \frac{1}{(p^2 + \lambda - n^2)^2}, \end{aligned}$$

where  $C$  is a constant that can change between lines. Moreover,

$$\sum_{p \geq N} p^{2r} \sum_{n < p} \frac{1}{(p^2 + \lambda - n^2)^2} \leq \sum_{p \geq N} p^{2r-2} \sum_{n < p} \frac{1}{(p-n)^2} < +\infty,$$

and, using (5.31),

$$\begin{aligned} \sum_{p \geq N} p^{2r} \sum_{p < n} \frac{1}{(p^2 + \lambda - n^2)^2} &\leq \sum_{p \geq N} p^{2r} \sum_{p < n} \frac{4}{(n^2 - p^2)^2} \\ &\leq \sum_{p \geq N} p^{2r-2} \sum_{p < n} \frac{4}{(p-n)^2} < +\infty. \end{aligned}$$

From the above proof we find that the condition  $r < 1/2$  is sharp.

**(6) Smoothing effect in  $H_1^{-1+m+r}$  for  $r \in [0, 1/2)$**

$$\begin{aligned} \sum_n \|n(q_n - a_n f_n / \lambda)\|_{H_1^{-1+m+r}}^2 &= \sum_n \left\| \sum_{p \neq n} \frac{n a_p f_p}{p^2 + \lambda - n^2} \right\|_{H_1^{-1+m+r}}^2, \\ &\leq C \sum_n \sum_{p \neq n} \frac{p^{2r-2} n^2}{(p^2 + \lambda - n^2)^2}, \\ &= C \sum_p p^{2r-2} \sum_{n \neq p} \frac{n^2}{(p^2 + \lambda - n^2)^2}, \\ &\leq C + C \sum_{p \geq N} p^{2r-2} \sum_{n \neq p} \frac{n^2}{(p^2 + \lambda - n^2)^2}. \end{aligned}$$

Moreover,

$$\sum_{p \geq N} p^{2r-2} \sum_{n < p} \frac{n^2}{(p^2 + \lambda - n^2)^2} \leq \sum_{p \geq N} p^{2r-2} \sum_{n < p} \frac{1}{(p-n)^2} < +\infty,$$

and, still using (5.31),

$$\sum_{p \geq N} p^{2r-2} \sum_{p < n} \frac{n^2}{(p^2 + \lambda - n^2)^2} \leq \sum_{p \geq N} p^{2r-2} \sum_{p < n} \frac{16}{(p-n)^2} < +\infty.$$

**(7) The quantity  $\sum_{n \in \mathbb{N}^*} (q_n - \frac{a_n f_n}{\lambda})$  belongs to  $H_1^r$  for  $r \in [0, 1/2)$ .**

At first we point out that since  $\{nq_n\}_{n \in \mathbb{N}^*}$  is a Riesz basis of  $H_1^{-1}$ , and  $\{nf_n\}_{n \in \mathbb{N}^*}$  is an orthonormal basis of  $H_1^{-1}$ , the candidate  $\sum_{n \in \mathbb{N}^*} (q_n - \frac{a_n f_n}{\lambda})$  belongs to  $H_1^{-1}$ . We need to show that it actually belongs to more regular spaces  $H_1^r$  for  $r \in [0, 1/2)$ . Moreover, both  $\sum_{n \in \mathbb{N}^*} q_n$  and  $\sum_{n \in \mathbb{N}^*} \frac{a_n f_n}{\lambda}$  belong to  $H^{-1/2-}$  but not to  $H^{-1/2}$ .

In the following, we mainly focus on the case  $r = 0$ . Since  $m = 0$  we notice that the lemma is equivalent to

$$\left\| \sum_n \sum_{p \neq n} \frac{a_p f_p}{p^2 + \lambda - n^2} \right\|_{L^2}^2 < +\infty. \quad (5.65)$$

Note that this **cannot be directly deduced from** the quadratically close inequality that we proved in Lemma 5.1 (3) (case  $m = s = 0$ ),

$$\sum_n \left\| \sum_{p \neq n} \frac{a_p f_p}{p^2 + \lambda - n^2} \right\|_{L^2}^2 < +\infty. \quad (5.66)$$

Indeed, we need more delicate estimates. We know from the fact that  $\sum_{n \in \mathbb{N}^*} (q_n - \frac{a_n f_n}{\lambda})$  belongs to  $H_1^{-1}$  that

$$\sum_{n \in \mathbb{N}^*} (q_n - \frac{a_n f_n}{\lambda}) = \sum_n \sum_{p \neq n} \frac{a_p f_p}{p^2 + \lambda - n^2} = \sum_p \sum_{n \neq p} \frac{a_p f_p}{p^2 + \lambda - n^2} \text{ in } H_1^{-1}, \quad (5.67)$$

the last equality can be obtained from the ‘‘distribution sense’’: we observe that the inner product of both quantities with  $f_k$  are equivalent, which implies that those two quantities are equivalent. Or, alternatively, since

$$\begin{aligned} & \sum_n \sum_{p \neq n} \left\| \frac{a_p f_p}{p^2 + \lambda - n^2} \right\|_{H_1^{-1}} \\ & \leq C \sum_n \sum_{p \neq n} \frac{1}{p|p^2 - n^2|} \\ & \leq C \sum_n \left( \sum_{p < n} + \sum_{n < p < 2n} + \sum_{p > 2n} \right) \frac{1}{p|p^2 - n^2|} \\ & \leq C \sum_n \left( \sum_{p < n} \frac{1}{n^2} \left( \frac{1}{p} + \frac{1}{n-p} \right) + \sum_{n < p < 2n} \frac{1}{n^2} \frac{1}{p-n} + \sum_{p > 2n} \frac{1}{p^3} \right) \end{aligned}$$

$$\leq C \sum_n \frac{1 + \log n}{n^2} < +\infty,$$

then thanks to Fubini we have that

$$\sum_n \sum_{p \neq n} \frac{a_p f_p}{p^2 + \lambda - n^2} = \sum_p \sum_{n \neq p} \frac{a_p f_p}{p^2 + \lambda - n^2} \text{ in } H_1^{-1}.$$

Hence, it suffices to show the last quantity in equation (5.67) belongs to  $L_1^2$ , which is equivalent to

$$\left\| \sum_p a_p f_p \sum_{n \neq p} \frac{1}{p^2 + \lambda - n^2} \right\|_{L_1^2}^2 = \sum_p a_p^2 \left( \sum_{n \neq p} \frac{1}{p^2 + \lambda - n^2} \right)^2 < +\infty, \quad (5.68)$$

which, to be combined with the growth condition  $a_n \sim 1$ , is equivalent to

$$\sum_p \left( \sum_{n \neq p} \frac{1}{p^2 + \lambda - n^2} \right)^2 < +\infty. \quad (5.69)$$

It further suffices to prove that

$$\sum_p \left( \sum_{n \neq p} \left| \frac{1}{p^2 + \lambda - n^2} \right| \right)^2 < +\infty, \quad (5.70)$$

which further reduces to showing that

$$\sum_{p \geq N} \left( \left( \sum_{n < p} + \sum_{p+1 < n < 2p} + \sum_{n \geq 2p} \right) \left| \frac{1}{p^2 + \lambda - n^2} \right| \right)^2 =: \sum_{p \geq N} (S_p^1 + S_p^2 + S_p^3)^2 < +\infty. \quad (5.71)$$

Next we estimate the  $S_p^i$  one by one.

i) Estimation of  $S_p^1$ :

$$\begin{aligned} S_p^1 &= \sum_{n \leq p-1} \frac{1}{p^2 + \lambda - n^2}, \\ &\leq \sum_{n \leq p-1} \frac{1}{p^2 - n^2}, \\ &\leq \frac{1}{p} \sum_{n \leq p-1} \frac{1}{p-n}, \\ &\leq C \frac{1 + \log p}{p}, \end{aligned}$$

where  $C$  is a constant independent of  $p$ .

ii) Estimation of  $S_p^2$ : by the choice of  $p \geq N$  and the fact that  $\lambda = N$ , we know that

$$S_p^2 = \sum_{p < n < 2p} \frac{1}{n^2 - p^2 - \lambda},$$

$$\begin{aligned}
&\leq \sum_{p < n < 2p} \frac{2}{n^2 - p^2}, \\
&\leq \frac{1}{p} \sum_{p < n < 2p} \frac{1}{n - p}, \\
&\leq C \frac{1 + \log p}{p}.
\end{aligned}$$

iii) Estimation of  $S_p^3$ : the last part can be estimated by

$$\begin{aligned}
S_p^3 &= \sum_{2p \leq n} \frac{1}{n^2 - p^2 - \lambda}, \\
&\leq \sum_{2p \leq n} \frac{2}{n^2 - p^2}, \\
&\leq \sum_{2p \leq n} \frac{4}{n^2}, \\
&\leq \frac{3}{p}.
\end{aligned}$$

Consequently,

$$\sum_{p \geq N} (S_p^1 + S_p^2 + S_p^3)^2 \leq \sum_{p \geq N} C \left( \frac{1 + \log p}{p} \right)^2 < +\infty \quad (5.72)$$

Finally, for the case  $r \in [0, 1/2)$ , by slightly modifying (5.68) it suffices to show that

$$\sum_{p \geq N} p^{2r} (S_p^1 + S_p^2 + S_p^3)^2 \leq C \sum_{p \geq N} p^{2r} \left( \frac{1 + \log p}{p} \right)^2 < +\infty, \quad (5.73)$$

where  $r < 1/2$  is sharp. This ends the proof of Lemma 5.1.  $\square$

**In the following, in order to simplify the calculation, we only consider the special case  $m = 0$ , thus**

$$c < |a_n| < C,$$

**the other cases can be treated exactly similarly.**

### 5.3 On the choice of the backstepping candidate

According to Section 5.1 we know that every sequence  $\{K_n\}_{n \in \mathbb{N}^*}$  determines a unique transformation  $T$ , and at least formally, it satisfies

$$TA + \phi K = AT - \lambda T. \quad (5.74)$$

Thanks to Section 5.2 we know that when  $\{K_n\}_{\mathbb{N}^*}$  verifies

$$|K_n| < C, \quad \forall n \in \mathbb{N}^*, \quad (5.75)$$

this unique transformation  $T$  is bounded on  $H^s$  with  $s \in (-3/2, 3/2)$ .

Moreover, if  $\{K_n\}_{n \in \mathbb{N}^*}$  further satisfies

$$c < |K_n| < C, \quad \forall n \in \mathbb{N}^*, \quad (5.76)$$

then  $T$  is an isomorphism on  $H^s$  with  $s \in (-3/2, 3/2)$ .

However, until now we have not yet treated the second condition:

$$T\phi = \phi. \tag{5.77}$$

This condition is called in the literature the “ $TB = B$  condition” (here the function  $\phi$  represent what is usually formally denoted  $B$ ), which is now becoming a standard requirement for Fredholm type backstepping transformations, [11, 12, 13, 14, 22, 41]. The aim of this section is to determine a precise candidate of  $\{K_n\}_{n \in \mathbb{N}^*}$  such that

- (i) The “TB=B condition” (5.77) holds, in a suitable space to be found.
- (ii) The boundedness condition (5.75) holds.
- (iii) The operator equality (5.74) holds, in a suitable space to be found.

The proofs of (i) and (ii) are provided by Sections 5.3.1–5.3.3. Then Section 5.3.4 is devoted to the proof of (iii).

We also remark here that, in this section, we only prove the condition (5.75) instead of the stronger (5.76), as it is sufficient to conclude about the conditions (5.77) and (5.74). The proof of (5.76), needed for  $T$  to be an isomorphism and not only a bounded operator, is left to Sections 5.4–5.5.

### 5.3.1 $T\phi = \phi$ condition does not hold in $L_1^2$ space

For any given  $\phi$  satisfying

$$\phi = \sum_{n \in \mathbb{N}^*} a_n f_n, \text{ with } c < |a_n| < C, \tag{5.78}$$

we want to find  $K$  such that

$$K = \sum_{n \in \mathbb{N}^*} K_n f_n, \text{ with } c < |K_n| < C, \tag{5.79}$$

$$T\phi = \phi, \text{ in a suitable sense.} \tag{5.80}$$

Since the solutions of the closed-loop system are expected to live in  $C^0([0, \tau]; L_1^2)$  space, it is natural to start by considering that  $T\phi = \phi$  holds in  $L_1^2$  sense. For any given  $\{a_n\}_{n \in \mathbb{N}^*}$  verifying (5.78) we have that Lemma 5.1 (2) holds. Meanwhile the strong  $T\phi = \phi$  condition for (5.80) reads as

$$\sum_p a_p f_p = \phi = T\phi = \sum_n a_n h_n = \sum_n -a_n K_n q_n, \tag{5.81}$$

which is equivalent to

$$\phi = - \sum_n a_n K_n q_n. \tag{5.82}$$

However, we can not use Lemma 5.1 (3) with  $s = 0$  directly: because  $\phi \in H_1^{-\frac{1}{2}-}$  instead of  $L_1^2$ . Otherwise, formally the preceding equation admits a solution.



**5.3.2 On the uniqueness and existence of  $\{K_n\}_{n \in \mathbb{N}^*}$  such that  $T\phi = \phi$  in weaker space:  $H_1^{-1/2-}$**

In this subsection, we show that the condition (5.77) holds in  $H^{-1/2-}$ . We notice that

$$\sum_n a_n K_n q_n = \sum_n \frac{a_n K_n}{n} (n q_n). \quad (5.83)$$

As  $\{n q_n\}_{n \in \mathbb{N}^*}$  is a Riesz basis of  $H_1^{-1}$ , and as  $\phi$  also belongs to  $H_1^{-1}$ , Equation (5.82) can be solved in  $H_1^{-1}$  sense. Indeed, since the span of  $\{n q_n\}_{n \in \mathbb{N}^*}$  is dense in  $H_1^{-1}$ , and since  $\phi$  also belongs to  $H_1^{-1}$ , we can decompose  $\phi$  in the form given (5.82) with a unique  $\{a_n K_n/n\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*)$ .

As a consequence there exists  $K_n$  satisfying

$$\left\{ \frac{a_n K_n}{n} \right\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*)$$

such that (5.82) holds in  $H_1^{-1}$ . A priori, for  $a_n \sim 1$  and  $K_n \sim 1$  the above conditions hold. Next, it suffices to find the exact value of  $K_n$  to guarantee the growth conditions.

Moreover, let  $\varepsilon > 0$ , because  $\{n^{1/2+\varepsilon} q_n\}$  is a Riesz basis of  $H_1^{-1/2-\varepsilon}$ , and that  $\phi \in H_1^{-1/2-\varepsilon}$ , we know that the equation can be also solved in the  $H_1^{-1/2-\varepsilon}$  sense, and that the unique solution  $\{K_n\}_{\mathbb{N}^*}$  satisfies

$$\left\{ \frac{a_n K_n}{n^{1/2+\varepsilon}} \right\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*). \quad (5.84)$$

Therefore, we have found  $\{K_n\}_{n \in \mathbb{N}^*}$  such that  $T\phi = \phi$  is held in  $H^{-1/2-}$  sense, thus (5.77). We remark here that (5.84) is not enough to conclude that  $\{K_n\}_{n \in \mathbb{N}^*}$  are uniformly bounded, though, conversely, for any uniformly bounded sequence  $\{K_n\}_{n \in \mathbb{N}^*}$  the condition (5.84) holds. Indeed, let  $b_n := \log n$ , we can easily observe that

$$\left\{ \frac{b_n}{n^{1/2+\varepsilon}} \right\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*), \quad \forall \varepsilon > 0. \quad (5.85)$$

In the next section, we will prove that the sequence  $\{K_n\}_{n \in \mathbb{N}^*}$  that we have found in this section is actually uniformly bounded.

**REMARK 5.3.** *We also remark here that even if a sequence  $\{b_n\}_{n \in \mathbb{N}^*}$  satisfies*

$$\left\{ \frac{b_n}{n^\varepsilon} \right\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*), \quad \forall \varepsilon > 0, \quad (5.86)$$

*we are not able to conclude that the sequence  $\{b_n\}_{n \in \mathbb{N}^*}$  is uniformly bounded. For example, by defining*

$$M(n) = [e^n] + 1, \quad \forall n \in \mathbb{N}^*, \quad (5.87)$$

$$b_m := n, \quad \text{if } m = M(n), \quad (5.88)$$

$$b_m := 0, \quad \text{if } m \notin \{M(n) : n \in \mathbb{N}^*\}, \quad (5.89)$$

*we know that for any  $\varepsilon > 0$ ,*

$$\sum_{m \in \mathbb{N}^*} \left( \frac{b_m}{m^\varepsilon} \right)^2 = \sum_{m \in \{M(n) : n \in \mathbb{N}^*\}} \left( \frac{b_m}{m^\varepsilon} \right)^2 \leq \sum_{n \in \mathbb{N}^*} \left( \frac{n}{e^{\varepsilon n}} \right)^2 < +\infty. \quad (5.90)$$

*Clearly, such  $\{b_n\}_{n \in \mathbb{N}^*}$  verifying (5.86) is not uniformly bounded.*

### 5.3.3 On the uniform boundedness of the solution $\{K_n\}_{n \in \mathbb{N}^*}$ : boundedness of the transformation $T$ on $H_1^s$ with $s \in (-3/2, 3/2)$

In order to prove the condition (5.75), namely the uniform boundedness of the candidate  $\{K_n\}_{n \in \mathbb{N}^*}$ , we come back to the  $T\phi = \phi$  equation

$$\sum_n -a_n K_n q_n = \sum_n a_n f_n. \quad (5.91)$$

Thanks to the preceding Section, we have found a unique  $\{K_n\}_{n \in \mathbb{N}^*}$  verifying (5.84) such that equation (5.91) is held in  $H^{-1/2-}$  sense (in particular, in  $H^{-1}$  sense).

Motivated by Lemma 5.1 (7), we define  $c_n$  as

$$-a_n K_n =: \lambda + c_n. \quad (5.92)$$

Then the  $T\phi = \phi$  condition in  $H_1^{-1}$  sense is equivalent to

$$\sum_n (\lambda + c_n) q_n = \sum_n a_n f_n \text{ in } H_1^{-1}. \quad (5.93)$$

As  $\{nq_n\}_{n \in \mathbb{N}^*}$  is a Riesz basis of  $H_1^{-1}$ , we know that

$$\sum_n \lambda q_n = \lambda \sum_n \frac{1}{n} (nq_n) \in H_1^{-1}. \quad (5.94)$$

Thus,  $T\phi = \phi$  condition in  $H_1^{-1}$  sense is equivalent to

$$\sum_n c_n q_n = \sum_n (a_n f_n - \lambda q_n) \text{ in } H_1^{-1}. \quad (5.95)$$

Thanks to Lemma 5.1 (7) in the case  $r = 0$ , we know that the right hand side of (5.95) lives in  $L_1^2$ , thus

$$\sum_n c_n q_n = \sum_n (a_n f_n - \lambda q_n) \in L_1^2. \quad (5.96)$$

Considering the fact that  $\{q_n\}$  is a Riesz basis of  $L_1^2$  (Lemma 5.1 (3) for the case  $m = 0, s = 0$ ), there exist a unique sequence  $\{c_n\}_{n \in \mathbb{N}^*}$  such that the preceding equation holds in  $L_1^2$  space (*i.e.* the left hand side series converges in  $L_1^2$  to the right hand side under the Cauchy sense), moreover,

$$\{c_n\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*). \quad (5.97)$$

Conversely, for such  $\{c_n\}_{n \in \mathbb{N}^*}$  belonging to  $l^2(\mathbb{N}^*)$ , recall that the precise value of  $K_n$  is given by

$$K_n = -\frac{\lambda + c_n}{a_n}. \quad (5.98)$$

We also know from the definition of  $\{c_n\}_{n \in \mathbb{N}^*}$  that equation (5.96) holds in  $L_1^2$ , thus it automatically holds in  $H_1^{-1}$ . Thanks to (5.92), we have

$$-\sum_n a_n K_n q_n = \sum_n a_n f_n \text{ in } H_1^{-1}. \quad (5.99)$$

Furthermore, since  $\{c_n\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*) \subset l^\infty(\mathbb{N}^*)$ , we know that  $K_n$  is uniformly bounded.

Now, from Lemma 5.1 (4) in the case  $m = 0$ , this allows us to deduce that  $T$  is bounded on  $H_1^s$  for  $s \in (-3/2, 3/2)$ . Thus

$$\phi = \sum_{n \in \mathbb{N}^*} a_n f_n \text{ in } H_1^{-1}, \quad (5.100)$$

$$T\phi = T\left(\sum_{n \in \mathbb{N}^*} a_n f_n\right) = \sum_{n \in \mathbb{N}^*} T(a_n f_n) = \sum_{n \in \mathbb{N}^*} -a_n K_n q_n \text{ in } H_1^{-1}, \quad (5.101)$$

which, together with (5.99), imply

$$T\phi = \phi \text{ in } H_1^{-1}. \quad (5.102)$$

Moreover, even for any  $r \in [0, 1/2)$ , thanks to Lemma 5.1 (7), we know that

$$\sum_n (c_n n^r)(n^{-r} q_n) = \sum_n c_n q_n = \sum_n (a_n f_n - \lambda q_n) \in H_1^r. \quad (5.103)$$

By Lemma 5.1 (3) in the case  $m = 0$  and  $s = r$ , we know that  $\{n^{-r} q_n\}$  is a Riesz basis of  $H_1^r$ , thus the same reasoning as the case  $r = 0$  implies that

$$\{c_n n^r\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*), \quad \forall r \in [0, 1/2), \quad (5.104)$$

and that

$$T\phi = \phi \text{ in } H_1^{r-1}. \quad (5.105)$$

Finally, by applying Lemma 5.1 (4) in the case  $m = 0$ , combined with the fact that  $\{K_n\}_{n \in \mathbb{N}^*}$  is uniformly bounded, we deduce that  $T$  is a bounded operator from  $L_1^2$  to itself. Moreover,  $T$  is also a bounded operator from  $H_1^s$  to itself with  $s \in (-3/2, 3/2)$ .

**REMARK 5.4.** *We notice that the equation (5.91) is equivalent to*

$$\sum_{n \in \mathbb{N}^*} (a_n K_n) \frac{1}{p^2 + \lambda - n^2} = -1, \quad \forall p \in \mathbb{N}^*. \quad (5.106)$$

*Hence the value of  $\{-a_n K_n\}_{n \in \mathbb{N}^*} = \{\lambda + c_n\}_{n \in \mathbb{N}^*}$  is independent of the choice of  $\{a_n\}_{n \in \mathbb{N}^*}$  satisfying*

$$c \leq |a_n| \leq C.$$

### 5.3.4 On the operator equality

In the previous subsection we proved the following lemma

**LEMMA 5.5.** *By the choice of  $\{a_n\}_{n \in \mathbb{N}^*}$  and  $\{K_n\}_{n \in \mathbb{N}^*}$  from Section 5.3.3, we have*

$$T\phi = \phi \text{ in } H_1^{-1/2-\varepsilon}, \quad \forall \varepsilon > 0, \quad (5.107)$$

*in particular it holds in  $H_1^{-1}$  sense. However, it does not hold in  $L_1^2$  sense.*

Now, we show the operator equality and the spaces in which this equality holds

**LEMMA 5.6.** *Let  $r \in (-1/2, 1/2)$ . By the choice of  $\{a_n\}_{n \in \mathbb{N}^*}$  and  $\{K_n\}_{n \in \mathbb{N}^*}$  from Section 5.3.3, for any  $\varphi \in H_1^{r+1}$  we have*

$$(TA + T\phi K)\varphi = (AT - \lambda T)\varphi \text{ in } H_1^{r-1}. \quad (5.108)$$

*In particular we can consider  $r = 0$  then the equality holds in  $H_1^{-1}$  sense. Moreover, the range  $r \in (-1/2, 1/2)$  is sharp.*

*Proof.* Recalling that, thanks to Section 5.3.3,  $T$  is a bounded operator from  $H^s$  to itself with  $s \in (-3/2, 3/2)$ . Though so far we do not know whether  $T$  is an isomorphism on  $H^s$  –this will be proved in the next section– this is now sufficient to conclude the required operator equality.

At first we consider each candidate of the operator equality (5.108) separately.

- i)  $TA\varphi$ : we know that  $A\varphi \in H_1^{r-1}$ . Notice that  $r - 1 \in (-3/2, -1/2)$ , which, combined with Lemma 5.1 (4) in the case  $m = 0$  and  $s := r - 1$ , implies that  $TA\varphi \in H_1^{r-1}$ . Moreover, as  $A : H_1^{r+1} \rightarrow H_1^{r-1}$  and  $T : H_1^{r-1} \rightarrow H_1^{r-1}$  are bounded, the linear operator  $TA : H_1^{r+1} \rightarrow H_1^{r-1}$  is bounded.  
Concerning the sharpness of  $r \in (-1/2, 1/2)$ , as we can see above, if  $r$  is chosen such that  $r \leq -1/2$  then the operator  $T$  is no longer bounded on  $H_1^{r-1}$ , thus the linear operator  $TA : H_1^{r+1} \rightarrow H_1^{r-1}$  is not bounded.
- ii)  $T\phi K\varphi$ : given that  $\varphi \in H_1^{r+1}$  with  $r > -1/2$ , and that  $K : H_1^{r+1} \rightarrow \mathbb{R}$  is a bounded operator, we know that  $|K\varphi| < +\infty$  is well-defined. Then, thanks to Lemma 5.5,  $T\phi K\varphi \in H_1^{-1/2-} \subset H_1^{r-1}$ . Moreover, as  $T\phi : \mathbb{R} \rightarrow H_1^{r-1}$  and  $K$  are bounded, the linear operator  $T\phi K : H_1^{r+1} \rightarrow H_1^{r-1}$  is bounded.
- iii)  $T\varphi$ : thanks to Lemma 5.1 (4) in the case  $m = 0$  and  $s = r + 1 \in (1/2, 3/2)$ , and the fact that  $\varphi \in H_1^{r+1}$ , we know that  $T\varphi \in H_1^{r+1} \subset H_1^{r-1}$ . Clearly,  $T : H_1^{r+1} \rightarrow H_1^{r-1}$  is bounded (in fact  $T$  is even bounded from  $H_1^{r+1}$  in itself).
- iv)  $AT\varphi$ : because  $T\varphi \in H_1^{r+1}$ , we have that  $AT\varphi \in H_1^{r-1}$ . As  $T : H_1^{r+1} \rightarrow H_1^{r+1}$  and  $A : H_1^{r+1} \rightarrow H_1^{r-1}$  are bounded, the linear operator  $AT : H_1^{r+1} \rightarrow H_1^{r-1}$  is bounded.  
Concerning the sharpness of  $r \in (-1/2, 1/2)$ , as we can see above, if  $r$  is chosen such that  $r \geq 1/2$  then the operator  $T$  is no longer bounded on  $H_1^{r+1}$ , thus the linear operator  $AT : H_1^{r+1} \rightarrow H_1^{r-1}$  is not bounded.

By adapting i)–iv) we know that

$$TA, T\phi K, AT, \lambda T : H_1^{r+1} \rightarrow H_1^{r-1},$$

are bounded linear operators. We also know from the choice of  $K_n$  that  $(TA + T\phi K - AT + \lambda T)f_n = 0$ . Indeed, according to the definition of  $q_n$  given in Section 5.1, we have in the  $H^{r-1}$  space

$$\begin{aligned} (TA + T\phi K - AT + \lambda T)f_n &= (-A - n^2 + \lambda)Tf_n + \phi K_n, \\ &= K_n(\phi - (-A - n^2 + \lambda)q_n), \\ &= 0. \end{aligned}$$

Since the finite linear combinations of  $\{f_n\}_{n \in \mathbb{N}^*}$  are dense in  $H_1^{r+1}$ , then for any  $\varphi \in H_1^{r+1}$  we have that,

$$(TA + T\phi K - AT + \lambda T)\varphi = 0 \text{ in } H_1^{r-1}.$$

Indeed, for any  $\varepsilon$  we can find a finite linear combination of  $\{f_n\}_{n \in \mathbb{N}^*}$ ,  $\tilde{f}$ , such that

$$\|\varphi - \tilde{f}\|_{H_1^{r+1}} \leq \varepsilon. \quad (5.109)$$

Then

$$(TA + T\phi K - AT + \lambda T)\varphi = (TA + T\phi K - AT + \lambda T)(\varphi - \tilde{f}) \text{ in } H_1^{r-1},$$

thus by the boundedness of  $TA + T\phi K - AT + \lambda T$  from  $H_1^{r+1}$  to  $H_1^{r-1}$  we know that

$$\|(TA + T\phi K - AT + \lambda T)\varphi\|_{H_1^{r-1}} \leq C\varepsilon. \quad (5.110)$$

□

#### 5.4 Invertibility of the transformation $T$ on the space $H_1^{-1}$ .

So far we know, thanks to Lemma 5.1, that  $T$  is a bounded operator on  $H_1^s$ . But we do not know yet that it is an isomorphism. To prove this we will first prove in this section that  $T$  is invertible on  $H_1^{-1}$  and then, we will show in Section 5.5 that it implies that it is invertible also on  $H^s$  for  $s \in (-3/2, 3/2)$ .

We start by the following lemma.

**LEMMA 5.7.** *Let  $r \in [0, 1/2)$ . The operator*

$$\begin{aligned} \tilde{T} &:= T - Id : L_1^2 \rightarrow H_1^r, \\ (\text{resp. } \tilde{T} &:= T - Id : H_1^{-1} \rightarrow H_1^{-1+r}), \end{aligned}$$

*is a continuous operator. Hence  $\tilde{T}$  is a compact operator from  $L_1^2$  to itself (resp. from  $H_1^{-1}$  to itself), and  $T$  is a Fredholm operator on  $L_1^2$  (resp. on  $H_1^{-1}$ ).*

For that purpose we need the following lemma.

**LEMMA 5.8.** *Let  $r \in [0, 1/2)$ . There exists a constant  $C > 0$  such that*

$$\left\| \sum_n b_n \left( q_n - \frac{a_n f_n}{\lambda} \right) \right\|_{H_1^r}^2 \leq C \sum_n b_n^2. \quad (5.111)$$

*Proof of Lemma 5.8.* In fact, we have from the definition of  $\{q_n\}_{n \in \mathbb{N}^*}$ , the Riesz basis property of  $\{f_p\}_{p \in \mathbb{N}^*}$  and the uniform boundedness of  $\{a_p\}_{p \in \mathbb{N}^*}$  by assumption.

$$\begin{aligned} \left\| \sum_n b_n \left( q_n - \frac{a_n f_n}{\lambda} \right) \right\|_{H_1^r}^2 &= \left\| \sum_p a_p f_p \sum_{n \neq p} \frac{b_n}{p^2 + \lambda - n^2} \right\|_{H^r}^2, \\ &\leq C \sum_p p^{2r} \left( \sum_{n \neq p} \frac{b_n}{p^2 + \lambda - n^2} \right)^2, \\ &\leq C \sum_p p^{2r} \left( \sum_{n \neq p} b_n^2 \right) \left( \sum_{n \neq p} \frac{1}{(p^2 + \lambda - n^2)^2} \right), \end{aligned}$$

$$\begin{aligned}
&\leq C \left( \sum_n b_n^2 \right) \sum_p p^{2r} \left( \sum_{n \neq p} \frac{1}{(p^2 + \lambda - n^2)^2} \right), \\
&\leq C \sum_n b_n^2,
\end{aligned}$$

where in the last step we have used Lemma 5.1 (5) (the case  $m = 0$  and  $a_n = 1$ ).  $\square$

*Proof of Lemma 5.7.* i) We first prove the results on  $L_1^2$ . We notice that

$$\tilde{T}f_n = -K_n q_n - f_n = \frac{\lambda + c_n}{a_n} q_n - f_n = \left( \frac{\lambda}{a_n} q_n - f_n \right) + \frac{c_n}{a_n} q_n.$$

Therefore, for any  $f = \sum_n b_n f_n \in L_1^2$  satisfying  $\{b_n\}_{n \in \mathbb{N}^*} \in l_{n \in \mathbb{N}^*}^2$ , we have that

$$\left\| \tilde{T} \sum_n b_n f_n \right\|_{H_1^r}^2 = \left\| \sum_n \frac{b_n}{a_n} (\lambda q_n - a_n f_n) + \frac{b_n}{a_n} c_n q_n \right\|_{H_1^r}^2.$$

On the one hand, according to Lemma 5.1 (5) the case that  $m = 0$ , we know that

$$\left\| \sum_n \frac{b_n}{a_n} (\lambda q_n - a_n f_n) \right\|_{H_1^r}^2 \leq C \left( \sum_n b_n^2 \right) \left( \sum_n \|\lambda q_n - f_n\|_{H_1^r}^2 \right) \leq C \sum_n b_n^2.$$

On the other hand, according to Lemma 5.1 (3) the case that  $m = 0$  and  $s = r$ , we know that  $\{n^{-r} q_n\}$  is a Riesz basis of  $H_1^r$ . Thus

$$\begin{aligned}
\left\| \sum_n \frac{b_n}{a_n} c_n q_n \right\|_{H_1^r}^2 &= \left\| \sum_n (b_n a_n^{-1} c_n n^r) (n^{-r} q_n) \right\|_{H_1^r}^2, \\
&\leq C \sum_n (b_n a_n^{-1} c_n n^r)^2, \\
&\leq C \sum_n (b_n)^2 (c_n n^r)^2, \\
&\leq C \sum_n b_n^2,
\end{aligned}$$

where we have used the fact that  $\{c_n n^r\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*) \subset l^\infty(\mathbb{N}^*)$  which is proved in Section 5.3.3, equation (5.104). Hence, the operator  $\tilde{T} : L_1^2 \rightarrow H_1^r$  is continuous, which further turned out to be a compact operator on  $L_1^2$ .

ii) Next we provide the proof on  $H_1^{-1}$  space. Therefore, for any

$$f = \sum_n b_n f_n = \sum_n (b_n/n) (n f_n) \in H_1^{-1}.$$

we know that  $\{b_n/n\}_{n \in \mathbb{N}^*} \in l_{n \in \mathbb{N}^*}^2$ , thus

$$\left\| \tilde{T} \sum_n b_n f_n \right\|_{H_1^{-1+r}}^2 = \left\| \sum_n \frac{b_n}{a_n} (\lambda q_n - a_n f_n) + \frac{b_n}{a_n} c_n q_n \right\|_{H_1^{-1+r}}^2.$$

On the one hand, according to Lemma 5.1 (6) the case that  $m = 0$ , we know that

$$\begin{aligned}
& \left\| \sum_n \frac{b_n}{na_n} (n(\lambda q_n - a_n f_n)) \right\|_{H_1^{-1+r}}^2 \\
& \leq C \left( \sum_n \left( \frac{b_n}{n} \right)^2 \right) \left( \sum_n \|n(\lambda q_n - a_n f_n)\|_{H_1^{-1+r}}^2 \right) \\
& \leq C \sum_n \left( \frac{b_n}{n} \right)^2 \\
& \leq C \|f\|_{H_1^{-1}}^2.
\end{aligned}$$

On the other hand, according to Lemma 5.1 (3) the case that  $m = 0$  and  $s = -1 + r$ , we know that  $\{n^{1-r}q_n\}$  is a Riesz basis of  $H_1^{-1+r}$ . Thus

$$\begin{aligned}
& \left\| \sum_n b_n a_n^{-1} c_n q_n \right\|_{H_1^{-1+r}}^2 = \left\| \sum_n (b_n a_n^{-1} c_n n^{r-1}) (n^{1-r} q_n) \right\|_{H_1^{-1+r}}^2, \\
& \leq C \sum_n \left( \frac{b_n}{n} a_n^{-1} c_n n^r \right)^2, \\
& \leq C \sum_n \left( \frac{b_n}{n} \right)^2 \\
& \leq C \|f\|_{H_1^{-1}}^2,
\end{aligned}$$

where, again, we have used the fact that  $\{c_n n^r\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*) \subset l^\infty(\mathbb{N}^*)$ .

Therefore,  $T$  is a continuous operator from  $H_1^{-1}$  to  $H_1^{-1+r}$  with  $r \in (0, 1/2)$ . Since the inclusion  $H_1^{-1+r} \rightarrow H_1^{-1}$  is compact, we conclude the proof of Lemma 5.7.  $\square$

We now prove the following invertibility result on  $H_1^{-1}$ .

**LEMMA 5.9.**  *$T$  is invertible from  $H_1^{-1}$  into itself.*

*Proof of Lemma 5.9.* Since  $T : H_1^{-1} \rightarrow H_1^{-1}$  is a Fredholm operator of index 0 (i.e. can be written as the sum of the identity and a compact operator), thanks to Lemma 5.7, it suffices to show that  $\text{Ker } T^* = \{0\}$  to conclude the invertibility. By the definition of Fredholm operator, we know that both  $\text{Ker } T$  and  $\text{Coker } T$  have finite dimension. Moreover, since the dual of a Fredholm operator is still a Fredholm operator (coming from the fact that the dual of a compact operator is still compact, namely Schauder's theorem),  $\text{Ker } T^*$  is of finite dimension. More precisely, we have

$$\dim \text{Ker } T = \dim \text{Coker } T = \dim \text{Ker } T^* = \dim \text{Coker } T^* < +\infty.$$

In the following we mimic the method of [11] to show that for the operator  $T : H_1^{-1} \rightarrow H_1^{-1}$  we have that  $\text{Ker } T^* = \{0\}$ . The proof is divided by three steps

1) There exists  $\rho \in \mathbb{C}$  such that

$$\begin{aligned}
& A + \phi_1 K + \lambda Id + \rho Id : H_1^1 \rightarrow H_1^{-1} \text{ is invertible,} \\
& A + \rho Id : H_1^1 \rightarrow H_1^{-1} \text{ is invertible.}
\end{aligned}$$

2) For such a complex number  $\rho$ ,  $\text{Ker } T^*$  is stable under  $(A + \rho Id)^{-1}$ , thus there exists an eigenfunction to  $(A + \rho Id)^{-1}$ , *i.e.* there exists  $(h, \mu)$  with  $\mu \neq 0$  such that  $(A + \rho Id)^{-1}h = \mu h$ . Hence  $h = f_k$  for some  $k \in \mathbb{N}^*$ .

3) By adapting the  $T\phi_1 = \phi_1$  condition, we show that  $h = f_k$  is not in  $\text{Ker } T^*$ .

1) As the spectrum of  $A + \rho Id$  is clear and discrete, in the following we mainly focus on the study of the spectrum of  $A + \phi K + \lambda Id + \rho Id$ . By denoting  $z := \lambda + \rho$ , we try to investigate the invertibility of  $Id + A^{-1}\phi K + zA^{-1}$  in the  $H_1^1$  space, with some  $z \in \mathbb{C}$ , which further gives some  $\rho = z - \lambda$  such that the investigated two operators are invertible. For such a purpose we consider two cases.

i) If  $K(A^{-1}\phi) \neq -1$ , then we know that the bounded operator  $Id + A^{-1}\phi K$  is invertible. In fact, for any  $f \in H_1^1$ , we can check that

$$\varphi := f - \frac{A^{-1}\phi(Kf)}{1 + K(A^{-1}\phi)} \in H_1^1, \quad (5.112)$$

solves

$$(Id + A^{-1}\phi K)\varphi = f. \quad (5.113)$$

Since  $Id + A^{-1}\phi K$  is invertible, and since  $A^{-1}$  is also a bounded operator (indeed even compact) on  $H_1^1$ , according to the openness of invertible operator, there exists a small ball  $B_\varepsilon(0)$  around 0, such that

$$(Id + A^{-1}\phi K) + zA^{-1} \text{ is invertible in } H_1^1, \forall z \in B_\varepsilon(0). \quad (5.114)$$

ii) If  $K(A^{-1}\phi) = -1$ , then we can easily verify that 0 is an eigenvalue of  $Id + A^{-1}\phi K$  with multiplicity 1 and the eigenspace is generated by  $A^{-1}\phi$ .

According to the perturbation theory, see for example [34], there exist small open neighborhoods  $\Omega$  and  $\tilde{\Omega}$  of  $0 \in \mathbb{C}$  satisfying

$$\begin{aligned} (Id + A^{-1}\phi K + zA^{-1})y(z) &= \lambda(z)y(z), \\ y(z) : z \in \Omega &\mapsto y(z) \in H_1^1 \text{ is holomorphic,} \\ \lambda(z) : z \in \Omega &\mapsto \lambda(z) \in \tilde{\Omega} \subset \mathbb{C} \text{ is holomorphic,} \\ \lambda(0) &= 0, \quad y_0 := y(0) = A^{-1}\phi, \end{aligned}$$

in such a way that for any  $z \in \Omega$ ,  $\lambda(z)$  is the unique eigenvalue inside  $\tilde{\Omega}$ . Recall that  $\lambda(0) = 0$  and that the zero points of any non-trivial (not identically zero) holomorphic function are isolated.

If further there exists a small neighborhood  $\omega$  of  $0 \in \mathbb{C}$  such that  $\lambda(z) = 0$  for any  $z \in \omega$ , then we are able to decompose  $y(z) \in H_1^1$  in power series as

$$y(z) = y_0 + \sum_{k=1}^{+\infty} z^k y_k, \quad y_k \in H_1^1. \quad (5.115)$$

By matching the coefficients of the power series, we get

$$(Id + y_0 K)y_k + A^{-1}y_{k-1} = 0, \quad \forall k \geq 1. \quad (5.116)$$

By adapting  $A^{-1}$  and  $K$  to the preceding equation we conclude that

$$K(A^{-1}y_k) = 0, \quad \forall k \geq 0. \quad (5.117)$$



Then by successively adapting  $A^{-1}$  and  $K$  to the same equation we arrive at

$$K(A^{-n}y_k) = 0, \forall k \geq 0, \forall n \geq 1, \quad (5.118)$$

which in particular yields

$$K(A^{-n}y_0) = 0, \forall n \geq 1. \quad (5.119)$$

The preceding equality implies

$$\sum_n \frac{a_n K_n}{n^{2l}} = 0, \forall l \geq 2. \quad (5.120)$$

Again using the holomorphic function technique, we conclude that  $a_n K_n = 0$ , which is a contradiction.

Therefore, there exists a sequence of  $\{z_k\}$  converging to 0 such that  $\lambda(z_k) \neq 0$ . Indeed, thanks to the fact that  $\lambda$  is holomorphic with 0 being a zero point, in this case we even have that  $\lambda(z) \neq 0$  in  $\omega_1 \setminus \{0\}$  with  $\omega_1$  being a small neighborhood of 0. Then, since  $\lambda(z_k) \neq 0$  and  $\lambda(z_k)$  is the unique eigenvalue inside  $\tilde{\Omega}$ , for  $z_k$  sufficiently close to 0 we know that  $Id + A^{-1}\phi K + z_k A^{-1}$  is invertible. Thus  $Id + A^{-1}\phi K + \lambda A^{-1} + (z_k - \lambda)A^{-1}$  is invertible. As the spectrum of  $A + \rho Id$  is discrete, we can find a  $\rho$  (more precisely, some  $z_k - \lambda$ ), such that both  $A + \rho Id$  and  $A + \phi K + \lambda Id + \rho Id$  are invertible.

2) Because

$$T(A + \phi K + \lambda Id + \rho Id) = AT + \rho T : H_1^1 \rightarrow H_1^{-1}, \quad (5.121)$$

we have

$$(A + \rho Id)^{-1}T = T(A + \phi K + \lambda Id + \rho Id)^{-1} : H_1^{-1} \rightarrow H_1^1. \quad (5.122)$$

Suppose that  $h \in \text{Ker } T^*$ , then for any  $\varphi \in H_1^{-1}$  we deduce from the above operator equality that

$$\begin{aligned} 0 &= \langle (A + \rho Id)^{-1}T\varphi - T(A + \phi K + \lambda Id + \rho Id)^{-1}\varphi, h \rangle_{H_1^{-1}}, \\ &= \langle \varphi, T^*(A + \rho Id)^{-1}h \rangle_{H_1^{-1}} - \langle (A + \phi K + \lambda Id + \rho Id)^{-1}\varphi, T^*h \rangle_{H_1^{-1}}, \\ &= \langle \varphi, T^*(A + \rho Id)^{-1}h \rangle_{H_1^{-1}}. \end{aligned}$$

Since  $\varphi \in H_1^{-1}$  is chosen arbitrary, we know that  $(A + \rho Id)^{-1}h \in \text{Ker } T^*$ , thus

$$(A + \rho Id)^{-1} : \text{Ker } T^* \rightarrow \text{Ker } T^*. \quad (5.123)$$

Suppose that  $\text{Ker } T^*$  is not reduced to  $\{0\}$ . Therefore, because the space  $\text{Ker } T^*$  is of finite dimension and not reduced to  $\{0\}$  we can find an eigenfunction  $(h, \mu)$ ,  $h \neq 0$  and  $\mu \neq 0$ , such that

$$(A + \rho Id)^{-1}h = \mu h \text{ and } h \in \text{Ker } T^*. \quad (5.124)$$

We immediately deduce that  $h$  is an eigenfunction of  $A = \Delta$  in  $H_1^{-1}$ , thus there exists  $k \in \mathbb{N}^*$  and  $C \neq 0$  such that  $h = Cf_k$ . Indeed, we know from the definition of  $h$  that  $h = \mu(A + \rho Id)h$ , hence  $Ah = (1 - \rho)/\mu h$ , and  $h$  is a eigenfunction of  $A$ . We notice that the subspaces of  $H_1^1$  that are also eigenspaces of  $\Delta$  have dimension 1 (the eigenvalues are not degenerate in  $H_1^1$ ). In particular the dimension of the eigenspace of  $(1 - \rho)/\mu$  is one, and therefore  $h = Cf_k$  for some  $k \in \mathbb{N}^*$ .

3) By the definition of  $h = f_k \in \text{Ker } T^*$ , we know that

$$\langle T\varphi, f_k \rangle_{H_1^{-1}} = 0, \quad \forall \varphi \in H_1^{-1}. \quad (5.125)$$

Thanks to the fact that  $T\phi = \phi$  in  $H_1^{-1}$  sense, we can take  $\varphi := \phi$  to achieve

$$0 = \langle T\phi, f_k \rangle_{H_1^{-1}} = \langle \phi, f_k \rangle_{H_1^{-1}} = \frac{a_k}{k^2}, \quad (5.126)$$

which is in contradiction with the fact that  $a_n \neq 0$ .  $\square$

### 5.5 The non-zeroneess of the solution $\{K_n\}_{n \in \mathbb{N}^*}$ : invertibility of the transformation $T$ on $H_1^s$ for any $s \in (-3/2, 3/2)$

In the previous part we have proved the invertibility of the transformation  $T$  on  $H^{-1}$ . We will now show that this implies

$$K_n \neq 0, \quad \forall n \in \mathbb{N}^*,$$

which will in turn imply the isomorphism property in  $H_1^s$  for any  $s \in (-3/2, 3/2)$  thanks to Lemma 5.1.

Recall that

$$T : nf_n \mapsto -K_n(nq_n), \quad (5.127)$$

is a bounded operator from  $H_1^{-1}$  to itself. Suppose by contradiction that for some  $n_0$  we have that  $K_{n_0} = 0$ . As  $T$  is an isomorphism from  $H_1^{-1}$  in itself there exists  $h \in H_1^{-1}$  such that  $Th = n_0q_{n_0}$ . Thus there exists  $\{d_n\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*)$  such that

$$h = \sum_n d_n(nf_n) \in H_1^{-1}. \quad (5.128)$$

Then

$$Th = \sum_n -d_n K_n(nq_n) \quad (5.129)$$

which converges absolutely as  $\{nq_n\}_{n \in \mathbb{N}^*}$  is a Riesz basis of  $H_1^{-1}$ . From assumption  $Th = n_0q_{n_0}$ , hence  $d_{n_0}K_{n_0} = 1$ , which gives a contradiction. Consequently, for every  $n \in \mathbb{N}^*$  we know that  $K_n \neq 0$ , thus  $-a_n K_n \neq 0$ .

We also know that

$$-a_n K_n = \lambda + c_n \text{ with } \{c_n\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*), \quad (5.130)$$

which implies that at high frequency  $\{|a_n K_n|\}_{n \geq M}$  is uniformly away from 0 and also uniformly bounded. Therefore,  $\{|a_n K_n|\}_{n \in \mathbb{N}^*}$  is uniformly away from 0 and also uniformly bounded. Hence

$$c < |K_n| < C, \quad \forall n \in \mathbb{N}^*. \quad (5.131)$$

By applying Lemma 5.1 (4) in the case  $m = 0$  and  $s \in (-3/2, 3/2)$ , we get that  $T$  is an isomorphism from  $H_1^s$  to itself for any  $s \in (-3/2, 3/2)$ . The special case  $s = 0$  corresponds to the invertibility on the space  $L_1^2$ .

## 5.6 Proof of Proposition 4.1

*Proof.* The proof of Proposition 4.1 is a consequence of the above subsections: we choose  $K_n$  as in Section 5.3.3. We know from Section 5.3.3 that  $K$  is a bounded functional on  $H_1^{1/2+}$  and from Section 5.5 we know that there exists  $c > 0$  and  $C > 0$  such that

$$c < |K_n| < C, \quad \forall n \in \mathbb{N}^*.$$

Consequently, from Lemma 5.1 that  $T$  is an isomorphism on  $H_1^s$  for any  $s \in (-3/2, 3/2)$ . Moreover from Lemma 5.6 we know that for any  $r \in (-1/2, 1/2)$ , and  $\varphi \in H_1^{r+1}$  we have

$$(TA + TBK)\varphi = (AT - \lambda T)\varphi \text{ in } H_1^{r-1},$$

and from Lemma 5.5

$$T\phi_1 = \phi_1 \text{ in } H_1^{-1/2-}. \quad (5.132)$$

Finally from (5.104) and the definition of  $\{c_n\}_{n \in \mathbb{N}^*}$  given in (5.92), for any  $r \in [0, 1/2)$  we have that

$$\{(\lambda + a_n K_n)n^r\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*).$$

Hence the operator  $K$  and  $T$  satisfies the properties announced in Proposition 4.1 for  $m = 0$ . Thanks to Lemma 5.1 the same can be done identically in the case  $m \neq 0$ . This ends the proof of Proposition 4.1.  $\square$

## 6 Proof of Lemma 4.4, Lemma 4.8 and Corollary 4.5

In this section we prove Lemma 4.4, Lemma 4.8 and Corollary 4.5 dealing with the well-posedness of the closed-loop systems. We focus on the first one.

*Proof of Lemma 4.4.* By denoting  $S(t)$  the free heat flow evolution:  $y := S(t)y_0$  as the solution of

$$\begin{cases} \partial_t y - \Delta y = 0 \text{ in } \mathbb{T}, \\ y(0) = y_0, \end{cases} \quad (6.1)$$

we know from integration by parts that

$$\|S(t)y_0\|_{C^0([0,T];L_1^2)} \leq \|y_0\|_{L_1^2}, \quad (6.2)$$

$$\|S(t)y_0\|_{L^2([0,T];H_1^1)} \leq \|y_0\|_{L_1^2}. \quad (6.3)$$

We also know from integration by parts that the solution  $g$  of the inhomogeneous heat equation,

$$\begin{cases} \partial_t g - \Delta g = f \text{ in } \mathbb{T}, \\ g(0) = 0, \end{cases} \quad (6.4)$$

verifies

$$\|g(t)\|_{C^0([0,T];L_1^2)} \leq \|f\|_{L^2(0,T;H_1^{-1})}, \quad (6.5)$$

$$\|g(t)\|_{L^2([0,T];H_1^1)} \leq \|f\|_{L^2(0,T;H_1^{-1})}. \quad (6.6)$$

For any given  $T > 0$ . Let us define

$$\mathcal{B}_T := \{y \in C^0([0, T]; L_1^2) \cap L^2([0, T]; H_1^1)\}, \quad (6.7)$$

with its norm given by

$$\|y\|_{\mathcal{B}_T} := \|y\|_{C^0([0, T]; L_1^2)} + \|y\|_{L^2([0, T]; H_1^1)}. \quad (6.8)$$

For  $M > 0$ , we also define  $\mathcal{B}_T(M)$  as

$$\mathcal{B}_T(M) := \{y \in \mathcal{B}_T \mid \|y\|_{\mathcal{B}_T} \leq M\}. \quad (6.9)$$

Suppose that  $\|y_0\|_{L_1^2} = R$ . Then, we consider the map

$$\mathcal{L} : z \in \mathcal{B}_T(3R) \mapsto y \in \mathcal{B}_T \quad (6.10)$$

that is defined as

$$\begin{cases} \partial_t y - \Delta y = \phi K(z) \text{ in } \mathbb{T}, \\ y(0) = y_0. \end{cases} \quad (6.11)$$

We immediately knows that  $y \in \mathcal{B}_T$ . However, in order to show that  $\mathcal{L}$  is a contraction on  $\mathcal{B}_T(3R)$  we need more delicate estimates. Indeed, we benefit from the fact that  $K$  is a functional on  $H_1^{3/4}$  instead of on  $H_1^1$ .

We know that the solution  $y = \mathcal{L}z$  verifies

$$\begin{aligned} \|y\|_{\mathcal{B}_T} &\leq 2R + 2\|\phi(Kz)\|_{L^2(0, T; H_1^{-1})}, \\ &\leq 2R + C\|Kz\|_{L^2(0, T)}, \\ &\leq 2R + C\|z\|_{L^2(0, T; H_1^{\frac{3}{4}})}, \\ &\leq 2R + CT^{\frac{1}{8}}\|z\|_{L^{\frac{8}{3}}(0, T; H_1^{\frac{3}{4}})}, \\ &\leq 2R + CT^{\frac{1}{8}}\|z\|_{\mathcal{B}_T}, \\ &\leq 2R + 3CRT^{\frac{1}{8}}, \end{aligned} \quad (6.12)$$

where we have used the following technical lemma.

**LEMMA 6.1.**

$$\|z\|_{L^{\frac{8}{3}}(0, T; H_1^{\frac{3}{4}})} \leq \|z\|_{L^\infty(0, T; L_1^2)}^{\frac{1}{4}} \|z\|_{L^2(0, T; H_1^1)}^{\frac{3}{4}} \leq \|z\|_{\mathcal{B}_T}. \quad (6.13)$$

*Proof.* As we know from Sobolev interpolation that

$$\|f\|_{H_1^{3/4}} \leq \|f\|_{L_1^2}^{\frac{1}{4}} \|f\|_{H_1^1}^{\frac{3}{4}}, \quad (6.14)$$

then further thanks Hölder inequality,

$$\|z\|_{L^{\frac{8}{3}}(0, T; H_1^{\frac{3}{4}})} \leq \| \|z\|_{L_1^2}^{\frac{1}{4}} \|z\|_{H_1^1}^{\frac{3}{4}} \|_{L^{\frac{8}{3}}(0, T)},$$

$$\begin{aligned}
&\leq \| \|z\|_{L^2_1}^{\frac{1}{4}} \|L^\infty(0,T)\| \| \|z\|_{H^1_1}^{\frac{3}{4}} \|L^{\frac{8}{3}}(0,T)\|, \\
&\leq \|z\|_{L^\infty(0,T;L^2_1)}^{\frac{1}{4}} \|z\|_{L^2(0,T;H^1_1)}^{\frac{3}{4}}, \\
&\leq \|z\|_{\mathcal{B}_T}.
\end{aligned}$$

□

Therefore, for  $T > 0$  sufficiently small we know that

$$\mathcal{L} : \mathcal{B}_T(3R) \rightarrow \mathcal{B}_T(3R). \quad (6.15)$$

Next, we show that, by choosing  $T$  even small if necessary, the map  $\mathcal{L}$  is actually a contraction map. For any  $z_1, z_2 \in \mathcal{B}_T(3R)$ , suppose that  $y_1 := \mathcal{L}z_1, y_2 := \mathcal{L}z_2$ , thus

$$\begin{cases} \partial_t y_1 - \Delta y_1 = \phi K(z_1) \text{ in } \mathbb{T}, \\ y_1(0) = y_0, \end{cases} \quad (6.16)$$

and

$$\begin{cases} \partial_t y_2 - \Delta y_2 = \phi K(z_2) \text{ in } \mathbb{T}, \\ y_2(0) = y_0. \end{cases} \quad (6.17)$$

This implies that  $w := y_1 - y_2 = \mathcal{L}(z_1 - z_2)$  verifies

$$\begin{cases} \partial_t w - \Delta w = \phi K(z_1 - z_2) \text{ in } \mathbb{T}, \\ w(0) = 0, \end{cases} \quad (6.18)$$

which further yields

$$\|\mathcal{L}(z_1 - z_2)\|_{\mathcal{B}_T} = \|w\|_{\mathcal{B}_T} \leq 2\|\phi K(z_1 - z_2)\|_{L^2(0,T;H^{-1}_1)} \leq CT^{\frac{1}{8}}\|z_1 - z_2\|_{\mathcal{B}_T}. \quad (6.19)$$

Therefore,  $\mathcal{L}$  is actually a contraction on  $\mathcal{B}_T(3R)$  for  $T$  sufficiently small. Banach fixed point theorem gives the existence and uniqueness of the solution in a small time interval.

Finally, it is standard to extend the solution to a large time domain. It actually suffices to show the existence on  $[0, 1]$ , thus it does not blow up in this domain. Concerning the solution  $y(t)$  of the system (4.22), integration by parts, which together with Sobolev interpolation and Young's inequality, yield,

$$\begin{aligned}
\frac{d}{dt} \|y(t)\|_{L^2_1}^2 &\leq -2\|y(t)\|_{H^1_1}^2 + C\|y(t)\|_{H^1_1} |Ky(t)|, \\
&\leq -2\|y(t)\|_{H^1_1}^2 + C\|y(t)\|_{H^1_1} \|y(t)\|_{H^{3/4}_1}, \\
&\leq -2\|y(t)\|_{H^1_1}^2 + C\|y(t)\|_{H^1_1}^{\frac{7}{4}} \|y(t)\|_{L^2_1}^{\frac{1}{4}}, \\
&\leq -2\|y(t)\|_{H^1_1}^2 + C\left(\frac{7\varepsilon}{8}\|y(t)\|_{H^1_1}^2 + \frac{1}{8\varepsilon^7}\|y(t)\|_{L^2_1}^2\right), \\
&\leq -\|y(t)\|_{H^1_1}^2 + C\|y(t)\|_{L^2_1}^2.
\end{aligned}$$

The preceding *a priori* estimate, to be combined with standard arguments, indicate the existence of solution on  $[0, 1]$  and further on  $[0, +\infty)$ . □

*Proof of Corollary 4.5.* Concerning Corollary 4.5, again, we only prove the case that  $k = 1$  as the other case that  $k = 2$  is similar. Observe that  $r - 1 < -1/2$ , meaning  $\phi_1 \in H_1^{r-1}$ . Meanwhile, the choice of  $r$  also tells us that  $K$  is bounded on  $H_1^{r+1}$ .

At first we investigate the related open-loop system, *i.e.* we replace  $K_1(y)$  by  $u(t) \in L_{loc}^2(0, +\infty)$ . The equation

$$\begin{cases} \partial_t y - \Delta y = \phi_1 u(t), \\ y(0) = y_0 \in H_1^r, \end{cases} \quad (6.20)$$

has a unique solution in the space (4.24), satisfying

$$\|y(t)\|_{C^0([0,T];H_1^r)}^2 \leq \|y_0\|_{H_1^r}^2 + \|\phi_1 u(t)\|_{L^2(0,T;H_1^{r-1})}^2, \quad (6.21)$$

$$\|y(t)\|_{L^2(0,T;H_1^{r+1})}^2 \leq \|y_0\|_{H_1^r}^2 + \|\phi_1 u(t)\|_{L^2(0,T;H_1^{r-1})}^2. \quad (6.22)$$

Indeed,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|y(t)\|_{H_1^r}^2 &= \langle y(t), \dot{y}(t) \rangle_{H_1^r} \\ &= -\|y(t)\|_{H_1^{r+1}}^2 + \langle y(t), u(t)\phi_1 \rangle_{H_1^r} \\ &\leq -\|y(t)\|_{H_1^{r+1}}^2 + \|y(t)\|_{H_1^{r+1}} \|u(t)\phi_1\|_{H_1^{r-1}} \\ &\leq -\|y(t)\|_{H_1^{r+1}}^2 + \frac{1}{2} \|y(t)\|_{H_1^{r+1}}^2 + \frac{1}{2} \|u(t)\phi_1\|_{H_1^{r-1}}^2 \\ &\leq -\frac{1}{2} \|y(t)\|_{H_1^{r+1}}^2 + \frac{1}{2} |u(t)|^2 \|\phi_1\|_{H_1^{r-1}}^2, \\ &\leq -\frac{1}{2} \|y(t)\|_{H_1^{r+1}}^2 + C|u(t)|^2, \end{aligned}$$

where we have used the fact that for  $h, g \in \mathcal{S}_1$  (thus extends to related Sobolev spaces)

$$\begin{aligned} h &:= \sum_{n \in \mathbb{N}^*} h_n \sin nx, \quad g := \sum_{n \in \mathbb{N}^*} g_n \sin nx, \\ \langle h, g \rangle_{H_1^r} &= \sum_{n \in \mathbb{N}^*} (n^r h_n)(n^r g_n) = \sum_{n \in \mathbb{N}^*} (n^{r+1} h_n)(n^{r-1} g_n) \leq \|h\|_{H_1^{r+1}} \|g\|_{H_1^{r-1}}, \end{aligned}$$

as well as that

$$\langle h, \Delta h \rangle_{H_1^r} = - \sum_{n \in \mathbb{N}^*} (n^r h_n)(n^{r+2} h_n) = - \sum_{n \in \mathbb{N}^*} (n^{r+1} h_n)(n^{r+1} h_n) = -\|h\|_{H_1^{r+1}}^2.$$

Next, for the closed-loop system (4.22), by the choice of  $r$  there exists some  $s_0, s_1$  satisfying  $r < 1/2 < s_0 < r+1$  such that  $H_1^{r+1} \subset H_1^{s_0} \subset H_1^r$ , and that  $K$  is bounded on  $H_1^{s_0}$ . Consequently, the same proof of Lemma 4.4 adapts here. For instance, in Lemma 4.4 the value of  $s_0$  is chosen as  $3/4$  (see Lemma 6.1 in Section 6). Indeed,

**LEMMA 6.2.** *For  $p := \frac{2}{s_0-r} > 2$ , we know that*

$$\|z\|_{L^p(0,T;H_1^{s_0})} \leq \|z\|_{L^\infty(0,T;H_1^r)}^{r+1-s_0} \|z\|_{L^2(0,T;H_1^{r+1})}^{s_0-r}. \quad (6.23)$$

*Proof.* Since

$$\|z\|_{H_1^{s_0}} \leq \|z\|_{H_1^{r+1-s_0}} \|z\|_{H_1^{s_0-r}}, \quad (6.24)$$

we know that

$$\begin{aligned} \|z\|_{L^p(0,T;H_1^{s_0})} &\leq \| \|z\|_{H_1^{r+1-s_0}} \|z\|_{H_1^{s_0-r}} \|_{L^p(0,T)}, \\ &\leq \| \|z\|_{H_1^{r+1-s_0}} \|_{L^\infty(0,T)} \| \|z\|_{H_1^{s_0-r}} \|_{L^p(0,T)}, \\ &= \|z\|_{L^\infty(0,T;H_1^{r+1-s_0})} \|z\|_{L^p(s_0-r)(0,T;H_1^{r+1})}. \end{aligned}$$

□

For any given  $T > 0$ . Let us define

$$\mathcal{B}_T^r := \{y \in C^0([0, T]; H_1^r) \cap L^2([0, T]; H_1^{r+1})\}, \quad (6.25)$$

with its norm given by

$$\|y\|_{\mathcal{B}_T^r} := \|y\|_{C^0([0,T];H_1^r)} + \|y\|_{L^2([0,T];H_1^{r+1})}, \quad (6.26)$$

and

$$\mathcal{B}_T^r(M) := \{y \in \mathcal{B}_T^r \mid \|y\|_{\mathcal{B}_T^r} \leq M\}. \quad (6.27)$$

Let  $\|y_0\|_{H_1^r} = R$ . For any  $T > 0$ , we further consider the map  $\mathcal{L}^r : z \in \mathcal{B}_T^r(3R) \mapsto y \in \mathcal{B}_T^r$  defined as

$$\begin{cases} \partial_t y - \Delta y = \phi K(z) \text{ in } \mathbb{T}, \\ y(0) = y_0. \end{cases} \quad (6.28)$$

When  $T$  is sufficiently small, by adapting

$$\|\phi(Kz)\|_{L^2(0,T;H_1^{r-1})} \leq C\|Kz\|_{L^2(0,T)} \leq C\|z\|_{L^2(0,T;H_1^{s_0})} \leq CT^{\frac{1}{2}-\frac{1}{p}}\|z\|_{L^p(0,T;H_1^{s_0})},$$

and the fixed point argument, there is a unique solution in the space (4.24) in a small interval of time  $[0, T]$ , more precisely as the unique fixed point of  $\mathcal{L}^r$  in  $\mathcal{B}_T^r(3R)$ .

Next, the *a priori* estimate further implies the existence of a unique solution in large interval of time. □

**REMARK 6.3.** For the case that  $k = 2$  instead of 1, similar approach leads to the same well-posedness result. The only place that needs to be (slightly) modified is that the inequalities (6.21)–(6.22) should be replaced by,

$$\begin{aligned} \|y(t)\|_{C^0([0,T];H_2^r)}^2 &\leq e^{2T} \left( \|y_0\|_{H_2^r}^2 + \|\phi_1 u(t)\|_{L^2(0,T;H_2^{r-1})}^2 \right), \\ \|y(t)\|_{L^2(0,T;H_2^{r+1})}^2 &\leq e^{2T} \left( \|y_0\|_{H_2^r}^2 + \|\phi_1 u(t)\|_{L^2(0,T;H_2^{r-1})}^2 \right). \end{aligned}$$

That is because

$$\langle y(t), \Delta y(t) \rangle_{H_2^r, H_2^r} = -\|y(t)\|_{H_2^{r+1}}^2 + (\langle y, f_0^2 \rangle)^2 \leq -\|y(t)\|_{H_2^{r+1}}^2 + \|y(t)\|_{H_2^r}^2,$$

which leads to

$$\frac{1}{2} \frac{d}{dt} \|y(t)\|_{H_2^r}^2 \leq -\|y(t)\|_{H_2^{r+1}}^2 + \|y(t)\|_{H_2^r}^2 + \langle y(t), u(t)\phi_1 \rangle_{H_2^r}$$

$$\leq \|y(t)\|_{H_2^r}^2 - \frac{1}{2}\|y(t)\|_{H_2^{r+1}}^2 + \frac{1}{2}\|u(t)\phi_1\|_{H_2^{r-1}}^2.$$

Thus

$$\|y(t)\|_{H_2^r}^2 + \|y\|_{L^2(0,t;H_2^{r+1})}^2 \leq e^{2t}\|y(0)\|_{H_2^r}^2 + e^{2t}\|u(t)\phi\|_{L^2(0,t;H_2^{r-1})}^2.$$

*Proof of Lemma 4.8.* Finally, we simply comment on the proof of Lemma 4.5 whose proof is essentially the same as the proof of Lemma 4.4.

Indeed, concerning the existence of the solution in a small interval of time, it suffices to treat the nonlinear term as a perturbation using Gagliardo–Nirenberg interpolation inequality, which is standard.

$$\begin{aligned} \|\partial_x(y^2)\|_{L^2(0,T;H^{-1})}^2 &\leq \|yy\|_{L^2(0,T;L^2)}^2, \\ &\leq \int_0^T \|y(t,\cdot)\|_{L^2}^2 \|y(t,\cdot)\|_{L^\infty}^2 dt, \\ &\leq C \int_0^T \|y(t,\cdot)\|_{L^2}^2 \|y(t,\cdot)\|_{H^{1/2}}^2 dt, \\ &\leq C \int_0^T \|y(t,\cdot)\|_{L^2}^3 \|y(t,\cdot)\|_{H^1} dt, \\ &\leq CT^{\frac{1}{2}} \|y\|_{C^0([0,T];L^2)}^3 \|y\|_{L^2(0,T;H^1)}. \end{aligned}$$

Next, classical energy estimates, benefiting from the fact that

$$\langle y, \partial_x(y^2) \rangle = 0, \tag{6.29}$$

lead to the existence of solution in large interval of time,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|y(t)\|_{L^2}^2 &= \langle y(t), \Delta y(t) + \phi_1 K_1(y(t)) + \phi_2 K_2(y(t)) \rangle_{H^1, H^{-1}} \\ &\leq -\|y(t)\|_{H^1}^2 + \|y(t)\|_{L^2}^2 + C\|y(t)\|_{H^1} (|K_1 y(t)| + |K_2 y(t)|), \\ &\leq -\frac{1}{2}\|y(t)\|_{H^1}^2 + C\|y(t)\|_{L^2}^2, \end{aligned}$$

where, slightly different from the calculation on  $L_1^2$  and  $H_1^1$ , for  $H^1$  norm (just as for  $H_2^1$  norm) we have

$$\langle y(t), \Delta y(t) \rangle_{H^1, H^{-1}} = -\|y(t)\|_{H^1}^2 + (\langle y, f_0^2 \rangle)^2 \leq -\|y(t)\|_{H^1}^2 + \|y(t)\|_{L^2}^2.$$

□

## 7 Proof of Theorems 1.1–1.4

### 7.1 The heat equation: on the well-posedness of the transformed system and its stability

In order to show Theorem 1.1, we need to show the well-posedness of the closed-loop system and its exponential stability. Also, to simplify the notation we assume that  $m = 0$  even though the exact same can be done with  $m \neq 0$ . Let  $y_0 \in L_1^2$ . Under the assumption of Theorem 1.1 we can define

$$B = (\phi_1, \phi_2), \quad K = (K_1, K_2)^T, \quad T = T_1 + T_2$$



given by Proposition 4.1 and Corollary 4.2.

Let  $\tau > 0$ . Considering the fact that for  $k \in \{1, 2\}$   $\phi_k \in H_k^{-1}$  and that  $K_1 : H_k^1 \rightarrow \mathbb{R}$ , for any  $\varphi(t) \in L^2(0, \tau; H^1)$  we have that  $BK\varphi(t) \in L^2(0, \tau; H^{-1})$ . Then, concerning the closed-loop system

$$y_t - \Delta y = BK(y), \quad y(0) = y_0, \quad (7.1)$$

from Corollary 4.7, we get a unique solution of the closed-loop system

$$y(t) \in C^0([0, \tau]; L^2(\mathbb{T}^1)) \cap L^2(0, \tau; H^1(\mathbb{T}^1)) \cap H^1(0, \tau; H^{-1}(\mathbb{T}^1)), \quad (7.2)$$

which indicates that the equation (7.1) (thus each item of it) is held in the  $L^2(0, \tau; H^{-1}(\mathbb{T}^1))$  sense.

Since the operator  $T$  is bounded in  $H^l(\mathbb{T}^1)$  space with  $l = -1, 0, 1$  (from Proposition 4.1 and Corollary 4.2), we know that

$$z(t) := Ty(t) \in C^0([0, \tau]; L^2(\mathbb{T})) \cap L^2(0, \tau; H^1(\mathbb{T})) \cap H^1(0, \tau; H^{-1}(\mathbb{T})). \quad (7.3)$$

Moreover, by applying  $T$  to (7.1) we know that

$$Ty_t - TAy = TBK(y) \quad \text{in } L^2(0, \tau; H^{-1}(\mathbb{T}^1)). \quad (7.4)$$

By applying Proposition 4.1 and Corollary 4.2 with  $s = 0$ , together with the fact that  $y(t) \in L^2(0, \tau; H_1^1(\mathbb{T}^1))$ , we arrive at

$$Ty_t = ATy - \lambda Ty, \quad \text{in } L^2(0, \tau; H^{-1}(\mathbb{T})). \quad (7.5)$$

Hence,

$$z_t - \Delta z + \lambda z = 0 \quad \text{in } L^2(0, \tau; H^{-1}(\mathbb{T})). \quad (7.6)$$

Consequently,

$$\frac{1}{2} \frac{d}{dt} \|z(t)\|_{L^2(\mathbb{T})}^2 = \langle z(t), \dot{z}(t) \rangle_{H^1, H^{-1}} \leq -\lambda \|z(t)\|_{L^2(\mathbb{T})}^2 \quad (7.7)$$

holds in the  $L^1(0, T)$  sense, which further implies the required decay property of the solution

$$\|z(t)\|_{L^2} \leq e^{-\lambda t} \|z(0)\|_{L^2}, \quad \forall t \in [0, \tau]. \quad (7.8)$$

Finally, as  $T$  is an isomorphism on  $L^2$  from Proposition 4.1 and Corollary 4.2, we conclude that

$$\|y(t)\|_{L^2} \leq C(\lambda) e^{-\lambda t} \|y^0\|_{L^2}, \quad \forall t \in [0, \tau], \quad (7.9)$$

Given that this is true for any  $\tau > 0$  and that  $C(\lambda)$  does not depend on  $\tau$ , it implies that  $y \in C^0([0, +\infty); L^2(\mathbb{T}))$  and

$$\|y(t)\|_{L^2} \leq C(\lambda) e^{-\lambda t} \|y^0\|_{L^2}, \quad \forall t \in [0, +\infty). \quad (7.10)$$

This nearly ends the proof of Theorem 1.1, the only thing left is to check that we have also a stabilization in  $H^r$  for any  $r \in (-1/2, 1/2)$ .

**REMARK 7.1.** *Let  $r \in (-1/2, 1/2)$ . The same feedback law  $K(y)$  also stabilizes the system (7.1) in  $H^r$ -space.*

Indeed, let  $r \in (-1/2, 1/2)$  and let  $y_0 \in H^r$ . As  $\phi_k \in H_k^{r-1}$  and  $K_k : H_k^{r+1} \rightarrow \mathbb{R}$ , for any  $\varphi(t) \in L^2(0, \tau; H^{r+1})$  we have that  $BK\varphi(t) \in L^2(0, \tau; H^{r-1})$ . Then equation (7.1) has a solution

$$y(t) \in C^0([0, \tau]; H^r(\mathbb{T})) \cap L^2(0, \tau; H^{r+1}(\mathbb{T})) \cap H^1(0, \tau; H^{r-1}(\mathbb{T})), \quad (7.11)$$

and is held in the  $L^2(0, \tau; H^{r-1}(\mathbb{T}))$  sense.

Because  $T$  is bounded in  $H^l$  with  $l = r - 1, r, r + 1$  (Lemma 5.1 part (4) and Section 5.3.3), we know that  $z(t) := Ty(t)$  lives in the same space of  $y(t)$  in (7.11).

By adapting  $T$  to the equation (7.1) we know that (7.4) holds in the  $L^2(0, \tau; H^{r-1}(\mathbb{T}))$  sense. Then by adapting Lemma 5.6 to the case that  $s = r$ , we get

$$Ty_t = ATy - \lambda Ty, \text{ in } L^2(0, \tau; H^{r-1}(\mathbb{T})). \quad (7.12)$$

Hence,

$$z_t - \Delta z + \lambda z = 0 \text{ in } L^2(0, \tau; H^{r-1}(\mathbb{T})), \quad (7.13)$$

which leads to the required exponential decay of  $z$  in  $H^r$ ,

$$\frac{1}{2} \frac{d}{dt} \|z(t)\|_{H^r}^2 = \langle z(t), \dot{z}(t) \rangle_{H^r} \leq -\lambda \|z(t)\|_{H^r}^2. \quad (7.14)$$

Consequently, using again that  $T$  is an isomorphism in  $H^r$ ,

$$\|y(t)\|_{H^r} \leq C_r(\lambda) e^{-\lambda t} \|y(0)\|_{H^r}, \quad (7.15)$$

with  $C$  and  $C_r(\lambda)$  depending on  $r \in (-1/2, 1/2)$  and  $\lambda \notin \mathcal{N}$ . This ends the proof of Theorem 1.1.

## 7.2 The viscous Burgers equation: on the well-posedness of the target system and the stability of the closed-loop system (1.8)

The proof of Theorem 1.4 dealing with the viscous Burgers equation is very similar to the proof of Theorem 1.1 dealing with the heat equation with  $m = 0, r = 0$ . Let  $\tau > 0$  and  $y_0 \in L_1^2$  such that  $\|y_0\|_{L^2} < \delta$ , where  $\delta$  is a constant to be chosen. Lemma 4.8 implies that the closed-loop system (1.8) has a unique solution  $y(t)$ , provided that  $\delta$  is sufficiently small (depending on  $\tau$ ), and

$$y(t) \in C^0([0, \tau]; L^2(\mathbb{T})) \cap L^2(0, \tau; H^1(\mathbb{T})) \cap H^1(0, \tau; H^{-1}(\mathbb{T})), \quad (7.16)$$

which is held in the  $L^2(0, \tau; H^{-1}(\mathbb{T}))$  sense. Next, again, thanks to the fact that the operator  $T$  is bounded in  $H_1^l(\mathbb{T})$  space with  $l = -1, 0, 1$ ,

$$z(t) := Ty(t) \in C^0([0, \tau]; L^2(\mathbb{T})) \cap L^2(0, \tau; H^1(\mathbb{T})) \cap H^1(0, \tau; H^{-1}(\mathbb{T})). \quad (7.17)$$

Next, by applying  $T$  to (1.8) we know that in the  $L^2(0, \tau; H^{-1}(\mathbb{T}))$  sense,

$$Ty_t - T Ay + T \partial_x(y^2/2) = TBK(y), \quad (7.18)$$

where we used the fact that  $T \partial_x(y^2/2) \in L^2(0, T; H^{-1})$ .

By applying Lemma 5.6 for the case  $s = 0$ , combined with the fact that  $y(t) \in L^2(0, \tau; H^1(\mathbb{T}^1))$ , we get that

$$Ty_t = ATy - \lambda Ty - T \partial_x(y^2)/2, \text{ in } L^2(0, \tau; H^{-1}(\mathbb{T})). \quad (7.19)$$

Hence,

$$z_t - \Delta z + \lambda z + T \partial_x(T^{-1}z)^2/2 = 0 \text{ in } L^2(0, \tau; H_1^{-1}(\mathbb{T})). \quad (7.20)$$

Therefore, the system is locally stable in  $L^2(\mathbb{T})$  space, provided that  $\|z\|_{L^2}$  is small enough, indeed

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|z(t)\|_{L^2(\mathbb{T})}^2 &= \langle z(t), \dot{z}(t) \rangle_{H^1, H^{-1}} \\
&= \langle z(t), \Delta z - \lambda z - T \partial_x (T^{-1} z)^2 / 2 \rangle_{H^1, H^{-1}}, \\
&\leq -\|z\|_{H^1}^2 + \|z\|_{L^2}^2 - \lambda \|z\|_{L^2}^2 + C \|z\|_{H^1} \|T \partial_x (T^{-1} z)^2\|_{H^{-1}} \\
&\leq -\|z\|_{H^1}^2 - (\lambda - 1) \|z\|_{L^2}^2 + C \|z\|_{H^1} \|(T^{-1} z)^2\|_{L^2} \\
&\leq -\|z\|_{H^1}^2 - (\lambda - 1) \|z\|_{L^2}^2 + C \|z\|_{H^1} \|T^{-1} z\|_{L^2} \|T^{-1} z\|_{H^1} \\
&\leq -\|z\|_{H^1}^2 - (\lambda - 1) \|z\|_{L^2}^2 + C \|z\|_{H^1}^2 \|z\|_{L^2}.
\end{aligned}$$

Hence, provided that  $\sup_{[0, \tau]} (\|z\|_{L^2})$  is small enough (depending on  $\lambda \in (1, +\infty)$ ) one has

$$\|z(t)\|_{L^2} \leq e^{-(\lambda-1)t} \|z(0)\|_{L^2}, \forall t \in [0, \tau], \quad (7.21)$$

which also implies, using  $T^{-1}$ , that

$$\|y(t)\|_{L^2} \leq C(\lambda) e^{-(\lambda-1)t} \|y(0)\|_{L^2}, \forall t \in [0, \tau], \quad (7.22)$$

provided that  $\sup_{[0, \tau]} (\|z\|_{L^2})$  small, or equivalently that  $\sup_{[0, \tau]} (\|y\|_{L^2})$  small, from the isomorphism property of  $T$ . Finally, from Lemma 4.8 and (4.30) it suffices to have  $\|y_0\|_{L^2}$  small. This means that there exists  $\delta_1(\tau, \lambda)$  such that for any  $\delta \in (0, \delta_1(\tau, \lambda))$  the solution  $y$  satisfies (7.16) and the exponential stability estimate (7.22) holds.

So far the constant  $\delta$  depends on  $\tau$  but, leveraging the exponential stability estimate (7.22), it can be made independent of  $\tau$  using a very classical argument: let  $\tau_1 > 0$  such that  $e^{-(\lambda-1)\tau_1/2} < (C(\lambda))^{-1}$ , and select  $\delta = \delta_1(\tau_1, \lambda)$ , then  $y$  exists and (7.22) holds on  $[0, \tau_1]$ , therefore

$$\|y(\tau_1, \cdot)\|_{L^2} \leq e^{-(\lambda-1)\tau_1/2} \|y_0\|_{L^2}. \quad (7.23)$$

As the system (1.8) is autonomous, studying it on  $[\tau_1, 2\tau_1]$  is the same as studying it on  $[0, \tau_1]$  with initial condition  $y(\tau_1, \cdot)$ . And from (7.23),  $\|y(\tau_1, \cdot)\|_{L^2} \leq \|y_0\|_{L^2} \leq \delta_1(\tau_1, \lambda)$ , hence the solution exists on  $[\tau_1, 2\tau_1]$  and

$$\|y(t)\|_{L^2} \leq C(\lambda) e^{-(\lambda-1)(t-\tau_1)} \|y(\tau_1)\|_{L^2}, \forall t \in [\tau_1, 2\tau_1], \quad (7.24)$$

which together with (7.22) gives

$$\|y(t)\|_{L^2} \leq C(\lambda) e^{-\frac{(\lambda-1)}{2}t} \|y(0)\|_{L^2}, \forall t \in [0, 2\tau_1], \quad (7.25)$$

Hence, iterating this procedure, for any  $n \in \mathbb{N}^*$   $y$  exists on  $[0, n\tau_1]$

$$\|y(t)\|_{L^2} \leq C(\lambda) e^{-\frac{(\lambda-1)}{2}t} \|y(0)\|_{L^2}, \forall t \in [0, n\tau_1], \quad (7.26)$$

hence  $y$  exists on  $[0, +\infty)$  and

$$\|y(t)\|_{L^2} \leq C(\lambda) e^{-\frac{(\lambda-1)}{2}t} \|y(0)\|_{L^2}, \forall t \in [0, +\infty). \quad (7.27)$$

This ends the proof of Theorem 1.4.

## 8 Conclusion

### 8.1 Quantitative studies on $C_r(\lambda, m)$

Thanks to the precise analysis introduced in this paper. The next step could also be on the quantitative study of the stabilization cost, namely on the value of the constant  $C_r(\lambda, m)$  in Theorem 1.1. For example, let  $m = 0$ . For any fixed  $\lambda \notin \mathcal{N}$ , even if we do not have enough information on the exact value of  $C_r(\lambda)$ , it can be conjectured that the optimal value of  $C_r(\lambda)$  (at least obeying our feedback law) tends to  $+\infty$  as  $|r|$  tends to  $1/2^-$ .

The appearance of the critical set  $\mathcal{N}$  also indicates that, for any  $r \in (-1/2, 1/2)$  fixed, as  $\lambda$  tends to  $\mathcal{N}$  the value of  $C_r(\lambda)$  tends to  $+\infty$ . Therefore, it seems that by adapting this feedback we are not able to achieve  $e^{C\sqrt{\lambda}}$  type estimates, at least not uniformly on  $\lambda \in \mathbb{R}^+$ . However, we believe that with the precise functional settings treated in this paper, we are much more closed to such quantitative results. Indeed, it is still possible and reasonable to expect  $e^{C\sqrt{\lambda}}$  estimate on  $\{\lambda = 4N + 2; N \in \mathbb{N}^*\}$ .

### 8.2 General parabolic equations

It is natural to ask whether our framework also adapts general parabolic equations, namely

$$\begin{cases} \partial_t y - \Delta y + a_1(x)\partial_x y + a_2(x)y = \phi_1 K_1(y) + \phi_2 K_2(y), \\ y(0) = y_0, \end{cases} \quad (8.1)$$

with  $a_i(x)$  satisfying suitable regularity assumption.

By regarding the lower order operators  $a_1(x)\partial_x + a_2(x)$  as source, the same feedback law and transformation yields the operator equality. However, on the next step, when applying  $T$  to the evolution equation, the source  $a_1(x)\partial_x y + a_2(x)y$  turned out to be  $T(a_1(x)\partial_x(T^{-1}z) + a_2(x)(T^{-1}z))$  which may become even stronger than the  $-\lambda z$  damping produced by backstepping.

Therefore, it seems that we need to perform backstepping directly on the elliptic operator  $-\Delta + a_1(x)\partial_x + a_2(x)$ . In the case that  $a_1(x) = 0$ , the analysis is probably simpler as the operator is remained to be self-adjoint. However, losing those explicit formulation of eigenvalues and eigenfunctions make it more challenging to conclude Lemma 5.1. While the other case that  $a_1(x) \neq 0$  is of course more delicate, maybe the perturbation theory of resolvent estimates should be applied, a good news is that due to the spectral gaps between different eigenvalues are increasing, it is possible that no smallness of  $a_1(x)\partial_x + a_2(x)$  should be assumed. Technically speaking, due to the appearance of eigenvalues admitting double multiplicity the bifurcation phenomenon when splitting those eigenvalues should appear, the resolvent analysis involved would be more interesting and more delicate to some related works such as [5, 12].

### 8.3 Stabilization with one scalar control

It is proved in this work that two scalar controls are necessary and sufficient for the rapid stabilization of the heat equation provided some decay information, because of those double eigenvalues. But if we work on more general parabolic equations, for which it is possible that every eigenvalues are simple and isolated, then probably one scalar control, of course always admitting suitable decay properties, is sufficient to conclude controllability and rapid stabilization.

According to the “return philosophy” introduced by Coron [9], it is still possible to stabilize nonlinear system even if the linearized system is not stabilizable. Therefore, it is also of interest to consider the rapid stabilization of the viscous Burgers equation with one scalar control.

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