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Multistability of saxophone oscillation regimes and its influence on sound production

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1 Summary

The lowest fingerings of the saxophone can lead to several different regimes, depending on the musician's control and the characteristics of the instrument. This is explored in this paper through a physical model of saxophone. The Harmonic Balance Method shows that for many combinations of musician control parameters, several regimes are stable. Time-domain synthesis is used to show how different regimes can be selected through initial conditions and the initial evolution (rising time) of the blowing pressure, which is explained by studying the attraction basin of each stable regime. These considerations are then applied to study how the produced regimes are affected by properties of the resonator. The inharmonicity between the first two resonances is varied in order to find the value leading to the best suppression of unwanted overblowing. Overlooking multistability in this description can lead to biased conclusions. Results for all the lowest fingerings show that a slightly positive inharmonicity, close to that measured on a saxophone, leads to first register oscillations for the greatest range of control parameters. A perfect harmonicity (integer ratio between the first two resonances) decreases first register production, which adds nuance to one of Benade's guidelines for understanding sound production. Thus, this study provides some *a posteriori* insight into empirical design choices relative to the saxophone.

1 Introduction

A classic endeavor in musical acoustics consists in the systematic study of sound production features of a musical instrument. Early studies use an artificial mouth to replace the musician (on the clarinet [45, 1, 60, 20] or the bassoon [28]) in order to better describe and understand the physical phenomena at play during sound production. Later on, artificial mouths have been robotized to provide a complete mapping of the instrument's behavior, aiming at understanding how the instrument must be acted on to produce different sounds [34, 42] or describing the

influence of an acoustical parameter of the resonator on sound production [23]. The objective of this last study is shared by other works using a rather different approach to systematic description of the instrument's behavior: using a physical model. Based on oscillation thresholds for instance [27], some conclusions can be drawn as to the acoustical characteristics facilitating the production of sound. Numerical resolution of the model's equations also constitute a repeatable way to map the produced sound to the characteristics of the instruments, which has direct applications in instrument making [58, 25]. However, from a mathematical perspective, as nonlinear dynamical systems, wind instruments models often admit multiple solutions for a given set of parameters. The question of the stability of each of these solutions holds great importance when aiming to describe or predict the playability of an instrument based on its physical model. But some important questions remain unanswered, even for ideal cases where the stability or instability of each regime would be known. For instance, which regime is produced if two regimes are stable for the same control parameters combination?

In the case of such coexistence of stable solutions, denominated *multistability* hereafter, the convergence towards one or the other solution depends on the initial conditions. Indeed, each solution is associated with a *region of attraction* or *attraction basin*, defined as the region of the phase space where all initial conditions converge towards this solution [52, 50]. For instance, attraction basins are studied in walking models [51, 43], where the 'walking' (periodic) regime almost always coexists with a stable equilibrium, corresponding to falling. In this case, describing attraction basins informs control strategies in robotics [61, 62]. Attraction basins are also studied for classic dynamical oscillators, such as Chua's circuit [44], with experimental explorations of the attraction basins [48] as well as numerical investigations [55]. As strongly nonlinear self-oscillating systems capable of multiple oscillating regimes, wind instrument models are among the systems for which studying attraction basins can shed light on their rich behavior and help understand control strategies used by musicians. However, to our

86 knowledge, no study on the attraction basins of mu-
 87 sical instruments has been produced, although sev-
 88 eral studies explore their multistability. Experimen-
 89 tal work on the clarinet [37] and a numerical study of
 90 several idealized woodwind resonators [56] illustrate
 91 in particular the hysteresis between regimes, which
 92 is a consequence of multistability. On the flute, con-
 93 tinuation and synthesis have been used to investigate
 94 the hysteresis between regimes, notably depending on
 95 inharmonicity [57].

96 Describing the attraction basins and comparing
 97 their sizes is expected to give information on which
 98 regime is most likely produced, assuming some prob-
 99 abilistic repartition of the initial conditions in the
 100 phase space [10]. However, an exhaustive description
 101 is almost impossible for a complete model of instru-
 102 ment, where the phase space is of very large dimen-
 103 sion. In such cases, attraction basins may be partially
 104 explored, based on a reduction of the phase space to
 105 one or two dimensions. For instance, the infinite-
 106 dimensional phase space of a delayed system can be
 107 partially described along two dimensions [59]. In the
 108 case of musical instruments, a reduction of the phase
 109 space is proposed in this paper, based on knowledge
 110 of typical musical scenarios. Throughout this work,
 111 the case of a model of saxophone is considered, and
 112 two scenarios are studied: transition from another es-
 113 tablished limit-cycle (scenario number 1), and first
 114 attack transient of a note, where the blowing pres-
 115 sure parameter goes from 0 to a certain final value
 116 (scenario number 2).

117 Section 2 presents the physical saxophone model
 118 and the two numerical methods used to solve its
 119 equations: the Harmonic Balance Method and time-
 120 domain synthesis. Next, multistability is introduced
 121 by computing the bifurcation diagram with the har-
 122 monic balance method and continuation (Asymptotic
 123 Numerical Method) and exhibiting hysteresis cycles
 124 using time-domain synthesis in Section 3 (control sce-
 125 nario number 1). Then, in Section 4, a simple test-
 126 case of scenario number two is presented to study
 127 sound production, where the blowing pressure in-
 128 creases from 0 to its final value over different du-
 129 rations. We show how this duration can influence
 130 the final regime in multistability regions, and explain
 131 these results by presenting the attraction basin of each
 132 regime. Section 5 demonstrates how the awareness of
 133 multistability can lead to a better description of the
 134 behavior of the model. Depending on the inharmonic-
 135 ity of the resonator, the size of the control param-
 136 eter regions where each regime appears in synthesis
 137 is described, taking into account multistability. This
 138 provides an interpretation to the inharmonicity value
 139 measured on the saxophone by showing that it corre-
 140 sponds to an optimum in periodic regime production.

2 Numerical simulation frame- 141 work 142

2.1 Saxophone model 143

144 The saxophone model used in this study is comprised
 145 of three main elements: a one degree-of-freedom os-
 146 cillator representing the reed, a regularized nonlin-
 147 ear characteristic giving the flow through the reed
 148 channel, and a modal description of the measured
 149 impedance of the resonator. Similar models solved
 150 by time-domain synthesis (Section 2.2) are used in
 151 conjunction with analytical techniques to study the
 152 playing frequency [19] and spectrum [49] of clarinets,
 153 as well as their radiated power with a comparison to
 154 measurements [33]. The Harmonic Balance Method
 155 (Section 2.3) can also be applied to this model to
 156 study its dynamic behavior, for instance to quantify
 157 the effect of neglecting reed contact [18].

Dimensionless [28, 35] acoustical Kirchhoff vari-
 ables (p, u) are used in this work:

$$p = \frac{\hat{p}}{p_M}, \quad u = Z_c \frac{\hat{u}}{p_M}, \quad (1)$$

where the hat notation indicates the variable in phys-
 ical unit, p_M is the static pressure necessary to close
 the reed channel completely and Z_c is the characteris-
 tic input impedance of the resonator for plane waves.
 Similarly, the reed displacement from equilibrium is
 given in dimensionless form

$$x = \frac{\hat{x}}{H} \quad (2)$$

where H is the distance between the reed and the
 mouthpiece lay at rest. With this formalism, the reed
 channel is closed when $x \leq -1$. In this work, the only
 time-varying control parameter [60] is the dimension-
 less blowing pressure γ :

$$\gamma(t) = \frac{p_m(t)}{p_M}, \quad (3)$$

158 where p_m is the physical value of the pressure in the
 159 mouth of the musician. We leave all other control pa-
 160 rameters constant in order to limit the dimensionality
 161 of the study. The values and names of the parameters
 162 are summarized in Table 1 and detailed below through
 163 the model description. Their values are drawn from
 164 [46] for the reed parameters q_r and ω_r , from [13] for
 165 the order of magnitude of the contact stiffness K_c .

2.1.1 The reed model 166

Following [13], the reed is modeled by a single de-
 gree of freedom oscillator including a nonlinear con-
 tact force accounting for the mouthpiece lay

$$\frac{\ddot{x}}{\omega_r^2} + \frac{q_r}{\omega_r} \dot{x} + x = p - \gamma + F_c(x + 1), \quad (4)$$

Name	Symbol	Value
Blowing pressure	γ	Variable
Reed opening at rest	ζ	0.6
Reed damping	q_r	1
Reed eigenfrequency	ω_r	4224 rad.s ⁻¹
Mouthpiece lay stiffness	K_c	100
Contact regularization	η	10 ⁻³
Number of modes	N_m	8
Number of harmonics	H	20

Table 1: Parameters of the numerical model: musician control parameters γ and ζ , reed parameters q_r and ω_r , contact parameter K_c and parameters inherent to the numerical implementation η , N_m and H .

where the two parameters of the reed are its angular eigenfrequency ω_r and its damping coefficient q_r , and the contact force is a function of the dimensionless reed opening $x + 1$ and is taken from [8],

$$F_c(x + 1) = K_c \min(x + 1, 0)^2, \quad (5)$$

where $K_c = 100$. Since $x + 1$ is the distance between the reed and the mouthpiece lay, F_c can be interpreted as a quadratic stiffness activated whenever the reed touches the lay. The ramp function $\min(x + 1, 0)$ is regularized using a parameter $\eta = 10^{-3}$ to avoid non-differentiability at $x = -1$ (reed closure)

$$\min(x + 1, 0) \simeq \frac{x + 1 - \sqrt{(x + 1)^2 + \eta}}{2}. \quad (6)$$

167 The regularization controlled by parameter η is necessary for the system to fit the quadratic formalism
168 required by the implementation of the harmonic balance method and asymptotic numerical continuation
169 in the MANLAB software, which produces the bifurcation diagrams of the present article. Although the
170 parameter η is not necessary for the time-domain synthesis method to function, it is kept for comparison
171 purposes.
172
173
174
175

176 2.1.2 The reed channel

The flow at the input of the resonator is deduced from Bernoulli's law [2, 36] applied to the reed channel and turbulent mixing into the mouthpiece

$$u = \zeta \max(x + 1, 0) \text{sign}(\gamma - p) \sqrt{|\gamma - p|} \quad (7)$$

where ζ is the dimensionless control parameter accounting for reed opening at rest

$$\zeta = Z_c w H \sqrt{\frac{2}{\rho}}, \quad (8)$$

w being the effective width of the reed channel and ρ the density of the medium. We choose to ignore the flow due to the speed of the reed [19, 21] in the present

model, as it only has a small effect on the playing frequency, which is not discussed here. The absolute value and ramp function in Eq. (7) are regularized with the same parameter η as in Eq. (6)

$$|\gamma - p| \simeq \sqrt{(\gamma - p)^2 + \eta}, \quad (9)$$

$$\max(x + 1, 0) \simeq \frac{x + 1 + \sqrt{(x + 1)^2 + \eta}}{2}. \quad (10)$$

2.1.3 The resonator

The input impedance is used to represent the resonator's acoustical response. The dimensionless input impedance $Z(\omega)$ of a Buffet-Crampon Senzo alto saxophone is measured with the CTTM impedance sensor [22]. The saxophone is measured without its mouthpiece, placing the reference plane of the impedance measurement at the input cross-section of the crook. A cylindrical tube is added by Transfer Matrix Method [12] in post-processing to represent the mouthpiece, before using it in synthesis. The length of the cylinder is 60 mm and the radius is the same as the input radius of the crook, 6 mm. The total volume of the added cylinder approximately fits that of the missing cone apex, as per a classical academic approximation [38]. In order to use this input impedance for the two numerical synthesis methods presented above, it is decomposed into modes [19] so that

$$Z(\omega) = \sum_{n=1}^{N_m} \frac{C_n}{j\omega - s_n} + \frac{C_n^*}{j\omega - s_n^*}, \quad (11)$$

where C_n and s_n are the estimated complex modal residues and poles [54] and N_m is the number of modes retained in the simulation. In this paper $N_m = 8$ modes are used. This translates into the time domain by describing the pressure as a sum of complex modal components p_n , whose evolution depends on the modal coefficients, such that

$$\dot{p}_n(t) - s_n p_n(t) = C_n u(t), \quad \forall n \in [1, N_m], \quad (12)$$

$$p(t) = 2 \sum_{n=1}^{N_m} \text{Re}(p_n(t)). \quad (13)$$

The flow u in (12) is given by (7). Figure 1 displays the measured impedance and the associated modal reconstruction according to Eq. (11) for the D \sharp fingering used throughout the rest of this article. The corresponding modal coefficients C_n and poles s_n are summarized in Table 2. Note that the choice of a modal formalism over a direct resolution of partial differential equations in the resonator is made here because of its lower computational cost and number of variables, which facilitates large-scale numerical studies such as those presented in Section 5. Additionally, the modal formalism involves a limited number of parameters, which are directly tied to the acoustics

Residues C_n (unitless)	Poles s_n (rad.s ⁻¹)
176.1	$-17.59 + j1195$
470.5	$-35.50 + j2483$
649.4	$-65.30 + j3727$
328.7	$-269.34 + j4405$
541.5	$-70.32 + j5153$
224.9	$-166.0 + j6177$
382.2	$-94.49 + j6749$
409.9	$-116.5 + j7987$

Table 2: Modal parameters for the fingering D \sharp of the alto saxophone.

191 of the resonator, instead of the complete geometri-
 192 cal description that a wave propagation model would
 193 require.

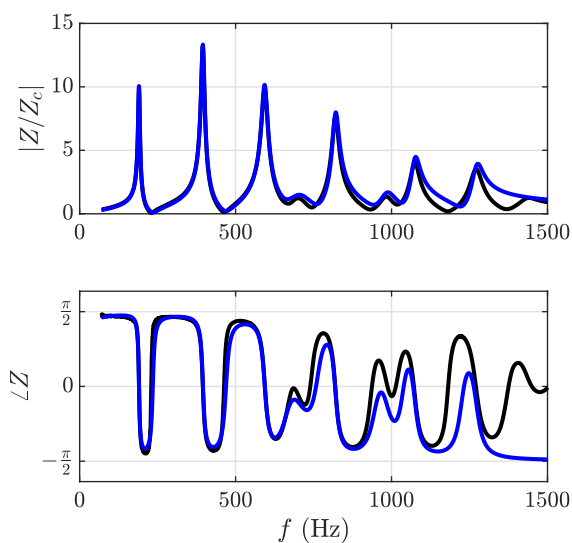


Figure 1: Measured input impedance (solid) and modal reconstruction (dashed) for the fingering D \sharp of the alto saxophone.

194 2.2 Time-domain synthesis

195 Equations (4), (7) and (12) are discretized using
 196 finite-difference approximations for the time-domain
 197 derivatives, following a discretization scheme first ap-
 198 plied to simple waveguides [29] and then adapted to
 199 a modal formalism [19]. The reader can find a de-
 200 tailed description of this discretization scheme in a
 201 recent document [16]. The sampling rate used in the
 202 simulation is $F_s = 176400$ Hz, four times higher than
 203 the standard audio sampling rate. Such a high sam-
 204 pling rate is required, given the chosen finite differ-
 205 ence scheme, to give precise results that match those
 206 obtained with the harmonic balance method.

207 As an illustrative result, Figure 2 shows an exam-
 208 ple of the synthesized pressure signal and its spectro-
 209 gram. Note that the signal shown is a portion of the

210 signal used in Figure 4, with control scenario num-
 211 ber one. It corresponds to the first occurrence of
 212 the oscillations at a blowing pressure value $\gamma \simeq 0.45$.
 213 At this point, the system jumps from equilibrium to
 214 the first register and passes through fleeting second
 215 register and quasi periodic regimes. The spectrogram
 216 (Figure 2 (b)) shows the second register to be the oc-
 217 tave (double the fundamental frequency) of the first
 218 register. The quasi-periodic portion of the signal dis-
 219 plays amplitude variations, seen in the envelope of the
 220 signal (Figure 2 (a)) and on the odd harmonic com-
 221 ponents of the spectrogram. Quasi-periodic regimes
 222 are well-known on saxophone-like instrument models,
 223 documented for instance in [20, 24, 23].

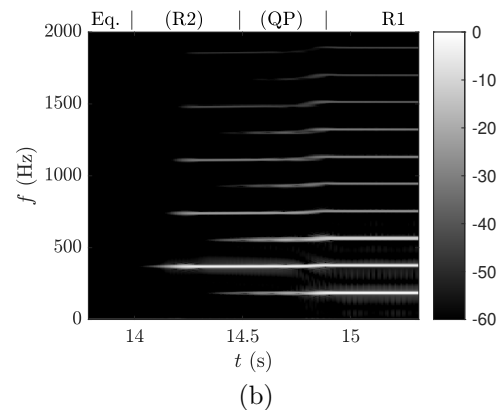
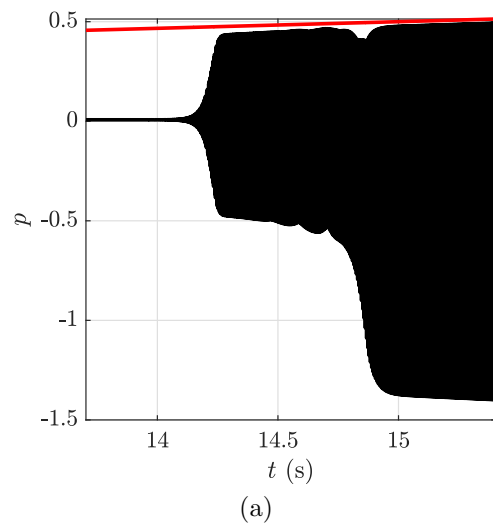


Figure 2: Time-domain synthesized pressure signal. (a) temporal envelope (black) and blowing pressure parameter γ (red). (b) Normalized spectrogram (dB) with regime names indicated (unstable ones between parentheses) : equilibrium (Eq.), second register, quasi-periodic and first register. The signal is extracted from the same blowing pressure ramp as in Figure 4, between $\gamma = 0.45$ and $\gamma = 0.51$ (at first occurrence of oscillation).

2.3 Harmonic balance and numerical continuation

The harmonic balance method is an analysis method particularly adapted to the study of musical instrument models [26], since it focuses on periodic solutions, which correspond to the produced notes. Assuming periodicity of the solution allows expanding all variables in Fourier series [39, 47] up to order H , such that the i -th variable X_i expands to

$$X_i(t) \simeq \sum_{h=-H}^H X_{i,h} \exp(jh\omega_0 t), \quad (14)$$

where there are $2H + 1$ complex Fourier coefficients $X_{i,h}$ per variable, and ω_0 is the fundamental angular frequency of the signal. In this work, the number of harmonics retained is $H = 20$. Applying the method to a differential system transforms it into an algebraic system of which the unknowns are the Fourier coefficients of the variables and the solution's fundamental frequency, of the form

$$R(\{X_{i,h}\}, \omega_0) = 0. \quad (15)$$

A numerical continuation method such as the Asymptotic Numerical Method can then be applied to the resulting algebraic system [15, 31] to find how the solution changes for other constant values of a chosen control parameter, for instance the blowing pressure parameter γ . More precisely, knowing a solution to the system for a given value of γ , the continuation method finds a solution for a slightly higher or lower values of γ , and therefore progressively maps out the solutions of the system for a given range of blowing pressures. In this work, simulations were carried out using the MANLAB software (<http://manlab.lma.cnrs-mrs.fr/>). This yields the value of the Fourier coefficients of the oscillating solutions along several values of a control parameter. The Fourier coefficients can then be used to reconstruct the time-domain solutions. This evolution can be summarized by a bifurcation diagram, which represents the variation of some descriptor, for instance the amplitude, of the solutions of the system with respect to the chosen control parameter. In addition, the stability of the solutions is determined using Floquet theory (for more details refer to [41, 5, 32]).

3 Multistability

This section presents the blowing pressure ranges where the model can produce each regime by studying their stability with the harmonic balance method. This result is summarized in the bifurcation diagram on which multistability zones appear as intervals where several regimes are stable. Signals are

also synthesized with time-domain synthesis to exhibit how multistability leads to hysteresis. The correspondance between the two methods (the harmonic balance method and time-domain synthesis) is also checked.

3.1 Overlapping stability zones on the bifurcation diagram

The bifurcation diagram is computed for the (written) low D \sharp fingering of an alto saxophone. The written D \sharp produces the heard note F \sharp 3 at 185 Hz as its first register regime. This intermediate fingering of the first register is chosen as test case because it exhibits both first and second register regimes, but no stable third register regime and few double two-step phenomena [17]. Figure 3 shows the L²-norm of the pressure signal

$$\|p\|_2 = \frac{1}{T} \sqrt{\int_0^T p(t)^2 dt}, \quad (16)$$

where T is the period of the signal, and identifies which regime each branch corresponds to. The blowing pressure parameter γ spans the interval between 0 and 2. This bifurcation diagram contains branches corresponding to the so-called equilibrium, where no sound is produced, for the lowest and highest γ values. The equilibrium at low γ corresponds to the musician not blowing hard enough into the instrument to obtain a sound, while at high γ equilibrium means that the reed channel remains closed due to the large pressure difference between mouth and mouthpiece. For intermediate γ values, the first and second register both appear. The first register is the fundamental pitch obtained with a given fingering, and the second register, sometimes referred to as *overblowing*, is pitched one octave higher than the first register. Even though the saxophone has an octave key facilitating the production of the second register, musicians know how to produce second register regimes without activating it. Note that contrary to the clarinet, the register key of the saxophone controls the two register holes, opening one or the other depending on all the pressed keys on the instrument. It is therefore not surprising that both regimes appear on the same fingering. In the present case, the γ interval between 0 and 2 contains all the studied limit cycles of the model, and at its bounds, only the equilibrium solution exists and is stable. The diagram in Figure 3 displays several zones of coexistence between stable regimes (i.e. multistability). These regions of coexistence are often bounded by bifurcation points, that mark qualitative changes in the oscillating regimes. In the present work, the system encounters several types of bifurcations, which we define succinctly in terms of the regime changes they correspond to. More formal definitions and characterizations of these bifurcations,

299 notably in terms of critical values of the eigenvalues
 300 of the jacobian matrix of the system, can be found
 301 in [40, 7]. The Hopf bifurcation marks the emergence
 302 of an oscillating solution from equilibrium. The fold
 303 bifurcation corresponds to a stable and unstable solu-
 304 tion branch colliding and disappearing, which can be
 305 better seen on the bifurcation diagrams as limit points
 306 in the solution branches. Neimark-Sacker bifurcations
 307 correspond to a periodic regime becoming unstable
 308 and being replaced by a quasi-periodic regime. A de-
 309 generate case of the Neimark-Sacker bifurcations is
 310 the period-doubling bifurcation, where a periodic so-
 311 lution of halved frequency emerges from an oscillating
 312 solution. On saxophone models, period doubling bi-
 313 furcations transform a second register regime into a
 314 first register. As is discussed in the next paragraph
 315 for the saxophone, Hopf and fold bifurcations often
 316 delimit coexistence between the equilibrium and an
 317 oscillating regime, while Neimark-Sacker and period-
 318 doubling bifurcations mark the limits of multistability
 319 regions between two oscillating regimes.

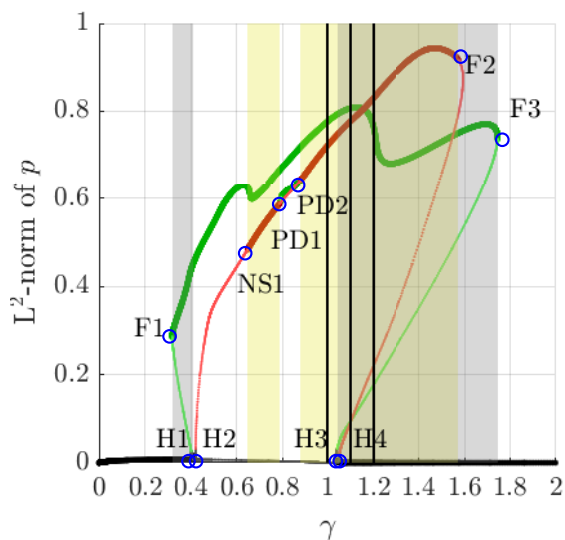


Figure 3: Bifurcation diagram obtained with harmonic balance method and numerical continuation: L^2 -norm of the acoustical pressure depending on the blowing pressure parameter γ for the low written D \sharp fingering of an alto saxophone. Thick lines: stable solution, thin lines: unstable solutions. Black: equilibrium, green: 1st register regimes, red: 2nd register regimes. Multistability zones are shaded: light yellow where 1st and 2nd register coexist, darker gray for equilibrium and 1st register. Blue circles specify the location of bifurcations. Vertical black lines correspond to those in Figure 6 (c), and (from left to right) to phase diagrams 7 (a), (b) and (c).

320 Starting from low blowing pressure values, the first
 321 coexistence zone appears between the first register
 322 and the equilibrium. It is delimited by the fold bi-

323 furcation F1 of the first register around $\gamma = 0.3$ and
 324 the inverse Hopf bifurcation H1 at $\gamma = 0.4$ where
 325 the equilibrium becomes unstable. The second coexis-
 326 tence zone is between first and second register, in the
 327 interval where the second register is stable between
 328 the Neimark-Sacker bifurcation NS1 and the period-
 329 doubling bifurcation PD1 (respectively at $\gamma = 0.66$
 330 and $\gamma = 0.79$). The Neimark-Sacker bifurcation NS1
 331 mark the destabilization of the second register and
 332 the emergence of a quasi-periodic regime (not repre-
 333 sented here), sometimes called multiphonics by musi-
 334 cians. The next coexistence zone occurs in the in-
 335 terval between the two period-doubling bifurcations
 336 PD1 and PD2 on the second register branch, where a
 337 stable double two-step solution [17] emerges. This co-
 338 existence zone is not shaded on the figure, as it could
 339 represent less of a musical issue, since double two-step
 340 regimes have roughly the same frequency as standard
 341 first register regimes. The fourth coexistence zone is
 342 more complicated: it starts between first and second
 343 register at the period-doubling bifurcation PD2, and
 344 then the equilibrium also becomes stable at the Hopf
 345 bifurcation H4. The limit of the last coexistence zone
 346 is made by the two fold bifurcations F2 and F3 where
 347 the first and second register solutions cease to exist.

The diagram in Figure 3 shows that coexistence
 348 zones between stable regimes span most of the range
 349 in γ where oscillating solutions exist, including ar-
 350 guably crucial γ values like the lowest for which an
 351 oscillating regime exists. Multistability is not an iso-
 352 lated phenomenon, but rather corresponds to the gen-
 353 eral situation, at least for this fingering.

3.2 Time-domain synthesis with blowing pressure ramps (control scenario number one)

355 Once multistability zones are identified, time-domain
 356 synthesis can be used to exhibit their role when play-
 357 ing the instrument. One of the main phenomena multi-
 358 stability entails is hysteresis: for several values of the
 359 blowing pressure, a different regime is produced de-
 360 pending on whether the blowing pressure is increasing
 361 or decreasing. Various multistable regimes are exhib-
 362 ited using this method in [56] on woodwind models.
 363 Figure 4 shows the hysteresis cycles obtained by using
 364 ramps of γ : in control scenario number one, the pa-
 365 rameter γ is progressively increased from 0 to 2 and
 366 then decreased back to 0. Each one of the increasing
 367 and decreasing phases of the synthesis has a duration
 368 of 60 s. This duration was chosen after several trials,
 369 sufficiently long to let stable regimes establish while
 370 keeping a γ slope steep enough to limit dynamical
 371 bifurcation delays [6].

372 Figure 4 shows that the synthesized signal starts
 373 from $\gamma = 0$ at equilibrium, its L^2 -norm being zero.
 374 Then, at Hopf bifurcation H1, the equilibrium be-
 375 comes unstable, which causes the system to start os-
 376 cillating. The first register regime is established at
 377 the fold bifurcation F1. The second register regime
 378 appears at the fold bifurcation F2. The equilibrium
 379 becomes stable again at the Hopf bifurcation H4. The
 380 second register regime disappears at the fold bifurca-
 381 tion F3. The first register regime disappears at the
 382 fold bifurcation F3. The equilibrium becomes stable
 383 again at the Hopf bifurcation H4. The first register
 384 regime disappears at the fold bifurcation F3. The
 385 equilibrium becomes stable again at the Hopf bifurca-
 386 tion H4. The first register regime disappears at the
 387 fold bifurcation F3. The equilibrium becomes stable
 388 again at the Hopf bifurcation H4.

379 cillating. At this point, the synthesis goes through a
 380 transient represented in Figure 2 (between $t = 14.2$ s
 381 and $t = 14.8$ s), shortly passing by unstable second
 382 register and quasi-periodic regimes before reaching
 383 the first register. Once the first register is estab-
 384 lished, the branch is followed all the way to extinc-
 385 tion (around $\gamma = 1.75$ on Figure 4), because the first
 386 register does not become unstable until the fold bifur-
 387 cation F3. At this point, the system returns to equi-
 388 librium until the highest value of the ramp, $\gamma = 2$.
 389 The blowing pressure γ then starts descreasing, and
 390 the system stays at equilibrium until Hopf bifurca-
 391 tion H3 is reached, for $\gamma \simeq 1.05$. There, the sys-
 392 tem jumps to the stable second register regime. The
 393 second register branch is followed until the period-
 394 doubling bifurcation PD2, where the system briefly
 395 follows the double two-step branch. The double two-
 396 step branch appears on the L^2 -norm very close to the
 397 second register branch, and looks like a small bump.
 398 The two branches cross again at period-doubling bi-
 399 furcation point PD1. Then, something rather surpris-
 400 ing occurs: the L^2 -norm of the time-synthesized
 401 signal seems to follow the second register branch of the
 402 bifurcation diagram further than Neimarck-Sacker bi-
 403 furcation NS1, although the second register branch
 404 has become unstable. This is because the quasi-periodic
 405 regime emerging at NS1 is actually a stable attractor,
 406 and the associated L^2 norm happens to be close to
 407 that of the second register. Branches of quasi-periodic
 408 regimes have not been computed so as not to clutter
 409 Figure 3, but note that it is possible with harmonic
 410 balance method and Manlab [31]. When the quasi-
 411 periodic regime becomes unstable (around $\gamma = 0.55$)
 412 the system jumps back onto the first register branch,
 413 which is followed until fold bifurcation F1, beyond
 414 which the stable equilibrium is the unique solution of
 415 the model. The path described precedently is highly
 416 hysteretic: the sequence of regimes produced for in-
 417 creasing and decreasing γ are very different. Actu-
 418 ally, the two paths only coincide in three regions: the low-
 419 est and highest γ intervals, for which only the equi-
 420 librium is stable, and a very small region around $\gamma = 0.5$
 421 where only the first register is stable.

422 The hysteresis phenomenon observed here in time-
 423 domain synthesis can be interpreted as the first step
 424 in attraction basin description: once a certain stable
 425 regime is reached, it is followed until extinction or
 426 loss of stability, even when other regimes are simul-
 427 taneously stable. This confirms that a stable peri-
 428 odic regime is part of its own attraction basin. Re-
 429 constructing stable parts of the bifurcation diagram
 430 using time-domain synthesis and comparing them to
 431 those obtained using the harmonic balance method
 432 also provides validation for the numerical discretiza-
 433 tion scheme in this context. Here, it shows that time-
 434 domain synthesized signals are not perturbed by nu-
 435 merical artifact due to time discretization and can
 436 be trusted to describe properties of the model. Note

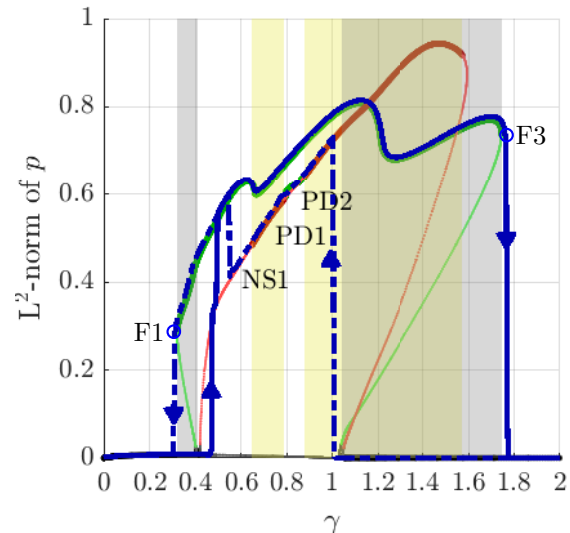


Figure 4: L^2 -norm of the time-domain synthesis signal (dark blue line) for control scenario number one, a ramp of the blowing pressure parameter γ from 0 to 2 and down, overlaid with the bifurcation diagram of Figure 3 (thick stable solutions and thin unstable solutions, of the first register in green and second register in red). Solid dark blue line indicates the first part of the signal (increasing γ) and dashed dark blue line the second (decreasing γ).

437 that even subtle details of the behavior such as the
 438 tiny branch of double two-step solution between pe-
 439 riod doubling PD1 and PD2 are found by the time-
 440 domain synthesis.

441 This exploration of the blowing pressure space us-
 442 ing a long ramp is very useful to exhibit the hysteresis
 443 phenomenon, as well as test the coherence between
 444 the two synthesis methods. However, this kind of
 445 sound is extremely artificial and far from anything a
 446 musician would use in everyday practice (provided it
 447 is even possible for a musician to produce it). There-
 448 fore, we frame the conditions of the rest of the study
 449 so that they can be interpreted in terms of selection
 450 of one regime over another.

4 Regime emergence in a multistable context

451 It is very likely that musicians learn to select between
 452 coexisting stable regimes, adjusting their control so
 453 that the established regime in a multistability region
 454 is the one they desire. This idea provides the layout
 455 for control scenario number two: we study the effect
 456 of a parametrized transient control or initial condi-
 457 tions on the established steady-state regime that fol-
 458 lows when the control is constant. To provide an-
 459 other way to interpret the selection process between
 460
 461

462 multistable regime, a third control scenario is used,
 463 where only initial conditions are varied and all control
 464 parameters are constant.

465 4.1 Control scenario number two: in- 466 creasing blowing pressure

One way to study the attraction basins more thoroughly is to run many simulations with initial conditions spanning the whole phase space. However, since the considered model has a $2N_m + 2$ dimensional phase space, a complete exploration is not possible. Moreover, many of the possible initial conditions are unlikely to be created by the musician. More interesting is the exploration of the regions of the phase space that are crossed by the system when a given control pattern is varied. Here, we focus on a monotonic increase of the blowing pressure γ at the attack: without using the tongue, the player starts blowing progressively harder into the instrument. Such a scenario was proposed in [53]. Note that instrumented mouthpiece measurements performed on the saxophone, such as those presented in [30], often show a different profile including a pressure overshoot before the apparition of the oscillations. However, in the present study, we omit that overshoot so that the control scenario is entirely defined by a single parameter. In control scenario number two, the blowing pressure starts from 0 and rises up until stabilizing at a certain value γ_f , during a certain time determined by the parameter τ_g . The temporal variation of γ is given by the

$$\gamma(t) = \frac{\gamma_f}{2} \left(1 + \tanh \left(\frac{t - 5\tau_g}{\tau_g} \right) \right), \quad (17)$$

467 which is differentiable infinitely many times. Figure
 468 5 displays four examples of such transients.

469 Other envelopes (sigmoid, sine branch) were tested,
 470 only causing small quantitative changes to the results.
 471 Figures 6 shows which established regimes appear in
 472 time-domain synthesis depending on τ_g , for final val-
 473 ues γ_f belonging to the multistability zones described
 474 in Figure 3. Each dot on the figure represents the
 475 type of established regime after five seconds of time-
 476 domain synthesis. This synthesis duration was chosen
 477 to be sufficiently long so that the transient is com-
 478 pleted and the established regimes can be observed.
 479 Regime types are estimated using an energy-based
 480 criterion for equilibrium (if the energy of the pres-
 481 sure signals in the last ten periods is less than that of
 482 the first ten, the regime is classified as equilibrium)
 483 and a fundamental frequency estimator to determine
 484 the first and second register. To detect quasi-periodic
 485 regimes, an attribute that can be observed in Figure
 486 2 is used: the fact that the amplitudes of the harmon-
 487 ics of a quasi-periodic signal vary temporally. For the
 488 classification, if the mean variance of the harmonic

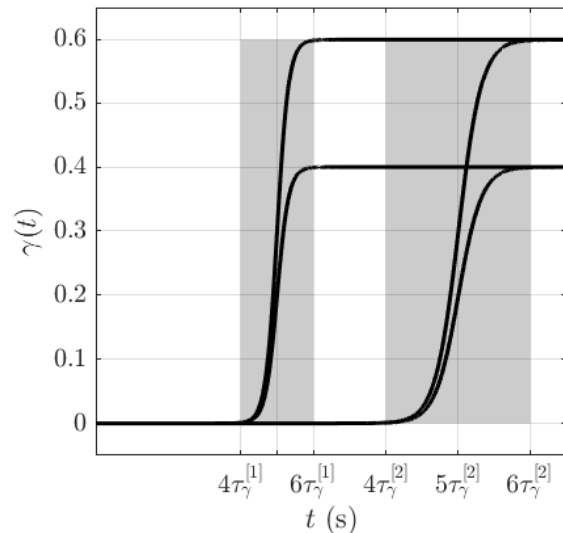


Figure 5: Four examples of blowing pressure variations in control scenario number two, as described by Eq. (17), for $\gamma_f = 0.4$ and 0.6 and $\tau_g = \tau_g^{[1]} = 10$ ms and $\tau_g^{[2]} = 20$ ms.

491 amplitudes is more than a certain threshold (here set
 492 to 10^{-6}), then the regime is considered quasi-periodic.

493 Figure 6 (a) focuses on the first multistability region
 494 (highlighted in gray in Figure 3), near the first
 495 Hopf bifurcation H1. The two stable regimes in this
 496 region are the equilibrium (\circ) and the first register
 497 (\bullet). For final values γ_f between 0.38 and 0.4 ,
 498 the system can converge to both regime depending on
 499 the characteristic rising time τ_g . It is interesting to
 500 note that equilibrium is reached for the longest rising
 501 times, i.e. the slowest γ variation, whereas the oscil-
 502 lating regime is reached for the shortest rising times.
 503 This is understandable as a quick γ increase tends to
 504 drive the system away from equilibrium, and there-
 505 fore possibly out of its attraction basin. Note that
 506 some of the oscillating regimes near the limit, for the
 507 longest attack times, are classified as quasi-periodic.
 508 This is due to the transient being extremely long in
 509 this particular region: the steady-state regime is not
 510 yet established at the end of the synthesized signals.
 511 This classification particularity, which could be seen
 512 as an error, is not corrected because, from a musi-
 513 cian's perspective, a regime still varying five seconds
 514 after the start of the attack will arguably not be con-
 515 sidered periodic.

516 The second zone of multistability is explored in Fig-
 517 ure 6 (b). The first and second register are separated
 518 by some stable quasi-periodic regimes (\blacktriangle). This
 519 is the same quasi-periodic regime that appears in time-
 520 domain synthesis in Figure 2, which overlays the un-
 521 stable portion of the second register branch. There is
 522 a particular range of characteristic time τ_g that seems
 523 to produce the first register for a larger range of γ_f .

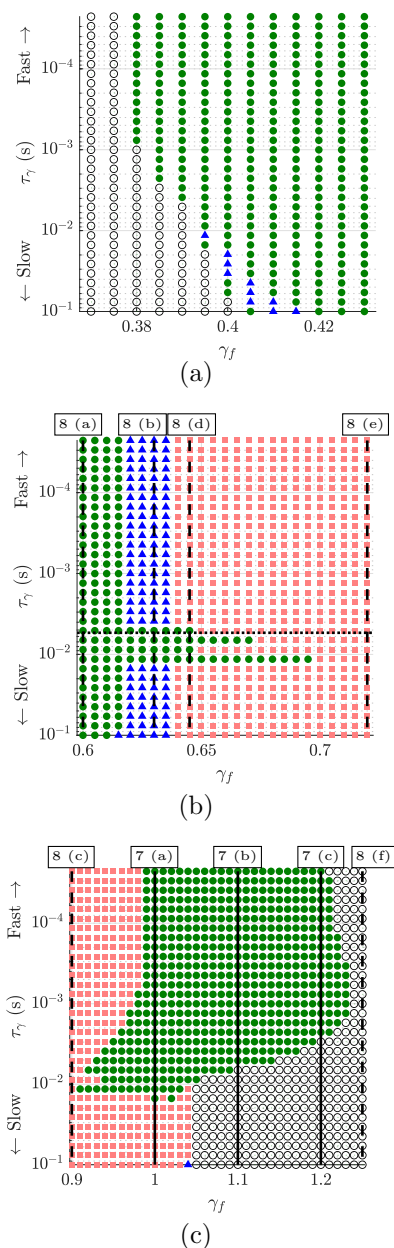


Figure 6: Classification of the steady-state regimes produced depending on the blowing pressure transient parameters: final value γ_f and characteristic time τ_g . Multistability zones (deduced from Figure 3): (a) equilibrium \circ and first register \bullet (b) first register \bullet and second register \blacksquare (c) all three regime types. Blue triangles \blacktriangle indicate quasi-periodic regimes. The horizontal line on graph (b) shows $\tau = 1/f_1$. The vertical lines highlight the γ values of phase diagrams in Figure 7 and 8.

The corresponding values of τ_g are close to the period of the first register, represented by the horizontal line in 6 (b).

In the last multistability ($0.9 \leq \gamma \leq 1.25$), three regimes may be stable for the same parameter values, as shown on Figure 3. However, Figure 6 reveals that there is no γ_f region where all three are

529 produced. This can be explained by analyzing the
530 attraction basins (see Section 4.2 Figure 8).

4.2 Control scenario number three: 531 varying initial conditions 532

The results concerning the influence of the blowing pressure parameter can be better understood by examining the region of the phase space leading to each regime. A point in the phase space represents the current state of the system, meaning the value of the state variables and their derivatives. Since the system is deterministic, a given point in the phase space will always lead to the same stable established regime. Therefore regions of the phase space can be associated with each regime. These regions are called *attraction basins*. The attraction basins are only rigorously defined in a context where all control parameters are constant. However, to interpret the results obtained with a control parameter transient such as control scenario number two, it is useful to observe the layout of attraction basins obtained for the final blowing pressure value γ_f . Specifically, at the beginning of control scenario number two, the blowing pressure is subject to fast variations, making a direct approach based on attraction basins ill-defined. However, after the transient, the blowing pressure stabilizes around its final value γ_f . To elucidate the behavior of the system at this moment, subsection 4.2 performs a systematical analysis of its convergence with constant control parameters, in which case the attraction basins representations are relevant. Therefore, a third control scenario is devised, where the control parameters are kept strictly constant and only the initial values of certain state variables of the system are modified.

Because the phase space is of dimension $2N_m + 2$ (all modal components and their derivatives, plus reed position and speed), it is necessary to choose a projection to represent the attraction basins. After some trials, a projection of the phase space on the two first modal components (see Eq. (12)) and the derivative of the second one, (p_1, p_2, \dot{p}_2) , was chosen as a three-dimensional projection. These variables were chosen not only because of their physical or mathematical meaning, as they relate respectively to the first and second register, but also because they allow for the clearest visual separation of the limit cycles and attraction basins that could be obtained by the authors. To estimate the attraction basins, time-domain synthesis is launched with initial conditions spanning the projected phase space and constant control parameters (control scenario number three). A total of 256 initial conditions are scattered in a Latin hypercube sampling into a rectangular parallelepiped such that

$$p_1^I \in [-0.2, 0.2], \quad p_2^I \in [-2, 2], \quad \dot{p}_2^I \in [-707, 707]. \quad (18)$$

These bounds should be understood with respect to 562

the amplitude of the limit cycle along each dimension (that can be seen in Figures 7 and 8). They were chosen so that whenever a regime is stable, it is obtained in synthesis at least once. All the other modal pressure components and their derivatives, as well as the reed speed \dot{x} , are initially zero. So that there is no discontinuity when starting the synthesis, the initial values of the variables p , then x and u are computed accordingly through equations (13), (4) and (7). All control parameters, including the blowing pressure γ , are kept constant during these simulations: recall that the attraction basin analysis is only strictly valid for constant values of the control parameters. Therefore, the deductions that are made using the projected attraction basins (control scenario number three) on the results of control scenario number two are subject to caution.

Figure 7 shows these initial conditions in the plane (p_2, \dot{p}_2) , associated with the regimes they lead to, for three values of the blowing pressure parameter γ . All the phase space points the system passes through during the transient are also part of the attraction basin, so they are represented in the figure as well. The last period of synthesized signals is plotted as the limit cycles. The three values of γ are chosen near the last Hopf bifurcation (H3 and H4 in Figure 3), where the equilibrium becomes stable again. Graph 7 (a) is computed at $\gamma = 1$, before the Hopf bifurcations, so only register one and two are stable. Therefore, none of the initial conditions lead to equilibrium (no black dots on the figure). Each attraction basin is located around the corresponding limit cycle. The attraction basins seem to overlap, but it is merely an effect of the projection of the phase space. Graph 7 (b) corresponds to $\gamma = 1.1$, right above the Hopf bifurcations H3 and H4. One can see that some initial conditions located near the origin now lead to equilibrium. This plot can seem surprising when compared to Figure 6 (c), which only shows equilibrium and register one for this value of γ_f using the control scenario number two, although the attraction basin of the second register (red points in Figure 7 (b)) seems larger than that of the other regimes. This is due to two effects. First, the projection spreads the second register but shrinks the first register. Secondly, control scenario number two starts from the origin of the phase space: it is ensured by setting all variables and their derivatives at 0 at the start of the synthesis. Therefore, attraction basins surrounding the origin, such as that of the first register, are more likely to be entered. Here we see that considering only the size of the attraction basins and ignoring their position provides only little information. This data needs to be interpreted in terms of musician action, notably pertaining to the ability of the musician to actually reach the region of the phase space. Here, the fact that the attraction basin is vast is effectively countered by the fact that it is far from the origin, making it hard to reach using

the studied control scenario number two. Graph 7 (c) represents the same results for $\gamma = 1.2$, where Figure 6 (c) showed more occurrences of the equilibrium than for $\gamma = 1.1$. This is explained by the attraction basin of the equilibrium being larger – there are many more black dots on Figure 7 (c) than on (b). Notice that the attraction basin of the first register expands (also many more green dots than on Figure 7 (b)), while the attraction basin of the second register (red dots) shrinks. This process (not represented here) continues until the second register ceases to be stable at fold bifurcation F3 (see Figure 3).

Figure 8 shows the attraction basins and limit cycles, in a three-dimensional projection of the phase space (p_1, p_2, \dot{p}_2) , at particular values of γ highlighted in Figure 6. Graphs 8 (a), (b), (d) and (e) should be read as further information on the regime map 6 (b), at the beginning of the multistability zone between first and second registers.

Graph 8 (a) corresponds to $\gamma = 0.6$, and confirms that the first register is the only stable regime: it was the only one to appear in the regime map 6 (b) (for $\gamma_f = 0.6$). Then, a quasi-periodic attractor appears in Graph 8 (b), for $\gamma = 0.63$. Although the associated attraction basin seems smaller than that of the first register, it seems to almost surround the origin of the phase diagram. It was observed in synthesis that the transient of control scenario number two does not send the system very far from the origin in the phase space – compared, for instance, with the size of the limit cycles of register one and two. This is linked to it necessarily starting from the origin of the phase space. The fact that control scenario number two tends to lead the system to phase space points near the origin explains why regime map 6 (b) displays more quasi-periodic regimes than first register. A similar interpretation can be formulated with regards to graphs 8 (d) and (e), for $\gamma = 0.645$ and $\gamma = 0.72$ respectively. For these values of γ on Figure 6 (b), there is more second register than first register. On graphs 8 (d) and (e), it can be seen that although the size of the first register's attraction basin seems comparable to that of the second register, the latter clearly holds a central position around the origin of the phase space. When the second register attraction basin grows in Graph 8 (e), this translates to the disappearance of the first register from the regime map 6 (b). Note that Graph 8 (e) confirms that the first register is still stable, as announced by the harmonic balance method in Figure 3. Graph 8 (c) $\gamma = 0.9$ illustrates a slightly different explanation of a similar case of only second register appearing in the regime map 6 (c): in this case, the attraction basin of the first register is just too small, it only makes up for a few points in Figure 8 (c). Graph 8 (f) ($\gamma = 1.25$) is comparable to (d) in that many regimes are stable, but only the one with the most central attraction basin appears in the corresponding regime map on Figure 6

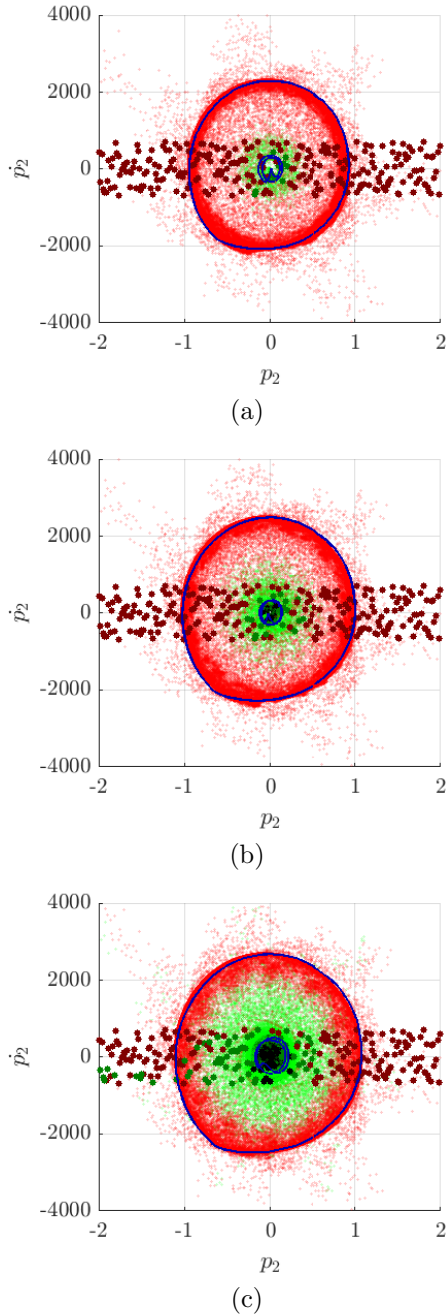


Figure 7: Projection of the attraction basins obtained with control scenario number three. Large dots are initial conditions, small dots are points the synthesis goes through. The dots' colors indicate the final regime they lead to (black: equilibrium, green: first register, red: second register). Blue lines represent the limit cycles. The small one on the inside is the first register and the one on the outside is the second register. The blowing pressure γ is (a) 1, (b) 1.1, (c) 1.2 (highlighted in figures 6 and 3).

(c). In this case this regime is the equilibrium, whose attraction basin in Graph 8 (f) surrounds the origin, although it appears smaller than the others. To complete the study, the full evolution sequence of the at-

traction basins obtained with control scenario number three can be found as an animation in multimedia file Supp1.mp4. The authors suggest frequently pausing the animation to observe precisely how the attraction basins develop along multistability zones.

5 Effect of the resonator's inharmonicity on regime production

The concept of multistability and the attraction basins are presented and explored here because they seem to be a very important part of the observed behavior of a saxophone model. The present section offers a succinct description of the behavior of the model, applied to an instrument design problematic. It also illustrates how ignoring multistability can affect the description of the behavior of a model.

Before using the analysis of a woodwind physical model to develop new instruments, it can be very informative to apply it to existing instruments, in the idea of a reverse engineering procedure. If the analysis method can explain *a posteriori* some design choices made on instruments with satisfying sound production characteristics, then it might help guide further innovative design choices in the right direction. In the present case, the produced regimes are studied for the seven lowest first register fingerings, and one acoustical parameter is varied artificially: the inharmonicity between first and second resonance. The original data corresponds to measured impedance for the corresponding fingerings of a Buffet-Crampon Senzo professional alto saxophone. According to the so-called *Bouasse-Benade prescription* [9, 4, 3], near perfect inharmonicity between the resonances is cited as a condition for good playability of the instrument. This prescription is also discussed in recent studies [27, 11]. On the saxophone, experimental studies using an artificial mouth have shown that varying inharmonicity greatly affects regime production [23, 20]. In this work we define inharmonicity as the ratio between the second and first resonance f_2/f_1 , or $\Im(s_2)/\Im(s_1)$ in terms of the parameters of Eq. (11). On a saxophone this ratio is close to two. Many definitions of the inharmonicity can be devised, possibly taking into account more resonances. The present definition of the harmonicity has the advantage of being very short, and easy to modify in the modal formalism by adjusting only one modal frequency.

Specifically for the purpose of the following study, optimal regime production conditions are defined crudely in terms of how often each regime appears in synthesis. This definition differs from that employed in [27]. On the lowest fingerings, in this study, we simply consider that optimizing regime production means maximizing the appearance of the first register while minimizing that of the second register

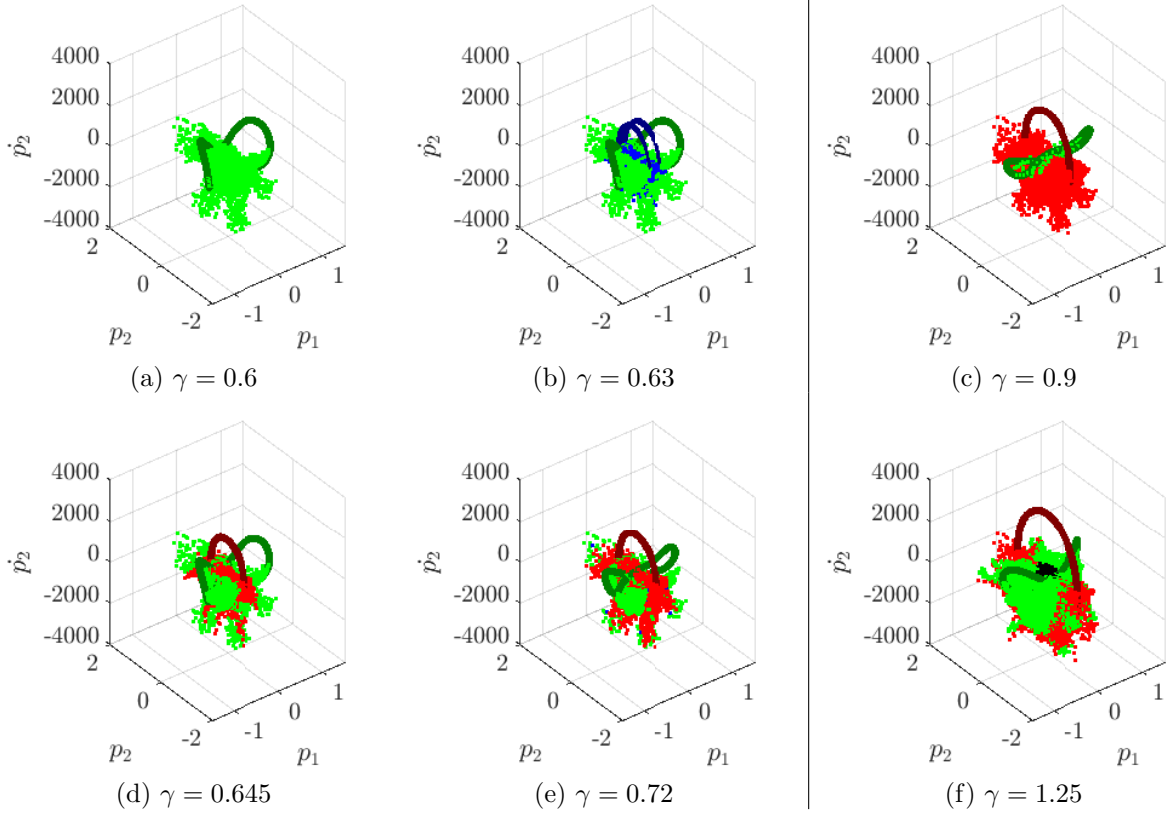


Figure 8: Attraction basins (dots) and limit cycles (lines) in a 3D projection of the phase space obtained with control scenario number three for different blowing pressures γ . Black: equilibrium, green: first register, red: second register, blue: quasi-periodic.

and quasi-periodic regimes. Indeed, one of the challenges many beginner saxophone players face on the lowest fingerings is controlling the instrument so that the first register can be produced, and not another regime. Quasi-periodic regimes are largely considered undesirable in common musical practice, however they are a common issue on the lowest fingerings of the saxophone.

5.1 Regime production regions

Expanding on the idea in Figure 6, one can study the produced regime across the two-dimensional parameter space (γ_f, ζ) , while still varying the characteristic time τ_g of control scenario number two. Figure 9 shows the classification of obtained regimes for several combinations (γ_f, ζ) and several characteristic times. Note that for a musician using their lower lip to control the instrument, it is difficult to control the reed opening parameter ζ without also varying the properties of the reed ω_r and q_r . Leaving the reed parameters constant in this study amounts to a simplification of the musician control. For readability reasons, the resolution of the cartography presented here is rather coarse: only eight values of γ_f and ζ and three characteristic times τ_g , for a total of 192 synthesized signals. The range in parameter ζ is inspired

by the range measured in [23] for artificial mouth experiments, using the method proposed in [21]. This method relies on measuring the flow rate as a function of the mouth pressure, and estimating ζ based on the maximum flow using the nonlinear characteristic of Eq. (7). As a case study, two maps computed with different inharmonicity values are presented on Figure 9 (a) and (b), so that they can be compared. In the modal formalism, the inharmonicity is changed very simply by modifying the value of the second modal frequency. Note that changing the length of the saxophone mouthpiece constitutes another method to vary the inharmonicity. However, it modifies all the modal frequencies simultaneously, which makes some interpretations less robust. Therefore, only the results obtained by varying the second modal frequency are presented here. Similar results can be obtained by varying the length of the mouthpiece. Two typical values of inharmonicity are chosen: one that could be called null, $f_2/f_1 = 2$; and the value measured on the saxophone which is slightly higher, $f_2/f_1 = 2.065$.

Focusing on Figure 9 (a), several features can be described, and recognized from the situations explored in Section 4 with a fixed ζ . Coexistence regions can be noticed on most of the map, with a given (γ_f, ζ) couple leading to different regimes depending on the char-

acteristic time. This further demonstrates that multistability is a very common phenomenon across the control parameter space in woodwind models. A particular case of coexistence occurs on all regime maps, where long attack times lead to the system remaining at equilibrium, while fast attacks can trigger oscillations. This is the same phenomenon as in Figure 6 (a). These phenomena are located, as for Figure 6 (a), at the boundaries between equilibrium and oscillation regimes, which correspond to the Hopf bifurcations of the model (where the equilibrium becomes unstable). On the regime maps, points containing both equilibrium regimes and oscillating regimes are seen on the vertical threshold near $\gamma = 0.4$, the horizontal threshold near $\zeta = 0.2$ and the extinction threshold around $\gamma = 1.1$.

Coexistence situations similar to Figure 6 (b) can also be seen on Figure 9 (a), for example at $\zeta = 1.2$ and $\gamma_f \simeq 0.8$, where the short and long characteristic times lead to the second register, while the medium time leads to a first register. This possibly indicates that the second register attraction basin almost surrounds the origin, as seen on Figure 8, in all the multistability zones between first and second register for that fingering.

Figure 9 (b) ($f_2/f_1 = 2$) displays a lot more second register (■) than Figure 9 (a) ($f_2/f_1 = 2.065$). Contrary to what could be expected, a null inharmonicity, where the second resonance frequency is twice the first, does not lead to more first register production. Since an exact integer ratio between resonances does not facilitate the production of the first register, one can ask if the model shows a particular value of inharmonicity which favors the production of first register.

5.2 Rate of produced regimes: influence of the rise time on global regime production

To study the question of the inharmonicity favoring first register production, regime maps are computed for all the fingerings of the saxophone that should produce first register regimes, meaning those where the register holes are closed. For each fingering, the second modal frequency f_2 is varied from $1.96f_1$ to $2.15f_1$, by steps of $0.01f_1$. A regime map containing $N_p = 192$ points, as for Figure 9, is then computed for each value of inharmonicity. The produced regimes are counted for the whole map, and a rate is computed for each of them with respect to the total number of oscillating regime as

$$R_i = \frac{N_{p,i}}{N_p}, \quad (19)$$

where regime i can either be first register, second register or quasi-periodic regimes, $N_{p,i}$ is the number of points corresponding to regime i in the regime map

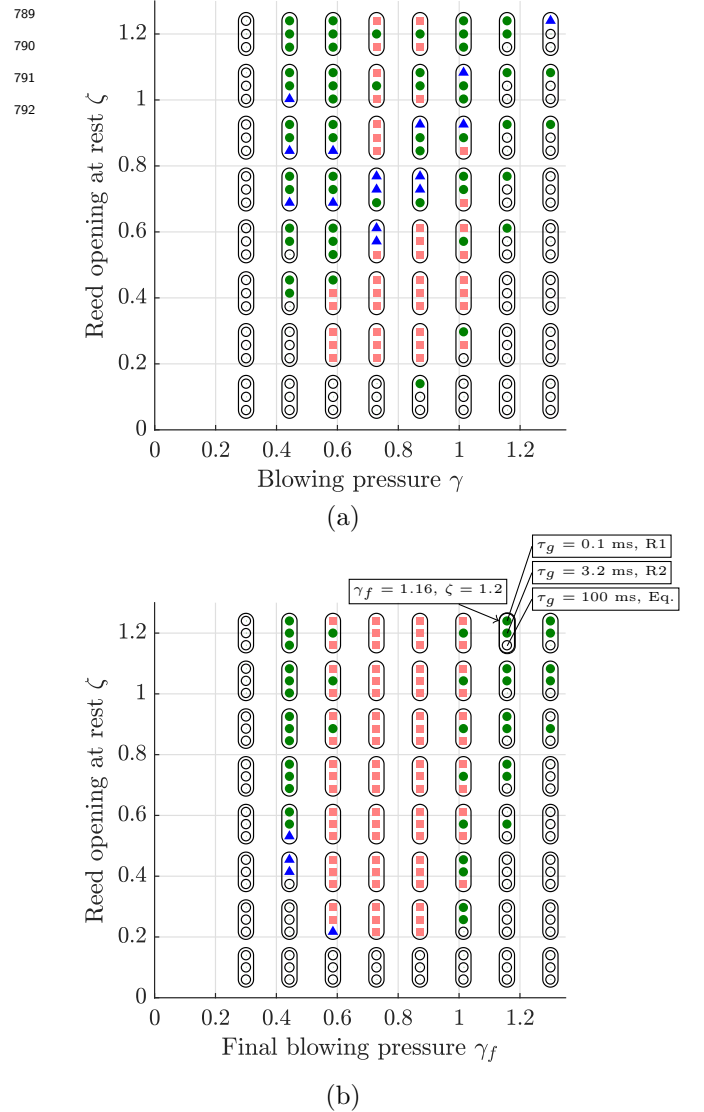


Figure 9: Classification of the regimes produced (○ equilibrium, ● first register, ■ second register, ▲ quasi-periodic) depending on control parameters: γ_f and ζ . Each rectangle corresponds to a couple (γ_f, ζ) and the points inside indicate the regime for each characteristic time τ_g (bottom 0.1 ms, middle 3 ms and top 100 ms). One rectangle on graph (b) is annotated as an example. Graphs correspond to two inharmonicities for low written D# fingering (a) $f_2 = 2.065f_1$, the value measured on a real saxophone and (b) $f_2 = 2f_1$.

and $N_{p,o}$ is the total number of points corresponding to any oscillating regime (i.e. all regimes but equilibrium). Note that this description ignores non-oscillating regimes.

Figure 10 depicts the rate of each oscillating regime produced depending on inharmonicity for the lowest fingering of the first register, the written Bb which produces the heard note C3 at 131 Hz. On this figure the regimes are counted separately for each characteristic time τ_g before being summed for the whole

map to produce an averaged rate. On the averaged 872 839ing on the chosen attack time. Notice that the lowest rate, optimal points are highlighted by triangles, corresponding to the maximum of first register and minimum 874 840rate of first register corresponds to the longest attack time $\tau_g = 100$ ms. Figure 9 shows that the first register 875 841time $\tau_g = 100$ ms. Figure 9 shows that the first register regimes are produced on the edges of the zone of 876 oscillation, in a multistability region between the first 877 register and the equilibrium. The longest attack times in this region tend not to lead to a first register, but 878 instead to an equilibrium due to its attraction basin 879 surrounding the origin (see for instance Figure 7 (c) or 880 multimedia file `Supp1.mp4`). Figure 11 (b) shows the 881 results for the written D fingering. This case exhibits 882 an outlier: the shortest attack time yields a optimal 883 inharmonicity value of 2.08, whereas the others point 884 to 2.04. In this case, considering several attack times 885 is a way to smooth out outliers due to a particular 886 value of the attack time. 887

Note that Graph 11 (b) can also be subject to an interesting interpretation in terms of musician control strategies: the fact that certain attack time values seem to markedly decrease the rate of production of a certain regime could be used by the musician to avoid producing it. 888 889 890 891 892 893

5.3 Inharmonicity of the saxophone

In this section, the optimal inharmonicity in terms of regime production is studied for the seven lowest fingerings of the instrument. Higher fingerings are not represented because they add no relevant information: first register regime production rates are close to 100% for all the studied inharmonicity values. This corresponds to the saxophonists' experience that the high notes of the instrument's first register are often easier to produce than the low notes, and to the fact that the first impedance peak is much higher than the others on the high fingerings [14]. The optimums are compared with the inharmonicity value measured on the saxophone on which the model is based. Figure 12 summarizes the production ratios for all the fingerings. The optimal inharmonicity seems to vary across the fingerings. It is always greater than two: null inharmonicity does not favor first register production on the low fingerings of the saxophone. The two optimums are close to the measured inharmonicity. Additionally, the trend is respected, with optimal and measured harmonicities increasing for higher fingerings. Note that the optimum for the E fingering is very far from the measured inharmonicity, but the production ratios are almost constant. Overall, the simulation shows that the most first register and least second register is produced by the model for values of inharmonicity near those measured on the saxophone. This result sheds some light on the empirical choice of the acoustical properties on the saxophone. Indeed, if the inharmonicity was far from the values observed on saxophones, the model predicts more second register would be produced. This effect is arguably undesirable. However, this choice comes with a compromise,

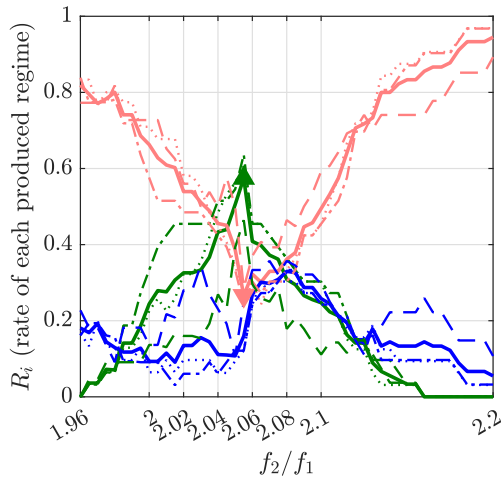


Figure 10: Rate of produced regimes (Eq. (19)) for 914 written low Bb fingering. Green: first register, red: 915 second register, blue: quasi-periodic. Linestyles indicate the characteristic time. Dotted: $\tau_g = 0.1$ ms, 916 dash-dot: $\tau_g = 3.2$ ms, dashed: $\tau_g = 100$ ms, solid: 917 averaged rate. An upward triangle marks the maximum 918 first register averaged rate, a downward triangle 919 marks the minimum second register rate. 920 921 922

For the D# fingering (Figure 11 (a)), one can see 923 that depending on the chosen increase duration τ_g the 924 production ratio varies greatly (from 20% to 60%). If 925 any quantitative interpretation is to be expected from 926 these results, it can be changed dramatically depend- 927

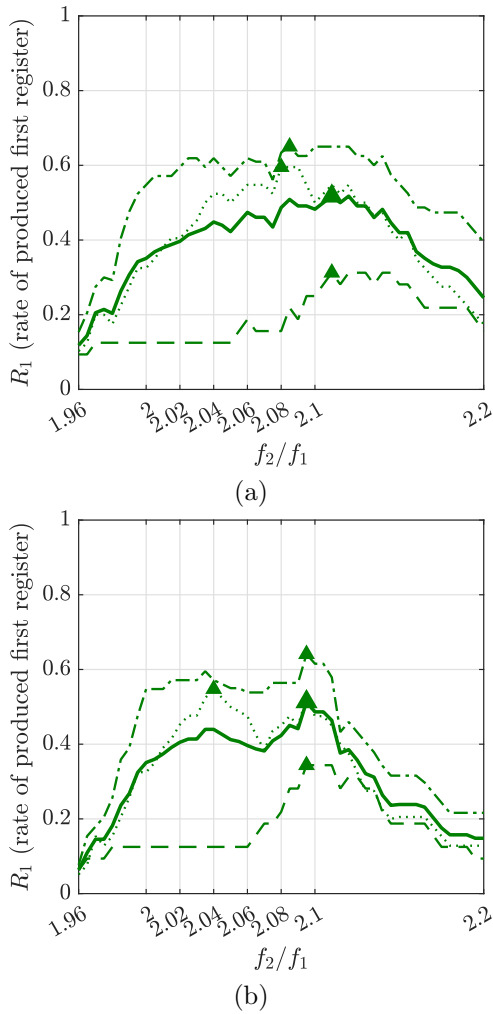


Figure 11: Rate of first register regime (Eq. (19)) produced for (a) written low $D\sharp$ fingering and (b) written low D fingering. Linestyles indicate the characteristic time. Dotted: $\tau_g = 0.1$ ms, dash-dot: $\tau_g = 3.2$ ms, dashed: $\tau_g = 100$ ms, solid: averaged rate. An upward triangle marks the maximums of first register rates.

as it also favors quasi-periodic regime production (see Figure 10), which are a known issue in low saxophone fingerings.

6 Conclusion

In the studied saxophone model, stable regimes coexist throughout large regions in the musician control parameter space. Thus, even an exhaustive description of the stability or instability of each oscillating regime in the control parameter space is only an incomplete answer, as it doesn't suffice to predict which regime emerges in these multistability zones. In particular, in the event that the attraction basin of a stable regime proves too small or unattainable by the musician, this regime should be considered unreachable, almost like an unstable regime. This work demon-

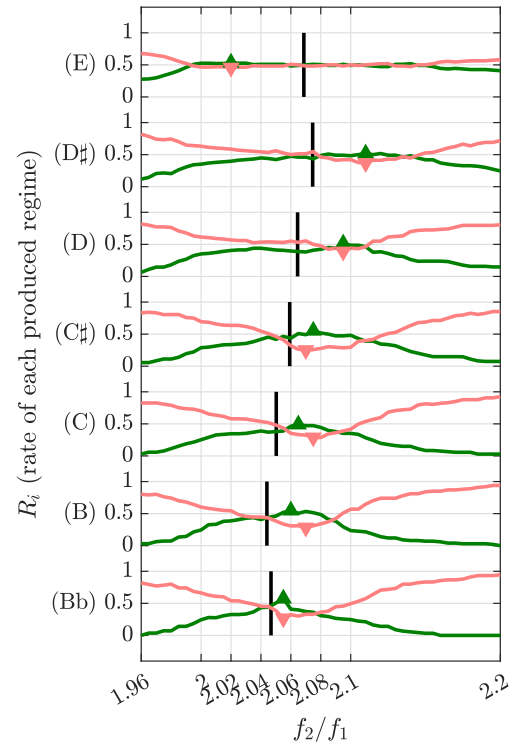


Figure 12: Rate of produced regimes for the lowest fingerings of the alto saxophone (written pitch). Green: first register, red: second register. An upward triangle marks the maximum first register averaged rate, a downward triangle marks the minimum second register rate. Vertical lines mark the measured inharmonicity values.

strates that saxophone regime stability is a nuanced topic.

An adapted combination of two radically different numerical methods provides a more complete description of the model's behavior: the stability study performed by harmonic balance method is completed by time-domain synthesis, which quantifies multistability by outlining the attraction basin of each regime. A varying control scenario explores which regime is produced in multistability zones, with the advantage that it can be tied to plausible musician actions. However, a varying control scenario only provides a very partial view of the attraction basins, and its results deserve to be explicated by representing the attraction basins in the phase space using different initial conditions. Dedicated experimental work, out of the scope of this paper, could help design more realistic control scenarios.

Accounting for multistability, the study of synthesized regimes may explain an acoustical choice made by instruments makers: the inharmonicity of the saxophone. An integer ratio between the first and second resonance frequencies does not favor the production of first register. Note that this result adds nuance one of Benade's guidelines stating that an oscillation

is favored if the impedance is large at its fundamental¹⁰¹⁸ 968
 and its harmonic frequencies [4]. This work shows¹⁰¹⁹ 969
 that competition between registers also comes into⁹⁷⁰
 play, depending on more than solely the impedance¹⁰²⁰ 971
 magnitude at the playing frequency and its harmon-¹⁰²¹ 972
 ics. Instead, an integer ratio between the first and¹⁰²² 973
 second resonance frequencies tends to favor the pro-
 duction of second register, which is arguably unde-
 sirable for a first register fingering. Carefully tuned
 inharmonic resonances, where the second frequency
 is higher than twice the first, can lead to more first
 register production. The optimal inharmonicity value
 found on the model is close to harmonicities measured
 on saxophone resonators. This result provides an *a*
posteriori interpretation of the acoustical characteris-
 tics of the saxophone, as chosen empirically by instru-
 ment makers, as the acoustical characteristic leading
 to easier production of the first register. Such results
 are among the first steps towards applying numerical
 simulations as predictive tools to estimate playability
 in instrument design.

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