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Teo Banica

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# Quantum permutation groups

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ABSTRACT. The permutation group  $S_N$  has a quantum analogue  $S_N^+$ , which is infinite at  $N \geq 4$ . We review the known facts regarding  $S_N^+$ , and its versions  $S_F^+$ , with  $F$  being a finite quantum space. We discuss then the structure of the closed subgroups  $G \subset S_N^+$  and  $G \subset S_F^+$ , with particular attention to the quantum reflection groups.

## Preface

One of the most puzzling discoveries in quantum algebra, going back to work of Wang from the late 90s, in answer to a question of Connes, is that the set  $X = \{1, \dots, 4\}$ , and more generally the set  $X = \{1, \dots, N\}$  with  $N \geq 4$ , has an infinity of quantum permutations. At the first glance, this looks as one of these physicists' crazy things, which might be worth attention or not. But please don't go away, and listen to what I have so say. Yes, all this is related to physics, and even to quite crazy physics, to put it this way. But the mathematics behind is extremely simple, and worth some attention.

Let us first look at the symmetric group  $S_N$ . When regarding this group geometrically, as the group of permutations of the  $N$  coordinate axes of  $\mathbb{R}^N$ , the standard coordinates  $u_{ij} \in C(S_N)$  are given by a very simple formula, namely  $u_{ij}(\sigma) = \delta_{\sigma(j)i}$ . It follows that these coordinates  $u_{ij} \in C(S_N)$  form a matrix  $u = (u_{ij})$  which is "magic", in the sense that its entries are projections,  $p^2 = p = p^*$ , which sum up to 1 on each row and each column. Moreover, by Stone-Weierstrass we have  $C(S_N) = \langle u_{ij} \rangle$ , and with a bit more work, by using the Gelfand theorem, we conclude that  $C(S_N)$  is isomorphic to the universal commutative  $C^*$ -algebra generated by the entries of a  $N \times N$  magic matrix.

This is quite interesting, and suggests looking at the universal  $C^*$ -algebra  $C(S_N^+)$  generated by the entries of a  $N \times N$  magic matrix. In analogy with what happens for  $C(S_N)$ , this algebra has a comultiplication  $\Delta$ , a counit  $\varepsilon$ , and an antipode  $S$ , and so according to the general compact quantum group theory developed by Woronowicz, its abstract spectrum  $S_N^+$  is a compact quantum group, called quantum permutation group. And the point is that the inclusion  $S_N \subset S_N^+$  is not an isomorphism at  $N \geq 4$ , because diagonally joining magic matrices of size  $\geq 2$  leads to the conclusion that  $S_N^+$  is a non-classical, infinite compact quantum group, substantially bigger than  $S_N$ .

Summarizing, some interesting mathematics going on here, and with this digested, the first thought goes to physics. Can such beasts be of help in connection with statistical mechanics, along the lines suggested by Jones? What about quarks and the Standard Model, along the lines suggested by Connes? What about statistical mechanics and nuclear physics alike, via random matrices and freeness in the sense of Voiculescu? And also, in tune with our times, what about applications to quantum information?

These questions are all old and difficult, going back to Wang's discovery of  $S_N^+$  in the late 90s, and the few couple of years afterwards. So, perhaps more modestly, we should start with some pure mathematics, before getting into such questions. The point indeed is that the symmetric group  $S_N$  and its various subgroups  $G \subset S_N$  have a lot of interesting mathematical properties, so before anything, we should understand what the analogous theory of  $S_N^+$  and of its closed quantum subgroups  $G \subset S_N^+$  is. And with a bit of luck, we will get in this way precisely into the mathematics of Connes, Jones, Voiculescu, putting us on the right track for doing some physics afterwards.

The present book is precisely about this, the mathematics of  $S_N^+$  and of its closed subgroups  $G \subset S_N^+$ , with physics motivations in mind. We will be interested in mathematics in a large sense, mixing algebra, geometry, analysis and probability. At the level of subgroups, we will mostly insist on the quantum reflection groups, which are quite fundamental objects, mathematically speaking, and which are expected as well to be useful in physics. Finally, we will discuss also the more general case of the quantum permutation groups  $S_F^+$  of the arbitrary "finite quantum spaces"  $F$ , and their closed subgroups  $G \subset S_F^+$ , with special attention to the quantum reflection subgroups.

There is a lot of material to be explained here, and the present book lies somewhere between graduate textbook and research monograph. Some prior familiarity with the quantum groups, although not really needed, is recommended, and a way of getting started is by quickly reading first my graduate textbook "Introduction to quantum groups". However, the present book, while being sometimes quick and heavy, remains a self-contained text, and can be used as such as a basis for a 1-year graduate course.

Getting back to physics, there will be not much of it in this book, which is meant to be a purely mathematical text. However, the relation with the work of Connes, Jones, Voiculescu, potentially leading to physics, will be carefully explained. In short, you will learn in this way everything that is needed for doing good physics afterwards.

This book is heavily based on a number of research papers on quantum permutations and reflections, and I am particularly grateful to Julien Bichon, Benoît Collins and Steve Curran, for substantial joint work on the subject. Many thanks go as well to my cats, for precious support and advice, during the preparation of the present book.

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Part I

Quantum permutations

*Carry me, caravan, take me away  
Take me to Portugal, take me to Spain  
Andalucia with fields full of grain  
I have to see you again and again*

## CHAPTER 1

### Quantum groups

#### 1a. Quantum groups

Generally speaking, a quantum group is a “quantum space” with group type structure. The quantum permutation groups, that we will be interested in here, are not finite, but rather compact. This might seem a bit surprising, but remember that what we are doing here is quantum mechanics, where particles do not have clear positions and speeds, and with this being a well-established fact. Along the same lines, the space  $\{1, \dots, N\}$  can be shown to have an infinity of “quantum permutations”, and this is how things are.

Long story short, to start with, we need a good formalism for the “compact quantum spaces”. There are several such formalisms, and a particularly simple and beautiful one, which is exactly what we need for our quantum permutation group purposes, is provided by the  $C^*$ -algebra theory. The starting definition here is as follows:

**DEFINITION 1.1.** *A  $C^*$ -algebra is a complex algebra  $A$ , having a norm  $\|\cdot\|$  making it a Banach algebra, and an involution  $*$ , related to the norm by the formula*

$$\|aa^*\| = \|a\|^2$$

*which must hold for any  $a \in A$ .*

As a basic example, the usual matrix algebra  $M_N(\mathbb{C})$  is a  $C^*$ -algebra, with the usual matrix norm and involution, namely:

$$\|M\| = \sup_{\|x\|=1} \|Mx\|$$

$$(M^*)_{ij} = \bar{M}_{ji}$$

More generally, any  $*$ -subalgebra  $A \subset M_N(\mathbb{C})$  is automatically closed, and so is a  $C^*$ -algebra. In fact, in finite dimensions, the situation is as follows:

**THEOREM 1.2.** *The finite dimensional  $C^*$ -algebras are exactly the algebras of type*

$$A = M_{N_1}(\mathbb{C}) \oplus \dots \oplus M_{N_k}(\mathbb{C})$$

*with norm  $\|(a_1, \dots, a_k)\| = \sup_i \|a_i\|$ , and involution  $(a_1, \dots, a_k)^* = (a_1^*, \dots, a_k^*)$ .*

PROOF. In one sense this is clear, either by standard direct sum arguments, or because with  $N = N_1 + \dots + N_k$  we have an embedding of  $*$ -algebras  $A \subset M_N(\mathbb{C})$ . In the other sense, this is something more subtle, coming by breaking the unit of our finite dimensional  $C^*$ -algebra  $A$  as a sum of central minimal projections, as follows:

$$1 = p_1 + \dots + p_k$$

Indeed, when doing so, each of the  $*$ -algebras  $A_i = p_i A p_i$  follows to be a matrix algebra,  $A_i \simeq M_{N_i}(\mathbb{C})$ , and this gives the direct sum decomposition in the statement.  $\square$

In general now, the main theoretical result about the  $C^*$ -algebras, due to Gelfand, Naimark and Segal, and called GNS representation theorem, is as follows:

**THEOREM 1.3.** *Given a Hilbert space  $H$ , the algebra  $B(H)$  of linear bounded operators  $T : H \rightarrow H$  is a  $C^*$ -algebra, with norm and involution given by:*

$$\begin{aligned} \|T\| &= \sup_{\|x\|=1} \|Tx\| \\ \langle Tx, y \rangle &= \langle x, T^*y \rangle \end{aligned}$$

*More generally, and norm closed  $*$ -subalgebra of this full operator algebra*

$$A \subset B(H)$$

*is a  $C^*$ -algebra. Any  $C^*$ -algebra appears in this way, for a certain Hilbert space  $H$ .*

PROOF. There are several statements here, with the first ones being standard operator theory, and with the last one being the GNS theorem, the idea being as follows:

(1) First of all, the full operator algebra  $B(H)$  is a Banach algebra. Indeed, given a Cauchy sequence  $\{T_n\}$  inside  $B(H)$ , we can set  $Tx = \lim_{n \rightarrow \infty} T_n x$ , for any  $x \in H$ . It is then routine to check that we have  $T \in B(H)$ , and that  $T_n \rightarrow T$  in norm.

(2) Regarding the involution, the point is that we must have  $\langle Tx, y \rangle = \langle x, T^*y \rangle$ , for a certain vector  $T^*y \in H$ . But this can serve as a definition for  $T^*$ , and the fact that  $T^*$  is indeed linear, and bounded, with the bound  $\|T^*\| = \|T\|$ , is routine. As for the formula  $\|TT^*\| = \|T\|^2$ , this is elementary as well, coming by double inequality.

(3) The assertion about the subalgebras  $A \subset B(H)$  which are closed under the norm and the involution is clear from definitions.

(4) Finally, the fact that any  $C^*$ -algebra appears as  $A \subset B(H)$ , for a certain Hilbert space  $H$ , is advanced. The idea is that each  $a \in A$  acts on  $A$  by multiplication,  $T_a(b) = ab$ . Thus, we are more or less led to the result, provided that we are able to convert our algebra  $A$ , regarded as a complex vector space, into a Hilbert space  $H = L^2(A)$ . But this latter conversion can be done, by using advanced functional analysis techniques.  $\square$

As a third and last basic result about the  $C^*$ -algebras, which will be of particular interest for us, we have the following well-known theorem of Gelfand:

**THEOREM 1.4.** *Given a compact space  $X$ , the algebra  $C(X)$  of continuous functions  $f : X \rightarrow \mathbb{C}$  is a  $C^*$ -algebra, with norm and involution as follows:*

$$\|f\| = \sup_{x \in X} |f(x)|$$

$$f^*(x) = \overline{f(x)}$$

*This algebra is commutative, and any commutative  $C^*$ -algebra  $A$  is of this form, with  $X = \text{Spec}(A)$  appearing as the space of Banach algebra characters  $\chi : A \rightarrow \mathbb{C}$ .*

**PROOF.** Once again, there are several statements here, some of them being trivial, and some of them being advanced, the idea being as follows:

(1) First of all, the fact that  $C(X)$  is indeed a Banach algebra is clear, because a uniform limit of continuous functions must be continuous.

(2) Regarding now for the formula  $\|ff^*\| = \|f\|^2$ , this is something trivial for functions, because on both sides we obtain  $\sup_{x \in X} |f(x)|^2$ .

(3) Given a commutative  $C^*$ -algebra  $A$ , the character space  $X = \{\chi : A \rightarrow \mathbb{C}\}$  is compact, and we have an evaluation morphism  $ev : A \rightarrow C(X)$ .

(4) The tricky point, which follows from basic spectral theory in Banach algebras, is to prove that  $ev$  is indeed isometric. This gives the last assertion.  $\square$

In what follows, we will be mainly using Definition 1.1 and Theorem 1.4, as general theory. To be more precise, in view of Theorem 1.4, let us formulate:

**DEFINITION 1.5.** *Given an arbitrary  $C^*$ -algebra  $A$ , we agree to write*

$$A = C(X)$$

*and call the abstract space  $X$  a compact quantum space.*

In other words, we can define the category of compact quantum spaces  $X$  as being the category of the  $C^*$ -algebras  $A$ , with the arrows reversed. A morphism  $f : X \rightarrow Y$  corresponds by definition to a morphism  $\Phi : C(Y) \rightarrow C(X)$ , a product of spaces  $X \times Y$  corresponds by definition to a product of algebras  $C(X) \otimes C(Y)$ , and so on.

Finally, no discussion here would be complete without a word about von Neumann algebras. These are operator algebras of more advanced type, that we will use later on, in connection with more advanced questions. Their basic theory is as follows:

**THEOREM 1.6.** *For a  $*$ -algebra  $A \subset B(H)$  the following conditions are equivalent, and if they are satisfied, we say that  $A$  is a von Neumann algebra:*

- (1)  *$A$  is closed with respect to the weak topology, making each  $T \rightarrow Tx$  continuous.*
- (2)  *$A$  is equal to its algebraic bicommutant,  $A = A''$ , computed inside  $B(H)$ .*

*As basic examples, we have the algebras  $A = L^\infty(X)$ , acting on  $H = L^2(X)$ . Such algebras are commutative, any any commutative von Neumann algebra is of this form.*

PROOF. There are several assertions here, the idea being as follows:

(1) The equivalence (1)  $\iff$  (2) is the well-known bicommutant theorem of von Neumann, which can be proved by using an amplification trick,  $H \rightarrow \mathbb{C}^N \otimes H$ .

(2) Given a measured space  $X$ , we have indeed an embedding  $L^\infty(X) \subset B(L^2(X))$ , with weakly closed image, given by  $T_f : g \rightarrow fg$ , as in the proof of the GNS theorem.

(3) Given a commutative von Neumann algebra  $A \subset B(H)$  we can write  $A = \langle T \rangle$  with  $T$  being a normal operator, and the Spectral Theorem gives  $A \simeq L^\infty(X)$ .  $\square$

In the context of a  $C^*$ -algebra representation  $A \subset B(H)$  we can consider the weak closure, or bicommutant  $A'' \subset B(H)$ , which is a von Neumann algebra. In the commutative case,  $C(X) \subset B(L^2(X))$ , the weak closure is  $L^\infty(X)$ . In general, we agree to write:

$$A'' = L^\infty(X)$$

We are ready now to introduce the compact quantum groups. The axioms here, due to Woronowicz [98], and slightly modified for our purposes, are as follows:

DEFINITION 1.7. *A Woronowicz algebra is a  $C^*$ -algebra  $A$ , given with a unitary matrix  $u \in M_N(A)$  whose coefficients generate  $A$ , such that the formulae*

$$\begin{aligned} \Delta(u_{ij}) &= \sum_k u_{ik} \otimes u_{kj} \\ \varepsilon(u_{ij}) &= \delta_{ij} \\ S(u_{ij}) &= u_{ji}^* \end{aligned}$$

define morphisms of  $C^*$ -algebras as follows,

$$\begin{aligned} \Delta : A &\rightarrow A \otimes A \\ \varepsilon : A &\rightarrow \mathbb{C} \\ S : A &\rightarrow A^{opp} \end{aligned}$$

called comultiplication, counit and antipode.

In the above definition the tensor product used in the definition of  $\Delta$  can be any  $C^*$ -algebra tensor product. In order to get rid of redundancies, coming from this and from amenability issues, we will divide everything by an equivalence relation, as follows:

DEFINITION 1.8. *We agree to identify two Woronowicz algebras,  $(A, u) = (B, v)$ , when we have an isomorphism of  $*$ -algebras as follows,*

$$\langle u_{ij} \rangle \simeq \langle v_{ij} \rangle$$

mapping standard coordinates to standard coordinates,  $u_{ij} \rightarrow v_{ij}$ .

We say that  $A$  is cocommutative when  $\Sigma\Delta = \Delta$ , where  $\Sigma(a \otimes b) = b \otimes a$  is the flip. We have then the following key result, from [98], providing us with examples:

PROPOSITION 1.9. *The following are Woronowicz algebras:*

(1)  $C(G)$ , with  $G \subset U_N$  compact Lie group. Here the structural maps are:

$$\Delta(\varphi) = (g, h) \rightarrow \varphi(gh)$$

$$\varepsilon(\varphi) = \varphi(1)$$

$$S(\varphi) = g \rightarrow \varphi(g^{-1})$$

(2)  $C^*(\Gamma)$ , with  $F_N \rightarrow \Gamma$  finitely generated group. Here the structural maps are:

$$\Delta(g) = g \otimes g$$

$$\varepsilon(g) = 1$$

$$S(g) = g^{-1}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

PROOF. In both cases, we have to exhibit a certain matrix  $u$ , and then prove that we have indeed a Woronowicz algebra. The constructions are as follows:

(1) For the first assertion, we can use the matrix  $u = (u_{ij})$  formed by the standard matrix coordinates of  $G$ , which is by definition given by:

$$g = \begin{pmatrix} u_{11}(g) & \dots & u_{1N}(g) \\ \vdots & & \vdots \\ u_{N1}(g) & \dots & u_{NN}(g) \end{pmatrix}$$

(2) For the second assertion, we can use the diagonal matrix formed by generators:

$$u = \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_N \end{pmatrix}$$

Finally, regarding the last assertion, in the commutative case this follows from the Gelfand result, Theorem 1.4 above. In the cocommutative case this is something more complicated, requiring as well an amenability discussion. We will be back to this.  $\square$

In order to get now to quantum groups, we will need as well:

PROPOSITION 1.10. *Assuming that  $G \subset U_N$  is abelian, we have an identification of Woronowicz algebras  $C(G) = C^*(\Gamma)$ , with  $\Gamma$  being the Pontrjagin dual of  $G$ :*

$$\Gamma = \{\chi : G \rightarrow \mathbb{T}\}$$

*Conversely, assuming that  $F_N \rightarrow \Gamma$  is abelian, we have an identification of Woronowicz algebras  $C^*(\Gamma) = C(G)$ , with  $G$  being the Pontrjagin dual of  $\Gamma$ :*

$$G = \{\chi : \Gamma \rightarrow \mathbb{T}\}$$

*Thus, the Woronowicz algebras which are both commutative and cocommutative are exactly those of type  $A = C(G) = C^*(\Gamma)$ , with  $G, \Gamma$  being abelian, in Pontrjagin duality.*



PROOF. All this follows from Gelfand duality, Theorem 1.4 above, because the characters of a group algebra are in correspondence with the characters of the group.  $\square$

We have the following definition, complementing Definition 1.7 and Definition 1.8:

DEFINITION 1.11. *Given a Woronowicz algebra, we write it as follows, and call  $G$  a compact quantum Lie group, and  $\Gamma$  a finitely generated discrete quantum group:*

$$A = C(G) = C^*(\Gamma)$$

Also, we say that  $G, \Gamma$  are dual to each other, and write  $G = \widehat{\Gamma}, \Gamma = \widehat{G}$ .

Summarizing, we have a nice axiomatic framework, for both the compact and the discrete quantum groups.

Let us discuss now some tools for studying the Woronowicz algebras, and the underlying quantum groups. First, we have the following result:

PROPOSITION 1.12. *Let  $(A, u)$  be a Woronowicz algebra.*

(1)  $\Delta, \varepsilon$  satisfy the usual axioms for a comultiplication and a counit, namely:

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$$

(2)  $S$  satisfies the antipode axiom, on the  $*$ -algebra generated by entries of  $u$ :

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(\cdot)1$$

(3) In addition, the square of the antipode is the identity,  $S^2 = id$ .

PROOF. As a first observation, the result holds in the commutative case,  $A = C(G)$  with  $G \subset U_N$ . Indeed, here we know from Proposition 1.9 that  $\Delta, \varepsilon, S$  appear as functional analytic transposes of the multiplication, unit and inverse maps  $m, u, i$ :

$$\Delta = m^t \quad , \quad \varepsilon = u^t \quad , \quad S = i^t$$

With these remark in hand, the various conditions in the statement on  $\Delta, \varepsilon, S$  come by transposition from the group axioms satisfied by  $m, u, i$ , namely:

$$m(m \times id) = m(id \times m)$$

$$m(u \times id) = m(id \times u) = id$$

$$m(i \times id)\delta = m(id \times i)\delta = 1$$

Observe that the condition  $S^2 = id$  is satisfied too, coming by transposition from the formula  $i^2 = id$ , which corresponds to the following formula, for group elements:

$$(g^{-1})^{-1} = g$$

The result holds as well in the cocommutative case,  $A = C^*(\Gamma)$  with  $F_N \rightarrow \Gamma$ , trivially. In general now, the two comultiplication axioms follow from:

$$(\Delta \otimes id)\Delta(u_{ij}) = (id \otimes \Delta)\Delta(u_{ij}) = \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj}$$

$$(\varepsilon \otimes id)\Delta(u_{ij}) = (id \otimes \varepsilon)\Delta(u_{ij}) = u_{ij}$$

As for the antipode axiom, the verification here is similar. First, we have the following computation, by using the fact that the matrix  $u = (u_{ij})$  is unitary:

$$m(S \otimes id)\Delta(u_{ij}) = \sum_k u_{ki}^* u_{kj} = (u^* u)_{ij} = \delta_{ij}$$

On the other hand, we have as well the following computation:

$$m(id \otimes S)\Delta(u_{ij}) = \sum_k u_{ik} u_{jk}^* = (u u^*)_{ij} = \delta_{ij}$$

Finally, we have  $S^2(u_{ij}) = u_{ij}$ , and so  $S^2 = id$  everywhere, as claimed.  $\square$

### 1b. Peter-Weyl theory

In the compact Lie group case, in order to reach to more advanced results, the idea is to do either representation theory, or Lie algebras. In what regards the compact quantum Lie groups, there is no Lie algebra that can be defined, at least in some elementary sense, and we are left with doing representation theory.

Following [98], let us start with the following definition:

**DEFINITION 1.13.** *Given a Woronowicz algebra  $A$ , we call corepresentation of it any unitary matrix  $v \in M_n(A)$  satisfying the same conditions are those satisfied by  $u$ , namely:*

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

*We also say that  $v$  is a representation of the underlying compact quantum group  $G$ , and a corepresentation of the underlying discrete quantum group  $\Gamma$ .*

In the commutative case,  $A = C(G)$  with  $G \subset U_n$ , we obtain in this way the finite dimensional unitary smooth representations  $v : G \rightarrow U_n$ , as follows:

$$v(g) = \begin{pmatrix} v_{11}(g) & \cdots & v_{1n}(g) \\ \vdots & & \vdots \\ v_{n1}(g) & \cdots & v_{nn}(g) \end{pmatrix}$$

In the cocommutative case,  $A = C^*(\Gamma)$  with  $F_N \rightarrow \Gamma$ , we will see in a moment that we obtain in this way the formal sums of elements of  $\Gamma$ , possibly rotated by a unitary. As a first result now regarding the representations, we have:

PROPOSITION 1.14. *The corepresentations are subject to the following operations:*

- (1) *Making sums,  $v + w = \text{diag}(v, w)$ .*
- (2) *Making tensor products,  $(v \otimes w)_{ia,jb} = v_{ij}w_{ab}$ .*
- (3) *Taking conjugates,  $(\bar{v})_{ij} = v_{ij}^*$ .*
- (4) *Rotating by a unitary,  $v \rightarrow UvU^*$ .*

PROOF. We first check the fact that the matrices in the statement are unitaries:

(1) The fact that  $v + w$  is unitary is clear.

(2) Regarding now  $v \otimes w$ , this can be written in standard leg-numbering notation as  $v \otimes w = v_{13}w_{23}$ , and with this interpretation in mind, the unitarity is clear as well.

(3) In order to check that  $\bar{v}$  is unitary, we can use the antipode. Indeed, by regarding the antipode as an antimultiplicative map  $S : A \rightarrow A$ , we have:

$$(\bar{v}v^t)_{ij} = \sum_k v_{ik}^* v_{jk} = \sum_k S(v_{kj}^* v_{ki}) = S((v^*v)_{ji}) = \delta_{ij}$$

We have as well the following computation:

$$(v^t\bar{v})_{ij} = \sum_k v_{ki} v_{kj}^* = \sum_k S(v_{jk} v_{ik}^*) = S((vv^*)_{ji}) = \delta_{ij}$$

(4) Finally, the fact that  $UvU^*$  is unitary is clear. As for the verification of the comultiplicativity axioms, involving  $\Delta, \varepsilon, S$ , this is routine, in all cases.  $\square$

As a consequence of the above result, we can formulate:

DEFINITION 1.15. *We denote by  $u^{\otimes k}$ , with  $k = \circ \bullet \bullet \circ \dots$  being a colored integer, the various tensor products between  $u, \bar{u}$ , indexed according to the rules*

$$u^{\otimes \emptyset} = 1 \quad , \quad u^{\otimes \circ} = u \quad , \quad u^{\otimes \bullet} = \bar{u}$$

*and multiplicativity,  $u^{\otimes kl} = u^{\otimes k} \otimes u^{\otimes l}$ , and call them Peter-Weyl corepresentations.*

Here are a few examples of such corepresentations, namely those coming from the colored integers of length 2, to be often used in what follows:

$$u^{\otimes \circ \circ} = u \otimes u$$

$$u^{\otimes \circ \bullet} = u \otimes \bar{u}$$

$$u^{\otimes \bullet \circ} = \bar{u} \otimes u$$

$$u^{\otimes \bullet \bullet} = \bar{u} \otimes \bar{u}$$

In order to do representation theory, we first need to know how to integrate over  $G$ . And we have here the following key result, due to Woronowicz [98]:

THEOREM 1.16. Any Woronowicz algebra  $A = C(G)$  has a unique Haar integration,

$$\left( \int_G \otimes id \right) \Delta = \left( id \otimes \int_G \right) \Delta = \int_G (\cdot) 1$$

which can be constructed by starting with any faithful positive form  $\varphi \in A^*$ , and setting

$$\int_G = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}$$

where  $\phi * \psi = (\phi \otimes \psi) \Delta$ . Moreover, for any corepresentation  $v \in M_n(\mathbb{C}) \otimes A$  we have

$$\left( id \otimes \int_G \right) v = P$$

where  $P$  is the orthogonal projection onto  $Fix(v) = \{\xi \in \mathbb{C}^n \mid v\xi = \xi\}$ .

PROOF. Following [98], this can be done in 3 steps, as follows:

(1) Given  $\varphi \in A^*$ , our claim is that the following limit converges, for any  $a \in A$ :

$$\int_{\varphi} a = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}(a)$$

Indeed, by linearity we can assume that  $a$  is the coefficient of certain corepresentation,  $a = (\tau \otimes id)v$ . But in this case, an elementary computation gives the following formula, with  $P_{\varphi}$  being the orthogonal projection onto the 1-eigenspace of  $(id \otimes \varphi)v$ :

$$\left( id \otimes \int_{\varphi} \right) v = P_{\varphi}$$

(2) Since  $v\xi = \xi$  implies  $[(id \otimes \varphi)v]\xi = \xi$ , we have  $P_{\varphi} \geq P$ , where  $P$  is the orthogonal projection onto the following fixed point space:

$$Fix(v) = \left\{ \xi \in \mathbb{C}^n \mid v\xi = \xi \right\}$$

The point now is that when  $\varphi \in A^*$  is faithful, by using a standard positivity trick, one can prove that we have  $P_{\varphi} = P$ . Assume indeed  $P_{\varphi}\xi = \xi$ , and let us set:

$$a = \sum_i \left( \sum_j v_{ij}\xi_j - \xi_i \right) \left( \sum_k v_{ik}\xi_k - \xi_i \right)^*$$

We must prove that we have  $a = 0$ . Since  $v$  is biunitary, we have:

$$\begin{aligned}
a &= \sum_i \left( \sum_j \left( v_{ij} \xi_j - \frac{1}{N} \xi_i \right) \right) \left( \sum_k \left( v_{ik}^* \bar{\xi}_k - \frac{1}{N} \bar{\xi}_i \right) \right) \\
&= \sum_{ijk} v_{ij} v_{ik}^* \xi_j \bar{\xi}_k - \frac{1}{N} v_{ij} \xi_j \bar{\xi}_i - \frac{1}{N} v_{ik}^* \xi_i \bar{\xi}_k + \frac{1}{N^2} \xi_i \bar{\xi}_i \\
&= \sum_j |\xi_j|^2 - \sum_{ij} v_{ij} \xi_j \bar{\xi}_i - \sum_{ik} v_{ik}^* \xi_i \bar{\xi}_k + \sum_i |\xi_i|^2 \\
&= \|\xi\|^2 - \langle v\xi, \xi \rangle - \overline{\langle v\xi, \xi \rangle} + \|\xi\|^2 \\
&= 2(\|\xi\|^2 - \operatorname{Re}(\langle v\xi, \xi \rangle))
\end{aligned}$$

By using now our assumption  $P_\varphi \xi = \xi$ , we obtain from this:

$$\begin{aligned}
\varphi(a) &= 2\varphi(\|\xi\|^2 - \operatorname{Re}(\langle v\xi, \xi \rangle)) \\
&= 2(\|\xi\|^2 - \operatorname{Re}(\langle P_\varphi \xi, \xi \rangle)) \\
&= 2(\|\xi\|^2 - \|\xi\|^2) \\
&= 0
\end{aligned}$$

Now since  $\varphi$  is faithful, this gives  $a = 0$ , and so  $v\xi = \xi$ . Thus  $\int_\varphi$  is independent of  $\varphi$ , and is given on coefficients  $a = (\tau \otimes id)v$  by the following formula:

$$\left( id \otimes \int_\varphi \right) v = P$$

(3) With the above formula in hand, the left and right invariance of  $\int_G = \int_\varphi$  is clear on coefficients, and so in general, and this gives all the assertions. See [98].  $\square$

With these integration results in hand, we can now develop a Peter-Weyl type theory for the corepresentations, in analogy with the theory from the classical case. We will need a number of straightforward definitions and results. Let us begin with:

**DEFINITION 1.17.** *Given two corepresentations  $v \in M_n(A), w \in M_m(A)$ , we set*

$$\operatorname{Hom}(v, w) = \left\{ T \in M_{m \times n}(\mathbb{C}) \mid Tv = wT \right\}$$

and we use the following conventions:

- (1) We use the notations  $\operatorname{Fix}(v) = \operatorname{Hom}(1, v)$ , and  $\operatorname{End}(v) = \operatorname{Hom}(v, v)$ .
- (2) We write  $v \sim w$  when  $\operatorname{Hom}(v, w)$  contains an invertible element.
- (3) We say that  $v$  is irreducible, and write  $v \in \operatorname{Irr}(G)$ , when  $\operatorname{End}(v) = \mathbb{C}1$ .

In the classical case, where  $A = C(G)$  with  $G \subset U_N$  being a closed subgroup, we obtain in this way the usual notions concerning the representations. Observe also that in

the group dual case we have  $g \sim h$  when  $g = h$ . Finally, observe that  $v \sim w$  means that  $v, w$  are conjugated by an invertible matrix.

Here are now a few basic results, regarding the above Hom spaces:

PROPOSITION 1.18. *We have the following results:*

- (1)  $T \in \text{Hom}(u, v), S \in \text{Hom}(v, w) \implies ST \in \text{Hom}(u, w)$ .
- (2)  $S \in \text{Hom}(p, q), T \in \text{Hom}(v, w) \implies S \otimes T \in \text{Hom}(p \otimes v, q \otimes w)$ .
- (3)  $T \in \text{Hom}(v, w) \implies T^* \in \text{Hom}(w, v)$ .

*In other words, the Hom spaces form a tensor  $*$ -category.*

PROOF. The proofs are all elementary, as follows:

- (1) Assume indeed that we have  $Tu = vT, Sv = Ws$ . We obtain, as desired:

$$STu = SvT = wST$$

- (2) Assume indeed that we have  $Sp = qS, Tv = wT$ . We have then:

$$(S \otimes T)(p \otimes v) = S_1 T_2 p_{13} v_{23} = (Sp)_{13} (Tv)_{23}$$

On the other hand, we have as well the following computation:

$$(q \otimes w)(S \otimes T) = q_{13} w_{23} S_1 T_2 = (qS)_{13} (wT)_{23}$$

The quantities on the right being equal, this gives the result.

- (3) By conjugating, and then using the unitarity of  $v, w$ , we obtain, as desired:

$$\begin{aligned} Tv = wT &\implies v^* T^* = T^* w^* \\ &\implies vv^* T^* w = v T^* w^* w \\ &\implies T^* w = v T^* \end{aligned}$$

Finally, the last assertion follows from definitions, and from the obvious fact that, in addition to (1,2,3) above, the Hom spaces are linear spaces, and contain the units.  $\square$

Finally, in order to formulate the Peter-Weyl results, we will need as well:

PROPOSITION 1.19. *The characters of the corepresentations, given by*

$$\chi_v = \sum_i v_{ii}$$

*behave as follows, in respect to the various operations:*

$$\chi_{v+w} = \chi_v + \chi_w \quad , \quad \chi_{v \otimes w} = \chi_v \chi_w \quad , \quad \chi_{\bar{v}} = \chi_v^*$$

*In addition, given two equivalent corepresentations,  $v \sim w$ , we have  $\chi_v = \chi_w$ .*

PROOF. The three formulae in the statement are all clear from definitions. Regarding now the last assertion, assuming that we have  $v = T^{-1}wT$ , we obtain:

$$\chi_v = \text{Tr}(v) = \text{Tr}(T^{-1}wT) = \text{Tr}(w) = \chi_w$$

We conclude that  $v \sim w$  implies  $\chi_v = \chi_w$ , as claimed.  $\square$

Consider the dense  $*$ -subalgebra  $\mathcal{A} \subset A$  generated by the coefficients of the fundamental corepresentation  $u$ , and endow it with the following scalar product:

$$\langle a, b \rangle = \int_G ab^*$$

With this convention, we have the following fundamental result, from [98]:

THEOREM 1.20. *We have the following Peter-Weyl type results:*

- (1) *Any corepresentation decomposes as a sum of irreducible corepresentations.*
- (2) *Each irreducible corepresentation appears inside a certain  $u^{\otimes k}$ .*
- (3)  *$\mathcal{A} = \bigoplus_{v \in \text{Irr}(A)} M_{\dim(v)}(\mathbb{C})$ , the summands being pairwise orthogonal.*
- (4) *The characters of irreducible corepresentations form an orthonormal system.*

PROOF. All these results are from [98], the idea being as follows:

- (1) Given a corepresentation  $v \in M_n(A)$ , consider its interwiner algebra:

$$\text{End}(v) = \left\{ T \in M_n(\mathbb{C}) \mid Tv = vT \right\}$$

We know from Proposition 1.18 that this is a finite dimensional  $C^*$ -algebra, and by using Theorem 1.2 above, we have a decomposition as follows:

$$\text{End}(v) = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C})$$

To be more precise, such a decomposition appears by writing the unit of our algebra as a sum of minimal projections, as follows, and then working out the details:

$$1 = p_1 + \dots + p_r$$

But this decomposition allows us to define subcorepresentations  $v_i \subset v$ , which are irreducible, so we obtain, as desired, a decomposition as follows:

$$v = v_1 + \dots + v_r$$

- (2) To any corepresentation  $v \in M_n(A)$  we associate its space of coefficients, given by  $C(v) = \text{span}(v_{ij})$ . The construction  $v \rightarrow C(v)$  is then functorial, in the sense that it maps subcorepresentations into subspaces. Observe also that we have:

$$\mathcal{A} = \sum_{k \in \mathbb{N} * \mathbb{N}} C(u^{\otimes k})$$

Now given an arbitrary corepresentation  $v \in M_n(A)$ , the corresponding coefficient space is a finite dimensional subspace  $C(v) \subset \mathcal{A}$ , and so we must have, for certain positive integers  $k_1, \dots, k_p$ , an inclusion of vector spaces, as follows:

$$C(v) \subset C(u^{\otimes k_1} \oplus \dots \oplus u^{\otimes k_p})$$

We deduce from this that we have an inclusion of corepresentations, as follows:

$$v \subset u^{\otimes k_1} \oplus \dots \oplus u^{\otimes k_p}$$

Thus, by using (1), we are led to the conclusion in the statement.

(3) By using (1) and (2), we obtain a linear space decomposition as follows:

$$\mathcal{A} = \sum_{v \in Irr(A)} C(v) = \sum_{v \in Irr(A)} M_{\dim(v)}(\mathbb{C})$$

In order to conclude, it is enough to prove that for any two irreducible corepresentations  $v, w \in Irr(A)$ , the corresponding spaces of coefficients are orthogonal:

$$v \not\sim w \implies C(v) \perp C(w)$$

As a first observation, which follows from an elementary computation, for any two corepresentations  $v, w$  we have a Frobenius type isomorphism, as follows:

$$Hom(v, w) \simeq Fix(\bar{v} \otimes w)$$

Now let us set  $P_{ia,jb} = \int_G v_{ij} w_{ab}^*$ . According to Theorem 1.16, the matrix  $P$  is the orthogonal projection onto the following vector space:

$$Fix(v \otimes \bar{w}) \simeq Hom(\bar{v}, \bar{w}) = \{0\}$$

Thus we have  $P = 0$ , and so  $C(v) \perp C(w)$ , which gives the result.

(4) The algebra  $\mathcal{A}_{central}$  contains indeed all the characters, because we have:

$$\Sigma \Delta(\chi_v) = \sum_{ij} v_{ji} \otimes v_{ij} = \Delta(\chi_v)$$

The fact that the characters span  $\mathcal{A}_{central}$ , and form an orthogonal basis of it, follow from (3). Finally, regarding the norm 1 assertion, consider the following integrals:

$$P_{ik,jl} = \int_G v_{ij} v_{kl}^*$$



We know from Theorem 1.16 that these integrals form the orthogonal projection onto  $Fix(v \otimes \bar{v}) \simeq End(\bar{v}) = \mathbb{C}1$ . By using this fact, we obtain the following formula:

$$\begin{aligned} \int_G \chi_v \chi_v^* &= \sum_{ij} \int_G v_{ii} v_{jj}^* \\ &= \sum_i \frac{1}{N} \\ &= 1 \end{aligned}$$

Thus the characters have indeed norm 1, and we are done.  $\square$

Observe that in the cocommutative case, we obtain from (4) above that we must have  $A = C^*(\Gamma)$  for some discrete group  $\Gamma$ , as mentioned in Proposition 1.9.

As another consequence of the above results, following [98] and then [48], we have the following result, dealing with amenability and functional analysis aspects:

**THEOREM 1.21.** *Let  $A_{full}$  be the enveloping  $C^*$ -algebra of  $\mathcal{A}$ , and let  $A_{red}$  be the quotient of  $A$  by the null ideal of the Haar integration. The following are then equivalent:*

- (1) *The Haar functional of  $A_{full}$  is faithful.*
- (2) *The projection map  $A_{full} \rightarrow A_{red}$  is an isomorphism.*
- (3) *The counit map  $\varepsilon : A_{full} \rightarrow \mathbb{C}$  factorizes through  $A_{red}$ .*
- (4) *We have  $N \in \sigma(Re(\chi_u))$ , the spectrum being taken inside  $A_{red}$ .*

*If this is the case, we say that the underlying discrete quantum group  $\Gamma$  is amenable.*

**PROOF.** This is well-known in the group dual case,  $A = C^*(\Gamma)$ , with  $\Gamma$  being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1)  $\iff$  (2) This simply follows from the fact that the GNS construction for the algebra  $A_{full}$  with respect to the Haar functional produces the algebra  $A_{red}$ .

(2)  $\iff$  (3) Here  $\implies$  is trivial, and conversely, a counit map  $\varepsilon : A_{red} \rightarrow \mathbb{C}$  produces an isomorphism  $\Phi : A_{red} \rightarrow A_{full}$ , via a formula of type  $\Phi = (\varepsilon \otimes id)\Delta'$ .

(3)  $\iff$  (4) Here  $\implies$  is clear, coming from  $\varepsilon(N - Re(\chi(u))) = 0$ , and the converse can be proved by doing some functional analysis. See [48], [98].  $\square$

With these results in hand, we can formulate, as a refinement of Definition 1.11:

**DEFINITION 1.22.** *Given a Woronowicz algebra  $A$ , we formally write as before*

$$A = C(G) = C^*(\Gamma)$$

*and by GNS construction with respect to the Haar functional, we write as well*

$$A'' = L^\infty(G) = L(\Gamma)$$

*with  $G$  being a compact quantum group, and  $\Gamma$  being a discrete quantum group.*

Now back to Theorem 1.21, as in the discrete group case, the most interesting criterion for amenability, leading to some interesting mathematics and physics, is the Kesten one, from Theorem 1.21 (4). This leads us into computing character laws:

PROPOSITION 1.23. *Given a Woronowicz algebra  $(A, u)$ , consider its main character:*

$$\chi = \sum_i u_{ii}$$

- (1) *The moments of  $\chi$  are the numbers  $M_k = \dim(\text{Fix}(u^{\otimes k}))$ .*
- (2) *When  $u \sim \bar{u}$  the law of  $\chi$  is a real measure, supported by  $\sigma(\chi)$ .*
- (3) *The notion of coamenability of  $A$  depends only on  $\text{law}(\chi)$ .*

PROOF. All this is elementary, the idea being as follows:

- (1) This follows indeed from Peter-Weyl theory.
- (2) When  $u \sim \bar{u}$  we have  $\chi = \chi^*$ , which gives the result.
- (3) This follows from Theorem 1.21 (4), and from (2) applied to  $u + \bar{u}$ . □

All this is quite interesting, because it tells us that, regardless on whether we want to understand the representation theory of our compact quantum group  $G$ , or the analytic aspects of its discrete dual  $\Gamma$ , we must compute the fixed point spaces  $\text{Fix}(u^{\otimes k})$ .

The computation of these spaces is a delicate algebra problem, related to results of Schur-Weyl, Brauer and Tannaka. In order to get started, the first idea is to replace the series of fixed point spaces  $F_k = \text{Fix}(u^{\otimes k})$  by the double series of Hom spaces:

$$C_{kl} = \text{Hom}(u^{\otimes k}, u^{\otimes l})$$

Indeed, by Frobenius duality, computing the sequence of spaces  $\{F_k\}$  is the same as computing the family of spaces  $\{C_{kl}\}$ . But computing the spaces  $\{C_{kl}\}$  is simpler than computing the spaces  $\{F_k\}$ , because these former spaces form a category. And we can use here the following version of Tannakian duality, due to Woronowicz [99]:

THEOREM 1.24. *The following operations are inverse to each other:*

- (1) *The construction  $A \rightarrow C$ , which associates to any Woronowicz algebra  $A$  the tensor category formed by the intertwiner spaces  $C_{kl} = \text{Hom}(u^{\otimes k}, u^{\otimes l})$ .*
- (2) *The construction  $C \rightarrow A$ , which associates to any tensor category  $C$  the Woronowicz algebra  $A$  presented by the relations  $T \in \text{Hom}(u^{\otimes k}, u^{\otimes l})$ , with  $T \in C_{kl}$ .*

PROOF. This is something quite deep, going back to [99] in a slightly different form, and to [76] in the simplified form presented above. The idea is that we have indeed a construction  $A \rightarrow C$  as above, whose output is a tensor  $C^*$ -subcategory with duals of the tensor  $C^*$ -category of Hilbert spaces. We have as well a construction  $C \rightarrow A$  as above, simply by dividing the free  $*$ -algebra on  $N^2$  variables by the relations in the statement. Regarding now the bijection claim, some elementary algebra shows that  $C = C_{A_C}$  implies

$A = A_{C_A}$ , and also that  $C \subset C_{A_C}$  is automatic. Thus we are left with proving  $C_{A_C} \subset C$ . But this latter inclusion can be proved indeed, by doing a lot of algebra, and using von Neumann's bicommutant theorem, in finite dimensions. See [76].  $\square$

As a last piece of general theory, let us discuss fusion rules, and Cayley graphs:

**PROPOSITION 1.25.** *Let  $(A, u)$  be a Woronowicz algebra, and assume, by enlarging if necessary  $u$ , that we have  $1 \in u = \bar{u}$ . The formula*

$$d(v, w) = \min \left\{ k \in \mathbb{N} \mid 1 \subset \bar{v} \otimes w \otimes u^{\otimes k} \right\}$$

*defines then a distance on  $\text{Irr}(A)$ , which coincides with the geodesic distance on the associated Cayley graph. Moreover, the moments of the main character,*

$$\int_G \chi^k = \dim(\text{Fix}(u^{\otimes k}))$$

*count the loops based at 1, having length  $k$ , on the corresponding Cayley graph.*

**PROOF.** Observation first the result holds indeed in the group dual case, where  $A = C^*(\Gamma)$  with  $\Gamma = \langle S \rangle$  being a finitely generated discrete group. In general, the fact that the lengths are finite follows from Peter-Weyl theory. The symmetry axiom is clear as well, and the triangle inequality is elementary to establish as well. Finally, the last assertion, regarding the moments, is elementary as well.  $\square$

### 1c. Basic examples

Let us discuss now the basic examples of compact and discrete quantum groups. We know so far that the compact quantum groups include the usual compact Lie groups,  $G \subset U_N$ , and the abstract duals  $G = \widehat{\Gamma}$  of the finitely generated groups  $F_N \rightarrow \Gamma$ . Equivalently, we know that the discrete quantum groups include the finitely generated groups  $F_N \rightarrow \Gamma$ , and the abstract duals  $\Gamma = \widehat{G}$  of the compact Lie groups,  $G \subset U_N$ .

We can combine these examples by performing basic operations, as follows:

**PROPOSITION 1.26.** *The class of Woronowicz algebras is stable under taking:*

- (1) *Tensor products,  $A = A' \otimes A''$ , with  $u = u' + u''$ . At the quantum group level we obtain usual products,  $G = G' \times G''$  and  $\Gamma = \Gamma' \times \Gamma''$ .*
- (2) *Free products,  $A = A' * A''$ , with  $u = u' + u''$ . At the quantum group level we obtain dual free products  $G = G' \hat{*} G''$  and free products  $\Gamma = \Gamma' * \Gamma''$ .*

**PROOF.** Everything here is clear from definitions. In addition to this, let us mention as well that we have  $\int_{A' \otimes A''} = \int_{A'} \otimes \int_{A''}$  and  $\int_{A' * A''} = \int_{A'} * \int_{A''}$ . Also, the corepresentations of the products can be explicitly computed. See [94].  $\square$

Here are some further basic operations, once again from [94]:

PROPOSITION 1.27. *The class of Woronowicz algebras is stable under taking:*

- (1) *Subalgebras  $A' = \langle u'_{ij} \rangle \subset A$ , with  $u'$  being a corepresentation of  $A$ . At the quantum group level we obtain quotients  $G \rightarrow G'$  and subgroups  $\Gamma' \subset \Gamma$ .*
- (2) *Quotients  $A \rightarrow A' = A/I$ , with  $I$  being a Hopf ideal,  $\Delta(I) \subset A \otimes I + I \otimes A$ . At the quantum group level we obtain subgroups  $G' \subset G$  and quotients  $\Gamma \rightarrow \Gamma'$ .*

PROOF. Once again, everything is clear, and we have as well some straightforward supplementary results, regarding integration and corepresentations. See [94].  $\square$

Finally, here are two more operations, which are of key importance:

PROPOSITION 1.28. *The class of Woronowicz algebras is stable under taking:*

- (1) *Projective versions,  $PA = \langle w_{ia,jb} \rangle \subset A$ , where  $w = u \otimes \bar{u}$ . At the quantum group level we obtain projective versions,  $G \rightarrow PG$  and  $P\Gamma \subset \Gamma$ .*
- (2) *Free complexifications,  $\hat{A} = \langle zu_{ij} \rangle \subset C(\mathbb{T}) * A$ . At the quantum group level we obtain free complexifications, denoted  $\tilde{G}$  and  $\tilde{\Gamma}$ .*

PROOF. This is clear from the previous results. For details here, we refer to [94].  $\square$

Once again following [94], as well as [37], let us discuss now a number of truly “new” quantum groups, obtained by liberating and half-liberating. We first have:

THEOREM 1.29. *The following universal algebras are Woronowicz algebras,*

$$\begin{aligned} C(O_N^+) &= C^* \left( (u_{ij})_{i,j=1,\dots,N} \mid u = \bar{u}, u^t = u^{-1} \right) \\ C(U_N^+) &= C^* \left( (u_{ij})_{i,j=1,\dots,N} \mid u^* = u^{-1}, u^t = \bar{u}^{-1} \right) \end{aligned}$$

and the same goes for the following quotient algebras,

$$\begin{aligned} C(O_N^*) &= C(O_N^+) / \left\langle abc = cba \mid \forall a, b, c \in \{u_{ij}\} \right\rangle \\ C(U_N^*) &= C(U_N^+) / \left\langle abc = cba \mid \forall a, b, c \in \{u_{ij}, u_{ij}^*\} \right\rangle \end{aligned}$$

so the underlying spaces  $O_N^+, U_N^+$  and  $O_N^*, U_N^*$  are compact quantum groups.

PROOF. The first assertion follows from the elementary fact that if a matrix  $u = (u_{ij})$  is orthogonal or biunitary, then so must be the following matrices:

$$u_{ij}^\Delta = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u_{ij}^\varepsilon = \delta_{ij} \quad , \quad u_{ij}^S = u_{ji}^*$$

Thus, we can define morphisms  $\Delta, \varepsilon, S$  as in Definition 1.7, by using the universality property of  $C(O_N^+), C(U_N^+)$ . As for the second assertion, the proof here is similar, based on the fact that if the entries of  $u$  satisfy  $abc = cba$ , then so do the entries of  $u^\Delta, u^\varepsilon, u^S$ .  $\square$

Our first task is to verify that Theorem 1.29 provides us indeed with new quantum groups. For this purpose, we can use the notion of diagonal torus, from [39]:

PROPOSITION 1.30. *Given a closed subgroup  $G \subset U_N^+$ , consider its diagonal torus, which is the closed subgroup  $T \subset G$  constructed as follows:*

$$C(T) = C(G) / \left\langle u_{ij} = 0 \mid \forall i \neq j \right\rangle$$

*This torus is then a group dual,  $T = \widehat{\Lambda}$ , where  $\Lambda = \langle g_1, \dots, g_N \rangle$  is the discrete group generated by the elements  $g_i = u_{ii}$ , which are unitaries inside  $C(T)$ .*

PROOF. Since  $u$  is unitary, its diagonal entries  $g_i = u_{ii}$  are unitaries inside  $C(T)$ . Moreover, from  $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$  we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have  $C(T) = C^*(\Lambda)$ , modulo identifying as usual the  $C^*$ -completions of the various group algebras, and so that we have  $T = \widehat{\Lambda}$ , as claimed.  $\square$

We can now distinguish between our various quantum groups, as follows:

THEOREM 1.31. *The diagonal tori of the basic unitary quantum groups, namely*

$$\begin{array}{ccccc} U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \end{array}$$

*are the following discrete group duals,*

$$\begin{array}{ccccc} \widehat{\mathbb{Z}^N} & \longrightarrow & \widehat{\mathbb{Z}^{\circ N}} & \longrightarrow & \widehat{\mathbb{Z}^{*N}} \\ \uparrow & & \uparrow & & \uparrow \\ \widehat{\mathbb{Z}_2^N} & \longrightarrow & \widehat{\mathbb{Z}_2^{\circ N}} & \longrightarrow & \widehat{\mathbb{Z}_2^{*N}} \end{array}$$

*with  $\circ$  standing for the half-classical product operation for groups.*

PROOF. This is clear for  $U_N^+$ , where on the diagonal we obtain the biggest group dual, namely  $\widehat{F}_N$ . For the other quantum groups this follows by taking quotients, which corresponds to taking quotients as well, at the level of the diagonal torus dual  $\Lambda = \widehat{T}$ .  $\square$

### 1d. Laws of characters

Let us discuss now the representation theory of these quantum groups. In order to formulate our results, we use the modern notion of “easiness”, from [37]:

DEFINITION 1.32. A closed subgroup  $G \subset U_N^+$  is called easy when we have

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{span} \left( T_\pi \Big|_{\pi \in D(k, l)} \right)$$

for any colored integers  $k, l$ , for certain sets of partitions  $D(k, l) \subset P(k, l)$ , where

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l}$$

with the Kronecker type symbols  $\delta_\pi \in \{0, 1\}$  depending on whether the indices fit or not.

To be more precise here, let  $P(k, l)$  be the set of partitions between an upper row of  $k$  points, and a lower row of  $l$  points. Our claim is that given  $N \in \mathbb{N}$ , any partition  $\pi \in P(k, l)$  produces a linear map between tensor powers of  $\mathbb{C}^N$ , as follows:

$$T_\pi : (\mathbb{C}^N)^{\otimes k} \rightarrow (\mathbb{C}^N)^{\otimes l}$$

Indeed, if we denote by  $e_1, \dots, e_N$  the standard basis of  $\mathbb{C}^N$ , we can define  $T_\pi$  by the formula in Definition 1.32, with the Kronecker symbols appearing there being computed by putting the multi-indices  $i, j$  on the legs of  $\pi$ , in the obvious way. If all the blocks of  $\pi$  contain equal indices we set  $\delta_\pi = 1$ , and if not, we set  $\delta_\pi = 0$ . We have then:

THEOREM 1.33. The basic unitary quantum groups are all easy, with

$$\begin{array}{ccccc} U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+ & & \mathcal{P}_2 & \longleftarrow & \mathcal{P}_2^* & \longleftarrow & \mathcal{NC}_2 \\ \uparrow & & \uparrow & & \uparrow & : & \downarrow & & \downarrow & & \downarrow \\ O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ & & P_2 & \longleftarrow & P_2^* & \longleftarrow & NC_2 \end{array}$$

being the associated categories of partitions  $D \subset P$ .

PROOF. This is something that requires some work, the idea being as follows:

(1)  $O_N^+$ . Consider the set  $NC_2$  of all noncrossing pairings. It is routine to check that  $\text{span}(T_\pi | \pi \in NC_2)$  is a Tannakian category, and also that this category is the smallest possible one allowed by the Tannakian axioms, in the  $u = \bar{u}$  setting. Thus, the associated quantum group must be the biggest subgroup  $G \subset O_N^+$ , which is  $O_N^+$  itself.

(2)  $O_N$ . Since  $O_N \subset O_N^+$  appears by adding the commutation relations  $ab = ba$  between coordinates, which are implemented by the linear map  $T_\chi$  coming from the basic crossing  $\chi$ , we obtain here the category  $\langle NC_2, \chi \rangle = P_2$  of all pairings.

(3)  $O_N^*$ . Here we obtain the category  $\langle NC_2, \chi \rangle = P_2^*$  of pairings having the property that, when legs are labelled clockwise  $\circ \bullet \circ \bullet \dots$ , each string connects  $\circ - \bullet$ .

(4)  $U_N^+, U_N, U_N^*$ . The situation is similar here, but due to  $u \neq \bar{u}$  everything is now colored, and we obtain in all cases pairings which are “matching”, in the sense that the vertical strings connect  $\circ - \circ$  or  $\bullet - \bullet$ , and the horizontal ones connect  $\circ - \bullet$ .  $\square$

Here are some concrete consequences of the above result, following [1], [23]:

**THEOREM 1.34.** *The quantum groups  $O_N^+, U_N^+$  have the following properties:*

(1) *We have an isomorphism as follows, up to the standard equivalence relation:*

$$\tilde{O}_N^+ = U_N^+$$

(2) *We have as well an isomorphism as follows, once again up to equivalence:*

$$PO_N^+ = PU_N^+$$

(3) *The fusion rules for  $O_N^+$  are the same as the Clebsch-Gordan rules for  $SU_2$ :*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+2} + \dots + r_{k+l}$$

(4) *Those for  $U_N^+$  are as follows, with the representations being indexed by  $\mathbb{N} * \mathbb{N}$ :*

$$r_k \otimes r_l = \sum_{k=xy, l=\bar{y}z} r_{xz}$$

(5) *The main characters follow the Wigner semicircle and Voiculescu circular law:*

$$\chi \sim \begin{cases} \gamma_1 & \text{for } O_N^+, N \geq 2 \\ \Gamma_1 & \text{for } U_N^+, N \geq 2 \end{cases}$$

(6) *With  $N \rightarrow \infty$ , the truncated characters follow the  $t$ -versions of these laws:*

$$\chi_t \sim \begin{cases} \gamma_t & \text{for } O_N^+, N \rightarrow \infty \\ \Gamma_t & \text{for } U_N^+, N \rightarrow \infty \end{cases}$$

**PROOF.** All this follows from our Brauer type results, via standard techniques. There is actually quite some work to be done here, the idea being as follows:

(1) As a first observation, by using the universal property of  $U_N^+$ , as being the biggest  $N \times N$  compact matrix quantum group, we have an inclusion as follows:

$$\tilde{O}_N^+ \subset U_N^+$$

Now by using the easiness results from Theorem 1.33, we can compute the tensor categories for both these quantum groups, with the conclusion that these tensor categories are equal. Thus, our inclusion is an isomorphism, up to the equivalence relation.

(2) This follows from (1) above, via the following computation:

$$PU_N^+ = P\tilde{O}_N^+ = PO_N^+$$

(3) This is something more complicated, the idea being that from the Brauer result for  $O_N^+$  we obtain, after some work, the following formula, valid at any  $N \geq 2$ :

$$\begin{aligned} \dim(\text{End}(u^{\otimes k})) &= |NC_2(k, k)| \\ &= \frac{1}{k+1} \binom{2k}{k} \end{aligned}$$

Now since the same formula is well-known to hold for the fundamental representation of  $SU_2$ , we obtain the same combinatorics, as claimed. We will be back to this.

(4) This follows from (1) and (3). Indeed, the fusion rules for the quantum group  $U_N^+ = \widetilde{O}_N^+$  can be computed starting from the knowledge of those of  $O_N^+$ , and we end up with a “free complexification” of the Clebsch-Gordan rules, namely:

$$r_k \otimes r_l = \sum_{k=xy, l=\bar{y}z} r_{xz}$$

(5) This follows once again from (1) and (3). Indeed, in what regards  $O_N^+$ , we can convert our combinatorial results into a moment formula, as follows:

$$\int_{O_N^+} \chi^{2k} = \frac{1}{k+1} \binom{2k}{k}$$

But this shows precisely that  $\chi$  must follow the Wigner semicircle law  $\gamma_1$ , and by complexifying, we obtain the result for  $U_N^+$  as well. We will be back to this.

(6) This is something more technical. Given a parameter  $t \in (0, 1]$ , we can define a truncation of the main character, as follows:

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

The point now is that our Brauer theorems allow us to explicitly integrate over  $O_N^+, U_N^+$ , via a combinatorial formula, and in the  $N \rightarrow \infty$  limit the combinatorics simplifies, and in what regards  $\chi_t$ , we obtain the laws  $\gamma_t, \Gamma_t$ . We will be back to this.  $\square$

The above presentation was of course quite short, but all this can be found in any good quantum group book. Some similar results regarding  $O_N^*, U_N^*$  are available as well, and we can twist everything at  $q = -1$  too. We will be back to this.

## 1e. Exercises

There has been a lot of mathematics in this chapter, and for more details, more theory, and for exercises as well regarding the general  $C^*$ -algebra theory, and the general compact quantum group theory, we refer to any standard textbook on the subject. In what follows we will take all this material for granted, and rather focus on Theorem 1.34, which is more specialized. Here is a first exercise, in connection with the statement (1) there:



EXERCISE 1.35. *Prove that the quantum group inclusion*

$$\tilde{O}_N^+ \subset U_N^+$$

*is an isomorphism at the level of the corresponding diagonal tori.*

This follows of course from Theorem 1.34 (1), but the problem is that of proving this directly, in an elementary way, via some discrete group computations.

As a second exercise, in connection with Theorem 1.34 (2), we have:

EXERCISE 1.36. *Prove that the quantum group inclusion*

$$PO_N^+ \subset PU_N^+$$

*is an isomorphism, by showing that the corresponding tensor categories coincide.*

This is something which is related to the proof of Theorem 1.34 (1) and (2), the full problem being that of proving all this the other way around, namely (2), then (1).

In connection with the fusion rule statements, Theorem 1.34 (3) and (4), we have:

EXERCISE 1.37. *Prove that the irreducible representations of  $SU_2$  can be labelled by positive integers, such that the fusion rules are the Clebsch-Gordan rules*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+2} + \dots + r_{k+l}$$

*exactly as for the quantum group  $O_N^+$  with  $N \geq 2$ .*

This is certainly something difficult, but the answer can be found a bit everywhere, in any good book on Lie groups or physics. There are actually several proofs here, all non-trivial, beautiful, and very instructive, so just take a look, pick the one that you like the most, and rewrite it in a concise way. And as a supplementary exercise here, once this done, and  $SU_2$  understood, go back Theorem 1.34 (3) and (4), for a second read.

Finally, in connection with probability, Theorem 1.34 (5) and (6), we have:

EXERCISE 1.38. *Prove that the Catalan numbers, namely*

$$C_k = \frac{1}{k+1} \binom{2k}{k}$$

*are indeed the moments of the semicircle law  $\gamma_1$ .*

This is something that we used, in the proof of Theorem 1.34 (5) above.

## CHAPTER 2

### Quantum permutations

#### 2a. Magic matrices

Welcome to quantum permutations. The rest of this book is dedicated to them. And, good news, the presentation will be far less intense than that in the previous chapter, which was meant to be a quick introduction to the quantum groups, survey style.

In order to get started, let us look at the usual symmetric group  $S_N$ . We have:

**PROPOSITION 2.1.** *Consider the symmetric group  $S_N$ , viewed as the permutation group of the  $N$  coordinate axes of  $\mathbb{R}^N$ . The coordinate functions on  $S_N \subset O_N$  are then given by*

$$u_{ij} = \chi \left( \sigma \in G \mid \sigma(j) = i \right)$$

*and the matrix  $u = (u_{ij})$  that these functions form is magic, in the sense that its entries are projections ( $p^2 = p^* = p$ ), summing up to 1 on each row and each column.*

**PROOF.** The action of  $S_N$  on the standard basis  $e_1, \dots, e_N \in \mathbb{R}^N$  is given by  $\sigma : e_j \rightarrow e_{\sigma(j)}$ . Thus a permutation  $\sigma \in S_N$ , viewed as matrix  $\sigma \in O_N$ , is given by:

$$\sigma = \sum_j e_{\sigma(j)j}$$

The coefficients of this matrix,  $u_{ij}(\sigma) = \sigma_{ij}$ , are then given by:

$$u_{ij}(\sigma) = \delta_{\sigma(j)i}$$

But this gives the formula of  $u_{ij}$  in the statement. As for the fact that the matrix  $u = (u_{ij})$  that these functions form is magic, this is clear.  $\square$

With a bit more effort, we obtain the following nice characterization of  $S_N$ :

**THEOREM 2.2.** *The algebra of functions on  $S_N$  has the following presentation,*

$$C(S_N) = C_{comm}^* \left( (u_{ij})_{i,j=1,\dots,N} \mid u = \text{magic} \right)$$

*and the multiplication, unit and inversion map of  $S_N$  appear from the maps*

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

*defined at the algebraic level, of functions on  $S_N$ , by transposing.*

PROOF. This is something elementary as well. Indeed, the universal algebra  $A$  in the statement is a commutative  $C^*$ -algebra, so by the Gelfand theorem it must be of the following form, with  $X$  being a certain compact space:

$$A = C(X)$$

Now since we have coordinates  $u_{ij} : X \rightarrow \mathbb{R}$ , we have an embedding  $X \subset M_N(\mathbb{R})$ . Also, since we know that these coordinates form a magic matrix, the elements  $g \in X$  must be 0-1 matrices, having exactly one 1 on each row and each column, and so:

$$X = S_N$$

Thus we have proved the first assertion, and the second assertion is clear as well, by using the general theory from section 1. To be more precise, the multiplication, unit and inverse map of  $S_N \subset O_N$  are the standard ones for the orthogonal matrices, namely:

$$(gh)_{ij} = \sum_k g_{ik}h_{kj} \quad , \quad 1_{ij} = \delta_{ij} \quad , \quad (g^{-1})_{ij} = g_{ji}$$

Now by transposing, we obtain the formulae of  $\Delta, \varepsilon, S$  in the statement. □

Following now Wang [95], we can liberate  $S_N$ , simply by lifting the commutativity condition in Theorem 2.2. To be more precise, we have the following result:

**THEOREM 2.3.** *The following universal  $C^*$ -algebra, with magic meaning as usual formed by projections ( $p^2 = p^* = p$ ), summing up to 1 on each row and each column,*

$$C(S_N^+) = C^* \left( (u_{ij})_{i,j=1,\dots,N} \mid u = \text{magic} \right)$$

*is a Woronowicz algebra, with comultiplication, counit and antipode given by*

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

*and so the underlying compact quantum space  $S_N^+$  is a compact quantum group, called quantum permutation group.*

PROOF. As a first observation, the universal  $C^*$ -algebra in the statement is indeed well-defined, because the conditions  $p^2 = p^* = p$  satisfied by the coordinates give:

$$\|u_{ij}\| \leq 1$$

In order to prove now that we have a Woronowicz algebra, we must construct maps  $\Delta, \varepsilon, S$  given by the formulae in the statement. Consider the following matrices:

$$u_{ij}^\Delta = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u_{ij}^\varepsilon = \delta_{ij} \quad , \quad u_{ij}^S = u_{ji}$$

Our claim is that, since  $u$  is magic, so are these three matrices. Indeed, regarding  $u^\Delta$ , its entries are idempotents, as shown by the following computation:

$$\begin{aligned} (u_{ij}^\Delta)^2 &= \sum_{kl} u_{ik} u_{il} \otimes u_{kj} u_{lj} \\ &= \sum_{kl} \delta_{kl} u_{ik} \otimes \delta_{kl} u_{kj} \\ &= u_{ij}^\Delta \end{aligned}$$

These elements are self-adjoint as well, as shown by the following computation:

$$\begin{aligned} (u_{ij}^\Delta)^* &= \sum_k u_{ik}^* \otimes u_{kj}^* \\ &= \sum_k u_{ik} \otimes u_{kj} \\ &= u_{ij}^\Delta \end{aligned}$$

The row sums for the matrix  $u^\Delta$  can be computed as follows:

$$\begin{aligned} \sum_j u_{ij}^\Delta &= \sum_{jk} u_{ik} \otimes u_{kj} \\ &= \sum_k u_{ik} \otimes 1 \\ &= 1 \end{aligned}$$

As for the computation of the column sums, this is similar, as follows:

$$\begin{aligned} \sum_i u_{ij}^\Delta &= \sum_{ik} u_{ik} \otimes u_{kj} \\ &= \sum_k 1 \otimes u_{kj} \\ &= 1 \end{aligned}$$

Thus,  $u^\Delta$  is magic. Regarding now  $u^\varepsilon, u^S$ , these matrices are magic too, and this for obvious reasons. Thus, all our three matrices  $u^\Delta, u^\varepsilon, u^S$  are magic, and so we can define  $\Delta, \varepsilon, S$  by the formulae in the statement, by using the universality property of  $C(S_N^+)$ .

As a conclusion, the algebra  $C(S_N^+)$  satisfies Woronowicz's axioms from chapter 1 above, and so its abstract spectrum  $S_N^+$  is a compact quantum group, as claimed.  $\square$

Summarizing, we have constructed a quantum group  $S_N^+$ , which is quite similar to the usual permutation group  $S_N$ , but whose standard coordinates do not longer commute.

Our first task now is to make sure that Theorem 2.3 produces indeed a new quantum group, which does not collapse to  $S_N$ . Following [95], we have here:

THEOREM 2.4. *We have an embedding  $S_N \subset S_N^+$ , given at the algebra level by:*

$$u_{ij} \rightarrow \chi \left( \sigma \mid \sigma(j) = i \right)$$

*This is an isomorphism at  $N \leq 3$ , but not at  $N \geq 4$ , where  $S_N^+$  is not classical, nor finite.*

PROOF. The fact that we have indeed an embedding as above follows from Theorem 2.2. Observe that in fact more is true, because Theorem 2.2 and Theorem 2.3 give:

$$C(S_N) = C(S_N^+) / \langle ab = ba \rangle$$

Thus, the inclusion  $S_N \subset S_N^+$  is a “liberation”, in the sense that  $S_N$  is the classical version of  $S_N^+$ . We will often use this basic fact, in what follows. Regarding now the second assertion, we can prove this in four steps, as follows:

Case  $N = 2$ . The fact that  $S_2^+$  is indeed classical, and hence collapses to  $S_2$ , is trivial, because the  $2 \times 2$  magic matrices are as follows, with  $p$  being a projection:

$$U = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

Indeed, this shows that the entries of  $U$  commute. Thus  $C(S_2^+)$  is commutative, and so equals its biggest commutative quotient, which is  $C(S_2)$ . Thus,  $S_2^+ = S_2$ .

Case  $N = 3$ . By using the same argument as in the  $N = 2$  case, and the symmetries of the problem, it is enough to check that  $u_{11}, u_{22}$  commute. But this follows from:

$$\begin{aligned} u_{11}u_{22} &= u_{11}u_{22}(u_{11} + u_{12} + u_{13}) \\ &= u_{11}u_{22}u_{11} + u_{11}u_{22}u_{13} \\ &= u_{11}u_{22}u_{11} + u_{11}(1 - u_{21} - u_{23})u_{13} \\ &= u_{11}u_{22}u_{11} \end{aligned}$$

Indeed, by applying the involution to this formula, we obtain from this that we have as well  $u_{22}u_{11} = u_{11}u_{22}u_{11}$ . Thus, we obtain  $u_{11}u_{22} = u_{22}u_{11}$ , as desired.

Case  $N = 4$ . Consider the following matrix, with  $p, q$  being projections:

$$U = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

This matrix is magic, and we can choose  $p, q$  as for the algebra  $\langle p, q \rangle$  to be non-commutative and infinite dimensional. We conclude that  $C(S_4^+)$  is noncommutative and infinite dimensional as well, and so  $S_4^+$  is non-classical and infinite, as claimed.

Case  $N \geq 5$ . Here we can use the standard embedding  $S_4^+ \subset S_N^+$ , obtained at the level of the corresponding magic matrices in the following way:

$$u \rightarrow \begin{pmatrix} u & 0 \\ 0 & 1_{N-4} \end{pmatrix}$$

Indeed, with this in hand, the fact that  $S_4^+$  is a non-classical, infinite compact quantum group implies that  $S_N^+$  with  $N \geq 5$  has these two properties as well.  $\square$

The above result is quite surprising, and understanding all this will be our next goal. As a first observation, we are not wrong with our formalism, because as explained once again in [95], we have as well the following alternative picture for  $S_N^+$ :

**THEOREM 2.5.** *The quantum permutation group  $S_N^+$  acts on the set  $X = \{1, \dots, N\}$ , the corresponding coaction map  $\Phi : C(X) \rightarrow C(X) \otimes C(S_N^+)$  being given by:*

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

*In fact,  $S_N^+$  is the biggest compact quantum group acting on  $X$ , by leaving the counting measure invariant, in the sense that  $(tr \otimes id)\Phi = tr(\cdot)1$ , where  $tr(e_i) = \frac{1}{N}, \forall i$ .*

**PROOF.** Our claim is that given a compact matrix quantum group  $G$ , the following formula defines a morphism of algebras, which is a coaction map, leaving the trace invariant, precisely when the matrix  $u = (u_{ij})$  is a magic corepresentation of  $C(G)$ :

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

Indeed, let us first determine when  $\Phi$  is multiplicative. We have:

$$\begin{aligned} \Phi(e_i)\Phi(e_k) &= \sum_{jl} e_j e_l \otimes u_{ji} u_{lk} \\ &= \sum_j e_j \otimes u_{ji} u_{jk} \end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned} \Phi(e_i e_k) &= \delta_{ik} \Phi(e_i) \\ &= \delta_{ik} \sum_j e_j \otimes u_{ji} \end{aligned}$$

We conclude that the multiplicativity of  $\Phi$  is equivalent to the following conditions:

$$u_{ji} u_{jk} = \delta_{ik} u_{ji} \quad , \quad \forall i, j, k$$

Regarding now the unitality of  $\Phi$ , we have the following formula:

$$\begin{aligned}\Phi(1) &= \sum_i \Phi(e_i) \\ &= \sum_{ij} e_j \otimes u_{ji} \\ &= \sum_j e_j \otimes \left( \sum_i u_{ji} \right)\end{aligned}$$

Thus  $\Phi$  is unital when  $\sum_i u_{ji} = 1, \forall j$ . Finally, the fact that  $\Phi$  is a  $*$ -morphism translates into  $u_{ij} = u_{ij}^*, \forall i, j$ . Summing up, in order for  $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$  to be a morphism of  $C^*$ -algebras, the elements  $u_{ij}$  must be projections, summing up to 1 on each row of  $u$ . Regarding now the preservation of the trace condition, observe that we have:

$$(tr \otimes id)\Phi(e_i) = \frac{1}{N} \sum_j u_{ji}$$

Thus the trace is preserved precisely when the elements  $u_{ij}$  sum up to 1 on each of the columns of  $u$ . We conclude from this that  $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$  is a morphism of  $C^*$ -algebras preserving the trace precisely when  $u$  is magic, and since the coaction conditions on  $\Phi$  are equivalent to the fact that  $u$  must be a corepresentation, this finishes the proof of our claim. But this claim proves all the assertions in the statement.  $\square$

## 2b. Diagrams, easiness

In order to study now  $S_N^+$ , we can use our various methods developed in chapter 1 above. Let us begin with some basic algebraic results, as follows:

**THEOREM 2.6.** *The quantum groups  $S_N^+$  have the following properties:*

- (1) *We have  $S_N^+ \hat{*} S_M^+ \subset S_{N+M}^+$ , for any  $N, M$ .*
- (2) *In particular, we have an embedding  $\widehat{D_\infty} \subset S_4^+$ .*
- (3)  *$S_4 \subset S_4^+$  are distinguished by their spinned diagonal tori.*
- (4) *The half-classical version  $S_N^* = S_N^+ \cap O_N^*$  collapses to  $S_N$ .*

**PROOF.** These results are all elementary, the proofs being as follows:

(1) If we denote by  $u, v$  the fundamental corepresentations of  $C(S_N^+), C(S_M^+)$ , the fundamental corepresentation of  $C(S_N^+ \hat{*} S_M^+)$  is by definition:

$$w = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

But this matrix is magic, because both  $u, v$  are magic. Thus by universality of  $C(S_{N+M}^+)$  we obtain a quotient map as follows, as desired:

$$C(S_{N+M}^+) \rightarrow C(S_N^+ \hat{*} S_M^+)$$

(2) This result, which refines our  $N = 4$  trick from the proof of Theorem 2.4, follows from (1) with  $N = M = 2$ . Indeed, we have the following computation:

$$\begin{aligned}
S_2^+ \hat{*} S_2^+ &= S_2 \hat{*} S_2 \\
&= \mathbb{Z}_2 \hat{*} \mathbb{Z}_2 \\
&\simeq \widehat{\mathbb{Z}_2 \hat{*} \mathbb{Z}_2} \\
&= \widehat{\mathbb{Z}_2 * \mathbb{Z}_2} \\
&= \widehat{D_\infty}
\end{aligned}$$

This might seem a bit wizarding, but we will be back to this, with further details.

(3) As a first observation here, the quantum groups  $S_4 \subset S_4^+$  are not distinguished by their diagonal torus, which is  $\{1\}$  for both of them. However, according to the general results of Woronowicz in [98], the group dual  $\widehat{D_\infty} \subset S_4^+$  that we found in (2) must be a subgroup of the diagonal torus of the following compact quantum group, with the standard unitary representations being spinned by a certain unitary  $F \in U_4$ :

$$(S_4^+, FuF^*)$$

Now since this group dual  $\widehat{D_\infty}$  is not classical, it cannot be a subgroup of the diagonal torus of  $(S_4, FuF^*)$ . Thus, the diagonal torus spinned by  $F$  distinguishes  $S_4 \subset S_4^+$ .

(4) Consider the following compact quantum group, with the intersection operation being taken inside  $U_N^+$ , whose coordinates satisfy  $abc = cba$ :

$$S_N^* = S_N^+ \cap O_N^*$$

In order to prove that we have  $S_N^* = S_N$ , it is enough to prove that  $S_N^*$  is classical. And here, we can use the fact that for a magic matrix, the entries in each row sum up to 1. Indeed, by making  $c$  vary over a full row of  $u$ , we obtain  $abc = cba \implies ab = ba$ .  $\square$

Summarizing, we have some advances on the quantum permutations, including a more conceptual explanation for our main observation so far, namely  $S_4^+ \neq S_4$ . Let us further build on this material. The construction in Theorem 2.6 (2), which is something quite tricky, and is our main result so far, has the following generalization:

**PROPOSITION 2.7.** *Consider a discrete group generated by elements of finite order, written as a quotient group, as follows:*

$$\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k} \rightarrow \Gamma$$

*We have then an embedding of quantum groups  $\widehat{\Gamma} \subset S_N^+$ , where  $N = N_1 + \dots + N_k$ .*



PROOF. We have a sequence of embeddings and isomorphisms as follows:

$$\begin{aligned}
\widehat{\Gamma} &\subset \widehat{\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k}} \\
&= \widehat{\mathbb{Z}_{N_1}} \hat{*} \dots \hat{*} \widehat{\mathbb{Z}_{N_k}} \\
&\simeq \mathbb{Z}_{N_1} \hat{*} \dots \hat{*} \mathbb{Z}_{N_k} \\
&\subset S_{N_1} \hat{*} \dots \hat{*} S_{N_k} \\
&\subset S_{N_1}^+ \hat{*} \dots \hat{*} S_{N_k}^+ \\
&\subset S_N^+
\end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

The above result is quite abstract, and it is worth working out the details, with an explicit formula for the associated magic matrix. Let us start with a study of the simplest situation, where  $k = 1$ , and where  $\Gamma = \mathbb{Z}_{N_1}$ . The result here is as follows:

PROPOSITION 2.8. *The magic matrix for the quantum permutation group*

$$\widehat{\mathbb{Z}_N} \simeq \mathbb{Z}_N \subset S_N \subset S_N^+$$

with standard Fourier isomorphism on the left, is given by the formula

$$u = FIF^*$$

where  $F = \frac{1}{\sqrt{N}}(w^{ij})$  with  $w = e^{2\pi i/N}$  is the Fourier matrix, and where

$$I = \begin{pmatrix} 1 & & & \\ & g & & \\ & & \ddots & \\ & & & g^{N-1} \end{pmatrix}$$

is the diagonal matrix formed by the elements of  $\mathbb{Z}_N$ , regarded as elements of  $C^*(\mathbb{Z}_N)$ .

PROOF. The magic matrix for the quantum group  $\mathbb{Z}_N \subset S_N \subset S_N^+$  is given by:

$$\begin{aligned}
v_{ij} &= \chi\left(\sigma \in \mathbb{Z}_N \mid \sigma(j) = i\right) \\
&= \delta_{i-j}
\end{aligned}$$

Let us apply now the Fourier transform. According to our Pontrjagin duality conventions from section 1 above, we have a pair of inverse isomorphisms, as follows:

$$\begin{aligned}
\Phi : C(\mathbb{Z}_N) &\rightarrow C^*(\mathbb{Z}_N) \quad , \quad \delta_i \rightarrow \frac{1}{N} \sum_k w^{ik} g^k \\
\Psi : C^*(\mathbb{Z}_N) &\rightarrow C(\mathbb{Z}_N) \quad , \quad g^i \rightarrow \sum_k w^{-ik} \delta_k
\end{aligned}$$

Here  $w = e^{2\pi i/N}$ , and we use the standard Fourier analysis convention that the indices are  $0, 1, \dots, N-1$ . With  $F = \frac{1}{\sqrt{N}}(w^{ij})$  and  $I = \text{diag}(g^i)$  as above, we have:

$$\begin{aligned}
u_{ij} &= \Phi(v_{ij}) \\
&= \frac{1}{N} \sum_k w^{(i-j)k} g^k \\
&= \frac{1}{N} \sum_k w^{ik} g^k w^{-jk} \\
&= \sum_k F_{ik} I_{kk} (F^*)_{kj} \\
&= (FIF^*)_{ij}
\end{aligned}$$

Thus, the magic matrix that we are looking for is  $u = FIF^*$ , as claimed.  $\square$

With the above result in hand, we can improve Proposition 2.7, as follows:

**THEOREM 2.9.** *Given a quotient group  $\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k} \rightarrow \Gamma$ , we have a closed subgroup  $\widehat{\Gamma} \subset S_N^+$ , with  $N = N_1 + \dots + N_k$ , whose magic matrix is given by*

$$u = \begin{pmatrix} F_{N_1} I_1 F_{N_1}^* & & \\ & \ddots & \\ & & F_{N_k} I_k F_{N_k}^* \end{pmatrix}$$

where  $F_N = \frac{1}{\sqrt{N}}(w_N^{ij})$  with  $w_N = e^{2\pi i/N}$  are Fourier matrices, and where

$$I_r = \begin{pmatrix} 1 & & & \\ & g_r & & \\ & & \ddots & \\ & & & g_r^{N_r-1} \end{pmatrix}$$

with  $g_1, \dots, g_k$  being the standard generators of  $\Gamma$ .

**PROOF.** This follows indeed from Proposition 2.7 and Proposition 2.8.  $\square$

We will be back to this on several occasions, and notably in chapter 13 below, where we will prove that all the group dual subgroups  $\widehat{\Gamma} \subset S_N^+$  appear as above.

Let us discuss now the representation theory of  $S_N^+$ , which will eventually lead to a clarification of all this. Our main result here, which is quite conceptual, will be the fact that  $S_N \subset S_N^+$  is a liberation of easy quantum groups. Following [37], let us formulate:

DEFINITION 2.10. Let  $P(k, l)$  be the set of partitions between an upper row of  $k$  points, and a lower row of  $l$  points. A collection of sets

$$D = \bigsqcup_{k, l} D(k, l)$$

with  $D(k, l) \subset P(k, l)$  is called a category of partitions when it has the following properties:

- (1) Stability under the horizontal concatenation,  $(\pi, \sigma) \rightarrow [\pi\sigma]$ .
- (2) Stability under the vertical concatenation,  $(\pi, \sigma) \rightarrow [\sigma]$ .
- (3) Stability under the upside-down turning,  $\pi \rightarrow \pi^*$ .
- (4) Each set  $P(k, k)$  contains the identity partition  $|| \dots ||$ .
- (5) The set  $P(0, 2)$  contains the semicircle partition  $\cap$ .

As a basic example, we have the category of all partitions  $P$  itself. Other basic examples include the category of pairings  $P_2$ , or the categories  $NC, NC_2$  of noncrossing partitions, and pairings. There are many other examples, and we will be back to this.

Following [37], the relation with the Tannakian categories and duality comes from:

PROPOSITION 2.11. Each partition  $\pi \in P(k, l)$  produces a linear map

$$T_\pi : (\mathbb{C}^N)^{\otimes k} \rightarrow (\mathbb{C}^N)^{\otimes l}$$

given by the following formula, with  $e_1, \dots, e_N$  being the standard basis of  $\mathbb{C}^N$ ,

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l}$$

and with the Kronecker type symbols  $\delta_\pi \in \{0, 1\}$  depending on whether the indices fit or not. The assignment  $\pi \rightarrow T_\pi$  is categorical, in the sense that we have

$$T_\pi \otimes T_\sigma = T_{[\pi\sigma]} \quad , \quad T_\pi T_\sigma = N^{c(\pi, \sigma)} T_{[\sigma]} \quad , \quad T_\pi^* = T_{\pi^*}$$

where  $c(\pi, \sigma)$  are certain integers, coming from the erased components in the middle.

PROOF. This follows from the elementary computations, as follows:

(1) The concatenation axiom follows from the following computation:

$$\begin{aligned} & (T_\pi \otimes T_\sigma)(e_{i_1} \otimes \dots \otimes e_{i_p} \otimes e_{k_1} \otimes \dots \otimes e_{k_r}) \\ &= \sum_{j_1 \dots j_q} \sum_{l_1 \dots l_s} \delta_\pi \begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_q \end{pmatrix} \delta_\sigma \begin{pmatrix} k_1 & \dots & k_r \\ l_1 & \dots & l_s \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_q} \otimes e_{l_1} \otimes \dots \otimes e_{l_s} \\ &= \sum_{j_1 \dots j_q} \sum_{l_1 \dots l_s} \delta_{[\pi\sigma]} \begin{pmatrix} i_1 & \dots & i_p & k_1 & \dots & k_r \\ j_1 & \dots & j_q & l_1 & \dots & l_s \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_q} \otimes e_{l_1} \otimes \dots \otimes e_{l_s} \\ &= T_{[\pi\sigma]}(e_{i_1} \otimes \dots \otimes e_{i_p} \otimes e_{k_1} \otimes \dots \otimes e_{k_r}) \end{aligned}$$

(2) The composition axiom follows from the following computation:

$$\begin{aligned}
& T_\pi T_\sigma(e_{i_1} \otimes \dots \otimes e_{i_p}) \\
&= \sum_{j_1 \dots j_q} \delta_\sigma \begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_q \end{pmatrix} \sum_{k_1 \dots k_r} \delta_\pi \begin{pmatrix} j_1 & \dots & j_q \\ k_1 & \dots & k_r \end{pmatrix} e_{k_1} \otimes \dots \otimes e_{k_r} \\
&= \sum_{k_1 \dots k_r} N^{c(\pi, \sigma)} \delta_{[\sigma]} \begin{pmatrix} i_1 & \dots & i_p \\ k_1 & \dots & k_r \end{pmatrix} e_{k_1} \otimes \dots \otimes e_{k_r} \\
&= N^{c(\pi, \sigma)} T_{[\sigma]}(e_{i_1} \otimes \dots \otimes e_{i_p})
\end{aligned}$$

(3) Finally, the involution axiom follows from the following computation:

$$\begin{aligned}
& T_\pi^*(e_{j_1} \otimes \dots \otimes e_{j_q}) \\
&= \sum_{i_1 \dots i_p} \langle T_\pi^*(e_{j_1} \otimes \dots \otimes e_{j_q}), e_{i_1} \otimes \dots \otimes e_{i_p} \rangle e_{i_1} \otimes \dots \otimes e_{i_p} \\
&= \sum_{i_1 \dots i_p} \delta_\pi \begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_q \end{pmatrix} e_{i_1} \otimes \dots \otimes e_{i_p} \\
&= T_{\pi^*}(e_{j_1} \otimes \dots \otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical.  $\square$

In relation with the quantum groups, we have the following result, from [37]:

**THEOREM 2.12.** *Each category of partitions  $D = (D(k, l))$  produces a family of compact quantum groups  $G = (G_N)$ , one for each  $N \in \mathbb{N}$ , via the formula*

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{span} \left( T_\pi \Big|_{\pi \in D(k, l)} \right)$$

*which produces a Tannakian category, and the Tannakian duality correspondence.*

**PROOF.** This follows indeed from Woronowicz's Tannakian duality, in its "soft" form from [76], as explained in chapter 1 above. Indeed, let us set:

$$C(k, l) = \text{span} \left( T_\pi \Big|_{\pi \in D(k, l)} \right)$$

By using the axioms in Definition 2.10, and the categorical properties of the operation  $\pi \rightarrow T_\pi$ , from Proposition 2.11 above, we deduce that  $C = (C(k, l))$  is a Tannakian category. Thus the Tannakian duality applies, and gives the result.  $\square$

We can now formulate the following key definition:

**DEFINITION 2.13.** *A compact quantum group  $G_N$  is called easy when we have*

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{span} \left( T_\pi \Big|_{\pi \in D(k, l)} \right)$$

*for any colored integers  $k, l$ , for a certain category of partitions  $D \subset P$ .*

In other words, a compact quantum group is called easy when its Tannakian category appears in the simplest possible way: from a category of partitions. The terminology is quite natural, because Tannakian duality is basically our only serious tool.

Observe that the category  $D$  is not unique, for instance because at  $N = 1$  all the categories of partitions produce the same easy quantum group, namely  $G_1 = \{1\}$ . We will be back to this issue on several occasions, with various results about it.

In relation now with our quantum permutation groups, and with the orthogonal quantum groups too, here is our main result, coming from [1], [2] and then [23], [24], [37]:

**THEOREM 2.14.** *The quantum permutation and rotation groups are all easy,*

$$\begin{array}{ccc}
 S_N^+ & \longrightarrow & O_N^+ \\
 \uparrow & & \uparrow \\
 S_N & \longrightarrow & O_N
 \end{array}
 \quad : \quad
 \begin{array}{ccc}
 NC & \longleftarrow & NC_2 \\
 \downarrow & & \downarrow \\
 P & \longleftarrow & P_2
 \end{array}$$

with the corresponding categories of partitions being those on the right.

**PROOF.** This is something quite fundamental, the proof being as follows:

(1)  $O_N^+$ . Consider the Tannakian category of  $O_N^+$ , formed by the following spaces:

$$C_{kl} = \text{Hom}(u^{\otimes k}, u^{\otimes l})$$

By using Proposition 2.11, consider as well the following Tannakian category:

$$D = \text{span} \left( T_\pi \mid \pi \in NC_2 \right)$$

We want to prove that we have  $C = D$ . In one sense, this follows from:

$$\begin{aligned}
 u^t = u^{-1} & \implies T_\cap \in C \\
 & \implies \langle T_\cap \rangle \subset C \\
 & \implies \text{span} \left( T_\pi \mid \pi \in \langle \cap \rangle \right) \subset C \\
 & \implies D \subset C
 \end{aligned}$$

In the other sense, Tannakian duality tells us that associated to  $D$  is a certain closed subgroup  $G \subset O_N^+$ . But since Tannakian duality is contravariant, at the level of categories  $G \subset O_N^+$  translates into  $C \subset D$ . Thus we have  $C = D$ , and we are done.

(2)  $O_N$ . Since  $O_N \subset O_N^+$  appears by adding the commutation relations  $ab = ba$  between coordinates, which are implemented by the linear map  $T_\chi$  coming from the basic crossing  $\chi$ , this group is indeed easy, coming from the following category:

$$\langle NC_2, \chi \rangle = P_2$$

Alternatively, if this argument was too fast, the above proof for  $O_N^+$  can be simply rewritten, by adding at each step the basic crossing  $\lambda$ , next to the semicircle  $\cap$ .

(3)  $S_N^+$ . We know that the algebra  $C(S_N^+)$  appears as follows:

$$C(S_N^+) = C(O_N^+) / \langle u = \text{magic} \rangle$$

In order to interpret the magic condition, consider the fork partition:

$$Y \in P(2, 1)$$

The linear map associated to this fork partition  $Y$  is then given by:

$$T_Y(e_i \otimes e_j) = \delta_{ij} e_i$$

Thus, in usual matrix notation, this linear map is given by:

$$T_Y = (\delta_{ijk})_{i,j,k}$$

Now given a corepresentation  $u$ , we have the following formula:

$$(T_Y u^{\otimes 2})_{i,j,k} = \sum_{lm} (T_Y)_{i,lm} (u^{\otimes 2})_{lm,jk} = u_{ij} u_{ik}$$

On the other hand, we have as well the following formula:

$$(u T_Y)_{i,j,k} = \sum_l u_{il} (T_Y)_{l,j,k} = \delta_{jk} u_{ij}$$

We conclude that we have the following equivalence:

$$T_Y \in \text{Hom}(u^{\otimes 2}, u) \iff u_{ij} u_{ik} = \delta_{jk} u_{ij}, \forall i, j, k$$

The condition on the right being equivalent to the magic condition, we obtain:

$$C(S_N^+) = C(O_N^+) / \langle T_Y \in \text{Hom}(u^{\otimes 2}, u) \rangle$$

Thus  $S_N^+$  is indeed easy, the corresponding category of partitions being:

$$D = \langle Y \rangle = NC$$

(4)  $S_N$ . Here there is no need for new computations, because we have:

$$S_N = S_N^+ \cap O_N$$

At the categorial level means that  $S_N$  is easy, coming from:

$$\langle NC, P_2 \rangle = P$$

Alternatively, we can rewrite if we want the proof for  $S_N^+$  or  $O_N$ , by adding at each step the basic crossing/fork next respectively to the fork/basic crossing.  $\square$

As explained in chapter 1, in the context of the unitary quantum groups, this kind of easiness result has a massive number of applications. We will explore these applications in what follows, gradually. Let us start with something philosophical:

**THEOREM 2.15.** *The inclusions  $O_N \subset O_N^+$  and  $S_N \subset S_N^+$  are liberation operations in the easy quantum group sense, given by*

$$D_{G^+} = D_G \cap NC$$

*at the level of the associated categories of partitions.*

**PROOF.** This is clear indeed from Theorem 2.14 above, and from the following trivial equalities connecting the categories found there:

$$NC_2 = P_2 \cap NC \quad , \quad NC = P \cap NC$$

Indeed, these equalities correspond to the formulae in the statement.  $\square$

### 2c. Gram determinants

Let us get now into the real thing, namely classification of the irreducible representations, fusion rules, Cayley graphs, laws of characters, and other probabilistic questions. As explained in chapter 1, all these problems are related, and their solution basically requires the knowledge of the associated Tannakian category, given by:

$$C_{kl} = \text{Hom}(u^{\otimes k}, u^{\otimes l})$$

But in the easy case, where our quantum group  $G$  comes from a category of partitions  $D$ , and which covers our 4 main examples, this problem is half-solved, because:

$$C_{kl} = \text{span} \left( T_\pi \mid \pi \in D(k, l) \right)$$

The remaining half-problem to be solved is that of investigating the linear independence properties of the maps  $T_\pi$ . Let us begin with some standard combinatorics:

**DEFINITION 2.16.** *Let  $P(k)$  be the set of partitions of  $\{1, \dots, k\}$ , and let  $\pi, \sigma \in P(k)$ .*

- (1) *We write  $\pi \leq \sigma$  if each block of  $\pi$  is contained in a block of  $\sigma$ .*
- (2) *We let  $\pi \vee \sigma \in P(k)$  be the partition obtained by superposing  $\pi, \sigma$ .*

*Also, we denote by  $|\cdot|$  the number of blocks of the partitions  $\pi \in P(k)$ .*

As an illustration here, at  $k = 2$  we have  $P(2) = \{||, \square\}$ , and we have:

$$|| \leq \square$$

Also, at  $k = 3$  we have  $P(3) = \{|||, \square|, \square, |\square, \square\square\}$ , and the order relation is as follows:

$$||| \leq \square|, \square, |\square \leq \square\square$$

Observe also that we have  $\pi, \sigma \leq \pi \vee \sigma$ , and that  $\pi \vee \sigma$  is the smallest partition with this property. Due to this fact,  $\pi \vee \sigma$  is called supremum of  $\pi, \sigma$ .

Now back to quantum groups, and to the questions that we want to solve, by Frobenius duality it is enough to study the partitions having no upper legs. We have:

PROPOSITION 2.17. *The vectors  $\xi_\pi = T_\pi$  with  $\pi \in P(k)$  are given by*

$$\xi_\pi = \sum_{i_1 \dots i_k} \delta_\pi(i_1, \dots, i_k) e_{i_1} \otimes \dots \otimes e_{i_k}$$

and their scalar products are given by the formula

$$\langle \xi_\pi, \xi_\sigma \rangle = N^{|\pi \vee \sigma|}$$

where  $\vee$  is the superposition operation, and  $|\cdot|$  is the number of blocks.

PROOF. According to the formula of the vectors  $\xi_\pi$ , we have:

$$\begin{aligned} \langle \xi_\pi, \xi_\sigma \rangle &= \sum_{i_1 \dots i_k} \delta_\pi(i_1, \dots, i_k) \delta_\sigma(i_1, \dots, i_k) \\ &= \sum_{i_1 \dots i_k} \delta_{\pi \vee \sigma}(i_1, \dots, i_k) \\ &= N^{|\pi \vee \sigma|} \end{aligned}$$

Thus, we have obtained the formula in the statement.  $\square$

In order to study the Gram matrix  $G_k(\pi, \sigma) = N^{|\pi \vee \sigma|}$ , and more specifically to compute its determinant, we will use several standard facts about the partitions. We have:

DEFINITION 2.18. *The Möbius function of any lattice, and so of  $P$ , is given by*

$$\mu(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi = \sigma \\ -\sum_{\pi \leq \tau < \sigma} \mu(\pi, \tau) & \text{if } \pi < \sigma \\ 0 & \text{if } \pi \not\leq \sigma \end{cases}$$

with the construction being performed by recurrence.

As an illustration here, let us go back to the set of 2-point partitions,  $P(2) = \{||, \square\}$ . We have by definition:

$$\mu(||, ||) = \mu(\square, \square) = 1$$

Also, we know that we have  $|| < \square$ , with no intermediate partition in between, and so the above recurrence procedure gives:

$$\mu(||, \square) = -\mu(||, ||) = -1$$

Finally, we have  $\square \not\leq ||$ , and so we have as well the following formula:

$$\mu(\square, ||) = 0$$

Thus, as a conclusion, we have computed the Möbius matrix  $M_2(\pi, \sigma) = \mu(\pi, \sigma)$  of the lattice  $P(2) = \{||, \square\}$ , the formula being as follows:

$$M_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$



The computation for  $P(3) = \{|||, |\square|, \square|, |\square, \square|\}$  is similar, and leads to the following formula for the associated Möbius matrix:

$$M_3 = \begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Back to the general case now, the main interest in the Möbius function comes from the Möbius inversion formula, which states that the following happens:

$$f(\sigma) = \sum_{\pi \leq \sigma} g(\pi) \quad \implies \quad g(\sigma) = \sum_{\pi \leq \sigma} \mu(\pi, \sigma) f(\pi)$$

In linear algebra terms, the statement and proof of this formula are as follows:

**THEOREM 2.19.** *The inverse of the adjacency matrix of  $P(k)$ , given by*

$$A_k(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi \leq \sigma \\ 0 & \text{if } \pi \not\leq \sigma \end{cases}$$

is the Möbius matrix of  $P$ , given by  $M_k(\pi, \sigma) = \mu(\pi, \sigma)$ .

**PROOF.** This is well-known, coming for instance from the fact that  $A_k$  is upper triangular. Indeed, when inverting, we are led into the recurrence from Definition 2.18.  $\square$

As a first illustration, for  $P(2)$  the formula  $M_2 = A_2^{-1}$  appears as follows:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$$

Also, for  $P(3) = \{|||, |\square|, \square|, |\square, \square|\}$  the formula  $M_3 = A_3^{-1}$  reads:

$$\begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

In general, the inversion formula  $M_k = A_k^{-1}$  is something of similar nature, and this is best seen in concrete applications of the Möbius inversion formula, in its functional form, as formulated before Theorem 2.19. We will be back to this, with examples, later on.

Now back to our Gram matrix considerations, we have the following key result:

PROPOSITION 2.20. *The Gram matrix of the vectors  $\xi_\pi$  with  $\pi \in P(k)$ ,*

$$G_{\pi\sigma} = N^{|\pi \vee \sigma|}$$

*decomposes as a product of upper/lower triangular matrices,  $G_k = A_k L_k$ , where*

$$L_k(\pi, \sigma) = \begin{cases} N(N-1)\dots(N-|\pi|+1) & \text{if } \sigma \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

*and where  $A_k$  is the adjacency matrix of  $P(k)$ .*

PROOF. We have the following computation, by using the formula of the Gram matrix in the statement from Proposition 2.17 above:

$$\begin{aligned} G_k(\pi, \sigma) &= N^{|\pi \vee \sigma|} \\ &= \# \left\{ i_1, \dots, i_k \in \{1, \dots, N\} \mid \ker i \geq \pi \vee \sigma \right\} \\ &= \sum_{\tau \geq \pi \vee \sigma} \# \left\{ i_1, \dots, i_k \in \{1, \dots, N\} \mid \ker i = \tau \right\} \\ &= \sum_{\tau \geq \pi \vee \sigma} N(N-1)\dots(N-|\tau|+1) \end{aligned}$$

According now to the definition of  $A_k, L_k$ , this formula reads:

$$\begin{aligned} G_k(\pi, \sigma) &= \sum_{\tau \geq \pi} L_k(\tau, \sigma) \\ &= \sum_{\tau} A_k(\pi, \tau) L_k(\tau, \sigma) \\ &= (A_k L_k)(\pi, \sigma) \end{aligned}$$

Thus, we are led to the formula in the statement.  $\square$

As an illustration for the above result, at  $k = 2$  we have  $P(2) = \{|\,|, \square\}$ , and the above decomposition  $G_2 = A_2 L_2$  appears as follows:

$$\begin{pmatrix} N^2 & N \\ N & N \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N^2 - N & 0 \\ N & N \end{pmatrix}$$

At  $k = 3$  now, we have  $P(3) = \{|||, \square|, \square|, \square, \square\square\}$ , and the Gram matrix is:

$$G_3 = \begin{pmatrix} N^3 & N^2 & N^2 & N^2 & N \\ N^2 & N^2 & N & N & N \\ N^2 & N & N^2 & N & N \\ N^2 & N & N & N^2 & N \\ N & N & N & N & N \end{pmatrix}$$

Regarding  $L_3$ , this can be computed by writing down the matrix  $E_3(\pi, \sigma) = \delta_{\sigma \leq \pi} |\pi|$ , and then replacing each entry by the corresponding polynomial in  $N$ . We reach to the conclusion that the product  $A_3 L_3$  is as follows, producing the above matrix  $G_3$ :

$$A_3 L_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} N^3 - 3N^2 + 2N & 0 & 0 & 0 & 0 \\ N^2 - N & N^2 - N & 0 & 0 & 0 \\ N^2 - N & 0 & N^2 - N & 0 & 0 \\ N^2 - N & 0 & 0 & N^2 - N & 0 \\ N & N & N & N & N \end{pmatrix}$$

In general, the formula  $G_k = A_k L_k$  appears a bit in the same way, with  $A_k$  being binary and upper triangular, and with  $L_k$  depending on  $N$ , and being lower triangular.

We are led in this way to the following formula, due to Lindstöm [74]:

**THEOREM 2.21.** *The determinant of the Gram matrix  $G_k$  is given by*

$$\det(G_k) = \prod_{\pi \in P(k)} \frac{N!}{(N - |\pi|)!}$$

with the convention that in the case  $N < k$  we obtain 0.

**PROOF.** If we order  $P(k)$  as usual, with respect to the number of blocks, and then lexicographically, then  $A_k$  is upper triangular, and  $L_k$  is lower triangular. Thus, we have:

$$\begin{aligned} \det(G_k) &= \det(A_k) \det(L_k) \\ &= \det(L_k) \\ &= \prod_{\pi} L_k(\pi, \pi) \\ &= \prod_{\pi} N(N-1) \dots (N - |\pi| + 1) \end{aligned}$$

Thus, we are led to the formula in the statement. □

We will be back to all this later, with further results, in chapter 3 below.

## 2d. Poisson laws

Now back to the laws of characters, we can formulate:

**THEOREM 2.22.** *For an easy quantum group  $G = (G_N)$ , coming from a category of partitions  $D = (D(k, l))$ , the asymptotic moments of the main character are given by*

$$\lim_{N \rightarrow \infty} \int_{G_N} \chi^k = |D(k)|$$

where  $D(k) = D(\emptyset, k)$ , with the limiting sequence on the left consisting of certain integers, and being stationary at least starting from the  $k$ -th term.

PROOF. According to the Peter-Weyl theory, and to the definition of easiness, the moments of the main character are given by the following formula:

$$\begin{aligned} \int_{G_N} \chi^k &= \int_{G_N} \chi_{u^{\otimes k}} \\ &= \dim(\text{Fix}(u^{\otimes k})) \\ &= \dim\left(\text{span}\left(\xi_\pi \mid \pi \in D(k)\right)\right) \end{aligned}$$

Now since by Theorem 2.21 the vectors  $\xi_\pi$  are linearly independent with  $N \geq k$ , and in particular with  $N \rightarrow \infty$ , we obtain the formula in the statement.  $\square$

In order to work out consequences, we will need the following standard result:

THEOREM 2.23. *The Catalan numbers  $C_k = |NC_2(2k)|$  satisfy*

$$C_{k+1} = \sum_{a+b=k} C_a C_b$$

their generating series  $f(z) = \sum_{k \geq 0} C_k z^k$  satisfies  $z f^2 - f + 1 = 0$ , and we have:

$$C_k = \frac{1}{k+1} \binom{2k}{k}$$

PROOF. We must count the noncrossing pairings of  $\{1, \dots, 2k\}$ . Now observe that such a pairing appears by pairing 1 to an odd number,  $2a+1$ , and then inserting a noncrossing pairing of  $\{2, \dots, 2a\}$ , and a noncrossing pairing of  $\{2a+2, \dots, 2k\}$ . We conclude that we have the following recurrence formula for the Catalan numbers:

$$C_k = \sum_{a+b=k-1} C_a C_b$$

In terms of the generating series  $f(z) = \sum_{k \geq 0} C_k z^k$ , this recurrence gives:

$$\begin{aligned} z f^2 &= \sum_{a,b \geq 0} C_a C_b z^{a+b+1} \\ &= \sum_{k \geq 1} \sum_{a+b=k-1} C_a C_b z^k \\ &= \sum_{k \geq 1} C_k z^k \\ &= f - 1 \end{aligned}$$

Thus  $z f^2 - f + 1 = 0$ , and by solving this equation and choosing the solution which is bounded at  $z = 0$ , we obtain the following formula for  $f$ :

$$f(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

By using now the Taylor formula for  $\sqrt{x}$ , we obtain the following formula:

$$f(z) = \sum_{k \geq 0} \frac{1}{k+1} \binom{2k}{k} z^k$$

But this gives the formula in the statement, for the coefficients  $C_k$ . □

Now back to quantum groups, we have the following result, from [23], [24]:

**THEOREM 2.24.** *The asymptotic  $k$ -moments for the main quantum permutation and rotation groups are the double factorials, and Bell and Catalan numbers,*

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array} \quad : \quad \begin{array}{ccc} C_k & \longrightarrow & C_{k/2} \\ \downarrow & & \downarrow \\ B_k & \longrightarrow & k!! \end{array}$$

the precise formulae being as follows,

- (1)  $k!! = 1.3.5 \dots (k-3)(k-1)$ ,
- (2)  $B_k = |P(k)|$  are the Bell numbers,
- (3)  $C_k = \frac{1}{k+1} \binom{2k}{k}$  are the Catalan numbers,

with the conventions  $k!! = 0$  and  $C_{k/2} = 0$  for  $k \notin 2\mathbb{N}$ .

**PROOF.** Consider indeed the quantum groups in the statement. According to the easiness result from Theorem 2.14, and to the character formula in Theorem 2.22, the asymptotic moments in question appear by counting the following sets of partitions:

$$\begin{array}{ccc} NC(k) & \longleftarrow & NC_2(k) \\ \downarrow & & \downarrow \\ P(k) & \longleftarrow & P_2(k) \end{array}$$

But these counting questions are all standard, as follows:

(1) Regarding  $k!! = |P_2(k)|$ , this formula is clear, because we have  $k-1$  choices for the pair of 1, then  $k-3$  choices for the pair of the next number, and so on.

(2) Regarding  $B_k = |P(k)|$ , there is nothing much to be done here, because these numbers, called Bell numbers, cannot be explicitly computed.

(3) Regarding now the numbers  $C_{k/2} = |NC_2(k)|$ , which are the Catalan numbers, these can be explicitly computed by recurrence, as explained in Theorem 2.23.

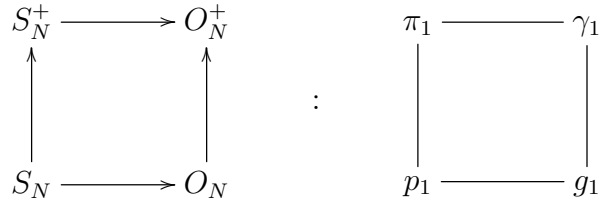
(4) Regarding  $C_k = |NC(k)|$ , this can be established either by recurrence, or deduced from (3), via fattening/shrinking. Indeed, by fattening the pairings into partitions, and shrinking the partitions into pairings, we have a correspondence as follows:

$$NC_2(2k) \simeq NC(k)$$

We conclude from this that we have  $|NC(k)| = C_k$ , as claimed. □

Once again following [23], [24], we have as well the following result:

**THEOREM 2.25.** *The asymptotic laws of characters for the quantum permutation and rotation groups are the Gaussian, Poisson, Wigner and Marchenko-Pastur laws,*



the precise formulae being as follows:

- (1)  $g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$  is the Gaussian law of parameter 1.
- (2)  $p_1 = \frac{1}{e} \sum_p \frac{\delta_p}{p!}$  is the Poisson law of parameter 1.
- (3)  $\gamma_1 = \frac{1}{2\pi} \sqrt{4-x^2} dx$  is the Wigner semicircle law of parameter 1.
- (4)  $\pi_1 = \frac{1}{2\pi} \sqrt{4x^{-1}-1} dx$  is the Marchenko-Pastur law of parameter 1.

**PROOF.** This follows indeed from Theorem 2.24, by doing some calculus:

(1) By partial integration, we have the following formula:

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^k e^{-x^2/2} dx = (k-1) \times \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^{k-2} e^{-x^2/2} dx$$

Thus the moments of  $g_1$  satisfy the same recurrence as the numbers  $k!!$ .

(2) The moments of the Poisson law  $p_1$  are the following numbers:

$$M_k = \frac{1}{e} \sum_{p \in \mathbb{N}} \frac{p^k}{k!}$$

Computations show that the recurrence is the same as for the Bell numbers  $B_k$ .

(3) The moment generating function for the semicircle law  $\gamma_1$  is given by:

$$f(z) = \frac{1}{2\pi} \int_{-2}^2 \frac{\sqrt{4-x^2}}{1-zx} dx$$

By doing some computations, the coefficients of  $f$  are the Catalan numbers.

(4) The moment generating function for the Marchenko-Pastur law  $\pi_1$  is:

$$f(z) = \frac{1}{2\pi} \int_0^4 \frac{\sqrt{4x^{-1} - 1}}{1 - zx} dx$$

By computation, we obtain the generating series of the Catalan numbers.  $\square$

The above proof was of course quite short, but all this is standard material, and we will be back to it with full details in chapter 5 below, when doing analysis.

As a conclusion now, the representation theory of our basic quantum groups is something extremely simple and fundamental, in the  $N \rightarrow \infty$  limit.

We will see in the next chapter that the results in the free case can be improved, with the convergences there being actually stationary, starting from  $N = 2$ . Also, we will see in chapter 5 below that the above results can be extended to the case of truncated characters, with the limiting  $N \rightarrow \infty$  measures being  $p_t, g_t, \pi_t, \gamma_t$ , with  $t \in (0, 1]$ .

## 2e. Exercises

Here is a first exercise, regarding the isomorphism  $S_3 = S_3^+$ , which is one of our most basic results, at the level of the general theory:

EXERCISE 2.26. *Find a new proof for the equality*

$$S_3 = S_3^+$$

*by doing manipulations with the  $3 \times 3$  magic matrices, as to end up with the conclusion that the coefficients must commute.*

To be more precise, we have already seen such a proof in the above, and the problem is that of finding a new proof, a bit in the same spirit, based on new computations. This should normally not be very difficult, because there are many possible such proofs.

Along the same lines, here is something more conceptual:

EXERCISE 2.27. *Prove that we have  $S_3 = S_3^+$  by looking at the coaction*

$$\Phi : \mathbb{C}^3 \rightarrow \mathbb{C}^3 \otimes C(S_3^+)$$

*written in terms of the Fourier basis of  $\mathbb{C}^3$ .*

To be more precise, the question here is that of changing the basis of  $\mathbb{C}^3$ , by using the Fourier transform over the group  $\mathbb{Z}_3$ , then reformulating the magic condition at  $N = 3$  in terms of this new basis, and then deducing that the coefficients must commute.

Here is now another exercise, this time at  $N = 4$ :

EXERCISE 2.28. *Prove that the quantum group inclusion*

$$\widehat{D}_\infty \subset S_4^+$$

*is not an isomorphism.*

This is something which normally should not be difficult.

At  $N = 5$  now, here is an exercise which is more advanced:

EXERCISE 2.29. *Prove that the discrete quantum group*

$$\Gamma = \widehat{S}_5^+$$

*is not amenable, in the discrete quantum group sense.*

This requires of course some good knowledge of the notion of amenability. As a hint here, try finding a quantum subgroup  $G \subset S_5^+$  whose dual is not amenable.

In relation now with easiness, here is a good exercise:

EXERCISE 2.30. *Prove that  $S_N$  is easy, directly.*

To be more precise, our proof for  $S_N$  was based on the fact that  $S_N^+$  is easy. The problem is that of finding a direct proof, with no reference to quantum groups.

Here is an instructive exercise in relation with partitions:

EXERCISE 2.31. *Write down explicitly the matrices*

$$A_4, M_4, G_4, L_4$$

*appearing in the various computations from this chapter, at  $k = 4$ .*

This is certainly quite time-consuming, and might look a bit futile. However, having everything working fine requires some knowledge, which is useful, and that can only be learned in this way. Welcome to research, where doing such things is commonplace.

In the same spirit, we have something even better, namely:

EXERCISE 2.32. *Verify the Lindstöm formula, namely*

$$\det(G_k) = \prod_{\pi \in P(k)} \frac{N!}{(N - |\pi|)!}$$

*at  $k = 2, 3, 4$ , explicitly, without any previous knowledge or trick.*

Here  $k = 2$  is something clear,  $k = 3$  is a must-do computation, as an initiation to the  $5 \times 5$  matrices, and  $k = 4$  is probably something quite difficult, but worth a try.

In relation with double factorials and Bell and Catalan numbers, we have:



EXERCISE 2.33. *Prove that the number of elements of*

$$\begin{array}{ccc} NC(k) & \longleftarrow & NC_2(k) \\ \downarrow & & \downarrow \\ P(k) & \longleftarrow & P_2(k) \end{array}$$

*are the double factorials, and Bell and Catalan numbers,*

$$\begin{array}{ccc} C_k & \longleftarrow & C_{k/2} \\ \downarrow & & \downarrow \\ B_k & \longleftarrow & k!! \end{array}$$

*with the conventions  $k!! = 0$  and  $C_{k/2} = 0$  for  $k \notin 2\mathbb{N}$ .*

This is something that we talked about, in the above, and then used afterwards, and the problem now is that of working out all this, in detail.

Finally, in relation with the Poisson laws, we have:

EXERCISE 2.34. *Prove that for the symmetric group  $S_N$  we have*

$$\chi \sim p_1$$

*with  $N \rightarrow \infty$ , directly, without using representations and easiness.*

This looks quite doable, because this is after all something about permutations, and their number of fixed points. As a hint, try using inclusion-exclusion.

## CHAPTER 3

### Representation theory

#### 3a. Basic results

We have seen so far that the inclusion  $S_N \subset S_N^+$ , as well as its companion inclusion  $O_N \subset O_N^+$ , are both liberations in the sense of easy quantum group theory, and that some interesting representation theory consequences, in the  $N \rightarrow \infty$  limit, can be derived from this. We discuss here the case where  $N \in \mathbb{N}$  is fixed, which is more technical.

Let us first discuss the representation theory of the quantum group  $O_N^+$ . Here the result, from [1], which is elementary, is as follows:

**THEOREM 3.1.** *The quantum groups  $O_N^+$  with  $N \geq 2$  have the following properties:*

- (1) *The odd moments of the main character vanish, and the even moments are:*

$$\int_{O_N^+} \chi^{2k} = C_k$$

- (2) *The main character follows the Wigner semicircle law of parameter 1:*

$$\chi \sim \gamma_1$$

- (3) *The fusion rules for irreducible representations are the same as for  $SU_2$ :*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+2} + \dots + r_{k+l}$$

- (4) *The dimensions of the representations are as follows, with  $q + q^{-1} = N$ :*

$$\dim(r_k) = \frac{q^{k+1} - q^{-k-1}}{q - q^{-1}}$$

**PROOF.** There are several proofs for this fact, the simplest one being via purely algebraic methods, based on the easiness property of  $O_N^+$ , as follows:

- (1) Our claim is that we can define, by recurrence on  $k \in \mathbb{N}$ , a sequence  $r_0, r_1, r_2, \dots$  of irreducible, self-adjoint and distinct representations of  $O_N^+$ , satisfying:

$$r_0 = 1$$

$$r_1 = u$$

$$r_k + r_{k-2} = r_{k-1} \otimes r_1$$

(2) Indeed, at  $k = 0$  this is clear, and at  $k = 1$  this is clear as well, with the irreducibility of  $r_1 = u$  coming from the embedding  $O_N \subset O_N^+$ . So assume now that  $r_0, \dots, r_{k-1}$  as above are constructed, and let us construct  $r_k$ . We have, by recurrence:

$$r_{k-1} + r_{k-3} = r_{k-2} \otimes r_1$$

In particular we have an inclusion of representations, as follows:

$$r_{k-1} \subset r_{k-2} \otimes r_1$$

Now since  $r_{k-2}$  is irreducible, by Frobenius reciprocity we have:

$$r_{k-2} \subset r_{k-1} \otimes r_1$$

Thus, there exists a certain representation  $r_k$  such that:

$$r_k + r_{k-2} = r_{k-1} \otimes r_1$$

(3) As a first observation, this representation  $r_k$  is self-adjoint. Indeed, our recurrence formula  $r_k + r_{k-2} = r_{k-1} \otimes r_1$  for the representations  $r_0, r_1, r_2, \dots$  shows that the characters of these representations are polynomials in  $\chi_u$ . Now since  $\chi_u$  is self-adjoint, all the characters that we can obtain via our recurrence are self-adjoint as well.

(4) It remains to prove that  $r_k$  is irreducible, and non-equivalent to  $r_0, \dots, r_{k-1}$ . For this purpose, observe that according to our recurrence formula,  $r_k + r_{k-2} = r_{k-1} \otimes r_1$ , we can now split  $u^{\otimes k}$ , as a sum of the following type, with positive coefficients:

$$u^{\otimes k} = c_k r_k + c_{k-2} r_{k-2} + \dots$$

We conclude by Peter-Weyl that we have an inequality as follows, with equality precisely when  $r_k$  is irreducible, and non-equivalent to the other summands  $r_i$ :

$$\sum_i c_i^2 \leq \dim(\text{End}(u^{\otimes k}))$$

(5) Now let us use the easiness property of  $O_N^+$ . This gives us an upper bound for the number on the right, that we can add to our inequality, as follows:

$$\sum_i c_i^2 \leq \dim(\text{End}(u^{\otimes k})) \leq C_k$$

The point now is that the coefficients  $c_i$  come straight from the Clebsch-Gordan rules, and their combinatorics shows that  $\sum_i c_i^2$  equals the Catalan number  $C_k$ , with the remark that this follows as well from the known theory of  $SU_2$ . Thus, we have global equality in the above estimate, and in particular we have equality at left, as desired.

(6) In order to finish the proof of our claim, observe that  $r_k$  is non-equivalent to  $r_{k-1}, r_{k-3}, \dots$ , for instance because of the embedding  $O_N \subset O_N^+$ , which shows that the even and odd tensor powers of  $u$  cannot have common irreducible components.

(7) Since by Peter-Weyl any irreducible representation of  $O_N^+$  must appear in some tensor power  $u^{\otimes k}$ , and we know how to decomposing each  $u^{\otimes k}$  into sums of representations  $r_k$ , these representations  $r_k$  are all the irreducible representations of  $O_N^+$ .

(8) In what regards now the law of the main character, we obtain here the Wigner law  $\gamma_1$ , as stated, due to the fact that the equality in (5) gives us the even moments of this law, and that the observation in (6) tells us that the odd moments vanish.

(9) Finally, from the Clebsch-Gordan rules we have  $r_k r_1 = r_{k-1} + r_{k+1}$ , and we obtain from this, by recurrence, with  $q > 0$  being such that  $q + q^{-1} = N$ :

$$\dim r_k = q^k + q^{k-2} + \dots + q^{-k+2} + q^{-k}$$

But this gives the dimension formula in the statement, and we are done.  $\square$

The above result has some interesting combinatorial consequences, as follows:

**PROPOSITION 3.2.** *The following are linearly independent, for any  $N \geq 2$ :*

- (1) *The linear maps  $\{T_\pi | \pi \in NC_2(k, l)\}$ , with  $k + l \in 2\mathbb{N}$ .*
- (2) *The vectors  $\{\xi_\pi | \pi \in NC_2(2k)\}$ , with  $k \in \mathbb{N}$ .*
- (3) *The linear maps  $\{T_\pi | \pi \in NC_2(k, k)\}$ , with  $k \in \mathbb{N}$ .*

**PROOF.** All this follows from the dimension equalities established in the proof of Theorem 3.1, because in all cases, the number of partitions is a Catalan number.  $\square$

In order to pass now to quantum permutations, we can use the following well-known trick, relating noncrossing pairings to arbitrary noncrossing partitions:

**PROPOSITION 3.3.** *We have a bijection  $NC(k) \simeq NC_2(2k)$ , constructed by fattening and shrinking, as follows:*

- (1) *The application  $NC(k) \rightarrow NC_2(2k)$  is the “fattening” one, obtained by doubling all the legs, and doubling all the strings as well.*
- (2) *Its inverse  $NC_2(2k) \rightarrow NC(k)$  is the “shrinking” application, obtained by collapsing pairs of consecutive neighbors.*

**PROOF.** The fact that the two operations in the statement are indeed inverse to each other is clear, by computing the corresponding two compositions, with the remark that the construction of the fattening operation requires the partitions to be noncrossing.  $\square$

At the level of the associated Gram matrices, the result is as follows:

**PROPOSITION 3.4.** *The Gram matrices of the sets of partitions*

$$NC_2(2k) \simeq NC(k)$$

*are related by the following formula, where  $\pi \rightarrow \pi'$  is the shrinking operation,*

$$G_{2k,n}(\pi, \sigma) = n^k (\Delta_{kn}^{-1} G_{k,n^2} \Delta_{kn}^{-1})(\pi', \sigma')$$

*and where  $\Delta_{kn}$  is the diagonal of  $G_{kn}$ .*

PROOF. In the context of the general fattening and shrinking bijection from Proposition 3.3 above, it is elementary to see that we have:

$$|\pi \vee \sigma| = k + 2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|$$

We therefore have the following formula, valid for any  $n \in \mathbb{N}$ :

$$n^{|\pi \vee \sigma|} = n^{k+2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|}$$

Thus, we obtain the formula in the statement. Now by applying the determinant, we obtain from this a formula of the following type, with  $C > 0$  being a constant:

$$\det(G_{2k,n}) = C \cdot \det(G_{k,n^2})$$

Thus, we are led to the formula in the statement.  $\square$

We can now formulate a “projective” version of Proposition 3.2, as follows:

PROPOSITION 3.5. *The following are linearly independent, for any  $N = n^2$  with  $n \geq 2$ :*

- (1) *The linear maps  $\{T_\pi | \pi \in NC(k, l)\}$ , with  $k, l \in 2\mathbb{N}$ .*
- (2) *The vectors  $\{\xi_\pi | \pi \in NC(k)\}$ , with  $k \in \mathbb{N}$ .*
- (3) *The linear maps  $\{T_\pi | \pi \in NC(k, k)\}$ , with  $k \in \mathbb{N}$ .*

PROOF. This follows indeed from the various results from Proposition 3.2, by using the Gram determinant formula from Proposition 3.4.  $\square$

Following [2], we can now work out the representation theory of the quantum group  $S_4^+$ , and more generally of any  $S_N^+$  with  $N = n^2$  and  $n \geq 2$ , as follows:

THEOREM 3.6. *The quantum groups  $S_N^+$  with  $N = n^2$  and  $n \geq 2$  are as follows:*

- (1) *The moments of the main character are the Catalan numbers:*

$$\int_{S_N^+} \chi^k = C_k$$

- (2) *The main character follows the Marchenko-Pastur law of parameter 1:*

$$\chi \sim \pi_1$$

- (3) *The fusion rules for irreducible representations are the same as for  $SO_3$ :*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+1} + \dots + r_{k+l}$$

- (4) *The dimensions of the representations are as follows, with  $q + q^{-1} = N - 2$ :*

$$\dim(r_k) = \frac{q^{k+1} - q^{-k}}{q - 1}$$

PROOF. This is quite similar to the proof of Theorem 3.1 above, by using the linear independence result from Proposition 3.5 as main ingredient, as follows:

(1) We have indeed the following computation, based on the above:

$$\begin{aligned} \int_{S_N^+} \chi^k &= \dim(\text{Fix}(u^{\otimes k})) \\ &= \#NC(k) \\ &= \#NC_2(2k) \\ &= C_k \end{aligned}$$

(2) This follows from (1), as explained in chapter 1 above.

(3) This is standard, by using the moment formula in (1), and the known theory of  $SO_3$ . Let indeed  $A = \text{span}(\chi_k | k \in \mathbb{N})$  be the algebra of characters of  $SO_3$ . We can define a morphism as follows,  $f$  being the character of the fundamental representation of  $S_N^+$ :

$$\Psi : A \rightarrow C(S_N^+) \quad , \quad \chi_1 \rightarrow f - 1$$

The elements  $f_k = \Psi(\chi_k)$  verify then the Clebsch-Gordan rules, namely:

$$f_k f_l = f_{|k-l|} + f_{|k-l|+1} + \dots + f_{k+l}$$

We prove now by recurrence that each  $f_k$  is the character of an irreducible corepresentation  $r_k$  of  $C(S_N^+)$ , non-equivalent to  $r_0, \dots, r_{k-1}$ . At  $k = 0, 1$  this is clear. So, assume now that the result holds at  $k - 1$ . By integrating characters we have, as for  $SO_3$ :

$$r_{k-2}, r_{k-1} \subset r_{k-1} \otimes r_1$$

Thus there exists a corepresentation  $r_k$  such that:

$$r_{k-1} \otimes r_1 = r_{k-2} + r_{k-1} + r_k$$

Once again by integrating characters, we conclude that  $r_k$  is irreducible, and non-equivalent to  $r_1, \dots, r_{k-1}$ , as for  $SO_3$ , and this proves our claim. Finally, since any irreducible representation of  $S_N^+$  must appear in some tensor power of  $u$ , and we have a formula for decomposing each  $u^{\otimes k}$  into sums of representations  $r_l$ , we conclude that these representations  $r_l$  are all the irreducible representations of  $S_N^+$ .

(4) Finally, the dimension formula is clear by recurrence. □

### 3b. Meander determinants

Let us discuss now the extension of the above result, to all the quantum groups  $S_N^+$  with  $N \geq 4$ . For this purpose we need an extension of the linear independence results from Proposition 3.5. There are several approaches here, none being trivial, and we will use in what follows a method which is long, but elementary and rock-solid, namely getting the linear independence by computing the associated Gram determinant.

We already know, from chapter 2 above, that for the group  $S_N$  the formula of the corresponding Gram matrix determinant, due to Lindstöm [74], is as follows:

**THEOREM 3.7.** *The determinant of the Gram matrix of  $S_N$  is given by*

$$\det(G_{kN}) = \prod_{\pi \in P(k)} \frac{N!}{(N - |\pi|)!}$$

with the convention that in the case  $N < k$  we obtain 0.

**PROOF.** This is something that we know from chapter 2, the idea being that  $G_{kN}$  naturally decomposes as a product of an upper triangular and lower triangular matrix.  $\square$

Although we will not need this here, let us discuss as well, for the sake of completeness, the case of the orthogonal group  $O_N$ . Here the combinatorics is that of the Young diagrams. We denote by  $|\cdot|$  the number of boxes, and we use quantity  $f^\lambda$ , which gives the number of standard Young tableaux of shape  $\lambda$ . The result is then as follows:

**THEOREM 3.8.** *The determinant of the Gram matrix of  $O_N$  is given by*

$$\det(G_{kN}) = \prod_{|\lambda|=k/2} f_N(\lambda)^{f^{2\lambda}}$$

where the quantities on the right are  $f_N(\lambda) = \prod_{(i,j) \in \lambda} (N + 2j - i - 1)$ .

**PROOF.** This follows from the results of Zinn-Justin [100]. Indeed, it is known from there that the Gram matrix is diagonalizable, as follows:

$$G_{kN} = \sum_{|\lambda|=k/2} f_N(\lambda) P_{2\lambda}$$

Here  $1 = \sum P_{2\lambda}$  is the standard partition of unity associated to the Young diagrams having  $k/2$  boxes, and the coefficients  $f_N(\lambda)$  are those in the statement. Now since we have  $\text{Tr}(P_{2\lambda}) = f^{2\lambda}$ , this gives the result. See [27], [100].  $\square$

For the free orthogonal and symmetric groups, the results, from [58], are substantially more complicated. Let us begin with some examples. We first have:

**PROPOSITION 3.9.** *At  $k = 1$  the set of partitions for  $S_N^+$  is  $NC(1) = \{\emptyset\}$ , and the corresponding Gram matrix and its determinant are:*

$$\det(N) = N$$

*Also, at  $k = 2$  the set of partitions for  $O_N^+$  is  $NC_2(2) = \{\square\}$ , and the corresponding Gram matrix and its determinant are:*

$$\det(N) = N$$

**PROOF.** This is indeed clear from definitions.  $\square$

Next in line, we have the following result:

PROPOSITION 3.10. *At  $k = 2$  the set of partitions for  $S_N^+$  is  $NC(2) = \{||, \sqcap\}$ , and the corresponding Gram matrix and its determinant are:*

$$\det \begin{pmatrix} N^2 & N \\ N & N \end{pmatrix} = N^2(N - 1)$$

*Also, at  $k = 4$  the set of partitions for  $O_N^+$  is  $NC_2(4) = \{\sqcap\sqcap, \sqcup\sqcup\}$ , and the corresponding Gram matrix and its determinant are:*

$$\det \begin{pmatrix} N^2 & N \\ N & N^2 \end{pmatrix} = N^2(N^2 - 1)$$

PROOF. This is once again clear from definitions.  $\square$

With a few tricks, we can work out as well the next computation, as follows:

PROPOSITION 3.11. *At  $k = 3$  the partition set for  $S_N^+$  is  $NC(3) = \{|||, \sqcap|, \sqcap\sqcap, |\sqcap, \sqcap\sqcap\}$ , and the corresponding Gram matrix and its determinant are:*

$$\det \begin{pmatrix} N^3 & N^2 & N^2 & N^2 & N \\ N^2 & N^2 & N & N & N \\ N^2 & N & N^2 & N & N \\ N^2 & N & N & N^2 & N \\ N & N & N & N & N \end{pmatrix} = N^5(N - 1)^4(N - 2)$$

*Also, at  $k = 6$  the set of partitions for  $O_N^+$  is  $NC_2(6) \simeq NC(3)$ , and the corresponding Gram matrix and its determinant are:*

$$\det \begin{pmatrix} N^3 & N^2 & N^2 & N^2 & N \\ N^2 & N^3 & N & N & N^2 \\ N^2 & N & N^3 & N & N^2 \\ N^2 & N & N & N^3 & N^2 \\ N & N^2 & N^2 & N^2 & N^3 \end{pmatrix} = N^5(N^2 - 1)^4(N^2 - 2)$$

PROOF. We have two formulae to be proved, the idea being as follows:

(1) In what regards  $S_N^+$ , the set of partitions here is  $NC(3) = P(3)$ , and so the corresponding Gram matrix is the one in the statement, exactly as for  $S_N$ , that we know well from chapter 2 above, and from Theorem 3.7. By using the Lindstöm formula, from Theorem 3.7, the determinant of this matrix is, as claimed:

$$\begin{aligned} \det &= \prod_{\pi \in P(3)} \frac{N!}{(N - |\pi|)!} \\ &= \frac{N!}{(N - 3)!} \left( \frac{N!}{(N - 2)!} \right)^3 \frac{N!}{(N - 1)!} \\ &= N(N - 1)(N - 2)N^3(N - 1)^3N \\ &= N^5(N - 1)^4(N - 2) \end{aligned}$$



(2) Regarding now  $O_N^+$ , the set of partitions here is  $NC_2(6)$ , and by using the identification  $NC_2(6) \simeq NC(3)$  from Proposition 3.3, we can list this set as follows:

$$NC_2(6) = \left\{ \ker(abbcc), \ker(aabccb), \ker(abbacc), \ker(abccba), \ker(abbcca) \right\}$$

With this ordering convention for  $NC_2(6)$ , used for writing the matrix indices, we obtain the Gram matrix in the statement, namely:

$$G_{6N} = \begin{pmatrix} N^3 & N^2 & N^2 & N^2 & N \\ N^2 & N^3 & N & N & N^2 \\ N^2 & N & N^3 & N & N^2 \\ N^2 & N & N & N^3 & N^2 \\ N & N^2 & N^2 & N^2 & N^3 \end{pmatrix}$$

Regarding now the determinant, we can deduce it from (1), via the fattening/shrinking formula in Proposition 3.4, which in our case tells that  $G_{6N}$  is equal to:

$$\begin{pmatrix} \frac{1}{N\sqrt{N}} & & & & \\ & \frac{1}{\sqrt{N}} & & & \\ & & \frac{1}{\sqrt{N}} & & \\ & & & \frac{1}{\sqrt{N}} & \\ & & & & \sqrt{N} \end{pmatrix} \begin{pmatrix} N^6 & N^4 & N^4 & N^4 & N^2 \\ N^4 & N^4 & N^2 & N^2 & N^2 \\ N^4 & N^2 & N^4 & N^2 & N^2 \\ N^4 & N^2 & N^2 & N^4 & N^2 \\ N^2 & N^2 & N^2 & N^2 & N^2 \end{pmatrix} \begin{pmatrix} \frac{1}{N\sqrt{N}} & & & & \\ & \frac{1}{\sqrt{N}} & & & \\ & & \frac{1}{\sqrt{N}} & & \\ & & & \frac{1}{\sqrt{N}} & \\ & & & & \sqrt{N} \end{pmatrix}$$

Thus, by using (1), we obtain the following formula for the determinant:

$$\begin{aligned} \det &= \frac{1}{N^2\sqrt{N}} \times N^{10}(N^2 - 1)^4(N^2 - 2) \times \frac{1}{N^2\sqrt{N}} \\ &= N^5(N^2 - 1)^4(N^2 - 2) \end{aligned}$$

Thus, we have obtained the formula in the statement.  $\square$

In general, the above trick based on the Lindstöm formula will not work anymore, because  $NC(k)$  is strictly smaller than  $P(k)$  at  $k \geq 4$ . Thus, in order to solve our problems, we must do some new combinatorics.

Let us start with the following standard result:

**PROPOSITION 3.12.** *The set  $NC_2(k)$  is in bijection with the set  $L_k$  of loops on  $\mathbb{N}$  based at 0, with the loop associated to a pairing  $\pi \in NC_2(2k)$  being*

$$h_0 - h_1 - h_2 - \dots - h_k$$

where  $h_i$  with  $i \in \{0, \dots, k\}$  is the number of indices  $j \in \{1, 2, \dots, i\}$  which are joined by  $\pi$  to a number in  $\{i + 1, i + 2, \dots, k\}$ .

PROOF. When  $k$  is odd, both sets in the statement are empty. When  $k$  is even, let us first work out the case  $k = 4$ . Here the numbers  $h_0, \dots, h_4$  are easy to compute, and we obtain the following loops, which are the only 2 possible length 2 loops on  $\mathbb{N}$ :

$$\ker(aabb) \rightarrow [0 - 1 - 0 - 1 - 0]$$

$$\ker(abba) \rightarrow [0 - 1 - 2 - 1 - 0]$$

At  $k = 6$  now, the computation is elementary too, and we obtain the following loops, which are once again all the possible length 6 loops on  $\mathbb{N}$ :

$$\ker(aabbcc) \rightarrow [0 - 1 - 0 - 1 - 0 - 1 - 0]$$

$$\ker(aabccb) \rightarrow [0 - 1 - 0 - 1 - 2 - 1 - 0]$$

$$\ker(abbacc) \rightarrow [0 - 1 - 2 - 1 - 0 - 1 - 0]$$

$$\ker(abccba) \rightarrow [0 - 1 - 2 - 3 - 2 - 1 - 0]$$

$$\ker(abbcca) \rightarrow [0 - 1 - 2 - 1 - 2 - 1 - 0]$$

In general, the proof is quite similar. It is clear from definitions that we have  $h_0 = 0$ ,  $h_k = 0$ , so our loop candidates start and end at 0, as they should. Also, it is clear as well that we have  $h_1 = 1$ ,  $h_{k-1} = 1$ , and with a bit more care we obtain that we have  $h_i = h_{i+1} \pm 1$  for any  $i$ , and so we have indeed loops on  $\mathbb{N}$ . But these loops are easily seen to be distinct, and since their number is the correct one, namely the Catalan number  $C_{k/2} = |NC_2(k)|$ , we obtain in this way all the length  $k$  loops on  $\mathbb{N}$ , as claimed.  $\square$

The point now is that the above model for  $NC_2(k)$  can be used in order to derive an explicit interpretation of the corresponding Gram matrix. Let us start with:

PROPOSITION 3.13. *The Chebycheff polynomials, given by*

$$P_0 = 1 \quad , \quad P_1 = N \quad , \quad P_{r+1} = NP_r - P_{r-1}$$

*are orthogonal polynomials for the graph  $\mathbb{N}$ , in the sense that we have*

$$AP = NP$$

*where  $A_{ij} = \delta_{|i-j|,1}$  is the adjacency matrix for  $\mathbb{N}$ .*

PROOF. The fact that eigenvalue formula  $AP = NP$  holds is indeed clear from the definition of the polynomials  $P_r$ .  $\square$

Let us record as well the first few values of the Chebycheff polynomials:

$$P_0 = 1$$

$$P_1 = N$$

$$P_2 = N^2 - 1$$

$$P_3 = N(N^2 - 2)$$

$$\vdots$$

These formulae are quite interesting in connection with the Gram determinant formulae for  $O_N^+$  from Propositions 3.9, 3.10, 3.11, because in all cases there, we obtain a certain product of Chebycheff polynomials. We will prove now that this is the case in general. Let us start with the following fact, coming as a continuation of Proposition 3.12:

PROPOSITION 3.14. *The Gram matrix of  $NC_2(k)$  is given by the formula*

$$G_{kN}(\pi, \sigma) = \langle f_\pi, f_\sigma \rangle$$

where for  $\pi \in NC_2(k)$  we define a vector  $f_\pi \in H_k = \text{span}(L_k)$  by

$$f_\pi = \sum_{l \in L_k} \left( \prod_{i \sim_{\pi} j} \delta(l_i, l_j^o) \gamma(l_i) \right) l$$

where  $e \rightarrow e^o$  is the edge reversing, and where the “spin factor” is as follows,

$$\gamma = \sqrt{\frac{P_t}{P_s}}$$

where  $s, t$  are the source and target of the edges.

PROOF. This is something quite standard, which follows for instance from the general planar algebra theory in [70].  $\square$

The point now is that, in the above explicit picture, using loops on  $\mathbb{N}$ , the Gram matrix can be decomposed, in the following way:

PROPOSITION 3.15. *The Gram matrix of  $NC_2(k)$  decomposes as*

$$G_{kN} = T_{kN} T_{kN}^t$$

where  $T_{kN}$  is the following matrix, defined using the above conventions:

$$T_{kN}(\pi, \sigma) = \prod_{i \sim_{\pi} j} \delta(l(\sigma)_i, l(\sigma)_j^o) \gamma(l(\sigma)_i)$$

Moreover, this latter matrix  $T_{kN}$  is lower triangular.

PROOF. We have several assertions to be proved, the idea being as follows:

(1) By definition of  $T_{kN}$ , we have the following formula:

$$f_\pi = \sum_{\sigma} T_{kN}(\pi, \sigma) \cdot l(\sigma)$$

We therefore obtain, as desired:

$$G_{kN} = T_{kN} T_{kN}^t$$

(2) We show that  $T_{kN}$  is lower triangular. Indeed, consider the partial order on  $NC_2(k)$  given by  $\pi \leq \sigma$  if  $h_\pi(i) \leq h_\sigma(i)$  for  $i = 1, \dots, k$ . Our claim is that  $\sigma \not\leq \pi$  implies:

$$T_{kN}(\pi, \sigma) = 0$$

Indeed, suppose that  $\sigma \not\sim \pi$ , and let  $j$  be the least number with  $h_\sigma(j) > h_\pi(j)$ . Note that we must have:

$$\begin{aligned} h_\sigma(j-1) &= h_\pi(j-1) \\ h_\sigma(j) &= h_\pi(j) + 2 \end{aligned}$$

It follows that we have  $i \sim_\pi j$  for some  $i < j$ . From the definitions of  $T_{kn}$  and  $l(\sigma)$ , if  $T_{kn}(\pi, \sigma) \neq 0$  then we must have:

$$h_\sigma(i-1) = h_\sigma(j) = h_\pi(j) + 2$$

But we also have  $h_\pi(i-1) = h_\pi(j)$ , so that  $h_\sigma(i-1) = h_\pi(i-1) + 2$ , which contradicts the minimality of  $j$ . Thus,  $T_{kN}$  is lower triangular, as claimed.  $\square$

We can now formulate, following [58], the following result:

**THEOREM 3.16.** *The determinant of the Gram matrix for  $O_N^+$  is given by*

$$\det(G_{kN}) = \prod_{r=1}^{\lfloor k/2 \rfloor} P_r(N)^{d_{k/2,r}}$$

where  $P_r$  are the Chebycheff polynomials, given by

$$P_0 = 1 \quad , \quad P_1 = X \quad , \quad P_{r+1} = XP_r - P_{r-1}$$

and  $d_{kr} = f_{kr} - f_{k,r+1}$ , with  $f_{kr}$  being the following numbers, depending on  $k, r \in \mathbb{Z}$ ,

$$f_{kr} = \binom{2k}{k-r} - \binom{2k}{k-r-1}$$

with the convention  $f_{kr} = 0$  for  $k \notin \mathbb{Z}$ .

**PROOF.** Since  $T_{kN}$  is lower triangular we have:

$$\begin{aligned} \det(T_{kN}) &= \prod_{\pi} T_{kN}(\pi, \pi) \\ &= \prod_{\pi} \prod_{i \sim_{\pi} j} \sqrt{\frac{P_{h_{\pi}(i)}}{P_{h_{\pi}(i)-1}}} \\ &= \prod_{r=1}^{k/2} P_r^{e_{kr}/2} \end{aligned}$$

Here the exponents appearing on the right are by definition as follows:

$$e_{kr} = \sum_{\pi} \sum_{i \sim_{\pi} j} \delta_{h_{\pi}(i),r} - \delta_{h_{\pi}(i),r+1}$$

Our claim now, which finishes the proof, is that for  $1 \leq r \leq k/2$  we have:

$$\sum_{\pi} \sum_{i \sim_{\pi} j} \delta_{h_{\pi}(i)r} = f_{k/2,r}$$

Indeed, note that the left term counts the number of times that the edge  $(r, r+1)$  appears in all loops in  $L_k$ . Define a shift operator  $S$  on the edges of the graph  $\mathbb{N}$  by:

$$S(s, t) = (s+1, t+1)$$

Given a loop  $l = l_1 \dots l_k$  and  $1 \leq s \leq k$  with  $l_s = (r, r+1)$ , define a path as follows:

$$S^r(l_s) \dots S^r(l_k) l_{s-1}^o \dots l_1^o$$

Observe that this is a path on the graph  $\mathbb{N}$  from  $2r$  to  $0$  whose first edge is  $(2r, 2r+1)$  and first reaches  $r-1$  after  $k-s+1$  steps.

Conversely, given a path  $f_1 \dots f_k$  from  $2r$  to  $0$  whose first edge is  $(2r, 2r+1)$  and first reaches  $r-1$  after  $s$  steps, define a loop:

$$f_k^o \dots f_s^o S^{-r}(f_1) \dots S^{-r}(f_{s-1})$$

Observe that this is a loop based at  $0$  whose  $k-s+1$  edge is  $(r, r+1)$ .

These two operations are inverse to each other, so we have established a bijection between  $k$ -loops based at  $0$  whose  $s$ -th edge is  $(r, r+1)$  and  $k$ -paths from  $2r$  to  $0$  whose first edge is  $(2r, 2r+1)$  and which first reach  $r-1$  after  $k-s+1$  steps.

It follows from this that the left hand side is equal to the number of paths on the graph  $\mathbb{N}$  from  $2r$  to  $0$  whose first edge is  $(2r, 2r+1)$ . By the usual reflection trick, this is the difference of binomials defining  $f_{k/2,r}$ , and we are done.  $\square$

Regarding now the quantum group  $S_N^+$ , we have here the following formula, also established by Di Francesco in [58]:

**THEOREM 3.17.** *The determinant of the Gram matrix for  $S_N^+$  is given by*

$$\det(G_{kN}) = (\sqrt{N})^{a_k} \prod_{r=1}^k P_r(\sqrt{N})^{d_{kr}}$$

where  $P_r$  are the Chebycheff polynomials, given by

$$P_0 = 1 \quad , \quad P_1 = X \quad , \quad P_{r+1} = XP_r - P_{r-1}$$

and  $d_{kr} = f_{kr} - f_{k,r+1}$ , with  $f_{kr}$  being the following numbers, depending on  $k, r \in \mathbb{Z}$ ,

$$f_{kr} = \binom{2k}{k-r} - \binom{2k}{k-r-1}$$

with the convention  $f_{kr} = 0$  for  $k \notin \mathbb{Z}$ , and where  $a_k = \sum_{\pi \in \mathcal{P}(k)} (2|\pi| - k)$ .

PROOF. According to Proposition 3.4 above, if we denote by  $G'$  the Gram matrix for  $O_N^+$ , we have the following formula, with  $D_{kN} = \text{diag}(N^{|\tilde{\pi}|/2 - k/4})$ :

$$G_{kN} = D_{kN} G'_{2k, \sqrt{N}} D_{kN}$$

With this formula in hand, the result follows from Theorem 3.16.  $\square$

We refer to [27], [58], [64] for a further discussion on these topics.

### 3c. Clebsch-Gordan rules

Getting back now to our quantum group questions, we have the following consequence of Theorem 3.17, which generalizes the findings in Proposition 3.5 above:

PROPOSITION 3.18. *The following are linearly independent, for any  $N \geq 4$ :*

- (1) *The linear maps  $\{T_\pi | \pi \in NC(k, l)\}$ , with  $k, l \in 2\mathbb{N}$ .*
- (2) *The vectors  $\{\xi_\pi | \pi \in NC(k)\}$ , with  $k \in \mathbb{N}$ .*
- (3) *The linear maps  $\{T_\pi | \pi \in NC(k, k)\}$ , with  $k \in \mathbb{N}$ .*

PROOF. The statement is identical to Proposition 3.5, with the assumption  $N = n^2$  lifted. As for the proof, this comes from the formula in Theorem 3.17.  $\square$

With the above linear independence results in hand, we can now state and prove the following extension of Theorem 3.6 above:

THEOREM 3.19. *The quantum groups  $S_N^+$  with  $N \geq 4$  have the following properties:*

- (1) *The moments of the main character are the Catalan numbers:*

$$\int_{S_N^+} \chi^k = C_k$$

- (2) *The main character follows the Marchenko-Pastur law of parameter 1:*

$$\chi \sim \pi_1$$

- (3) *The fusion rules for irreducible representations are the same as for  $SO_3$ :*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+1} + \dots + r_{k+l}$$

- (4) *The dimensions of the representations are as follows, with  $q + q^{-1} = N - 2$ :*

$$\dim(r_k) = \frac{q^{k+1} - q^{-k}}{q - 1}$$

PROOF. The above statement is identical to Theorem 3.6, with the assumption  $N = n^2$  lifted. As for the proof, this is identical to the proof of Theorem 3.6, using this time the linear independence result from Proposition 3.18 as technical ingredient.  $\square$

Summarizing, we have now full representation theory results for both  $O_N^+, S_N^+$ . These results are quite surprising, and there are many things that can be said about  $O_N^+, S_N^+$ , in analogy with the known facts about  $SU_2, SO_3$ . However, all this is quite technical, and we defer the discussion to chapter 4 below. Let us record, however:

**THEOREM 3.20.** *The quantum groups  $O_N^+, S_N^+$  have the following properties:*

- (1)  $O_2^+, S_4^+$  are coamenable, and of polynomial growth.
- (2)  $O_N^+, S_N^+$  with  $N \geq 3, 5$  are not coamenable, and have exponential growth.

**PROOF.** The various coamenability assertions follow from the Kesten criterion from section 1 above, the support of the spectral measure of  $\chi$  being respectively:

$$\text{supp}(\gamma_1) = [-2, 2]$$

$$\text{supp}(\pi_1) = [0, 4]$$

As for the growth assertions, which can be of course improved with explicit exponents and so on, these follow from the fact that the corresponding Cayley graphs are  $\mathbb{N}$ .  $\square$

We will be back to amenability questions, later on.

### 3d. Planar algebras

In the remainder of this chapter we keep developing some useful general theory for  $O_N^+, S_N^+$  and their subgroups. We will present among others a general result from [7], refining the Tannakian duality for the quantum permutation groups  $G \subset S_N^+$ , stating that the following spaces form a planar algebra in the sense of Jones [69]:

$$P_k = \text{Fix}(u^{\otimes k})$$

To be more precise, we will show that these spaces form a planar subalgebra  $P = (P_k)$  of the Jones spin planar algebra  $\mathcal{S}_N$ , and that any planar subalgebra  $P \subset \mathcal{S}_N$  appears in this way, so that we have a refined Tannakian correspondence, as follows:

$$G \subset S_N^+ \longleftrightarrow P \subset \mathcal{S}_N$$

In order to get started, we need a lot of preliminaries, the lineup being von Neumann algebras,  $\text{II}_1$  factors, subfactors, and finally planar algebras. We already met von Neumann algebras, in section 1 above. The fundamental result regarding them is as follows:

**THEOREM 3.21.** *Any von Neumann algebra  $A \subset B(H)$  decomposes as*

$$A = \int_X A_x dx$$

with  $X$  being the measured space appearing as spectrum of the center,  $Z(A) = L^\infty(X)$ , and with the fibers  $A_x$  being “factors”, in the sense that  $Z(A_x) = \mathbb{C}$ .

PROOF. The decomposition result definitely holds in finite dimensions, where von Neumann algebra is the same as  $C^*$ -algebra, and where the algebras are as follows:

$$A = M_{N_1}(\mathbb{C}) \oplus \dots \oplus M_{N_k}(\mathbb{C})$$

Indeed, as explained in chapter 1 above, this decomposition is obtained by writing  $Z(A) = \mathbb{C}^k$ . In general, the decomposition result in the statement is von Neumann's "reduction theory" result, based on advanced functional analysis.  $\square$

At an even more advanced level now, we know from Theorem 3.21 that, at least in theory, things basically reduce to "factors". And, regarding these factors, we have:

THEOREM 3.22. *The von Neumann factors,  $Z(A) = \mathbb{C}$ , have the following properties:*

- (1) *They can be fully classified in terms of  $\text{II}_1$  factors, which are by definition those satisfying  $\dim A = \infty$ , and having a faithful trace  $\text{tr} : A \rightarrow \mathbb{C}$ .*
- (2) *The  $\text{II}_1$  factors enjoy the "continuous dimension geometry" property, in the sense that the traces of their projections can take any values in  $[0, 1]$ .*
- (3) *Among the  $\text{II}_1$  factors, the smallest one is the Murray-von Neumann hyperfinite factor  $R$ , obtained as an inductive limit of matrix algebras.*

PROOF. This is one again something heavy, the idea being as follows:

(1) This comes from results of Murray-von Neumann and Connes, the idea being that the other factors can be basically obtained via crossed product constructions.

(2) This is subtle functional analysis, with the rational traces being relatively easy to obtain, and with the irrational ones coming from limiting arguments.

(3) Once again, heavy results, by Murray-von Neumann and Connes, the idea being that any finite dimensional construction always leads to the same factor, called  $R$ .  $\square$

Full details on the above can be found in any operator algebra book.

Let us discuss now subfactor theory, following Jones' fundamental paper [66]. Jones looked at the inclusions of  $\text{II}_1$  factors  $A \subset B$ , called subfactors, which are quite natural objects physics. Given such an inclusion, we can talk about its index:

DEFINITION 3.23. *The index of an inclusion of  $\text{II}_1$  factors  $A \subset B$  is the quantity*

$$[B : A] = \dim_A B \in [1, \infty]$$

*constructed by using the Murray-von Neumann continuous dimension theory.*

In order to explain Jones' result [66], it is better to relabel our subfactor as:

$$A_0 \subset A_1$$

We can construct the orthogonal projection  $e_1 : A_1 \rightarrow A_0$ , and set:

$$A_2 = \langle A_1, e_1 \rangle$$



This remarkable procedure, called “basic construction”, can be iterated, and we obtain in this way a whole tower of  $\text{II}_1$  factors, as follows:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \dots$$

Quite surprisingly, this construction leads to a link with the Temperley-Lieb algebra  $TL_N = \text{span}(NC_2)$ , and with many other things.

These results can be summarized as follows:

**THEOREM 3.24.** *Let  $A_0 \subset A_1$  be an inclusion of  $\text{II}_1$  factors.*

- (1) *The sequence of projections  $e_1, e_2, e_3, \dots \in B(H)$  produces a representation of the Temperley-Lieb algebra of index  $N = [A_1, A_0]$ , as follows:*

$$TL_N \subset B(H)$$

- (2) *The index  $N = [A_1, A_0]$ , which is a Murray-von Neumann continuous quantity  $N \in [1, \infty]$ , must satisfy the following condition:*

$$N \in \left\{ 4 \cos^2 \left( \frac{\pi}{n} \right) \mid n \in \mathbb{N} \right\} \cup [4, \infty]$$

**PROOF.** This result, from [66], is something quite tricky, the idea being as follows:

(1) The idea here is that the functional analytic study of the basic construction leads to the conclusion that the sequence of projections  $e_1, e_2, e_3, \dots \in B(H)$  behaves algebraically, when rescaled, exactly as the sequence of diagrams  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots \in TL_N$  given by:

$$\begin{aligned} \varepsilon_1 &= \cup \\ \varepsilon_2 &= \left| \cup \right. \\ \varepsilon_3 &= \left| \left| \cup \right. \right. \\ &\vdots \end{aligned}$$

But these diagrams generate  $TL_N$ , and so we have an embedding  $TL_N \subset B(H)$ , where  $H$  is the Hilbert space where our subfactor  $A_0 \subset A_1$  lives, as claimed.

(2) This is something quite surprising, which follows from (1), via some clever positivity considerations, involving the Perron-Frobenius theorem. In fact, the subfactors having index  $N \in [1, 4]$  can be classified by ADE diagrams, and the obstruction  $N = 4 \cos^2(\frac{\pi}{n})$  itself comes from the fact that  $N$  must be the squared norm of such a graph.  $\square$

Quite remarkably, Theorem 3.24 is just the “tip of the iceberg”. One can prove indeed that the planar algebra structure of  $TL_N$ , taken in an intuitive sense, extends to a planar algebra structure on the following sequence of commutants:

$$P_k = A'_0 \cap A_k$$

In order to discuss this key result, from [69], that we will need as well, in connection with our quantum permutation group problems, let us start with:

DEFINITION 3.25. *The planar algebras are defined as follows:*

- (1) *A  $k$ -tangle, or  $k$ -box, is a rectangle in the plane, with  $2k$  marked points on its boundary, containing  $r$  small boxes, each having  $2k_i$  marked points, and with the  $2k + \sum 2k_i$  marked points being connected by noncrossing strings.*
- (2) *A planar algebra is a sequence of finite dimensional vector spaces  $P = (P_k)$ , together with linear maps  $P_{k_1} \otimes \dots \otimes P_{k_r} \rightarrow P_k$ , one for each  $k$ -box, such that the gluing of boxes corresponds to the composition of linear maps.*

As basic example of a planar algebra, we have the Temperley-Lieb algebra  $TL_N$ . Indeed, putting  $TL_N(k_i)$  diagrams into the small  $r$  boxes of a  $k$ -box clearly produces a  $TL_N(k)$  diagram, and so we have indeed a planar algebra, of somewhat “trivial” type.

Further examples of this type, with the elements of the planar algebra being diagrams, include the Fuss-Catalan algebra of Bisch and Jones [47], obtained by coloring the strings of the Temperley-Lieb diagrams. We will be back to this, later on.

In general, the planar algebras are more complicated than this, and we will be back later with some explicit examples. However, the idea is very simple, namely “the elements of a planar algebra are not necessarily diagrams, but they behave like diagrams”.

In relation now with subfactors, the result, which extends Theorem 3.24 (1) above, and which was found by Jones in [69], almost 20 years after [66], is as follows:

THEOREM 3.26. *Given a subfactor  $A_0 \subset A_1$ , the collection  $P = (P_k)$  of linear spaces*

$$P_k = A'_0 \cap A_k$$

*has a planar algebra structure, extending the planar algebra structure of  $TL_N$ .*

PROOF. As a first observation, since  $e_1 : A_1 \rightarrow A_0$  commutes with  $A_0$  we have  $e_1 \in P'_2$ . By translation we obtain  $e_1, \dots, e_{k-1} \in P_k$  for any  $k$ , and so:

$$TL_N \subset P$$

The point now is that the planar algebra structure of  $TL_N$ , obtained by composing diagrams, can be shown to extend into an abstract planar algebra structure of  $P$ . This is something quite heavy, and we will not get into details here. See [69].  $\square$

Getting back now to quantum groups, all this machinery is very interesting for us. We will need the construction of the spin planar algebra  $\mathcal{S}_N$ . Let us start with:

DEFINITION 3.27. *We write the standard basis of  $(\mathbb{C}^N)^{\otimes k}$  in  $2 \times k$  matrix form,*

$$e_{i_1 \dots i_k} = \begin{pmatrix} i_1 & i_1 & i_2 & i_2 & i_3 & \dots & \dots \\ i_k & i_k & i_{k-1} & \dots & \dots & \dots & \dots \end{pmatrix}$$

*by duplicating the indices, and then writing them clockwise, starting from top left.*

Now with this convention in hand for the tensors, we can formulate the construction of the spin planar algebra  $\mathcal{S}_N$ , also from [69], as follows:

DEFINITION 3.28. *The spin planar algebra  $\mathcal{S}_N$  is the sequence of vector spaces*

$$P_k = (\mathbb{C}^N)^{\otimes k}$$

*written as above, with the multilinear maps associated to the various  $k$ -tangles*

$$T_\pi : P_{k_1} \otimes \dots \otimes P_{k_r} \rightarrow P_k$$

*being given by the following formula, in multi-index notation,*

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_r}) = \sum_j \delta_\pi(i_1, \dots, i_r : j) e_j$$

*with the Kronecker symbols  $\delta_\pi$  being 1 if the indices fit, and being 0 otherwise.*

Here are some illustrating examples for the spin planar algebra calculus:

(1) The identity  $1_k$  is the  $(k, k)$ -tangle having vertical strings only. The solutions of  $\delta_{1_k}(x, y) = 1$  being the pairs of the form  $(x, x)$ , this tangle  $1_k$  acts by the identity:

$$1_k \begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix} = \begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix}$$

(2) The multiplication  $M_k$  is the  $(k, k, k)$ -tangle having 2 input boxes, one on top of the other, and vertical strings only. It acts in the following way:

$$M_k \left( \begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix} \otimes \begin{pmatrix} l_1 & \dots & l_k \\ m_1 & \dots & m_k \end{pmatrix} \right) = \delta_{j_1 m_1} \dots \delta_{j_k m_k} \begin{pmatrix} l_1 & \dots & l_k \\ i_1 & \dots & i_k \end{pmatrix}$$

(3) The inclusion  $I_k$  is the  $(k, k+1)$ -tangle which looks like  $1_k$ , but has one more vertical string, at right of the input box. Given  $x$ , the solutions of  $\delta_{I_k}(x, y) = 1$  are the elements  $y$  obtained from  $x$  by adding to the right a vector of the form  $\binom{l}{\cdot}$ , and so:

$$I_k \begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix} = \sum_l \begin{pmatrix} j_1 & \dots & j_k & l \\ i_1 & \dots & i_k & l \end{pmatrix}$$

Observe that  $I_k$  is an inclusion of algebras, and that the various  $I_k$  are compatible with each other. The inductive limit of the algebras  $\mathcal{S}_N(k)$  is a graded algebra, denoted  $\mathcal{S}_N$ .

(4) The expectation  $U_k$  is the  $(k+1, k)$ -tangle which looks like  $1_k$ , but has one more string, connecting the extra 2 input points, both at right of the input box:

$$U_k \begin{pmatrix} j_1 & \dots & j_k & j_{k+1} \\ i_1 & \dots & i_k & i_{k+1} \end{pmatrix} = \delta_{i_{k+1} j_{k+1}} \begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix}$$

Observe that  $U_k$  is a bimodule morphism with respect to  $I_k$ .

(5) The Jones projection  $E_k$  is a  $(0, k + 2)$ -tangle, having no input box. There are  $k$  vertical strings joining the first  $k$  upper points to the first  $k$  lower points, counting from left to right. The remaining upper 2 points are connected by a semicircle, and the remaining lower 2 points are also connected by a semicircle. We have the following formula:

$$E_k(1) = \sum_{ijl} \begin{pmatrix} i_1 & \cdots & i_k & j & j \\ i_1 & \cdots & i_k & l & l \end{pmatrix}$$

The elements  $e_k = N^{-1}E_k(1)$  are projections, and define a representation of the infinite Temperley-Lieb algebra of index  $N$  inside the inductive limit algebra  $\mathcal{S}_N$ .

(6) The rotation  $R_k$  is the  $(k, k)$ -tangle which looks like  $1_k$ , but the first 2 input points are connected to the last 2 output points, and the same happens at right:

$$R_k = \begin{array}{c} \cap \quad | \quad | \quad | \\ \parallel \quad \quad \parallel \\ \parallel \quad | \quad | \quad | \quad \cup \end{array}$$

The action of  $R_k$  on the standard basis is by rotation of the indices, as follows:

$$R_k(e_{i_1 \dots i_k}) = e_{i_2 i_3 \dots i_k i_1}$$

Thus  $R_k$  acts by an order  $k$  linear automorphism of  $\mathcal{S}_N(k)$ , also called rotation.

There are many other interesting examples of  $k$ -tangles, but in view of our present purposes, we can actually stop here, due to the following useful fact:

**THEOREM 3.29.** *The multiplications, inclusions, expectations, Jones projections, and rotations generate the set of all tangles, via the gluing operation.*

**PROOF.** This is something well-known and elementary, obtained by “chopping” the various planar tangles into small pieces, as in the above list. See [69].  $\square$

Finally, in order for our discussion to be complete, we must talk as well about the  $*$ -structure of the spin planar algebra. Once again this is constructed as in the easy quantum group calculus, by turning upside-down the diagrams, as follows:

$$\begin{pmatrix} j_1 & \cdots & j_k \\ i_1 & \cdots & i_k \end{pmatrix}^* = \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$$

Summarizing, the sequence of vector spaces  $\mathcal{S}_N(k) = C(X^k)$  has a planar  $*$ -algebra structure, called spin planar algebra of index  $N = |X|$ . See [69].

Getting back now to quantum groups, following [7], we have the following result:

THEOREM 3.30. *Given  $G \subset S_N^+$ , consider the tensor powers of the associated coaction map on  $C(X)$ , where  $X = \{1, \dots, N\}$ , which are the following linear maps:*

$$\begin{aligned} \Phi^k : C(X^k) &\rightarrow C(X^k) \otimes C(G) \\ e_{i_1 \dots i_k} &\rightarrow \sum_{j_1 \dots j_k} e_{j_1 \dots j_k} \otimes u_{j_1 i_1} \cdots u_{j_k i_k} \end{aligned}$$

The fixed point spaces of these coactions, which are by definition the spaces

$$P_k = \left\{ x \in C(X^k) \mid \Phi^k(x) = 1 \otimes x \right\}$$

are given by  $P_k = \text{Fix}(u^{\otimes k})$ , and form a subalgebra of the spin planar algebra  $\mathcal{S}_N$ .

PROOF. Since the map  $\Phi$  is a coaction, coming from the corepresentation  $u$ , its tensor powers  $\Phi^k$  are coactions too, coming from the corepresentations  $u^{\otimes k}$ , and at the level of the fixed point algebras we have the following formula, which is standard:

$$P_k = \text{Fix}(u^{\otimes k})$$

In order to prove now the planar algebra assertion, we will use Theorem 3.29.

Consider the rotation  $R_k$ . Rotating, then applying  $\Phi^k$ , and rotating backwards by  $R_k^{-1}$  is the same as applying  $\Phi^k$ , then rotating each  $k$ -fold product of coefficients of  $\Phi$ .

Thus the elements obtained by rotating, then applying  $\Phi^k$ , or by applying  $\Phi^k$ , then rotating, differ by a sum of Dirac masses tensored with commutators in  $A = C(G)$ :

$$\Phi^k R_k(x) - (R_k \otimes id)\Phi^k(x) \in C(X^k) \otimes [A, A]$$

Now let  $\int_A$  be the Haar functional of  $A$ , and consider the conditional expectation onto the fixed point algebra  $P_k$ , which is given by the following formula:

$$\phi_k = \left( id \otimes \int_A \right) \Phi^k$$

The square of the antipode being the identity, the Haar integration  $\int_A$  is a trace, so it vanishes on commutators. Thus  $R_k$  commutes with  $\phi_k$ :

$$\phi_k R_k = R_k \phi_k$$

The commutation relation  $\phi_k T = T \phi_l$  holds in fact for any  $(l, k)$ -tangle  $T$ . These tangles are called annular, and the proof is by verification on generators of the annular category. In particular we obtain, for any annular tangle  $T$ :

$$\phi_k T \phi_l = T \phi_l$$

We conclude from this that the annular category is contained in the suboperad  $\mathcal{P}' \subset \mathcal{P}$  of the planar operad consisting of tangles  $T$  satisfying the following condition, where  $\phi = (\phi_k)$ , and where  $i(\cdot)$  is the number of input boxes:

$$\phi T \phi^{\otimes i(T)} = T \phi^{\otimes i(T)}$$

On the other hand the multiplicativity of  $\Phi^k$  gives  $M_k \in \mathcal{P}'$ . Since  $\mathcal{P}$  is generated by multiplications and annular tangles, it follows that we have  $\mathcal{P}' = \mathcal{P}$ .

Thus for any tangle  $T$  the corresponding multilinear map between spaces  $P_k(X)$  restricts to a multilinear map between spaces  $P_k$ . In other words, the action of the planar operad  $\mathcal{P}$  restricts to  $P$ , and makes it a subalgebra of  $\mathcal{S}_N$ , as claimed.  $\square$

As a second result now, also from [7], completing our study, we have:

**THEOREM 3.31.** *Any planar subalgebra of the spin planar algebra*

$$Q \subset \mathcal{S}_N$$

*comes via the construction in Theorem 3.30 from a quantum permutation group*

$$G \subset S_N^+$$

*and the quantum groups  $\{1\}, S_N^+$  correspond in this way to the algebras  $\mathcal{S}_N, TL_N$ .*

**PROOF.** As a first observation, this statement generalizes what we know so far about  $S_N^+$ , because knowing that this quantum group is in correspondence with  $TL_N$  gives us right away its representation theory, and other invariants. Thus, what we have here is an advanced form of Tannakian duality, in the quantum permutation group case.

In practice now, the idea is that this will follow by applying Tannakian duality to the annular category over  $Q$ . Let  $n, m$  be positive integers. To any element  $T_{n+m} \in Q_{n+m}$  we can associate a linear map  $L_{nm}(T_{n+m}) : P_n(X) \rightarrow P_m(X)$  in the following way:

$$L_{nm} \left( \begin{array}{c} | | | \\ T_{n+m} \\ | | | \end{array} \right) : \left( \begin{array}{c} | \\ a_n \\ | \end{array} \right) \rightarrow \left( \begin{array}{c} | | \cap \\ T_{n+m} | \\ | | | | \\ a_n | | | | \\ \cup | | \end{array} \right)$$

That is, we consider the planar  $(n, n + m, m)$ -tangle having an small input  $n$ -box, a big input  $n + m$ -box and an output  $m$ -box, with strings as on the picture of the right. This defines a certain multilinear map, as follows:

$$P_n(X) \otimes P_{n+m}(X) \rightarrow P_m(X)$$

Now let us put the element  $T_{n+m}$  in the big input box. We obtain in this way a certain linear map  $P_n(X) \rightarrow P_m(X)$ , that we call  $L_{nm}$ . To be more precise:

- (1) The above picture corresponds to  $n = 1$  and  $m = 2$ . This is illustrating whenever  $n \leq m$ , it suffices to imagine that in the general case all strings are multiple.
- (2) If  $n > m$  there are  $n + m$  strings of  $a_n$  which connect to the  $n + m$  lower strings of  $T_{n+m}$ , and the remaining  $n - m$  ones go to the upper right side and connect to the  $n - m$

strings on top right of  $T_{n+m}$ . Here is the picture for  $n = 2$  and  $m = 1$ :

$$L_{nm} \left( \begin{array}{c} | \\ | \\ | \\ | \\ T_{n+m} \\ | \\ | \\ | \\ | \end{array} \right) : \left( \begin{array}{c} | \\ | \\ a_n \\ | \\ | \end{array} \right) \rightarrow \left( \begin{array}{c} | \\ | \\ T_{n+m} \\ | \\ | \\ | \\ | \\ a_n \\ | \\ | \\ \cup \\ | \end{array} \right)$$

Consider now the linear spaces formed by the maps constructed above:

$$Q_{nm} = \left\{ L_{nm}(T_{n+m}) : P_n(X) \rightarrow P_m(X) \mid T_{n+m} \in Q_{n+m} \right\}$$

These spaces form a Tannakian category, and so by [99] we obtain a Woronowicz algebra  $(A, u)$ , such that the following equalities hold, for any  $m, n$ :

$$\text{Hom}(u^{\otimes m}, u^{\otimes n}) = Q_{mn}$$

We prove that  $u$  is a magic unitary. We have  $\text{Hom}(1, u^{\otimes 2}) = Q_{02} = Q_2$ , so the unit of  $Q_2$  must be a fixed vector of  $u^{\otimes 2}$ . But  $u^{\otimes 2}$  acts on the unit of  $Q_2$  as follows:

$$\begin{aligned} u^{\otimes 2}(1) &= u^{\otimes 2} \left( \sum_i \begin{pmatrix} i & i \\ i & i \end{pmatrix} \right) \\ &= \sum_{ikl} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes u_{ki} u_{li} \\ &= \sum_{kl} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes (uu^t)_{kl} \end{aligned}$$

From  $u^{\otimes 2}(1) = 1 \otimes 1$  we get that  $uu^t$  is the identity matrix. Together with the unitarity of  $u$ , this gives the following formulae:

$$u^t = u^* = u^{-1}$$

Consider the Jones projection  $E_1 \in Q_3$ . After isotoping,  $L_{21}(E_1)$  looks as follows:

$$L_{21} \left( \begin{array}{c} | \\ | \\ \cup \\ | \\ \cap \\ | \end{array} \right) : \left( \begin{array}{c} | \\ | \\ i \\ | \\ j \\ | \\ i \\ | \\ j \\ | \end{array} \right) \rightarrow \left( \begin{array}{c} | \\ | \\ i \\ | \\ j \\ | \\ i \\ | \\ j \\ | \end{array} \right) \supset = \delta_{ij} \left( \begin{array}{c} | \\ i \\ | \\ i \end{array} \right)$$

In other words, the linear map  $M = L_{21}(E_1)$  is the multiplication  $\delta_i \otimes \delta_j \rightarrow \delta_{ij} \delta_i$ :

$$M \begin{pmatrix} i & i \\ j & j \end{pmatrix} = \delta_{ij} \begin{pmatrix} i \\ i \end{pmatrix}$$

Consider now the following element of  $C(X) \otimes A$ :

$$\begin{aligned} (M \otimes id)u^{\otimes 2} \left( \begin{pmatrix} i & i \\ j & j \end{pmatrix} \otimes 1 \right) &= (M \otimes id) \left( \sum_{kl} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes u_{ki}u_{lj} \right) \\ &= \sum_k \begin{pmatrix} k \\ k \end{pmatrix} \delta_k \otimes u_{ki}u_{kj} \end{aligned}$$

Since  $M \in Q_{21} = Hom(u^{\otimes 2}, u)$ , this equals the following element of  $C(X) \otimes A$ :

$$\begin{aligned} u(M \otimes id) \left( \begin{pmatrix} i & i \\ j & j \end{pmatrix} \otimes 1 \right) &= u \left( \delta_{ij} \begin{pmatrix} i \\ i \end{pmatrix} \delta_i \otimes 1 \right) \\ &= \sum_k \begin{pmatrix} k \\ k \end{pmatrix} \delta_k \otimes \delta_{ij}u_{ki} \end{aligned}$$

Thus  $u_{ki}u_{kj} = \delta_{ij}u_{ki}$  for any  $i, j, k$ . With  $i = j$  we get  $u_{ki}^2 = u_{ki}$ , and together with the formula  $u^t = u^*$  this shows that all entries of  $u$  are self-adjoint projections. With  $i \neq j$  we get  $u_{ki}u_{kj} = 0$ , so the projections on each row of  $u$  are orthogonal to each other. Together with  $u^t = u^{-1}$ , this shows that each row of  $u$  is a partition of unity with self-adjoint projections. The antipode is given by the formula  $(id \otimes S)u = u^*$ . But  $u^*$  is the transpose of  $u$ , so we can apply  $S$  to the formulae saying that rows of  $u$  are partitions of unity, and we get that columns of  $u$  are also partitions of unity. Thus  $u$  is a magic unitary.

Now if  $P$  is the planar algebra associated to  $u$ , we have  $Hom(1, v^{\otimes n}) = P_n = Q_n$ , as desired. As for the uniqueness, this is clear from the Peter-Weyl theory.  $\square$

The results established above, regarding the subgroups  $G \subset S_N^+$ , have several generalizations, to the subgroups  $G \subset O_N^+$  and  $G \subset U_N^+$ , as well as subfactor versions, going beyond the purely combinatorial level. We refer here to [5] and related papers, and we will be back to some of these questions in chapter 12 below.

### 3e. Exercises

Things have been quite technical in this chapter, and our exercises will be quite technical as well. In relation with the representation theory of  $O_N^+$ , we have:

EXERCISE 3.32. *Prove that the quantum group  $O_2^+$  appears as a twist of  $SU_2$ ,*

$$O_2^+ \simeq SU_2^{-1}$$

*and deduce the Clebsch-Gordan fusion rules for  $O_2^+$  by using this.*

All this is not exactly obvious, requiring a good knowledge of the cocycle twisting procedure. As a bonus exercise, you can try to understand as well the relation between  $O_N^+$  and  $SU_2$  at  $N \geq 3$ , with the indication here that all this is related to two technical topics, namely the ‘‘FRT deformation’’ procedure, and the ‘‘free symplectic groups’’.



Along the same lines, dealing this time with  $S_N^+$  at  $N = n^2$  with  $n \geq 2$ , we have:

EXERCISE 3.33. *Prove that the quantum group  $S_4^+$  appears as a twist of  $SO_3$ ,*

$$S_4^+ \simeq SO_3^{-1}$$

*and deduce the Clebsch-Gordan fusion rules for  $S_4^+$  by using this.*

As before with the previous exercise, all this requires a good knowledge of the cocycle twisting procedure, and basically the same comments as there apply. We will be actually back to all this in what follows, on several occasions.

In relation now with  $S_N^+$  at arbitrary values  $N \geq 4$ , we first have:

EXERCISE 3.34. *Prove the main technical result in this chapter, namely that the following linear maps are linearly independent at  $N \geq 4$ ,*

$$\left\{ T_\pi \mid \pi \in NC(k, l) \right\}$$

*by computing directly the Gram determinant for  $S_N^+$ , without using the passage through the Gram determinant of  $O_N^+$ , via fattening and shrinking.*

This is of course something quite long and technical, amounting in rewriting all that has been said in the above, directly in terms of  $S_N^+$ .

As a second exercise now, on the same topic, we have:

EXERCISE 3.35. *Prove the main technical result in this chapter, namely that the following linear maps are linearly independent at  $N \geq 4$ ,*

$$\left\{ T_\pi \mid \pi \in NC(k, l) \right\}$$

*by using a well-chosen Temperley-Lieb algebra and subfactor argument.*

To be more precise, the point is that the above linear independence result is known as well in subfactors, via a different proof, which is non-trivial either, and the problem is that of finding that proof in the literature, and writing a short account of it.

As a third and last exercise now, also on the same topic, we have:

EXERCISE 3.36. *Prove the main technical result in this chapter, namely that the following linear maps are linearly independent at  $N \geq 4$ ,*

$$\left\{ T_\pi \mid \pi \in NC(k, l) \right\}$$

*by developing a theory of generalized quantum groups, of type  $O_N^+$  with  $N \in [2, \infty)$ , then working out their representation theory, and then doing fattening and shrinking.*

As before with the previous exercises, this remains something quite technical. However, all these exercises are worth to be worked out, hiding many interesting things.

## CHAPTER 4

### Twisted permutations

#### 4a. Quantum spaces

In this chapter we investigate the quantum permutation groups  $S_F^+$  of the finite quantum spaces  $F$ . Besides providing a useful generalization of our results regarding  $S_N^+$ , this will eventually explain the connection with  $SO_3$ , in an elegant way. As a bonus, we will obtain as well a conceptual result on the connection between  $S_N^+$  and  $O_N^+$ .

In order to get started, let us first talk about finite quantum spaces. We have:

**DEFINITION 4.1.** *A finite quantum space  $F$  is the abstract dual of a finite dimensional  $C^*$ -algebra  $B$ , according to the following formula:*

$$C(F) = B$$

*The number of elements of such a space is by definition the number  $|F| = \dim B$ . By decomposing the algebra  $B$ , we have a formula of the following type:*

$$C(F) = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C})$$

*With  $n_1 = \dots = n_k = 1$  we obtain in this way the space  $F = \{1, \dots, k\}$ . Also, when  $k = 1$  the equation is  $C(F) = M_n(\mathbb{C})$ , and the solution will be denoted  $F = M_n$ .*

In order to talk now about the quantum symmetry group  $S_F^+$ , we must use universal coactions. As in chapter 2 above, when dealing with universal coactions on the space  $F = \{1, \dots, k\}$ , we must endow our space  $F$  with its counting measure:

**DEFINITION 4.2.** *We endow each finite quantum space  $F$  with its counting measure, corresponding as the algebraic level to the integration functional*

$$tr : C(F) \rightarrow B(l^2(F)) \rightarrow \mathbb{C}$$

*obtained by applying the regular representation, and then the normalized matrix trace.*

To be more precise, consider the algebra  $B = C(F)$ , which is by definition finite dimensional. We can make act  $B$  on itself, by left multiplication:

$$\pi : B \rightarrow \mathcal{L}(B) \quad , \quad a \rightarrow (b \rightarrow ab)$$

The target of  $\pi$  being a matrix algebra,  $\mathcal{L}(B) \simeq M_N(\mathbb{C})$  with  $N = \dim B$ , we can further compose with the normalized matrix trace, and we obtain  $tr$ :

$$tr = \frac{1}{N} Tr \circ \pi$$

As basic examples, for both  $F = \{1, \dots, N\}$  and  $F = M_N$  we obtain the usual trace. In general, with  $C(F) = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C})$ , the weights of  $tr$  are:

$$c_i = \frac{n_i^2}{\sum_i n_i^2}$$

Let us also mention that the canonical trace is precisely the one making  $\mathbb{C} \subset B$  a Markov inclusion. Equivalently, the counting measure is the one making  $F \rightarrow \{\cdot\}$  a Markov fibration. For a discussion of these facts, see [2], and also [5], [19].

Let us study now the quantum group actions  $G \curvearrowright F$ . It is convenient here to use, in order to get started, the no basis approach from [2]. If we denote by  $\mu, \eta$  the multiplication and unit map of the algebra  $C(F)$ , we have the following result, from [2]:

PROPOSITION 4.3. *Consider a linear map  $\Phi : C(F) \rightarrow C(F) \otimes C(G)$ , written as*

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

with  $\{e_i\}$  being a linear space basis of  $C(F)$ , chosen orthonormal with respect to  $tr$ .

- (1)  $\Phi$  is a linear space coaction  $\iff u$  is a corepresentation.
- (2)  $\Phi$  is multiplicative  $\iff \mu \in \text{Hom}(u^{\otimes 2}, u)$ .
- (3)  $\Phi$  is unital  $\iff \eta \in \text{Hom}(1, u)$ .
- (4)  $\Phi$  leaves invariant  $tr$   $\iff \eta \in \text{Hom}(1, u^*)$ .
- (5) If these conditions hold,  $\Phi$  is involutive  $\iff u$  is unitary.

PROOF. This is a bit similar to the proof for  $S_N^+$  from chapter 2, as follows:

(1) There are two axioms to be processed here. First, we have:

$$\begin{aligned} (id \otimes \Delta)\Phi &= (\Phi \otimes id)\Phi \iff \sum_j e_j \otimes \Delta(u_{ji}) = \sum_k \Phi(e_k) \otimes u_{ki} \\ &\iff \sum_j e_j \otimes \Delta(u_{ji}) = \sum_{jk} e_j \otimes u_{jk} \otimes u_{ki} \\ &\iff \Delta(u_{ji}) = \sum_k u_{jk} \otimes u_{ki} \end{aligned}$$

As for the axiom involving the counit, here we have as well, as desired:

$$\begin{aligned} (id \otimes \varepsilon)\Phi &= id \iff \sum_j \varepsilon(u_{ji})e_j = e_i \\ &\iff \varepsilon(u_{ji}) = \delta_{ji} \end{aligned}$$

(2) We have the following formula:

$$\begin{aligned}\Phi(e_i) &= \left( \sum_{ij} e_{ji} \otimes u_{ji} \right) (e_i \otimes 1) \\ &= u(e_i \otimes 1)\end{aligned}$$

By using this formula, we obtain the following identity:

$$\begin{aligned}\Phi(e_i e_k) &= u(e_i e_k \otimes 1) \\ &= u(\mu \otimes id)(e_i \otimes e_k \otimes 1)\end{aligned}$$

On the other hand, we have as well the following identity, as desired:

$$\begin{aligned}\Phi(e_i)\Phi(e_k) &= \sum_{jl} e_j e_l \otimes u_{ji} u_{lk} \\ &= (\mu \otimes id) \sum_{jl} e_j \otimes e_l \otimes u_{ji} u_{lk} \\ &= (\mu \otimes id) \left( \sum_{ijkl} e_{ji} \otimes e_{lk} \otimes u_{ji} u_{lk} \right) (e_i \otimes e_k \otimes 1) \\ &= (\mu \otimes id) u^{\otimes 2} (e_i \otimes e_k \otimes 1)\end{aligned}$$

(3) The formula  $\Phi(e_i) = u(e_i \otimes 1)$  found above gives by linearity:

$$\Phi(1) = u(1 \otimes 1)$$

But this shows that  $\Phi$  is unital precisely when  $u(1 \otimes 1) = 1 \otimes 1$ , as desired.

(4) This follows from the following computation, by applying the involution:

$$\begin{aligned}(tr \otimes id)\Phi(e_i) = tr(e_i)1 &\iff \sum_j tr(e_j)u_{ji} = tr(e_i)1 \\ &\iff \sum_j u_{ji}^* 1_j = 1_i \\ &\iff (u^*1)_i = 1_i \\ &\iff u^*1 = 1\end{aligned}$$

(5) Assuming that (1-4) are satisfied, and that  $\Phi$  is involutive, we have:

$$\begin{aligned}
(u^*u)_{ik} &= \sum_l u_{li}^* u_{lk} \\
&= \sum_{jl} \text{tr}(e_j^* e_l) u_{ji}^* u_{lk} \\
&= (\text{tr} \otimes \text{id}) \sum_{jl} e_j^* e_l \otimes u_{ji}^* u_{lk} \\
&= (\text{tr} \otimes \text{id})(\Phi(e_i)^* \Phi(e_k)) \\
&= (\text{tr} \otimes \text{id}) \Phi(e_i^* e_k) \\
&= \text{tr}(e_i^* e_k) 1 \\
&= \delta_{ik}
\end{aligned}$$

Thus  $u^*u = 1$ , and since we know from (1) that  $u$  is a corepresentation, it follows that  $u$  is unitary. The proof of the converse is standard too, by using similar tricks.  $\square$

Following now [2], we have the following result, extending the basic theory of  $S_N^+$  from the previous section to the present finite quantum space setting:

**THEOREM 4.4.** *Given a finite quantum space  $F$ , there is a universal compact quantum group  $S_F^+$  acting on  $F$ , leaving the counting measure invariant. We have*

$$C(S_F^+) = C(U_N^+) / \left\langle \mu \in \text{Hom}(u^{\otimes 2}, u), \eta \in \text{Fix}(u) \right\rangle$$

where  $N = |F|$  and where  $\mu, \eta$  are the multiplication and unit maps of  $C(F)$ . For  $F = \{1, \dots, N\}$  we have  $S_F^+ = S_N^+$ . Also, for the space  $F = M_2$  we have  $S_F^+ = SO_3$ .

**PROOF.** This result is from [2], the idea being as follows:

(1) This follows from Proposition 4.3 above, by using the standard fact that the complex conjugate of a corepresentation is a corepresentation too.

(2) Regarding now the main example, for  $F = \{1, \dots, N\}$  we obtain indeed the quantum permutation group  $S_N^+$ , due to the results in chapter 2 above.

(3) In order to do now the computation for  $F = M_2$ , we use some standard facts about  $SU_2, SO_3$ . We have an action by conjugation  $SU_2 \curvearrowright M_2(\mathbb{C})$ , and this action produces, via the canonical quotient map  $SU_2 \rightarrow SO_3$ , an action  $SO_3 \curvearrowright M_2(\mathbb{C})$ . On the other hand, it is routine to check, by using arguments like those from the proof of  $S_N^+ = S_N$  at  $N = 2, 3$ , from chapter 2 above, that any action  $G \curvearrowright M_2(\mathbb{C})$  must come from a classical group. We conclude that the action  $SO_3 \curvearrowright M_2(\mathbb{C})$  is universal, as claimed.  $\square$

We will see in a moment a more conceptual proof for (3) above, based on the fact that the inclusion  $SO_3 \subset S_{M_2}^+$  can be shown to have the property that it preserves the law of the main character, and so must be an isomorphism.

In practice, for our purposes here it will be very useful to have bases and indices. We will use a single index approach, based on the following formalism:

DEFINITION 4.5. *Given a finite quantum space  $F$ , we let  $\{e_i\}$  be the standard multimatrix basis of the algebra  $B = C(F)$ , so that the multiplication, involution and unit of  $B$  are given by the formulae*

$$e_i e_j = e_{ij} \quad , \quad e_i^* = e_{\bar{i}} \quad , \quad 1 = \sum_{i=\bar{i}} e_i$$

where  $(i, j) \rightarrow ij$  is the standard partially defined multiplication on the indices, with the convention  $e_\emptyset = 0$ , and where  $i \rightarrow \bar{i}$  is the standard involution on the indices.

To be more precise, let  $\{e_{ab}^r\} \subset B$  be the multimatrix basis. We set then  $i = (abr)$ , and with this convention, the multiplication, coming from  $e_{ab}^r e_{cd}^p = \delta_{rp} \delta_{bc} e_{ad}^r$ , is given by:

$$(abr)(cdp) = \begin{cases} (adr) & \text{if } r = p, b = c \\ \emptyset & \text{otherwise} \end{cases}$$

As for the involution, coming from  $(e_{ab}^r)^* = e_{ba}^r$ , this is given by:

$$\overline{(a, b, r)} = (b, a, r)$$

Finally, the unit formula comes from the following formula for the unit  $1 \in B$ :

$$1 = \sum_{ar} e_{aa}^r$$

We can now convert the main assertion in Theorem 4.4, namely the no indices formula of  $S_F^+$  from there into something more concrete, as follows:

THEOREM 4.6. *Given a finite quantum space  $F$ , with basis  $\{e_i\} \subset C(F)$  as above, the algebra  $C(S_F^+)$  is generated by variables  $u_{ij}$  with the following relations,*

$$\begin{aligned} \sum_{ij=p} u_{ik} u_{jl} &= u_{p,kl} \quad , \quad \sum_{kl=p} u_{ik} u_{jl} = u_{ij,p} \\ \sum_{i=\bar{i}} u_{ij} &= \delta_{j\bar{j}} \quad , \quad \sum_{j=\bar{j}} u_{ij} = \delta_{i\bar{i}} \\ u_{ij}^* &= u_{\bar{i}\bar{j}} \end{aligned}$$

with the fundamental corepresentation being the matrix  $u = (u_{ij})$ . We call a matrix  $u = (u_{ij})$  satisfying the above relations “generalized magic”.

PROOF. This can be deduced from any of the known presentations of  $C(S_F^+)$ :

(1) If we take the triple index presentation of  $C(S_F^+)$  from [95], and replace there the triple indices by single indices, we obtain the relations in the statement.

(2) Alternatively, if we take the double index presentation of  $C(S_F^+)$  from [5], and replace there the double indices by single indices, we obtain the relations in the statement.

(3) Also, when using Theorem 4.4,  $\mu \in \text{Hom}(u^{\otimes 2}, u)$  and  $\eta \in \text{Fix}(u)$  produce the 1st and 4th relations, then the biunitarity of  $u$  gives the 5th relation, and finally the 2nd and 3rd relations follow from the 1st and 4th relations, by using the antipode.  $\square$

As an illustration, consider the case  $F = \{1, \dots, N\}$ . Here the index multiplication is  $ii = i$  and  $ij = \emptyset$  for  $i \neq j$ , and the involution is  $\bar{i} = i$ . Thus, our relations read:

$$\begin{aligned} u_{ik}u_{il} &= \delta_{kl}u_{ik} \quad , \quad u_{ik}u_{jk} = \delta_{ij}u_{ik} \\ \sum_i u_{ij} &= 1 \quad , \quad \sum_j u_{ij} = 1 \\ u_{ij}^* &= u_{ij} \end{aligned}$$

We recognize here the standard magic conditions on a matrix  $u = (u_{ij})$ .

#### 4b. Symmetry groups

Let us develop now some basic theory for the quantum symmetry groups  $S_F^+$ , and their closed subgroups  $G \subset S_F^+$ . Some of the results here are well-known, some other are folklore, and some other are new. We first have the following result, from [2]:

**THEOREM 4.7.** *The quantum groups  $S_F^+$  have the following properties:*

- (1) *The associated Tannakian categories are  $TL_N$ , with  $N = |F|$ .*
- (2) *The main character follows the Marchenko-Pastur law  $\pi_1$ , when  $N \geq 4$ .*
- (3) *The fusion rules for  $S_F^+$  with  $|F| \geq 4$  are the same as for  $SO_3$ .*

**PROOF.** This result is from [2], the idea being as follows:

(1) Our first claim is that the fundamental representation is equivalent to its adjoint,  $u \sim \bar{u}$ . Indeed, let us go back to the coaction formula from Proposition 4.3:

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

We can pick our orthogonal basis  $\{e_i\}$  to be the standard multimatrix basis of  $C(F)$ , so that we have, for a certain involution  $i \rightarrow i^*$  on the index set:

$$e_i^* = e_{i^*}$$

With this convention made, by conjugating the above formula of  $\Phi(e_i)$ , we obtain:

$$\Phi(e_{i^*}) = \sum_j e_{j^*} \otimes u_{ji}^*$$

Now by interchanging  $i \leftrightarrow i^*$  and  $j \leftrightarrow j^*$ , this latter formula reads:

$$\Phi(e_i) = \sum_j e_j \otimes u_{j^*i^*}^*$$

We therefore conclude, by comparing with the original formula, that we have:

$$u_{ji}^* = u_{j^*i^*}$$

But this shows that we have an equivalence as follows, as claimed:

$$u \sim \bar{u}$$

Now with this result in hand, the proof goes as for the proof for  $S_N^+$ , from the previous chapter. To be more precise, the result follows from the fact that the multiplication and unit of any complex algebra, and in particular of the algebra  $C(F)$  that we are interested in here, can be modelled by the following two diagrams:

$$m = |\cup| \quad , \quad u = \cap$$

Indeed, this is certainly true algebraically, and this is something well-known. As in what regards the  $*$ -structure, things here are fine too, because our choice for the trace from Definition 4.2 leads to the following formula, which must be satisfied as well:

$$\mu\mu^* = N \cdot id$$

But the above diagrams  $m, u$  generate the Temperley-Lieb algebra  $TL_N$ , as stated.

(2) The proof here is exactly as for  $S_N^+$ , by using moments. To be more precise, according to (1) these moments are the Catalan numbers, which are the moments of  $\pi_1$ .

(3) Once again same proof as for  $S_N^+$ , by using the fact that the moments of  $\chi$  are the Catalan numbers, which naturally leads to the Clebsch-Gordan rules.  $\square$

It is quite clear now that our present formalism, and the above results, provide altogether a good and conceptual explanation for our  $SO_3$  result regarding  $S_N^+$ . To be more precise, we can merge and reformulate our main results so far in the following way:

**THEOREM 4.8.** *The quantum groups  $S_F^+$  have the following properties:*

- (1) For  $F = \{1, \dots, N\}$  we have  $S_F^+ = S_N^+$ .
- (2) For the space  $F = M_N$  we have  $S_F^+ = PO_N^+ = PU_N^+$ .
- (3) In particular, for the space  $F = M_2$  we have  $S_F^+ = SO_3$ .
- (4) The fusion rules for  $S_F^+$  with  $|F| \geq 4$  are independent of  $F$ .
- (5) Thus, the fusion rules for  $S_F^+$  with  $|F| \geq 4$  are the same as for  $SO_3$ .

**PROOF.** This is basically a compact form of what has been said above, with a new result added, and with some technicalities left aside:

- (1) This is something that we know from Theorem 4.4.



(2) This is new, the idea being as follows. First of all, we know from chapter 1 above that the inclusion  $PO_N^+ \subset PU_N^+$  is an isomorphism, with this coming from the free complexification formula  $\widetilde{O}_N^+ = U_N^+$ , but we will actually reprove this result. Consider indeed the standard vector space action of the free unitary group:

$$U_N^+ \curvearrowright \mathbb{C}^N$$

We associate to this action its adjoint action:

$$PU_N^+ \curvearrowright M_N(\mathbb{C})$$

By universality of  $S_{M_N}^+$ , we must have inclusions as follows:

$$PO_N^+ \subset PU_N^+ \subset S_{M_N}^+$$

On the other hand, the main character of  $O_N^+$  with  $N \geq 2$  being semicircular, the main character of  $PO_N^+$  must be Marchenko-Pastur. Thus the inclusion  $PO_N^+ \subset S_{M_N}^+$  has the property that it keeps fixed the law of main character, and by Peter-Weyl theory we conclude that this inclusion must be an isomorphism, as desired.

(3) This is something that we know from Theorem 4.4, and that can be deduced as well from (2), by using the formula  $PO_2^+ = SO_3$ , which is something elementary.

(4) This is something that we know from Theorem 4.7.

(5) This follows from (3,4), as already pointed out in Theorem 4.7. □

Summarizing, we have now a good explanation for the occurrence of  $SO_3$ , in connection with quantum permutation questions. Philosophically, the idea is that  $S_F^+$  does not depend that much on  $F$ , and so in order to obtain results, it is enough to take  $F = M_2$ , where the corresponding symmetry group is simply  $S_F^+ = SO_3$ , and then to conclude.

As another application of our extended formalism, the Cayley theorem for the finite quantum groups, which fails in the  $S_N^+$  setting, due to some subtle reasons, as explained in [20], holds in the  $S_F^+$  setting. We have indeed the following result:

**THEOREM 4.9.** *Any finite quantum group  $G$  has a Cayley embedding, as follows:*

$$G \subset S_G^+$$

*However, there are finite quantum groups which are not quantum permutation groups.*

**PROOF.** There are two statements here, the idea being as follows:

(1) We have an action  $G \curvearrowright G$ , which leaves invariant the Haar measure. Now since the counting measure is left and right invariant, so is the Haar measure, we conclude that  $G \curvearrowright G$  leaves invariant the counting measure, and so we have  $G \subset S_G^+$ , as claimed.

(2) Regarding the second assertion, this is something non-trivial, from [20], the simplest counterexample being a certain quantum group  $G$  appearing as a split abelian extension associated to the exact factorization  $S_4 = \mathbb{Z}_4 S_3$ , and having cardinality  $|G| = 24$ . □

Getting back now to the quantum groups  $S_F^+$  themselves, and to Theorem 4.8 above, it is quite hard to go beyond this result, with results truly matching the known theory of  $S_N^+$ . Some simplifications, however, appear in the “homogeneous” case:

DEFINITION 4.10. *We call homogeneous the finite quantum spaces of the following type:*

$$F = M_K \times \{1, \dots, L\}$$

*That is, the algebra  $B = C(F)$  must be a finite dimensional random matrix algebra:*

$$B = M_K(\mathbb{C}^L)$$

*The corresponding quantum permutation groups  $S_F^+$  are called homogeneous too.*

The above spaces generalize both the spaces  $X = M_K$  and  $X = \{1, \dots, L\}$ , where most of the known theory lies. The “random matrix” terminology comes from the fact that the random matrix algebras, in general, are the von Neumann algebras of type  $B = M_K(L^\infty(X))$ , with  $X$  being a measured space, and for such an algebra to be finite dimensional, we must have  $X = \{1, \dots, L\}$ . Thus, we are led to the above definition, up to changing the given measure  $X = \{1, \dots, L\}$  into the counting measure.

As a first result regarding such spaces, which is well-known, we have:

THEOREM 4.11. *The symmetry group of  $F = M_K \times \{1, \dots, L\}$  is given by*

$$G(F) = PU_K \wr S_L$$

*with on the right a wreath product, equal by definition to  $PU_K^L \rtimes S_L$ .*

PROOF. The fact that we have an inclusion  $PU_K \wr S_L \subset G(F)$  is standard, and this follows as well by taking the classical version of the inclusion  $PU_K^+ \wr S_L^+ \subset G^+(F)$ , established below. As for the fact that this inclusion  $PU_K \wr S_L \subset G(F)$  is an isomorphism, this can be proved by picking an arbitrary element  $g \in G^+(F)$ , and decomposing it.  $\square$

In order to discuss the quantum analogue of the above result, we will need a notion of free wreath product. The basic theory here, coming from [43], is as follows:

PROPOSITION 4.12. *Given closed subgroups  $G \subset U_N^+$ ,  $H \subset S_k^+$ , with fundamental corepresentations  $u, v$ , the following construction produces a closed subgroup of  $U_{Nk}^+$ :*

$$C(G \wr_* H) = (C(G)^{*k} * C(H)) / \langle [u_{ij}^{(a)}, v_{ab}] = 0 \rangle$$

*In the case where  $G, H$  are classical, the classical version of  $G \wr_* H$  is the usual wreath product  $G \wr H$ . Also, when  $G$  is a quantum permutation group, so is  $G \wr_* H$ .*

PROOF. Consider the matrix  $w_{ia,jb} = u_{ij}^{(a)} v_{ab}$ , over the quotient algebra in the statement. It is routine to check that  $w$  is unitary, and in the case  $G \subset S_N^+$ , our claim is that

this matrix is magic. Indeed, the entries are projections, because they appear as products of commuting projections, and the row sums are as follows:

$$\begin{aligned} \sum_{jb} w_{ia,jb} &= \sum_{jb} u_{ij}^{(a)} v_{ab} \\ &= \sum_b v_{ab} \sum_j u_{ij}^{(a)} \\ &= 1 \end{aligned}$$

As for the column sums, these can be computed as follows:

$$\begin{aligned} \sum_{ia} w_{ia,jb} &= \sum_{ia} u_{ij}^{(a)} v_{ab} \\ &= \sum_a v_{ab} \sum_i u_{ij}^{(a)} \\ &= 1 \end{aligned}$$

With these observations in hand, it is routine to check that  $G \wr_* H$  is indeed a quantum group, with fundamental corepresentation  $w$ , by constructing maps  $\Delta, \varepsilon, S$  as in section 1, and in the case  $G \subset S_N^+$ , we obtain in this way a closed subgroup of  $S_{Nk}^+$ . Finally, the assertion regarding the classical version is standard as well. See [43].  $\square$

We refer to [11], [43], [89] for more details regarding the above construction. With this notion in hand, we can now formulate the following result:

**THEOREM 4.13.** *The quantum symmetry group of a finite quantum space of type*

$$F = M_K \times \{1 \dots, L\}$$

*contains the free wreath product of the corresponding quantum symmetry groups:*

$$PU_K^+ \wr_* S_L^+ \subset G^+(F)$$

*At the level of the classical versions, this inclusion becomes an isomorphism. However, this inclusion itself is not an isomorphism at  $K, L \geq 2$ .*

**PROOF.** We have several assertions to be proved, the idea being as follows:

(1) The fact that we have  $PU_K^+ \wr_* S_L^+ \subset G^+(F)$  is well-known and routine, by checking the fact that the matrix  $w_{ija,klb} = u_{ij,kl}^{(a)} v_{ab}$  is a generalized magic unitary.

(2) At the level of the classical versions we obtain an inclusion  $PU_K \wr S_L \subset G(F)$ , that we know from Theorem 4.11 above to be an isomorphism.

(2) The inclusion  $PU_K^+ \wr_* S_L^+ \subset G^+(F)$  is not an isomorphism, for instance by using [89], along with the fact that  $\pi_1 \boxtimes \pi_1 \neq \pi_1$  where  $\pi_1$  is the Marchenko-Pastur distribution.  $\square$

There are many other things that can be said about the quantum symmetry groups  $G^+(F)$ , especially in the homogeneous case,  $F = M_K \times \{1, \dots, L\}$ , where some of the results regarding the cases  $F = M_K$  and  $F = \{1, \dots, L\}$  can be unified. One interesting question for instance is the study of the group dual subgroups of  $G^+(F)$ , and we will be back to this later, after solving first the problem in the case  $F = \{1, \dots, L\}$ .

#### 4c. Twisting results

Let us focus now on the case  $N = 4$ . According to our various considerations above, the link between  $S_4^+$  and  $SO_3$  basically comes as follows:

$$\{1, 2, 3, 4\} \sim M_2 \implies S_4^+ \sim SO_3$$

It is possible to get beyond this, with a very precise result, stating that  $S_4^+$  is a twist of  $SO_3$ . Let us start with the following definition, from [12]:

DEFINITION 4.14.  $C(SO_3^{-1})$  is the universal  $C^*$ -algebra generated by the entries of a  $3 \times 3$  orthogonal matrix  $a = (a_{ij})$ , with the following relations:

- (1) *Skew-commutation*:  $a_{ij}a_{kl} = \pm a_{kl}a_{ij}$ , with sign  $+$  if  $i \neq k, j \neq l$ , and  $-$  otherwise.
- (2) *Twisted determinant condition*:  $\sum_{\sigma \in S_3} a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} = 1$ .

Normally, our first task would be to prove that  $C(SO_3^{-1})$  is a Woronowicz algebra. This is of course possible, by doing some computations, but we will not need to do these computations, because the result follows from the following result, from [12]:

THEOREM 4.15. *We have an isomorphism of compact quantum groups*

$$S_4^+ = SO_3^{-1}$$

given by the Fourier transform over the Klein group  $K = \mathbb{Z}_2 \times \mathbb{Z}_2$ .

PROOF. Consider indeed the following matrix, corresponding to the standard vector space action of  $SO_3^{-1}$  on  $\mathbb{C}^4$ :

$$a^+ = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

We apply to this matrix the Fourier transform over the Klein group  $K = \mathbb{Z}_2 \times \mathbb{Z}_2$ :

$$u = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & a_{13} \\ 0 & a_{21} & a_{22} & a_{23} \\ 0 & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

It is routine to check that this matrix is magic, and vice versa, i.e. that the Fourier transform over  $K$  converts the relations in Definition 4.14 into the magic relations. Thus, we obtain the identification from the statement.  $\square$

We recall now that the subgroups of  $SO_3$  are subject to an ADE type classification. By twisting, we obtain the following classification result, also from [12]:

THEOREM 4.16. *The closed subgroups of  $S_4^+ = SO_3^{-1}$  are as follows:*

- (1) *Infinite quantum groups:  $S_4^+, O_2^{-1}, \widehat{D}_\infty$ .*
- (2) *Finite groups:  $S_4$ , and its subgroups.*
- (3) *Finite group twists:  $S_4^{-1}, A_5^{-1}$ .*
- (4) *Series of twists:  $D_{2n}^{-1}$  ( $n \geq 3$ ),  $DC_{2n}^{-1}$  ( $n \geq 2$ ).*
- (5) *A group dual series:  $\widehat{D}_n$ , with  $n \geq 3$ .*

Moreover, these quantum groups are subject to an ADE classification result, with the graphs coming from the representation theory of the quantum groups.

PROOF. The idea here is that the classification can be obtained by taking some inspiration from the McKay classification of the subgroups of  $SO_3$ , by twisting everything using the cocycle twisting method. As for the last assertion, the idea here is that the moments of the main character count the loops based at 1 on the graph. See [12].  $\square$

An interesting extension of the  $S_4^+ = SO_3^{-1}$  result comes by looking at the general case  $N = n^2$ , with  $n \in \mathbb{N}$ . We will prove that we have a twisting result, as follows:

$$PO_n^+ = (S_N^+)^{\sigma}$$

In order to explain this material, from [19], which is quite technical, requiring good algebraic knowledge, let us begin with some generalities. We first have:

PROPOSITION 4.17. *Given a finite group  $F$ , the algebra  $C(S_F^+)$  is isomorphic to the abstract algebra presented by generators  $x_{gh}$  with  $g, h \in F$ , with the following relations:*

$$\begin{aligned} x_{1g} &= x_{g1} = \delta_{1g} \\ x_{s,gh} &= \sum_{t \in F} x_{st^{-1},g} x_{th} \\ x_{gh,s} &= \sum_{t \in F} x_{gt^{-1}} x_{h,ts} \end{aligned}$$

The comultiplication, counit and antipode are given by the formulae

$$\begin{aligned} \Delta(x_{gh}) &= \sum_{s \in F} x_{gs} \otimes x_{sh} \\ \varepsilon(x_{gh}) &= \delta_{gh} \\ S(x_{gh}) &= x_{h^{-1}g^{-1}} \end{aligned}$$

on the standard generators  $x_{gh}$ .

PROOF. This follows indeed from a direct verification, based either on Theorem 4.4 above, or on its equivalent formulation from Wang's paper [94].  $\square$

Let us discuss now the twisted version of the above result. Consider a 2-cocycle on  $F$ , which is by definition a map  $\sigma : F \times F \rightarrow \mathbb{C}^*$  satisfying:

$$\begin{aligned}\sigma_{gh,s}\sigma_{gh} &= \sigma_{g,hs}\sigma_{hs} \\ \sigma_{g1} &= \sigma_{1g} = 1\end{aligned}$$

Given such a cocycle, we can construct the associated twisted group algebra  $C(\widehat{F}_\sigma)$ , as being the vector space  $C(\widehat{F}) = C^*(F)$ , with product  $e_g e_h = \sigma_{gh} e_{gh}$ . We have then the following generalization of Proposition 4.17:

PROPOSITION 4.18. *The algebra  $C(S_{\widehat{F}_\sigma}^+)$  is isomorphic to the abstract algebra presented by generators  $x_{gh}$  with  $g, h \in G$ , with the relations  $x_{1g} = x_{g1} = \delta_{1g}$  and:*

$$\begin{aligned}\sigma_{gh} x_{s,gh} &= \sum_{t \in F} \sigma_{st^{-1},t} x_{st^{-1},g} x_{th} \\ \sigma_{gh}^{-1} x_{gh,s} &= \sum_{t \in F} \sigma_{t^{-1},ts}^{-1} x_{gt^{-1}} x_{h,ts}\end{aligned}$$

The comultiplication, counit and antipode are given by the formulae

$$\begin{aligned}\Delta(x_{gh}) &= \sum_{s \in F} x_{gs} \otimes x_{sh} \\ \varepsilon(x_{gh}) &= \delta_{gh} \\ S(x_{gh}) &= \sigma_{h^{-1}h} \sigma_{g^{-1}g}^{-1} x_{h^{-1}g^{-1}}\end{aligned}$$

on the standard generators  $x_{gh}$ .

PROOF. Once again, this follows from a direct verification. Note that by using the cocycle identities we obtain  $\sigma_{gg^{-1}} = \sigma_{g^{-1}g}$ , needed in the proof.  $\square$

In what follows, we will prove that the quantum groups  $S_{\widehat{F}}^+$  and  $S_{\widehat{F}_\sigma}^+$  are related by a cocycle twisting operation. Let  $H$  be a Hopf algebra. We recall that a left 2-cocycle is a convolution invertible linear map  $\sigma : H \otimes H \rightarrow \mathbb{C}$  satisfying:

$$\begin{aligned}\sigma_{x_1 y_1} \sigma_{x_2 y_2, z} &= \sigma_{y_1 z_1} \sigma_{x, y_2 z_2} \\ \sigma_{x1} &= \sigma_{1x} = \varepsilon(x)\end{aligned}$$

Note that  $\sigma$  is a left 2-cocycle if and only if  $\sigma^{-1}$ , the convolution inverse of  $\sigma$ , is a right 2-cocycle, in the sense that we have:

$$\begin{aligned}\sigma_{x_1 y_1, z}^{-1} \sigma_{x_1 y_2}^{-1} &= \sigma_{x, y_1 z_1}^{-1} \sigma_{y_2 z_2}^{-1} \\ \sigma_{x1}^{-1} &= \sigma_{1x}^{-1} = \varepsilon(x)\end{aligned}$$

Given a left 2-cocycle  $\sigma$  on  $H$ , one can form the 2-cocycle twist  $H^\sigma$  as follows. As a coalgebra,  $H^\sigma = H$ , and an element  $x \in H$ , when considered in  $H^\sigma$ , is denoted  $[x]$ . The product in  $H^\sigma$  is defined, in Sweedler notation, by:

$$[x][y] = \sum \sigma_{x_1 y_1} \sigma_{x_3 y_3}^{-1} [x_2 y_2]$$

Note that the cocycle condition ensures the fact that we have indeed a Hopf algebra. Note also that the coalgebra isomorphism  $H \rightarrow H^\sigma$  given by  $x \rightarrow [x]$  commutes with the respective Haar integrals, as soon as  $H$  has a Haar integral.

We can now state and prove a main theorem from [19], as follows:

**THEOREM 4.19.** *If  $F$  is a finite group and  $\sigma$  is a 2-cocycle on  $F$ , the Hopf algebras*

$$C(S_{\widehat{F}}^+) \quad , \quad C(S_{\widehat{F}^\sigma}^+)$$

*are 2-cocycle twists of each other, in the above sense.*

**PROOF.** In order to prove this result, we use the following Hopf algebra map:

$$\pi : C(S_{\widehat{F}}^+) \rightarrow C(\widehat{F})$$

$$x_{gh} \rightarrow \delta_{gh} e_g$$

Our 2-cocycle  $\sigma : F \times F \rightarrow \mathbb{C}^*$  can be extended by linearity into a linear map as follows, which is a left and right 2-cocycle in the above sense:

$$\sigma : C(\widehat{F}) \otimes C(\widehat{F}) \rightarrow \mathbb{C}$$

Consider now the following composition:

$$\alpha = \sigma(\pi \otimes \pi) : C(S_{\widehat{F}}^+) \otimes C(S_{\widehat{F}}^+) \rightarrow C(\widehat{F}) \otimes C(\widehat{F}) \rightarrow \mathbb{C}$$

Then  $\alpha$  is a left and right 2-cocycle, because it is induced by a cocycle on a group algebra, and so is its convolution inverse  $\alpha^{-1}$ . Thus we can construct the twisted algebra  $C(S_{\widehat{F}}^+)^{\alpha^{-1}}$ , and inside this algebra we have the following computation:

$$\begin{aligned} [x_{gh}][x_{rs}] &= \alpha^{-1}(x_g, x_r) \alpha(x_h, x_s) [x_{gh} x_{rs}] \\ &= \sigma_{gr}^{-1} \sigma_{hs} [x_{gh} x_{rs}] \end{aligned}$$

By using this, we obtain the following formula:

$$\begin{aligned} \sum_{t \in F} \sigma_{st^{-1}, t} [x_{st^{-1}, g}] [x_{th}] &= \sum_{t \in F} \sigma_{st^{-1}, t} \sigma_{st^{-1}, t}^{-1} \sigma_{gh} [x_{st^{-1}, g} x_{th}] \\ &= \sigma_{gh} [x_{s, gh}] \end{aligned}$$

Similarly, we have the following formula:

$$\sum_{t \in F} \sigma_{t^{-1}, ts}^{-1} [x_{g, t^{-1}}] [x_{h, ts}] = \sigma_{gh}^{-1} [x_{gh, s}]$$

We deduce from this that there exists a Hopf algebra map, as follows:

$$\begin{aligned}\Phi : C(S_{\widehat{F}\sigma}^+) &\rightarrow C(S_{\widehat{F}}^+)^{\alpha^{-1}} \\ x_{gh} &\rightarrow [x_{g,h}]\end{aligned}$$

This map is clearly surjective, and is injective as well, by a standard fusion semiring argument, because both Hopf algebras have the same fusion semiring.  $\square$

Summarizing, we have proved our main twisting result. Our purpose in what follows will be that of working out versions and particular cases of it. We first have:

**PROPOSITION 4.20.** *If  $F$  is a finite group and  $\sigma$  is a 2-cocycle on  $F$ , then*

$$\Phi(x_{g_1 h_1} \dots x_{g_m h_m}) = \Omega(g_1, \dots, g_m)^{-1} \Omega(h_1, \dots, h_m) x_{g_1 h_1} \dots x_{g_m h_m}$$

with the coefficients on the right being given by the formula

$$\Omega(g_1, \dots, g_m) = \prod_{k=1}^{m-1} \sigma_{g_1 \dots g_k, g_{k+1}}$$

is a coalgebra isomorphism  $C(S_{\widehat{F}\sigma}^+) \rightarrow C(S_{\widehat{F}}^+)$ , commuting with the Haar integrals.

**PROOF.** This is indeed just a technical reformulation of Theorem 4.19.  $\square$

Here is another useful result from [19], that we will need in what follows:

**THEOREM 4.21.** *Let  $X \subset F$  be such that  $\sigma_{gh} = 1$  for any  $g, h \in X$ , and consider the subalgebra*

$$B_X \subset C(S_{\widehat{F}\sigma}^+)$$

generated by the elements  $x_{gh}$ , with  $g, h \in X$ . Then we have an injective algebra map

$$\Phi_0 : B_X \rightarrow C(S_{\widehat{F}}^+)$$

given by  $x_{g,h} \rightarrow x_{g,h}$ .

**PROOF.** With the notations in the proof of Theorem 4.19, we have the following equality in  $C(S_{\widehat{F}}^+)^{\alpha^{-1}}$ , for any  $g_i, h_i, r_i, s_i \in X$ :

$$[x_{g_1 h_1} \dots x_{g_p h_p}] \cdot [x_{r_1 s_1} \dots x_{r_q s_q}] = [x_{g_1 h_1} \dots x_{g_p h_p} x_{r_1 s_1} \dots x_{r_q s_q}]$$

Now  $\Phi_0$  can be defined to be the composition of  $\Phi|_{B_X}$  with the linear isomorphism  $C(S_{\widehat{F}}^+)^{\alpha^{-1}} \rightarrow C(S_{\widehat{F}}^+)$  given by  $[x] \rightarrow x$ , and is clearly an injective algebra map.  $\square$

Let us discuss now some concrete applications of the general results established above. Consider the group  $F = \mathbb{Z}_n^2$ , let  $w = e^{2\pi i/n}$ , and consider the following map:

$$\begin{aligned}\sigma : F \times F &\rightarrow \mathbb{C}^* \\ \sigma_{(ij)(kl)} &= w^{jk}\end{aligned}$$



It is easy to see that  $\sigma$  is a bicharacter, and hence a 2-cocycle on  $F$ . Thus, we can apply our general twisting result, to this situation. In order to understand what is the formula that we obtain, we must do some computations. Let  $E_{ij}$  with  $i, j \in \mathbb{Z}_n$  be the standard basis of  $M_n(\mathbb{C})$ . Following [19], we first have the following result:

PROPOSITION 4.22. *The linear map given by*

$$\psi(e_{(i,j)}) = \sum_{k=0}^{n-1} w^{ki} E_{k,k+j}$$

defines an isomorphism of algebras  $\psi : C(\widehat{F}_\sigma) \simeq M_n(\mathbb{C})$ .

PROOF. Consider indeed the following linear map:

$$\psi'(E_{ij}) = \frac{1}{n} \sum_{k=0}^{n-1} w^{-ik} e_{(k,j-i)}$$

It is routine to check that both  $\psi, \psi'$  are morphisms of algebras, and that these maps are inverse to each other. In particular,  $\psi$  is an isomorphism of algebras, as stated.  $\square$

Next in line, we have the following result:

PROPOSITION 4.23. *The algebra map given by*

$$\varphi(u_{ij}u_{kl}) = \frac{1}{n} \sum_{a,b=0}^{n-1} w^{ai-bj} x_{(a,k-i),(b,l-j)}$$

defines a Hopf algebra isomorphism  $\varphi : C(S_{M_n}^+) \simeq C(S_{\widehat{F}_\sigma}^+)$ .

PROOF. Consider the universal coactions on the two algebras in the statement:

$$\begin{aligned} \alpha : M_n(\mathbb{C}) &\rightarrow M_n(\mathbb{C}) \otimes C(S_{M_n}^+) \\ \beta : C(\widehat{F}_\sigma) &\rightarrow C(\widehat{F}_\sigma) \otimes C(S_{\widehat{F}_\sigma}^+) \end{aligned}$$

In terms of the standard bases, these coactions are given by:

$$\begin{aligned} \alpha(E_{ij}) &= \sum_{kl} E_{kl} \otimes u_{ki}u_{lj} \\ \beta(e_{(i,j)}) &= \sum_{kl} e_{(k,l)} \otimes x_{(k,l),(i,j)} \end{aligned}$$

We use now the identification  $C(\widehat{F}_\sigma) \simeq M_n(\mathbb{C})$  from Proposition 4.22. This identification produces a coaction map, as follows:

$$\gamma : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \otimes C(S_{\widehat{F}_\sigma}^+)$$

Now observe that this map is given by the following formula:

$$\gamma(E_{ij}) = \frac{1}{n} \sum_{ab} E_{ab} \otimes \sum_{kr} w^{ar-ik} x_{(r,b-a),(k,j-i)}$$

By comparing with the formula of  $\alpha$ , we obtain the isomorphism in the statement.  $\square$

We will need one more result of this type, as follows:

PROPOSITION 4.24. *The algebra map given by*

$$\rho(x_{(a,b),(i,j)}) = \frac{1}{n^2} \sum_{klrs} w^{ki+lj-ra-sb} p_{(r,s),(k,l)}$$

defines a Hopf algebra isomorphism  $\rho : C(S_{\widehat{F}}^+) \simeq C(S_F^+)$ .

PROOF. We have a Fourier transform isomorphism, as follows:

$$C(\widehat{F}) \simeq C(F)$$

Thus the algebras in the statement are indeed isomorphic.  $\square$

As a conclusion to all this, we have the following result, from [19]:

THEOREM 4.25. *Let  $n \geq 2$  and  $w = e^{2\pi i/n}$ . Then*

$$\Theta(u_{ij}u_{kl}) = \frac{1}{n} \sum_{ab=0}^{n-1} w^{-a(k-i)+b(l-j)} p_{ia,jb}$$

defines a coalgebra isomorphism

$$C(PO_n^+) \rightarrow C(S_{n^2}^+)$$

commuting with the Haar integrals.

PROOF. We recall from Theorem 4.23 (2) that we have identifications as follows, where the projective version of  $(A, u)$  is the pair  $(PA, v)$ , with  $PA = \langle v_{ij} \rangle$  and  $v = u \otimes \bar{u}$ :

$$PO_n^+ = PU_n^+ = S_{M_n}^+$$

With this in hand, the result follows from Theorem 4.18 and Proposition 4.20, by combining them with the various isomorphisms established above.  $\square$

Here is a useful version of the above result, that we will need later on:

THEOREM 4.26. *The following two algebras are isomorphic, via  $u_{ij}^2 \rightarrow X_{ij}$ :*

- (1) *The algebra generated by the variables  $u_{ij}^2 \in C(O_n^+)$ .*
- (2) *The algebra generated by  $X_{ij} = \frac{1}{n} \sum_{a,b=1}^n p_{ia,jb} \in C(S_{n^2}^+)$*

PROOF. We have  $\Theta(u_{ij}^2) = X_{ij}$ , so it remains to prove that if  $B$  is the subalgebra of  $C(S_{M_n}^+)$  generated by the variables  $u_{ij}^2$ , then  $\Theta|_B$  is an algebra morphism. Let us set:

$$X = \{(i, 0) | i \in \mathbb{Z}_n\} \subset \mathbb{Z}_n^2$$

Then  $X$  satisfies the assumption in Theorem 4.20, and  $\varphi(B) \subset B_X$ . Thus by Theorem 4.20, the map  $\Theta|_B = \rho F_0 \varphi|_B$  is indeed an algebra morphism.  $\square$

We will be back to this in chapter 8 below, with some probabilistic consequences.

As an overall conclusion, the twisting formula  $S_4^+ = SO_3^{-1}$  ultimately comes from something of type  $X_4 \simeq M_2$ , where  $X_4 = \{1, 2, 3, 4\}$  and  $M_2 = \text{Spec}(M_2(\mathbb{C}))$ , and at  $N \geq 5$  there are some extensions of this, and notably when  $N = n^2$  with  $n \geq 3$ .

#### 4d. Planar algebras

We discuss now the representation theory of the quantum groups  $S_F^+$ , and of their subgroups  $G \subset S_F^+$ , with planar algebra results generalizing those established in chapter 3, in the case  $F = \{1, \dots, N\}$ . Let us start with a general definition, as follows:

DEFINITION 4.27. *Let  $(B, tr)$  be a finite dimensional  $\mathbb{C}^*$ -algebra together with a positive trace. A coaction of a Woronowicz algebra  $A$  on  $(B, tr)$  is a morphism of  $*$ -algebras  $\Phi : B \rightarrow B \otimes A$ , subject to the following conditions:*

- (1) *Coassociativity condition:  $(\Phi \otimes id)\Phi = (id \otimes \Delta)\Phi$ .*
- (2) *Counitality condition:  $(id \otimes \varepsilon)\Phi = id$ .*
- (3) *Copreservation of  $tr$  condition:  $(tr \otimes id)\Phi = tr(\cdot)1$ .*

In order to study such coactions, we will need to use some suitable bases and indices. Following [5], we use the following conventions:

DEFINITION 4.28. *Let  $(B, tr)$  be as above. Choose a system of matrix units  $X \subset B$  making  $tr$  diagonal, with the following multiplication convention:*

$$\begin{pmatrix} i \\ j \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} = \delta_{jk} \begin{pmatrix} i \\ l \end{pmatrix}$$

We denote by  $q_i$  the fourth roots of the weights of  $tr$ , chosen positive:

$$\varphi \begin{pmatrix} i \\ j \end{pmatrix} = \delta_{ij} q_i^4 \quad , \quad q_i > 0$$

Any linear map  $\Phi : B \rightarrow B \otimes A$  will be written in the following form:

$$\Phi \begin{pmatrix} i \\ j \end{pmatrix} = \sum_{kl} \begin{pmatrix} k \\ l \end{pmatrix} \otimes q_k^{-1} q_i q_j q_l^{-1} V \begin{pmatrix} l & j \\ k & i \end{pmatrix}$$

This is, to any linear map  $v$  we associate in this way a matrix  $V$ , and vice versa.

It is convenient to define the coefficients  $V\left(\begin{smallmatrix} l & j \\ k & i \end{smallmatrix}\right)$  for all indices  $i, j, k, l$ , by saying that they are equal to zero if  $\left(\begin{smallmatrix} j \\ i \end{smallmatrix}\right)$  or  $\left(\begin{smallmatrix} l \\ k \end{smallmatrix}\right)$  don't exist. We have:

PROPOSITION 4.29. *A linear map  $\Phi : B \rightarrow B \otimes A$  is a coaction of  $A$  on  $B$  if and only if the matrix  $V$  constructed above satisfies the following conditions:*

$$\begin{aligned} \varepsilon V\left(\begin{smallmatrix} l & j \\ k & i \end{smallmatrix}\right) &= \delta_{ki} \delta_{lj} 1 \\ \Delta V\left(\begin{smallmatrix} l & j \\ k & i \end{smallmatrix}\right) &= \sum_{hg} V\left(\begin{smallmatrix} l & h \\ k & g \end{smallmatrix}\right) \otimes V\left(\begin{smallmatrix} h & j \\ g & i \end{smallmatrix}\right) \\ V\left(\begin{smallmatrix} l & j \\ k & i \end{smallmatrix}\right)^* &= V\left(\begin{smallmatrix} k & i \\ l & j \end{smallmatrix}\right) \\ \sum q_i^2 V\left(\begin{smallmatrix} l & i \\ k & i \end{smallmatrix}\right) &= \delta_{kl} q_k^2 \\ \sum_s q_s^{-2} V\left(\begin{smallmatrix} s & h \\ k & g \end{smallmatrix}\right) V\left(\begin{smallmatrix} l & j \\ s & i \end{smallmatrix}\right) &= \delta_{hi} q_i^{-2} V\left(\begin{smallmatrix} l & j \\ k & g \end{smallmatrix}\right) \end{aligned}$$

This sequence of five conditions will be denoted  $(\varepsilon)$ ,  $(\Delta)$ ,  $(*)$ ,  $(u^\circ)$ ,  $(m^\circ)$ .

PROOF. This is something that we basically know from Proposition 4.3 above and its proof, now rescaled by using the conventions in Definition 4.28.  $\square$

We have as well the following result:

PROPOSITION 4.30. *Assume that  $\Phi$  is a coaction of  $A$  on  $B$ . Then the following three conditions  $(S)$ ,  $({}^\circ u)$  and  $(m^\circ)$*

$$\begin{aligned} S V\left(\begin{smallmatrix} l & j \\ k & i \end{smallmatrix}\right) &= q_k^2 q_i^{-2} q_j^2 q_l^{-2} V\left(\begin{smallmatrix} i & k \\ j & l \end{smallmatrix}\right) \\ \sum_i q_i^2 V\left(\begin{smallmatrix} i & l \\ i & k \end{smallmatrix}\right) &= \delta_{kl} q_k^2 \\ \sum_s q_s^{-2} V\left(\begin{smallmatrix} h & s \\ k & g \end{smallmatrix}\right) V\left(\begin{smallmatrix} l & j \\ i & s \end{smallmatrix}\right) &= \delta_{hi} q_i^{-2} V\left(\begin{smallmatrix} l & j \\ k & g \end{smallmatrix}\right) \end{aligned}$$

are equivalent, and are satisfied if and only if  $\Phi$  preserves  $tr$ .

PROOF. As before, this is something that we basically know from Proposition 4.3 above and its proof, now rescaled by using the conventions in Definition 4.28.  $\square$

With the above results in hand, let us construct now the tensor powers of the coaction map  $\Phi$ , a bit in the same way as we did in chapter 3, in the case  $B = \mathbb{C}^N$ .

For any  $n$  the set  $X^n$  is a basis of the linear space  $B^{\otimes n}$ . We use “loop” notations for this basis, depending on the parity of  $n$ :

$$\begin{aligned} \begin{pmatrix} i_1 \\ j_1 \end{pmatrix} \otimes \begin{pmatrix} i_2 \\ j_2 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} i_{2s} \\ j_{2s} \end{pmatrix} &= \begin{pmatrix} j_{2s} & i_{2s} & \dots & i_{s+1} \\ i_1 & j_1 & \dots & j_s \end{pmatrix} \\ \begin{pmatrix} i_1 \\ j_1 \end{pmatrix} \otimes \begin{pmatrix} i_2 \\ j_2 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} i_{2s-1} \\ j_{2s-1} \end{pmatrix} &= \begin{pmatrix} j_{2s-1} & i_{2s-1} & \dots & j_s \\ i_1 & j_1 & \dots & i_s \end{pmatrix} \end{aligned}$$

We have then a  $C^*$ -algebra structure on the linear space  $B^{\otimes n}$ , which is not exactly the standard one, with multiplication and involution defined as follows:

$$\begin{aligned} \begin{pmatrix} j_1 & \dots & j_n \\ i_1 & \dots & i_n \end{pmatrix} \begin{pmatrix} l_1 & \dots & l_n \\ k_1 & \dots & k_n \end{pmatrix} &= \delta_{j_1 k_1} \dots \delta_{j_n k_n} \begin{pmatrix} l_1 & \dots & l_n \\ i_1 & \dots & i_n \end{pmatrix} \\ \begin{pmatrix} j_1 & \dots & j_n \\ i_1 & \dots & i_n \end{pmatrix}^* &= \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix} \end{aligned}$$

The conditions  $(\varepsilon)$  and  $(\Delta)$  show that the following matrix is a corepresentation:

$$u = \sum_{ijkl} e_{(k)(i)}^{(l)(j)} \otimes V \begin{pmatrix} l & j \\ k & i \end{pmatrix} \in \mathcal{L}(B) \otimes A$$

Consider the tensor powers of this corepresentation:

$$u^{\otimes n} = u_{1,n+1} u_{2,n+1} \dots u_{n,n+1} \in \mathcal{L}(B^{\otimes n}) \otimes A$$

Let  $V_n$  be the matrix of coefficients of  $u^{\otimes n}$ , defined by:

$$u^{\otimes n} = \sum_{ijkl} e_{(k_1 \dots k_n)(i_1 \dots i_n)}^{(l_1 \dots l_n)(j_1 \dots j_n)} \otimes V_n \begin{pmatrix} l_1 & \dots & l_n & j_1 & \dots & j_n \\ k_1 & \dots & k_n & i_1 & \dots & i_n \end{pmatrix}$$

We define a linear form  $\tilde{tr}_n$  on the linear space  $B^{\otimes n}$  by:

$$\tilde{tr}_n \begin{pmatrix} j_1 & \dots & j_n \\ i_1 & \dots & i_n \end{pmatrix} = \delta_{(i_1 \dots i_n)(j_1 \dots j_n)} q_{(i_1 \dots i_n)}^4$$

Here the weights are given by the following function, where  $\pm 1 = (-1)^n$ :

$$q_{(i_1 \dots i_n)} = q_{i_1} q_{i_2}^{-1} q_{i_3} \dots q_{i_n}^{\mp 1}$$

Note that we have  $q_{(i)} = q_i$ , and so  $\tilde{tr}_1 = tr$  on the algebra  $B^{\otimes 1} = B$ .

With these conventions, we have the following result:

PROPOSITION 4.31. *If  $\Phi : A \rightarrow A \otimes H$  is a coaction of  $A$  on  $B$  which preserves  $tr$  then the linear map  $\Phi_n : B^{\otimes n} \rightarrow B^{\otimes n} \otimes A$  given by*

$$\begin{aligned} \Phi_n \begin{pmatrix} j_1 & \cdots & j_n \\ i_1 & \cdots & i_n \end{pmatrix} &= \sum_{kl} \begin{pmatrix} l_1 & \cdots & l_n \\ k_1 & \cdots & k_n \end{pmatrix} \otimes q_{(k_1 \dots k_n)}^{-1} q_{(i_1 \dots i_n)} q_{(j_1 \dots j_n)} q_{(l_1 \dots l_n)}^{-1} \\ &\cdot V_n \begin{pmatrix} l_1 & \cdots & l_n & j_1 & \cdots & j_n \\ k_1 & \cdots & k_n & i_1 & \cdots & i_n \end{pmatrix} \end{aligned}$$

is a coaction of  $A$  on  $B^{\otimes n}$  which preserves  $\tilde{tr}_n$ .

PROOF. This follows indeed from a routine computation. See [5].  $\square$

With the above family of coaction maps  $\Phi_n$  constructed, we say that a linear map  $T : B^{\otimes n} \rightarrow B^{\otimes m}$  is  $\Phi_\infty$ -equivariant if the following diagram commutes:

$$\begin{array}{ccc} B^{\otimes n} & \xrightarrow{T} & B^{\otimes m} \\ \Phi_n \downarrow & & \downarrow \Phi_m \\ B^{\otimes n} \otimes A & \xrightarrow{T \otimes id} & B^{\otimes m} \otimes A \end{array}$$

With this convention, we have the following result, also from [5]:

PROPOSITION 4.32. *The following linear maps*

$$\begin{aligned} I_n \begin{pmatrix} j_1 & \cdots & j_{n-1} \\ i_1 & \cdots & i_{n-1} \end{pmatrix} &= \sum_l \begin{pmatrix} j_1 & \cdots & j_{n-1} & l \\ i_1 & \cdots & i_{n-1} & l \end{pmatrix} \\ \tilde{e}_n &= \sum_{g^{ij}} q_i^{\pm 2} q_j^{\pm 2} \begin{pmatrix} g_1 & \cdots & g_{n-2} & j & j \\ g_1 & \cdots & g_{n-2} & i & i \end{pmatrix} \\ \tilde{E}_n \begin{pmatrix} j_1 & \cdots & j_n \\ i_1 & \cdots & i_n \end{pmatrix} &= \delta_{i_n j_n} q_{i_n}^{\mp 4} \begin{pmatrix} j_1 & \cdots & j_{n-1} \\ i_1 & \cdots & i_{n-1} \end{pmatrix} \end{aligned}$$

where  $\pm 1 = (-1)^n$ , are  $\Phi_\infty$ -equivariant.

PROOF. As before, this follows from a routine computation. See [5].  $\square$

If  $\Phi : B \rightarrow B \otimes A$  is a coaction then  $\Gamma_n = (id \otimes \int_A) \Phi_n$  is an idempotent of  $\mathcal{L}(B^{\otimes n})$ , whose image are the fixed points of  $\Phi_n$ . This follows from the following computation:

$$\begin{aligned} \Phi_n \Gamma_n(x) &= \Phi_n \left( id \otimes \int_A \right) \Phi_n(x) \\ &= \left( id \otimes id \otimes \int_A \right) (id \otimes \Delta) \Phi_n(x) \\ &= \Gamma_n(x) \otimes 1 \end{aligned}$$

In order to get now to planar algebra results, we say that a pair of linear maps  $(T, T^q) : B^{\otimes n} \rightarrow B^{\otimes m}$  is weakly  $\Phi_\infty$ -equivariant if the following diagram commutes:

$$\begin{array}{ccc} B^{\otimes n} & \xrightarrow{T^q} & B^{\otimes m} \\ \Gamma_n \downarrow & & \downarrow \Gamma_m \\ B^{\otimes n} & \xrightarrow{T} & B^{\otimes m} \end{array}$$

The interest in this notion is that it makes the following diagram factorize:

$$\begin{array}{ccc} B^{\otimes n} & \xrightarrow{T} & B^{\otimes m} \\ \cup & & \cup \\ \text{Im}(\Gamma_n) & \longrightarrow & \text{Im}(\Gamma_m) \end{array}$$

We say that an operator  $T$  is weakly  $\Phi_\infty$ -equivariant if the pair  $(T, T)$  is weakly  $\Phi_\infty$ -equivariant, in the above sense. This happens for instance if  $T$  is  $\Phi_\infty$ -equivariant, because we can glue the  $\Phi_\infty$ -equivariance diagram of  $T$  to the following trivial diagram:

$$\begin{array}{ccc} B^{\otimes n} \otimes A & \xrightarrow{T \otimes id} & B^{\otimes m} \otimes A \\ id \otimes \int_A \downarrow & & \downarrow id \otimes \int_A \\ B^{\otimes n} & \xrightarrow{T} & B^{\otimes m} \end{array}$$

With these conventions, we have the following result:

**PROPOSITION 4.33.** *Assuming that  $(B, tr)$  is as before, with  $tr$  being the canonical trace, the following happen:*

- (1) *The following pair of maps is weakly  $\Phi_\infty$ -equivariant:*

$$J_n \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \cdots & i_{n+1} \end{pmatrix} = \sum_{kl} \begin{pmatrix} l & k & j_3 & \cdots & j_{n+1} \\ l & k & i_3 & \cdots & i_{n+1} \end{pmatrix}$$

$$J_n^q \begin{pmatrix} j_3 & \cdots & j_{n+1} \\ i_3 & \cdots & i_{n+1} \end{pmatrix} = \sum_{kl} q_l^{-8} q_k^8 \begin{pmatrix} l & k & j_3 & \cdots & j_{n+1} \\ l & k & i_3 & \cdots & i_{n+1} \end{pmatrix}$$

- (2) *The map  $J_n$  is weakly  $\Phi_\infty$ -equivariant.*  
(3) *If  $A$  is commutative then  $J_n$  is  $\Phi_\infty$ -equivariant.*

**PROOF.** As before, this follows from a routine computation. See [5]. □

We have now all the needed ingredients for formulating the planar algebra result. We recall that the annular category  $\mathcal{A}$  is defined as follows:

- (1) The objects are the positive integers.
- (2) The space of arrows  $\mathcal{A}(i, j)$  is formed by tangles in the planar operad  $\mathcal{P}$  with “output” disc having  $2j$  marked points and one “input” disc, having  $2i$  marked points.
- (3) The composition of arrows is given by gluing of annuli.

Now let  $(B, tr)$  be as before, and let  $P(B)$  be the planar algebra associated to the bipartite graph of  $B$ , with spin vector  $a \rightarrow tr(1_a)$ . The sequence of vector spaces of  $P(B)$  will be canonically identified with the sequence of tensor powers of  $B$ .

With these conventions, we have the following result:

**THEOREM 4.34.** *We have the following results:*

- (1) *The linear maps in the image of  $\mathcal{A}$  are weakly  $\Phi_\infty$ -equivariant.*
- (2) *If  $A$  is commutative, these linear maps are  $\Phi_\infty$ -equivariant.*
- (3) *The spaces of fixed points of the coactions  $\Phi_n$  form a subalgebra of  $P(B)$ .*
- (4) *Any planar subalgebra of  $P(B)$  appears in this way.*

**PROOF.** We have several things to be proved, the idea being as follows:

(1) The gluing of commutative diagrams shows that weak  $\Phi_\infty$ -equivariance is stable by composition, so the annular tangles whose image is weakly  $\Phi_\infty$ -equivariant form a certain subcategory  $\mathcal{B} \subset \mathcal{A}$ . We want to prove that we have  $\mathcal{B} = \mathcal{A}$ .

For this purpose, consider the inclusion tangle in  $\mathcal{A}(n-1, n)$ , expectation tangle in  $\mathcal{A}(n, n-1)$ , Jones projection tangle in  $\mathcal{A}(0, n)$  and shift tangle in  $\mathcal{A}(n-1, n+1)$ . The images of these tangles are given by the various formulae above, suitably rescaled, and we deduce from this that these tangles are in the above subcategory  $\mathcal{B}$ .

The sets  $\mathcal{A}(0, n)$  of Temperley-Lieb tangles being generated by inclusions and Jones projections, these sets are in  $\mathcal{B}$ . For  $x, y \in \mathcal{A}(0, n)$  let  $M(x, y) \in \mathcal{A}(n, n)$  be the 3-multiplication  $n$ -tangle of  $\mathcal{P}$  with the upper circle filled with  $y$  and the lower circle filled with  $x$ . The corresponding linear map is given by  $\widetilde{M}(x, y) : p \rightarrow \widetilde{x}p\widetilde{y}$ , and since the fixed points of  $\Phi_n$  are stable under multiplication,  $M(x, y)$  is in  $\mathcal{B}$ .

Let now  $T \in \mathcal{A}(i, j)$ . By using boxes instead of discs, isotope  $T$ , then cut it horizontally in three parts such that the middle part contains the inner box plus vertical strings only. By adding contractible circles at right, we can arrange such that the number of points on the middle cuts is greater than  $j$ . By adding more contractible circles at right, each of them consisting of “up” and “down” semicircles plus two outside “expectation” strings connecting them, we get an equality of the form  $T^{\circ \dots \circ} = EM(x, y)IJ$ , where  $I$  is a composition of inclusion tangles,  $J$  is a composition of shift tangles,  $E$  is a composition



of expectation tangles,  $x$  and  $y$  are in  $A(k, k)$  for some big  $k$  and  $T^{\circ\circ\dots\circ}$  is obtained from  $T$  by adding contractible circles. Thus  $T^{\circ\circ\dots\circ}$  is in  $\mathcal{B}$ , so  $T$  is in  $\mathcal{B}$ , as desired.

(2) The same proof as above works in the case where  $A$  is commutative, with “weakly  $\Phi_\infty$ -equivariant” replaced by “ $\Phi_\infty$ -equivariant”, and this gives the result.

(3) Since the weakly  $\Phi_\infty$ -equivariant maps send fixed points to fixed points, the result follows from (1) above. See [5].

(4) This is something more technical, that can be obtained along the lines of the proof in chapter 3, in the case  $B = \mathbb{C}^N$ , and we refer here to [89].  $\square$

We will be back to this later, in chapter 12 below, with a result which is even more general, dealing with quantum reflection groups and their subgroups.

#### 4e. Exercises

Things have been quite technical in this chapter, especially towards the end, and our exercises here will focus on the first part, which is more elementary. First, we have:

EXERCISE 4.35. *Write down a complete, elementary proof of*

$$S_{M_2}^+ = SO_3$$

*and then unify this with the other such result that we have, namely  $S_3^+ = S_3$ .*

This is something quite tricky, the key word here being “elementary”. Indeed, as we have seen in the above, all this can be understood via heavy representation theory.

As a second exercise now, dealing this time with  $S_4^+$ , we have:

EXERCISE 4.36. *Come up with some explicit constructions for the subgroups of*

$$S_4^+ = SO_3^{-1}$$

*listed in the above, and explain what the associated ADE diagrams should be.*

In short, the exercise here is that of doing the “easy” part of the ADE classification theorem, namely constructing the subgroups, and computing, or at least guessing, the corresponding ADE graphs. In what regards the “hard” part, namely proving the uniqueness, and working out the ADE details, we refer here to the literature.

Finally, in relation now with the quantum groups  $S_F^+$ , we have:

EXERCISE 4.37. *Perform a study of the group dual subgroups*

$$\widehat{\Gamma} \subset S_F^+$$

*in the homogeneous space case, where  $F = M_K \times \{1, \dots, L\}$ .*

We will be back to such subgroup questions later on, when analyzing the quantum subgroups  $G \subset S_F^+$  coming from the finite quantum graphs with vertex set  $F$ .

## Part II

# Analytic aspects

*I'm gonna make me a good sharp axe  
Shining steel tempered in the fire  
I'll chop you down like an old dead tree  
Dirty old town, dirty old town*

## CHAPTER 5

### Laws of characters

#### 5a. Truncated characters

In this second part of this book, this chapter and the next 3 ones, we discuss a number of analytic questions, for the most in relation with free probability. We will be mainly interested in  $S_N, S_N^+$ , but in view of the subtle relationship between  $S_N^+$  and  $O_N^+$ , we will include  $O_N, O_N^+$  as well in our discussion. We will comment on  $S_F^+$  extensions, too.

Let us begin with some character basics. We have the following result:

**THEOREM 5.1.** *Given a Woronowicz algebra  $(A, u)$ , with fundamental corepresentation  $u \in M_N(A)$ , the law of the main character*

$$\chi = \sum_{i=1}^N u_{ii}$$

*with respect to the Haar integration has the following properties:*

- (1) *The moments of  $\chi$  are the numbers  $M_k = \dim(\text{Fix}(u^{\otimes k}))$ .*
- (2)  *$M_k$  counts as well the length  $k$  loops at 1, on the Cayley graph of  $A$ .*
- (3)  *$\text{law}(\chi)$  is the Kesten measure of the associated discrete quantum group.*
- (4) *When  $u \sim \bar{u}$  the law of  $\chi$  is a usual measure, supported on  $[-N, N]$ .*
- (5) *The algebra  $A$  is amenable precisely when  $N \in \text{supp}(\text{law}(\text{Re}(\chi)))$ .*
- (6) *Any morphism  $f : (A, u) \rightarrow (B, v)$  must increase the numbers  $M_k$ .*
- (7) *Such a morphism  $f$  is an isomorphism when  $\text{law}(\chi_u) = \text{law}(\chi_v)$ .*

**PROOF.** All this is very standard, basically coming from the Peter-Weyl theory developed in [98], and explained in chapter 1 above, the idea being as follows:

(1) This comes from the Peter-Weyl type theory, which tells us the number of fixed points of  $v = u^{\otimes k}$  can be recovered by integrating the character  $\chi_v = \chi_u^k$ .

(2) This is something true, and well-known, for  $A = C^*(\Gamma)$ , with  $\Gamma = \langle g_1, \dots, g_N \rangle$  being a discrete group. In general, the proof is quite similar.

(3) This is actually the definition of the Kesten measure, in the case  $A = C^*(\Gamma)$ , with  $\Gamma = \langle g_1, \dots, g_N \rangle$  being a discrete group. In general, this follows from (2).

(4) The equivalence  $u \sim \bar{u}$  translates into  $\chi_u = \chi_u^*$ , and this gives the first assertion. As for the support claim, this follows from  $uu^* = 1 \implies \|u_{ii}\| \leq 1$ , for any  $i$ .

(5) This is the Kesten amenability criterion, which can be established as in the classical case,  $A = C^*(\Gamma)$ , with  $\Gamma = \langle g_1, \dots, g_N \rangle$  being a discrete group.

(6) This is something elementary, which follows from (1) above, and from the fact that the morphisms of Woronowicz algebras increase the spaces of fixed points.

(7) This follows by using (6), and the Peter-Weyl type theory, the idea being that if  $f$  is not injective, then it must strictly increase one of the spaces  $Fix(u^{\otimes k})$ .  $\square$

As a conclusion to this, regardless of our precise motivations, computing the law of  $\chi = \sum_i u_{ii}$  is a central question, and the “main problem” to be solved. Let us also mention, in addition to what has been said above, that in the context where  $G$  acts on some basic physical system, such as a statistical mechanical lattice model, computing the law of  $\chi = \sum_i u_{ii}$  is known to be as well the central question to be solved.

In what regards the quantum rotation and permutation groups, that we are interested in here, we have already solved this problem for them, at  $N = \infty$ , as follows:

**THEOREM 5.2.** *The main character laws for the basic quantum groups are the Poisson, Gaussian, Marchenko-Pastur and Wigner laws  $p_1, g_1, \pi_1, \gamma_1$*

$$\begin{array}{ccc}
 S_N^+ & \longrightarrow & O_N^+ \\
 \uparrow & & \uparrow \\
 S_N & \longrightarrow & O_N
 \end{array}
 \quad : \quad
 \begin{array}{ccc}
 \pi_1 & \longrightarrow & \gamma_1 \\
 \uparrow & & \uparrow \\
 p_1 & \longrightarrow & g_1
 \end{array}$$

in the  $N \rightarrow \infty$  limit. Moreover, the convergence is stationary starting from  $N = 2$  for  $O_N^+$ , starting from  $N = 4$  for  $S_N^+$ , and is not stationary for  $O_N, S_N$ .

**PROOF.** This is something that we know from chapters 1-4, as follows:

(1) For an easy quantum group  $G = (G_N)$ , coming from a category of partitions  $D = (D(k, l))$ , the asymptotic moments of the main character are given by:

$$\begin{aligned}
 \lim_{N \rightarrow \infty} \int_{G_N} \chi^k &= \lim_{N \rightarrow \infty} \dim (Fix(u^{\otimes k})) \\
 &= \lim_{N \rightarrow \infty} \dim \left( span \left( \xi_\pi \mid \pi \in D(k) \right) \right) \\
 &= |D(k)|
 \end{aligned}$$

(2) This result applies to our 4 quantum groups, which are all easy, the corresponding categories of partitions, and asymptotic moments of  $\chi$ , being as follows:

$$\begin{array}{ccc}
 NC & \longleftarrow & NC_2 \\
 \downarrow & & \downarrow \\
 P & \longleftarrow & P_2 \\
 & & : \\
 & & C_k \longleftarrow C_{k/2} \\
 & & \downarrow \quad \downarrow \\
 & & B_k \longleftarrow k!!
 \end{array}$$

But these numbers being the moments of  $p_1, g_1, \pi_1, \gamma_1$ , we obtain the result.

(3) Regarding the stationarity claims, these are more advanced, and come for  $O_N^+, S_N^+$  from the results in chapter 3 above. As for the non-stationarity claims for  $O_N, S_N$ , these come either via direct computations, or from the Kesten amenability criterion.  $\square$

All this is very nice, but the lack of symmetry between the classical and quantum results, in what concerns the stationarity, remains an issue.

As a piece of an answer here, standard free probability theory, to be explained in a moment, based on partitions as above, shows that the measures  $\pi_1, \gamma_1$  are the free analogues of the measures  $p_1, g_1$ . However, at a more advanced level, that of the Bercovici-Pata bijection [40], the correct statement is that the free convolution semigroups  $\{\pi_t\}, \{\gamma_t\}$  are the free analogues of the convolution semigroups  $\{p_t\}, \{g_t\}$ .

All this suggests that, in order to fix things, we need a parameter  $t > 0$ .

In order to do so, the idea will be that of looking at truncated characters of the quantum group, with respect to a parameter  $t \in (0, 1]$ . Let us formulate indeed:

DEFINITION 5.3. *Associated to any Woronowicz algebra  $(A, u)$  are the variables*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

*depending on a parameter  $t \in (0, 1]$ , called truncations of the main character.*

With this definition in hand, we will see that for our basic quantum groups, the asymptotic laws of these truncated characters are respectively  $p_t, g_t, \pi_t, \gamma_t$ , and that the convergence at generic  $t \in (0, 1]$  is not stationary.

Thus, we will have our desired fix for Theorem 5.2. Also, as a bonus, all this will get us into advanced representation theory and free probability, that we will explore later.

In order to understand what the variables  $\chi_t$  are about, let us first investigate the symmetric group  $S_N$ . The result here, which is something very classical, is as follows:

**THEOREM 5.4.** *Consider the symmetric group  $S_N$ , regarded as a compact group of matrices,  $S_N \subset O_N$ , via the standard permutation matrices.*

- (1) *The main character  $\chi \in C(S_N)$ , defined as usual as  $\chi = \sum_i u_{ii}$ , counts the number of fixed points,  $\chi(\sigma) = \#\{i \mid \sigma(i) = i\}$ .*
- (2) *The probability for a permutation  $\sigma \in S_N$  to be a derangement, meaning to have no fixed points at all, becomes, with  $N \rightarrow \infty$ , equal to  $1/e$ .*
- (3) *The law of the main character  $\chi \in C(S_N)$  becomes with  $N \rightarrow \infty$  a Poisson law of parameter 1, with respect to the counting measure.*
- (4) *In fact, the law of any truncated character  $\chi_t = \sum_{i=1}^{[tN]} u_{ii}$  becomes with  $N \rightarrow \infty$  a Poisson law of parameter  $t$ , with respect to the counting measure.*

**PROOF.** This is something very classical, the proof being as follows:

- (1) We have indeed the following computation, which gives the result:

$$\begin{aligned} \chi(\sigma) &= \sum_i u_{ii}(\sigma) \\ &= \sum_i \delta_{\sigma(i)i} \\ &= \#\{i \mid \sigma(i) = i\} \end{aligned}$$

- (2) We use the inclusion-exclusion principle. Consider the following sets:

$$S_N^i = \left\{ \sigma \in S_N \mid \sigma(i) = i \right\}$$

The set of permutations having no fixed points is then:

$$X_N = \left( \bigcup_i S_N^i \right)^c$$

In order to compute  $|X_N|$ , consider as well the following sets:

$$S_N^{i_1 \dots i_k} = \left\{ \sigma \in S_N \mid \sigma(i_1) = i_1, \dots, \sigma(i_k) = i_k \right\}$$

The inclusion-exclusion principle tells us that we have:

$$\begin{aligned} & \left| \left( \bigcup_i S_N^i \right)^c \right| \\ &= |S_N| - \sum_i |S_N^i| + \sum_{i < j} |S_N^i \cap S_N^j| - \dots + (-1)^N \sum_{i_1 < \dots < i_N} |S_N^{i_1} \cup \dots \cup S_N^{i_N}| \\ &= |S_N| - \sum_i |S_N^i| + \sum_{i < j} |S_N^{ij}| - \dots + (-1)^N \sum_{i_1 < \dots < i_N} |S_N^{i_1 \dots i_N}| \end{aligned}$$

Thus, the probability that we are interested in is given by:

$$\mathbb{P}(\chi = 0) = \frac{1}{N!} \left( |S_N| - \sum_i |S_N^i| + \sum_{i < j} |S_N^{ij}| - \dots + (-1)^N \sum_{i_1 < \dots < i_N} |S_N^{i_1 \dots i_N}| \right)$$

Now observe that for any  $i_1 < \dots < i_k$  we have  $|S_N^{i_1 \dots i_k}| = (N - k)!$ . We obtain:

$$\begin{aligned} \mathbb{P}(\chi = 0) &= \frac{1}{N!} \sum_{k=0}^N (-1)^k \sum_{i_1 < \dots < i_k} |S_N^{i_1 \dots i_k}| \\ &= \frac{1}{N!} \sum_{k=0}^N (-1)^k \sum_{i_1 < \dots < i_k} (N - k)! \\ &= \frac{1}{N!} \sum_{k=0}^N (-1)^k \binom{N}{k} (N - k)! \\ &= \sum_{k=0}^N \frac{(-1)^k}{k!} \\ &= 1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^{N-1} \frac{1}{(N-1)!} + (-1)^N \frac{1}{N!} \end{aligned}$$

Since we have here the expansion of  $\frac{1}{e}$ , we conclude that we have, as desired:

$$\lim_{N \rightarrow \infty} \mathbb{P}(\chi = 0) = \frac{1}{e}$$

(3) This follows by generalizing the computation in (2). To be more precise, a similar application of the inclusion-exclusion principle gives the following formula:

$$\lim_{N \rightarrow \infty} \mathbb{P}(\chi = k) = \frac{1}{k!e}$$

Thus, we obtain in the limit a Poisson law of parameter 1, as stated.

(4) As a first observation, and in analogy with the formula in (1) above, the truncated characters count as well certain fixed points, as follows:

$$\begin{aligned} \chi(\sigma) &= \sum_{i=1}^{[tN]} u_{ii}(\sigma) \\ &= \sum_{i=1}^{[tN]} \delta_{\sigma(i)i} \\ &= \# \left\{ i \in \{1, \dots, [tN]\} \mid \sigma(i) = i \right\} \end{aligned}$$



Regarding now the computation of the law of  $\chi_t$ , this follows by generalizing the computation in (3). Indeed, an application of the inclusion-exclusion principle gives:

$$\lim_{N \rightarrow \infty} \mathbb{P}(\chi_t = k) = \frac{t^k}{k!e^t}$$

Thus, we obtain in the limit a Poisson law of parameter  $t$ , as stated.  $\square$

The above result will be something quite fundamental for us, and is worth a second proof, with the remark that in what concerns the case  $t = 1$ , that we already discussed in Theorem 5.2, using easiness, this will be actually a third proof of it. We can use indeed the following integration formula over  $S_N$ , which has its own interest:

**THEOREM 5.5.** *Consider the symmetric group  $S_N$ , with its standard coordinates:*

$$u_{ij} = \chi \left( \sigma \in S_N \mid \sigma(j) = i \right)$$

*The integrals over  $S_N$  are given, modulo linearity, by the formula*

$$\int_{S_N} u_{i_1 j_1} \dots u_{i_k j_k} = \begin{cases} \frac{(N - |\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

*where  $\ker i$  is the partition of  $\{1, \dots, k\}$  whose blocks collect the equal indices of  $i$ .*

**PROOF.** According to the definition of  $u_{ij}$ , the integrals in the statement are given by:

$$\int_{S_N} u_{i_1 j_1} \dots u_{i_k j_k} = \frac{1}{N!} \# \left\{ \sigma \in S_N \mid \sigma(j_1) = i_1, \dots, \sigma(j_k) = i_k \right\}$$

Now observe that the existence of  $\sigma \in S_N$  as above requires:

$$i_m = i_n \iff j_m = j_n$$

Thus, the above integral vanishes when  $\ker i \neq \ker j$ . Regarding now the case  $\ker i = \ker j$ , if we denote by  $b \in \{1, \dots, k\}$  the number of blocks of this partition  $\ker i = \ker j$ , we have  $N - b$  points to be sent bijectively to  $N - b$  points, and so  $(N - b)!$  solutions, and the integral in the statement follows to be  $\frac{(N-b)!}{N!}$ , as claimed.  $\square$

As an illustration for the above formula, we can recover the computation of the asymptotic laws of the truncated characters  $\chi_t$ . We have indeed:

**THEOREM 5.6.** *For the symmetric group  $S_N \subset O_N$ , regarded as a compact group of matrices,  $S_N \subset O_N$ , via the standard permutation matrices, the truncated character*

$$\chi_t(g) = \sum_{i=1}^{[tN]} u_{ii}$$

*counts the number of fixed points among  $\{1, \dots, [tN]\}$ , and its law with respect to the counting measure becomes, with  $N \rightarrow \infty$ , a Poisson law of parameter  $t$ .*

PROOF. The first assertion is something that we already know. For the second assertion, we use the formula in Theorem 5.5. With  $S_{kb}$  being the Stirling numbers, counting the partitions in  $P(k)$  having exactly  $b$  blocks, we have:

$$\begin{aligned} \int_{S_N} \chi_t^k &= \sum_{i_1 \dots i_k=1}^{[tN]} \int_{S_N} u_{i_1 i_1} \dots u_{i_k i_k} \\ &= \sum_{\pi \in P(k)} \frac{[tN]!}{([tN] - |\pi|)!} \cdot \frac{(N - |\pi|)!}{N!} \\ &= \sum_{b=1}^{[tN]} \frac{[tN]!}{([tN] - b)!} \cdot \frac{(N - b)!}{N!} \cdot S_{kb} \end{aligned}$$

In particular with  $N \rightarrow \infty$  we obtain the following formula:

$$\lim_{N \rightarrow \infty} \int_{S_N} \chi_t^k = \sum_{b=1}^k S_{kb} t^b$$

But this is the  $k$ -th moment of the Poisson law  $p_t$ , and so we are done.  $\square$

Summarizing, the truncated characters for  $S_N$  are definitely interesting objects. However, in what regards  $O_N, S_N^+, O_N^+$ , things are quite tricky, and we need a good motivation, coming on top of what we know about  $S_N$ , for getting into computations here.

For this purpose, recall from our comments preceding Definition 5.3 that the need for a parameter  $t > 0$  basically comes from theoretical probability, and more precisely from the classical/free bijection there, at the semigroup level. So, let us explain this now.

### 5b. Free probability

In order to get started, recall that the Gaussian laws  $g_t$  and Poisson laws  $p_t$  appear via the Central Limit Theorem (CLT) and the Poisson Limit Theorem (PLT).

Our first task will be that of explaining these results. The first of them is as follows:

**THEOREM 5.7 (CLT).** *Given a sequence of real variables  $f_1, f_2, f_3, \dots \in L^\infty(X)$  which are i.i.d., centered, and with variance  $t > 0$ , we have, with  $n \rightarrow \infty$ , in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n f_i \sim g_t$$

where  $g_t$  is the Gaussian law of parameter  $t$ , given by:

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

PROOF. This is something very classical, which can be done in two steps:

(1) Our first claim is that the log of the Fourier transform  $F_f(x) = \mathbb{E}(e^{ixf})$  linearizes the convolution. Indeed, assuming that  $f, g \in L^\infty(X)$  are independent, we have:

$$\begin{aligned}
 F_{f+g}(x) &= \int_{\mathbb{R}} e^{ixy} d\mu_{f+g}(y) \\
 &= \int_{\mathbb{R}} e^{ixy} d(\mu_f * \mu_g)(y) \\
 &= \int_{\mathbb{R} \times \mathbb{R}} e^{ix(y+z)} d\mu_f(y) d\mu_g(z) \\
 &= \int_{\mathbb{R}} e^{ixy} d\mu_f(y) \int_{\mathbb{R}} e^{ixz} d\mu_g(z) \\
 &= F_f(x) F_g(x)
 \end{aligned}$$

Thus, we have  $F_{f+g} = F_f F_g$ , which proves our linearization claim.

(2) The Fourier transform of the variable in the statement is:

$$\begin{aligned}
 F(x) &= \left[ F_f \left( \frac{x}{\sqrt{n}} \right) \right]^n \\
 &= \left[ 1 - \frac{tx^2}{2n} + O(n^{-2}) \right]^n \\
 &\simeq e^{-tx^2/2}
 \end{aligned}$$

On the other hand, the Fourier transform of  $g_t$  is given by:

$$\begin{aligned}
 F_{g_t}(x) &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-y^2/2t + ixy} dy \\
 &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-(y/\sqrt{2t} - \sqrt{t/2}ix)^2 - tx^2/2} dy \\
 &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-z^2 - tx^2/2} \sqrt{2t} dz \\
 &= \frac{1}{\sqrt{\pi}} e^{-tx^2/2} \int_{\mathbb{R}} e^{-z^2} dz \\
 &= \frac{1}{\sqrt{\pi}} e^{-tx^2/2} \cdot \sqrt{\pi} \\
 &= e^{-tx^2/2}
 \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Regarding now the Poisson Limit Theorem (PLT), this is as follows:

THEOREM 5.8 (PLT). *We have the following convergence, in moments,*

$$\left( \left( 1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1 \right)^{*n} \rightarrow p_t$$

where  $p_t$  is the Poisson law of parameter  $t > 0$ , given by:

$$p_t = e^{-t} \sum_k \frac{t^k \delta_k}{k!}$$

PROOF. Once again, we use the fact the log of the Fourier transform  $F_f(z) = \mathbb{E}(e^{izf})$  linearizes the convolution. The Fourier transform of the variable in the statement is:

$$\begin{aligned} F(x) &= \lim_{n \rightarrow \infty} \left( \left( 1 - \frac{t}{n} \right) + \frac{t}{n} e^{ix} \right)^n \\ &= \lim_{n \rightarrow \infty} \left( 1 + \frac{(e^{ix} - 1)t}{n} \right)^n \\ &= \exp((e^{ix} - 1)t) \end{aligned}$$

On the other hand, the Fourier transform of  $p_t$  is given by:

$$\begin{aligned} F_{p_t}(x) &= e^{-t} \sum_k \frac{t^k}{k!} F_{\delta_k}(x) \\ &= e^{-t} \sum_k \frac{t^k}{k!} e^{ikx} \\ &= e^{-t} \sum_k \frac{(e^{ix}t)^k}{k!} \\ &= \exp(-t) \exp(e^{ix}t) \\ &= \exp((e^{ix} - 1)t) \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

In order to discuss now the free version of the above results, we first need to talk about moments, laws and freeness, in the general operator algebra setting.

Let us start with the following definition:

DEFINITION 5.9. *Let  $A$  be a  $C^*$ -algebra, given with a trace  $tr$ .*

- (1) *The elements  $a \in A$  are called random variables.*
- (2) *The moments of such a variable are the numbers  $M_k(a) = tr(a^k)$ .*
- (3) *The law of such a variable is the functional  $\mu_a : P \rightarrow tr(P(a))$ .*

Here  $k = \circ \bullet \bullet \circ \dots$  is as usual a colored integer, and the powers  $a^k$  are defined by the following formulae, and multiplicativity:

$$a^\emptyset = 1 \quad , \quad a^\circ = a \quad , \quad a^\bullet = a^*$$

As for the polynomial  $P$ , this is a noncommuting  $*$ -polynomial in one variable:

$$P \in \mathbb{C} \langle X, X^* \rangle$$

Observe that the law is uniquely determined by the moments, because:

$$P(X) = \sum_k \lambda_k X^k \implies \mu_a(P) = \sum_k \lambda_k M_k(a)$$

Let us discuss now the independence, and its noncommutative versions.

As a starting point here, we have the following straightforward definition:

**DEFINITION 5.10.** *We call two subalgebras  $B, C \subset A$  independent when the following condition is satisfied, for any  $b \in B$  and  $c \in C$ :*

$$\text{tr}(bc) = \text{tr}(b)\text{tr}(c)$$

*Equivalently, the following condition must be satisfied, for any  $b \in B$  and  $c \in C$ :*

$$\text{tr}(b) = \text{tr}(c) = 0 \implies \text{tr}(bc) = 0$$

*Also, two variables  $b, c \in A$  are called independent when the algebras that they generate*

$$B = \langle b \rangle, C = \langle c \rangle$$

*are independent inside  $A$ , in the above sense.*

Observe that the above two independence conditions are indeed equivalent. In one sense this is clear, and in the other sense, with  $a' = a - \text{tr}(a)$ , this follows from:

$$\begin{aligned} \text{tr}(bc) &= \text{tr}[(b' + \text{tr}(b))(c' + \text{tr}(c))] \\ &= \text{tr}(b'c') + \text{tr}(b')\text{tr}(c) + \text{tr}(b)\text{tr}(c') + \text{tr}(b)\text{tr}(c) \\ &= \text{tr}(b'c') + \text{tr}(b)\text{tr}(c) \\ &= \text{tr}(b)\text{tr}(c) \end{aligned}$$

It is possible to develop some theory here, but this is ultimately not very interesting, being just an abstract generalization of usual probability theory.

As a much more interesting notion now, coming from [91], and that we will study next, we have the notion of freeness, which can be introduced as follows:

DEFINITION 5.11. We call two subalgebras  $B, C \subset A$  free when the following condition is satisfied, for any  $b_i \in B$  and  $c_i \in C$ :

$$\text{tr}(b_i) = \text{tr}(c_i) = 0 \implies \text{tr}(b_1 c_1 b_2 c_2 \dots) = 0$$

Also, two variables  $b, c \in A$  are called free when the algebras that they generate

$$B = \langle b \rangle, C = \langle c \rangle$$

are free inside  $A$ , in the above sense.

In short, freeness appears by definition as a kind of “free analogue” of independence, taking into account the fact that the variables do not necessarily commute.

As a first observation, of theoretical nature, there is actually a certain lack of symmetry between Definition 5.10 and Definition 5.11, because in contrast to the former, the latter does not include an explicit formula for the quantities of the following type:

$$\text{tr}(b_1 c_1 b_2 c_2 \dots)$$

However, this is not an issue, and is simply due to the fact that the formula in the free case is something more complicated, the result being as follows:

PROPOSITION 5.12. Assuming that  $B, C \subset A$  are free, the restriction of  $\text{tr}$  to  $\langle B, C \rangle$  can be computed in terms of the restrictions of  $\text{tr}$  to  $B, C$ . To be more precise,

$$\text{tr}(b_1 c_1 b_2 c_2 \dots) = P\left(\{\text{tr}(b_{i_1} b_{i_2} \dots)\}_i, \{\text{tr}(c_{j_1} c_{j_2} \dots)\}_j\right)$$

where  $P$  is certain polynomial in several variables, depending on the length of the word  $b_1 c_1 b_2 c_2 \dots$ , and having as variables the traces of products of type

$$b_{i_1} b_{i_2} \dots, \quad c_{j_1} c_{j_2} \dots$$

with the indices being chosen increasing,  $i_1 < i_2 < \dots$  and  $j_1 < j_2 < \dots$

PROOF. This is something quite theoretical, so let us begin with an example. Our claim is that if  $b, c$  are free then, exactly as in the case where we have independence:

$$\text{tr}(bc) = \text{tr}(b)\text{tr}(c)$$

But this follows from the computation performed after Definition 5.10. Indeed, the only non-trivial equality is the last one, which follows from:

$$\text{tr}(b') = \text{tr}(c') = 0 \implies \text{tr}(b'c') = 0$$

In general now, the situation is of course more complicated, but the same trick applies. To be more precise, we can start our computation as follows:

$$\begin{aligned} \text{tr}(b_1 c_1 b_2 c_2 \dots) &= \text{tr}[(b'_1 + \text{tr}(b_1))(c'_1 + \text{tr}(c_1))(b'_2 + \text{tr}(b_2))(c'_2 + \text{tr}(c_2)) \dots] \\ &= \text{tr}(b'_1 c'_1 b'_2 c'_2 \dots) + \text{other terms} \\ &= \text{other terms} \end{aligned}$$

Observe that we have used here the freeness condition, in the following form:

$$\operatorname{tr}(b'_i) = \operatorname{tr}(c'_i) = 0 \implies \operatorname{tr}(b'_1 c'_1 b'_2 c'_2 \dots) = 0$$

Now regarding the “other terms”, those which are left, each of them will consist of a product of traces of type  $\operatorname{tr}(b_i)$  and  $\operatorname{tr}(c_i)$ , and then a trace of a product still remaining to be computed, which is of the following form, with  $\beta_i \in B$  and  $\gamma_i \in C$ :

$$\operatorname{tr}(\beta_1 \gamma_1 \beta_2 \gamma_2 \dots)$$

To be more precise, the variables  $\beta_i \in B$  appear as ordered products of those  $b_i \in B$  not getting into individual traces  $\operatorname{tr}(b_i)$ , and the variables  $\gamma_i \in C$  appear as ordered products of those  $c_i \in C$  not getting into individual traces  $\operatorname{tr}(c_i)$ . Now since the length of each such alternating product  $\beta_1 \gamma_1 \beta_2 \gamma_2 \dots$  is smaller than the length of the original alternating product  $b_1 c_1 b_2 c_2 \dots$ , we are led into a recurrence, and this gives the result.  $\square$

As a basic result now regarding the notions of independence and freeness, and providing us with examples, we have:

**PROPOSITION 5.13.** *We have the following results, valid for group algebras:*

- (1)  $C^*(\Gamma), C^*(\Lambda)$  are independent inside  $C^*(\Gamma \times \Lambda)$ .
- (2)  $C^*(\Gamma), C^*(\Lambda)$  are free inside  $C^*(\Gamma * \Lambda)$ .

**PROOF.** In order to prove these results, we can use the fact that each group algebra is spanned by the corresponding group elements. Thus, it is enough to check the independence and freeness formulae on group elements, and this is in turn trivial.  $\square$

In short, we have now a notion of freeness, dealing with noncommutativity itself, in its most pure form, where there are no algebraic relations at all. This is very nice, but in practice now, we need a free analogue of the Fourier transform, or rather of the log of the Fourier transform. The result here, due to Voiculescu [91], is as follows:

**THEOREM 5.14.** *Given a real probability measure  $\mu$ , consider its Cauchy transform*

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t}$$

*and then define its  $R$ -transform as the solution of the following equation:*

$$G_\mu \left( R_\mu(\xi) + \frac{1}{\xi} \right) = \xi$$

*The free convolution operation is then linearized by the  $R$ -transform.*

**PROOF.** The proof here, which is quite tricky, is in four steps, as follows:

(1) In order to model the free convolution, we can use the algebra of creation operators on the free Fock space  $F(\mathbb{R}^2)$ . This is the same as the semigroup algebra  $C^*(\mathbb{N} * \mathbb{N})$ , and we have some freeness here, a bit in the same way as for group algebras.

(2) In what concerns single distributions, the point here is that the variables of type  $S^* + f(S)$ , with  $S \in C^*(\mathbb{N})$  being the shift, and with  $f \in \mathbb{C}[X]$  being a polynomial, are easily seen to model in moments all the distributions  $\mu : \mathbb{C}[X] \rightarrow \mathbb{C}$ .

(3) Now let  $f, g \in \mathbb{C}[X]$  and consider the variables  $S^* + f(S)$  and  $T^* + g(T)$ , where  $S, T \in C^*(\mathbb{N} * \mathbb{N})$  are the shifts corresponding to the generators of  $\mathbb{N} * \mathbb{N}$ . These variables are free, and by using a 45° argument, their sum has the same law as  $S^* + (f + g)(S)$ .

(4) Thus the operation  $\mu \rightarrow f$  linearizes the free convolution. We are therefore left with a computation inside  $C^*(\mathbb{N})$ , which is standard, and whose conclusion is that  $R_\mu = f$  can be recaptured from  $\mu$  via the Cauchy transform  $G_\mu$ , as in the statement.  $\square$

We refer to [91] or [93] for full details on the above. Now with the above technology in hand, we are ready to state and prove the free CLT, once again following [91]:

**THEOREM 5.15 (FCLT).** *Given noncommutative self-adjoint variables  $x_1, x_2, x_3, \dots \in A$  which are f.i.d., centered, with variance  $t > 0$ , we have, with  $n \rightarrow \infty$ , in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim \gamma_t$$

where  $\gamma_t$  is the Wigner semicircle law of parameter  $t$ , given by:

$$\gamma_t = \frac{1}{2\pi t} \sqrt{4t^2 - x^2} dx$$

**PROOF.** We follow the same idea as in the proof of the CLT, explained before:

(1) At  $t = 1$ , the  $R$ -transform of the variable in the statement can be computed by using the linearization property from Theorem 5.14, and is given by:

$$R(\xi) = nR_x \left( \frac{\xi}{\sqrt{n}} \right) \simeq \xi$$

(2) On the other hand, standard computations show that the Cauchy transform of the Wigner semicircle law  $\gamma_1$  satisfies the following equation:

$$G_{\gamma_1} \left( \xi + \frac{1}{\xi} \right) = \xi$$

Thus we have the following formula, which by the way follows as well from  $S + S^* \sim \gamma_1$ , which is clear from the proof of Theorem 5.14 above:

$$R_{\gamma_1}(\xi) = \xi$$

(3) But this gives the result, and so we are done with the case  $t = 1$ . The passage to the general case,  $t > 0$ , is routine, by dilation.  $\square$

We can state and prove as well a free PLT, as follows:



THEOREM 5.16 (FPLT). *We have the following convergence, in moments,*

$$\left( \left( 1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1 \right)^{\boxplus n} \rightarrow \pi_t$$

*the limiting measure being the Marchenko-Pastur law of parameter  $t > 0$ ,*

$$\pi_t = \max(1 - t, 0) \delta_0 + \frac{\sqrt{4t - (x - 1 - t)^2}}{2\pi x} dx$$

*also called free Poisson law of parameter  $t > 0$ .*

PROOF. Consider the measure in the statement, under the convolution sign:

$$\mu = \left( 1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1$$

The Cauchy transform of this measure is given by:

$$G_\mu(\xi) = \left( 1 - \frac{t}{n} \right) \frac{1}{\xi} + \frac{t}{n} \cdot \frac{1}{\xi - 1}$$

We want to compute the following  $R$ -transform:

$$R = R_{\mu^{\boxplus n}}(y) = nR_\mu(y)$$

By Theorem 5.14, the equation satisfied by  $R$  is as follows:

$$\left( 1 - \frac{t}{n} \right) \frac{1}{y^{-1} + R/n} + \frac{t}{n} \cdot \frac{1}{y^{-1} + R/n - 1} = y$$

By multiplying by  $n/y$ , and rearranging terms, this equation can be written as:

$$\frac{t + yR}{1 + yR/n} = \frac{t}{1 + yR/n - y}$$

With  $n \rightarrow \infty$  the equation simplifies, and we obtain the following formula:

$$t + yR = \frac{t}{1 - y}$$

Thus we have  $R = \frac{t}{1-y}$ , which equals  $R_{\pi_t}$ , and we obtain the result.  $\square$

As a conclusion to all this, let us formulate the following statement:

THEOREM 5.17. *The main limiting results in classical and free probability are*

$$\begin{array}{ccc} \text{FPLT} & \text{---} & \text{FCLT} \\ \downarrow & & \downarrow \\ \text{PLT} & \text{---} & \text{CLT} \end{array} \quad \rightarrow \quad \begin{array}{ccc} \pi_t & \text{---} & \gamma_t \\ \downarrow & & \downarrow \\ p_t & \text{---} & g_t \end{array}$$

*the limiting measures being Gaussian, Poisson, Wigner and Marchenko-Pastur.*

PROOF. This follows indeed by putting together all the above results, classical and free, and with  $g_t, p_t, \gamma_t, \pi_t$  being respectively the measures in the statement.  $\square$

Now back to our permutation and rotation questions, the above result makes a clear connection with our quantum group scheme, from Theorem 5.2 above, namely:

$$\begin{array}{ccc}
 S_N^+ & \longrightarrow & O_N^+ \\
 \uparrow & & \uparrow \\
 S_N & \longrightarrow & O_N
 \end{array}
 \quad : \quad
 \begin{array}{ccc}
 \pi_1 & \longrightarrow & \gamma_1 \\
 \downarrow & & \downarrow \\
 p_1 & \longrightarrow & g_1
 \end{array}$$

In order to get beyond this, and reach to the parameter  $t > 0$ , as suggested by both our quantum group and free probability results, we must do some further probability.

Following [87], given a noncommutative random variable  $a$ , we can define its classical cumulants  $k_n(a)$  and its free cumulants  $\kappa_n(a)$  by the following formulae:

$$\begin{aligned}
 \log F_a(\xi) &= \sum_n k_n(a) \xi^n \\
 R_a(\xi) &= \sum_n \kappa_n(a) \xi^n
 \end{aligned}$$

With this notion in hand, we can define then more general quantities  $k_\pi(a), \kappa_\pi(a)$ , depending on partitions  $\pi \in P(k)$ , by multiplicativity over blocks, and we have:

THEOREM 5.18. *We have the classical and free moment-cumulant formulae*

$$\begin{aligned}
 M_k(a) &= \sum_{\pi \in P(k)} k_\pi(a) \\
 M_k(a) &= \sum_{\pi \in NC(k)} \kappa_\pi(a)
 \end{aligned}$$

where  $k_\pi(a), \kappa_\pi(a)$  are the generalized cumulants and free cumulants of  $a$ .

PROOF. This is standard, by using the formulae of  $F_a, R_a$ , or by doing some direct combinatorics, based on the Möbius inversion formula from chapter 2 above. See [87].  $\square$

Now back to our liberation questions, we can use Theorem 5.18, and the cumulant technology in general, in order to axiomatize the liberation operation in free probability. Indeed, we can say that a random variable  $\alpha$  appears as the free analogue of a random variable  $a$  when the following equalities happen, for any  $n \in \mathbb{N}$ :

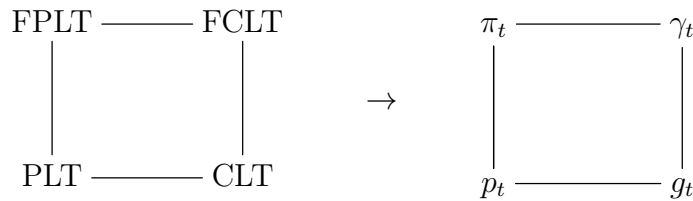
$$k_n(a) = \kappa_n(\alpha)$$

This is something that depends only of the corresponding laws, and in this case we also say that the law  $\mu = \mu_\alpha$  appears as the free analogue of the law  $m = \mu_a$ .

The liberation operation  $m \rightarrow \mu$  constructed in this way is called Bercovici-Pata bijection, and for the full story here, which actually involves semigroups of measures, with the final bijection being something of type  $\{m_t\} \rightarrow \{\mu_t\}$ , we refer to [40].

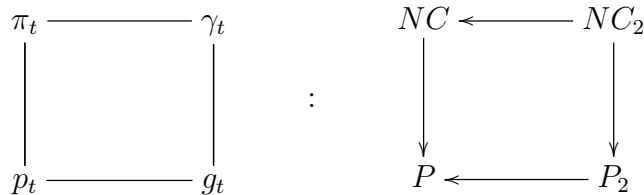
Now with this advanced free probability technology in hand, we can reformulate Theorem 5.17 above in a more conceptual way, as follows:

**THEOREM 5.19.** *The main central limiting results in classical and free probability are as follows, with  $g_t, p_t$  being the Gaussian and Poisson convolution semigroups, and with  $\gamma_t, \pi_t$  being the Wigner and Marchenko-Pastur free convolution semigroups,*



which are related by the Bercovici-Pata bijection, in the sense that “the classical cumulants of the classical measures are equal to the free cumulants of the free measures”.

**PROOF.** We already know the main assertion, from Theorem 5.17, so we just have to discuss the assertions regarding the convolution semigroup properties, the cumulants and the Bercovici-Pata bijection [40]. For this purpose, let us recall from the above that at  $t = 1$  the moments of the limiting measures in the statement appear by counting certain diagrams, according to the following scheme:



The point now is that at  $t > 0$  the moments of the measures in the statement can be recaptured as well from the above diagrams, according to the following formula:

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

Now by putting this into the classical and free cumulant machinery from Theorem 5.18, we obtain the conclusions in the statement, in relation with [40]. See [81].  $\square$

### 5c. Weingarten integration

As a conclusion to the above various probabilistic considerations, we have now a good understanding of the limiting character measures appearing from  $O_N, O_N^+, S_N, S_N^+$ , and with the remark that the presence of a parameter  $t > 0$  would be desirable.

But we already have our parameter  $t > 0$  in the quantum group setting, coming from truncated characters, introduced in Definition 5.3 above:

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

So, following [23], [24], let us discuss now the computation of the law of  $\chi_t$ . In general, and in particular in what regards  $O_N, S_N^+, O_N^+$ , there is no simple trick as for  $S_N$ , and we must use general integration methods, from [23], [52].

First, we have the following well-known formula:

**THEOREM 5.20.** *Assuming that  $A = C(G)$  has Tannakian category  $C = (C(k, l))$ , the Haar integration over  $G$  is given by the Weingarten type formula*

$$\int_G u_{i_1 j_1}^{e_1} \cdots u_{i_k j_k}^{e_k} = \sum_{\pi, \sigma \in D_k} \delta_\pi(i) \delta_\sigma(j) W_k(\pi, \sigma)$$

for any colored integer  $k = e_1 \dots e_k$  and indices  $i, j$ , where  $D_k$  is a linear basis of  $C(\emptyset, k)$ ,

$$\delta_\pi(i) = \langle \pi, e_{i_1} \otimes \dots \otimes e_{i_k} \rangle$$

and  $W_k = G_k^{-1}$ , with  $G_k(\pi, \sigma) = \langle \pi, \sigma \rangle$ .

**PROOF.** We know from chapter 1 above that the integrals in the statement form altogether the orthogonal projection  $P^k$  onto the following space:

$$Fix(u^{\otimes k}) = span(D_k)$$

Consider now the following linear map, with  $D_k = \{\xi_k\}$  being as in the statement:

$$E(x) = \sum_{\pi \in D_k} \langle x, \xi_\pi \rangle \xi_\pi$$

By a standard linear algebra computation, it follows that we have  $P = WE$ , where  $W$  is the inverse on  $span(T_\pi | \pi \in D_k)$  of the restriction of  $E$ . But this restriction is the linear map given by  $G_k$ , and so  $W$  is the linear map given by  $W_k$ , and this gives the result.  $\square$

In the easy quantum group case, the above formula simplifies, as follows:

**THEOREM 5.21.** *For an easy quantum group  $G \subset O_N^+$ , coming from a category of partitions  $D = (D(k, l))$ , we have the Weingarten integration formula*

$$\int_G u_{i_1 j_1} \cdots u_{i_k j_k} = \sum_{\pi, \sigma \in D(k)} \delta_\pi(i) \delta_\sigma(j) W_{kN}(\pi, \sigma)$$

where  $D(k) = D(\emptyset, k)$ ,  $\delta$  are usual Kronecker symbols, and  $W_{kN} = G_{kN}^{-1}$ , with

$$G_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma|}$$

where  $|\cdot|$  is the number of blocks.

PROOF. With notations from Theorem 5.20, the Kronecker symbols are given by:

$$\begin{aligned}\delta_{\xi_\pi}(i) &= \langle \xi_\pi, e_{i_1} \otimes \dots \otimes e_{i_k} \rangle \\ &= \delta_\pi(i_1, \dots, i_k)\end{aligned}$$

The Gram matrix being as well the correct one, we obtain the result. See [23].  $\square$

In practice, the above formula is something very useful, because it allows the study of integration problems over the easy quantum groups just by using a basic laptop. We will present in what follows various applications of this formula, on various occasions.

### 5d. Poisson semigroups

With the above formula in hand, we can go back now to the question of computing the laws of truncated characters. First, we have the following moment formula, from [23]:

THEOREM 5.22. *The moments of truncated characters are given by the formula*

$$\int_G (u_{11} + \dots + u_{ss})^k = \text{Tr}(W_{kN} G_{ks})$$

where  $G_{kN}$  and  $W_{kN} = G_{kN}^{-1}$  are the associated Gram and Weingarten matrices.

PROOF. We have indeed the following computation:

$$\begin{aligned}\int_G (u_{11} + \dots + u_{ss})^k &= \sum_{i_1=1}^s \dots \sum_{i_k=1}^s \int u_{i_1 i_1} \dots u_{i_k i_k} \\ &= \sum_{\pi, \sigma \in D(k)} W_{kN}(\pi, \sigma) \sum_{i_1=1}^s \dots \sum_{i_k=1}^s \delta_\pi(i) \delta_\sigma(i) \\ &= \sum_{\pi, \sigma \in D(k)} W_{kN}(\pi, \sigma) G_{ks}(\sigma, \pi) \\ &= \text{Tr}(W_{kN} G_{ks})\end{aligned}$$

Thus, we have obtained the formula in the statement.  $\square$

In order to process now the above formula, things are quite technical, and won't work well in general. We must impose here a uniformity condition, as follows:

THEOREM 5.23. *For an easy quantum group  $G = (G_N)$ , coming from a category of partitions  $D \subset P$ , the following conditions are equivalent:*

- (1)  $G_{N-1} = G_N \cap U_{N-1}^+$ , via the embedding  $U_{N-1}^+ \subset U_N^+$  given by  $u \rightarrow \text{diag}(u, 1)$ .
- (2)  $G_{N-1} = G_N \cap U_{N-1}^+$ , via the  $N$  possible diagonal embeddings  $U_{N-1}^+ \subset U_N^+$ .
- (3)  $D$  is stable under the operation which consists in removing blocks.

If these conditions are satisfied, we say that  $G = (G_N)$  is “uniform”.

PROOF. We use the general easiness theory from chapter 2 above.

(1)  $\iff$  (2) This is something standard, coming from the inclusion  $S_N \subset G_N$ , which makes everything  $S_N$ -invariant. The result follows as well from the proof of (1)  $\iff$  (3) below, which can be converted into a proof of (2)  $\iff$  (3), in the obvious way.

(1)  $\iff$  (3) Given  $K \subset U_{N-1}^+$ , with fundamental corepresentation  $u$ , consider the  $N \times N$  matrix  $v = \text{diag}(u, 1)$ . Our claim is that for any  $\pi \in P(k)$  we have:

$$\xi_\pi \in \text{Fix}(v^{\otimes k}) \iff \xi_{\pi'} \in \text{Fix}(v^{\otimes k'}), \forall \pi' \in P(k'), \pi' \subset \pi$$

In order to prove this, we must study the condition on the left. We have:

$$\begin{aligned} & \xi_\pi \in \text{Fix}(v^{\otimes k}) \\ \iff & (v^{\otimes k} \xi_\pi)_{i_1 \dots i_k} = (\xi_\pi)_{i_1 \dots i_k}, \forall i \\ \iff & \sum_j (v^{\otimes k})_{i_1 \dots i_k, j_1 \dots j_k} (\xi_\pi)_{j_1 \dots j_k} = (\xi_\pi)_{i_1 \dots i_k}, \forall i \\ \iff & \sum_j \delta_\pi(j_1, \dots, j_k) v_{i_1 j_1} \dots v_{i_k j_k} = \delta_\pi(i_1, \dots, i_k), \forall i \end{aligned}$$

Now let us recall that our corepresentation has the special form  $v = \text{diag}(u, 1)$ . We conclude from this that for any index  $a \in \{1, \dots, k\}$ , we must have:

$$i_a = N \implies j_a = N$$

With this observation in hand, if we denote by  $i', j'$  the multi-indices obtained from  $i, j$  obtained by erasing all the above  $i_a = j_a = N$  values, and by  $k' \leq k$  the common length of these new multi-indices, our condition becomes:

$$\sum_{j'} \delta_\pi(j_1, \dots, j_k) (v^{\otimes k'})_{i' j'} = \delta_\pi(i_1, \dots, i_k), \forall i$$

Here the index  $j$  is by definition obtained from  $j'$  by filling with  $N$  values. In order to finish now, we have two cases, depending on  $i$ , as follows:

Case 1. Assume that the index set  $\{a | i_a = N\}$  corresponds to a certain subpartition  $\pi' \subset \pi$ . In this case, the  $N$  values will not matter, and our formula becomes:

$$\sum_{j'} \delta_\pi(j'_1, \dots, j'_{k'}) (v^{\otimes k'})_{i' j'} = \delta_\pi(i'_1, \dots, i'_{k'})$$

Case 2. Assume now the opposite, namely that the set  $\{a | i_a = N\}$  does not correspond to a subpartition  $\pi' \subset \pi$ . In this case the indices mix, and our formula reads:

$$0 = 0$$

Thus, we are led to  $\xi_{\pi'} \in \text{Fix}(v^{\otimes k'})$ , for any subpartition  $\pi' \subset \pi$ , as claimed. Now with this claim in hand, the result follows from Tannakian duality.  $\square$

By getting back now to the truncated characters, we have the following result:

THEOREM 5.24. *For a uniform easy quantum group  $G = (G_N)$ , we have the formula*

$$\lim_{N \rightarrow \infty} \int_{G_N} \chi_t^k = \sum_{\pi \in D(k)} t^{|\pi|}$$

with  $D \subset P$  being the associated category of partitions.

PROOF. We use the general moment formula for truncated characters from Theorem 5.22 above. With  $s = [tN]$ , this formula becomes:

$$\int_{G_N} \chi_t^k = \text{Tr}(W_{kN} G_{k[tN]})$$

The point now is that in the uniform case the Gram matrix, and so the Weingarten matrix too, are asymptotically diagonal, and so we have formulae as follows:

$$G_{kN} \simeq \text{diag}(N^{|\pi|})$$

$$W_{kN} \simeq \text{diag}(N^{-|\pi|})$$

Thus, the asymptotic moments of the truncated characters are given by:

$$\begin{aligned} \int_{G_N} \chi_t^k &= \text{Tr}(W_{kN} G_{k[tN]}) \\ &\simeq \sum_{\pi \in D(k)} N^{-|\pi|} [tN]^{|\pi|} \\ &\simeq \sum_{\pi \in D(k)} t^{|\pi|} \end{aligned}$$

Thus, we are led to the conclusion in the statement. See [23], [29], [37]. □

We can now improve our quantum group results, as follows:

THEOREM 5.25. *The main truncated character laws for the basic quantum groups are the Poisson, Gaussian, Marchenko-Pastur and Wigner laws  $p_t, g_t, \pi_t, \gamma_t$*

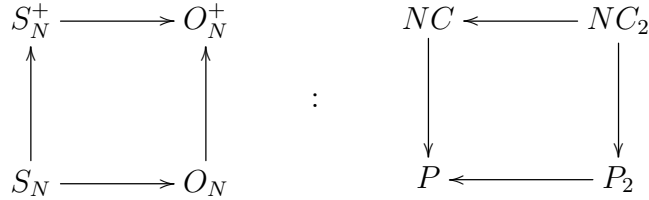
$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array} \quad : \quad \begin{array}{ccc} \pi_t & \longrightarrow & \gamma_t \\ \downarrow & & \downarrow \\ p_t & \longrightarrow & g_t \end{array}$$

in the  $N \rightarrow \infty$  limit. Also, the convergences are not stationary at generic  $t \in (0, 1]$ .

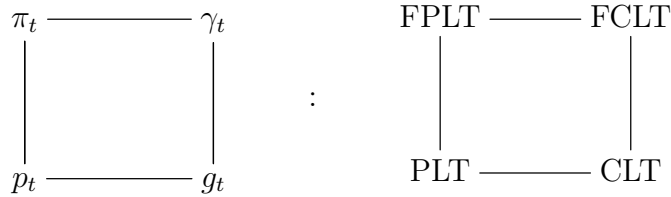
PROOF. This follows indeed from the easiness property of our quantum groups, and from Theorem 5.24, which produces the moments in Theorem 5.19. As for the last assertion, this follows from some standard computations, based on the formulae above. □

As an umbrella result now, summarizing all our knowledge, we have:

**THEOREM 5.26.** *The asymptotic truncated character laws for the basic quantum permutation and rotation groups, which are all easy, as follows,*



are the Poisson, Gaussian, Marchenko-Pastur and Wigner laws  $p_t, g_t, \pi_t, \gamma_t$ , which appear from the main limiting laws in classical and free probability,



and which form semigroups related by the Bercovici-Pata bijection, “the classical cumulants of the classical measures are equal to the free cumulants of the free measures”.

**PROOF.** This follows from Theorem 5.19 and Theorem 5.25, and from the various results leading to them. In fact, the present result summarizes our probabilistic knowledge in the  $N \rightarrow \infty$  limit, with the only things left being the technical stationarity results for  $S_N^+, O_N^+$ , which from the present  $N \rightarrow \infty$  perspective look rather “accidental”.  $\square$

There are many ways of further extending the above results, and for a basic computation here, in the spirit of [59], we refer to [29]. Also, we will be back to this later, in chapters 11-12 below, when doing reflection groups, with the result that the various square diagrams in Theorem 5.26 can be suitably modified, and then completed into cubes.

### 5e. Exercises

We have seen a lot of basic theory and fundamental results in this chapter, and our exercises here will be mostly about this. First, we have:

**EXERCISE 5.27.** *Work out all the details of Theorem 5.1 above, and in particular the claim that any morphism of Woronowicz algebras*

$$f : (A, u) \rightarrow (B, v)$$

*increases the moments of the main character, with  $f$  being an isomorphism precisely when all these moments, or equivalently, the corresponding character laws, coincide.*



As already mentioned before, all this can only come via a straightforward application of the Peter-Weyl theory. The problem is that of working out all the details.

In relation now with permutations and rotations, we have:

EXERCISE 5.28. *Prove that for  $S_N, O_N$ , the convergences*

$$\text{law}(\chi) \rightarrow p_1, g_1$$

*with  $N \rightarrow \infty$ , established above, are not stationary.*

Here both results are actually quite obvious, or at least intuitive, and the problem is that of finding the simplest argument which proves the non-stationarity, in each case.

In relation now with free probability, as a complement to the above, we have:

EXERCISE 5.29. *Formulate and prove complex analogues of the CLT and FCLT, and then prove that the corresponding limiting laws  $G_t, \Gamma_t$  appear as limiting laws for the main truncated characters of  $U_N, U_N^+$ .*

Here in the classical case  $G_t$  can only be the complex normal law, and everything regarding it, meaning both CLT and character theory for  $U_N$ , can only be quite standard. As for the free case, here things are more tricky, and waiting to be discussed, with the resulting measure  $\Gamma_t$  being called the Voiculescu circular law.

In relation now with Weingarten integration, we first have:

EXERCISE 5.30. *Write down the explicit formulae of  $G_{kN}$  and of its inverse*

$$W_{kN} = G_{kN}^{-1}$$

*for the quantum groups  $S_N, O_N, S_N^+, O_N^+$ , at small values of  $k \in \mathbb{N}$ .*

Here the inversion question can be quite tricky, and so once you get to reasonably high values of  $k \in \mathbb{N}$ , better look for estimates of  $W_{kN}$ , and of its entries.

As a second exercise now regarding Weingarten integration, we have:

EXERCISE 5.31. *Work out all the details for the integration formula*

$$\lim_{N \rightarrow \infty} \int_{G_N} \chi_t^k = \sum_{\pi \in D(k)} t^{|\pi|}$$

*in the uniform case.*

This is something that we already discussed in the above, with the needed details concerning the fact that in the uniform case, the Gram matrix, and so the Weingarten matrix too, is asymptotically diagonal. We will be actually back to all this in chapters 7-8 below, when doing some more advanced Weingarten computations.

## CHAPTER 6

### Partial permutations

#### 6a. Partial permutations

We discuss in this chapter an extension of some of the results that we have seen so far, both of algebraic and analytic nature, from the case of the basic quantum permutation and rotation groups, to their “partial semigroup” analogues:

$$\begin{array}{ccc}
 S_N^+ & \longrightarrow & O_N^+ \\
 \uparrow & & \uparrow \\
 S_N & \longrightarrow & O_N
 \end{array}
 \quad \longrightarrow \quad
 \begin{array}{ccc}
 \tilde{S}_N^+ & \longrightarrow & \tilde{O}_N^+ \\
 \uparrow & & \uparrow \\
 \tilde{S}_N & \longrightarrow & \tilde{O}_N
 \end{array}$$

As we will see, on several occasions in what follows, this extension goes beyond the theoretical interest, with the study of the quantum semigroups  $\tilde{G}_N$  on the right often leading to new results about the quantum groups  $G_N$  on the left themselves.

Let us start with a discussion concerning the semigroup  $\tilde{S}_N$  of partial permutations of  $\{1, \dots, N\}$ . This is a quite familiar object in combinatorics, defined as follows:

DEFINITION 6.1.  $\tilde{S}_N$  is the semigroup of partial permutations of  $\{1, \dots, N\}$ ,

$$\tilde{S}_N = \left\{ \sigma : X \simeq Y \mid X, Y \subset \{1, \dots, N\} \right\}$$

with the usual composition operation for such partial permutations, given by:

$$\sigma' \sigma : \sigma^{-1}(X' \cap Y) \rightarrow \sigma'(X' \cap Y)$$

Observe that  $\tilde{S}_N$  is not simplifiable, because the null permutation  $\emptyset \in \tilde{S}_N$ , having the empty set as domain/range, satisfies  $\emptyset \sigma = \sigma \emptyset = \emptyset$ , for any  $\sigma \in \tilde{S}_N$ . Observe also that  $\tilde{S}_N$  has a “subinverse” map, sending  $\sigma : X \rightarrow Y$  to its usual inverse  $\sigma^{-1} : Y \rightarrow X$ .

A first interesting result about this semigroup  $\tilde{S}_N$ , which shows that we are dealing here with some non-trivial combinatorics, is as follows:

PROPOSITION 6.2. *The number of partial permutations is given by*

$$|\tilde{S}_N| = \sum_{k=0}^N k! \binom{N}{k}^2$$

that is, 1, 2, 7, 34, 209, ..., and with  $N \rightarrow \infty$  we have:

$$|\tilde{S}_N| \simeq N! \sqrt{\frac{\exp(4\sqrt{N} - 1)}{4\pi\sqrt{N}}}$$

PROOF. The first assertion is clear, because in order to construct a partial permutation  $\sigma : X \rightarrow Y$  we must choose an integer  $k = |X| = |Y|$ , then we must pick two subsets  $X, Y \subset \{1, \dots, N\}$  having cardinality  $k$ , and there are  $\binom{N}{k}$  choices for each, and finally we must construct a bijection  $\sigma : X \rightarrow Y$ , and there are  $k!$  choices here:

$$\begin{aligned} |\tilde{S}_N| &= \sum_{k=0}^N \#\{X : |X| = k\} \#\{Y : |Y| = k\} \#\{\sigma : X \simeq Y\} \\ &= \sum_{k=0}^N \binom{N}{k} \cdot \binom{N}{k} \cdot k! \\ &= \sum_{k=0}^N k! \binom{N}{k}^2 \end{aligned}$$

As for the estimate, which is non-trivial, this is however something standard, and well-known, and we refer here to the combinatorics literature.  $\square$

Another result, which is trivial, but quite fundamental, is as follows:

PROPOSITION 6.3. *We have a semigroup embedding  $u : \tilde{S}_N \subset M_N(0, 1)$ , defined by*

$$u_{ij}(\sigma) = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

whose image are the matrices having at most one nonzero entry, on each row and column.

PROOF. This is trivial from definitions, with  $u : \tilde{S}_N \subset M_N(0, 1)$  extending the standard embedding  $u : S_N \subset M_N(0, 1)$ , that we have been heavily using, so far.  $\square$

Let us discuss now some probabilistic aspects, related to the Poisson computations in chapter 5. We denote by  $\kappa : \tilde{S}_N \rightarrow \mathbb{N}$  the cardinality of the domain/range, which in terms of the standard coordinates  $u_{ij}$  from Proposition 6.3 is given by:

$$\kappa = \sum_{ij} u_{ij}$$

Let also  $\chi : \tilde{S}_N \rightarrow \mathbb{N}$  be the number of fixed points. As before in terms of the standard coordinates  $u_{ij}$  from Proposition 6.3, this variable is given by:

$$\chi = \sum_i u_{ii}$$

More generally, given a number  $l \leq N$ , we denote by  $\chi_l : \tilde{S}_N \rightarrow \mathbb{N}$  the number of fixed points among  $\{1, \dots, l\}$ . The formula of this variable is:

$$\chi_l = \sum_{i=1}^l u_{ii}$$

Generally speaking, we are interested in computing the joint law of  $(\chi_l, \kappa)$ . There are many interesting questions here, and as a main result on this subject, we have:

**THEOREM 6.4.** *The real probability measures*

$$\mu_k^l = \text{law} \left( \chi_l \mid \kappa = k \right)$$

are given by the formula

$$\mu_k^l = \sum_{q \geq 0} \binom{k}{q} \binom{l}{q} \binom{N}{q}^{-2} \frac{(\delta_1 - \delta_0)^{*q}}{q!}$$

and become Poisson (st) in the  $k = sN, l = tN, N \rightarrow \infty$  limit.

**PROOF.** Observe first that at  $k = l = N$  this result corresponds to the well-known fact that the number of fixed points  $\chi : S_N \rightarrow \mathbb{N}$  becomes Poisson (1), in the  $N \rightarrow \infty$  limit. More generally, at  $k = N$  this corresponds to the fact that the truncated character  $\chi_l : S_N \rightarrow \mathbb{N}$  becomes Poisson ( $t$ ), in the  $l = tN \rightarrow \infty$  limit.

In general, we can use the same method as for  $S_N$ , as explained in chapter 5 above, namely the inclusion-exclusion principle. Let us set indeed:

$$\tilde{S}_N^{(k)} = \left\{ \sigma \in \tilde{S}_N \mid \kappa(\sigma) = k \right\}$$

By inclusion-exclusion, we obtain the following formula:

$$\begin{aligned} & P \left( \chi_l = p \mid \kappa = k \right) \\ &= \frac{1}{|\tilde{S}_N^{(k)}|} \binom{l}{p} \# \left\{ \sigma \in \tilde{S}_{N-p}^{(k-p)} \mid \sigma(i) \neq i, \forall i \leq l-p \right\} \\ &= \frac{1}{|\tilde{S}_N^{(k)}|} \binom{l}{p} \sum_{r \geq 0} (-1)^r \binom{l-p}{r} \left| \tilde{S}_{N-p-r}^{(k-p-r)} \right| \end{aligned}$$

Here the index  $r$ , which counts the fixed points among  $\{1, \dots, l-p\}$ , runs a priori up to  $\min(k, l) - p$ . However, since the binomial coefficient or the cardinality of the set on the right vanishes by definition at  $r > \min(k, l) - p$ , we can sum over  $r \geq 0$ .

We have the following formula:

$$|\tilde{S}_N^{(k)}| = k! \binom{N}{k}^2$$

By using this and then by cancelling various factorials, and grouping back into binomial coefficients, we obtain the following formula:

$$\begin{aligned} & P(\chi_l = p \mid \kappa = k) \\ &= \frac{1}{k! \binom{N}{k}^2} \binom{l}{p} \sum_{r \geq 0} (-1)^r \binom{l-p}{r} (k-p-r)! \binom{N-p-r}{k-p-r}^2 \\ &= \sum_{r \geq 0} \frac{(-1)^r}{p!r!} \binom{k}{p+r} \binom{l}{p+r} \binom{N}{p+r}^{-2} \end{aligned}$$

We can now compute the measure itself. With  $p = q - r$ , we obtain:

$$\begin{aligned} & law(\chi_l \mid \kappa = k) \\ &= \sum_{p \geq 0} \sum_{r \geq 0} \frac{(-1)^r}{p!r!} \binom{k}{p+r} \binom{l}{p+r} \binom{N}{p+r}^{-2} \delta_p \\ &= \sum_{q \geq 0} \sum_{r \geq 0} \frac{(-1)^r}{(q-r)!r!} \binom{k}{q} \binom{l}{q} \binom{N}{q}^{-2} \delta_{q-r} \\ &= \sum_{q \geq 0} \binom{k}{q} \binom{l}{q} \binom{N}{q}^{-2} \cdot \frac{1}{q!} \sum_{r \geq 0} (-1)^r \binom{q}{r} \delta_{q-r} \end{aligned}$$

The sum at right being  $(\delta_1 - \delta_0)^{*q}$ , this gives the formula in the statement.

Regarding now the asymptotics, in the regime  $k = sN, l = tN, N \rightarrow \infty$  from the statement, the coefficient of  $(\delta_1 - \delta_0)^{*q}/q!$  in the formula of  $\mu_k^l$  is:

$$\begin{aligned} c_q &= \binom{k}{q} \binom{l}{q} \binom{N}{q}^{-2} \\ &= \frac{\binom{k}{q}}{\binom{N}{q}} \cdot \frac{\binom{l}{q}}{\binom{N}{q}} \\ &\simeq \left(\frac{k}{N}\right)^q \left(\frac{l}{N}\right)^q \\ &= (st)^q \end{aligned}$$

We deduce that the Fourier transform of  $\mu_k^l$  is given by:

$$\begin{aligned} \mathcal{F}(\mu_k^l)(y) &\simeq \sum_{q \geq 0} (st)^q \frac{(e^y - 1)^q}{q!} \\ &= e^{st(e^y - 1)} \end{aligned}$$

But this is the Fourier transform of Poisson  $(st)$ , and we are done.  $\square$

Observe that the formula in Theorem 6.4 shows in particular that we have the following equality, valid for any integers  $k, l \in \mathbb{N}$ :

$$\mu_k^l = \mu_l^k$$

This is an interesting equality, which seems to be quite unobvious to prove, with bare hands. We will be back to this with more comments, later on.

## 6b. Quantum semigroups

Let us discuss now the construction and main properties of the quantum semigroup of quantum partial permutations  $\tilde{S}_N^+$ , in analogy with the above classical considerations. For this purpose, let us go back to the embedding in Proposition 6.3, namely:

$$u : \tilde{S}_N \subset M_N(0, 1)$$

$$u_{ij}(\sigma) = \delta_{i\sigma(j)}$$

The image of this embedding being formed by the matrices  $\Sigma \in M_N(0, 1)$  having at most one nonzero entry, on each row and column, the matrix  $u = (u_{ij})$  is “submagic”, in the sense that its entries are projections, which are pairwise orthogonal on each row and column. In fact, Gelfand duality shows that we have the following formula:

$$C(\tilde{S}_N) = C_{comm}^* \left( (u_{ij})_{i,j=1,\dots,N} \mid u = \text{submagic} \right)$$

This suggests the following definition, given in [36]:

DEFINITION 6.5.  $C(\tilde{S}_N^+)$  is the universal  $C^*$ -algebra generated by the entries of a  $N \times N$  submagic matrix  $u$ , with comultiplication and counit maps given by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$$

$$\varepsilon(u_{ij}) = \delta_{ij}$$

where “submagic” means formed of projections, which are pairwise orthogonal on rows and columns. We call  $\tilde{S}_N^+$  semigroup of quantum partial permutations of  $\{1, \dots, N\}$ .

Here the fact that the morphisms of algebras  $\Delta, \varepsilon$  as above exist indeed follows from the universality property of  $C(\tilde{S}_N^+)$ , with the needed submagic checks being nearly identical to the magic checks for  $C(S_N^+)$ , from chapter 2 above.

Observe that the morphisms  $\Delta, \varepsilon$  constructed in this way satisfy the usual axioms for a comultiplication and antipode, namely:

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$$

Thus, we have a bialgebra structure of  $C(\tilde{S}_N^+)$ , which tells us that the underlying noncommutative space  $\tilde{S}_N^+$  is a compact quantum semigroup. This semigroup is of quite special type, because  $C(\tilde{S}_N^+)$  has as well a subantipode map, defined by:

$$S(u_{ij}) = u_{ji}$$

To be more precise here, this map exists because the transpose of a submagic matrix is submagic too. As for the subantipode axiom satisfied by it, this is as follows, where  $m^{(3)}$  is the triple multiplication, and  $\Delta^{(2)}$  is the double comultiplication:

$$m^{(3)}(S \otimes id \otimes S)\Delta^{(2)} = S$$

Observe also that  $\Delta, \varepsilon, S$  restrict to  $C(\tilde{S}_N)$ , and correspond there, via Gelfand duality, to the usual multiplication, unit element, and subinversion map of  $\tilde{S}_N$ .

As a conclusion to this discussion, the basic properties of the quantum semigroup  $\tilde{S}_N^+$  that we constructed in Definition 6.5 can be summarized as follows:

PROPOSITION 6.6. *We have maps as follows*

$$\begin{array}{ccc} C(\tilde{S}_N^+) & \rightarrow & C(S_N^+) & & \tilde{S}_N^+ & \supset & S_N^+ \\ \downarrow & & \downarrow & : & \cup & & \cup \\ C(\tilde{S}_N) & \rightarrow & C(S_N) & & \tilde{S}_N & \supset & S_N \end{array}$$

with the bialgebras at left corresponding to the quantum semigroups at right.

PROOF. This is clear from the above discussion, and from the well-known fact that projections which sum up to 1 are pairwise orthogonal. See [36].  $\square$

As a first example, we have  $\tilde{S}_1^+ = \tilde{S}_1$ . At  $N = 2$  now, recall that the algebra generated by two free projections  $p, q$  is isomorphic to the group algebra of  $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ . We denote by  $\varepsilon : C^*(D_\infty) \rightarrow \mathbb{C}1$  the counit map, given by the following formulae:

$$\varepsilon(1) = 1$$

$$\varepsilon(\dots pqpq \dots) = 0$$

With these conventions, we have the following result, from [36]:

PROPOSITION 6.7. *We have an isomorphism*

$$C(\tilde{S}_2^+) \simeq \left\{ (x, y) \in C^*(D_\infty) \oplus C^*(D_\infty) \mid \varepsilon(x) = \varepsilon(y) \right\}$$

which is given by the formula

$$u = \begin{pmatrix} p \oplus 0 & 0 \oplus r \\ 0 \oplus s & q \oplus 0 \end{pmatrix}$$

where  $p, q$  and  $r, s$  are the standard generators of the two copies of  $C^*(D_\infty)$ .

PROOF. Consider an arbitrary  $2 \times 2$  matrix formed by projections:

$$u = \begin{pmatrix} P & R \\ S & Q \end{pmatrix}$$

This matrix is submagic when the following conditions are satisfied:

$$PR = PS = QR = QS = 0$$

Now observe that these conditions tell us that the non-unital algebras  $X = \langle P, Q \rangle$  and  $Y = \langle R, S \rangle$  must commute, and must satisfy  $xy = 0$ , for any  $x \in X, y \in Y$ .

Thus, if we denote by  $Z$  the universal non-unital algebra generated by two projections, we have an isomorphism as follows:

$$C(\tilde{S}_2^+) \simeq \mathbb{C}1 \oplus Z \oplus Z$$

Now since we have  $C^*(D_\infty) = \mathbb{C}1 \oplus Z$ , we obtain an isomorphism as follows:

$$C(\tilde{S}_2^+) \simeq \left\{ (\lambda + a, \lambda + b) \mid \lambda \in \mathbb{C}, a, b \in Z \right\}$$

Thus, we are led to the conclusion in the statement. See [36].  $\square$

Let us extend now to our free setting the classical results. Proposition 6.2 has no free analogue, because  $\tilde{S}_N^+$  is infinite. Proposition 6.3 was already extended, as being part of Definition 6.5. Regarding now Theorem 6.4, we first have:



PROPOSITION 6.8. *The following two elements of  $C(\tilde{S}_N^+)$  are self-adjoint,*

$$\chi = \sum_i u_{ii}, \quad \kappa = \sum_{ij} u_{ij}$$

*satisfy  $0 \leq \chi, \kappa \leq N$ , and coincide with the usual  $\chi, \kappa$  on the quotient  $C(\tilde{S}_N)$ .*

PROOF. All the above assertions are clear from definitions, with the inequalities  $0 \leq \chi, \kappa \leq N$  being taken of course in an operator-theoretic sense.  $\square$

More generally, we can talk about truncations of the variable  $\chi$  constructed above, with respect to a parameter  $l \in \{1, \dots, N\}$ , which are constructed as follows:

$$\chi_l = \sum_{i=1}^l u_{ii}$$

Let us look now at Theorem 6.4. Since  $C(\tilde{S}_N^+)$  has no integration functional, we cannot talk about the joint law of  $(\chi, \kappa)$ . Thus, we need an alternative approach to  $\mu_k^l$ .

For this purpose, we use the following simple observation:

PROPOSITION 6.9. *Any partial permutation  $\sigma : X \simeq Y$  can be factorized as*

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & Y \\ \gamma \downarrow & & \uparrow \alpha \\ \{1, \dots, k\} & \xrightarrow{\beta} & \{1, \dots, k\} \end{array}$$

*with  $\alpha, \beta, \gamma \in S_k$  being certain non-unique permutations, where  $k = \kappa(\sigma)$ .*

PROOF. Since we have  $|X| = |Y| = k$ , in order to reach to the factorization in the statement, we can choose any two bijections  $X \simeq \{1, \dots, k\}$  and  $\{1, \dots, k\} \simeq Y$ , and then complete them up to permutations  $\gamma, \alpha \in S_N$ . The remaining permutation  $\beta \in S_k$  is then uniquely determined by the formula  $\sigma = \alpha\beta\gamma$ .  $\square$

We can now formulate an alternative definition for the measures  $\mu_k^l$ . We fix  $k \leq N$ , and we denote by  $p, q, r$  the magic matrices for  $S_N, S_k, S_N$ . We have:

PROPOSITION 6.10. *Consider the map  $\varphi : S_N \times S_k \times S_N \rightarrow \tilde{S}_N$ , sending  $(\alpha, \beta, \gamma)$  to the partial permutation  $\sigma : \gamma^{-1}\{1, \dots, k\} \simeq \alpha\{1, \dots, k\}$  given by:*

$$\sigma(\gamma^{-1}(t)) = \alpha(\beta(t))$$

- (1) *The image of  $\varphi$  is the set  $\tilde{S}_N^{(k)} = \{\sigma \in \tilde{S}_N \mid \kappa(\sigma) = k\}$ .*
- (2) *The transpose of  $\varphi$  is given by  $\varphi^*(u_{ij}) = \sum_{s,t \leq k} p_{is} \otimes q_{st} \otimes r_{tj}$ .*
- (3)  *$\mu_k^l$  equals the law of the variable  $\varphi^*(\chi_l) \in C(S_N \times S_k \times S_N)$ .*

PROOF. This is an elementary statement, whose proof goes as follows:

(1) Since  $\alpha, \gamma \in S_N$ , the domain and range of the associated element  $\sigma \in \tilde{S}_N$  have indeed cardinality  $k$ . The surjectivity follows from Proposition 6.9 above.

(2) For the element  $\sigma \in \tilde{S}_N$  in the statement, we have:

$$\begin{aligned}
u_{ij}(\sigma) = 1 &\iff \sigma(j) = i \\
&\iff \exists t \leq k, \gamma^{-1}(t) = j, \alpha(\beta(t)) = i \\
&\iff \exists s, t \leq k, \gamma^{-1}(t) = j, \beta(t) = s, \alpha(s) = i \\
&\iff \exists s, t \leq k, r_{tj}(\gamma) = 1, q_{st}(\beta) = 1, p_{is}(\alpha) = 1 \\
&\iff \exists s, t \leq k, (p_{is} \otimes q_{st} \otimes r_{tj})(\alpha, \beta, \gamma) = 1
\end{aligned}$$

Now since the numbers  $s, t \leq k$  are uniquely determined by  $\alpha, \beta, \gamma, i, j$ , if they exist, we conclude that we have the following formula:

$$u_{ij}(\sigma) = \sum_{s, t \leq k} (p_{is} \otimes q_{st} \otimes r_{tj})(\alpha, \beta, \gamma)$$

But this gives the formula in the statement, and we are done.

(3) This comes from the fact that the map  $\varphi_k : S_N \times S_k \times S_N \rightarrow \tilde{S}_N^{(k)}$  obtained by restricting the target of  $\varphi$  commutes with the normalized (mass one) counting measures. At  $k = N$  this follows from the well-known fact that given  $(\alpha, \beta, \gamma) \in S_N \times S_N \times S_N$  random, the product  $\alpha\beta\gamma \in S_N$  is random, and the general case is clear as well.  $\square$

The point now is that we can use the same trick, “ $\sigma = \alpha\beta\gamma$ ”, in the free case. The precise preliminary statement that we will need is as follows:

PROPOSITION 6.11. *Let  $p, q, r$  be the magic matrices for  $S_N^+, S_k^+, S_N^+$ .*

- (1) *The matrix  $U_{ij} = \sum_{s, t \leq k} p_{is} \otimes q_{st} \otimes r_{tj}$  is submagic.*
- (2) *We have a representation  $\pi : C(\tilde{S}_N^+) \rightarrow C(S_N^+ \times S_k^+ \times S_N^+)$ ,  $\pi(u_{ij}) = U_{ij}$ .*
- (3)  *$\pi$  factorizes through the algebra  $C(\tilde{S}_N^{+(k)}) = C(\tilde{S}_N^+) / \langle \kappa = k \rangle$ .*
- (4) *At  $k = N$ , this factorization  $\pi_k$  commutes with the Haar functionals.*

PROOF. Once again, this is an elementary statement, whose proof goes at follows:

(1) By using the fact that  $p, q, r$  are magic, we obtain:

$$\begin{aligned}
U_{ij}U_{il} &= \sum_{s,t \leq k} \sum_{v,w \leq k} p_{is}p_{iv} \otimes q_{st}q_{vw} \otimes r_{tj}r_{wl} \\
&= \sum_{s,t \leq k} \sum_{w \leq k} p_{is} \otimes q_{st}q_{sw} \otimes r_{tj}r_{wl} \\
&= \sum_{s,t \leq k} p_{is} \otimes q_{st} \otimes r_{tj}r_{tl} \\
&= \delta_{jl}U_{ij}
\end{aligned}$$

The proof of  $U_{ij}U_{lj} = \delta_{il}U_{ij}$  is similar, and we conclude that  $U$  is submagic.

(2) This follows from (1), and from the definition of  $C(\tilde{S}_N^+)$ .

(3) By using the fact that  $p, q, r$  are magic, we obtain indeed:

$$\begin{aligned}
\sum_{ij} U_{ij} &= \sum_{ij} \sum_{s,t \leq k} p_{is} \otimes q_{st} \otimes r_{tj} \\
&= \sum_{s,t \leq k} 1 \otimes q_{st} \otimes 1 \\
&= k
\end{aligned}$$

Thus the representation  $\pi$  factorizes indeed through the algebra in the statement.

(4) This is a well-known analogue of the fact that “the product of random permutations is a random permutation”, that we already used before. Here is a representation theory proof, using Peter-Weyl theory. With  $P = Proj(Fix(u^{\otimes n}))$ , we have:

$$\begin{aligned}
&\int_{S_N^+ \times S_N^+ \times S_N^+} U_{i_1 j_1} \cdots U_{i_n j_n} \\
&= \sum_{st} \int_{S_N^+} p_{i_1 s_1} \cdots p_{i_n s_n} \int_{S_N^+} q_{s_1 t_1} \cdots q_{s_n t_n} \int_{S_N^+} r_{t_1 j_1} \cdots r_{t_n j_n} \\
&= \sum_{st} P_{i_1 \dots i_n, s_1 \dots s_n} P_{s_1 \dots s_n, t_1 \dots t_n} P_{t_1 \dots t_n, j_1 \dots j_n} \\
&= (P^3)_{i_1 \dots i_n, j_1 \dots j_n} \\
&= P_{i_1 \dots i_n, j_1 \dots j_n} \\
&= \int_{S_N^+} u_{i_1 j_1} \cdots u_{i_n j_n}
\end{aligned}$$

Thus  $\pi_N$  commutes indeed with the Haar functionals, and we are done.  $\square$

Observe that, since  $\kappa$  is now continuous,  $0 \leq \kappa \leq N$ , the algebras  $C(\tilde{S}_N^{+(k)})$  constructed above don't sum any longer up to the algebra  $C(\tilde{S}_N^+)$  itself. Thus, in a certain sense, the above measures  $\mu_k^l$  encode only a part of the "probabilistic theory" of  $\tilde{S}_N^+$ .

We can however formulate a free analogue of Theorem 6.4, as follows:

**THEOREM 6.12.** *The measures  $\mu_k^l = \text{law}(\pi_k(\chi_l))$ , where  $\pi_k$  is defined as*

$$\pi_k : C(\tilde{S}_N^+) \rightarrow C(S_N^+ \times S_k^+ \times S_N^+)$$

$$u_{ij} \rightarrow \sum_{s,t \leq k} p_{is} \otimes q_{st} \otimes r_{tj}$$

become free Poisson ( $st$ ) in the  $k = sN, l = tN, N \rightarrow \infty$  limit.

**PROOF.** Observe first that at  $k = l = N$  this corresponds to the fact that the law of the main character  $\chi : S_N^+ \rightarrow \mathbb{N}$  becomes free Poisson (1), in the  $N \rightarrow \infty$  limit. Unlike in the classical case, the convergence here is stationary, starting from  $N = 4$ .

More generally, at  $k = N$  this corresponds to the fact that the truncated character  $\chi_l : S_N^+ \rightarrow \mathbb{N}$  becomes free Poisson ( $t$ ), in the  $l = tN \rightarrow \infty$  limit.

In general, we can use the same technique, namely the moment method, and the Weingarten formula. The variable that we are interested in,  $\chi_k^l = \pi_k(\chi_l)$ , is given by:

$$\chi_k^l = \sum_{i \leq l} \sum_{s,t \leq k} p_{is} \otimes q_{st} \otimes r_{ti}$$

By raising to the power  $n$  and integrating, we obtain the following formula:

$$\begin{aligned} & \int_{S_N^+ \times S_k^+ \times S_N^+} (\chi_k^l)^n \\ &= \sum_{i_a \leq l} \sum_{s_a, t_a \leq k} \int_{S_N^+} p_{i_1 s_1} \cdots p_{i_n s_n} \int_{S_k^+} q_{s_1 t_1} \cdots q_{s_n t_n} \int_{S_N^+} r_{t_1 i_1} \cdots r_{t_n i_n} \end{aligned}$$

By using now the Weingarten formula, the above moment is:

$$\begin{aligned}
&= \sum_{ist}^{c_n} \sum_{\alpha \dots \rho \in NC(n)} \delta_\alpha(i) \delta_\beta(s) W_{nN}(\alpha, \beta) \cdot \delta_\gamma(s) \delta_\delta(t) W_{nk}(\gamma, \delta) \cdot \delta_\varepsilon(t) \delta_\rho(i) W_{nN}(\varepsilon, \rho) \\
&= \sum_{\alpha \dots \rho \in NC(n)} W_{nN}(\alpha, \beta) W_{nk}(\gamma, \delta) W_{nN}(\varepsilon, \rho) \sum_{ist} \delta_\alpha(i) \delta_\beta(s) \delta_\gamma(s) \delta_\delta(t) \delta_\varepsilon(t) \delta_\rho(i) \\
&= \sum_{\alpha \dots \rho \in NC(n)} W_{nN}(\alpha, \beta) W_{nk}(\gamma, \delta) W_{nN}(\varepsilon, \rho) \sum_{ist} \delta_{\alpha \vee \rho}(i) \delta_{\beta \vee \gamma}(s) \delta_{\delta \vee \varepsilon}(t) \\
&= \sum_{\alpha \dots \rho \in NC(n)} W_{nN}(\alpha, \beta) W_{nk}(\gamma, \delta) W_{nN}(\varepsilon, \rho) \cdot l^{|\alpha \vee \rho|} k^{|\beta \vee \gamma|} k^{|\delta \vee \varepsilon|}
\end{aligned}$$

Let us examine now the asymptotic regime  $k = sN, l = tN, N \rightarrow \infty$  in the statement. We use here two standard facts from [24], namely the fact that in the  $N \rightarrow \infty$  limit the Gram and Weingarten matrices are concentrated on the diagonal, and the fact that we have the following estimate, with equality when  $\pi = \sigma$ :

$$|\pi \vee \sigma| \leq \frac{|\pi| + |\sigma|}{2}$$

By using this, we obtain, as in [24]:

$$\begin{aligned}
c_n &\simeq \sum_{\alpha, \gamma, \varepsilon \in NC(n)} N^{-|\alpha|} k^{-|\gamma|} N^{-|\varepsilon|} \cdot l^{|\alpha \vee \varepsilon|} k^{|\alpha \vee \gamma|} k^{|\gamma \vee \varepsilon|} \\
&\simeq \sum_{\alpha, \gamma, \varepsilon \in NC(n)} N^{-|\alpha| - |\gamma| - |\varepsilon| + |\alpha \vee \varepsilon| + |\alpha \vee \gamma| + |\gamma \vee \varepsilon|} \cdot s^{-|\gamma| + |\alpha \vee \gamma| + |\gamma \vee \varepsilon|} \cdot t^{|\alpha \vee \varepsilon|} \\
&\simeq \sum_{\alpha \in NC(n)} (st)^{|\alpha|}
\end{aligned}$$

We recognize at right the well-known formula for the moments of the free Poisson law of parameter  $st$ , and this finishes the proof.  $\square$

As a conclusion, with Theorem 6.4 and Theorem 6.12 in hand, and by using the well-known fact that Poisson  $(st) \rightarrow$  free Poisson  $(st)$  is indeed a liberation, in the sense of free probability [40], [93], we can now state that  $\tilde{S}_N \rightarrow \tilde{S}_N^+$  is a “correct” liberation.

As a final comment on this, we have seen in the above that the semigroups  $\tilde{S}_N$  and  $\tilde{S}_N^+$  naturally split into a number of components, which basically appear as homogeneous spaces over products of quantum groups  $S_N$  and  $S_N^+$ . It is possible to be a bit more precise here, and we will be back to this later on, with a number of comments on this.

### 6c. Partial isometries

We discuss now the continuous analogues of the above constructions, involving this time rotations and quantum rotations. Our starting point will be:

DEFINITION 6.13.  $\tilde{O}_N$  is the semigroup of partial linear isometries of  $\mathbb{R}^N$ ,

$$\tilde{O}_N = \left\{ T : A \rightarrow B \text{ isometry} \mid A, B \subset \mathbb{R}^N \right\}$$

with the usual composition operation for such maps, namely:

$$T'T : T^{-1}(A' \cap B) \rightarrow T'(A' \cap B)$$

As a first remark,  $\tilde{O}_N$  is indeed a semigroup, with respect to the operation in the statement, and this is best seen in the matrix model picture, as follows:

PROPOSITION 6.14. We have an embedding  $\tilde{O}_N \subset M_N(\mathbb{R})$ , obtained by completing maps  $T : A \rightarrow B$  into linear maps  $U : \mathbb{R}^N \rightarrow \mathbb{R}^N$ , by setting  $U|_{A^\perp} = 0$ . Moreover:

- (1) This embedding makes  $\tilde{O}_N$  correspond to the set of matrix-theoretic partial isometries, i.e. to the matrices  $U \in M_N(\mathbb{R})$  satisfying:

$$UU^tU = U$$

- (2) The semigroup operation on  $\tilde{O}_N$  corresponds in this way to the semigroup operation for matrix-theoretic partial isometries, namely:

$$U \circ V = U(U^tU \wedge VV^t)V$$

PROOF. All these assertions are well-known, and elementary. For a vector space  $C = A, B$  let indeed  $I_C : C \subset \mathbb{R}^N$  be the inclusion, and  $P_C : \mathbb{R}^N \rightarrow C$  be the projection. The correspondence  $T \leftrightarrow U$  in the statement is then given by:

$$\begin{array}{ccc} A & \xrightarrow{T} & B \\ \uparrow P_A & & \downarrow I_B \\ \mathbb{R}^N & \xrightarrow{U} & \mathbb{R}^N \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{T} & B \\ \downarrow I_A & & \uparrow P_B \\ \mathbb{R}^N & \xrightarrow{U} & \mathbb{R}^N \end{array}$$

The fact that the composition  $U \circ V$  is indeed a partial isometry comes from the fact that the projections  $U^tU$  and  $VV^t$  are absorbed when performing the product:

$$U(U^tU \wedge VV^t)V \cdot V^t(U^tU \wedge VV^t)U^t \cdot U(U^tU \wedge VV^t)V = U(U^tU \wedge VV^t)V$$

Thus, we are led to the conclusions in the statement.  $\square$

In general, the multiplication formula  $U \circ V = U(U^tU \wedge VV^t)V$  in Proposition 6.14 (2), while being quite complicated, is quite unavoidable.

In view of some future liberation purposes, we would need a functional analytic interpretation of it. We have here the following result:

PROPOSITION 6.15.  $C(\tilde{\mathcal{O}}_N)$  is the universal commutative  $C^*$ -algebra generated by the entries of a  $N \times N$  matrix  $u = (u_{ij})$  satisfying the relations

$$u = \bar{u} \quad , \quad uu^t u = u$$

with comultiplication given by the formula

$$(id \otimes \Delta)u = u_{12}(p_{13} \wedge q_{12})u_{13} = \lim_{n \rightarrow \infty} \underbrace{UU^t \dots U^t U}_{2n+1 \text{ terms}}$$

where  $p = uu^t$ ,  $q = u^t u$  and  $U_{ij} = \sum_k u_{ik} \otimes u_{kj}$ .

PROOF. The presentation assertion is standard, by using the Gelfand and Stone-Weierstrass theorems. Let us find now the comultiplication map of  $C(\tilde{\mathcal{O}}_N)$ . For this purpose, consider the following canonical isomorphism:

$$\Phi : C(\tilde{\mathcal{O}}_N) \otimes C(\tilde{\mathcal{O}}_N) \rightarrow C(\tilde{\mathcal{O}}_N \times \tilde{\mathcal{O}}_N)$$

Consider as well the following map:

$$L_{ij}(U, V) = (U \circ V)_{ij}$$

With these conventions, the comultiplication map of  $C(\tilde{\mathcal{O}}_N)$  is given by:

$$\Delta(u_{ij}) = \Phi^{-1}(L_{ij})$$

In order to write now the map  $L_{ij}$  in tensor product form, we can use the following well-known formula:

$$P \wedge Q = \lim_{n \rightarrow \infty} (PQ)^n$$

More precisely, with  $P = VV^t$  and  $Q = U^t U$ , we obtain the following formula:

$$\begin{aligned} (U \circ V)_{ij} &= \sum_{kl} U_{ik}(P \wedge Q)_{kl} V_{lj} \\ &= \lim_{n \rightarrow \infty} \sum_{kl} U_{ik}(PQ)^n_{kl} V_{lj} \end{aligned}$$

With  $a_0 = k$ ,  $a_{2n} = l$ , and by expanding the product, we obtain:

$$\begin{aligned} (U \circ V)_{ij} &= \lim_{n \rightarrow \infty} \sum_{a_0 \dots a_{2n}} U_{ia_0} P_{a_0 a_1} Q_{a_1 a_2} \dots P_{a_{2n-2} a_{2n-1}} Q_{a_{2n-1} a_{2n}} V_{a_{2n} j} \\ &= \lim_{n \rightarrow \infty} \sum_{a_0 \dots a_{2n}} U_{ia_0} Q_{a_1 a_2} \dots Q_{a_{2n-1} a_{2n}} \cdot P_{a_0 a_1} \dots P_{a_{2n-2} a_{2n-1}} V_{a_{2n} j} \end{aligned}$$

Now by getting back to  $\Delta(u_{ij}) = \Phi^{-1}(L_{ij})$ , with  $L_{ij}(U, V) = (U \circ V)_{ij}$ , we conclude that we have the following formula, with  $p = uu^t$  and  $q = u^t u$ :

$$\Delta(u_{ij}) = \lim_{n \rightarrow \infty} \sum_{a_0 \dots a_{2n}} u_{ia_0} q_{a_1 a_2} \dots q_{a_{2n-1} a_{2n}} \otimes p_{a_0 a_1} \dots p_{a_{2n-2} a_{2n-1}} u_{a_{2n} j}$$

Let us expand now both matrix products  $p = uu^t$  and  $q = u^t u$ . In terms of the element  $U_{ij} = \sum_k u_{ik} \otimes u_{kj}$  in the statement, the sum on the right, say  $S_{ij}^{(n)}$ , becomes:

$$\begin{aligned} S_{ij}^{(n)} &= \sum_{a_s} u_{ia_0} (u^t u)_{a_1 a_2} \dots (u^t u)_{a_{2n-1} a_{2n}} \otimes (uu^t)_{a_0 a_1} \dots (uu^t)_{a_{2n-2} a_{2n-1}} u_{a_{2n} j} \\ &= \sum_{a_s b_s c_s} u_{ia_0} u_{b_1 a_1} u_{b_1 a_2} \dots u_{b_n a_{2n-1}} u_{b_n a_{2n}} \otimes u_{a_0 c_1} u_{a_1 c_1} \dots u_{a_{2n-2} c_n} u_{a_{2n-1} c_n} u_{a_{2n} j} \\ &= \sum_{b_s c_s} U_{i c_1} U_{b_1 c_1} U_{b_1 c_2} \dots U_{b_n c_n} U_{b_n j} \\ &= \underbrace{(UU^t \dots U^t U)}_{2n+1 \text{ terms}}_{ij} \end{aligned}$$

Thus we have obtained the second formula in the statement. Regarding now the first formula, observe that we have  $U = u_{12} u_{13}$ . This gives:

$$\begin{aligned} \underbrace{UU^t \dots U^t U}_{2n+1 \text{ terms}} &= (u_{12} u_{13})(u_{13}^t u_{12}^t) \dots (u_{13}^t u_{12}^t)(u_{12} u_{13}) \\ &= u_{12} (u_{13} u_{13}^t) (u_{12}^t u_{12}) \dots (u_{13} u_{13}^t) (u_{12}^t u_{12}) u_{13} \\ &= u_{12} p_{13} q_{12} \dots p_{13} q_{12} u_{13} \end{aligned}$$

Now since the product on the right converges in the  $n \rightarrow \infty$  limit to  $u_{12}(p_{13} \wedge q_{12})u_{13}$ , this gives the first formula in the statement as well, and we are done.  $\square$

Observe that if we further assume that  $u$  is unitary, or that its entries satisfy the condition  $u_{ij}^2 = p_{ij}$  (projection) with  $p = (p_{ij})$  magic, then  $UU^t U = U$ , so the convergence in the formula of  $\Delta$  is stationary, and we obtain  $\Delta(u_{ij}) = U_{ij}$ .

Thus, we can recover in this way the fact that both the inclusions  $O_N \subset \tilde{O}_N \subset M_N(\mathbb{R})$  are semigroup maps, with respect to the usual multiplication of the  $N \times N$  matrices. We will be back to this observation, directly in the free case, in Proposition 6.18 below.

Let us construct now the liberations. We have here the following definition:

DEFINITION 6.16. *To any  $N \in \mathbb{N}$  we associate the following algebra,*

$$C(\tilde{O}_N^+) = C^* \left( (u_{ij})_{i,j=1,\dots,N} \mid u_{ij} = u_{ij}^*, uu^t = p, \text{ projection} \right)$$

*and we call the underlying object  $\tilde{O}_N^+$  space of quantum partial isometries.*



As a first observation, due to the presentation results in Proposition 6.15, we have an inclusions  $\tilde{O}_N \subset \tilde{O}_N^+$ . We have as well a liberated version of Proposition 6.14, or rather of the last assertion there, the rest being already known.

These functoriality statements are best summarized as follows:

PROPOSITION 6.17. *We have embeddings of compact quantum spaces as follows,*

$$\begin{array}{ccc} O_N^+ & \longrightarrow & \tilde{O}_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & \tilde{O}_N \end{array}$$

and the compact quantum spaces on the right produce the compact quantum groups on the left by dividing, at the algebra level, by the relations

$$p = p' = q = q' = 1$$

where  $q = u^*u$  and  $q' = u^t\bar{u}$ , as in Definition 6.16.

PROOF. It follows from definitions that we have embeddings as above. Regarding now the second assertion, in the case of  $\tilde{O}_N^+$ , the relations  $p = p' = q = q' = 1$  read:

$$uu^t = \bar{u}u^t = u^t u = u^t \bar{u} = 1$$

We deduce that both  $u, u^t$  are unitaries, and so when dividing by these relations we obtain the quantum group  $O_N^+$ . As for the result regarding the classical versions, this is clear too, by dividing by the commutation relations  $ab = ba$ .  $\square$

Let us discuss now the multiplicative structure. We have here:

PROPOSITION 6.18.  *$\tilde{O}_N^+$  has a non-associative multiplication given by*

$$(id \otimes \Delta)u = u_{12}(p_{13} \wedge q_{12})u_{13} = \lim_{n \rightarrow \infty} \underbrace{UU^t \dots U^t U}_{2n+1 \text{ terms}}$$

where  $p = uu^t, q = u^t u$  and  $U_{ij} = \sum_k u_{ik} \otimes u_{kj}$ . The embeddings

$$\tilde{O}_N, O_N^+ \subset \tilde{O}_N^+$$

commute with the multiplications.

PROOF. First of all, the equality between the two matrices on the right in the statement follows as in the proof of Proposition 6.15. Let us call  $W = (W_{ij})$  this matrix.

In order to check that  $\Delta(u_{ij}) = W_{ij}$  defines indeed a morphism, we must verify that  $W = (W_{ij})$  satisfies the conditions in Definition 6.16. We have:

$$\begin{aligned} WW^tW &= u_{12}(p_{13} \wedge q_{12})u_{13} \cdot u_{13}^t(p_{13} \wedge q_{12})u_{12}^t \cdot u_{12}(p_{13} \wedge q_{12})u_{13} \\ &= u_{12}(p_{13} \wedge q_{12})p_{13}(p_{13} \wedge q_{12})p_{12}(p_{13} \wedge q_{12})u_{13} \\ &= u_{12}(p_{13} \wedge q_{12})u_{13} = W \end{aligned}$$

Regarding now the last assertion, for the inclusion  $\tilde{O}_N \subset \tilde{O}_N^+$  this is clear. For the inclusion  $O_N^+ \subset \tilde{O}_N^+$  this is clear too, because with  $p = q = 1$  we obtain  $(id \otimes \Delta)u = U$ , which is the usual comultiplication formula for  $C(O_N^+)$ .  $\square$

Let us discuss now probabilistic aspects. We will see that, while our space  $\tilde{O}_N^+$  is not a semigroup, the Bercovici-Pata bijection criterion is satisfied for it.

We use the same method as for  $\tilde{S}_N^+$ , namely a “ $\sigma = \alpha\beta\gamma$ ” type trick. So, pick an exponent  $\circ \in \{\emptyset, +\}$ , set  $\kappa = \sum_{ij} u_{ij}u_{ij}^t$ , and consider the following algebra:

$$C(\tilde{O}_N^{\circ(k)}) = C(\tilde{O}_N^\circ) / \langle \kappa = k \rangle$$

With this convention, we have the following result:

PROPOSITION 6.19. *For any  $\circ \in \{\emptyset, +\}$  we have a representation*

$$\begin{aligned} \pi_k : C(\tilde{O}_N^{\circ(k)}) &\rightarrow C(O_N^\circ \times O_k^\circ \times O_N^\circ) \\ \pi_k(u_{ij}) &= \sum_{s,t \leq k} p_{is} \otimes q_{st} \otimes r_{tj} \end{aligned}$$

which commutes with the Haar functionals at  $k = N$ .

PROOF. In the classical case, the first observation is that any partial isometry  $T : A \rightarrow B$ , with the linear spaces  $A, B \subset \mathbb{R}^N$  having dimension  $\dim(A) = \dim(B) = k$ , decomposes as  $T = UVW$ , with  $U, W \in O_N$  and  $V \in O_k$ :

$$\begin{array}{ccc} A & \xrightarrow{T} & B \\ \downarrow W & & \uparrow U \\ \mathbb{R}^k & \xrightarrow{V} & \mathbb{R}^k \end{array}$$

We conclude that we have a surjection  $\varphi : O_N \times O_k \times O_N \rightarrow \tilde{O}_N^{\circ(k)}$  mapping  $(U, V, W)$  to the partial isometry  $T : W^{-1}(\mathbb{R}^k) \rightarrow U(\mathbb{R}^k)$  given by  $T(W^{-1}x) = U(Vx)$ . By proceeding now as in the proof of Proposition 6.10 (2) above, we see that the transpose map  $\pi = \varphi^*$  is the representation in the statement, and we are done with the classical case.

In the free case, this is a straightforward extension of Proposition 6.11 above. Let us first check that the matrix  $U = (U_{ij})$  formed by the elements appearing on the right in the statement satisfies the partial isometry condition. We have:

$$\begin{aligned}
(UU^tU)_{ij} &= \sum_{kl} U_{ik}U_{lk}U_{lj} \\
&= \sum_{kl} \sum_{s,t \leq k} \sum_{v,w \leq k} \sum_{y,z \leq k} p_{is}p_{lv}p_{ly} \otimes q_{st}q_{vw}q_{yz} \otimes r_{tk}r_{wk}r_{zj} \\
&= \sum_{s,t \leq k} \sum_{y,z \leq k} p_{is} \otimes q_{st}q_{yt}q_{yz} \otimes r_{zj} \\
&= \sum_{s,z \leq k} p_{is} \otimes q_{sz} \otimes r_{zj} \\
&= U_{ij}
\end{aligned}$$

Since  $u_{ij} = u_{ij}^*$  implies  $U_{ij} = U_{ij}^*$ , this proves the partial isometry condition.

Let us check now that the representation that we have just constructed vanishes on the ideal  $\langle \kappa = k \rangle$ . We have:

$$\begin{aligned}
\sum_{ij} U_{ij}U_{ij}^t &= \sum_{ij} \sum_{s,t \leq k} \sum_{v,w \leq k} p_{is}p_{iv} \otimes q_{st}q_{vw} \otimes r_{tj}r_{wj} \\
&= \sum_{s,t \leq k} 1 \otimes q_{st}q_{st} \otimes 1 \\
&= k
\end{aligned}$$

Thus we have a representation  $\pi_k$  as in the statement. Finally, the last assertion is already known, from the proof of Proposition 6.11 (3).  $\square$

With the above result in hand, we can construct variables  $\chi_k^l$  and then real probability measures  $\mu_k^l$  exactly as in the discrete case, in the following way:

$$\begin{aligned}
\chi_k^l &= \pi_k(\chi_l) \\
\mu_k^l &= \text{law}(\chi_k^l)
\end{aligned}$$

With these conventions, we have the following result, which is similar to the analytic liberation result obtained in the above for the liberation operation  $\tilde{S}_N \rightarrow \tilde{S}_N^+$ :

**THEOREM 6.20.** *The partial isometry semigroup liberation operation*

$$\tilde{O}_N \rightarrow \tilde{O}_N^+$$

*is an analytic liberation, in the sense that we have the Bercovici-Pata bijection for*

$$\mu_k^l = \pi_k(\chi_l)$$

*in the  $k = sN, l = tN, N \rightarrow \infty$  limit.*

PROOF. This follows by using standard integration technology, from [23], [37], [52]. More precisely, the Weingarten computation in the proof of Theorem 6.12 above gives the following formula, in the  $k = sN$ ,  $l = tN$ ,  $N \rightarrow \infty$  limit, where  $D(n) \subset P(n)$  denotes the set of partitions associated to the quantum group  $O_N^\circ$  under consideration:

$$\lim_{N \rightarrow \infty} \int_{O_N^\circ \times O_k^\circ \times O_N^\circ} (\chi_k^l)^n = \sum_{\alpha \in D(n)} (st)^{|\alpha|}$$

On the other hand, we know from [23], [37], [52] that the law of the truncated character  $\chi_l$  is given by the following formula, in the  $l = tN$ ,  $N \rightarrow \infty$  limit:

$$\lim_{N \rightarrow \infty} \int_{O_N^\circ} (\chi_l)^n = \sum_{\alpha \in D(n)} t^{|\alpha|}$$

We conclude that in the  $k = sN$ ,  $l = tN$ ,  $N \rightarrow \infty$  limit, we have the following equality of distributions:

$$\lim_{N \rightarrow \infty} \mu_k^l = \lim_{N \rightarrow \infty} \mu_N^{st}$$

Thus, we are led to the conclusion in the statement.  $\square$

With the above results in hand, the same comments as in the discrete case apply. In particular, we have seen that the semigroups  $\tilde{O}_N$  and  $\tilde{O}_N^+$  naturally split into a number of components, which basically appear as homogeneous spaces over products of quantum groups  $O_N$  and  $O_N^+$ . We refer here to the literature on the subject.

## 6d. Configuration spaces

Let us go back now to permutations and quantum permutations, and discuss a number of more specialized questions. Let us start with the following definition:

DEFINITION 6.21. *The diagonal algebra of  $C(S_N)$  is defined as*

$$D(S_N) = \langle u_{11}, \dots, u_{NN} \rangle \subset C(S_N)$$

with  $u_{ij}$  being as usual the standard generators of the algebra  $C(S_N)$ .

As a first observation, the diagonal algebra  $D(S_N)$  constructed above is generated by the characteristic functions  $u_{ii}$  of the following sets:

$$S^i = \left\{ \sigma \in S_N \mid \sigma(i) = i \right\}$$

Our claim now is that we have the following formula:

$$\dim D(S_N) = 2^N - N$$

Indeed, the sets  $S^i$  are in “almost” generic position, leading to a dimension of  $2^N$ , but up to the following constraint, which lowers the dimension by  $N$ :

$$S^{i_1} \cap \dots \cap S^{i_{N-1}} = S^1 \cap \dots \cap S^N$$

We have in fact the following result:

PROPOSITION 6.22. *The “configuration space”  $S_N \rightarrow E_N$  given by*

$$C(E_N) = D(S_N)$$

*appears as follows, where  $T_N = \mathbb{Z}_2$  is the standard real torus, or cube,*

$$E_N = T_N^\circ$$

*and where  $T_N^\circ \subset T_N$  is a certain subspace, obtained by removing  $N$  points.*

PROOF. Since the diagonal coordinates  $u_{ii}$  are commuting projections, we have a quotient map  $C^*(\mathbb{Z}_2^N) \rightarrow D(S_N)$ , obtained via Fourier transform, which at the dual level corresponds to an embedding of spaces  $E_N \subset T_N$ , as indicated in the statement.  $\square$

Let us discuss now the semigroup case. With  $C(\tilde{S}_N) = \langle u_{ij} \rangle$  as before, we can define the diagonal subalgebra of  $C(\tilde{S}_N)$  as follows:

$$D(\tilde{S}_N) = \langle u_{11}, \dots, u_{NN} \rangle \subset C(\tilde{S}_N)$$

This algebra is generated by the characteristic functions  $u_{ii}$  of the following sets:

$$S^i = \left\{ \sigma \in \tilde{S}_N \mid \sigma(i) = i \right\}$$

Since the sets  $S^i$  are now in generic position, we have the following formula:

$$\dim D(\tilde{S}_N) = 2^N$$

We have in fact the following result:

PROPOSITION 6.23. *The “configuration space”  $\tilde{S}_N \rightarrow \tilde{E}_N$  given by*

$$C(\tilde{E}_N) = D(\tilde{S}_N)$$

*appears as the standard real torus, or cube:*

$$\tilde{E}_N = T_N$$

PROOF. As before, we have a quotient map  $C^*(\mathbb{Z}_2^N) \rightarrow D(\tilde{S}_N)$ , which at the dual level corresponds to an embedding of spaces  $\tilde{E}_N \subset T_N$ , which must be an equality.  $\square$

We can perform the same construction in the free case, as follows:

$$D(S_N^+) = \langle u_{11}, \dots, u_{NN} \rangle \subset C(S_N^+)$$

Here the situation changes, and becomes much more interesting, because by [2] the irreducible representations have the same fusion rules as  $SO_3$ , and in particular their characters are polynomials in the main character  $\chi = \sum_i u_{ii}$ . Thus, we have:

$$C(S_N^+)_{\text{central}} \subset D(S_N^+)$$

In relation with tori now, we have the following result:

PROPOSITION 6.24. *The “configuration space”  $S_N^+ \rightarrow E_N^+$  given by*

$$C(E_N^+) = D(S_N^+)$$

*appears as a subspace of the free real torus:*

$$E_N^+ \subset T_N^+$$

PROOF. As before, we have a quotient map  $C^*(\mathbb{Z}_2^{*N}) \rightarrow D(S_N^+)$ , which at the dual level corresponds to an embedding of spaces  $E_N^+ \subset T_N^+$ , as indicated in the statement.  $\square$

Finally, in order to complete the picture, we can perform the same construction in the free semigroup case, in a similar way, as follows:

$$D(\tilde{S}_N^+) = \langle u_{11}, \dots, u_{NN} \rangle \subset C(\tilde{S}_N^+)$$

In relation with tori now, we have the following result:

PROPOSITION 6.25. *The “configuration space”  $\tilde{S}_N^+ \rightarrow \tilde{E}_N^+$  given by*

$$C(\tilde{E}_N^+) = D(\tilde{S}_N^+)$$

*appears as a subspace of the free real torus:*

$$\tilde{E}_N^+ \subset T_N^+$$

PROOF. As before, we have a quotient map  $C^*(\mathbb{Z}_2^{*N}) \rightarrow D(\tilde{S}_N^+)$ , which at the dual level corresponds to an embedding of spaces  $\tilde{E}_N^+ \subset T_N^+$ , as indicated in the statement.  $\square$

As a conclusion to all this, we have a diagram as follows, with all maps being inclusions, and with the vertical inclusions being liberation operations:

$$\begin{array}{ccc} E_N^+ & \longrightarrow & \tilde{E}_N^+ \\ \uparrow & & \uparrow \\ E_N & \longrightarrow & \tilde{E}_N \end{array}$$

This diagram sits inside the diagram formed by the real tori:

$$\begin{array}{ccc} T_N^+ & \longrightarrow & T_N^+ \\ \uparrow & & \uparrow \\ T_N^\circ & \longrightarrow & T_N \end{array}$$

We already know that on the bottom, the two tori there are the correct ones. We will see now that the tori on top are the correct ones at well, except for the torus on top left, which at  $N = 2, 3$  collapses to  $T_N^\circ$ . To be more precise, we have the following result:

**THEOREM 6.26.** *The inclusion  $E_N^+ \subset T_N^+$  is an equality at  $N \geq 4$ .*

**PROOF.** The problem here is that of computing the following functional, and comparing it to the Haar functional of  $T_N^+$ :

$$\varphi : C(T_N^+) \rightarrow C(E_N^+) \subset C(S_N^+) \rightarrow \mathbb{C}$$

But this leads to the conclusion that  $\varphi$  is strictly positive, giving the faithfulness of the quotient map  $C(T_N^+) \rightarrow C(E_N^+)$ .  $\square$

As a consequence of the above result, in what regards the quantum partial permutations, we have  $\tilde{E}_N^+ = T_N^+$  as well at  $N \geq 4$ . This equality holds in fact in general:

**THEOREM 6.27.** *The inclusion  $\tilde{E}_N^+ \subset T_N^+$  is an equality at any  $N \geq 2$ .*

**PROOF.** As already mentioned, Theorem 6.26 shows that the result holds at  $N \geq 4$ . Thus, we are a priori led to a study at  $N = 2, 3$ . However, in view of some further use, later on, we will need to know the formula of the following formal composition:

$$\psi : C(T_N^+) \rightarrow C(\tilde{E}_N^+) \subset C(\tilde{S}_N^+) \rightarrow \mathbb{C}$$

The problem is that the functional on the right is not well-defined, but we can deal with this via an homogeneous space “chopping” of  $\tilde{S}_N^+$ , and we obtain the result.  $\square$

There are several interesting probabilistic questions, in relation with all this, appearing at  $N = \infty$ . Consider indeed the following “diagonal” variables:

$$\chi_S = \sum_{i \in S} u_{ii}$$

Here  $S \subset \mathbb{N}$  is a subset, and in order for things to work, we will assume that  $S$  is “uniform”, in the sense that we have the following formula, for a certain  $t \in (0, 1]$ :

$$\lim_{N \rightarrow \infty} \frac{|S \cap \{1, \dots, N\}|}{N} = t$$

Stricly speaking we have  $\chi_s = \infty$ , so we will be doing something formal here. We start by picking numbers  $a_N > 0$  with  $\sum_N a_N = 1$ , so that we can define a probability measure on  $S_\infty$  by the following formula:

$$\mu_\infty = a_0\mu_0 + a_1\mu_1 + a_2\mu_2 + \dots$$

We have then the following computation, in a formal sense:

$$\begin{aligned}\mathbb{P}_\infty(\chi_S = k) &= \sum_N a_N \mathbb{P}_N(\chi_S = k) \\ &= \sum_N a_N \frac{t^k}{k!} \left(1 - \frac{t}{1!} + \frac{t^2}{2!} + \dots + (-1)^N \frac{t^N}{N!}\right) \\ &= \frac{t^k}{k!} \sum_N (a_N + a_{N+1} + \dots) \frac{(-t)^N}{N!}\end{aligned}$$

Now observe that with  $a_N = x^N(1-x)$  with  $x \in (0, 1)$ , this formula becomes:

$$\begin{aligned}\mathbb{P}_\infty(\chi_S = k) &= \frac{t^k}{k!} \sum_N (1-x)(x^N + x^{N+1} + \dots) \frac{(-t)^N}{N!} \\ &= \frac{t^k}{k!} \sum_N \frac{(-tx)^N}{N!} \\ &= \frac{t^k e^{-tx}}{k!}\end{aligned}$$

With  $x \rightarrow 1$  we therefore obtain the following formula:

$$\lim_{x \rightarrow 1} \mathbb{P}_\infty(\chi_S = k) = \frac{t^k}{k! e^t}$$

Summarizing, we have reached to the formal conclusion that the “variable”  $\chi_S = \infty$  is Poisson( $t$ ) with respect to the following “measure” on  $S_\infty$ :

$$\mu_\infty = \lim_{x \rightarrow 1} \sum_N x^N (1-x) \mu_N$$

In order to further clarify this, let us start with the classical case, that of the group  $S_\infty$ . Here, in order to specify the integration functional, we would like to compute integrals of products of diagonal coordinates  $u_{ii}$ . The problem however is that all these integrals vanish, no matter of our choice for the weights  $a_N > 0$  of the integration:

$$\int_{S_\infty} u_{i_1 i_1} \dots u_{i_k i_k} = 0$$

Thus, things are quite tricky here, and instead of considering such integrals, which are all 0 and with this being the end of the story, we should rather consider integrals of products of sums of variables  $u_{ii}$ , with these sums being suitably “big”, as for the integral to be non-vanishing. But we are led in this way precisely into truncated characters, in the modified sense explained above, with the resulting integrals being:

$$I_{S_1, \dots, S_k} = \int_{S_\infty} \chi_{S_1} \dots \chi_{S_k}$$



To be more precise, here we have  $S_i \subset \mathbb{N}$  for any  $i$ , and with these subsets being “uniform” with corresponding parameters  $t_1, \dots, t_k \in (0, 1]$ , in the following sense:

$$\lim_{N \rightarrow \infty} \frac{|S_i \cap \{1, \dots, N\}|}{N} = t_i$$

But the above integrals are not very far from those computed in above, and so a clarification of the material above, along with a free extension, will lead us not very far from a solution to the key problem of constructing an algebra of type  $D(S_\infty^+)$ .

### 6e. Exercises

Things in this chapter have often been related to various research questions, and our exercises here will be research-level as well. First, we have:

EXERCISE 6.28. *Construct an explicit embedding of type*

$$\tilde{S}_N \subset S_{2N}$$

*and then variables over  $S_{2N}$  extending our variables  $\chi, \kappa$ .*

Here there are of course many possible answers to the first question, but assuming that the answer to this first question is found in the most straightforward way, our variables  $\chi, \kappa : \tilde{S}_N \rightarrow \mathbb{N}$  should then correspond to the variables  $\chi_{left}, \chi_{right} : S_{2N} \rightarrow \mathbb{N}$  counting respectively the number of fixed points in  $\{1, \dots, N\}$ , and in  $\{N + 1, \dots, 2N\}$ .

As a second exercise now, which is something longer, dealing with most of what we have been doing here, both in the discrete and the continuous case, we have:

EXERCISE 6.29. *Reformulate all the probabilistic computations for*

$$\begin{array}{ccc} \tilde{S}_N^+ & \longrightarrow & \tilde{O}_N^+ \\ \uparrow & & \uparrow \\ \tilde{S}_N & \longrightarrow & \tilde{O}_N \end{array}$$

*in terms of suitable homogeneous spaces over the corresponding quantum groups.*

The answer can be actually found in the literature, so the question is that of finding that literature, and making a brief account of it, in the cases that we are interested in.

Finally, in relation with diagonal algebras and configuration spaces, we have:

EXERCISE 6.30. *Develop some theory for the diagonal algebra  $D(S_\infty^+)$ .*

To be more precise here,  $S_\infty^+$  itself is not defined, and the problem is that of talking however about  $D(S_\infty^+)$ , based on the computations did in the above.

## CHAPTER 7

### De Finetti theorems

#### 7a. Invariance questions

An interesting question, which often appears in theoretical probability, as well in connection with various questions coming from physics, is the study of the sequences of random variables  $x_1, x_2, x_3, \dots \in L^\infty(X)$  which are exchangeable, in the sense that their joint distribution is invariant under the permutations  $\sigma \in S_\infty$ :

$$\mu_{x_1, x_2, x_3, \dots} = \mu_{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \dots}$$

This question is solved by the classical De Finetti theorem, which basically says that the variables  $x_1, x_2, x_3, \dots$  must be i.i.d., in some asymptotic sense.

The De Finetti theorem has many generalizations. One can replace for instance the action of the group  $S_\infty = \cup_N S_N$  by the action of the bigger group  $O_\infty = \cup_N O_N$ , and the sequences  $x_1, x_2, x_3, \dots \in L^\infty(X)$  which are invariant in this stronger sense, which are called “rotatable”, can be characterized as well, via a De Finetti type theorem.

All this is interesting for us, in connection with what we have been doing so far, in this book. On one hand the groups  $S_N, O_N$  are easy, and we would like to understand how the above-mentioned De Finetti theorems, involving  $S_N, O_N$ , as well as their various technical generalizations, follow from the easiness property of  $S_N, O_N$ . On the other hand, we would like to understand as well what happens for  $S_N^+, O_N^+$ .

Long story short, we would like to discuss here probabilistic invariance questions with respect to the basic quantum permutation and rotation groups, namely:

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array}$$

We will discuss all this following the classical theory of the De Finetti theorem, then the paper [73], then the paper [55], and then the paper [30].

Let us start by fixing some notations. As we did before, on several occasions, we will use here the formalism of the orthogonal quantum groups, which best covers the 4 main examples that we are interested in. We first have the following definition:

DEFINITION 7.1. *Given a closed subgroup  $G \subset O_N^+$ , we denote by*

$$\alpha : \mathbb{C} \langle t_1, \dots, t_N \rangle \rightarrow \mathbb{C} \langle t_1, \dots, t_N \rangle \otimes C(G)$$

$$t_i \rightarrow \sum_j t_j \otimes u_{ji}$$

*the standard coaction of  $C(G)$  on the free complex algebra on  $N$  variables.*

Observe that the map  $\alpha$  constructed above is indeed a coaction, in the sense that it satisfies the following coassociativity and counitality conditions:

$$(id \otimes \Delta)\alpha = (\alpha \otimes id)\alpha$$

$$(id \otimes \varepsilon)\alpha = id$$

With the above notion of coaction in hand, we can now talk about invariant sequences of random variables, as follows:

DEFINITION 7.2. *Let  $(B, tr)$  be a  $C^*$ -algebra with a trace, and  $x_1, \dots, x_N \in B$ . We say that  $x = (x_1, \dots, x_N)$  is invariant under  $G \subset O_N^+$  if the distribution functional*

$$\mu_x : \mathbb{C} \langle t_1, \dots, t_N \rangle \rightarrow \mathbb{C}$$

$$P \rightarrow tr(P(x_1, \dots, x_N))$$

*is invariant under the coaction  $\alpha$ , in the sense that we have*

$$(\mu_x \otimes id)\alpha(P) = \mu_x(P)$$

*for any noncommuting polynomial  $P \in \mathbb{C} \langle t_1, \dots, t_N \rangle$ .*

In the classical case, where  $G \subset O_N$  is a usual group, we recover in this way the usual invariance notion from classical probability. This is clear indeed from definitions.

We have the following equivalent formulation of the above invariance condition:

PROPOSITION 7.3. *Let  $(B, tr)$  be a  $C^*$ -algebra with a trace, and  $x_1, \dots, x_N \in B$ . Then  $x = (x_1, \dots, x_N)$  is invariant under  $G \subset O_N^+$  precisely when*

$$tr(x_{i_1} \dots x_{i_k}) = \sum_{j_1 \dots j_k} tr(x_{j_1} \dots x_{j_k}) u_{j_1 i_1} \dots u_{j_k i_k}$$

*as an equality in  $C(G)$ , for any  $k \in \mathbb{N}$ , and any  $i_1, \dots, i_k \in \{1, \dots, N\}$ .*

PROOF. By linearity, in order for a sequence  $x = (x_1, \dots, x_N)$  to be  $G$ -invariant in the sense of Definition 7.2 above, the formula there must be satisfied for any noncommuting monomial  $P \in \mathbb{C} \langle t_1, \dots, t_N \rangle$ . But an arbitrary such monomial can be written as follows, for a certain  $k \in \mathbb{N}$ , and certain indices  $i_1, \dots, i_k \in \{1, \dots, N\}$ :

$$P = t_{i_1} \dots t_{i_k}$$

Now with this formula for  $P$ , we have the following computation:

$$\begin{aligned} (\mu_x \otimes id)\alpha(P) &= (\mu_x \otimes id) \sum_{j_1, \dots, j_k} t_{j_1} \dots t_{j_k} \otimes u_{j_1 i_1} \dots u_{j_k i_k} \\ &= \sum_{j_1, \dots, j_k} \mu_x(t_{j_1} \dots t_{j_k}) u_{j_1 i_1} \dots u_{j_k i_k} \\ &= \sum_{j_1 \dots j_k} tr(x_{j_1} \dots x_{j_k}) u_{j_1 i_1} \dots u_{j_k i_k} \end{aligned}$$

On the other hand, by definition of the distribution  $\mu_x$ , we have:

$$\begin{aligned} \mu_x(P) &= \mu_x(t_{i_1} \dots t_{i_k}) \\ &= tr(x_{i_1} \dots x_{i_k}) \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

As already mentioned after Definition 7.2, in the classical case, where  $G \subset O_N$  is a usual compact group, our notion of  $G$ -invariance coincides with the usual  $G$ -invariance notion from classical probability. We have in fact the following result:

PROPOSITION 7.4. *In the classical group case,  $G \subset O_N$ , a sequence  $(x_1, \dots, x_N)$  is  $G$ -invariant in the above sense if and only if*

$$tr(x_{i_1} \dots x_{i_k}) = \sum_{j_1 \dots j_k} g_{j_1 i_1} \dots g_{j_k i_k} tr(x_{j_1} \dots x_{j_k})$$

for any  $k \in \mathbb{N}$ , any  $i_1, \dots, i_k \in \{1, \dots, N\}$ , and any  $g = (g_{ij}) \in G$ , and this coincides with the usual notion of  $G$ -invariance for a sequence of classical random variables.

PROOF. According to Proposition 7.3, the invariance property happens precisely when we have the following equality, for any  $k \in \mathbb{N}$ , and any  $i_1, \dots, i_k \in \{1, \dots, N\}$ :

$$tr(x_{i_1} \dots x_{i_k}) = \sum_{j_1 \dots j_k} tr(x_{j_1} \dots x_{j_k}) u_{j_1 i_1} \dots u_{j_k i_k}$$

Now by evaluating both sides of this equation at a given group element  $g \in G$ , this leads to the conclusion in the statement.  $\square$

Summarizing, we have so far a quite general notion of invariance, generalizing the classical notions of exchangeability and rotatability, than we can use for reformulating the classical De Finetti problematics, and its various generalizations.

In order to formulate De Finetti type theorems, that we can try to prove afterwards, we are still in need of some theory, and of many definitions. Indeed, in the classical De Finetti theorem, the independence occurs after conditioning. Likewise, we can expect the free De Finetti theorem to be a statement about freeness with amalgamation.

Both these concepts may be expressed in terms of operator-valued probability theory, that we will recall now. There are many things to be said here, and in what follows we will mainly present the main definitions and theorems, with some brief explanations.

First, we have the following definition:

DEFINITION 7.5. *An operator-valued probability space consists of:*

- (1) *A unital algebra  $A$ .*
- (2) *A unital subalgebra  $B \subset A$ .*
- (3) *An expectation  $E : A \rightarrow B$ , which must be unital,  $E(1) = 1$ , and satisfying*

$$E(b_1 a b_2) = b_1 E(a) b_2$$

*for any  $a \in A$ , and any  $b_1, b_2 \in B$ .*

Given such an operator-valued probability space, the joint distribution of a family of variables  $(x_i)_{i \in I}$  in the algebra  $A$  is by definition the following functional:

$$\begin{aligned} \mu_x : B \langle (t_i)_{i \in I} \rangle &\rightarrow B \\ P &\rightarrow E(P(x)) \end{aligned}$$

We refer to [87] and related papers for more on all this, general results and examples, in relation with the operator-valued probability theory.

Next in line, we have the following key definition:

DEFINITION 7.6. *Let  $(A, B, E)$  be as above, and  $(x_i)_{i \in I}$  be a family of variables.*

- (1) *These variables are called independent if the following algebra is commutative*

$$\langle B, (x_i)_{i \in I} \rangle \subset A$$

*and for  $i_1, \dots, i_k \in I$  distinct and  $P_1, \dots, P_k \in B \langle t \rangle$ , we have:*

$$E(P_1(x_{i_1}) \dots P_k(x_{i_k})) = E(P_1(x_{i_1})) \dots E(P_k(x_{i_k}))$$

- (2) *These variables are called free if for any  $i_1, \dots, i_k \in I$  such that  $i_l \neq i_{l+1}$ , and any  $P_1, \dots, P_k \in B \langle t \rangle$  such that  $E(P_l(x_{i_l})) = 0$ , we have:*

$$E(P_1(x_{i_1}) \dots P_k(x_{i_k})) = 0$$

The above notions are straightforward extensions of the usual notions of independence and freeness, that we discussed in chapter 5, which correspond to the case  $B = \mathbb{C}$ .

As in the scalar case,  $B = \mathbb{C}$ , in order to deal with invariance questions, we will need the theory of classical and free cumulants, in this setting.

Let us start with the following definition:

DEFINITION 7.7. *Let  $(A, B, E)$  be an operator-valued probability space.*

- (1) *A  $B$ -functional is a  $N$ -linear map  $\rho : A^N \rightarrow B$  such that:*

$$\rho(b_0 a_1 b_1, a_2 b_2 \dots, a_N b_N) = b_0 \rho(a_1, b_1 a_2, \dots, b_{N-1} a_N) b_N$$

*Equivalently,  $\rho$  is a linear map of the following type*

$$A^{\otimes_B N} \rightarrow B$$

*where the tensor product is taken with respect to the natural  $B - B$  bimodule structure on the algebra  $A$ .*

- (2) *Suppose that  $B$  is commutative. For  $k \in \mathbb{N}$  let  $\rho^{(k)}$  be a  $B$ -functional. Given  $\pi \in P(n)$ , we define a  $B$ -functional  $\rho^{(\pi)} : A^N \rightarrow B$  by the formula*

$$\rho^{(\pi)}(a_1, \dots, a_N) = \prod_{V \in \pi} \rho(V)(a_1, \dots, a_N)$$

*where if  $V = (i_1 < \dots < i_s)$  is a block of  $\pi$  then:*

$$\rho(V)(a_1, \dots, a_N) = \rho_s(a_{i_1}, \dots, a_{i_s})$$

As before with the notions of independence and freeness, these are classical extensions of the notions that we discussed in chapter 5 above.

If  $B$  is noncommutative, there is no natural order in which to compute the product appearing in the above formula for  $\rho^{(\pi)}$ .

However, the nesting property of the noncrossing partitions allows for a natural definition of  $\rho^{(\pi)}$  for  $\pi \in NC(N)$ , which we now recall:

DEFINITION 7.8. *For  $k \in \mathbb{N}$  let  $\rho^{(k)} : A^k \rightarrow B$  be a  $B$ -functional. Given  $\pi \in NC(N)$ , define a  $B$ -functional  $\rho^{(N)} : A^N \rightarrow B$  recursively as follows:*

- (1) *If  $\pi = 1_N$  is the partition having one block, define  $\rho^{(\pi)} = \rho^{(N)}$ .*  
 (2) *Otherwise, let  $V = \{l + 1, \dots, l + s\}$  be an interval of  $\pi$  and define:*

$$\rho^{(\pi)}(a_1, \dots, a_N) = \rho^{(\pi-V)}(a_1, \dots, a_l \rho^{(s)}(a_{l+1}, \dots, a_{l+s}), a_{l+s+1}, \dots, a_N)$$

As before, we refer to [81], [87] for more on all this.

Finally, we have the following definition:

DEFINITION 7.9. Let  $(x_i)_{i \in I}$  be a family of random variables in  $A$ .

- (1) The operator-valued classical cumulants  $c_E^{(k)} : A^k \rightarrow B$  are the  $B$ -functionals defined by the following classical moment-cumulant formula:

$$E(a_1 \dots a_N) = \sum_{\pi \in P(N)} c_E^{(\pi)}(a_1, \dots, a_N)$$

- (2) The operator-valued free cumulants  $\kappa_E^{(k)} : A^k \rightarrow B$  are the  $B$ -functionals defined by the following free moment-cumulant formula:

$$E(a_1, \dots, a_N) = \sum_{\pi \in NC(N)} \kappa_E^{(\pi)}(a_1, \dots, a_N)$$

As basic illustrations here, in the scalar case, where the subalgebra is  $B = \mathbb{C}$ , we recover in this way the classical and free cumulants, as discussed in chapter 5 above. In general, we refer to [87] for more on the above notions.

We have the following result, which is well-known in the classical case, and which in the free case is due to Speicher [87]:

THEOREM 7.10. Let  $(x_i)_{i \in I}$  a family of random variables in  $A$ .

- (1) If the algebra  $\langle B, (x_i)_{i \in I} \rangle$  is commutative, then  $(x_i)_{i \in I}$  are conditionally independent given  $B$  if and only if when there are  $1 \leq k, l \leq N$  such that  $i_k \neq i_l$ :

$$c_E^{(N)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

- (2) The variables  $(x_i)_{i \in I}$  are free with amalgamation over  $B$  if and only if when there are  $1 \leq k, l \leq N$  such that  $i_k \neq i_l$ :

$$\kappa_E^{(N)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

Note that the condition in (1) is equivalent to the statement that if  $\pi \in P(N)$ , then the following happens, unless  $\pi \leq \ker i$ :

$$c_E^{(\pi)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

Similarly, the condition (2) is equivalent to the statement that if  $\pi \in NC(N)$ , then the following happens, unless  $\pi \leq \ker i$ :

$$\kappa_E^{(\pi)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

Observe also that in the case  $B = \mathbb{C}$  we obtain the usual notions of independence and freeness. As before, we refer to [81], [87] for more on all this.

Stronger characterizations of the joint distribution of  $(x_i)_{i \in I}$  can be given by specifying what types of partitions may contribute nonzero cumulants.

To be more precise, we have here the following result:

THEOREM 7.11. *Let  $(x_i)_{i \in I}$  be a family of random variables in  $A$ .*

- (1) *Suppose that  $\langle B, (x_i)_{i \in I} \rangle$  is commutative. The  $B$ -valued joint distribution of  $(x_i)_{i \in I}$  is independent for  $D = P$  and independent centered Gaussian for  $D = P_2$  if and only if, for any  $\pi \in P(N)$ , unless  $\pi \in D(N)$  and  $\pi \leq \ker i$ :*

$$c_E^{(\pi)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

- (2) *The  $B$ -valued joint distribution of  $(x_i)_{i \in I}$  is freely independent for  $D = NC$  and freely independent centered semicircular for  $D = NC_2$  if and only if, for any  $\pi \in NC(N)$ , unless  $\pi \in D(N)$  and  $\pi \leq \ker i$ :*

$$\kappa_E^{(\pi)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

PROOF. These results are well-known, coming from the definition of the classical and free cumulants, in the present setting, via some combinatorics.  $\square$

For the detailed proofs, examples and comments on all this, we refer as before to [81], [87]. Finally, here is one more basic result that we will need:

THEOREM 7.12. *Let  $(x_i)_{i \in I}$  be a family of random variables. Define the  $B$ -valued moment functionals  $E^{(N)}$  by the following formula:*

$$E^{(N)}(a_1, \dots, a_N) = E(a_1 \dots a_N)$$

- (1) *If  $B$  is commutative, then for any  $\sigma \in P(N)$  and  $a_1, \dots, a_N \in A$  we have:*

$$c_E^{(\sigma)}(a_1, \dots, a_N) = \sum_{\pi \in P(N), \pi \leq \sigma} \mu_{P(N)}(\pi, \sigma) E^{(\pi)}(a_1, \dots, a_N)$$

- (2) *For any  $\sigma \in NC(N)$  and  $a_1, \dots, a_N \in A$  we have:*

$$\kappa_E^{(\sigma)}(a_1, \dots, a_N) = \sum_{\pi \in NC(N), \pi \leq \sigma} \mu_{NC(N)}(\pi, \sigma) E^{(\pi)}(a_1, \dots, a_N)$$

PROOF. This follows indeed from the Möbius inversion formula.  $\square$

This was the general operator-valued free probability theory that we will need, in what follows. For the detailed proofs, examples and comments on all the above, and more operator-valued free probability in general, we refer to [81], [87].

## 7b. Reverse De Finetti

With the above ingredients in hand, we can now investigate invariance questions for the sequences of random variables, with respect to the main quantum permutation and



rotation groups that we are interested in here, namely:

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array}$$

To be more precise, we can first prove a reverse De Finetti theorem, as follows:

**THEOREM 7.13.** *Let  $(x_1, \dots, x_N)$  be a sequence in  $A$ .*

- (1) *If  $x_1, \dots, x_N$  are freely independent and identically distributed with amalgamation over  $B$ , then the sequence is  $S_N^+$ -invariant.*
- (2) *If  $x_1, \dots, x_N$  are freely independent and identically distributed with amalgamation over  $B$ , and have centered semicircular distributions with respect to  $E$ , then the sequence is  $O_N^+$ -invariant.*
- (3) *If  $\langle B, x_1, \dots, x_N \rangle$  is commutative and  $x_1, \dots, x_N$  are conditionally independent and identically distributed given  $B$ , then the sequence is  $S_N$ -invariant.*
- (4) *If  $\langle x_1, \dots, x_N \rangle$  is commutative and  $x_1, \dots, x_N$  are conditionally independent and identically distributed given  $B$ , and have centered Gaussian distributions with respect to  $E$ , then the sequence is  $O_N$ -invariant.*

**PROOF.** Suppose that the joint distribution of  $(x_1, \dots, x_N)$  satisfies one of the conditions in the statement, and let  $D$  be the partition family associated to the corresponding easy quantum group. We have then the following computation:

$$\begin{aligned} & \sum_{j_1 \dots j_k} \text{tr}(x_{j_1} \dots x_{j_k}) u_{j_1 i_1} \dots u_{j_k i_k} \\ &= \sum_{j_1 \dots j_k} \text{tr}(E(x_{j_1} \dots x_{j_k})) u_{j_1 i_1} \dots u_{j_k i_k} \\ &= \sum_{j_1 \dots j_k} \sum_{\pi \leq \ker j} \text{tr}(\xi_E^{(\pi)}(x_1, \dots, x_1)) u_{j_1 i_1} \dots u_{j_k i_k} \\ &= \sum_{\pi \in D(k)} \text{tr}(\xi_E^{(\pi)}(x_1, \dots, x_1)) \sum_{\ker j \geq \pi} u_{j_1 i_1} \dots u_{j_k i_k} \end{aligned}$$

Here  $\xi$  denotes the free and classical cumulants in the cases (1,2) and (3,4) respectively. On the other hand, it follows from a direct computation that if  $\pi \in D(k)$  then we have the following formula, in each of the 4 cases under investigation:

$$\sum_{\ker j \geq \pi} u_{j_1 i_1} \dots u_{j_k i_k} = \begin{cases} 1 & \text{if } \pi \leq \ker i \\ 0 & \text{otherwise} \end{cases}$$

By using this formula, we can finish our computation, as follows:

$$\begin{aligned}
& \sum_{j_1 \dots j_k} \text{tr}(x_{j_1} \dots x_{j_k}) u_{j_1 i_1} \dots u_{j_k i_k} \\
&= \sum_{\pi \in D(k)} \text{tr}(\xi_E^{(\pi)}(x_1, \dots, x_1)) \delta_{\pi \leq \ker i} \\
&= \sum_{\pi \leq \ker i} \text{tr}(\xi_E^{(\pi)}(x_1, \dots, x_1)) \\
&= \text{tr}(x_{i_1} \dots x_{i_k})
\end{aligned}$$

Thus, we are led to the conclusions in the statement.  $\square$

Summarizing, we have so far a reverse De Finetti theorem, for the quantum groups that we are interested in here. Our goal in what follows will be that of proving the corresponding De Finetti theorems, which are converse to the above theorem.

This will be something quite technical, getting us, among others, into certain technical aspects of the Weingarten integration formula, and of Weingarten combinatorics.

Let us begin with some technical results, in view to establish the above-mentioned converse De Finetti theorems. We will use the following simple fact:

**PROPOSITION 7.14.** *Suppose that a sequence  $(x_1, \dots, x_N)$  is  $G$ -invariant. Then there is a coaction*

$$\tilde{\alpha} : M_N(\mathbb{C}) \rightarrow M_N(\mathbb{C}) \otimes C(G)$$

*determined by the following formula:*

$$\tilde{\alpha}(p(x)) = (ev_x \otimes \pi_N)\alpha(p)$$

*Moreover, the fixed point algebra of  $\tilde{\alpha}$  is the  $G$ -invariant subalgebra  $B_N$ .*

**PROOF.** This follows indeed after identifying the GNS representation of the algebra  $\mathbb{C} \langle t_1, \dots, t_N \rangle$  for the state  $\mu_x$  with the morphism  $ev_x : \mathbb{C} \langle t_1, \dots, t_N \rangle \rightarrow M_N(\mathbb{C})$ .  $\square$

There is a natural conditional expectation given by integrating the coaction  $\tilde{\alpha}$  with respect to the Haar state, as follows:

$$\begin{aligned}
E_N : M_N(\mathbb{C}) &\rightarrow B_N \\
E_N(m) &= \left( id \otimes \int_G \right) \tilde{\alpha}(m)
\end{aligned}$$

The point now is that by using the Weingarten calculus, we can give a simple combinatorial formula for the moment functionals with respect to  $E_N$ , in the case where  $G$  is one of the easy quantum groups under consideration.

To be more precise, we have the following result:

THEOREM 7.15. *Suppose that  $(x_1, \dots, x_N)$  is  $G$ -invariant, and that either  $G = O_N^+, S_N^+$ , or that  $G = O_N, S_N$  and  $(x_1, \dots, x_N)$  commute. We have then*

$$E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k) = \frac{1}{N^{|\pi|}} \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots b x_{i_k} b_k$$

for any  $\pi$  in the partition category  $D(k)$  for  $G$ , and any  $b_0, \dots, b_k \in B_N$ .

PROOF. We prove this by recurrence on the number of blocks of  $\pi$ . First suppose that  $\pi = 1_k$  is the partition with only one block. Then:

$$\begin{aligned} & E_N^{(1_k)}(b_0 x_1 b_1, \dots, x_1 b_k) \\ &= E_N(b_0 x_1 \dots x_1 b_k) \\ &= \sum_{i_1 \dots i_k} b_0 x_{i_1} \dots x_{i_k} b_k \int_G u_{i_1 1} \dots u_{i_k 1} \end{aligned}$$

Here we have used the fact that  $b_0, \dots, b_k$  are fixed by the coaction  $\tilde{\alpha}$ . Applying the Weingarten integration formula, we have:

$$\begin{aligned} & E_N(b_0 x_1 \dots x_1 b_k) \\ &= \sum_{i_1 \dots i_k} b_0 x_{i_1} \dots x_{i_k} b_k \sum_{\pi \leq \ker i} \sum_{\sigma} W_{kN}(\pi, \sigma) \\ &= \sum_{\pi \in D(k)} \left( \sum_{\sigma \in D(k)} W_{kN}(\pi, \sigma) \right) \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \end{aligned}$$

Now observe that for any  $\sigma \in D(k)$  we have:

$$G_{kN}(\sigma, 1_k) = N^{|\sigma \vee 1_k|} = N$$

It follows that for any  $\pi \in D(k)$ , we have:

$$\begin{aligned} & N \sum_{\sigma \in D(k)} W_{kN}(\pi, \sigma) \\ &= \sum_{\sigma \in D(k)} W_{kN}(\pi, \sigma) G_{kN}(\sigma, 1_k) \\ &= \delta_{\pi 1_k} \end{aligned}$$

Applying this above, we find, as desired:

$$\begin{aligned}
& E_N(b_0 x_1 \dots x_1 b_k) \\
&= \sum_{\pi \in D(k)} \frac{1}{N} \delta_{\pi 1_k} \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \\
&= \frac{1}{N} \sum_{i=1}^N b_0 x_i \dots x_i b_k
\end{aligned}$$

If the condition (3) or (4) is satisfied, then the general case follows from:

$$E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k) = b_1 \dots b_k \prod_{V \in \pi} E_N(V)(x_1, \dots, x_1)$$

The one thing we must check here is that if  $\pi \in D(k)$  and  $V$  is a block of  $\pi$  with  $s$  elements, then  $1_s \in D(s)$ . This is easily verified, in each case.

Suppose now that the condition (1) or (2) is satisfied. Let  $\pi \in D(k)$ . Since  $\pi$  is noncrossing,  $\pi$  contains an interval  $V = \{l+1, \dots, l+s+1\}$ . We then have:

$$\begin{aligned}
& E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k) \\
&= E_N^{(\pi-V)}(b_0 x_1 b_1, \dots, E_N(x_1 b_{l+1} \dots x_1 b_{l+s}) x_1, \dots, x_1 b_k)
\end{aligned}$$

To apply induction, we must check that  $\pi - V \in D(k-s)$  and  $1_s \in D(s)$ . Indeed, this is easily verified for  $NC, NC_2$ . Applying induction, we have:

$$\begin{aligned}
& E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k) \\
&= \frac{1}{N^{|\pi|-1}} \sum_{\pi-V \leq \ker i} b_0 x_{i_1} \dots b_l (E_n(x_1 b_{l+1} \dots x_1 b_{l+s})) x_{i_{l+s}} \dots x_{i_k} b_k \\
&= \frac{1}{N^{|\pi|-1}} \sum_{\pi-V \leq \ker i} b_0 x_{i_1} \dots b_l \left( \frac{1}{N} \sum_{i=1}^N x_i b_{l+1} \dots b x_i b_{l+s} \right) x_{i_{l+s}} \dots x_{i_k} b_k \\
&= \frac{1}{N^{|\pi|}} \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k
\end{aligned}$$

This completes the proof. □

Summarizing, we have so far reverse De Finetti theorems for the quantum groups that we are interested in here, along with some technical results, connecting the corresponding potential De Finetti theorems to the Weingarten integration combinatorics.

### 7c. Weingarten estimates

In order to advance, we will need some standard Weingarten estimates for our quantum groups, which have their own interest, and that we will discuss now. So, consider the diagram formed by the main quantum permutation and rotation groups, namely:

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array}$$

Regarding the symmetric group  $S_N$ , the situation here is very simple, because we can explicitly compute the Weingarten function, and then estimate it, as follows:

PROPOSITION 7.16. *For  $S_N$  the Weingarten function is given by*

$$W_{kN}(\pi, \sigma) = \sum_{\tau \leq \pi \wedge \sigma} \mu(\tau, \pi) \mu(\tau, \sigma) \frac{(N - |\tau|)!}{N!}$$

and satisfies the following estimate,

$$W_{kN}(\pi, \sigma) = N^{-|\pi \wedge \sigma|} (\mu(\pi \wedge \sigma, \pi) \mu(\pi \wedge \sigma, \sigma) + O(N^{-1}))$$

with  $\mu$  being the Möbius function of  $P(k)$ .

PROOF. The first assertion follows from the Weingarten formula, namely:

$$\int_{S_N} u_{i_1 j_1} \cdots u_{i_k j_k} = \sum_{\pi, \sigma \in P(k)} \delta_\pi(i) \delta_\sigma(j) W_{kN}(\pi, \sigma)$$

Indeed, in this formula the integrals on the left are known, from the explicit integration formula over  $S_N$  that we established in chapter 5, namely:

$$\int_{S_N} g_{i_1 j_1} \cdots g_{i_k j_k} = \begin{cases} \frac{(N - |\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

But this allows the computation of the right term, via the Möbius inversion formula, explained in chapter 2. As for the second assertion, this follows from the first one.  $\square$

The above result is of course something very special, coming from the fact that the integration over  $S_N$  is something very simple.

Regarding now the quantum group  $S_N^+$ , that we are particularly interested in, let us begin with some explicit computations.

We first have the following simple and final result at  $k = 2, 3$ , directly in terms of the quantum group integrals:

PROPOSITION 7.17. *At  $k = 2, 3$  we have the following estimate:*

$$\int_{S_N^+} u_{i_1 j_1} \cdots u_{i_k j_k} = \begin{cases} 0 & (\ker i \neq \ker j) \\ \simeq N^{-|\ker i|} & (\ker i = \ker j) \end{cases}$$

PROOF. Since at  $k \leq 3$  we have  $NC(k) = P(k)$ , the Weingarten integration formulae for  $S_N^+$  and  $S_N$  coincide, and we obtain, by using the above formula for  $S_N$ :

$$\begin{aligned} \int_{S_N^+} u_{i_1 j_1} \cdots u_{i_k j_k} &= \int_{S_N} u_{i_1 j_1} \cdots u_{i_k j_k} \\ &= \delta_{\ker i, \ker j} \frac{(N - |\ker i|)!}{N!} \end{aligned}$$

But this gives the formula in the statement.  $\square$

In general now, the idea will be that of working out a “master estimate” for the Weingarten function, as above. Before starting here, let us record the formulae at  $k = 2, 3$ , which will be useful, as illustrations. At  $k = 2$ , with indices  $||, \square$ , and with the convention that  $\approx$  means componentwise dominant term, we have:

$$W_{2N} \approx \begin{pmatrix} N^{-2} & -N^{-2} \\ -N^{-2} & N^{-1} \end{pmatrix}$$

At  $k = 3$  now, with indices  $|||, |\square, \square|, \square\square, \square\square$ , and same meaning for  $\approx$ , we have:

$$W_{3N} \approx \begin{pmatrix} N^{-3} & -N^{-3} & -N^{-3} & -N^{-3} & 2N^{-3} \\ -N^{-3} & N^{-2} & N^{-3} & N^{-3} & -N^{-2} \\ -N^{-3} & N^{-3} & N^{-2} & N^{-3} & -N^{-2} \\ -N^{-3} & N^{-3} & N^{-3} & N^{-2} & -N^{-2} \\ 2N^{-3} & -N^{-2} & -N^{-2} & -N^{-2} & N^{-1} \end{pmatrix}$$

These formulae follow indeed from the plain formulae for  $W_{kN}$  at  $k = 2, 3$  from [23], after rearranging the matrix indices as above.

Observe in particular that we have the following formula, which will be of interest in what follows:

$$W_{3N}(|\square, \square|) \simeq N^{-3}$$

In order to deal now with the general case, let us start with:

PROPOSITION 7.18. *The following happen, regarding the partitions in  $P(k)$ :*

- (1)  $|\pi| + |\sigma| \leq |\pi \vee \sigma| + |\pi \wedge \sigma|$ .
- (2)  $|\pi \vee \tau| + |\tau \vee \sigma| \leq |\pi \vee \sigma| + |\tau|$ .
- (3)  $d(\pi, \sigma) = \frac{|\pi| + |\sigma|}{2} - |\pi \vee \sigma|$  is a distance.

PROOF. All this is well-known, the idea being as follows:

(1) This formula is well-known, coming from the fact that  $P(k)$  is a semi-modular lattice.

(2) This follows from (1), as explained for instance in [30].

(3) This follows from (2), which says that the following holds:

$$\begin{aligned} & \frac{|\pi| + |\tau|}{2} - d(\pi, \tau) + \frac{|\tau| + |\sigma|}{2} - d(\tau, \sigma) \\ & \leq \frac{|\pi| + |\sigma|}{2} - d(\pi, \sigma) + |\tau| \end{aligned}$$

Thus, we obtain the triangle inequality:

$$d(\pi, \tau) + d(\tau, \sigma) \geq d(\pi, \sigma)$$

As for the other distance conditions, these are all clear.  $\square$

Actually in what follows we will only need (3) in the above statement. For more on this, and on the geometry and combinatorics of partitions, see [81].

As a main result now regarding the Weingarten matrix, we have:

**THEOREM 7.19.** *The Weingarten matrix  $W_{kN}$  has a series expansion in  $N^{-1}$ ,*

$$W_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma| - |\pi| - |\sigma|} \sum_{g=0}^{\infty} K_g(\pi, \sigma) N^{-g}$$

where the objects on the right are defined as follows:

(1) A path from  $\pi$  to  $\sigma$  is a sequence as follows:

$$p = [\pi = \tau_0 \neq \tau_1 \neq \dots \neq \tau_r = \sigma]$$

(2) The signature of such a path is  $+$  when  $r$  is even, and  $-$  when  $r$  is odd.

(3) The geodesicity defect of such a path is:

$$g(p) = \sum_{i=1}^r d(\tau_{i-1}, \tau_i) - d(\pi, \sigma)$$

(4)  $K_g$  counts the signed paths from  $\pi$  to  $\sigma$ , with geodesicity defect  $g$ .

PROOF. We recall that the Weingarten matrix  $W_{kN}$  appears by definition as the inverse of the Gram matrix  $G_{kN}$ , which is given by the following formula:

$$G_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma|}$$

Now observe that the Gram matrix can be written as follows:

$$\begin{aligned} G_{kN}(\pi, \sigma) &= N^{|\pi \vee \sigma|} \\ &= N^{\frac{|\pi|}{2}} N^{|\pi \vee \sigma| - \frac{|\pi| + |\sigma|}{2}} N^{\frac{|\sigma|}{2}} \\ &= N^{\frac{|\pi|}{2}} N^{-d(\pi, \sigma)} N^{\frac{|\sigma|}{2}} \end{aligned}$$

Consider now the following diagonal matrix:

$$\Delta = \text{diag}(N^{\frac{|\pi|}{2}})$$

Consider as well the following matrix:

$$H(\pi, \sigma) = \begin{cases} 0 & (\pi = \sigma) \\ N^{-d(\pi, \sigma)} & (\pi \neq \sigma) \end{cases}$$

In terms of these two matrices, the above formula simply reads:

$$G_{kN} = \Delta(1 + H)\Delta$$

Thus, the Weingarten matrix is given by the following formula:

$$W_{kN} = \Delta^{-1}(1 + H)^{-1}\Delta^{-1}$$

In order to compute the inverse of  $1 + H$ , we will use the following formula:

$$(1 + H)^{-1} = 1 - H + H^2 - H^3 + \dots$$

Consider indeed the set  $P_r(\pi, \sigma)$  of length  $r$  paths between  $\pi$  and  $\sigma$ . The powers of  $H$  are then given by:

$$\begin{aligned} H^r(\pi, \sigma) &= \sum_{p \in P_r(\pi, \sigma)} H(\tau_0, \tau_1) \dots H(\tau_{r-1}, \tau_r) \\ &= \sum_{p \in P_r(\pi, \sigma)} N^{-d(\pi, \sigma) - g(p)} \end{aligned}$$

Thus by using the formula  $(1 + H)^{-1} = 1 - H + H^2 - H^3 + \dots$  we obtain:

$$\begin{aligned} (1 + H)^{-1}(\pi, \sigma) &= \sum_{r=0}^{\infty} (-1)^r H^r(\pi, \sigma) \\ &= N^{-d(\pi, \sigma)} \sum_{r=0}^{\infty} \sum_{p \in P_r(\pi, \sigma)} (-1)^r N^{-g(p)} \end{aligned}$$



It follows that the Weingarten matrix is given by:

$$\begin{aligned}
W_{kN}(\pi, \sigma) &= \Delta^{-1}(\pi)(1 + H)^{-1}(\pi, \sigma)\Delta^{-1}(\sigma) \\
&= N^{-\frac{|\pi|}{2} - \frac{|\sigma|}{2} - d(\pi, \sigma)} \sum_{r=0}^{\infty} \sum_{p \in P_r(\pi, \sigma)} (-1)^r N^{-g(p)} \\
&= N^{|\pi \vee \sigma| - |\pi| - |\sigma|} \sum_{r=0}^{\infty} \sum_{p \in P_r(\pi, \sigma)} (-1)^r N^{-g(p)}
\end{aligned}$$

Now by rearranging the various terms of the above double sum according to their geodesicity defect  $g = g(p)$ , this gives the following formula:

$$W_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma| - |\pi| - |\sigma|} \sum_{g=0}^{\infty} K_g(\pi, \sigma) N^{-g}$$

Thus, we are led to the conclusion in the statement.  $\square$

As an illustration, we have the following explicit estimates:

**THEOREM 7.20.** *Consider an easy quantum group  $G = (G_N)$ , coming from a category of partitions  $D = (D(k))$ . For any  $\pi \leq \sigma$  we have the estimate*

$$W_{kN}(\pi, \sigma) = N^{-|\pi|}(\mu(\pi, \sigma) + O(N^{-1}))$$

and for  $\pi, \sigma$  arbitrary we have

$$W_{kN}(\pi, \sigma) = O(N^{|\pi \vee \sigma| - |\pi| - |\sigma|})$$

with  $\mu$  being the Möbius function of  $D(k)$ .

**PROOF.** We have two assertions here, the idea being as follows:

(1) The first estimate is clear from the general formula established in Theorem 7.19 above, namely:

$$W_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma| - |\pi| - |\sigma|} \sum_{g=0}^{\infty} K_g(\pi, \sigma) N^{-g}$$

(2) In the case  $\pi \leq \sigma$  it is known that  $K_0$  coincides with the Möbius function of  $NC(k)$ , as explained for instance in [30], so we obtain once again from Theorem 7.19 the fine estimate as well, namely:

$$W_{kN}(\pi, \sigma) = N^{-|\pi|}(\mu(\pi, \sigma) + O(N^{-1})) \quad \forall \pi \leq \sigma$$

Observe that, by symmetry of  $W_{kN}$ , we obtain as well the following estimate:

$$W_{kN}(\pi, \sigma) = N^{-|\sigma|}(\mu(\sigma, \pi) + O(N^{-1})) \quad \forall \pi \geq \sigma$$

Thus, we are led to the conclusions in the statement.  $\square$

When  $\pi, \sigma$  are not comparable by  $\leq$ , the situation is quite unclear. The simplest example appears at  $k = 3$ , where we have the following formula, which is elementary:

$$W_{3N}(|\sqcap, \sqcap|) \simeq N^{-3}$$

Observe that the exponent  $-3$  is precisely the dominant one, because:

$$|\!|\!|\sqcap \vee \sqcap|\!|\!| - |\!|\!|\sqcap|\!|\!| - |\!|\!|\sqcap|\!|\!| = 1 - 2 - 2 = -3$$

As for the corresponding coefficient,  $K_0(|\sqcap, \sqcap|) = 1$ , this is definitely not the Möbius function, which vanishes for partitions which are not comparable by  $\leq$ . According to Theorem 7.19, this is rather the number of signed geodesic paths from  $|\sqcap$  to  $|\sqcap|$ .

In relation to this, observe that geometrically,  $NC(5)$  consists of the partitions  $|\sqcap, \sqcap|, |\sqcap|$ , which form an equilateral triangle with edges worth 1, and then the partitions  $|\!|\!|\!|, \sqcap\sqcap$ , which are at distance 1 apart, and each at distance  $1/2$  from each of the vertices of the triangle.

It is not exactly obvious how to recover the formula  $K_0(|\sqcap, \sqcap|) = 1$  from this.

Finally, we will need as well the following result:

**PROPOSITION 7.21.** *We have the following results:*

- (1) *If  $D = NC, NC_2$ , then  $\mu_{D(k)}(\pi, \sigma) = \mu_{NC(k)}(\pi, \sigma)$ .*
- (2) *If  $D = P, P_2$  then  $\mu_{D(k)}(\pi, \sigma) = \mu_{P(k)}(\pi, \sigma)$ .*

**PROOF.** Let  $Q = NC, P$  according to the cases (1,2) in the statement. It is easy to see in each case that  $D(k)$  is closed under taking intervals in  $Q(k)$ , in the sense that if  $\pi_1, \pi_2 \in D(k)$ ,  $\sigma \in Q(k)$  and  $\pi_1 < \sigma < \pi_2$  then  $\sigma \in D(k)$ . The result now follows from the definition of the Möbius function. □

### 7d. De Finetti theorems

With the above combinatorial ingredients in hand, let us go back now to invariance questions with respect to the main quantum permutation and rotation groups that we are interested in here, namely:

$$\begin{array}{ccc}
 S_N^+ & \longrightarrow & O_N^+ \\
 \uparrow & & \uparrow \\
 S_N & \longrightarrow & O_N
 \end{array}$$

We can now state and prove an approximation result for finite sequences, from [30], as follows:

**THEOREM 7.22.** *Suppose that  $(x_1, \dots, x_N)$  is  $G_N$ -invariant, and that  $G_N = O_N^+, S_N^+$ , or that  $G_N = O_N, S_N$  and  $(x_1, \dots, x_N)$  commute. Let  $(y_1, \dots, y_N)$  be a sequence of  $B_N$ -valued random variables with  $B_N$ -valued joint distribution determined as follows:*

- (1)  $G = O^+$ : *Free semicircular, centered with same variance as  $x_1$ .*
- (2)  $G = S^+$ : *Freely independent,  $y_i$  has same distribution as  $x_1$ .*
- (3)  $G = O$ : *Independent Gaussian, centered with same variance as  $x_1$ .*
- (4)  $G = S$ : *Independent,  $y_i$  has same distribution as  $x_1$ .*

Then if  $1 \leq j_1, \dots, j_k \leq N$  and  $b_0, \dots, b_k \in B_N$ , we have the following estimate,

$$\|E_N(b_0 x_{j_1} \dots x_{j_k} b_k) - E(b_0 y_{j_1} \dots y_{j_k} b_k)\| \leq \frac{C_k(G)}{N} \|x_1\|^k \|b_0\| \dots \|b_k\|$$

with  $C_k(G)$  being a constant depending only on  $k$  and  $G$ .

**PROOF.** First we note that it suffices to prove the result for  $N$  sufficiently large. We will assume that  $N$  is sufficiently large as for the Gram matrix  $G_{kN}$  to be invertible.

Let  $1 \leq j_1, \dots, j_k \leq N$  and  $b_0, \dots, b_k \in B_N$ . We have:

$$\begin{aligned} & E_N(b_0 x_{j_1} \dots x_{j_k} b_k) \\ &= \sum_{i_1 \dots i_k} b_0 x_{i_1} \dots x_{i_k} b_k \int u_{i_1 j_1} \dots u_{i_k j_k} \\ &= \sum_{i_1 \dots i_k} b_0 x_{i_1} \dots x_{i_k} b_k \sum_{\pi \leq \ker i} \sum_{\sigma \leq \ker j} W_{kN}(\pi, \sigma) \\ &= \sum_{\sigma \leq \ker j} \sum_{\pi} W_{kN}(\pi, \sigma) \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \end{aligned}$$

On the other hand, it follows from the assumptions on  $(y_1, \dots, y_N)$  and the various moment-cumulant formulae, that we have:

$$E(b_0 y_{j_1} \dots y_{j_k} b_k) = \sum_{\sigma \leq \ker j} \xi_{E_N}^{(\sigma)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

Here, and in what follows,  $\xi$  are the relevant free or classical cumulants:

The right hand side can be expanded, via Möbius inversion, in terms of expectation functionals as follows, with  $\pi$  being a partition in  $NC, P$  according to the cases (1,2) or (3,4), and with  $\pi \leq \sigma$  for some  $\sigma \in D(k)$ :

$$E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

Now if  $\pi \notin D(k)$  then we claim that this expectation functional is zero.

Indeed this is only possible if  $D = NC_2, P_2$  and  $\pi$  has a block with an odd number of legs. But it is easy to see that in these cases  $x_1$  has an even distribution with respect to

$E_N$ , and therefore we have, as claimed:

$$E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k) = 0$$

This observation allows to rewrite the above equation as:

$$E(b_0 y_{j_1} \dots y_{j_k} b_k) = \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

We therefore obtain the following formula:

$$E(b_0 y_{j_1} \dots y_{j_k} b_k) = \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) N^{-|\pi|} \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k$$

Comparing these two equations, we find that:

$$\begin{aligned} & E_N(b_0 x_{j_1} \dots x_{j_k} b_k) - E(b_0 y_{j_1} \dots y_{j_k} b_k) \\ &= \sum_{\sigma \leq \ker j} \sum_{\pi} (W_{kN}(\pi, \sigma) - \mu_{D(k)}(\pi, \sigma) N^{-|\pi|}) \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \end{aligned}$$

Now since  $x_1, \dots, x_N$  are identically distributed with respect to the faithful state  $\varphi$ , it follows that these variables have the same norm. Therefore, for any  $\pi \in D(k)$ :

$$\left\| \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \right\| \leq N^{|\pi|} \|x_1\|^k \|b_0\| \dots \|b_k\|$$

Combining this with the former equation, we obtain:

$$\begin{aligned} & \left| E_N(b_0 x_{j_1} \dots x_{j_k} b_k) - E(b_0 y_{j_1} \dots y_{j_k} b_k) \right| \\ & \leq \sum_{\sigma \leq \ker j} \sum_{\pi} |W_{kN}(\pi, \sigma) N^{|\pi|} - \mu_{D(k)}(\pi, \sigma)| \|x_1\|^k \|b_0\| \dots \|b_k\| \end{aligned}$$

Let us set now:

$$C_k(G) = \sup_{N \in \mathbb{N}} N \times \sum_{\sigma, \pi \in D(k)} |W_{kN}(\pi, \sigma) N^{|\pi|} - \mu_{D(k)}(\pi, \sigma)|$$

But this is finite by our main estimate, which completes the proof.  $\square$

We will make use of the inclusions  $G_N \subset G_M$  for  $N < M$ , which correspond to the Hopf algebra morphisms  $\omega_{N,M} : C(G_M) \rightarrow C(G_N)$  determined by:

$$\omega_{N,M}(u_{ij}) = \begin{cases} u_{ij} & \text{if } 1 \leq i, j \leq N \\ \delta_{ij} & \text{if } \max(i, j) > N \end{cases}$$

We begin by extending the notion of  $G_N$ -invariance to infinite sequences:

DEFINITION 7.23. Let  $(x_i)_{i \in \mathbb{N}}$  be a sequence in a noncommutative probability space  $(A, \varphi)$ . We say that  $(x_i)_{i \in \mathbb{N}}$  is  $G$ -invariant if

$$(x_1, \dots, x_N)$$

is  $G_N$ -invariant for each  $N \in \mathbb{N}$ .

In other words, the condition is that the joint distribution functional of  $(x_1, \dots, x_N)$  should be invariant under the following coaction, for each  $N \in \mathbb{N}$ :

$$\alpha_N : \mathbb{C} \langle t_1, \dots, t_N \rangle \rightarrow \mathbb{C} \langle t_1, \dots, t_N \rangle \otimes C(G_N)$$

It is convenient to extend these coactions to a coaction on the algebra of noncommutative polynomials on an infinite number of variables, as follows:

$$\beta_N : \mathbb{C} \langle t_i | i \in \mathbb{N} \rangle \rightarrow \mathbb{C} \langle t_i | i \in \mathbb{N} \rangle \otimes C(G_N)$$

Indeed, we can define  $\beta_N$  to be the unique unital morphism such that:

$$\beta_N(t_j) = \begin{cases} \sum_{i=1}^N t_i \otimes u_{ij} & \text{if } 1 \leq j \leq N \\ t_j \otimes 1 & \text{if } j > N \end{cases}$$

It is clear that  $\beta_N$  is an action of  $G_N$ . Also, we have the following relations, where  $\iota_N : \mathbb{C} \langle t_1, \dots, t_N \rangle \rightarrow \mathbb{C} \langle t_i | i \in \mathbb{N} \rangle$  is the natural inclusion:

$$(id \otimes \omega_{N,M})\beta_M = \beta_N$$

$$(\iota_N \otimes id)\alpha_N = \beta_N \iota_N$$

By using these compatibility relations, we have the following result:

PROPOSITION 7.24. An infinite sequence of random variables  $(x_i)_{i \in \mathbb{N}}$  is  $G$ -invariant if and only if the joint distribution functional

$$\mu_x : \mathbb{C} \langle t_i | i \in \mathbb{N} \rangle \rightarrow \mathbb{C}$$

$$P \rightarrow tr(P(x))$$

is invariant under the coaction  $\beta_N$ , for each  $N \in \mathbb{N}$ .

PROOF. This is clear indeed from the above discussion.  $\square$

In what follows  $(x_i)_{i \in \mathbb{N}}$  will be a sequence of self-adjoint random variables in a von Neumann algebra  $(M, tr)$ . We will assume that  $M$  is generated by  $(x_i)_{i \in \mathbb{N}}$ .

We denote by  $L^2(M, tr)$  the corresponding GNS Hilbert space, with inner product which is by definition as follows:

$$\langle m_1, m_2 \rangle = tr(m_1 m_2^*)$$

Also, the strong topology on  $M$ , that we will use in what follows, will be taken by definition with respect to the faithful representation on  $L^2(M, tr)$ .

We let  $P_N$  be the fixed point algebra of the action  $\beta_N$ , and we set:

$$B_N = \left\{ p(x) \mid p \in P_N \right\}''$$

We have then the following formula:

$$(id \otimes \omega_{N,N+1})\beta_{N+1} = \beta_N$$

Thus we have an inclusion as follows, for any  $n \geq 1$ :

$$B_{N+1} \subset B_N$$

We then define the  $G$ -invariant subalgebra by:

$$B = \bigcap_{N \geq 1} B_N$$

With these conventions, we have the following result:

**PROPOSITION 7.25.** *If an infinite sequence of random variables  $(x_i)_{i \in \mathbb{N}}$  is  $G$ -invariant, then for each  $N \in \mathbb{N}$  there is a coaction*

$$\tilde{\beta}_N : M \rightarrow M \otimes L^\infty(G_N)$$

*determined by the following formula, for any  $p \in \mathcal{P}_\infty$ :*

$$\tilde{\beta}_N(p(x)) = (ev_x \otimes \pi_N)\beta_N(p)$$

*The fixed point algebra of  $\tilde{\beta}_N$  is then  $B_N$ .*

**PROOF.** This is indeed clear from definitions, and from the various compatibility formulae above, between the coactions  $\alpha_N$  and  $\beta_N$ .  $\square$

We have as well the following result, which is clear as well:

**PROPOSITION 7.26.** *In the above context, of an infinite sequence of random variables belonging to an arbitrary von Neumann algebra  $M$  with a trace*

$$(x_i)_{i \in \mathbb{N}}$$

*which is  $G$ -invariant, for each  $N \in \mathbb{N}$  there is a trace-preserving conditional expectation  $E_N : M \rightarrow B_N$  given by integrating the action  $\tilde{\beta}_N$ :*

$$E_N(m) = \left( id \otimes \int_G \right) \tilde{\beta}_N(m)$$

*By taking the limit of these expectations as  $N \rightarrow \infty$ , we obtain a trace-preserving conditional expectation onto the  $G$ -invariant subalgebra.*

**PROOF.** Once again, this is clear from definitions, and from the various compatibility formulae above, between the coactions  $\alpha_N$  and  $\beta_N$ .  $\square$

Next, we have the following result:

PROPOSITION 7.27. *Suppose that  $(x_i)_{i \in \mathbb{N}}$  is  $G$ -invariant. Then:*

- (1) *For any  $m \in M$ , the sequence  $E_N(m)$  converges in 2-norm and with respect to the strong topology to a limit  $E(m) \in B$ .*
- (2)  *$E$  is a trace-preserving conditional expectation of  $M$  onto  $B$ .*
- (3) *For  $\pi \in NC(k)$  and  $m_1, \dots, m_k \in M$  we have, with strong convergence:*

$$E^{(\pi)}(m_1 \otimes \dots \otimes m_k) = \lim_{n \rightarrow \infty} E_N^{(\pi)}(m_1 \otimes \dots \otimes m_k)$$

PROOF. This is again clear from definitions. Note that (1) is just a simple noncommutative reversed martingale convergence theorem.  $\square$

We are now prepared to state and prove the main theorem, from [30], which comes as a complement to the reverse De Finetti theorem that we already established:

THEOREM 7.28. *Let  $(x_i)_{i \in \mathbb{N}}$  be a  $G$ -invariant sequence of self-adjoint random variables in  $(M, tr)$ , and assume that  $M = \langle (x_i)_{i \in \mathbb{N}} \rangle$ . Then there is a subalgebra  $B \subset M$  and a trace-preserving conditional expectation  $E : M \rightarrow B$  such that:*

- (1) *If  $G = (S_N)$ , then  $(x_i)_{i \in \mathbb{N}}$  are conditionally independent and identically distributed given  $B$ .*
- (2) *If  $G = (S_N^+)$ , then  $(x_i)_{i \in \mathbb{N}}$  are freely independent and identically distributed with amalgamation over  $B$ .*
- (3) *If  $G = (O_N)$ , then  $(x_i)_{i \in \mathbb{N}}$  are conditionally independent, and have Gaussian distributions with mean zero and common variance, given  $B$ .*
- (4) *If  $G = (O_N^+)$ , then  $(x_i)_{i \in \mathbb{N}}$  form a  $B$ -valued free semicircular family with mean zero and common variance.*

PROOF. We use the various partial results and formulae established above. Let  $j_1, \dots, j_k \in \mathbb{N}$  and  $b_0, \dots, b_k \in B$ . We have the following computation:

$$\begin{aligned} & E(b_0 x_{j_1} \dots x_{j_k} b_k) \\ &= \lim_{N \rightarrow \infty} E_N(b_0 x_{j_1} \dots x_{j_k} b_k) \\ &= \lim_{N \rightarrow \infty} \sum_{\sigma \leq \ker j} \sum_{\pi} W_{kN}(\pi, \sigma) \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \\ &= \lim_{N \rightarrow \infty} \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) N^{-|\pi|} \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \end{aligned}$$

Let us recall now from the above that we have the following compatibility formula, where  $\tilde{\iota}_N : W^*(x_1, \dots, x_N) \rightarrow M$  is the obvious inclusion, and  $\tilde{\alpha}_N$  is as before:

$$(\tilde{\iota}_N \otimes id)\tilde{\alpha}_N = \tilde{\beta}_N \tilde{\iota}_N$$

By using this, and the above cumulant results, we have:

$$E(b_0 x_{j_1} \dots x_{j_k} b_k) = \lim_{N \rightarrow \infty} \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

We therefore obtain the following formula:

$$E(b_0 x_{j_1} \dots x_{j_k} b_k) = \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) E^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

We can replace the sum of expectation functionals by cumulants to obtain:

$$E(b_0 x_{j_1} \dots x_{j_k} b_k) = \sum_{\sigma \leq \ker j} \xi_E^{(\sigma)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

Here and in what follows  $\xi$  denotes the relevant free or classical cumulants, depending on the quantum group that we are dealing with, free or classical.

Now since the cumulants are determined by the moment-cumulant formulae, we find that we have the following formula:

$$\xi_E^{(\sigma)}(b_0 x_{j_1} b_1, \dots, x_{j_k} b_k) = \begin{cases} \xi_E^{(\sigma)}(b_0 x_1 b_1, \dots, x_1 b_k) & \text{if } \sigma \in D(k) \text{ and } \sigma \leq \ker j \\ 0 & \text{otherwise} \end{cases}$$

The result then follows from the characterizations of these joint distributions in terms of cumulants.  $\square$

We refer to [30] and related papers for more on the above.

## 7e. Exercises

Things have been quite technical in this chapter, dealing with advanced probability theory, and so will be our exercises here. As a first exercise, we have:

**EXERCISE 7.29.** *Formulate and prove the classical De Finetti theorem, concerning sequences which are invariant under  $S_\infty$ , without using representation theory methods.*

This is something very standard, and is a must-do exercise, the point being that all the Weingarten technology used in this chapter, which is something quite heavy, was motivated by the fact that we want to deal with several quantum groups at the same time, in a “uniform” way. In the case of the symmetric group itself things are in fact much simpler, and the exercise is about understanding how this works.

We have as well the following exercise, regarding now the free case:

**EXERCISE 7.30.** *Formulate and prove the free De Finetti theorem, concerning sequences which are invariant under  $(S_N^+)$ , without using representation theory methods.*

The same comments as for the previous exercise apply, the idea being that, once again, the Weingarten function machinery can be avoided in this case.

We have as well several exercises in connection with the Weingarten function estimates, that we needed in the proof of the De Finetti theorems, and which are of independent interest as well. First, we have the following exercise:



EXERCISE 7.31. *Work out the full proof of the explicit formula for the Weingarten function for  $S_N$ , namely*

$$W_{kN}(\pi, \sigma) = \sum_{\tau \leq \pi \wedge \sigma} \mu(\tau, \pi) \mu(\tau, \sigma) \frac{(N - |\tau|)!}{N!}$$

*then of the main estimate for this function, namely*

$$W_{kN}(\pi, \sigma) = N^{-|\pi \wedge \sigma|} (\mu(\pi \wedge \sigma, \pi) \mu(\pi \wedge \sigma, \sigma) + O(N^{-1}))$$

*where  $\mu$  is the Möbius function of  $P(k)$ .*

This was something that was already discussed in the above, the idea being that all this comes from the explicit knowledge of the integrals over  $S_N$ , via the Möbius inversion formula, and the problem now is that of working out all the details.

As a second exercise now, which is quite technical, we have:

EXERCISE 7.32. *Work out estimates for the integrals of type*

$$\int_{S_N^+} u_{i_1 j_1} u_{i_2 j_2} u_{i_3 j_3} u_{i_4 j_4}$$

*and then for the Weingarten function of  $S_N^+$  at  $k = 4$ .*

Once again, this was something partly discussed in the above, with the comment that things are clear at  $k = 2, 3$ , due to the formula  $P(k) = NC(k)$  valid here. The problem now is that of working out what happens at  $k = 4$ , where things are non-trivial.

Finally, we have the following very standard exercise:

EXERCISE 7.33. *Prove directly that the function*

$$d(\pi, \sigma) = \frac{|\pi| + |\sigma|}{2} - |\pi \vee \sigma|$$

*is a distance on  $P(k)$ .*

To be more precise here, this is something that we talked about in the above, with the idea being that this follows from a number of well-known facts regarding the partitions in  $P(k)$ . The problem now is that of proving directly this result.

## CHAPTER 8

### Hypergeometric laws

#### 8a. Hyperspherical laws

We have seen so far that, in what concerns the probability theory on classical or quantum groups, the first problem which appears, and which is of key importance, is that of computing the laws of characters, and more generally of truncated characters:

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

For the quantum rotation and permutation groups, this problem can be investigated by using easiness and combinatorics, and satisfactory results in this sense, which are in tune with free probability theory, can be obtained in the  $N \rightarrow \infty$  limit.

That was for the basic theory. In this chapter we discuss more advanced aspects, regarding the case where  $N \in \mathbb{N}$  is fixed, or variables which are more general than the truncated characters  $\chi_t$ , or regarding both, more advanced variables at fixed  $N \in \mathbb{N}$ .

As in the previous chapters, we will investigate such problems for the main quantum permutation and rotation groups, namely:

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array}$$

Let us first discuss the case of the orthogonal group  $O_N$ . In certain situations, we can use the following elementary and well-known fact:

**PROPOSITION 8.1.** *Each row of coordinates on  $O_N$  has the same joint distribution as the sequence of coordinates on the real sphere  $S_{\mathbb{R}}^{N-1}$ ,*

$$(u_{i1}, \dots, u_{iN}) \sim (x_1, \dots, x_N)$$

*and the same happens for the columns.*

PROOF. Given an index  $i \in \{1, \dots, N\}$ , our claim is that we have an embedding as follows, which commutes with the corresponding uniform integration functionals:

$$C(S_{\mathbb{R}}^{N-1}) \subset C(O_N)$$

$$x_j \rightarrow u_{ij}$$

In order to prove this claim, consider the subalgebra  $C(S) \subset C(O_N)$  generated by the variables  $u_{ij}$ , with  $i$  being fixed, and with  $j = 1, \dots, N$ . Since these  $N$  variables are real, and their squares sum up to 1, we have a quotient map, as follows:

$$C(S_{\mathbb{R}}^{N-1}) \rightarrow C(S) \subset C(O_N)$$

$$x_j \rightarrow u_{ij}$$

Now observe that  $S \subset S_{\mathbb{R}}^{N-1}$  must be an isomorphism, because by Gram-Schmidt we can complete any vector of  $S_{\mathbb{R}}^{N-1}$  into an orthogonal matrix. Thus, the above composition of morphisms is an embedding. As for the commutation with the uniform integration functionals, this follows from the fact that we have an action  $O_N \curvearrowright S$ .  $\square$

Motivated by the above observation, let us compute now the hyperspherical laws at fixed values of  $N \in \mathbb{N}$ . We begin with a full discussion in the classical case.

At  $N = 2$  the sphere is the unit circle  $\mathbb{T}$ , and with  $z = e^{it}$  the coordinates are  $\cos t, \sin t$ . The integrals of the arbitrary products of such coordinates can be computed by using standard calculus, the result being well-known, as follows:

THEOREM 8.2. *We have the following formula,*

$$\int_0^{\pi/2} \cos^p t \sin^q t dt = \left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)} \frac{p!!q!!}{(p+q+1)!!}$$

where  $\varepsilon(p) = 1$  if  $p$  is even, and  $\varepsilon(p) = 0$  if  $p$  is odd, and where

$$m!! = (m-1)(m-3)(m-5)\dots$$

with the product ending at 2 if  $m$  is odd, and ending at 1 if  $m$  is even.

PROOF. This is something known to everyone loving and teaching calculus. Let us first discuss the case  $q = 0$ . Consider the integral in the statement at  $p = 0$ , namely:

$$I_p = \int_0^{\pi/2} \cos^p t dt$$

We compute this integral by partial integration. For this purpose, we use:

$$\begin{aligned} (\cos^p t \sin t)' &= p \cos^{p-1} t (-\sin t) \sin t + \cos^p t \cos t \\ &= p \cos^{p+1} t - p \cos^{p-1} t + \cos^{p+1} t \\ &= (p+1) \cos^{p+1} t - p \cos^{p-1} t \end{aligned}$$

By integrating between 0 and  $\pi/2$ , we obtain the following formula:

$$(p+1)I_{p+1} = pI_{p-1}$$

Thus we can compute  $I_p$  by recurrence, and we obtain:

$$\begin{aligned} I_p &= \frac{p-1}{p} I_{p-2} \\ &= \frac{p-1}{p} \cdot \frac{p-3}{p-2} I_{p-4} \\ &= \frac{p-1}{p} \cdot \frac{p-3}{p-2} \cdot \frac{p-5}{p-4} I_{p-6} \\ &\quad \vdots \\ &= \frac{p!!}{(p+1)!!} I_{1-\varepsilon(p)} \end{aligned}$$

Together with  $I_0 = \frac{\pi}{2}$  and  $I_1 = 1$ , which are both clear, we obtain:

$$I_p = \left(\frac{\pi}{2}\right)^{\varepsilon(p)} \frac{p!!}{(p+1)!!}$$

Summarizing, we have proved the following formula, with one equality coming from the above computation, and with the other equality coming from this, via  $t = \frac{\pi}{2} - s$ :

$$\int_0^{\pi/2} \cos^p t \, dt = \int_0^{\pi/2} \sin^p t \, dt = \left(\frac{\pi}{2}\right)^{\varepsilon(p)} \frac{p!!}{(p+1)!!}$$

In relation with the formula in the statement, we are therefore done with the case  $p = 0$  or  $q = 0$ . Let us investigate now the general case. We must compute:

$$I_{pq} = \int_0^{\pi/2} \cos^p t \sin^q t \, dt$$

In order to do the partial integration, observe that we have:

$$\begin{aligned} (\cos^p t \sin^q t)' &= p \cos^{p-1} t (-\sin t) \sin^q t \\ &\quad + \cos^p t \cdot q \sin^{q-1} t \cos t \\ &= -p \cos^{p-1} t \sin^{q+1} t + q \cos^{p+1} t \sin^{q-1} t \end{aligned}$$

By integrating between 0 and  $\pi/2$ , we obtain, for  $p, q > 0$ :

$$pI_{p-1, q+1} = qI_{p+1, q-1}$$

Thus, we can compute  $I_{pq}$  by recurrence. When  $q$  is even we have:

$$\begin{aligned}
 I_{pq} &= \frac{q-1}{p+1} I_{p+2, q-2} \\
 &= \frac{q-1}{p+1} \cdot \frac{q-3}{p+3} I_{p+4, q-4} \\
 &= \frac{q-1}{p+1} \cdot \frac{q-3}{p+3} \cdot \frac{q-5}{p+5} I_{p+6, q-6} \\
 &\quad \vdots \\
 &= \frac{p!!q!!}{(p+q)!!} I_{p+q}
 \end{aligned}$$

But the last term was already computed above, and we obtain the result:

$$\begin{aligned}
 I_{pq} &= \frac{p!!q!!}{(p+q)!!} I_{p+q} \\
 &= \frac{p!!q!!}{(p+q)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(p+q)} \frac{(p+q)!!}{(p+q+1)!!} \\
 &= \left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)} \frac{p!!q!!}{(p+q+1)!!}
 \end{aligned}$$

Observe that this gives the result for  $p$  even as well, by symmetry. Indeed, we have  $I_{pq} = I_{qp}$ , by using the following change of variables:

$$t = \frac{\pi}{2} - s$$

In the remaining case now, where both  $p, q$  are odd, we can use once again the formula  $pI_{p-1, q+1} = qI_{p+1, q-1}$  established above, and the recurrence goes as follows:

$$\begin{aligned}
 I_{pq} &= \frac{q-1}{p+1} I_{p+2, q-2} \\
 &= \frac{q-1}{p+1} \cdot \frac{q-3}{p+3} I_{p+4, q-4} \\
 &= \frac{q-1}{p+1} \cdot \frac{q-3}{p+3} \cdot \frac{q-5}{p+5} I_{p+6, q-6} \\
 &\quad \vdots \\
 &= \frac{p!!q!!}{(p+q-1)!!} I_{p+q-1, 1}
 \end{aligned}$$

In order to compute the last term, observe that we have:

$$\begin{aligned} I_{p1} &= \int_0^{\pi/2} \cos^p t \sin t \, dt \\ &= -\frac{1}{p+1} \int_0^{\pi/2} (\cos^{p+1} t)' \, dt \\ &= \frac{1}{p+1} \end{aligned}$$

Thus, we can finish our computation in the case  $p, q$  odd, as follows:

$$\begin{aligned} I_{pq} &= \frac{p!!q!!}{(p+q-1)!!} I_{p+q-1,1} \\ &= \frac{p!!q!!}{(p+q-1)!!} \cdot \frac{1}{p+q} \\ &= \frac{p!!q!!}{(p+q+1)!!} \end{aligned}$$

Thus, we obtain the formula in the statement, the exponent of  $\pi/2$  appearing there being in the present case:

$$\varepsilon(p)\varepsilon(q) = 0 \cdot 0 = 0$$

Thus, we are led to the conclusion in the statement.  $\square$

In order to discuss the higher spheres, we will use spherical coordinates, which are constructed as follows:

**THEOREM 8.3.** *We have spherical coordinates in  $N$  dimensions,*

$$\begin{cases} x_1 &= r \cos t_1 \\ x_2 &= r \sin t_1 \cos t_2 \\ \vdots & \\ x_{N-1} &= r \sin t_1 \sin t_2 \dots \sin t_{N-2} \cos t_{N-1} \\ x_N &= r \sin t_1 \sin t_2 \dots \sin t_{N-2} \sin t_{N-1} \end{cases}$$

*the corresponding Jacobian being given by the following formula:*

$$J(r, t) = r^{N-1} \sin^{N-2} t_1 \sin^{N-3} t_2 \dots \sin^2 t_{N-3} \sin t_{N-2}$$

**PROOF.** The fact that we have coordinates is clear. Regarding the Jacobian, the proof is similar to the one from 2 or 3 dimensions, by developing the determinant over the last

column, and then by proceeding by recurrence. Indeed, by developing, we have:

$$\begin{aligned}
J_N &= r \sin t_1 \dots \sin t_{N-2} \sin t_{N-1} \times \sin t_{N-1} J_{N-1} \\
&+ r \sin t_1 \dots \sin t_{N-2} \cos t_{N-1} \times \cos t_{N-1} J_{N-1} \\
&= r \sin t_1 \dots \sin t_{N-2} (\sin^2 t_{N-1} + \cos^2 t_{N-1}) J_{N-1} \\
&= r \sin t_1 \dots \sin t_{N-2} J_{N-1}
\end{aligned}$$

Thus, we obtain the formula in the statement, by recurrence.  $\square$

Let us first compute the volume of the sphere. The result here is as follows:

**THEOREM 8.4.** *The volume of the unit sphere in  $\mathbb{R}^N$  is given by*

$$V = \left(\frac{\pi}{2}\right)^{[N/2]} \frac{2^N}{(N+1)!!}$$

with the convention

$$N!! = (N-1)(N-3)(N-5)\dots$$

with the product ending at 2 if  $N$  is odd, and ending at 1 if  $N$  is even.

**PROOF.** Let us denote by  $B^+$  the positive part of the unit sphere, or rather unit ball  $B$ , obtained by cutting this unit ball in  $2^N$  parts. At the level of volumes, we have:

$$V = 2^N V^+$$

We have the following computation, using spherical coordinates, and Fubini:

$$\begin{aligned}
&V^+ \\
&= \int_{B^+} 1 \\
&= \int_0^1 \int_0^{\pi/2} \dots \int_0^{\pi/2} r^{N-1} \sin^{N-2} t_1 \dots \sin t_{N-2} dr dt_1 \dots dt_{N-1} \\
&= \int_0^1 r^{N-1} dr \int_0^{\pi/2} \sin^{N-2} t_1 dt_1 \dots \int_0^{\pi/2} \sin t_{N-2} dt_{N-2} \int_0^{\pi/2} 1 dt_{N-1} \\
&= \frac{1}{N} \times \left(\frac{\pi}{2}\right)^{[N/2]} \times \frac{(N-2)!!}{(N-1)!!} \cdot \frac{(N-3)!!}{(N-2)!!} \dots \frac{2!!}{3!!} \cdot \frac{1!!}{2!!} \cdot 1 \\
&= \frac{1}{N} \times \left(\frac{\pi}{2}\right)^{[N/2]} \times \frac{1}{(N-1)!!} \\
&= \left(\frac{\pi}{2}\right)^{[N/2]} \frac{1}{(N+1)!!}
\end{aligned}$$

Here we have used the following formula, for computing the exponent of  $\pi/2$ :

$$\begin{aligned} & \varepsilon(0) + \varepsilon(1) + \varepsilon(2) + \dots + \varepsilon(N-2) \\ &= 1 + 0 + 1 + \dots + \varepsilon(N-2) \\ &= \left[ \frac{N-2}{2} \right] + 1 \\ &= \left[ \frac{N}{2} \right] \end{aligned}$$

Thus, we obtain the formula in the statement.  $\square$

With the above ingredients in hand, we can now compute arbitrary polynomial integrals, over the spheres of arbitrary dimension, the result being is as follows:

**THEOREM 8.5.** *The spherical integral of  $x_{i_1} \dots x_{i_k}$  vanishes, unless each  $a \in \{1, \dots, N\}$  appears an even number of times in the sequence  $i_1, \dots, i_k$ . We have*

$$\int_{S_{\mathbb{R}}^{N-1}} x_{i_1} \dots x_{i_k} dx = \frac{(N-1)!! l_1!! \dots l_N!!}{(N + \sum l_i - 1)!!}$$

with  $l_a$  being this number of occurrences.

**PROOF.** We can restrict attention to the case  $l_a \in 2\mathbb{N}$ , since the other integrals vanish. The integral in the statement can be written in spherical coordinates, as follows:

$$I = \frac{2^N}{V} \int_0^{\pi/2} \dots \int_0^{\pi/2} x_1^{l_1} \dots x_N^{l_N} J dt_1 \dots dt_{N-1}$$

In this formula  $V$  is the volume of the sphere,  $J$  is the Jacobian, and the  $2^N$  factor comes from the restriction to the  $1/2^N$  part of the sphere where all the coordinates are positive. The normalization constant in front of the integral is:

$$\frac{2^N}{V} = \frac{2^N}{N\pi^{N/2}} \cdot \Gamma\left(\frac{N}{2} + 1\right) = \left(\frac{2}{\pi}\right)^{[N/2]} (N-1)!!$$

As for the unnormalized integral, this is given by:

$$\begin{aligned} I' &= \int_0^{\pi/2} \dots \int_0^{\pi/2} (\cos t_1)^{l_1} (\sin t_1 \cos t_2)^{l_2} \\ &\quad \vdots \\ &\quad (\sin t_1 \sin t_2 \dots \sin t_{N-2} \cos t_{N-1})^{l_{N-1}} \\ &\quad (\sin t_1 \sin t_2 \dots \sin t_{N-2} \sin t_{N-1})^{l_N} \\ &\quad \sin^{N-2} t_1 \sin^{N-3} t_2 \dots \sin^2 t_{N-3} \sin t_{N-2} \\ &\quad dt_1 \dots dt_{N-1} \end{aligned}$$



By rearranging the terms, we obtain:

$$\begin{aligned}
I' &= \int_0^{\pi/2} \cos^{l_1} t_1 \sin^{l_2+\dots+l_N+N-2} t_1 dt_1 \\
&\quad \int_0^{\pi/2} \cos^{l_2} t_2 \sin^{l_3+\dots+l_N+N-3} t_2 dt_2 \\
&\quad \vdots \\
&\quad \int_0^{\pi/2} \cos^{l_{N-2}} t_{N-2} \sin^{l_{N-1}+l_N+1} t_{N-2} dt_{N-2} \\
&\quad \int_0^{\pi/2} \cos^{l_{N-1}} t_{N-1} \sin^{l_N} t_{N-1} dt_{N-1}
\end{aligned}$$

Now by using the formula at  $N = 2$ , from Theorem 8.2, this gives:

$$\begin{aligned}
I' &= \frac{l_1!!(l_2 + \dots + l_N + N - 2)!!}{(l_1 + \dots + l_N + N - 1)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(N-2)} \\
&\quad \frac{l_2!!(l_3 + \dots + l_N + N - 3)!!}{(l_2 + \dots + l_N + N - 2)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(N-3)} \\
&\quad \vdots \\
&\quad \frac{l_{N-2}!!(l_{N-1} + l_N + 1)!!}{(l_{N-2} + l_{N-1} + l_N + 2)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(1)} \\
&\quad \frac{l_{N-1}!!l_N!!}{(l_{N-1} + l_N + 1)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(0)}
\end{aligned}$$

Now observe that the various double factorials multiply up to quantity in the statement, modulo a  $(N - 1)!!$  factor, and that the  $\frac{\pi}{2}$  factors multiply up to  $F = \left(\frac{\pi}{2}\right)^{[N/2]}$ . Thus by multiplying with the normalization constant, we obtain the result.  $\square$

In connection now with our probabilistic questions, we have:

**THEOREM 8.6.** *The even moments of the hyperspherical variables are*

$$\int_{S_{\mathbb{R}}^{N-1}} x_i^k dx = \frac{(N - 1)!!k!!}{(N + k - 1)!!}$$

and the variables  $y_i = x_i/\sqrt{N}$  become normal and independent with  $N \rightarrow \infty$ .

PROOF. The moment formula in the statement follows from Theorem 8.5. Now observe that with  $N \rightarrow \infty$  we have the following estimate:

$$\begin{aligned} \int_{S_{\mathbb{R}}^{N-1}} x_i^k dx &= \frac{(N-1)!!}{(N+k-1)!!} \times k!! \\ &\simeq N^{k/2} \times k!! \\ &= N^{k/2} M_k(g_1) \end{aligned}$$

Thus, we have  $x_i/\sqrt{N} \sim g_1$ , as claimed. Finally, the independence assertion follows as well from the formula in Theorem 8.5, via standard probability theory.  $\square$

In the case of the free sphere now, the computations are substantially more complicated. Let us start with the following result, that we know from chapter 5:

THEOREM 8.7. *For the free sphere  $S_{\mathbb{R},+}^{N-1}$ , the rescaled coordinates*

$$y_i = \sqrt{N}x_i$$

*become semicircular and free, in the  $N \rightarrow \infty$  limit.*

PROOF. As explained in chapter 5 above, the Weingarten formula for the free sphere, together with the standard fact that the Gram matrix, and hence the Weingarten matrix too, is asymptotically diagonal, gives the following estimate:

$$\int_{S_{\mathbb{R},+}^{N-1}} x_{i_1} \dots x_{i_k} dx \simeq N^{-k/2} \sum_{\sigma \in NC_2(k)} \delta_{\sigma}(i_1, \dots, i_k)$$

With this formula in hand, we can compute the asymptotic moments of each coordinate  $x_i$ . Indeed, by setting  $i_1 = \dots = i_k = i$ , all Kronecker symbols are 1, and we obtain:

$$\int_{S_{\mathbb{R},+}^{N-1}} x_i^k dx \simeq N^{-k/2} |NC_2(k)|$$

Thus the rescaled coordinates  $y_i = \sqrt{N}x_i$  become semicircular in the  $N \rightarrow \infty$  limit, as claimed. As for the asymptotic freeness result, this follows as well from the above general joint moment estimate, via standard free probability theory. See [23].  $\square$

Summarizing, we have fully satisfactory results in the  $N \rightarrow \infty$  limit.

## 8b. Free coordinates

The problem now, which is highly non-trivial, is that of computing the moments of the coordinates of the free sphere at fixed values of  $N \in \mathbb{N}$ . The answer here, from [26], based on advanced quantum group and calculus techniques, is as follows:

THEOREM 8.8. *The moments of the free hyperspherical law are given by*

$$\int_{S_{\mathbb{R},+}^{N-1}} x_1^{2l} dx = \frac{1}{(N+1)^l} \cdot \frac{q+1}{q-1} \cdot \frac{1}{l+1} \sum_{r=-l-1}^{l+1} (-1)^r \binom{2l+2}{l+r+1} \frac{r}{1+q^r}$$

where  $q \in [-1, 0)$  is such that  $q + q^{-1} = -N$ .

PROOF. The idea is that  $x_1 \in C(S_{\mathbb{R},+}^{N-1})$  has the same law as  $u_{11} \in C(O_N^+)$ , which has the same law as a certain variable  $w \in C(SU_2^q)$ , which can be in turn modelled by an explicit operator on  $l^2(\mathbb{N})$ , whose law can be computed by using advanced calculus.

Let us first explain the relation between  $O_N^+$  and  $SU_2^q$ . To any matrix  $F \in GL_N(\mathbb{R})$  satisfying  $F^2 = 1$  we associate the following universal algebra:

$$C(O_F^+) = C^* \left( (u_{ij})_{i,j=1,\dots,N} \mid u = F\bar{u}F = \text{unitary} \right)$$

Observe that  $O_{I_N}^+ = O_N^+$ . In general, the above algebra satisfies Woronowicz's generalized axioms in [98], which do not include the strong antipode axiom  $S^2 = id$ .

At  $N = 2$ , up to a trivial equivalence relation on the matrices  $F$ , and on the quantum groups  $O_F^+$ , we can assume that  $F$  is as follows, with  $q \in [-1, 0)$ :

$$F = \begin{pmatrix} 0 & \sqrt{-q} \\ 1/\sqrt{-q} & 0 \end{pmatrix}$$

Our claim is that for this matrix we have:

$$O_F^+ = SU_2^q$$

Indeed, the relations  $u = F\bar{u}F$  tell us that  $u$  must be of the following special form:

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

Thus  $C(O_F^+)$  is the universal algebra generated by two elements  $\alpha, \gamma$ , with the relations making the above matrix  $u$  unitary. But these unitarity conditions are:

$$\begin{aligned} \alpha\gamma &= q\gamma\alpha \\ \alpha\gamma^* &= q\gamma^*\alpha \\ \gamma\gamma^* &= \gamma^*\gamma \\ \alpha^*\alpha + \gamma^*\gamma &= 1 \\ \alpha\alpha^* + q^2\gamma\gamma^* &= 1 \end{aligned}$$

We recognize here the relations in [98] defining the algebra  $C(SU_2^q)$ , and it follows that we have an isomorphism of Hopf  $C^*$ -algebras, as follows:

$$C(O_F^+) \simeq C(SU_2^q)$$

Now back to the general case, let us try to understand the integration over  $O_F^+$ . Given  $\pi \in NC_2(2k)$  and  $i = (i_1, \dots, i_{2k})$ , we set:

$$\delta_\pi^F(i) = \prod_{s \in \pi} F_{i_{s_l} i_{s_r}}$$

Here the product is over all strings  $s = \{s_l \curvearrowright s_r\}$  of  $\pi$ . Our claim is that the following family of vectors, with  $\pi \in NC_2(2k)$ , spans the space of fixed vectors of  $u^{\otimes 2k}$ :

$$\xi_\pi = \sum_i \delta_\pi^F(i) e_{i_1} \otimes \dots \otimes e_{i_{2k}}$$

Indeed, having  $\xi_\pi$  fixed by  $u^{\otimes 2}$  is equivalent to assuming that  $u = F\bar{u}F$  is unitary. By using now the above vectors, we obtain the following Weingarten formula:

$$\int_{O_F^+} u_{i_1 j_1} \dots u_{i_{2k} j_{2k}} = \sum_{\pi \sigma} \delta_\pi^F(i) \delta_\sigma^F(j) W_{kN}(\pi, \sigma)$$

With these preliminaries in hand, let us start the computation. Let  $N \in \mathbb{N}$ , and consider the number  $q \in [-1, 0)$  satisfying:

$$q + q^{-1} = -N$$

Our claim is that we have:

$$\int_{O_N^+} \varphi(\sqrt{N+2} u_{ij}) = \int_{SU_2^q} \varphi(\alpha + \alpha^* + \gamma - q\gamma^*)$$

Indeed, the moments of the variable on the left are given by:

$$\int_{O_N^+} u_{ij}^{2k} = \sum_{\pi \sigma} W_{kN}(\pi, \sigma)$$

On the other hand, the moments of the variable on the right, which in terms of the fundamental corepresentation  $v = (v_{ij})$  is given by  $w = \sum_{ij} v_{ij}$ , are given by:

$$\int_{SU_2^q} w^{2k} = \sum_{ij} \sum_{\pi \sigma} \delta_\pi^F(i) \delta_\sigma^F(j) W_{kN}(\pi, \sigma)$$

We deduce that  $w/\sqrt{N+2}$  has the same moments as  $u_{ij}$ , which proves our claim.

In order to do now the computation over  $SU_2^q$ , we can use a matrix model due to Woronowicz [97], where the standard generators  $\alpha, \gamma$  are mapped as follows:

$$\begin{aligned} \pi_u(\alpha) e_k &= \sqrt{1 - q^{2k}} e_{k-1} \\ \pi_u(\gamma) e_k &= u q^k e_k \end{aligned}$$

Here  $u \in \mathbb{T}$  is a parameter, and  $(e_k)$  is the standard basis of  $l^2(\mathbb{N})$ . The point with this representation is that it allows the computation of the Haar functional. Indeed, if  $D$  is the diagonal operator given by  $D(e_k) = q^{2k}e_k$ , then the formula is as follows:

$$\int_{SU_2^q} x = (1 - q^2) \int_{\mathbb{T}} \text{tr}(D\pi_u(x)) \frac{du}{2\pi i u}$$

With the above model in hand, the law of the variable that we are interested in is of the following form:

$$\int_{SU_2^q} \varphi(\alpha + \alpha^* + \gamma - q\gamma^*) = (1 - q^2) \int_{\mathbb{T}} \text{tr}(D\varphi(M)) \frac{du}{2\pi i u}$$

To be more precise, this formula holds indeed, with:

$$M(e_k) = e_{k+1} + q^k(u - qu^{-1})e_k + (1 - q^{2k})e_{k-1}$$

The point now is that the integral on the right can be computed, by using advanced calculus methods, and this gives the result. We refer here to [26].  $\square$

There are many interesting open questions in relation with the above result, the first of which being the problem of finding a simpler proof for it.

### 8c. Hypergeometric laws

Following now [19], let us discuss the free hypergeometric laws. We will use here a twisting result established in chapter 4 above, which is also from [19]. We know from that twisting result that we have, at the probabilistic level:

**THEOREM 8.9.** *The following two algebras are isomorphic, via  $u_{ij}^2 \rightarrow X_{ij}$ ,*

- (1) *The algebra generated by the variables  $u_{ij}^2 \in C(O_n^+)$ ,*
- (2) *The algebra generated by  $X_{ij} = \frac{1}{n} \sum_{a,b=1}^n p_{ia,jb} \in C(S_{n^2}^+)$ ,*

*and this isomorphism commutes with the respective Haar integration functionals.*

**PROOF.** This follows indeed from the general twisting result from chapter 4.  $\square$

As pointed out in [19], it is possible to derive as well this result directly, by using the Weingarten formula, and manipulations on the partitions:

**THEOREM 8.10.** *The following families of variables have the same joint law,*

- (1)  $\{u_{ij}^2\} \in C(O_n^+)$ ,
- (2)  $\{X_{ij} = \frac{1}{n} \sum_{ab} p_{ia,jb}\} \in C(S_{n^2}^+)$ ,

*where  $u = (u_{ij})$  and  $p = (p_{ia,jb})$  are the corresponding fundamental corepresentations.*

PROOF. As already mentioned, this result can be obtained via twisting methods. An alternative approach is by using the Weingarten formula for our two quantum groups, and the shrinking operation  $\pi \rightarrow \pi'$ . Indeed, we obtain the following moment formulae:

$$\int_{O_n^+} u_{ij}^{2k} = \sum_{\pi, \sigma \in NC_2(2k)} W_{2k,n}(\pi, \sigma)$$

$$\int_{S_{n^2}^+} X_{ij}^k = \sum_{\pi, \sigma \in NC_2(2k)} n^{|\pi'|+|\sigma'|-k} W_{k,n^2}(\pi', \sigma')$$

According to the fattening results in chapter 2 the summands coincide, and so the moments are equal, as desired. The proof for joint moments is similar.  $\square$

In what follows we will be interested in single variables. We have here:

DEFINITION 8.11. *The noncommutative random variable*

$$X(n, m, N) = \sum_{i=1}^n \sum_{j=1}^m u_{ij} \in C(S_N^+)$$

is called *free hypergeometric, of parameters*  $(n, m, N)$ .

The terminology comes from the fact that the variable  $X'(n, m, N)$ , defined as above, but over the algebra  $C(S_N)$ , follows a hypergeometric law of parameters  $(n, m, N)$ .

Following [19], here is an exploration of the basic asymptotic properties of these laws, by using standard cumulant machinery:

THEOREM 8.12. *The free hypergeometric laws have the following properties:*

- (1) *Let  $n, m, N \rightarrow \infty$ , with  $\frac{nm}{N} \rightarrow t \in (0, \infty)$ . Then the law of*

$$X(n, m, N)$$

*converges to Marchenko-Pastur law  $\pi_t$ .*

- (2) *Let  $n, m, N \rightarrow \infty$ , with  $\frac{n}{N} \rightarrow \nu \in (0, 1)$  and  $\frac{m}{N} \rightarrow 0$ . Then the law of*

$$S(n, m, N) = (X(n, m, N) - m\nu) / \sqrt{m\nu(1-\nu)}$$

*converges to the semicircle law  $\gamma_1$ .*

PROOF. This is standard, by using the Weingarten formula, as follows:

- (1) From the Weingarten formula, we have:

$$\int X(n, m, N)^k = \sum_{\pi, \sigma \in NC(k)} W_{kN}(\pi, \sigma) n^{|\pi|} m^{|\sigma|}$$

The point now is that we have the following estimate:

$$W_{kN}(\pi, \sigma) = \begin{cases} N^{-|\pi|} + O(N^{-|\pi|-1}) & \text{if } \pi = \sigma \\ O(N^{|\pi \vee \sigma| - |\pi| - |\sigma|}) & \text{if } \pi \neq \sigma \end{cases}$$

It follows that we have:

$$W_{kN}(\pi, \sigma) n^{|\pi|} m^{|\sigma|} \rightarrow \begin{cases} t^{|\pi|} & \text{if } \pi = \sigma \\ 0 & \text{if } \pi \neq \sigma \end{cases}$$

Thus the  $k$ -th moment of  $X(n, m, N)$  converges to:

$$\sum_{\pi \in NC(k)} \lambda^{|\pi|}$$

But this is the  $k$ -th moment of the Marchenko-Pastur law  $\pi_t$ , and we are done.

(2) We need to show that the free cumulants satisfy:

$$\kappa^{(p)}[S(n, m, N), \dots, S(n, m, N)] \rightarrow \begin{cases} 1 & \text{if } p = 2 \\ 0 & \text{if } p \neq 2 \end{cases}$$

The case  $p = 1$  is trivial, so suppose  $p \geq 2$ . We have:

$$\begin{aligned} & \kappa^{(p)}[S(n, m, N), \dots, S(n, m, N)] \\ &= (m\nu(1-\nu))^{-p/2} \kappa^{(p)}[X(n, m, N), \dots, X(n, m, N)] \end{aligned}$$

On the other hand, from the Weingarten formula, we have:

$$\begin{aligned} & \kappa^{(p)}[X(n, m, N), \dots, X(n, m, N)] \\ &= \sum_{w \in NC(p)} \mu_p(w, 1_p) \prod_{V \in w} \sum_{\pi_V, \sigma_V \in NC(V)} W_{NC(V), N}(\pi_V, \sigma_V) n^{|\pi_V|} m^{|\sigma_V|} \\ &= \sum_{w \in NC(p)} \mu_p(w, 1_p) \prod_{V \in w} \sum_{\pi_V, \sigma_V \in NC(V)} (N^{-|\pi_V|} \mu_{|V|}(\pi_V, \sigma_V) + O(N^{-|\pi_V|-1})) n^{|\pi_V|} m^{|\sigma_V|} \\ &= \sum_{\substack{\pi, \sigma \in NC(p) \\ \pi \leq \sigma}} (N^{-|\pi|} \mu_p(\pi, \sigma) + O(N^{-|\pi|-1})) n^{|\pi|} m^{|\sigma|} \sum_{\substack{w \in NC(p) \\ \sigma \leq w}} \mu_p(w, 1_p) \end{aligned}$$

We use now the following standard identity:

$$\sum_{\substack{w \in NC(p) \\ \sigma \leq w}} \mu_p(w, 1_p) = \begin{cases} 1 & \text{if } \sigma = 1_p \\ 0 & \text{if } \sigma \neq 1_p \end{cases}$$

This gives the following formula for the cumulants:

$$\begin{aligned} & \kappa^{(p)}[X(n, m, N), \dots, X(n, m, N)] \\ &= m \sum_{\pi \in NC(p)} (N^{-|\pi|} \mu_p(\pi, 1_p) + O(N^{-|\pi|-1})) n^{|\pi|} \end{aligned}$$

It follows that for  $p \geq 3$  we have, as desired:

$$\kappa^{(p)}[S(n, m, N), \dots, S(n, m, N)] \rightarrow 0$$

As for the remaining case  $p = 2$ , here we have:

$$\begin{aligned} \kappa^{(2)}[S(n, m, N), S(n, m, N)] &\rightarrow \frac{1}{\nu(1-\nu)} \sum_{\pi \in NC(2)} \nu^{|\pi|} \mu_2(\pi, 1_2) \\ &= \frac{1}{\nu(1-\nu)} (\nu - \nu^2) \\ &= 1 \end{aligned}$$

This gives the result. □

We refer to [19] for more results and comments regarding the hypergeometric and hyperspherical laws, and the relation between them.

### 8d. Traces of powers

As a final analytic topic, let us discuss now, following [29], the computation of the asymptotic laws of the following variables, depending on an integer  $k \in \mathbb{N}$ :

$$\chi^{(k)} = Tr(u^k)$$

These variables, called Diaconis-Shahshahani variables, following the paper [59], generalize the usual characters, which appear at  $k = 1$ :

$$\chi = Tr(u)$$

As before with the previous analytic questions that we investigated, we will do the computations for the main quantum permutation and rotation groups, namely:

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array}$$

In order to do our computation, let us start with the following standard definition:



DEFINITION 8.13. Associated to  $k_1, \dots, k_s \in \mathbb{N}$  is the permutation  $\gamma \in S_k$ , having cycles as follows, with  $k = \sum k_i$ ,

$$\begin{aligned} & (1, \dots, k_1) \\ & (k_1 + 1, \dots, k_1 + k_2) \\ & \vdots \\ & (k - k_s + 1, \dots, k) \end{aligned}$$

called trace permutation associated to  $k_1, \dots, k_s \in \mathbb{N}$ .

We denote by  $\gamma(\sigma)$  the partition given by:

$$i \sim_\sigma j \iff \gamma(i) \sim_{\gamma(\sigma)} \gamma(j)$$

With these conventions, we have the following result:

THEOREM 8.14. Given an easy quantum group  $G$ , we have:

$$\int_G \text{Tr}(u^{k_1}) \dots \text{Tr}(u^{k_s}) du = \# \left\{ \pi \in D_k \mid \pi = \gamma(\pi) \right\} + O(N^{-1})$$

If  $G$  is classical, this estimate is exact, without any lower order corrections.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Let  $I$  be the integral to be computed. According to the definition of  $\gamma$ , we have:

$$\begin{aligned} I &= \int_G \text{Tr}(u^{k_1}) \dots \text{Tr}(u^{k_s}) du \\ &= \sum_{i_1 \dots i_k} \int_G (u_{i_1 i_2} \dots u_{i_k i_1}) \dots (u_{i_{k-k_s+1} i_{k-k_s+2}} \dots u_{i_k i_{k-k_s+1}}) \\ &= \sum_{i_1 \dots i_k} \int_G u_{i_1 i_{\gamma(1)}} \dots u_{i_k i_{\gamma(k)}} \end{aligned}$$

We use now the Weingarten formula. We obtain:

$$\begin{aligned} I &= \sum_{i_1 \dots i_k} \sum_{\pi \leq \ker i} \sum_{\sigma \leq \ker i \gamma} W_{kN}(\pi, \sigma) \\ &= \sum_{i_1 \dots i_k} \sum_{\pi \leq \ker i} \sum_{\gamma(\sigma) \leq \ker i} W_{kN}(\pi, \sigma) \\ &= \sum_{\pi, \sigma \in D_k} N^{|\pi \vee \gamma(\sigma)|} W_{kN}(\pi, \sigma) \\ &= \sum_{\pi, \sigma \in D_k} N^{|\pi \vee \gamma(\sigma)|} N^{|\pi \vee \sigma| - |\pi| - |\sigma|} (1 + O(N^{-1})) \end{aligned}$$

Let us look now at the power of  $N$  in the above, namely:

$$N^{|\pi \vee \gamma(\sigma)| + |\pi \vee \sigma| - |\pi| - |\sigma|}$$

The leading order is  $N^0$ , which is achieved if and only if  $\sigma \geq \pi$  and  $\pi \geq \gamma(\sigma)$ , or equivalently when  $\pi = \sigma = \gamma(\sigma)$ . But this gives the formula.

(2) In the classical case, instead of using the approximation for  $W_{kN}(\pi, \sigma)$ , we can write  $N^{|\pi \vee \gamma(\sigma)|} = G_{kN}(\gamma(\sigma), \pi)$ , and we can continue as follows:

$$\begin{aligned} I &= \sum_{\pi, \sigma \in D_k} G_{kN}(\gamma(\sigma), \pi) W_{kN}(\pi, \sigma) \\ &= \sum_{\sigma \in D_k} \delta(\gamma(\sigma), \sigma) \\ &= \#\{\sigma \in D_k \mid \sigma = \gamma(\pi)\} \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

If  $c$  is a cycle we use the notation  $c^1 = c$ , and  $c^*$  = cycle opposite to  $c$ .

We have the following definition, generalizing Definition 8.13 above:

DEFINITION 8.15. *Associated to any  $k_1, \dots, k_s \in \mathbb{N}$  and any  $e_1, \dots, e_s \in \{1, *\}$  is the trace permutation  $\gamma \in S_k$ , with  $k = \sum k_i$ , having as cycles*

$$\begin{aligned} &(1, \dots, k_1)^{e_1} \\ &(k_1 + 1, \dots, k_1 + k_2)^{e_2} \\ &\quad \vdots \\ &(k - k_s + 1, \dots, k)^{e_s} \end{aligned}$$

*called trace permutation associated to  $k_1, \dots, k_s \in \mathbb{N}$  and  $e_1, \dots, e_s \in \{1, *\}$ .*

With this convention, Theorem 8.14 extends to this setting, as follows:

THEOREM 8.16. *Given an easy quantum group  $G$ , we have:*

$$\int_G \text{Tr}(u^{k_1})^{e_1} \dots \text{Tr}(u^{k_s})^{e_s} du = \#\{\pi \in D_k \mid \pi = \gamma(\pi)\} + O(N^{-1})$$

*If  $G$  is classical, this estimate is exact, without any lower order corrections.*

PROOF. This is similar to the proof of Theorem 8.14.  $\square$

In terms of cumulants, we have the following result, also from [29]:

THEOREM 8.17. For  $G = O_N, S_N$  we have the following cumulant formula:

$$c_r(\text{Tr}(u^{k_1})^{e_1}, \dots, \text{Tr}(u^{k_r})^{e_r}) = \#\left\{\pi \in D_k \mid \pi \vee \gamma = 1_k, \pi = \gamma(\pi)\right\}$$

Also, for  $G = O_N^+, S_N^+$  we have the following free cumulant formula:

$$\kappa_r(\text{Tr}(u^{k_1})^{e_1}, \dots, \text{Tr}(u^{k_r})^{e_r}) = \#\left\{\pi \in D_k \mid \pi \vee \gamma = 1_k, \pi = \gamma(\pi)\right\} + O(N^{-1})$$

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Let  $c_r$  be the considered cumulant. We write, for those partitions  $\pi \in P_k$  such that the restriction of  $\pi$  to a block of  $\sigma$  is an element in the corresponding set  $D_{|v|}$ :

$$D_\sigma = \left\{\pi \in P_k \mid p|_v \in D_{|v|} \forall v \in \sigma\right\}$$

We have then the following equivalent formula:

$$D_\sigma = \left\{\pi \in D_k \mid \pi \leq \sigma^\gamma\right\}$$

Then, by the definition of the classical cumulants via Möbius inversion, we get:

$$\begin{aligned} c_r &= \sum_{\sigma \in P(r)} \mu(\sigma, 1_r) \cdot \#\{\pi \in D_\sigma \mid \pi = \gamma(\pi)\} \\ &= \sum_{\sigma \in P(r)} \mu(\sigma, 1_r) \cdot \#\{\pi \in D_k \mid \pi \leq \sigma^\gamma, \pi = \gamma(\pi)\} \\ &= \sum_{\sigma \in P(r)} \mu(\sigma, 1_r) \sum_{\pi \leq \sigma^\gamma, \pi = \gamma(\pi)} 1 \end{aligned}$$

In order to exchange the two summations, we first have to replace the summation over  $\sigma \in P(r)$  by a summation over  $\tau = \sigma^\gamma \in P(k)$ . Note that the condition on the latter is exactly  $\tau \geq \gamma$  and that we have  $\mu(\sigma, 1_r) = \mu(\sigma^\gamma, 1_k)$ . Thus:

$$\begin{aligned} c_r &= \sum_{\tau \geq \gamma} \mu(\tau, 1_k) \sum_{\pi \leq \tau, \pi = \gamma(\pi)} 1 \\ &= \sum_{\pi = \gamma(\pi)} \sum_{\pi \vee \gamma \leq \tau} \mu(\tau, 1_k) \end{aligned}$$

The definition of the Möbius function gives for the second summation:

$$\sum_{\pi \vee \gamma \leq \tau} \mu(\tau, 1_k) = \begin{cases} 1 & \text{if } \pi \vee \gamma = 1_k \\ 0 & \text{otherwise} \end{cases}$$

With this formula in hand, the assertion follows.

(2) In the free case, the proof runs in the same way, by using free cumulants and the corresponding Möbius function on noncrossing partitions. Note that we have the analogue of our equation in this case only for noncrossing  $\sigma$ .  $\square$

We can now recover the theorem of Diaconis and Shahshahani in [59]:

**THEOREM 8.18.** *The variables  $u_k = \lim_{N \rightarrow \infty} \text{Tr}(u^k)$  are as follows:*

- (1) *For  $O_N$ , the  $u_k$  are real Gaussian variables, with variance  $k$  and mean 0 or 1, depending on whether  $k$  is odd or even. The  $u_k$  are independent.*
- (2) *For  $O_N^+$ , at  $k = 1, 2$  we get semicircular variables of variance 1 and mean 0 for  $u_1$  and mean 1 for  $u_2$ , and at  $k \geq 3$  we get circular variables of mean 0 and covariance 1. The  $u_k$  are  $*$ -free.*

**PROOF.** This follows by using the formula in Theorem 8.17, as follows:

(1) In this case  $D_k$  consists of all pairings of  $k$  elements. We have to count all pairings  $\pi$  with the properties that  $\pi \vee \gamma = 1_k$  and  $\pi = \gamma(\pi)$ .

Note that if  $\pi$  connects two different cycles of  $\gamma$ , say  $c_i$  and  $c_j$ , then the property  $\pi = \gamma(\pi)$  implies that each element from  $c_i$  must be paired with an element from  $c_j$ . Thus those cycles cannot be connected to other cycles and they must contain the same number of elements. This means that for  $s \geq 3$  there is no such  $\pi$ . Thus all cumulants of order 3 and higher vanish asymptotically and all traces are asymptotically Gaussian.

Since in the case  $s = 2$  we only have permissible pairings if the two cycles have the same number of elements, we also see that the covariance between traces of different powers vanishes and thus different powers are asymptotically independent. The variance of  $u_k$  is given by the number of matchings between  $\{1, \dots, k\}$  and  $\{k+1, \dots, 2k\}$  which are invariant under rotations. Since such a matching is determined by the partner of the first element 1, for which we have  $k$  possibilities, the variance of  $u_k$  is  $k$ . For the mean, if  $k$  is odd there is clearly no pairing at all, and if  $k = 2p$  is even then the only pairing of  $\{1, \dots, 2p\}$  which is invariant under rotations is  $(1, p+1), (2, p+2), \dots, (p, 2p)$ . Thus the mean of  $u_k$  is zero if  $k$  is odd and 1 if  $k$  is even.

(2) In the quantum case  $D_k$  consists of noncrossing pairings. We can essentially repeat the arguments from above but have to take care that only noncrossing pairings are counted. We also have to realize that for  $k \geq 3$ , the  $u_k$  are not selfadjoint any longer, thus we have to consider also  $u_k^*$  in these cases. This means that in our arguments we have to allow cycles which are rotated “backwards” under  $\gamma$ .

By the same reasoning as before we see that free cumulants of order three and higher vanish. The pairing which gave mean 1 in the classical case is only in the case  $k = 2$  a noncrossing one, thus the mean of  $u_2$  is 1, all other means are zero. For the variances, one has again that different powers allow no pairings at all and are asymptotically  $*$ -free. For the matchings between  $\{1, \dots, k\}$  and  $\{k+1, \dots, 2k\}$  one has to observe that there is only one non-crossing possibility, namely  $(1, 2k), (2, 2k-1), \dots, (k, k+1)$  and this satisfies  $\pi = \gamma(\pi)$  only if  $\gamma$  rotates both cycles in different directions.

For  $k = 1$  and  $k = 2$  there is no difference between both directions, but for  $k \geq 3$  this implies that we get only a non-vanishing covariance between  $u_k$  and  $u_k^*$ , with value 1. This shows that  $u_1$  and  $u_2$  are semicircular, whereas the higher  $u_k$  are circular.  $\square$

In order to discuss permutations and quantum permutations, let us start with:

PROPOSITION 8.19. *The cumulants of  $u_k = \lim_{N \rightarrow \infty} Tr(u^k)$  are as follows:*

(1) *For  $S_N$ , the classical cumulants are given by:*

$$c_r(u_{k_1}, \dots, u_{k_r}) = \sum_{q|k_i \forall i=1, \dots, r} q^{r-1}$$

(2) *For  $S_N^+$ , the free cumulants are given by:*

$$c_r(u_{k_1}^{e_1}, \dots, u_{k_r}^{e_r}) = \begin{cases} 2 & \text{if } r = 1, k_1 \geq 2 \\ 2 & \text{if } r = 2, k_1 = k_2, e_1 = e_2^* \\ 2 & \text{if } r = 2, k_1 = k_2 = 2 \\ 1 & \text{otherwise} \end{cases}$$

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Here  $D_k$  consists of all partitions. We have to count partitions  $\pi$  which have the properties that  $\pi \vee \gamma = 1_k$  and  $\pi = \gamma(\pi)$ .

Consider a partition  $\pi$  which connects different cycles of  $\gamma$ . Consider the restriction of  $\pi$  to one cycle. Let  $k$  be the number of elements in this cycle and  $t$  be the number of the points in the restriction. Then the orbit of those  $t$  points under  $\gamma$  must give a partition of that cycle, which means that  $t$  is a divisor of  $k$  and that the  $t$  points are equally spaced. The same must be true for all cycles of  $\gamma$  which are connected via  $\pi$ , and the ratio between  $t$  and  $k$  is the same for all those cycles.

But this means that if one block of  $\pi$  connects some cycles then the orbit under  $\gamma$  of this block connects exactly those cycles and exhausts all points of those cycles. So if we want to connect all cycles of  $\gamma$  then this can only happen in the way that we have one block of  $\pi$  intersecting each of the cycles of  $\gamma$ .

To be more precise, let us consider  $c_r(u_{k_1}, \dots, u_{k_r})$ . We have then to look for a common divisor  $q$  of all  $k_1, \dots, k_r$ , and a contributing  $\pi$  is then one the blocks of which are of the following form:  $k_1/q$  points in the first cycle, equally spaced, and so on up to  $k_r/q$  points in the last cycle, equally spaced.

We can specify this by saying to which points in the other cycles the first point in the first cycle is connected. There are  $q^{r-1}$  possibilities for such choices, and this gives the formula in the statement.

(2) For  $S_N^+$  we have to consider noncrossing partitions instead of all partitions. Most of the contributing partitions from the classical case are crossing, so do not count for the quantum case.

Actually, whenever a restriction of a block to one cycle has two or more elements then the corresponding partition is crossing, unless the restriction exhausts the whole group.

This is the case  $q = 1$  from the considerations above, corresponding to the partition which has only one block, giving a contribution 1 to each cumulant  $c_r(u_{k_1}, \dots, u_{k_r})$ .

For cumulants of order 3 or higher there are no other contributions. For cumulants of second order one might also have contributions coming from pairings, where each restriction of a block to a cycle has one element.

But this is the same problem as in the  $O_N^+$  case, and we only get an additional contribution for the second order cumulants  $c_2(u_k, u_k^*)$ . For first order cumulants, singletons can also appear and make an additional contribution.

Taking this all together gives the formula in the statement. □

With the above ingredients in hand, we can now formulate a result about permutations and quantum permutations:

**THEOREM 8.20.** *The variables  $u_k = \lim_{N \rightarrow \infty} \text{Tr}(u^k)$  are as follows:*

(1) *For  $S_N$  we have a decomposition of type*

$$u_k = \sum_{l|k} l C_l$$

*with the variables  $C_k$  being Poisson of parameter  $1/k$ , and independent.*

(2) *For  $S_N^+$  we have a decomposition of the type*

$$u_1 = C_1 \quad , \quad u_k = C_1 + C_k \quad (k \geq 2)$$

*where the variables  $C_l$  are \*-free,  $C_1$  is free Poisson,  $C_2$  is semicircular, and  $C_k$  with  $k \geq 3$  are circular.*

**PROOF.** We have several assertions to be proved, the idea being as follows:

(1) Let  $C_k$  be the number of cycles of length  $k$ . We have  $u_k = \sum_{l|k} l C_l$ . We are claiming now that the  $C_k$  are independent and each is a Poisson variable of parameter  $1/k$ , i.e., that  $c_r(C_{l_1}, \dots, C_{l_r})$  is zero unless all the  $l_i$  are the same, say  $= l$ , in which case it is  $1/l$ , independently of  $r$ . This is compatible with the cumulants for the  $u_k$ , according

to the following computation:

$$\begin{aligned} c_r(u_{k_1}, \dots, u_{k_r}) &= \sum_{l_1|k_1} \dots \sum_{l_r|k_r} l_1 \dots l_r c_r(C_{l_1}, \dots, C_{l_r}) \\ &= \sum_{l|k_i \forall i} l^r \frac{1}{l} \end{aligned}$$

Since the  $C_k$  are uniquely determined by the  $u_k$ , via some kind of Möbius inversion, this shows that the  $C_k$  are independent, and that  $C_k$  is Poisson with parameter  $1/k$ .

(2) In the classical case the random variable  $C_l$  can be defined by:

$$C_l = \frac{1}{l} \sum_{i_1 \dots i_l \text{ distinct}} u_{i_1 i_2} u_{i_2 i_3} \dots u_{i_l i_1}$$

Note that we divide by  $l$  because each term appears actually  $l$  times, in cyclically permuted versions, which are all the same because our variables commute.

Note that, by using commutativity and the monomial condition, in general the expression  $u_{i_1 i_2} u_{i_2 i_3} \dots u_{i_l i_1}$  has to be zero unless the indices  $(i_1, \dots, i_l)$  are of the form  $(i_1, \dots, i_l, i_1, \dots, i_l, \dots)$ , where  $l$  divides  $k$  and  $i_1, \dots, i_l$  are distinct. This yields then the following relation, which we used before to define  $C_l$ :

$$\begin{aligned} Tr(u^k) &= \sum_{i_1 \dots i_l} u_{i_1 i_2} u_{i_2 i_3} \dots u_{i_l i_1} \\ &= \sum_{l|k} \sum_{i_1 \dots i_l \text{ distinct}} (u_{i_1 i_2} u_{i_2 i_3} \dots u_{i_l i_1})^{k/l} \\ &= \sum_{l|k} l C_l \end{aligned}$$

This explicit form of  $C_l$  in terms of  $u_{ij}$  can be used to give a direct proof, by using the Weingarten formula, of the fact that the  $C_l$  are independent and Poisson.

(3) In the free case we define the “cycle”  $C_l$  by requiring neighboring indices to be different, as follows:

$$C_l = \sum_{i_1 \neq i_2 \neq \dots \neq i_l \neq i_1} u_{i_1 i_2} u_{i_2 i_3} \dots u_{i_l i_1}$$

Note that if two adjacent indices are the same in  $u_{i_1 i_2} u_{i_2 i_3} \dots u_{i_l i_1}$  then, because of the relation  $u_{ij} u_{ik} = 0$  for  $j \neq k$ , all must be the same or the term vanishes. For the case where all indices are the same we have:

$$\begin{aligned} \sum_i u_{ii} u_{ii} \dots u_{ii} &= \sum_i u_{ii} \\ &= C_1 \end{aligned}$$

But this gives then the following relation:

$$\text{Tr}(u^k) = C_k + C_1$$

Again, the  $C_l$  are uniquely determined by the  $\text{Tr}(u^k)$  and thus our calculations also show that the  $C_l$  defined by our equation are  $*$ -free and have the distributions as stated. Thus, we are led to the conclusion in the statement.  $\square$

### 8e. Exercises

There are many interesting exercises, in connection with the above material. Let us start with some standard calculus. First, we have:

EXERCISE 8.21. *Prove, with full details, that we have the equality of joint laws*

$$(u_{i1}, \dots, u_{iN}) \sim (x_1, \dots, x_N)$$

where  $u_{ij}$  are the coordinates of  $O_N$ , and  $x_i$  are the coordinates of  $S_{\mathbb{R}}^{N-1}$ .

This is something that we already discussed in the above, the idea being that of using the action  $O_N \curvearrowright S_{\mathbb{R}}^{N-1}$ . The problem is that of working out all the details.

Here is another classical exercise, which is instructive as well:

EXERCISE 8.22. *Prove that the volume of the unit sphere in  $\mathbb{R}^N$  is given by*

$$V \simeq \left(\frac{2\pi e}{N}\right)^{N/2} \frac{1}{\sqrt{\pi N}}$$

in the  $N \rightarrow \infty$  limit.

The obvious way to go here is with the Stirling formula, and the problem is that of working carefully what happens when  $N$  is odd, and when  $N$  is even, with the above formula, which is uniform in  $N$ , as a final conclusion.

Still in relation with such questions, we have:

EXERCISE 8.23. *Prove that the area of the unit sphere in  $\mathbb{R}^N$  is given by*

$$A = \left(\frac{\pi}{2}\right)^{[N/2]} \frac{2^N}{(N-1)!!}$$

with the our usual convention for the double factorials, namely

$$N!! = (N-1)(N-3)(N-5)\dots$$

and compute as well asymptotic estimates for  $A$ , in the  $N \rightarrow \infty$  limit.

As with the volume problem, the way to go here is via spherical coordinates.

Finally, here is one more classical exercise, of the same type:



EXERCISE 8.24. *Prove that we have the following integration formula over the sphere  $S_{\mathbb{R}}^{N-1}$ , with respect to the normalized measure, valid for any exponents  $k_i \in \mathbb{N}$ ,*

$$\int_{S_{\mathbb{R}}^{N-1}} |x_1^{k_1} \dots x_N^{k_N}| dx = \left(\frac{2}{\pi}\right)^{\Sigma(k_1, \dots, k_N)} \frac{(N-1)!! k_1!! \dots k_N!!}{(N + \Sigma k_i - 1)!!}$$

with  $\Sigma = [\text{odds}/2]$  if  $N$  is odd and  $\Sigma = [(\text{odds} + 1)/2]$  if  $N$  is even, where “odds” denotes the number of odd numbers in the sequence  $k_1, \dots, k_N$ .

As before with other such questions, the way to go is via spherical coordinates.

In relation now with the quantum group questions, we first have:

EXERCISE 8.25. *Find some evidence for the formula*

$$\int_{S_{\mathbb{R},+}^{N-1}} x_1^{2l} dx = \frac{1}{(N+1)^l} \cdot \frac{q+1}{q-1} \cdot \frac{1}{l+1} \sum_{r=-l-1}^{l+1} (-1)^r \binom{2l+2}{l+r+1} \frac{r}{1+q^r}$$

where  $q \in [-1, 0)$  is such that  $q + q^{-1} = -N$ , independent of the proof of this formula.

This is something quite tricky, involving some non-trivial computations.

Here is a more difficult question now, regarding the free hypergeometric laws:

EXERCISE 8.26. *Work out some basic theory for the free hypergeometric variables*

$$X(n, m, N) = \sum_{i=1}^n \sum_{j=1}^m u_{ij} \in C(S_N^+)$$

going beyond what was done in the above.

This is actually a research-level question, which nothing much being known.

Finally, in connection with the Diaconis-Shahshahani variables, we have:

EXERCISE 8.27. *Work out the basic theory of the Diaconis-Shahshahani variables*

$$\chi^{(k)} = \text{Tr}(u^k)$$

for the unitary group  $U_N$ , and for the free unitary quantum group  $U_N^+$ .

Here the problem is that of suitably adapting the above computations for  $O_N, O_N^+$ .

## Part III

# Quantum reflections

*I'm coming up only to hold you under  
I'm coming up only to show you wrong  
And to know you is hard, we wonder  
To know you all wrong, we warn*

## CHAPTER 9

### Finite graphs

#### 9a. Finite graphs

In this third part of this book, this chapter and the next 3 ones, we study the main examples of quantum subgroups  $G \subset S_N^+$ , or more generally  $G \subset S_F^+$ , with emphasis on the quantum reflection groups, which are the central examples of such subgroups.

Many interesting examples of quantum permutation groups appear as particular cases of the following general construction from [6], [7], involving finite graphs:

**PROPOSITION 9.1.** *Given a finite graph  $X$ , with adjacency matrix  $d \in M_N(0, 1)$ , the following construction produces a quantum permutation group,*

$$C(G^+(X)) = C(S_N^+) / \langle du = ud \rangle$$

whose classical version  $G(X)$  is the usual automorphism group of  $X$ .

**PROOF.** The fact that we have a quantum group comes from the fact that  $du = ud$  reformulates as  $d \in \text{End}(u)$ , which makes it clear that we are dividing by a Hopf ideal. Regarding the second assertion, we must establish here the following equality:

$$C(G(X)) = C(S_N) / \langle du = ud \rangle$$

For this purpose, recall that the coordinates are given by:

$$u_{ij}(\sigma) = \delta_{\sigma(j)i}$$

By using this formula, we have the following computation:

$$(du)_{ij}(\sigma) = \sum_k d_{ik} u_{kj}(\sigma) = \sum_k d_{ik} \delta_{\sigma(j)k} = d_{i\sigma(j)}$$

On the other hand, we have as well the following formula:

$$(ud)_{ij}(\sigma) = \sum_k u_{ik}(\sigma) d_{kj} = \sum_k \delta_{\sigma(k)i} d_{kj} = d_{\sigma^{-1}(i)j}$$

Thus the condition  $du = ud$  reformulates as  $d_{ij} = d_{\sigma(i)\sigma(j)}$ , and we are led to the usual notion of an action of a permutation group on  $X$ , as claimed.  $\square$

Let us work out some basic examples. We have the following result:

THEOREM 9.2. *The construction  $X \rightarrow G^+(X)$  has the following properties:*

- (1) *For the  $N$ -point graph, having no edges at all, we obtain  $S_N^+$ .*
- (2) *For the  $N$ -simplex, having edges everywhere, we obtain as well  $S_N^+$ .*
- (3) *We have  $G^+(X) = G^+(X^c)$ , where  $X^c$  is the complementary graph.*
- (4) *For a disconnected union, we have  $G^+(X) \hat{*} G^+(Y) \subset G^+(X \sqcup Y)$ .*
- (5) *For the square, we obtain a non-classical, proper subgroup of  $S_4^+$ .*

PROOF. All these results are elementary, the proofs being as follows:

(1) This follows from definitions, because here we have  $d = 0$ .

(2) Here  $d = \mathbb{I}$  is the all-one matrix, and the magic condition gives  $u\mathbb{I} = \mathbb{I}u = N\mathbb{I}$ . We conclude that  $du = ud$  is automatic in this case, and so  $G^+(X) = S_N^+$ .

(3) The adjacency matrices of  $X, X^c$  being related by the following formula:

$$d_X + d_{X^c} = \mathbb{I}$$

We can use here the above formula  $u\mathbb{I} = \mathbb{I}u = N\mathbb{I}$ , and we conclude that  $d_X u = u d_X$  is equivalent to  $d_{X^c} u = u d_{X^c}$ . Thus, we obtain, as claimed,  $G^+(X) = G^+(X^c)$ .

(4) The adjacency matrix of a disconnected union is given by:

$$d_{X \sqcup Y} = \text{diag}(d_X, d_Y)$$

Now let  $w = \text{diag}(u, v)$  be the fundamental corepresentation of  $G^+(X) \hat{*} G^+(Y)$ . Then  $d_X u = u d_X$  and  $d_Y v = v d_Y$  imply, as desired,  $d_{X \sqcup Y} w = w d_{X \sqcup Y}$ .

(5) We know from (3) that we have  $G^+(\square) = G^+(\mid \mid)$ . We know as well from (4) that we have  $\mathbb{Z}_2 \hat{*} \mathbb{Z}_2 \subset G^+(\mid \mid)$ . It follows that  $G^+(\square)$  is non-classical. Finally, the inclusion  $G^+(\square) \subset S_4^+$  is indeed proper, because  $S_4 \subset S_4^+$  does not act on the square.  $\square$

In order to further advance, and to explicitly compute various quantum automorphism groups, we can use the spectral decomposition of  $d$ , as follows:

PROPOSITION 9.3. *A closed subgroup  $G \subset S_N^+$  acts on a graph  $X$  precisely when*

$$P_\lambda u = u P_\lambda \quad , \quad \forall \lambda \in \mathbb{R}$$

where  $d = \sum_\lambda \lambda \cdot P_\lambda$  is the spectral decomposition of the adjacency matrix of  $X$ .

PROOF. Since  $d \in M_N(0, 1)$  is a symmetric matrix, we can consider indeed its spectral decomposition,  $d = \sum_\lambda \lambda \cdot P_\lambda$ . We have then the following formula:

$$\langle d \rangle = \text{span} \left\{ P_\lambda \mid \lambda \in \mathbb{R} \right\}$$

But this shows that we have the following equivalence:

$$d \in \text{End}(u) \iff P_\lambda \in \text{End}(u), \forall \lambda \in \mathbb{R}$$

Thus, we are led to the conclusion in the statement.  $\square$

In order to exploit this, we will often combine it with the following standard fact:

PROPOSITION 9.4. *Consider a closed subgroup  $G \subset S_N^+$ , with associated coaction map*

$$\Phi : \mathbb{C}^N \rightarrow \mathbb{C}^N \otimes C(G)$$

For a linear subspace  $V \subset \mathbb{C}^N$ , the following are equivalent:

- (1) *The magic matrix  $u = (u_{ij})$  commutes with  $P_V$ .*
- (2)  *$V$  is invariant, in the sense that  $\Phi(V) \subset V \otimes C(G)$ .*

PROOF. Let  $P = P_V$ . For any  $i \in \{1, \dots, N\}$  we have the following formula:

$$\begin{aligned} \Phi(P(e_i)) &= \Phi\left(\sum_k P_{ki}e_k\right) \\ &= \sum_{jk} P_{ki}e_j \otimes u_{jk} \\ &= \sum_j e_j \otimes (uP)_{ji} \end{aligned}$$

On the other hand the linear map  $(P \otimes id)\Phi$  is given by a similar formula:

$$\begin{aligned} (P \otimes id)(\Phi(e_i)) &= \sum_k P(e_k) \otimes u_{ki} \\ &= \sum_{jk} P_{jk}e_j \otimes u_{ki} \\ &= \sum_j e_j \otimes (Pu)_{ji} \end{aligned}$$

Thus  $uP = Pu$  is equivalent to  $\Phi P = (P \otimes id)\Phi$ , and the conclusion follows.  $\square$

We have as well the following useful complementary result, from [6]:

PROPOSITION 9.5. *Let  $p \in M_N(\mathbb{C})$  be a matrix, and consider its “color” decomposition, obtained by setting  $(p_c)_{ij} = 1$  if  $p_{ij} = c$  and  $(p_c)_{ij} = 0$  otherwise:*

$$p = \sum_{c \in \mathbb{C}} c \cdot p_c$$

Then  $u = (u_{ij})$  commutes with  $p$  if and only if it commutes with all matrices  $p_c$ .

PROOF. Consider the multiplication and counit maps of the algebra  $\mathbb{C}^N$ :

$$M : e_i \otimes e_j \rightarrow e_i e_j$$

$$C : e_i \rightarrow e_i \otimes e_i$$

Since  $M, C$  intertwine  $u, u^{\otimes 2}$ , their iterations  $M^{(k)}, C^{(k)}$  intertwine  $u, u^{\otimes k}$ , and so:

$$\begin{aligned} p^{(k)} &= M^{(k)} p^{\otimes k} C^{(k)} \\ &= \sum_{c \in \mathbb{C}} c^k p_c \\ &\in \text{End}(u) \end{aligned}$$

Let  $S = \{c \in \mathbb{C} | p_c \neq 0\}$ , and  $f(c) = c$ . By Stone-Weierstrass we have  $S = \langle f \rangle$ , and so for any  $e \in S$  the Dirac mass at  $e$  is a linear combination of powers of  $f$ :

$$\begin{aligned} \delta_e &= \sum_k \lambda_k f^k \\ &= \sum_k \lambda_k \left( \sum_{c \in S} c^k \delta_c \right) \\ &= \sum_{c \in S} \left( \sum_k \lambda_k c^k \right) \delta_c \end{aligned}$$

The corresponding linear combination of matrices  $p^{(k)}$  is given by:

$$\begin{aligned} \sum_k \lambda_k p^{(k)} &= \sum_k \lambda_k \left( \sum_{c \in S} c^k p_c \right) \\ &= \sum_{c \in S} \left( \sum_k \lambda_k c^k \right) p_c \end{aligned}$$

The Dirac masses being linearly independent, in the first formula all coefficients in the right term are 0, except for the coefficient of  $\delta_e$ , which is 1. Thus the right term in the second formula is  $p_e$ , and it follows that we have  $p_e \in \text{End}(u)$ , as claimed.  $\square$

The above results can be combined, and we are led to the following statement:

**THEOREM 9.6.** *A closed subgroup  $G \subset S_N^+$  acts on a graph  $X$  precisely when*

$$u = (u_{ij})$$

*commutes with all the matrices coming from the color-spectral decomposition of  $d$ .*

**PROOF.** This follows by combining Proposition 9.3 and Proposition 9.5, with the “color-spectral” decomposition in the statement referring to what comes out by successively doing the color and spectral decomposition, until the process stabilizes.  $\square$

The above statement might seem in need of some further discussion, and axiomatization, in what regards the two operations used there. In answer to all this, the point is that we are in fact doing planar algebras. We have the following result, from [7]:

**THEOREM 9.7.** *The planar algebra associated to  $G^+(X)$  is equal to the planar algebra generated by  $d$ , viewed as a 2-box in the spin planar algebra  $\mathcal{S}_N$ , with  $N = |X|$ .*

**PROOF.** We recall from chapter 3 above that any quantum permutation group  $G \subset S_N^+$  produces a subalgebra  $P \subset \mathcal{S}_N$  of the spin planar algebra, given by:

$$P_k = \text{Fix}(u^{\otimes k})$$

In our case, the idea is that  $G = G^+(X)$  comes via the relation  $d \in \text{End}(u)$ , but we can view this relation, via Frobenius duality, as a relation of type:

$$\xi_d \in \text{Fix}(u^{\otimes 2})$$

Indeed, let us view the adjacency matrix  $d \in M_N(0, 1)$  as a 2-box in  $\mathcal{S}_N$ , by using the canonical identification between  $M_N(\mathbb{C})$  and the algebra of 2-boxes  $\mathcal{S}_N(2)$ :

$$(d_{ij}) \leftrightarrow \sum_{ij} d_{ij} \begin{pmatrix} i & i \\ j & j \end{pmatrix}$$

Let  $P$  be the planar algebra associated to  $G^+(X)$  and let  $Q$  be the planar algebra generated by  $d$ . The action of  $u^{\otimes 2}$  on  $d$  viewed as a 2-box is given by:

$$\begin{aligned} u^{\otimes 2} \left( \sum_{ij} d_{ij} \begin{pmatrix} i & i \\ j & j \end{pmatrix} \right) &= \sum_{ijkl} d_{ij} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes u_{ki} u_{lj} \\ &= \sum_{kl} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes (udu^t)_{kl} \end{aligned}$$

Since  $v$  is a magic unitary commuting with  $d$  we have:

$$udu^t = duu^t = d$$

This means that  $d$ , viewed as a 2-box, is in the algebra  $P_2$  of fixed points of  $u^{\otimes 2}$ . Thus we have  $Q \subset P$ . For  $P \subset Q$  we use the duality found in chapter 3. Let indeed  $(B, v)$  be the pair whose associated planar algebra is  $Q$ . The same computation with  $v$  at the place of  $u$  shows that  $v$  commutes with  $d$ . Thus we have a morphism  $A \rightarrow B$ , given by  $u_{ij} \rightarrow v_{ij}$ , and it follows that we have  $P \subset Q$ , and we are done.  $\square$

With the above results in hand, it is quite clear that our assumption that  $d \in M_N(0, 1)$  is the adjacency matrix of a usual graph  $X$  is somehow unnatural, and that we can look at more general objects. We can consider for instance general permutation quantum groups of the following type, depending on an arbitrary matrix  $d \in M_N(\mathbb{C})$ :

$$C(G^+(X)) = C(S_N^+) / \langle du = ud \rangle$$

Here  $X$  stands for the combinatorial object associated to  $d$ , namely the complete graph having as vertices  $\{1, \dots, N\}$ , with each oriented edge  $i \rightarrow j$  colored by  $d_{ij} \in \mathbb{C}$ . Generally speaking, the theory extends well to this setting, and we have analogues of the



above results, some valid for any  $d \in M_N(\mathbb{C})$ , and some other valid under the assumption  $d = d^*$ . We refer to [7] and subsequent papers for a full discussion here.

### 9b. Circulant graphs

With these issues discussed, so let us get back now to concrete things. As a basic application of the above results, following [6], we can further study  $G^+(\square)$ , as follows:

**THEOREM 9.8.** *The quantum automorphism group of the  $N$ -cycle is as follows:*

- (1) *At  $N \neq 4$  we have  $G^+(X) = D_N$ .*
- (2) *At  $N = 4$  we have  $D_4 \subset G^+(X) \subset S_4^+$ , with proper inclusions.*

**PROOF.** We know that the results hold at  $N \leq 4$ , so let us assume  $N \geq 5$ . Given a  $N$ -th root of unity,  $w^N = 1$ , the vector  $\xi = (w^i)$  is an eigenvector of  $d$ , with eigenvalue:

$$\lambda = w + w^{N-1}$$

Now by taking  $w = e^{2\pi i/N}$ , it follows that the are eigenvectors of  $d$  are:

$$1, f, f^2, \dots, f^{N-1}$$

More precisely, the invariant subspaces of  $d$  are as follows, with the last subspace having dimension 1 or 2 depending on the parity of  $N$ :

$$\mathbb{C}1, \mathbb{C}f \oplus \mathbb{C}f^{N-1}, \mathbb{C}f^2 \oplus \mathbb{C}f^{N-2}, \dots$$

Consider now the associated coaction  $\Phi : \mathbb{C}^N \rightarrow \mathbb{C}^N \otimes C(G)$ , and write:

$$\Phi(f) = f \otimes a + f^{N-1} \otimes b$$

By taking the square of this equality we obtain:

$$\Phi(f^2) = f^2 \otimes a^2 + f^{N-2} \otimes b^2 + 1 \otimes (ab + ba)$$

It follows that  $ab = -ba$ , and that  $\Phi(f^2)$  is given by the following formula:

$$\Phi(f^2) = f^2 \otimes a^2 + f^{N-2} \otimes b^2$$

By multiplying this with  $\Phi(f)$  we obtain:

$$\Phi(f^3) = f^3 \otimes a^3 + f^{N-3} \otimes b^3 + f^{N-1} \otimes ab^2 + f \otimes ba^2$$

Now since  $N \geq 5$  implies that  $1, N-1$  are different from  $3, N-3$ , we must have  $ab^2 = ba^2 = 0$ . By using this and  $ab = -ba$ , we obtain by recurrence on  $k$  that:

$$\Phi(f^k) = f^k \otimes a^k + f^{N-k} \otimes b^k$$

In particular at  $k = N-1$  we obtain:

$$\Phi(f^{N-1}) = f^{N-1} \otimes a^{N-1} + f \otimes b^{N-1}$$

On the other hand we have  $f^* = f^{N-1}$ , so by applying  $*$  to  $\Phi(f)$  we get:

$$\Phi(f^{N-1}) = f^{N-1} \otimes a^* + f \otimes b^*$$

Thus  $a^* = a^{N-1}$  and  $b^* = b^{N-1}$ . Together with  $ab^2 = 0$  this gives:

$$\begin{aligned} (ab)(ab)^* &= abb^*a^* \\ &= ab^N a^{N-1} \\ &= (ab^2)b^{N-2}a^{N-1} \\ &= 0 \end{aligned}$$

From positivity we get from this  $ab = 0$ , and together with  $ab = -ba$ , this shows that  $a, b$  commute. On the other hand  $C(G)$  is generated by the coefficients of  $\Phi$ , which are powers of  $a, b$ , and so  $C(G)$  must be commutative, and we obtain the result.  $\square$

Summarizing, this was a bad attempt in understanding  $G^+(\square)$ , which appears to be “exceptional” among the quantum symmetry groups of the  $N$ -cycles. We will be back to this question later, with a second bad solution, and then with the good solution.

Following [17], let us keep discussing the case of the circulant graphs, which is quite interesting, due to the fact that we can use the Fourier transform. We first have:

**DEFINITION 9.9.** *Associated to any circulant graph  $X$  having  $N$  vertices are:*

- (1) *The set  $S \subset \mathbb{Z}_N$  given by  $i \sim_X j \iff j - i \in S$ .*
- (2) *The group  $E \subset \mathbb{Z}_N^*$  consisting of elements  $g$  such that  $gS = S$ .*
- (3) *The number  $k = |E|$ , called type of  $X$ .*

The interest in  $k$  comes from the fact that this is a good parameter measuring the “complexity of the spectral theory” of  $X$ . Here are a few basic examples:

- (1) The type can be  $2, 4, 6, 8, \dots$ . This is because  $\{\pm 1\} \subset E$ .
- (2) The cycle  $C_N$  is of type 2. Indeed, we have  $S = \{\pm 1\}$ ,  $E = \{\pm 1\}$ .
- (3) The empty graph  $X_N$  is of type  $\varphi(N)$ . Indeed, here  $S = \emptyset$ ,  $E = \mathbb{Z}_N^*$ .

Following [17], we can start by diagonalizing the adjacency matrix  $d \in M_N(0, 1)$ , by using the discrete Fourier transform, and we are led to the following result:

**THEOREM 9.10.** *A type  $k$  circulant graph having  $p \gg k$  vertices, with  $p$  prime, has no quantum symmetry.*

**PROOF.** This basically follows by using the same idea as above, for the cycle graph  $C_N$ , namely the spectral decomposition of the adjacency matrix  $d \in M_N(0, 1)$ , and then a number of arguments eventually converting the needed algebra into analysis.  $\square$

In the case where  $N$  is not prime, the way that the eigenvectors group into eigenspaces is something which is of subtle arithmetic nature, and in order to overcome the difficulties here, we must use a strategy different from the one in [17], namely doing as much algebra as we can, and in a “uniform” way, in our study of the actions  $G \curvearrowright X$ , and leaving the arithmetics for the end. In principle, this can lead to extensions of Theorem 9.10.

### 9c. The hypercube

Let us go back now to the square problem, that we left open in the above. An alternative approach to the understanding of  $G^+(\square)$  comes by regarding the square as the  $N = 2$  particular case of the  $N$ -hypercube  $\square_N$ . Indeed, the usual symmetry group of  $\square_N$  is the hyperoctahedral group  $H_N$ , so we should have a formula of type  $G(\square) = H_2^+$ .

Quite surprisingly, we will see that  $G^+(\square_N)$  is in fact a twist of  $O_N$ . In order to discuss this material, let us start with:

**THEOREM 9.11.** *There is a signature map  $\varepsilon : P_{\text{even}} \rightarrow \{-1, 1\}$ , given by*

$$\varepsilon(\tau) = (-1)^c$$

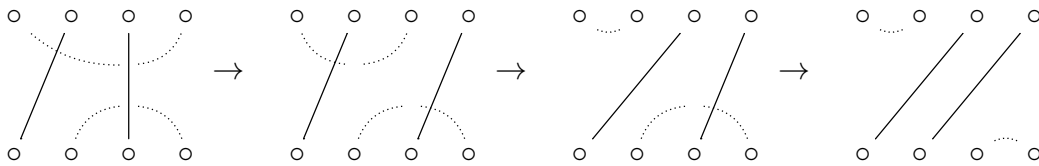
where  $c$  is the number of switches needed to make  $\tau$  noncrossing. In addition:

- (1) For  $\tau \in S_k$ , this is the usual signature.
- (2) For  $\tau \in P_2$  we have  $(-1)^c$ , where  $c$  is the number of crossings.
- (3) For  $\tau \leq \pi \in NC_{\text{even}}$ , the signature is 1.

**PROOF.** The fact that  $\varepsilon$  is indeed well-defined comes from the fact that the number  $c$  in the statement is well-defined modulo 2, which is standard combinatorics.

In order to prove the remaining assertion, observe that any partition  $\tau \in P(k, l)$  can be put in “standard form”, by ordering its blocks according to the appearance of the first leg in each block, counting clockwise from top left, and then by performing the switches as for block 1 to be at left, then for block 2 to be at left, and so on.

Here is an example of such an algorithmic switching operation, with block 1 being first put at left, by using two switches, then with block 2 left unchanged, and then with block 3 being put at left as well, but at right of blocks 1 and 2, with one switch:



With this convention, the proof of the remaining assertions is as follows:

(1) For  $\tau \in S_k$  the standard form is  $\tau' = id$ , and the passage  $\tau \rightarrow id$  comes by composing with a number of transpositions, which gives the signature.

(2) For a general  $\tau \in P_2$ , the standard form is of type  $\tau' = |\dots \cup_{\cap} \dots \cup_{\cap} \dots|$ , and the passage  $\tau \rightarrow \tau'$  requires  $c \bmod 2$  switches, where  $c$  is the number of crossings.

(3) Assuming that  $\tau \in P_{\text{even}}$  comes from  $\pi \in NC_{\text{even}}$  by merging a certain number of blocks, we can prove that the signature is 1 by proceeding by recurrence.  $\square$

With the above result in hand, we can now formulate:

DEFINITION 9.12. Associated to a partition  $\pi \in P_{\text{even}}(k, l)$  is the linear map

$$\bar{T}_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \bar{\delta}_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l}$$

where the signed Kronecker symbols

$$\bar{\delta}_\pi \in \{-1, 0, 1\}$$

are given by  $\bar{\delta}_\pi = \varepsilon(\tau)$  if  $\tau \geq \pi$ , and  $\bar{\delta}_\pi = 0$  otherwise, with  $\tau = \ker \binom{i}{j}$ .

In other words, what we are doing here is to add signatures to the usual formula of  $T_\pi$ . Indeed, observe that the usual formula for  $T_\pi$  can be written as follows:

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j: \ker \binom{i}{j} \geq \pi} e_{j_1} \otimes \dots \otimes e_{j_l}$$

Now by inserting signs, coming from the signature map  $\varepsilon : P_{\text{even}} \rightarrow \{\pm 1\}$ , we are led to the following formula, which coincides with the one from Definition 9.12:

$$\bar{T}_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{\tau \geq \pi} \varepsilon(\tau) \sum_{j: \ker \binom{i}{j} = \tau} e_{j_1} \otimes \dots \otimes e_{j_l}$$

We have the following key categorical result:

PROPOSITION 9.13. The assignment  $\pi \rightarrow \bar{T}_\pi$  is categorical, in the sense that

$$\begin{aligned} \bar{T}_\pi \otimes \bar{T}_\sigma &= \bar{T}_{[\pi\sigma]} \\ \bar{T}_\pi \bar{T}_\sigma &= N^{c(\pi, \sigma)} \bar{T}_{[\frac{\pi}{\sigma}]} \\ \bar{T}_\pi^* &= \bar{T}_{\pi^*} \end{aligned}$$

where  $c(\pi, \sigma)$  are certain positive integers.

PROOF. In order to prove this result we can go back to the proof from the easy case, and insert signs, where needed. We have to check three conditions, as follows:

1. Concatenation. It is enough to check the following formula:

$$\varepsilon \left( \ker \begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_q \end{pmatrix} \right) \varepsilon \left( \ker \begin{pmatrix} k_1 \dots k_r \\ l_1 \dots l_s \end{pmatrix} \right) = \varepsilon \left( \ker \begin{pmatrix} i_1 \dots i_p & k_1 \dots k_r \\ j_1 \dots j_q & l_1 \dots l_s \end{pmatrix} \right)$$

Let us denote by  $\tau, \nu$  the partitions on the left, so that the partition on the right is of the form  $\rho \leq [\tau\nu]$ . Now by switching to the noncrossing form,  $\tau \rightarrow \tau'$  and  $\nu \rightarrow \nu'$ , the partition on the right transforms into  $\rho \rightarrow \rho' \leq [\tau'\nu']$ . Now since  $[\tau'\nu']$  is noncrossing, we can use Theorem 9.11 (3), and we obtain the result.

2. Composition. Here we must establish the following formula:

$$\varepsilon \left( \ker \begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_q \end{pmatrix} \right) \varepsilon \left( \ker \begin{pmatrix} j_1 \dots j_q \\ k_1 \dots k_r \end{pmatrix} \right) = \varepsilon \left( \ker \begin{pmatrix} i_1 \dots i_p \\ k_1 \dots k_r \end{pmatrix} \right)$$

Let  $\tau, \nu$  be the partitions on the left, so that the partition on the right is of the form  $\rho \leq [\begin{smallmatrix} \tau \\ \nu \end{smallmatrix}]$ . Our claim is that we can jointly switch  $\tau, \nu$  to the noncrossing form. Indeed, we can first switch as for  $\ker(j_1 \dots j_q)$  to become noncrossing, and then switch the upper legs of  $\tau$ , and the lower legs of  $\nu$ , as for both these partitions to become noncrossing.

Now observe that when switching in this way to the noncrossing form,  $\tau \rightarrow \tau'$  and  $\nu \rightarrow \nu'$ , the partition on the right transforms into  $\rho \rightarrow \rho' \leq [\begin{smallmatrix} \tau' \\ \nu' \end{smallmatrix}]$ . Now since  $[\begin{smallmatrix} \tau' \\ \nu' \end{smallmatrix}]$  is noncrossing, we can apply Theorem 9.11 (3), and we obtain the result.

3. Involution. Here we must prove the following formula:

$$\bar{\delta}_\pi \begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_q \end{pmatrix} = \bar{\delta}_{\pi^*} \begin{pmatrix} j_1 \dots j_q \\ i_1 \dots i_p \end{pmatrix}$$

But this is clear from the definition of  $\bar{\delta}_\pi$ , and we are done. □

As a conclusion, our construction  $\pi \rightarrow \bar{T}_\pi$  has all the needed properties for producing quantum groups, via Tannakian duality. So, we can now formulate:

**THEOREM 9.14.** *Given a category of partitions  $D \subset P_{even}$ , the construction*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span \left( \bar{T}_\pi \Big| \pi \in D(k, l) \right)$$

*produces via Tannakian duality a quantum group  $\bar{G}_N \subset O_N^+$ , for any  $N \in \mathbb{N}$ .*

**PROOF.** This follows indeed from the Tannakian results from chapter 1 above, exactly as in the easy case, by using this time Proposition 9.13 as technical ingredient. □

We can unify the easy quantum groups, or at least the examples coming from categories  $D \subset P_{even}$ , with the quantum groups constructed above, as follows:

**DEFINITION 9.15.** *A closed subgroup  $G \subset O_N^+$  is called  $q$ -easy, or quizzy, with deformation parameter  $q = \pm 1$ , when its tensor category appears as follows,*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span \left( \dot{T}_\pi \Big| \pi \in D(k, l) \right)$$

*for a certain category of partitions  $D \subset P_{even}$ , where, for  $q = -1, 1$ :*

$$\dot{T} = \bar{T}, T$$

*The Schur-Weyl twist of  $G$  is the quizzy quantum group  $\bar{G} \subset O_N^+$  obtained via  $q \rightarrow -q$ .*

Let us compute now the twist of  $O_N$ . We recall that the Möbius function of any lattice, and in particular of  $P_{even}$ , is given by:

$$\mu(\sigma, \pi) = \begin{cases} 1 & \text{if } \sigma = \pi \\ -\sum_{\sigma \leq \tau < \pi} \mu(\sigma, \tau) & \text{if } \sigma < \pi \\ 0 & \text{if } \sigma \not\leq \pi \end{cases}$$

With this notation, we have the following result:

PROPOSITION 9.16. *For any partition  $\pi \in P_{\text{even}}$  we have the formula*

$$\bar{T}_\pi = \sum_{\tau \leq \pi} \alpha_\tau T_\tau$$

where  $\alpha_\sigma = \sum_{\sigma \leq \tau \leq \pi} \varepsilon(\tau) \mu(\sigma, \tau)$ , with  $\mu$  being the Möbius function of  $P_{\text{even}}$ .

PROOF. The linear combinations  $T = \sum_{\tau \leq \pi} \alpha_\tau T_\tau$  acts on tensors as follows:

$$\begin{aligned} T(e_{i_1} \otimes \dots \otimes e_{i_k}) &= \sum_{\tau \leq \pi} \alpha_\tau T_\tau(e_{i_1} \otimes \dots \otimes e_{i_k}) \\ &= \sum_{\tau \leq \pi} \alpha_\tau \sum_{\sigma \leq \tau} \sum_{j: \ker(j) = \sigma} e_{j_1} \otimes \dots \otimes e_{j_i} \\ &= \sum_{\sigma \leq \pi} \left( \sum_{\sigma \leq \tau \leq \pi} \alpha_\tau \right) \sum_{j: \ker(j) = \sigma} e_{j_1} \otimes \dots \otimes e_{j_i} \end{aligned}$$

Thus, in order to have  $\bar{T}_\pi = \sum_{\tau \leq \pi} \alpha_\tau T_\tau$ , we must have, for any  $\sigma \leq \pi$ :

$$\varepsilon(\sigma) = \sum_{\sigma \leq \tau \leq \pi} \alpha_\tau$$

But this problem can be solved by using the Möbius inversion formula, and we obtain the numbers  $\alpha_\sigma = \sum_{\sigma \leq \tau \leq \pi} \varepsilon(\tau) \mu(\sigma, \tau)$  in the statement.  $\square$

We can now twist the orthogonal group. The result here is as follows:

THEOREM 9.17. *The twist of  $O_N$  is obtained by replacing the relations  $ab = ba$  with*

$$ab = \pm ba$$

*with anticommutation on rows and columns, and commutation otherwise.*

PROOF. The basic crossing,  $\ker \begin{pmatrix} ij \\ ji \end{pmatrix}$  with  $i \neq j$ , comes from the transposition  $\tau \in S_2$ , so its signature is  $-1$ . As for its degenerated version  $\ker \begin{pmatrix} ii \\ ii \end{pmatrix}$ , this is noncrossing, so here the signature is  $1$ . We conclude that the linear map associated to the basic crossing is:

$$\bar{T}_\chi(e_i \otimes e_j) = \begin{cases} -e_j \otimes e_i & \text{for } i \neq j \\ e_j \otimes e_i & \text{otherwise} \end{cases}$$

We can proceed now as in the untwisted case, and since the intertwining relations coming from  $\bar{T}_\chi$  correspond to the relations defining  $\bar{O}_N$ , we obtain the result.  $\square$

Getting back now to graphs, we have the following result, from [17]:

THEOREM 9.18. *The quantum symmetry group of the  $N$ -hypercube is*

$$G^+(\square_N) = \bar{O}_N$$

with the corresponding coaction map on the vertex set being the map

$$\begin{aligned} \Phi : C^*(\mathbb{Z}_2^N) &\rightarrow C^*(\mathbb{Z}_2^N) \otimes C(\bar{O}_N) \\ g_i &\rightarrow \sum_j g_j \otimes u_{ji} \end{aligned}$$

via the standard identification  $\square_N = \widehat{\mathbb{Z}_2^N}$ .

PROOF. We use here the fact that the cube  $\square_N$ , when regarded as a graph, is the Cayley graph of the group  $\mathbb{Z}_2^N$ . The eigenvectors and eigenvalues of  $\square_N$  are as follows:

$$\begin{aligned} v_{i_1 \dots i_N} &= \sum_{j_1 \dots j_N} (-1)^{i_1 j_1 + \dots + i_N j_N} g_1^{j_1} \dots g_N^{j_N} \\ \lambda_{i_1 \dots i_N} &= (-1)^{i_1} + \dots + (-1)^{i_N} \end{aligned}$$

Modulo some standard computations, explained in [17], it is enough to construct a map  $\Phi$  as in the statement. For this purpose, consider the following variables:

$$G_i = \sum_j g_j \otimes u_{ji}$$

We must show that these variables satisfy the same relations as the generators  $g_j \in \mathbb{Z}_2^N$ . The self-adjointness being automatic, the relations to be checked are therefore:

$$G_i^2 = 1 \quad , \quad G_i G_j = G_j G_i$$

In what regards the squares, we have the following formula:

$$G_i^2 = \sum_{kl} g_k g_l \otimes u_{ik} u_{il} = 1 + \sum_{k < l} g_k g_l \otimes (u_{ik} u_{il} + u_{il} u_{ik})$$

Also, we have the following formula:

$$[G_i, G_j] = \sum_{k < l} g_k g_l \otimes (u_{ik} u_{jl} - u_{jk} u_{il} + u_{il} u_{jk} - u_{jl} u_{ik})$$

From the first relation we obtain  $ab = 0$  for  $a \neq b$  on the same row of  $u$ , and by using the antipode, the same happens for the columns. From the second relation we obtain:

$$[u_{ik}, u_{jl}] = [u_{jk}, u_{il}] \quad , \quad \forall k \neq l$$

Now by applying the antipode we obtain:

$$[u_{lj}, u_{ki}] = [u_{li}, u_{kj}]$$

By relabelling, this gives, for  $j \neq i$ :

$$[u_{ik}, u_{jl}] = [u_{il}, u_{jk}]$$

Thus for  $i \neq j, k \neq l$  we must have:

$$[u_{ik}, u_{jl}] = [u_{jk}, u_{il}] = 0$$

We are therefore led to  $G \subset \bar{O}_N$ , as claimed. □

In connection with the various extensions of our formalism, regarding colored graphs, or finite metric spaces, let us record as well the following result, also from [17]:

**THEOREM 9.19.** *The quantum isometry group of the  $N$ -hypercube, regarded as a finite metric subspace of  $\mathbb{R}^N$ , is:*

$$G^+(\square_N) = \bar{O}_N$$

*That is, we obtain the twisted orthogonal group  $\bar{O}_N$ .*

**PROOF.** The distance matrix of the cube has a color decomposition as follows:

$$d = d_1 + \sqrt{2}d_2 + \sqrt{3}d_3 + \dots + \sqrt{N}d_N$$

Since the powers of  $d_1$  can be computed by counting loops on the cube, we have formulae as follows, with  $x_{ij} \in \mathbb{N}$  being certain positive integers:

$$\begin{aligned} d_1^2 &= x_{21}1_N + x_{22}d_2 \\ d_1^3 &= x_{31}1_N + x_{32}d_2 + x_{33}d_3 \\ &\vdots \\ d_1^N &= x_{N1}1_N + x_{N2}d_2 + x_{N3}d_3 + \dots + x_{NN}d_N \end{aligned}$$

But this shows that we have the following equality of algebras:

$$\langle d \rangle = \langle d_1 \rangle$$

Now since  $d_1$  is the adjacency matrix of  $\square_N$ , viewed as graph, this proves our claim, and we obtain the result from Theorem 9.18. □

Our purpose now is to understand which representation of  $O_N$  produces by twisting the magic representation of  $\bar{O}_N$ . In order to solve this question, we will need:

**PROPOSITION 9.20.** *The Fourier transform over  $\mathbb{Z}_2^N$  is the map*

$$\begin{aligned} \alpha : C(\mathbb{Z}_2^N) &\rightarrow C^*(\mathbb{Z}_2^N) \\ \delta_{g_1^{i_1} \dots g_N^{i_N}} &\rightarrow \frac{1}{2^N} \sum_{j_1 \dots j_N} (-1)^{\langle i, j \rangle} g_1^{j_1} \dots g_N^{j_N} \end{aligned}$$

*with the usual convention  $\langle i, j \rangle = \sum_k i_k j_k$ , and its inverse is the map*

$$\begin{aligned} \beta : C^*(\mathbb{Z}_2^N) &\rightarrow C(\mathbb{Z}_2^N) \\ g_1^{i_1} \dots g_N^{i_N} &\rightarrow \sum_{j_1 \dots j_N} (-1)^{\langle i, j \rangle} \delta_{g_1^{j_1} \dots g_N^{j_N}} \end{aligned}$$

*with all the exponents being binary,  $i_1, \dots, i_N, j_1, \dots, j_N \in \{0, 1\}$ .*



PROOF. Observe first that the group  $\mathbb{Z}_2^N$  can be written as follows:

$$\mathbb{Z}_2^N = \left\{ g_1^{i_1} \dots g_N^{i_N} \mid i_1, \dots, i_N \in \{0, 1\} \right\}$$

Thus both  $\alpha, \beta$  are well-defined, and it is elementary to check that both are morphisms of algebras. We have as well  $\alpha\beta = \beta\alpha = id$ , coming from the standard formula:

$$\begin{aligned} \frac{1}{2^N} \sum_{j_1 \dots j_N} (-1)^{\langle i, j \rangle} &= \prod_{k=1}^N \left( \frac{1}{2} \sum_{j_r} (-1)^{i_r j_r} \right) \\ &= \delta_{i0} \end{aligned}$$

Thus we have indeed a pair of inverse Fourier morphisms, as claimed. □

As an illustration here, at  $N = 1$ , with  $\mathbb{Z}_2 = \{1, g\}$ , the map  $\alpha$  is given by:

$$\delta_1 \rightarrow \frac{1}{2}(1 + g) \quad , \quad \delta_g \rightarrow \frac{1}{2}(1 - g)$$

As for its inverse  $\beta$ , this is given by the following formulae:

$$1 \rightarrow \delta_1 + \delta_g \quad , \quad g \rightarrow \delta_1 - \delta_g$$

By using now these Fourier transforms, we obtain following formula:

**THEOREM 9.21.** *The magic unitary for the embedding  $\bar{O}_N \subset S_{2N}^+$  is given by*

$$w_{i_1 \dots i_N, k_1 \dots k_N} = \frac{1}{2^N} \sum_{j_1 \dots j_N} \sum_{b_1 \dots b_N} (-1)^{\langle i+k_b, j \rangle} \left( \frac{1}{N} \right)^{\#\{0 \in j\}} u_{1b_1}^{j_1} \dots u_{Nb_N}^{j_N}$$

where  $k_b = (k_{b_1}, \dots, k_{b_N})$ , with respect to multi-indices  $i, k \in \{0, 1\}^N$  as above.

PROOF. By composing the coaction map  $\Phi$  from Theorem 9.18 with the above Fourier transform isomorphisms  $\alpha, \beta$ , we have a diagram as follows:

$$\begin{array}{ccc} C^*(\mathbb{Z}_2^N) & \xrightarrow{\Phi} & C^*(\mathbb{Z}_2^N) \otimes C(\bar{O}_N) \\ \alpha \uparrow & & \downarrow \beta \otimes id \\ C(\mathbb{Z}_2^N) & \xrightarrow{\Psi} & C(\mathbb{Z}_2^N) \otimes C(\bar{O}_N) \end{array}$$

In order to compute the composition on the bottom  $\Psi$ , we first recall from Theorem 9.18 above that the coaction map  $\Phi$  is defined by the following formula:

$$\Phi(g_b) = \sum_a g_a \otimes u_{ab}$$

Now by making products of such quantities, we obtain the following global formula for  $\Phi$ , valid for any exponents  $i_1, \dots, i_N \in \{1, \dots, N\}$ :

$$\Phi(g_1^{i_1} \dots g_N^{i_N}) = \left(\frac{1}{N}\right)^{\#(0 \in i)} \sum_{b_1 \dots b_N} g_{b_1}^{i_1} \dots g_{b_N}^{i_N} \otimes u_{1b_1}^{i_1} \dots u_{Nb_N}^{i_N}$$

The term on the right can be put in “standard form” as follows:

$$g_{b_1}^{i_1} \dots g_{b_N}^{i_N} = g_1^{\sum_{b_x=1} i_x} \dots g_N^{\sum_{b_x=N} i_x}$$

We therefore obtain the following formula for the coaction map  $\Phi$ :

$$\Phi(g_1^{i_1} \dots g_N^{i_N}) = \left(\frac{1}{N}\right)^{\#(0 \in i)} \sum_{b_1 \dots b_N} g_1^{\sum_{b_x=1} i_x} \dots g_N^{\sum_{b_x=N} i_x} \otimes u_{1b_1}^{i_1} \dots u_{Nb_N}^{i_N}$$

Now by applying the Fourier transforms, we obtain the following formula:

$$\begin{aligned} & \Psi(\delta_{g_1^{i_1} \dots g_N^{i_N}}) \\ &= (\beta \otimes id)\Phi\left(\frac{1}{2^N} \sum_{j_1 \dots j_N} (-1)^{\langle i, j \rangle} g_1^{j_1} \dots g_N^{j_N}\right) \\ &= \frac{1}{2^N} \sum_{j_1 \dots j_N} \sum_{b_1 \dots b_N} (-1)^{\langle i, j \rangle} \left(\frac{1}{N}\right)^{\#(0 \in j)} \beta\left(g_1^{\sum_{b_x=1} j_x} \dots g_N^{\sum_{b_x=N} j_x}\right) \otimes u_{1b_1}^{j_1} \dots u_{Nb_N}^{j_N} \end{aligned}$$

By using now the formula of  $\beta$  from Proposition 9.20, we obtain:

$$\begin{aligned} \Psi(\delta_{g_1^{i_1} \dots g_N^{i_N}}) &= \frac{1}{2^N} \sum_{j_1 \dots j_N} \sum_{b_1 \dots b_N} \sum_{k_1 \dots k_N} \left(\frac{1}{N}\right)^{\#(0 \in j)} \\ & \quad (-1)^{\langle i, j \rangle} (-1)^{\langle (\sum_{b_x=1} j_x, \dots, \sum_{b_x=N} j_x), (k_1, \dots, k_N) \rangle} \\ & \quad \delta_{g_1^{k_1} \dots g_N^{k_N}} \otimes u_{1b_1}^{j_1} \dots u_{Nb_N}^{j_N} \end{aligned}$$

Now observe that, with the notation  $k_b = (k_{b_1}, \dots, k_{b_N})$ , we have:

$$\left\langle \left( \sum_{b_x=1} j_x, \dots, \sum_{b_x=N} j_x \right), (k_1, \dots, k_N) \right\rangle = \langle j, k_b \rangle$$

Thus, we obtain the following formula for our map  $\Psi$ :

$$\begin{aligned} & \Psi(\delta_{g_1^{i_1} \dots g_N^{i_N}}) \\ &= \frac{1}{2^N} \sum_{j_1 \dots j_N} \sum_{b_1 \dots b_N} \sum_{k_1 \dots k_N} (-1)^{\langle i+k_b, j \rangle} \left(\frac{1}{N}\right)^{\#(0 \in j)} \delta_{g_1^{k_1} \dots g_N^{k_N}} \otimes u_{1b_1}^{j_1} \dots u_{Nb_N}^{j_N} \end{aligned}$$

But this gives the formula in the statement for the corresponding magic unitary, with respect to the basis  $\{\delta_{g_1^{i_1} \dots g_N^{i_N}}\}$  of the algebra  $C(\mathbb{Z}_2^N)$ , and we are done.  $\square$

We can now solve our original question, namely understanding where the magic representation of  $\bar{O}_N$  really comes from, with the following final answer to it:

**THEOREM 9.22.** *The magic representation of  $\bar{O}_N$ , coming from its action on the  $N$ -cube, corresponds to the antisymmetric representation of  $O_N$ , via twisting.*

**PROOF.** This follows from the formula of  $w$  in Theorem 9.21, by computing the character, and then interpreting the result via twisting, as follows:

(1) By applying the trace to the formula of  $w$ , we obtain:

$$\chi = \sum_{j_1 \dots j_N} \sum_{b_1 \dots b_N} \left( \frac{1}{2^N} \sum_{i_1 \dots i_N} (-1)^{\langle i + i_b, j \rangle} \right) \left( \frac{1}{N} \right)^{\#(0 \in j)} u_{1b_1}^{j_1} \dots u_{Nb_N}^{j_N}$$

(2) By computing the Fourier sum in the middle, we are led to the following formula, with binary indices  $j_1, \dots, j_N \in \{0, 1\}$ , and plain indices  $b_1, \dots, b_N \in \{1, \dots, N\}$ :

$$\chi = \sum_{j_1 \dots j_N} \sum_{b_1 \dots b_N} \left( \frac{1}{N} \right)^{\#(0 \in j)} \delta_{j_1, \sum_{b_x=1} j_x} \dots \delta_{j_N, \sum_{b_x=N} j_x} u_{1b_1}^{j_1} \dots u_{Nb_N}^{j_N}$$

(3) With the notation  $r = \#(1 \in j)$  we obtain a decomposition of type:

$$\chi = \sum_{r=0}^N \chi_r$$

To be more precise, the variables  $\chi_r$  are as follows:

$$\chi_r = \frac{1}{N^{N-r}} \sum_{\#(1 \in j)=r} \sum_{b_1 \dots b_N} \delta_{j_1, \sum_{b_x=1} j_x} \dots \delta_{j_N, \sum_{b_x=N} j_x} u_{1b_1}^{j_1} \dots u_{Nb_N}^{j_N}$$

(4) Consider now the set  $A \subset \{1, \dots, N\}$  given by:

$$A = \{a | j_a = 1\}$$

The binary multi-indices  $j \in \{0, 1\}^N$  satisfying  $\#(1 \in j) = r$  being in bijection with such subsets  $A$ , satisfying  $|A| = r$ , we can replace the sum over  $j$  with a sum over such subsets  $A$ . We obtain a formula as follows, where  $j$  is the index corresponding to  $A$ :

$$\chi_r = \frac{1}{N^{N-r}} \sum_{|A|=r} \sum_{b_1 \dots b_N} \delta_{j_1, \sum_{b_x=1} j_x} \dots \delta_{j_N, \sum_{b_x=N} j_x} \prod_{a \in A} u_{ab_a}$$

(5) Let us identify  $b$  with the corresponding function  $b : \{1, \dots, N\} \rightarrow \{1, \dots, N\}$ , via  $b(a) = b_a$ . Then for any  $p \in \{1, \dots, N\}$  we have:

$$\delta_{j_p, \sum_{b_x=p} j_x} = 1 \iff |b^{-1}(p) \cap A| = \chi_A(p) \pmod{2}$$

We conclude that the multi-indices  $b \in \{1, \dots, N\}^N$  which effectively contribute to the sum are those coming from the functions satisfying  $b < A$ . Thus, we have:

$$\chi_r = \frac{1}{N^{N-r}} \sum_{|A|=r} \sum_{b < A} \prod_{a \in A} u_{ab_a}$$

(6) We can further split each  $\chi_r$  over the sets  $A \subset \{1, \dots, N\}$  satisfying:

$$|A| = r$$

The point is that for each of these sets we have:

$$\frac{1}{N^{N-r}} \sum_{b < A} \prod_{a \in A} u_{ab_a} = \sum_{\sigma \in S_N^A} \prod_{a \in A} u_{a\sigma(a)}$$

Thus, the magic character of  $\bar{O}_N$  splits as:

$$\chi = \sum_{r=0}^N \chi_r$$

To be more precise, the components are:

$$\chi_r = \sum_{|A|=r} \sum_{\sigma \in S_N^A} \prod_{a \in A} u_{a\sigma(a)}$$

(7) The twisting operation  $O_N \rightarrow \bar{O}_N$  makes correspond the following products:

$$\varepsilon(\sigma) \prod_{a \in A} u_{a\sigma(a)} \rightarrow \prod_{a \in A} u_{a\sigma(a)}$$

Now by summing over sets  $A$  and permutations  $\sigma$ , we conclude that the twisting operation  $O_N \rightarrow \bar{O}_N$  makes correspond the following quantities:

$$\sum_{|A|=r} \sum_{\sigma \in S_N^A} \varepsilon(\sigma) \prod_{a \in A} u_{a\sigma(a)} \rightarrow \sum_{|A|=r} \sum_{\sigma \in S_N^A} \prod_{a \in A} u_{a\sigma(a)}$$

Thus, we are led to the conclusion in the statement.  $\square$

We will be back to all this in chapter 10 below, with yet another solution to the square problem. The idea there will be that of regarding the square as the complement of the graph formed by 2 segments, and then looking at the quantum symmetry group  $H_N^+$  of the graph formed by  $N$  segments, which will be the correct generalization of  $H_N$ .

### 9d. Partial symmetries

As a final topic for this chapter, let us discuss now the partial automorphism groups of the finite graphs  $X$ , and their quantum analogues. Let us start with:

**PROPOSITION 9.23.** *Given a graph  $X$  with  $N$  vertices, and adjacency matrix  $d \in M_N(0, 1)$ , consider its partial automorphism semigroup, given by:*

$$\tilde{G}(X) = \left\{ \sigma \in \tilde{S}_N \mid d_{ij} = d_{\sigma(i)\sigma(j)}, \forall i, j \in \text{Dom}(\sigma) \right\}$$

We have then the following presentation formula,

$$C(\tilde{G}(X)) = C(\tilde{S}_N) / \left\langle R(du - ud)C = 0 \right\rangle$$

with  $R = \text{diag}(R_i)$ ,  $C = \text{diag}(C_j)$ , where  $R_i, C_j$  are the row and column sums of the associated submagic matrix  $u$ .

**PROOF.** With the convention  $i \sim j$  when  $i, j$  are connected by an edge of  $X$ , the definition of  $\tilde{G}(X)$  from the statement reformulates as follows:

$$\tilde{G}(X) = \left\{ \sigma \in \tilde{S}_N \mid i \sim j, \exists \sigma(i), \exists \sigma(j) \implies \sigma(i) \sim \sigma(j) \right\}$$

We have the following computations:

$$(du)_{ij}(\sigma) = \sum_k d_{ik} u_{kj}(\sigma) = \sum_{k \sim i} u_{kj}(\sigma) = \begin{cases} 1 & \text{if } \sigma(j) \sim i \\ 0 & \text{otherwise} \end{cases}$$

$$(ud)_{ij}(\sigma) = \sum_k u_{ik} d_{kj}(\sigma) = \sum_{k \sim j} u_{ik}(\sigma) = \begin{cases} 1 & \text{if } \sigma^{-1}(i) \sim j \\ 0 & \text{otherwise} \end{cases}$$

On the other hand, we have as well the following formulae:

$$R_i(\sigma) = \sum_j u_{ij}(\sigma) = \begin{cases} 1 & \text{if } \exists \sigma^{-1}(i) \\ 0 & \text{otherwise} \end{cases}$$

$$C_j(\sigma) = \sum_i u_{ij}(\sigma) = \begin{cases} 1 & \text{if } \exists \sigma(j) \\ 0 & \text{otherwise} \end{cases}$$

Now by multiplying the above formulae, we obtain the following formulae:

$$(R_i(du)_{ij}C_j)(\sigma) = \begin{cases} 1 & \text{if } \sigma(j) \sim i \text{ and } \exists \sigma^{-1}(i) \text{ and } \exists \sigma(j) \\ 0 & \text{otherwise} \end{cases}$$

$$(R_i(ud)_{ij}C_j)(\sigma) = \begin{cases} 1 & \text{if } \sigma^{-1}(i) \sim j \text{ and } \exists \sigma^{-1}(i) \text{ and } \exists \sigma(j) \\ 0 & \text{otherwise} \end{cases}$$

We conclude that the relations in the statement, which read  $R_i(du)_{ij}C_j = R_i(ud)_{ij}C_j$ , when applied to a given  $\sigma \in \tilde{S}_N$ , correspond to the following condition:

$$\exists \sigma^{-1}(i), \exists \sigma(j) \implies [\sigma(j) \sim i \iff \sigma^{-1}(i) \sim j]$$

But with  $i = \sigma(k)$ , this latter condition reformulates as follows:

$$\exists \sigma(k), \exists \sigma(j) \implies [\sigma(j) \sim \sigma(k) \iff k \sim j]$$

Thus we must have  $\sigma \in \tilde{G}(X)$ , and we obtain the presentation result for  $\tilde{G}(X)$ .  $\square$

With the above result in hand, we are led to the following statement:

**THEOREM 9.24.** *Given a graph  $X$ , with adjacency matrix  $d \in M_N(0, 1)$ , the following construction,  $R, C$  being the diagonal matrices formed by the row and column sums of  $u$ ,*

$$C(\tilde{G}^+(X)) = C(\tilde{S}_N^+) / \left\langle R(du - ud)C = 0 \right\rangle$$

*produces a quantum semigroup with subantipode  $\tilde{G}^+(X) \subset \tilde{S}_N^+$ , called quantum semigroup of quantum partial automorphisms of  $X$ , whose classical version is  $\tilde{G}(X)$ .*

**PROOF.** In order to construct  $\Delta$ , consider the elements  $U_{ij} = \sum_k u_{ik} \otimes u_{kj}$ . We must prove that the relations in the statement are satisfied by  $U = (U_{ij})$ . We have:

$$\begin{aligned} R_i^U (dU)_{ij} C_j^U &= \sum_k U_{ik} \sum_l d_{il} U_{lj} \sum_m U_{mj} \\ &= \sum_{kn} u_{in} \otimes u_{nk} \sum_{lo} d_{il} u_{lo} \otimes u_{oj} \sum_{mp} u_{mp} \otimes u_{pj} \\ &= \sum_{klmnop} d_{il} \cdot u_{in} u_{lo} u_{mp} \otimes u_{nk} u_{oj} u_{pj} \\ &= \sum_{klmno} d_{il} \cdot u_{in} u_{lo} u_{mo} \otimes u_{nk} u_{oj} \\ &= \sum_{kln} d_{il} \cdot u_{in} u_{lo} \otimes u_{nk} u_{oj} \end{aligned}$$

On the other hand, we have as well the following formula:

$$\begin{aligned} R_i(du)_{ij} C_j &= \sum_k u_{ik} \sum_l d_{il} u_{lj} \sum_m u_{mj} \\ &= \sum_{klm} d_{il} \cdot u_{ik} u_{lj} u_{mj} \\ &= \sum_{kl} d_{il} \cdot u_{ik} u_{lj} \end{aligned}$$

Now by getting back to our computation, we can finish it as follows:

$$\begin{aligned}
R_i^U(dU)_{ij}C_j^U &= \sum_{kln} d_{il} \cdot u_{in}u_{lo} \otimes u_{nk}u_{oj} \\
&= \sum_{kn} u_{in} \otimes u_{nk} \sum_{lo} d_{il} \cdot u_{lo} \otimes u_{oj} \\
&= \sum_{kl} d_{il} \cdot \Delta(u_{ik})\Delta(u_{lj}) \\
&= \Delta\left(\sum_{kl} d_{il} \cdot u_{ik}u_{lj}\right) \\
&= \Delta(R_i(du)_{ij}C_j)
\end{aligned}$$

The second computation that we need is similar. We first have:

$$\begin{aligned}
R_i^U(Ud)_{ij}C_j^U &= \sum_k U_{ik} \sum_l U_{il}d_{lj} \sum_m U_{mj} \\
&= \sum_{kn} u_{in} \otimes u_{nk} \sum_{lo} d_{lj}u_{io} \otimes u_{ol} \sum_{mp} u_{mp} \otimes u_{pj} \\
&= \sum_{klmnop} d_{lj} \cdot u_{in}u_{io}u_{mp} \otimes u_{nk}u_{ol}u_{pj} \\
&= \sum_{klmnp} d_{lj} \cdot u_{in}u_{mp} \otimes u_{nk}u_{nl}u_{pj} \\
&= \sum_{kmnp} d_{lj} \cdot u_{in}u_{mp} \otimes u_{nk}u_{pj}
\end{aligned}$$

On the other hand, we have as well the following formula:

$$\begin{aligned}
R_i(ud)_{ij}C_j &= \sum_k u_{ik} \sum_l u_{il}d_{lj} \sum_m u_{mj} \\
&= \sum_{klm} d_{lj} \cdot u_{ik}u_{il}u_{mj} \\
&= \sum_{km} d_{lj} \cdot u_{ik}u_{mj}
\end{aligned}$$

Now by getting back to our computation, we can finish it as follows:

$$\begin{aligned}
R_i^U(Ud)_{ij}C_j^U &= \sum_{kmnp} d_{lj} \cdot u_{in}u_{mp} \otimes u_{nk}u_{pj} \\
&= \sum_{kn} u_{in} \otimes u_{nk} \sum_{mp} d_{lj} \cdot u_{mp} \otimes u_{pj} \\
&= \sum_{km} d_{lj} \cdot \Delta(u_{ik})\Delta(u_{mj}) \\
&= \Delta\left(\sum_{km} d_{lj} \cdot u_{ik}u_{mj}\right) \\
&= \Delta(R_i(ud)_{ij}C_j)
\end{aligned}$$

We can now construct  $\Delta$ , based on the formulae found above.

Regarding now  $\varepsilon$ , the algebra in the statement has indeed a morphism  $\varepsilon$  defined by  $u_{ij} \rightarrow \delta_{ij}$ , because the following relations are trivially satisfied:

$$R_i(d1_N)_{ij}C_j = R_i(1_N d)_{ij}C_j$$

Regarding now  $S$ , we must prove that we have a morphism  $S$  given by  $u_{ij} \rightarrow u_{ji}$ . Here the best is to use the reformulation of the relations in the statement mentioned before the statement itself, which is as follows, with  $R = \text{diag}(R_i)$  and  $C = \text{diag}(C_j)$ :

$$R(du - ud)C = 0$$

Indeed, when transposing this formula, we obtain:

$$C^t(u^t d - du^t)R^t = 0$$

Now since  $C^t, R^t$  are respectively the diagonal matrices formed by the row sums and column sums of  $u^t$ , we conclude that the relations  $R(du - ud)C = 0$  are satisfied by the transpose matrix  $u^t$ , and this gives the existence of the subantipode map  $S$ .  $\square$

There are many interesting questions regarding the above semigroups.

### 9e. Exercises

The constructions discussed in this chapter produce many interesting examples of quantum permutation groups, and we will keep studying them, in the remainder of this book. As a first exercise about all this, in direct relation to the above, we have:

EXERCISE 9.25. *Work out the behavior of the operation*

$$X \rightarrow G^+(X)$$

*with respect to the basic product operations on the graphs.*



There are several possible product operations to be investigated here, and none of them is trivial, ultimately leading to the study of the eigenvalues of the corresponding adjacency matrices. We will be back to all this in the next chapter, with a detailed study of the disconnected unions, and of some related product operations.

In connection now with the circulant graphs, we have:

EXERCISE 9.26. *Work out some theoretical generalizations of the no quantum symmetry result for the  $N$ -cycle, in the  $N \rightarrow \infty$  limit.*

This is something that we already discussed in the above, with the comment that some technology is available in the case where  $N$  is prime, and that all this is in need of some further extensions, to the case where  $N$  is not necessarily prime.

In connection with the twists and hypercubes, we have:

EXERCISE 9.27. *Find graphs  $X$  related to the hypercube, for which the operation*

$$X \rightarrow G^+(X)$$

*produces twists of classical Lie groups.*

This is something quite tricky, requiring for instance going beyond the Schur-Weyl twisting procedure, say by using a twisted version of the determinant.

Finally, we have several theoretical questions. Let us start with:

EXERCISE 9.28. *Discuss some applications of the general planar algebra theory to the computation of the quantum symmetry groups of type  $G^+(X)$ .*

This is something that we briefly discussed in the above, with a general structure result for the planar algebra associated to  $G^+(X)$ . The problem is that of going beyond this, with explicit applications of the planar algebra theory to  $G^+(X)$  problems.

As a second theoretical exercise, we have:

EXERCISE 9.29. *Prove that any graph having  $N \geq 2$  vertices does have quantum partial symmetry, in the sense that  $\tilde{G}^+(X)$  is not classical. Also, try constructing the quantum partial automorphism group of an infinite graph.*

Here the first question is quite elementary, and the second one is quite tricky, leading to quite complicated functional analysis.

## CHAPTER 10

### Reflection groups

#### 10a. Wreath products

Let us go back now to the square problem. In order to present the correct, final solution to it, the idea will be that to look at the quantum group  $G^+(\sqcup \mid)$  instead, which is equal to it. We will need the following result, from [42]:

**THEOREM 10.1.** *Given closed subgroups  $G \subset U_N^+$ ,  $H \subset S_k^+$ , with fundamental corepresentations  $u, v$ , the following construction produces a closed subgroup of  $U_{Nk}^+$ :*

$$C(G \wr_* H) = (C(G)^{*k} * C(H)) / \langle [u_{ij}^{(a)}, v_{ab}] = 0 \rangle$$

*In the case where  $G, H$  are classical, the classical version of  $G \wr_* H$  is the usual wreath product  $G \wr H$ . Also, when  $G$  is a quantum permutation group, so is  $G \wr_* H$ .*

**PROOF.** Consider indeed the following matrix, written using double indices, over the quotient algebra in the statement:

$$w_{ia,jb} = u_{ij}^{(a)} v_{ab}$$

Then  $w$  is unitary, and in the case  $G \subset S_N^+$ , this matrix is magic.

With these observations in hand, it is routine to check that  $G \wr_* H$  is indeed a quantum group, with fundamental corepresentation  $w$ , by constructing maps  $\Delta, \varepsilon, S$  as in chapter 1 above, and in the case  $G \subset S_N^+$ , we obtain in this way a closed subgroup of  $S_{Nk}^+$ .

For details on all this, we refer to [42]. □

We refer to [11], [42], [89] for more details regarding the above construction. With this notion in hand, we can now formulate a non-trivial result, as follows:

**THEOREM 10.2.** *Given a connected graph  $X$ , and  $k \in \mathbb{N}$ , we have the formulae*

$$G(kX) = G(X) \wr S_k$$

$$G^+(kX) = G^+(X) \wr_* S_k^+$$

*where  $kX = X \sqcup \dots \sqcup X$  is the  $k$ -fold disjoint union of  $X$  with itself.*

PROOF. The first formula is something well-known, which follows as well from the second formula, by taking the classical version. Regarding now the second formula, it is elementary to check that we have an inclusion as follows, for any finite graph  $X$ :

$$G^+(X) \wr S_k^+ \subset G^+(kX)$$

Regarding now the reverse inclusion, which requires  $X$  to be connected, this follows by doing some matrix analysis, by using the commutation with  $u$ . To be more precise, let us denote by  $w$  the fundamental corepresentation of  $G^+(kX)$ , and set:

$$u_{ij}^{(a)} = \sum_b w_{ia,jb} \quad , \quad v_{ab} = \sum_i v_{ab}$$

It is then routine to check, by using the fact that  $X$  is indeed connected, that we have here magic unitaries, as in the definition of the free wreath products. Thus, we obtain:

$$G^+(kX) \subset G^+(X) \wr S_k^+$$

But this gives the result. See [11]. □

We are led in this way to the following result, from [17]:

THEOREM 10.3. *Consider the graph consisting of  $N$  segments.*

- (1) *Its symmetry group is the hyperoctahedral group  $H_N = \mathbb{Z}_2 \wr S_N$ .*
- (2) *Its quantum symmetry group is the quantum group  $H_N^+ = \mathbb{Z}_2 \wr S_N^+$ .*

PROOF. Here the first assertion is clear from definitions, with the remark that the relation with the formula  $H_N = G(\square_N)$  comes by viewing the  $N$  segments as being the  $[-1, 1]$  segments on each of the  $N$  coordinate axes of  $\mathbb{R}^N$ .

Indeed, a symmetry of the  $N$ -cube is the same as a symmetry of the  $N$  segments, and so, as desired:

$$G(\square_N) = \mathbb{Z}_2 \wr S_N$$

As for the second assertion, this follows from Theorem 10.2 above, applied to the segment graph. Observe also that (2) implies (1), by taking the classical version. □

Now back to the square, we have  $G^+(\square) = H_2^+$ , and our claim is that this is the “good” and final formula. In order to prove this, we must work out the easiness theory for  $H_N, H_N^+$ , and find a compatibility there. We first have the following result:

PROPOSITION 10.4. *The algebra  $C(H_N^+)$  can be presented in two ways, as follows:*

- (1) *As the universal algebra generated by the entries of a  $2N \times 2N$  magic unitary having the “sudoku” pattern  $w = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ , with  $a, b$  being square matrices.*
- (2) *As the universal algebra generated by the entries of a  $N \times N$  orthogonal matrix which is “cubic”, in the sense that  $u_{ij}u_{ik} = u_{ji}u_{ki} = 0$ , for any  $j \neq k$ .*

*As for  $C(H_N)$ , this has similar presentations, among the commutative algebras.*

PROOF. We must prove that the algebras  $A_s, A_c$  coming from (1,2) coincide. We can define a morphism  $A_c \rightarrow A_s$  by the following formula:

$$\varphi(u_{ij}) = a_{ij} - b_{ij}$$

We construct now the inverse morphism. Consider the following elements:

$$\alpha_{ij} = \frac{u_{ij}^2 + u_{ij}}{2}$$

$$\beta_{ij} = \frac{u_{ij}^2 - u_{ij}}{2}$$

These are projections, and the following matrix is a sudoku unitary:

$$M = \begin{pmatrix} (\alpha_{ij}) & (\beta_{ij}) \\ (\beta_{ij}) & (\alpha_{ij}) \end{pmatrix}$$

Thus we can define a morphism  $A_s \rightarrow A_c$  by the following formula:

$$\psi(a_{ij}) = \frac{u_{ij}^2 + u_{ij}}{2}$$

$$\psi(b_{ij}) = \frac{u_{ij}^2 - u_{ij}}{2}$$

We check now the fact that  $\psi, \varphi$  are indeed inverse morphisms:

$$\begin{aligned} \psi\varphi(u_{ij}) &= \psi(a_{ij} - b_{ij}) \\ &= \frac{u_{ij}^2 + u_{ij}}{2} - \frac{u_{ij}^2 - u_{ij}}{2} \\ &= u_{ij} \end{aligned}$$

As for the other composition, we have the following computation:

$$\begin{aligned} \varphi\psi(a_{ij}) &= \varphi\left(\frac{u_{ij}^2 + u_{ij}}{2}\right) \\ &= \frac{(a_{ij} - b_{ij})^2 + (a_{ij} - b_{ij})}{2} \\ &= a_{ij} \end{aligned}$$

A similar computation gives  $\varphi\psi(b_{ij}) = b_{ij}$ , as desired. As for the final assertion, regarding  $C(H_N)$ , this follows from the above results, by taking classical versions.  $\square$

We can now work out the easiness property of  $H_N, H_N^+$ , with respect to the cubic representations, and we are led to the following result, which is fully satisfactory:

**THEOREM 10.5.** *The quantum groups  $H_N, H_N^+$  are both easy, as follows:*

- (1)  $H_N$  corresponds to the category  $P_{\text{even}}$ .
- (2)  $H_N^+$  corresponds to the category  $NC_{\text{even}}$ .

PROOF. These assertions follow indeed from the fact that the cubic relations are implemented by the one-block partition in  $P(2, 2)$ , which generates  $NC_{even}$ .  $\square$

As a final conclusion now, to the long story told here, the correct analogue of the hyperoctahedral group  $H_N$  is the quantum group  $H_N^+$  constructed above, with  $H_N \rightarrow H_N^+$  being a liberation, in the sense of easy quantum group theory.

### 10b. Complex reflections

These quantum groups  $H_N$  and  $H_N^+$  belong in fact to series, depending on a parameter  $s \in \mathbb{N} \cup \{\infty\}$ , as follows:

$$\begin{aligned} H_N^s &= \mathbb{Z}_s \wr S_N \\ H_N^{s+} &= \mathbb{Z}_s \wr_* S_N^+ \end{aligned}$$

We discuss here, following [9], [38], the algebraic and analytic structure of these latter quantum groups. The main motivation comes from the cases  $s = 1, 2, \infty$ , where we recover respectively  $S_N, S_N^+$  and  $H_N, H_N^+$ , and the full reflection groups  $K_N, K_N^+$ .

Let us start with a brief discussion concerning the classical case. The result that we will need, which is well-known and elementary, is as follows:

PROPOSITION 10.6. *The group  $H_N^s = \mathbb{Z}_s \wr S_N$  of  $N \times N$  permutation-like matrices having as nonzero entries the  $s$ -th roots of unity is as follows:*

- (1)  $H_N^1 = S_N$  is the symmetric group.
- (2)  $H_N^2 = H_N$  is the hyperoctahedral group.
- (3)  $H_N^\infty = K_N$  is the group of unitary permutation-like matrices.

PROOF. Everything here is clear from definitions.  $\square$

The free analogues of the reflection groups  $H_N^s$  can be constructed as follows:

DEFINITION 10.7.  *$C(H_N^{s+})$  is the universal  $C^*$ -algebra generated by  $N^2$  normal elements  $u_{ij}$ , subject to the following relations,*

- (1)  $u = (u_{ij})$  is unitary,
- (2)  $u^t = (u_{ji})$  is unitary,
- (3)  $p_{ij} = u_{ij}u_{ij}^*$  is a projection,
- (4)  $u_{ij}^s = p_{ij}$ ,

*with Woronowicz algebra maps  $\Delta, \varepsilon, S$  constructed by universality.*

Here we allow the value  $s = \infty$ , with the convention that the last axiom simply disappears in this case. Observe that at  $s < \infty$  the normality condition is actually redundant. This is because a partial isometry  $a$  subject to the relation  $aa^* = a^s$  is normal. As a first result, making the connection with  $H_N^s$ , we have:

THEOREM 10.8. *We have an inclusion as follows,*

$$H_N^s \subset H_N^{s+}$$

*which is a liberation, in the sense that the classical version of  $H_N^{s+}$ , obtained by dividing by the commutator ideal, is  $H_N^s$ .*

PROOF. This follows as for  $O_N \subset O_N^+$  or  $S_N \subset S_N^+$ , by using the Gelfand theorem.  $\square$

In analogy with the results from the real case, we have the following result:

PROPOSITION 10.9. *The algebras  $C(H_N^{s+})$  with  $s = 1, 2, \infty$ , and their presentation relations in terms of the entries of the matrix  $u = (u_{ij})$ , are as follows.*

- (1) *For  $C(H_N^{1+}) = C(S_N^+)$ , the matrix  $u$  is magic: all its entries are projections, summing up to 1 on each row and column.*
- (2) *For  $C(H_N^{2+}) = C(H_N^+)$  the matrix  $u$  is cubic: it is orthogonal, and the products of pairs of distinct entries on the same row or the same column vanish.*
- (3) *For  $C(H_N^{\infty+}) = C(K_N^+)$  the matrix  $u$  is unitary, its transpose is unitary, and all its entries are normal partial isometries.*

PROOF. The idea here is as follows:

- (1) This follows from definitions and from standard operator algebra tricks.
- (2) This follows as well from definitions and from standard operator algebra tricks.
- (3) This is just a translation of the definition of  $C(H_N^{s+})$ , at  $s = \infty$ .  $\square$

Let us prove now that  $H_N^{s+}$  with  $s < \infty$  is a quantum permutation group. For this purpose, we must change the fundamental representation. Let us start with:

DEFINITION 10.10. *A  $(s, N)$ -sudoku matrix is a magic unitary of size  $sN$ , of the form*

$$m = \begin{pmatrix} a^0 & a^1 & \dots & a^{s-1} \\ a^{s-1} & a^0 & \dots & a^{s-2} \\ \vdots & \vdots & & \vdots \\ a^1 & a^2 & \dots & a^0 \end{pmatrix}$$

*where  $a^0, \dots, a^{s-1}$  are  $N \times N$  matrices.*

The basic examples of such sudoku matrices come from the group  $H_n^s$ . Indeed, with  $w = e^{2\pi i/s}$ , each of the  $N^2$  matrix coordinates  $u_{ij} : H_N^s \rightarrow \mathbb{C}$  takes values in the set:

$$S = \{0\} \cup \{1, w, \dots, w^{s-1}\}$$

Thus this coordinate decomposes as follows:

$$u_{ij} = \sum_{r=0}^{s-1} w^r a_{ij}^r$$

Here each  $a_{ij}^r$  is a function taking values in  $\{0, 1\}$ , and so a projection in the  $C^*$ -algebra sense, and it follows from definitions that these projections form a sudoku matrix. With this notion in hand, we have the following result:

**THEOREM 10.11.** *The following happen:*

- (1) *The algebra  $C(H_N^s)$  is isomorphic to the universal commutative  $C^*$ -algebra generated by the entries of a  $(s, N)$ -sudoku matrix.*
- (2) *The algebra  $C(H_N^{s+})$  is isomorphic to the universal  $C^*$ -algebra generated by the entries of a  $(s, N)$ -sudoku matrix.*

**PROOF.** The first assertion follows from the second one, via Theorem 10.8. In order to prove now the second assertion, consider the universal algebra in the statement, namely:

$$A = C^* \left( a_{ij}^p \mid (a_{ij}^{q-p})_{pi,qj} = (s, N) - \text{sudoku} \right)$$

Consider also the algebra  $C(H_N^{s+})$ . According to Definition 10.7, this is presented by certain relations  $R$ , that we will call here level  $s$  cubic conditions:

$$C(H_N^{s+}) = C^* \left( u_{ij} \mid u = N \times N \text{ level } s \text{ cubic} \right)$$

We will construct a pair of inverse morphisms between these algebras.

(1) Our first claim is that  $U_{ij} = \sum_p w^{-p} a_{ij}^p$  is a level  $s$  cubic unitary. Indeed, by using the sudoku condition, the verification of (1-4) in Definition 10.7 is routine.

(2) Our second claim is that the elements  $A_{ij}^p = \frac{1}{s} \sum_r w^{rp} u_{ij}^r$ , with the convention  $u_{ij}^0 = p_{ij}$ , form a level  $s$  sudoku unitary. Once again, the proof here is routine.

(3) According to the above, we can define a morphism  $\Phi : C(H_N^{s+}) \rightarrow A$  by the formula  $\Phi(u_{ij}) = U_{ij}$ , and a morphism  $\Psi : A \rightarrow C(H_N^{s+})$  by the formula  $\Psi(a_{ij}^p) = A_{ij}^p$ .

(4) We check now the fact that  $\Phi, \Psi$  are indeed inverse morphisms:

$$\begin{aligned} \Psi\Phi(u_{ij}) &= \sum_p w^{-p} A_{ij}^p \\ &= \frac{1}{s} \sum_p w^{-p} \sum_r w^{rp} u_{ij}^r \\ &= \frac{1}{s} \sum_{pr} w^{(r-1)p} u_{ij}^r \\ &= u_{ij} \end{aligned}$$

As for the other composition, we have the following computation:

$$\begin{aligned} \Phi\Psi(a_{ij}^p) &= \frac{1}{s} \sum_r w^{rp} U_{ij}^r \\ &= \frac{1}{s} \sum_r w^{rp} \sum_q w^{-rq} a_{ij}^q \\ &= \frac{1}{s} \sum_q a_{ij}^q \sum_r w^{r(p-q)} \\ &= a_{ij}^p \end{aligned}$$

Thus we have an isomorphism  $C(H_N^{s+}) = A$ , as claimed. □

Let us discuss now the interpretation of  $H_N^s, H_N^{s+}$  as classical and quantum symmetry groups of graphs.

We will need the following simple fact:

PROPOSITION 10.12. *A  $sN \times sN$  magic unitary commutes with the matrix*

$$\Sigma = \begin{pmatrix} 0 & I_N & 0 & \dots & 0 \\ 0 & 0 & I_N & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & I_N \\ I_N & 0 & 0 & \dots & 0 \end{pmatrix}$$

*if and only if it is a sudoku matrix in the sense of Definition 10.10.*

PROOF. This follows from the fact that commutation with  $\Sigma$  means that the matrix is circulant. Thus, we obtain the sudoku relations from Definition 10.10 above. □

Now let  $Z_s$  be the oriented cycle with  $s$  vertices, and consider the graph  $NZ_s$  consisting of  $N$  disjoint copies of it. Observe that, with a suitable labeling of the vertices, the adjacency matrix of this graph is the above matrix  $\Sigma$ . We obtain from this:

THEOREM 10.13. *We have the following results:*

- (1)  $H_N^s$  is the symmetry group of  $NZ_s$ .
- (2)  $H_N^{s+}$  is the quantum symmetry group of  $NZ_s$ .

PROOF. The idea here is as follows:

(1) This follows from definitions.

(2) This follows from Theorem 10.11 and Proposition 10.12, because  $C(H_N^{s+})$  is the quotient of  $C(S_{sN}^+)$  by the relations making the fundamental corepresentation commute with the adjacency matrix of  $NZ_s$ . □



Next in line, we must talk about wreath products. We have here:

THEOREM 10.14. *We have the following results:*

- (1)  $H_N^s = \mathbb{Z}_s \wr S_N$ .
- (2)  $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$ .

PROOF. This follows from the following formulae, valid for any connected graph  $X$ , and explained before, applied to  $Z_s$ :

$$\begin{aligned} G(NX) &= G(X) \wr S_N \\ G^+(NX) &= G^+(X) \wr_* S_N^+ \end{aligned}$$

Alternatively, (1) follows from definitions, and (2) can be proved directly, by constructing a pair of inverse morphisms. For details here, we refer to [38].  $\square$

Regarding now the easiness property of the quantum groups  $H_N^s, H_N^{s+}$ , we already know that this happens at  $s = 1, 2$ . In general, we have the following result, from [9]:

THEOREM 10.15. *The quantum groups  $H_N^s, H_N^{s+}$  are easy, the corresponding categories*

$$P^s \subset P$$

$$NC^s \subset NC$$

*consisting of partitions having the property*

$$\# \circ - \# \bullet = 0(s)$$

*as a weighted sum, in each block.*

PROOF. Observe that the result holds at  $s = 1$ , trivially, and at  $s = 2$  as well, where our condition is equivalent to  $\# \circ + \# \bullet = 0(2)$ , in each block. In general, this follows as in the proof for  $H_N, H_N^+$ , by using the one-block partition in  $P(s, s)$ . See [9].  $\square$

The above proof was of course quite brief, but we will not be really interested here in the case  $s \geq 3$ , which is quite technical. In fact, the above result, dealing with the general case  $s \in \mathbb{N}$ , is here for providing an introduction to the case  $s = \infty$ , where we have:

THEOREM 10.16. *The quantum groups  $K_N, K_N^+$  are easy, the corresponding categories*

$$\mathcal{P}_{\text{even}} \subset P$$

$$\mathcal{NC}_{\text{even}} \subset NC$$

*consisting of partitions having the property*

$$\# \circ = \# \bullet$$

*as a weighted equality, in each block.*

PROOF. This follows from Theorem 10.15, or rather by proving the result directly, a bit as in the  $s = 1, 2$  cases, because the  $s = \infty$  case is needed first, in order to discuss the general case,  $s \in \mathbb{N} \cup \{\infty\}$ . For details here, we refer once again to [9].  $\square$

### 10c. Fusion rules

Let us discuss now, following [38], the classification of the irreducible representations of  $H_N^{s+}$ , and the computation of their fusion rules.

For this purpose, let us go back to the elements  $u_{ij}, p_{ij}$  in Definition 10.7 above. We recall that, as a consequence of Proposition 10.9, the matrix  $p = (p_{ij})$  is a magic unitary. We first have the following result:

PROPOSITION 10.17. *The elements  $u_{ij}$  and  $p_{ij}$  satisfy:*

- (1)  $p_{ij}u_{ij} = u_{ij}$ .
- (2)  $u_{ij}^* = u_{ij}^{s-1}$ .
- (3)  $u_{ij}u_{ik} = 0$  for  $j \neq k$ .

PROOF. We use the fact that in a  $C^*$ -algebra,  $aa^* = 0$  implies  $a = 0$ .

- (1) This follows from the following computation, with  $a = (p_{ij} - 1)u_{ij}$ :

$$\begin{aligned} aa^* &= (p_{ij} - 1)p_{ij}(p_{ij} - 1) \\ &= 0 \end{aligned}$$

- (2) With  $a = u_{ij}^* - u_{ij}^{s-1}$  we have  $aa^* = 0$ , which gives the result.

- (3) With  $a = u_{ij}u_{ik}$  we have  $aa^* = 0$ , which gives the result.  $\square$

In what follows, we make the convention  $u_{ij}^0 = p_{ij}$ . We have then:

THEOREM 10.18. *The algebra  $C(H_N^{s+})$  has a family of  $N$ -dimensional corepresentations  $\{u_k | k \in \mathbb{Z}\}$ , satisfying the following conditions:*

- (1)  $u_k = (u_{ij}^k)$  for any  $k \geq 0$ .
- (2)  $u_k = u_{k+s}$  for any  $k \in \mathbb{Z}$ .
- (3)  $\bar{u}_k = u_{-k}$  for any  $k \in \mathbb{Z}$ .

PROOF. The idea here is as follows:

- (1) Let us set  $u_k = (u_{ij}^k)$ . By using Proposition 10.17 (3), we have:

$$\begin{aligned} \Delta(u_{ij}^k) &= \sum_{l_1 \dots l_k} u_{il_1} \dots u_{il_k} \otimes u_{l_1j} \dots u_{l_kj} \\ &= \sum_l u_{il}^k \otimes u_{lj}^k \end{aligned}$$

We have as well, trivially, the following two formulae:

$$\begin{aligned} \varepsilon(u_{ij}^k) &= \delta_{ij} \\ S(u_{ij}^k) &= u_{ji}^{*k} \end{aligned}$$

(2) This follows once again from Proposition 10.17 (3), as follows:

$$\begin{aligned} u_{ij}^{k+s} &= u_{ij}^k u_{ij}^s \\ &= u_{ij}^k p_{ij} \\ &= u_{ij}^k \end{aligned}$$

(3) This follows from Proposition 10.17 (2), and we are done.  $\square$

Let us compute now the intertwiners between the various tensor products between the above corepresentations  $u_i$ . For this purpose, we make the assumption  $N \geq 4$ , which brings linear independence. In order to simplify the notations, we will use:

DEFINITION 10.19. For  $i_1, \dots, i_k \in \mathbb{Z}$  we use the notation

$$u_{i_1 \dots i_k} = u_{i_1} \otimes \dots \otimes u_{i_k}$$

where  $\{u_i | i \in \mathbb{Z}\}$  are the corepresentations in Theorem 10.18.

Observe that in the particular case  $i_1, \dots, i_k \in \{\pm 1\}$ , we obtain in this way all the possible tensor products between  $u = u_1$  and  $\bar{u} = u_{-1}$ , known by [98] to contain any irreducible corepresentation of  $C(H_N^{s+})$ . Here is now our main result:

THEOREM 10.20. We have the following equality of linear spaces

$$\text{Hom}(u_{i_1 \dots i_k}, u_{j_1 \dots j_l}) = \text{span} \left\{ T_p \mid p \in NC_s(i_1 \dots i_k, j_1 \dots j_l) \right\}$$

where the set on the right consists of elements of  $NC(k, l)$  having the property that in each block, the sum of  $i$  indices equals the sum of  $j$  indices, modulo  $s$ .

PROOF. This result is from [38], the idea of the proof being as follows:

(1) Our first claim is that, in order to prove  $\supset$ , we may restrict attention to the case  $k = 0$ . This follows indeed from the Frobenius duality isomorphism.

(2) Our second claim is that, in order to prove  $\supset$  in the case  $k = 0$ , we may restrict attention to the one-block partitions. Indeed, this follows once again from a standard trick. Consider the following disjoint union:

$$NC_s = \bigcup_{k=0}^{\infty} \bigcup_{i_1 \dots i_k} NC_s(0, i_1 \dots i_k)$$

This is a set of labeled partitions, having property that each  $p \in NC_s$  is noncrossing, and that for  $p \in NC_s$ , any block of  $p$  is in  $NC_s$ . But it is well-known that under these assumptions, the global algebraic properties of  $NC_s$  can be checked on blocks.

(3) Proof of  $\supset$ . According to the above considerations, we just have to prove that the vector associated to the one-block partition in  $NC(l)$  is fixed by  $u_{j_1 \dots j_l}$ , when:

$$s \mid j_1 + \dots + j_l$$

Consider the standard generators  $e_{ab} \in M_N(\mathbb{C})$ , acting on the basis vectors by:

$$e_{ab}(e_c) = \delta_{bc}e_a$$

The corepresentation  $u_{j_1 \dots j_l}$  is given by the following formula:

$$u_{j_1 \dots j_l} = \sum_{a_1 \dots a_l} \sum_{b_1 \dots b_l} u_{a_1 b_1}^{j_1} \dots u_{a_l b_l}^{j_l} \otimes e_{a_1 b_1} \otimes \dots \otimes e_{a_l b_l}$$

As for the vector associated to the one-block partition, this is:

$$\xi_l = \sum_b e_b^{\otimes l}$$

By using now several times the relations in Proposition 10.17, we obtain, as claimed:

$$\begin{aligned} u_{j_1 \dots j_l}(1 \otimes \xi_l) &= \sum_{a_1 \dots a_l} \sum_b u_{a_1 b}^{j_1} \dots u_{a_l b}^{j_l} \otimes e_{a_1} \otimes \dots \otimes e_{a_l} \\ &= \sum_{ab} u_{ab}^{j_1 + \dots + j_l} \otimes e_a^{\otimes l} \\ &= 1 \otimes \xi_l \end{aligned}$$

(4) Proof of  $\subset$ . The spaces on the right in the statement form a Tannakian category in the sense of [99], so they correspond to a certain Woronowicz algebra  $A$ .

This algebra is by definition the maximal model for the Tannakian category. In other words, it comes with a family of corepresentations  $\{v_i\}$ , such that:

$$Hom(v_{i_1 \dots i_k}, v_{j_1 \dots j_l}) = \text{span} \left\{ T_p \mid p \in NC_s(i_1 \dots i_k, j_1 \dots j_l) \right\}$$

On the other hand, the inclusion  $\supset$  that we just proved shows that  $C(H_N^{s+})$  is a model for the category. Thus we have a quotient map as follows:

$$A \rightarrow C(H_N^{s+})$$

$$v_i \rightarrow u_i$$

But this latter map can be shown to be an isomorphism, by suitably adapting the proof from the  $s = 1$  case, for the quantum permutation group  $S_N^+$ . See [9], [38].  $\square$

As an illustration for the above result, we have the following statement:

**THEOREM 10.21.** *The basic corepresentations  $u_0, \dots, u_{s-1}$  are as follows:*

- (1)  $u_1, \dots, u_{s-1}$  are irreducible.
- (2)  $u_0 = 1 + r_0$ , with  $r_0$  irreducible.
- (3)  $r_0, u_1, \dots, u_{s-1}$  are distinct.

PROOF. We apply Theorem 10.20 with  $k = l = 1$  and  $i_1 = i, j_1 = j$ . This gives:

$$\dim(\text{Hom}(u_i, u_j)) = \#NC_s(i, j)$$

We have two candidates for the elements of  $NC_s(i, j)$ , namely the two partitions in  $NC(1, 1)$ . So, consider these two partitions, with the points labeled by  $i, j$ :

$$p = \left\{ \begin{array}{c} i \\ | \\ j \end{array} \right\} \quad q = \left\{ \begin{array}{c} i \\ | \\ | \\ j \end{array} \right\}$$

We have to check for each of these partitions if the sum of  $i$  indices equals or not the sum of  $j$  indices, modulo  $s$ , in each block. The answer is as follows:

$$\begin{aligned} p \in NC_s(i, j) &\iff i = j \\ q \in NC_s(i, j) &\iff i = j = 0 \end{aligned}$$

By collecting together these two answers, we obtain:

$$\#NC_s(i, j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \neq 0 \\ 2 & \text{if } i = j = 0 \end{cases}$$

We can now prove the various assertions, as follows:

- (1) This follows from the second equality.
- (2) This follows from the third equality and from the fact that we have  $1 \in u_s$ .
- (3) This follows from the first equality. □

Let us record as well, as a second consequence, the following result:

**THEOREM 10.22.** *We have the formula*

$$\#(1 \in u_{i_1} \otimes \dots \otimes u_{i_k}) = \#NC_s(i_1 \dots i_k)$$

where the set on the right consists of noncrossing partitions of  $\{1, \dots, k\}$  having the property that the sum of indices in each block is a multiple of  $s$ .

PROOF. This is clear indeed from Theorem 10.20 above. □

We can now compute the fusion rules for  $H_N^{s+}$ .

The result here, from [38], is as follows:

THEOREM 10.23. Let  $F = \langle \mathbb{Z}_s \rangle$  be the words over  $\mathbb{Z}_s$ , with involution

$$(i_1 \dots i_k)^- = (-i_k) \dots (-i_1)$$

and with fusion product given by:

$$(i_1 \dots i_k) \cdot (j_1 \dots j_l) = i_1 \dots i_{k-1} (i_k + j_1) j_2 \dots j_l$$

The irreducible representations of  $H_N^{s+}$  can then be labeled  $r_x$  with  $x \in F$ , such that the involution and fusion rules are  $\bar{r}_x = r_{\bar{x}}$  and

$$r_x \otimes r_y = \sum_{x=vz, y=\bar{z}w} r_{vw} + r_{v \cdot w}$$

and such that we have  $r_i = u_i - \delta_{i0}1$  for any  $i \in \mathbb{Z}_s$ .

PROOF. This basically follows from Theorem 10.20, the idea being as follows:

(1) Consider the monoid  $A = \{a_x | x \in F\}$ , with multiplication  $a_x a_y = a_{xy}$ . We denote by  $\mathbb{N}A$  the set of linear combinations of elements in  $A$ , with coefficients in  $\mathbb{N}$ , and we endow it with fusion rules as in the statement:

$$a_x \otimes a_y = \sum_{x=vz, y=\bar{z}w} a_{vw} + a_{v \cdot w}$$

With these notations,  $(\mathbb{N}A, +, \otimes)$  is a semiring. We will use as well the set  $\mathbb{Z}A$ , formed by the linear combinations of elements of  $A$ , with coefficients in  $\mathbb{Z}$ . The above tensor product operation extends to  $\mathbb{Z}A$ , and  $(\mathbb{Z}A, +, \otimes)$  is a ring.

(2) Our claim is that the fusion rules on  $\mathbb{Z}A$  can be uniquely described by conversion formulae as follows, with  $C$  being positive integers, and  $D$  being integers:

$$a_{i_1} \otimes \dots \otimes a_{i_k} = \sum_l \sum_{j_1 \dots j_l} C_{i_1 \dots i_k}^{j_1 \dots j_l} a_{j_1 \dots j_l}$$

$$a_{i_1 \dots i_k} = \sum_l \sum_{j_1 \dots j_l} D_{i_1 \dots i_k}^{j_1 \dots j_l} a_{j_1} \otimes \dots \otimes a_{j_l}$$

Indeed, the existence and uniqueness of such decompositions follow from the definition of the tensor product operation, and by recurrence over  $k$  for the  $D$  coefficients.

(3) Our claim is that there is a unique morphism of rings  $\Phi : \mathbb{Z}A \rightarrow R$ , such that  $\Phi(a_i) = r_i$  for any  $i$ . Indeed, consider the following elements of  $R$ :

$$r_{i_1 \dots i_k} = \sum_l \sum_{j_1 \dots j_l} D_{i_1 \dots i_k}^{j_1 \dots j_l} r_{j_1} \otimes \dots \otimes r_{j_l}$$

In case we have a morphism as claimed, we must have  $\Phi(a_x) = r_x$  for any  $x \in F$ . Thus our morphism is uniquely determined on  $A$ , so it is uniquely determined on  $\mathbb{Z}A$ .

In order to prove the existence, we can set  $\Phi(a_x) = r_x$  for any  $x \in F$ , then extend  $\Phi$  by linearity to the whole  $\mathbb{Z}A$ . Since  $\Phi$  commutes with the above conversion formulae, which describe the fusion rules, it is indeed a morphism.

(4) Our claim is that  $\Phi$  commutes with the linear forms  $x \rightarrow \#(1 \in x)$ . Indeed, by linearity we just have to check the following equality:

$$\#(1 \in a_{i_1} \otimes \dots \otimes a_{i_k}) = \#(1 \in r_{i_1} \otimes \dots \otimes r_{i_k})$$

Now remember that the elements  $r_i$  are defined as  $r_i = u_i - \delta_{i0}1$ . So, consider the elements  $c_i = a_i + \delta_{i0}1$ . Since the operations  $r_i \rightarrow u_i$  and  $a_i \rightarrow c_i$  are of the same nature, by linearity the above formula is equivalent to:

$$\#(1 \in c_{i_1} \otimes \dots \otimes c_{i_k}) = \#(1 \in u_{i_1} \otimes \dots \otimes u_{i_k})$$

Now by using Theorem 10.20, what we have to prove is:

$$\#(1 \in c_{i_1} \otimes \dots \otimes c_{i_k}) = \#NC_s(i_1 \dots i_k)$$

In order to prove this formula, consider the product on the left:

$$P = (a_{i_1} + \delta_{i_1 0}1) \otimes (a_{i_2} + \delta_{i_2 0}1) \otimes \dots \otimes (a_{i_k} + \delta_{i_k 0}1)$$

This quantity can be computed by using the fusion rules on  $A$ . A recurrence on  $k$  shows that the final components of type  $a_x$  will come from the different ways of grouping and summing the consecutive terms of the sequence  $(i_1, \dots, i_k)$ , and removing some of the sums which vanish modulo  $s$ , as to obtain the sequence  $x$ . But this can be encoded by families of noncrossing partitions, and in particular the 1 components will come from the partitions in  $NC_s(i_1 \dots i_k)$ . Thus  $\#(1 \in P) = \#NC_s(i_1 \dots i_k)$ , as claimed.

(5) Our claim now is that  $\Phi$  is injective. Indeed, this follows from the result in the previous step, by using a standard positivity argument:

$$\begin{aligned} \Phi(\alpha) = 0 &\implies \Phi(\alpha\alpha^*) = 0 \\ &\implies \#(1 \in \Phi(\alpha\alpha^*)) = 0 \\ &\implies \#(1 \in \alpha\alpha^*) = 0 \\ &\implies \alpha = 0 \end{aligned}$$

Here  $\alpha$  is arbitrary in the domain of  $\Phi$ , we use the notation  $a_x^* = a_{\bar{x}}$ , where  $a \rightarrow \#(1, a)$  is the unique linear extension of the operation consisting of counting the number of 1's. Observe that this latter linear form is indeed positive definite, according to the identity  $\#(1, a_x a_y^*) = \delta_{xy}$ , which is clear from the definition of the product of  $\mathbb{Z}A$ .

(6) Our claim is that we have  $\Phi(A) \subset R_{irr}$ . This is the same as saying that  $r_x \in R_{irr}$  for any  $x \in F$ , and we will prove it by recurrence on the length of  $x$ .

Assume that the assertion is true for all the words of length  $< k$ , and consider an arbitrary length  $k$  word,  $x = i_1 \dots i_k$ . We have:

$$a_{i_1} \otimes a_{i_2 \dots i_k} = a_x + a_{i_1+i_2, i_3 \dots i_k} + \delta_{i_1+i_2, 0} a_{i_3 \dots i_k}$$

By applying  $\Phi$  to this decomposition, we obtain:

$$r_{i_1} \otimes r_{i_2 \dots i_k} = r_x + r_{i_1+i_2, i_3 \dots i_k} + \delta_{i_1+i_2, 0} r_{i_3 \dots i_k}$$

We have the following computation, which is valid for  $y = i_1 + i_2, i_3 \dots i_k$ , as well as for  $y = i_3 \dots i_k$  in the case  $i_1 + i_2 = 0$ :

$$\begin{aligned} \#(r_y \in r_{i_1} \otimes r_{i_2 \dots i_k}) &= \#(1, r_{\bar{y}} \otimes r_{i_1} \otimes r_{i_2 \dots i_k}) \\ &= \#(1, a_{\bar{y}} \otimes a_{i_1} \otimes a_{i_2 \dots i_k}) \\ &= \#(a_y \in a_{i_1} \otimes a_{i_2 \dots i_k}) \\ &= 1 \end{aligned}$$

Moreover, we know from the previous step that we have  $r_{i_1+i_2, i_3 \dots i_k} \neq r_{i_3 \dots i_k}$ , so we conclude that the following formula defines an element of  $R^+$ :

$$\alpha = r_{i_1} \otimes r_{i_2 \dots i_k} - r_{i_1+i_2, i_3 \dots i_k} - \delta_{i_1+i_2, 0} r_{i_3 \dots i_k}$$

On the other hand, we have  $\alpha = r_x$ , so we conclude that we have  $r_x \in R^+$ . Finally, the irreducibility of  $r_x$  follows from the following computation:

$$\begin{aligned} \#(1 \in r_x \otimes \bar{r}_x) &= \#(1 \in r_x \otimes r_{\bar{x}}) \\ &= \#(1 \in a_x \otimes a_{\bar{x}}) \\ &= \#(1 \in a_x \otimes \bar{a}_x) \\ &= 1 \end{aligned}$$

(7) Summarizing, we have constructed an injective ring morphism:

$$\Phi : \mathbb{Z}A \rightarrow R$$

$$\Phi(A) \subset R_{irr}$$

The remaining fact to be proved, namely that we have  $\Phi(A) = R_{irr}$ , is clear from the general results in [98]. Indeed, since each element of  $\mathbb{N}A$  is a sum of elements in  $A$ , by applying  $\Phi$  we get that each element in  $\Phi(\mathbb{N}A)$  is a sum of irreducible corepresentations in  $\Phi(A)$ . But since  $\Phi(\mathbb{N}A)$  contains all the tensor powers between the fundamental corepresentation and its conjugate, we get  $\Phi(A) = R_{irr}$ , and we are done.  $\square$

As an illustration for the above result, at  $s = 1$  we obtain the Clebsch-Gordan rules for the irreducible representations of  $S_N^+$ . The cases  $s = 2, \infty$  are quite interesting too.

We refer to [39] for more on all the above.



### 10d. Bessel laws

Let us discuss now the computation of the asymptotic laws of characters. We begin with a discussion for  $H_N$ , from [17], which has its own interest:

**THEOREM 10.24.** *The asymptotic law of  $\chi_t$  for the group  $H_N$  is given by*

$$b_t = e^{-t} \sum_{k=-\infty}^{\infty} \delta_k \sum_{p=0}^{\infty} \frac{(t/2)^{|k|+2p}}{(|k|+p)!p!}$$

where  $\delta_k$  is the Dirac mass at  $k \in \mathbb{Z}$ .

**PROOF.** We regard the hyperoctahedral group  $H_N$  as being the symmetry group of the graph  $I_N = \{I^1, \dots, I^N\}$  formed by  $N$  segments. The diagonal coefficients are then:

$$u_{ii}(g) = \begin{cases} 0 & \text{if } g \text{ moves } I^i \\ +1 & \text{if } g \text{ fixes } I^i \\ -1 & \text{if } g \text{ returns } I^i \end{cases}$$

Let  $s = [tN]$ , and denote by  $\uparrow g, \downarrow g$  the number of segments among  $\{I^1, \dots, I^s\}$  which are fixed, respectively returned by an element  $g \in H_N$ . With this notation, we have:

$$u_{11} + \dots + u_{ss} = \uparrow g - \downarrow g$$

We denote by  $P_N$  probabilities computed over the group  $H_N$ . The density of the law of  $u_{11} + \dots + u_{ss}$  at a point  $k \geq 0$  is given by the following formula:

$$\begin{aligned} D(k) &= P_N(\uparrow g - \downarrow g = k) \\ &= \sum_{p=0}^{\infty} P_N(\uparrow g = k + p, \downarrow g = p) \end{aligned}$$

Assume first  $t = 1$ . We use the fact that the probability of  $\sigma \in S_N$  to have no fixed points is asymptotically  $P_0 = \frac{1}{e}$ . Thus the probability of  $\sigma \in S_N$  to have  $m$  fixed points is asymptotically  $P + m = \frac{1}{(em)!}$ . In terms of probabilities over  $H_N$ , we obtain:

$$\begin{aligned} \lim_{N \rightarrow \infty} D(k) &= \lim_{N \rightarrow \infty} \sum_{p=0}^{\infty} (1/2)^{k+2p} \binom{k+2p}{k+p} P_N(\uparrow g + \downarrow g = k+2p) \\ &= \sum_{p=0}^{\infty} (1/2)^{k+2p} \binom{k+2p}{k+p} \frac{1}{e(k+2p)!} \\ &= \frac{1}{e} \sum_{p=0}^{\infty} \frac{(1/2)^{k+2p}}{(k+p)!p!} \end{aligned}$$

The general case  $t \in (0, 1]$  follows by performing some modifications in the above computation. The asymptotic density is computed as follows:

$$\begin{aligned} \lim_{N \rightarrow \infty} D(k) &= \lim_{N \rightarrow \infty} \sum_{p=0}^{\infty} (1/2)^{k+2p} \binom{k+2p}{k+p} P_N(\uparrow g + \downarrow g = k+2p) \\ &= \sum_{p=0}^{\infty} (1/2)^{k+2p} \binom{k+2p}{k+p} \frac{t^{k+2p}}{e^t (k+2p)!} \\ &= e^{-t} \sum_{p=0}^{\infty} \frac{(t/2)^{k+2p}}{(k+p)!p!} \end{aligned}$$

On the other hand, we have  $D(-k) = D(k)$ , so we obtain the result.  $\square$

Observe that the measure found above is of the form:

$$b_t = e^{-t} \sum_{k=-\infty}^{\infty} \delta_k f_k(t/2)$$

Here  $f_k$  is the Bessel function of the first kind:

$$f_k(t) = \sum_{p=0}^{\infty} \frac{t^{|k|+2p}}{(|k|+p)!p!}$$

Next, we have the following result, once again from [17]:

**THEOREM 10.25.** *The Bessel laws  $b_t$  have the additivity property*

$$b_s * b_t = b_{s+t}$$

*so they form a truncated one-parameter semigroup with respect to convolution.*

**PROOF.** The Fourier transform of  $b_t$  is given by:

$$Fb_t(y) = e^{-t} \sum_{k=-\infty}^{\infty} e^{ky} f_k(t/2)$$

We compute now the derivative with respect to  $t$ :

$$Fb_t(y)' = -Fb_t(y) + \frac{e^{-t}}{2} \sum_{k=-\infty}^{\infty} e^{ky} f_k'(t/2)$$

On the other hand, the derivative of  $f_k$  with  $k \geq 1$  is given by:

$$\begin{aligned}
f'_k(t) &= \sum_{p=0}^{\infty} \frac{(k+2p)t^{k+2p-1}}{(k+p)!p!} \\
&= \sum_{p=0}^{\infty} \frac{(k+p)t^{k+2p-1}}{(k+p)!p!} + \sum_{p=0}^{\infty} \frac{pt^{k+2p-1}}{(k+p)!p!} \\
&= \sum_{p=0}^{\infty} \frac{t^{k+2p-1}}{(k+p-1)!p!} + \sum_{p=1}^{\infty} \frac{t^{k+2p-1}}{(k+p)!(p-1)!} \\
&= \sum_{p=0}^{\infty} \frac{t^{(k-1)+2p}}{((k-1)+p)!p!} + \sum_{p=1}^{\infty} \frac{t^{(k+1)+2(p-1)}}{((k+1)+(p-1))!(p-1)!} \\
&= f_{k-1}(t) + f_{k+1}(t)
\end{aligned}$$

This computation works in fact for any  $k$ , so we get:

$$\begin{aligned}
Fb_t(y)' &= -Fb_t(y) + \frac{e^{-t}}{2} \sum_{k=-\infty}^{\infty} e^{ky} (f_{k-1}(t/2) + f_{k+1}(t/2)) \\
&= -Fb_t(y) + \frac{e^{-t}}{2} \sum_{k=-\infty}^{\infty} e^{(k+1)y} f_k(t/2) + e^{(k-1)y} f_k(t/2) \\
&= -Fb_t(y) + \frac{e^y + e^{-y}}{2} Fb_t(y) \\
&= \left( \frac{e^y + e^{-y}}{2} - 1 \right) Fb_t(y)
\end{aligned}$$

Thus the log of the Fourier transform is linear in  $t$ , and we get the assertion.  $\square$

In order to discuss now the free analogue  $\beta_t$  of the above measure  $b_t$ , as well as the  $s$ -analogues  $b_t^s, \beta_t^s$  of the measures  $b_t, \beta_t$ , we need some free probability. We have the following notion, extending the Poisson limit theory from chapter 5 above:

**DEFINITION 10.26.** *Associated to any compactly supported positive measure  $\rho$  on  $\mathbb{R}$  are the probability measures*

$$\begin{aligned}
p_\rho &= \lim_{n \rightarrow \infty} \left( \left( 1 - \frac{c}{n} \right) \delta_0 + \frac{1}{n} \rho \right)^{*n} \\
\pi_\rho &= \lim_{n \rightarrow \infty} \left( \left( 1 - \frac{c}{n} \right) \delta_0 + \frac{1}{n} \rho \right)^{\boxplus n}
\end{aligned}$$

where  $c = \text{mass}(\rho)$ , called *compound Poisson* and *compound free Poisson* laws.

In what follows we will be interested in the case where  $\rho$  is discrete, as is for instance the case for  $\rho = \delta_t$  with  $t > 0$ , which produces the Poisson and free Poisson laws.

The following result allows one to detect compound Poisson/free Poisson laws:

PROPOSITION 10.27. *For  $\rho = \sum_{i=1}^s c_i \delta_{z_i}$  with  $c_i > 0$  and  $z_i \in \mathbb{R}$  we have*

$$F_{p\rho}(y) = \exp\left(\sum_{i=1}^s c_i(e^{iyz_i} - 1)\right)$$

$$R_{\pi\rho}(y) = \sum_{i=1}^s \frac{c_i z_i}{1 - yz_i}$$

where  $F, R$  denote respectively the Fourier transform, and Voiculescu's  $R$ -transform.

PROOF. Let  $\mu_n$  be the measure in Definition 10.26, under the convolution signs:

$$\mu_n = \left(1 - \frac{c}{n}\right) \delta_0 + \frac{1}{n}$$

In the classical case, we have the following computation:

$$F_{\mu_n}(y) = \left(1 - \frac{c}{n}\right) + \frac{1}{n} \sum_{i=1}^s c_i e^{iyz_i}$$

$$\implies F_{\mu_n^{*n}}(y) = \left(\left(1 - \frac{c}{n}\right) + \frac{1}{n} \sum_{i=1}^s c_i e^{iyz_i}\right)^n$$

$$\implies F_{p\rho}(y) = \exp\left(\sum_{i=1}^s c_i(e^{iyz_i} - 1)\right)$$

In the free case now, we use a similar method. The Cauchy transform of  $\mu_n$  is:

$$G_{\mu_n}(\xi) = \left(1 - \frac{c}{n}\right) \frac{1}{\xi} + \frac{1}{n} \sum_{i=1}^s \frac{c_i}{\xi - z_i}$$

Consider now the  $R$ -transform of the measure  $\mu_n^{\boxplus n}$ , which is given by:

$$R_{\mu_n^{\boxplus n}}(y) = nR_{\mu_n}(y)$$

The above formula of  $G_{\mu_n}$  shows that the equation for  $R = R_{\mu_n^{\boxplus n}}$  is as follows:

$$\left(1 - \frac{c}{n}\right) \frac{1}{y^{-1} + R/n} + \frac{1}{n} \sum_{i=1}^s \frac{c_i}{y^{-1} + R/n - z_i} = y$$

$$\implies \left(1 - \frac{c}{n}\right) \frac{1}{1 + yR/n} + \frac{1}{n} \sum_{i=1}^s \frac{c_i}{1 + yR/n - yz_i} = 1$$

Now by multiplying by  $n$ , rearranging the terms, and letting  $n \rightarrow \infty$ , we get:

$$\begin{aligned} \frac{c + yR}{1 + yR/n} &= \sum_{i=1}^s \frac{c_i}{1 + yR/n - yz_i} \\ \implies c + yR_{\pi_\rho}(y) &= \sum_{i=1}^s \frac{c_i}{1 - yz_i} \\ \implies R_{\pi_\rho}(y) &= \sum_{i=1}^s \frac{c_i z_i}{1 - yz_i} \end{aligned}$$

This finishes the proof in the free case, and we are done.  $\square$

We have as well the following result, providing an alternative to Definition 10.26:

**THEOREM 10.28.** *For  $\rho = \sum_{i=1}^s c_i \delta_{z_i}$  with  $c_i > 0$  and  $z_i \in \mathbb{R}$  we have*

$$p_\rho / \pi_\rho = \text{law} \left( \sum_{i=1}^s z_i \alpha_i \right)$$

where the variables  $\alpha_i$  are Poisson/free Poisson( $c_i$ ), independent/free.

**PROOF.** Let  $\alpha$  be the sum of Poisson/free Poisson variables in the statement. We will show that the Fourier/ $R$ -transform of  $\alpha$  is given by the formulae in Proposition 10.27.

Indeed, by using some well-known Fourier transform formulae, we have:

$$\begin{aligned} F_{\alpha_i}(y) &= \exp(c_i(e^{iy} - 1)) \\ \implies F_{z_i \alpha_i}(y) &= \exp(c_i(e^{iyz_i} - 1)) \\ \implies F_\alpha(y) &= \exp \left( \sum_{i=1}^s c_i(e^{iyz_i} - 1) \right) \end{aligned}$$

Also, by using some well-known  $R$ -transform formulae, we have:

$$\begin{aligned} R_{\alpha_i}(y) &= \frac{c_i}{1 - y} \\ \implies R_{z_i \alpha_i}(y) &= \frac{c_i z_i}{1 - yz_i} \\ \implies R_\alpha(y) &= \sum_{i=1}^s \frac{c_i z_i}{1 - yz_i} \end{aligned}$$

Thus we have indeed the same formulae as those in Proposition 10.27.  $\square$

Summarizing, we have now a full extension of the basic Poisson limit theory from chapter 5, in the classical and free cases.

We can go back now to quantum reflection groups, and we have:

**THEOREM 10.29.** *The asymptotic laws of truncated characters are as follows, where  $\varepsilon_s$  with  $s \in \{1, 2, \dots, \infty\}$  is the uniform measure on the  $s$ -th roots of unity:*

- (1) *For  $H_N^s$  we obtain the compound Poisson law  $b_t^s = p_{t\varepsilon_s}$ .*
- (2) *For  $H_N^{s+}$  we obtain the compound free Poisson law  $\beta_t^s = \pi_{t\varepsilon_s}$ .*

*These measures are in Bercovici-Pata bijection.*

**PROOF.** This follows from easiness, and from the Weingarten formula, exactly as for the classical and quantum permutation groups. To be more precise, at  $t = 1$  this follows by counting the partitions, and at  $t \in (0, 1]$  general, this follows in the usual way, for instance by using cumulants. For details here, we refer to [9]. □

The Bessel and free Bessel laws have particularly interesting properties at the parameter values  $s = 2, \infty$ . So, let us record the precise statement here:

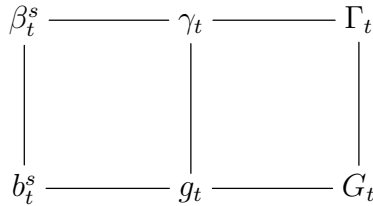
**THEOREM 10.30.** *The asymptotic laws of truncated characters are as follows:*

- (1) *For  $H_N$  we obtain the real Bessel law  $b_t = p_{t\varepsilon_2}$ .*
- (2) *For  $K_N$  we obtain the complex Bessel law  $B_t = p_{t\varepsilon_\infty}$ .*
- (3) *For  $H_N^+$  we obtain the free real Bessel law  $\beta_t = \pi_{t\varepsilon_2}$ .*
- (4) *For  $K_N^+$  we obtain the free complex Bessel law  $\mathfrak{B}_t = \pi_{t\varepsilon_\infty}$ .*

**PROOF.** This follows indeed from Theorem 10.29 above, at  $s = 2, \infty$ . □

We have the following result:

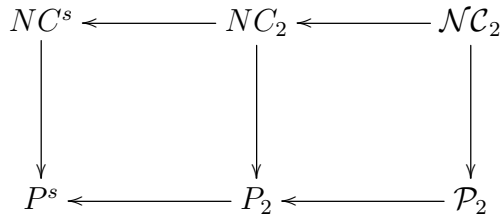
**THEOREM 10.31.** *The moments of the various central limiting measures, namely*



*are always given by the same formula, involving partitions, namely*

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

*where the sets of partitions  $D(k)$  in question are respectively*



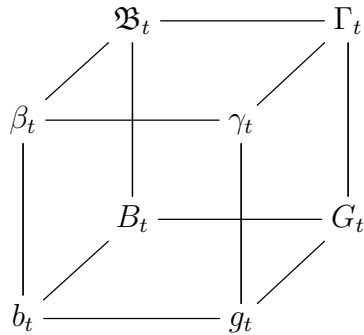
*and where  $|\cdot|$  is the number of blocks.*

PROOF. This follows by putting together the various moment results that we have.  $\square$

As already mentioned, in what regards the Bessel and free Bessel laws  $b_t^s, \beta_t^s$ , the important particular cases are  $s = 1, 2, \infty$ . It is therefore tempting to leave one of these 3 cases aside, and fold the corresponding diagram into a cube.

Quite surprisingly, in order to do so, in a correct way, the case which must be left aside is the most important one, namely  $s = 1$ , corresponding to the Poisson and free Poisson laws  $p_t, \pi_t$ . We will comment later on this, but let us just start by doing so:

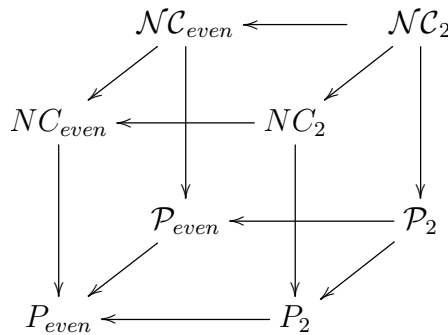
THEOREM 10.32. *The moments of the various central limiting measures, namely*



are always given by the same formula, involving partitions, namely

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

where the sets of partitions  $D(k)$  in question are respectively



and where  $|\cdot|$  is the number of blocks.

PROOF. This follows by putting together the various moment results that we have.  $\square$

In addition to what has been said above, there are as well some interesting results about the Bessel and free Bessel laws involving the multiplicative convolution  $\times$ , and the multiplicative free convolution  $\boxtimes$  from [92]. For details, we refer here to [9].

### 10e. Exercises

The present chapter was just an introduction to the quantum reflection groups, and there are many more basic things that can be said. As a first exercise, we have:

EXERCISE 10.33. *Prove that  $H_N, H_N^+$ , regarded as intermediate quantum groups*

$$S_N \subset H_N \subset O_N$$

$$S_N^+ \subset H_N^+ \subset O_N^+$$

*are indeed easy, the corresponding categories of partitions being as follows,*

$$P \supset P_{\text{even}} \supset P_2$$

$$NC \supset NC_{\text{even}} \supset NC_2$$

*with “even” standing for “with all the blocks having even size”.*

This is something that we briefly discussed in the above, and the problem now is that of working out all the details. As a hint here, the partition which generates the partitions having even blocks is  $\pi = \prod \prod$ , and so the problem is that of understanding the meaning of the condition  $T_\pi \in \text{Fix}(u^{\otimes 4})$ , and relating this to the definition of  $H_N, H_N^+$ .

As a continuation of the above exercise, we have:

EXERCISE 10.34. *Reformulate the Schur-Weyl twisting theory from chapter 9 in terms of the intermediate easy quantum groups*

$$H_N \subset G \subset U_N^+$$

*as a duality between such intermediate quantum groups.*

The point here is that, as explained in chapter 9, in order to perform the Schur-Weyl twisting operation we need a signature map for the partitions, and this signature map is only defined on  $P_{\text{even}}$ . Thus, we have a link here with  $H_N$ , and so the whole twisting material from chapter 9 above must be now reviewed, by taking this into account.

As a third exercise, on the same topic, we have:

EXERCISE 10.35. *Prove that the quantum reflection groups*

$$H_N, H_N^+$$

*equal their own Schur-Weyl twists, and then do the same for  $H_N^s, H_N^{s+}$ .*

This is not something very difficult, normally coming from definitions, one subtlety however coming from the fact that for the second question we must assume  $s \in 2\mathbb{N}$ , in order for the quantum groups  $H_N^s, H_N^{s+}$  to be indeed twistable in our sense.

At a more advanced level now, we have:



EXERCISE 10.36. *Prove that the intermediate easy quantum groups*

$$H_N \subset G \subset H_N^+$$

*are subject to a dichotomy, as follows,*

$$H_N \subset G \subset H_N^{[\infty]} \quad / \quad H_N^{[\infty]} \subset G \subset H_N^+$$

*where  $H_N^{[\infty]}$  is a certain suitably chosen such intermediate quantum group.*

As already mentioned, this is a rather difficult exercise.

Along the same lines, and as a final exercise on the subject, we have:

EXERCISE 10.37. *Fully classify the intermediate easy quantum groups*

$$H_N \subset G \subset H_N^+$$

*by solving the 2 classification problems raised by the previous exercise.*

As before, this is a difficult exercise, and the problem here is rather that of finding the relevant literature, and writing down a brief account of that.

## CHAPTER 11

### Liberation theory

#### 11a. Reflection subgroups

We have seen in the previous chapter that the basic reflection groups  $H_N^s = \mathbb{Z}_s \wr S_N$  have free analogues  $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$ , and that the theory of these quantum groups, both classical and free, is very interesting, algebraically and analytically speaking.

The world of quantum reflection groups is in fact much wider than this. In the classical case already, the classification theorem for the complex reflection groups, a celebrated result by Shephard and Todd [85], from the 50s, is as follows:

**THEOREM 11.1.** *The irreducible complex reflection groups are*

$$H_N^{sd} = \left\{ U \in H_N^s \mid (\det U)^d = 1 \right\}$$

*along with 34 exceptional examples.*

**PROOF.** This is something quite advanced, and we refer here to the paper of Shephard and Todd [85], and to the subsequent literature on the subject.  $\square$

In the general quantum case now, the axiomatization and classification of the quantum reflection groups is a key problem, which is not understood yet.

We will be interested in what follows in the “twistable” case, where the theory is more advanced than in the general case.

Let us start with the following definition:

**DEFINITION 11.2.** *A closed subgroup  $G \subset U_N^+$  is called:*

- (1) *Half-homogeneous, when it contains the alternating group,  $A_N \subset G$ .*
- (2) *Homogeneous, when it contains the symmetric group,  $S_N \subset G$ .*
- (3) *Twistable, when it contains the hyperoctahedral group,  $H_N \subset G$ .*

These notions are mostly motivated by the easy case. Here we have by definition  $S_N \subset G \subset U_N^+$ , and so our quantum group is automatically homogeneous.

The point now is that the twistability assumption corresponds to the following condition, at the level of the associated category of partitions  $D \subset P$ :

$$D \subset P_{\text{even}}$$

We recognize here the condition which is needed for performing the Schur-Weyl twisting operation, explained in chapter 9 above, and based on the signature map:

$$\varepsilon : P_{\text{even}} \rightarrow \{\pm 1\}$$

As a conclusion, in the easy case our notion of twistability is the correct one. In general, there are of course more general twisting methods, usually requiring  $\mathbb{Z}_2^N \subset G$  only. But in the half-homogeneous case, the condition  $\mathbb{Z}_2^N \subset G$  is equivalent to  $H_N \subset G$ .

With this discussion done, let us formulate now the following definition:

DEFINITION 11.3. *A twistable quantum reflection group is an intermediate subgroup*

$$H_N \subset K \subset K_N^+$$

*between the group  $H_N = \mathbb{Z}_2 \wr S_N$ , and the quantum group  $K_N^+ = \mathbb{T} \wr_* S_N^+$ .*

Here is now another definition, which is important for general compact quantum group purposes, and which provides motivations for our formalism from Definition 11.3:

DEFINITION 11.4. *Given a closed subgroup  $G \subset U_N^+$  which is twistable, in the sense that we have  $H_N \subset G$ , we define its associated reflection subgroup to be*

$$K = G \cap K_N^+$$

*with the intersection taken inside  $U_N^+$ . We say that  $G$  appears as a soft liberation of its classical version  $G_{\text{class}} = G \cap U_N$  when  $G = \langle G_{\text{class}}, K \rangle$ .*

These notions are important in the classification theory of compact quantum groups, and in connection with certain noncommutative geometry questions as well.

As a first observation, with  $K$  being as above, we have an intersection diagram, as follows:

$$\begin{array}{ccc} K & \longrightarrow & G \\ \uparrow & & \uparrow \\ K_{\text{class}} & \longrightarrow & G_{\text{class}} \end{array}$$

The soft liberation condition states that this diagram must be a generation diagram. We will be back to this in a moment, with some further theoretical comments. Let us work out some examples. As a basic result, we have:

THEOREM 11.5. *The reflection subgroups of the basic unitary quantum groups*

$$\begin{array}{ccccc}
 U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+
 \end{array}$$

are as follows,

$$\begin{array}{ccccc}
 K_N & \longrightarrow & K_N^* & \longrightarrow & K_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N & \longrightarrow & H_N^* & \longrightarrow & H_N^+
 \end{array}$$

and these unitary quantum groups all appear via soft liberation.

PROOF. The fact that the reflection subgroups of the quantum groups on the left are those on the right is clear in all cases, with the middle objects being by definition:

$$H_N^* = H_N \cap O_N^*$$

$$K_N^* = K_N \cap U_N^*$$

Regarding the second assertion, things are quite tricky here, as follows:

(1) In the classical case there is nothing to prove, because any classical group is by definition a soft liberation of itself.

(2) In the half-classical case the results are non-trivial, but can be proved by using the technology developed by Bichon and Dubois-Violette in [46].

(3) In the free case the results are highly non-trivial, and the only known proof so far uses the recurrence methods developed by Chirvasitu in [51].  $\square$

### 11b. Toral subgroups

Summarizing, we are here into recent and interesting quantum group theory. We will discuss a bit later the concrete applications of Theorem 11.5. There is a connection here as well with the notion of diagonal torus, introduced in chapter 1 above. We can indeed refine Definition 11.4, in the following way:

DEFINITION 11.6. Given  $H_N \subset G \subset U_N^+$ , the diagonal tori  $T = G \cap \mathbb{T}_N^+$  and reflection subgroups  $K = G \cap K_N^+$  for  $G$  and for  $G_{class} = G \cap U_N$  form a diagram as follows:

$$\begin{array}{ccccc}
 T & \longrightarrow & K & \longrightarrow & G \\
 \uparrow & & \uparrow & & \uparrow \\
 T_{class} & \longrightarrow & K_{class} & \longrightarrow & G_{class}
 \end{array}$$

We say that  $G$  appears as a soft/hard liberation when it is generated by  $G_{class}$  and by  $K/T$ , which means that the right square/whole rectangle should be generation diagrams.

It is in fact possible to further complicate the picture, by adding free versions as well, with these free versions being by definition given by the following formula:

$$G_{free} = \langle G, S_N^+ \rangle$$

Importantly, we can equally add the parameter  $N \in \mathbb{N}$  to the picture, the idea being that we have a kind of “ladder”, whose steps are the diagrams in Definition 11.6, perhaps extended with the free versions too, at fixed values of  $N \in \mathbb{N}$ .

The various generation and intersection properties of this ladder are important properties of  $G = (G_N)$  itself, with subtle relations between them. In fact, as already mentioned in the proof of Theorem 11.5 above, the proof of the soft generation property for  $O_N^+, U_N^+$  uses in fact this ladder, via the recurrence methods developed in [51].

All this is quite technical, so as a concrete result in connection with the above hard liberation notion, we have the following statement, improving Theorem 11.5:

THEOREM 11.7. *The diagonal tori of the basic unitary quantum groups*

$$\begin{array}{ccccc}
 U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+
 \end{array}$$

are as follows,

$$\begin{array}{ccccc}
 \mathbb{T}_N & \longrightarrow & \mathbb{T}_N^* & \longrightarrow & \mathbb{T}_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 T_N & \longrightarrow & T_N^* & \longrightarrow & T_N^+
 \end{array}$$

and these unitary quantum groups all appear via hard liberation.

PROOF. The first assertion is something that we already know, from chapter 1 above. As for the second assertion, this can be proved by carefully examining the proof of Theorem 11.5, and performing some suitable modifications, where needed.  $\square$

As an interesting remark, some subtleties appear in the following way:

PROPOSITION 11.8. *The diagonal tori of the basic quantum reflection groups*

$$\begin{array}{ccccc}
 K_N & \longrightarrow & K_N^* & \longrightarrow & K_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N & \longrightarrow & H_N^* & \longrightarrow & H_N^+
 \end{array}$$

are as follows,

$$\begin{array}{ccccc}
 \mathbb{T}_N & \longrightarrow & \mathbb{T}_N^* & \longrightarrow & \mathbb{T}_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 T_N & \longrightarrow & T_N^* & \longrightarrow & T_N^+
 \end{array}$$

and these quantum reflection groups do not all appear via hard liberation.

PROOF. The first assertion is clear, for instance as a consequence of Theorem 11.7, because the diagonal torus is the same for a quantum group, and for its reflection subgroup:

$$G \cap \mathbb{T}_N^+ = (G \cap K_N^+) \cap \mathbb{T}_N^+$$

Regarding the second assertion, things are quite tricky here, as follows:

(1) In the classical case the hard liberation property definitely holds, because any classical group is by definition a hard liberation of itself.

(2) In the half-classical case the answer is again positive, and this can be proved by using the technology developed by Bichon and Dubois-Violette in [46].

(3) In the free case the hard liberation property fails, due to some intermediate quantum groups  $H_N^{[\infty]}$ ,  $K_N^{[\infty]}$ , where “hard liberation stops”. We will be back to this.  $\square$

As a conjectural solution to these latter difficulties, coming from Proposition 11.8, we have the notion of Fourier liberation, that we will discuss now.

Let us first discuss the group dual subgroups of the arbitrary compact quantum groups  $G \subset U_N^+$ .

To start with, we have the following basic statement:

PROPOSITION 11.9. *Let  $G \subset U_N^+$  be a compact quantum group, and consider the group dual subgroups  $\widehat{\Lambda} \subset G$ , also called toral subgroups, or simply “tori”.*

- (1) *In the classical case, where  $G \subset U_N$  is a compact Lie group, these are the usual tori, where by torus we mean here closed abelian subgroup.*
- (2) *In the group dual case,  $G = \widehat{\Gamma}$  with  $\Gamma = \langle g_1, \dots, g_N \rangle$  being a discrete group, these are the duals of the various quotients  $\Gamma \rightarrow \Lambda$ .*

PROOF. Both these assertions are elementary, as follows:

(1) This follows indeed from the fact that a closed subgroup  $H \subset U_N^+$  is at the same time classical, and a group dual, precisely when it is classical and abelian.

(2) This follows from the general properties of the Pontrjagin duality, and more precisely from the fact that the subgroups  $\widehat{\Lambda} \subset \widehat{\Gamma}$  correspond to the quotients  $\Gamma \rightarrow \Lambda$ .  $\square$

At a more concrete level now, most of the tori that we met appear as diagonal tori. However, for certain quantum groups like the bistochastic ones, or the quantum permutation group ones, this torus collapses to  $\{1\}$ , and so it cannot be of use in the study of  $G$ . In order to deal with this issue, the idea, from [35], will be that of using:

PROPOSITION 11.10. *Given a closed subgroup  $G \subset U_N^+$  and a matrix  $Q \in U_N$ , we let  $T_Q \subset G$  be the diagonal torus of  $G$ , with fundamental representation spinned by  $Q$ :*

$$C(T_Q) = C(G) / \left\langle (QuQ^*)_{ij} = 0 \mid \forall i \neq j \right\rangle$$

*This torus is then a group dual,  $T_Q = \widehat{\Lambda}_Q$ , where  $\Lambda_Q = \langle g_1, \dots, g_N \rangle$  is the discrete group generated by the elements  $g_i = (QuQ^*)_{ii}$ , which are unitaries inside  $C(T_Q)$ .*

PROOF. This follows indeed from our results, because, as said in the statement,  $T_Q$  is by definition a diagonal torus. Equivalently, since  $v = QuQ^*$  is a unitary corepresentation, its diagonal entries  $g_i = v_{ii}$ , when regarded inside  $C(T_Q)$ , are unitaries, and satisfy:

$$\Delta(g_i) = g_i \otimes g_i$$

Thus  $C(T_Q)$  is a group algebra, and more specifically we have  $C(T_Q) = C^*(\Lambda_Q)$ , where  $\Lambda_Q = \langle g_1, \dots, g_N \rangle$  is the group in the statement, and this gives the result.  $\square$

Summarizing, associated to any closed subgroup  $G \subset U_N^+$  is a whole family of tori, indexed by the unitaries  $U \in U_N$ . As a first result regarding these tori, we have:

THEOREM 11.11. *Any torus  $T \subset G$  appears as follows, for a certain  $Q \in U_N$ :*

$$T \subset T_Q \subset G$$

*In other words, any torus appears inside a standard torus.*

PROOF. Given a torus  $T \subset G$ , we have an inclusion  $T \subset G \subset U_N^+$ . On the other hand, we know that each torus  $T = \widehat{\Lambda} \subset U_N^+$ , coming from a discrete group  $\Lambda = \langle g_1, \dots, g_N \rangle$ , has a fundamental corepresentation as follows, with  $Q \in U_N$ :

$$u = Q \text{diag}(g_1, \dots, g_N) Q^*$$

But this shows that we have  $T \subset T_Q$ , and this gives the result. □

Let us do now some computations. In the classical case, the result is as follows:

PROPOSITION 11.12. *For a closed subgroup  $G \subset U_N$  we have*

$$T_Q = G \cap (Q^* \mathbb{T}^N Q)$$

where  $\mathbb{T}^N \subset U_N$  is the group of diagonal unitary matrices.

PROOF. This is indeed clear at  $Q = 1$ , where  $\Gamma_1$  appears by definition as the dual of the compact abelian group  $G \cap \mathbb{T}^N$ . In general, this follows by conjugating by  $Q$ . □

In the group dual case now, we have the following result, from [41]:

PROPOSITION 11.13. *Given a discrete group  $\Gamma = \langle g_1, \dots, g_N \rangle$ , consider its dual compact quantum group  $G = \widehat{\Gamma}$ , diagonally embedded into  $U_N^+$ . We have then*

$$\Lambda_Q = \Gamma / \left\langle g_i = g_j \mid \exists k, Q_{ki} \neq 0, Q_{kj} \neq 0 \right\rangle$$

with the embedding  $T_Q \subset G = \widehat{\Gamma}$  coming from the quotient map  $\Gamma \rightarrow \Lambda_Q$ .

PROOF. Assume indeed that  $\Gamma = \langle g_1, \dots, g_N \rangle$  is a discrete group, with  $\widehat{\Gamma} \subset U_N^+$  coming via  $u = \text{diag}(g_1, \dots, g_N)$ . With  $v = QuQ^*$ , we have:

$$\begin{aligned} \sum_s \bar{Q}_{si} v_{sk} &= \sum_{st} \bar{Q}_{si} Q_{st} \bar{Q}_{kt} g_t \\ &= \sum_t \delta_{it} \bar{Q}_{kt} g_t \\ &= \bar{Q}_{ki} g_i \end{aligned}$$

Thus  $v_{ij} = 0$  for  $i \neq j$  gives  $\bar{Q}_{ki} v_{kk} = \bar{Q}_{ki} g_i$ , which is the same as saying that  $Q_{ki} \neq 0$  implies  $g_i = v_{kk}$ . But this latter equality reads:

$$g_i = \sum_j |Q_{kj}|^2 g_j$$

We conclude from this that  $Q_{ki} \neq 0, Q_{kj} \neq 0$  implies  $g_i = g_j$ , as desired. As for the converse, this is elementary to establish as well. □

In view of the above, we can expect the collection  $\{T_Q \mid Q \in U_N\}$  to encode various algebraic and analytic properties of  $G$ . We have the following result, from [35]:



**THEOREM 11.14.** *The following results hold, both for the compact Lie groups, and for the duals of the finitely generated discrete groups:*

- (1) *Generation: any closed quantum subgroup  $G \subset U_N^+$  has the generation property  $G = \langle T_Q | Q \in U_N \rangle$ . In other words,  $G$  is generated by its tori.*
- (2) *Characters: if  $G$  is connected, for any nonzero  $P \in C(G)_{\text{central}}$  there exists  $Q \in U_N$  such that  $P$  becomes nonzero, when mapped into  $C(T_Q)$ .*
- (3) *Amenability: a closed subgroup  $G \subset U_N^+$  is coamenable if and only if each of the tori  $T_Q$  is coamenable, in the usual discrete group sense.*
- (4) *Growth: assuming  $G \subset U_N^+$ , the discrete quantum group  $\widehat{G}$  has polynomial growth if and only if each the discrete groups  $\widehat{T_Q}$  has polynomial growth.*

**PROOF.** In the classical case, where  $G \subset U_N$ , the proof is elementary, based on standard facts from linear algebra, and goes as follows:

- (1) **Generation.** We use the following formula, established above:

$$T_Q = G \cap Q^* \mathbb{T}^N Q$$

Since any group element  $U \in G$  is diagonalizable,  $U = Q^* D Q$  with  $Q \in U_N, D \in \mathbb{T}^N$ , we have  $U \in T_Q$  for this value of  $Q \in U_N$ , and this gives the result.

- (2) **Characters.** We can take here  $Q \in U_N$  to be such that  $Q T Q^* \subset \mathbb{T}^N$ , where  $T \subset U_N$  is a maximal torus for  $G$ , and this gives the result.

- (3) **Amenability.** This conjecture holds trivially in the classical case,  $G \subset U_N$ , due to the fact that these latter quantum groups are all coamenable.

- (4) **Growth.** This is something nontrivial, well-known from the theory of compact Lie groups, and we refer here for instance to the literature.

Regarding now the group duals, here everything is trivial. Indeed, when the group duals are diagonally embedded we can take  $Q = 1$ , and when the group duals are embedded by using a spinning matrix  $Q \in U_N$ , we can use precisely this matrix  $Q$ . □

The various statements above are conjectured to hold for any compact quantum group. We refer to [35] and to subsequent papers for a number of verifications, notably covering many basic examples of easy quantum groups, as well as half-liberations.

Let us focus now on the generation property. We will need:

**PROPOSITION 11.15.** *Given a closed subgroup  $G \subset U_N^+$  and a matrix  $Q \in U_N$ , the corresponding standard torus and its Tannakian category are given by*

$$T_Q = G \cap \mathbb{T}_Q$$

$$C_{T_Q} = \langle C_G, C_{\mathbb{T}_Q} \rangle$$

where  $\mathbb{T}_Q \subset U_N^+$  is the dual of the free group  $F_N = \langle g_1, \dots, g_N \rangle$ , with the fundamental corepresentation of  $C(\mathbb{T}_Q)$  being the matrix  $u = Q \text{diag}(g_1, \dots, g_N) Q^*$ .

PROOF. The first assertion comes from the well-known fact that given two closed subgroups  $G, H \subset U_N^+$ , the corresponding quotient algebra  $C(U_N^+) \rightarrow C(G \cap H)$  appears by dividing by the kernels of both the quotient maps

$$C(U_N^+) \rightarrow C(G) \quad , \quad C(U_N^+) \rightarrow C(H)$$

Indeed, the construction of  $T_Q$  amounts precisely in performing this operation, with  $H = \mathbb{T}_Q$ , and so we obtain  $T_Q = G \cap \mathbb{T}_Q$ , as claimed. As for the Tannakian category formula, this follows from this, and from the general duality formula  $C_{G \cap H} = \langle C_G, C_H \rangle$ .  $\square$

We have the following Tannakian reformulation of the generation property:

**THEOREM 11.16.** *Given a closed subgroup  $G \subset U_N^+$ , the subgroup*

$$G' = \langle T_Q | Q \in U_N \rangle$$

*generated by its standard tori has the following Tannakian category:*

$$C_{G'} = \bigcap_{Q \in U_N} \langle C_G, C_{\mathbb{T}_Q} \rangle$$

*In particular we have  $G = G'$  when this intersection reduces to  $C_G$ .*

PROOF. Consider indeed the subgroup  $G' \subset G$  constructed in the statement. We have:

$$C_{G'} = \bigcap_{Q \in U_N} C_{T_Q}$$

Together with the formula in Proposition 11.15, this gives the result.  $\square$

The above result can be used for investigating the toral generation conjecture, but the combinatorics is quite difficult, and there are no results yet, along these lines. Let us further discuss now the toral generation property, with some modest results, regarding its behaviour with respect to product operations. We first have:

**PROPOSITION 11.17.** *Given two closed subgroups  $G, H \subset U_N^+$ , and  $Q \in U_N$ , we have:*

$$\langle T_Q(G), T_Q(H) \rangle \subset T_Q(\langle G, H \rangle)$$

*Also, the toral generation property is stable under the operation  $\langle, \rangle$ .*

PROOF. The first assertion can be proved either by using Theorem 11.16, or directly. For the direct proof, which is perhaps the simplest, we have:

$$T_Q(G) = G \cap \mathbb{T}_Q \subset \langle G, H \rangle \cap \mathbb{T}_Q = T_Q(\langle G, H \rangle)$$

We have as well the following computation:

$$T_Q(H) = H \cap \mathbb{T}_Q \subset \langle G, H \rangle \cap \mathbb{T}_Q = T_Q(\langle G, H \rangle)$$

Now since  $A, B \subset C$  implies  $\langle A, B \rangle \subset C$ , this gives the result. Regarding now the second assertion, we have the following computation:

$$\begin{aligned} \langle G, H \rangle &= \langle \langle T_Q(G)|Q \in U_N \rangle, \langle T_Q(H)|Q \in U_N \rangle \rangle \\ &= \langle T_Q(G), T_Q(H)|Q \in U_N \rangle \\ &= \langle \langle T_Q(G), T_Q(H) \rangle |Q \in U_N \rangle \\ &\subset \langle T_Q(\langle G, H \rangle)|Q \in U_N \rangle \end{aligned}$$

Thus the quantum group  $\langle G, H \rangle$  is generated by its tori, as claimed.  $\square$

We have as well the following result:

PROPOSITION 11.18. *We have the following formula, for any  $G, H$  and  $R, S$ :*

$$T_{R \otimes S}(G \times H) = T_R(G) \times T_S(H)$$

Also, the toral generation property is stable under usual products  $\times$ .

PROOF. The product formula is clear. Regarding now the second assertion, we have:

$$\begin{aligned} \langle T_Q(G \times H)|Q \in U_{MN} \rangle &\supset \langle T_{R \otimes S}(G \times H)|R \in U_M, S \in U_N \rangle \\ &= \langle T_R(G) \times T_S(H)|R \in U_M, S \in U_N \rangle \\ &= \langle T_R(G) \times \{1\}, \{1\} \times T_S(H)|R \in U_M, S \in U_N \rangle \\ &= \langle T_R(G)|R \in U_M \rangle \times \langle T_S(H)|S \in U_N \rangle \\ &= G \times H \end{aligned}$$

Thus the quantum group  $G \times H$  is generated by its tori, as claimed.  $\square$

### 11c. Fourier liberation

Let us go back now to the quantum permutation groups. In relation with the tori, let us start with the following basic fact, which generalizes the embedding  $\widehat{D}_\infty \subset S_4^+$  that we met in chapter 2 above, when proving that we have  $S_4^+ \neq S_4$ :

PROPOSITION 11.19. *Consider a discrete group generated by elements of finite order, written as a quotient group, as follows:*

$$\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k} \rightarrow \Gamma$$

We have then an embedding of quantum groups  $\widehat{\Gamma} \subset S_N^+$ , where  $N = N_1 + \dots + N_k$ .

PROOF. We have a sequence of embeddings and isomorphisms as follows:

$$\begin{aligned} \widehat{\Gamma} &\subset \widehat{\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k}} \\ &= \widehat{\mathbb{Z}_{N_1}} \hat{*} \dots \hat{*} \widehat{\mathbb{Z}_{N_k}} \\ &\simeq \mathbb{Z}_{N_1} \hat{*} \dots \hat{*} \mathbb{Z}_{N_k} \\ &\subset S_{N_1} \hat{*} \dots \hat{*} S_{N_k} \\ &\subset S_{N_1}^+ \hat{*} \dots \hat{*} S_{N_k}^+ \\ &\subset S_N^+ \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The above result is quite abstract, and it is worth working out the details, with an explicit formula for the associated magic matrix. Let us start with a study of the simplest situation, where  $k = 1$ , and where  $\Gamma = \mathbb{Z}_{N_1}$ . The result here is as follows:

PROPOSITION 11.20. *The magic matrix for the quantum permutation group*

$$\widehat{\mathbb{Z}_N} \simeq \mathbb{Z}_N \subset S_N \subset S_N^+$$

*with standard Fourier isomorphism on the left, is given by the formula*

$$u = FIF^*$$

where  $F = \frac{1}{\sqrt{N}}(w^{ij})$  with  $w = e^{2\pi i/N}$  is the Fourier matrix, and where

$$I = \begin{pmatrix} 1 & & & \\ & g & & \\ & & \ddots & \\ & & & g^{N-1} \end{pmatrix}$$

is the diagonal matrix formed by the elements of  $\mathbb{Z}_N$ , regarded as elements of  $C^*(\mathbb{Z}_N)$ .

PROOF. The magic matrix for the quantum group  $\mathbb{Z}_N \subset S_N \subset S_N^+$  is given by:

$$\begin{aligned} v_{ij} &= \chi\left(\sigma \in \mathbb{Z}_N \mid \sigma(j) = i\right) \\ &= \delta_{i-j} \end{aligned}$$

Let us apply now the Fourier transform. According to our Pontrjagin duality conventions from chapter 1 above, we have a pair of inverse isomorphisms, as follows:

$$\begin{aligned} \Phi : C(\mathbb{Z}_N) &\rightarrow C^*(\mathbb{Z}_N) \quad , \quad \delta_i \rightarrow \frac{1}{N} \sum_k w^{ik} g^k \\ \Psi : C^*(\mathbb{Z}_N) &\rightarrow C(\mathbb{Z}_N) \quad , \quad g^i \rightarrow \sum_k w^{-ik} \delta_k \end{aligned}$$

Here  $w = e^{2\pi i/N}$ , and we use the standard Fourier analysis convention that the indices are  $0, 1, \dots, N-1$ . With  $F = \frac{1}{\sqrt{N}}(w^{ij})$  and  $I = \text{diag}(g^i)$  as above, we have:

$$\begin{aligned} u_{ij} &= \Phi(v_{ij}) \\ &= \frac{1}{N} \sum_k w^{(i-j)k} g^k \\ &= \frac{1}{N} \sum_k w^{ik} g^k w^{-jk} \\ &= \sum_k F_{ik} I_{kk} (F^*)_{kj} \\ &= (FIF^*)_{ij} \end{aligned}$$

Thus, the magic matrix that we are looking for is  $u = FIF^*$ , as claimed.  $\square$

With the above result in hand, we can refine Proposition 11.19, as follows:

**THEOREM 11.21.** *Given a quotient group  $\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k} \rightarrow \Gamma$ , we have an embedding  $\widehat{\Gamma} \subset S_N^+$ , with  $N = N_1 + \dots + N_k$ , with magic matrix given by the formula*

$$u = \begin{pmatrix} F_{N_1} I_1 F_{N_1}^* & & \\ & \ddots & \\ & & F_{N_k} I_k F_{N_k}^* \end{pmatrix}$$

where  $F_N = \frac{1}{\sqrt{N}}(w_N^{ij})$  with  $w_N = e^{2\pi i/N}$  are Fourier matrices, and where

$$I_r = \begin{pmatrix} 1 & & & \\ & g_r & & \\ & & \ddots & \\ & & & g_r^{N_r-1} \end{pmatrix}$$

with  $g_1, \dots, g_k$  being the standard generators of  $\Gamma$ .

**PROOF.** This follows indeed from Proposition 11.19 and Proposition 11.20.  $\square$

As explained in [44], as a consequence of the orbit theory developed there, any group dual subgroup  $\widehat{\Gamma} \subset S_N^+$  appears in the above way, from a partition  $N = N_1 + \dots + N_k$ , and then a quotient group  $\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k} \rightarrow \Gamma$ .

We will be back to this in chapter 13 below, when discussing the orbit theory developed in [44], and its applications.

In the meantime, we can recover this result, that we will need in what follows, by using our maximal torus method.

Following [10], we have indeed the following result:

THEOREM 11.22. *For the quantum permutation group  $S_N^+$ , we have:*

(1) *Given  $Q \in U_N$ , the quotient  $F_N \rightarrow \Lambda_Q$  comes from the following relations:*

$$\begin{cases} g_i = 1 & \text{if } \sum_l Q_{il} \neq 0 \\ g_i g_j = 1 & \text{if } \sum_l Q_{il} Q_{jl} \neq 0 \\ g_i g_j g_k = 1 & \text{if } \sum_l Q_{il} Q_{jl} Q_{kl} \neq 0 \end{cases}$$

(2) *Given a decomposition  $N = N_1 + \dots + N_k$ , for the matrix  $Q = \text{diag}(F_{N_1}, \dots, F_{N_k})$ , where  $F_N = \frac{1}{\sqrt{N}}(\xi^{ij})_{ij}$  with  $\xi = e^{2\pi i/N}$  is the Fourier matrix, we obtain:*

$$\Lambda_Q = \mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k}$$

(3) *Given an arbitrary matrix  $Q \in U_N$ , there exists a decomposition  $N = N_1 + \dots + N_k$ , such that  $\Lambda_Q$  appears as quotient of  $\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k}$ .*

PROOF. This is more or less equivalent to the classification of the group dual subgroups  $\widehat{\Gamma} \subset S_N^+$  from [44], and can be proved by a direct computation, as follows:

(1) Fix a unitary matrix  $Q \in U_N$ , and consider the following quantities:

$$\begin{cases} c_i = \sum_l Q_{il} \\ c_{ij} = \sum_l Q_{il} Q_{jl} \\ d_{ijk} = \sum_l \bar{Q}_{il} \bar{Q}_{jl} Q_{kl} \end{cases}$$

We write  $w = QvQ^*$ , where  $v$  is the fundamental corepresentation of  $C(S_N^+)$ . Assume  $X \simeq \{1, \dots, N\}$ , and let  $\alpha$  be the coaction of  $C(S_N^+)$  on  $C(X)$ . Let us set:

$$\varphi_i = \sum_l \bar{Q}_{il} \delta_l \in C(X)$$

Also, let  $g_i = (QvQ^*)_{ii} \in C^*(\Lambda_Q)$ . If  $\beta$  is the restriction of  $\alpha$  to  $C^*(\Lambda_Q)$ , then:

$$\beta(\varphi_i) = \varphi_i \otimes g_i$$

Now recall that  $C(X)$  is the universal  $C^*$ -algebra generated by elements  $\delta_1, \dots, \delta_N$  which are pairwise orthogonal projections. Writing these conditions in terms of the linearly independent elements  $\varphi_i$  by means of the formulae  $\delta_i = \sum_l Q_{il} \varphi_l$ , we find that the universal relations for  $C(X)$  in terms of the elements  $\varphi_i$  are as follows:

$$\begin{cases} \sum_i c_i \varphi_i = 1 \\ \varphi_i^* = \sum_j c_{ij} \varphi_j \\ \varphi_i \varphi_j = \sum_k d_{ijk} \varphi_k \end{cases}$$

Let  $\tilde{\Lambda}_Q$  be the group in the statement. Since  $\beta$  preserves these relations, we get:

$$\begin{cases} c_i(g_i - 1) = 0 \\ c_{ij}(g_i g_j - 1) = 0 \\ d_{ijk}(g_i g_j - g_k) = 0 \end{cases}$$

We conclude from this that  $\Lambda_Q$  is a quotient of  $\tilde{\Lambda}_Q$ . On the other hand, it is immediate that we have a coaction map as follows:

$$C(X) \rightarrow C(X) \otimes C^*(\tilde{\Lambda}_Q)$$

Thus  $C(\tilde{\Lambda}_Q)$  is a quotient of  $C(S_N^+)$ . Since  $w$  is the fundamental corepresentation of  $S_N^+$  with respect to the basis  $\{\varphi_i\}$ , it follows that the generator  $w_{ii}$  is sent to  $\tilde{g}_i \in \tilde{\Lambda}_Q$ , while  $w_{ij}$  is sent to zero. We conclude that  $\tilde{\Lambda}_Q$  is a quotient of  $\Lambda_Q$ . Since the above quotient maps send generators on generators, we conclude that  $\Lambda_Q = \tilde{\Lambda}_Q$ , as desired.

(2) We apply the result found in (1), with the  $N$ -element set  $X$  used in the proof there chosen to be the following set:

$$X = \mathbb{Z}_{N_1} \sqcup \dots \sqcup \mathbb{Z}_{N_k}$$

With this choice, we have  $c_i = \delta_{i0}$  for any  $i$ . Also, we have  $c_{ij} = 0$ , unless  $i, j, k$  belong to the same block to  $Q$ , in which case  $c_{ij} = \delta_{i+j,0}$ , and also  $d_{ijk} = 0$ , unless  $i, j, k$  belong to the same block of  $Q$ , in which case  $d_{ijk} = \delta_{i+j,k}$ . We conclude from this that  $\Lambda_Q$  is the free product of  $k$  groups which have generating relations as follows:

$$g_i g_j = g_{i+j} \quad , \quad g_i^{-1} = g_{-i}$$

But this shows that our group is  $\Lambda_Q = \mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k}$ , as stated.

(3) This follows indeed from (2). See [41]. □

As already mentioned, there are many other possible ways of recovering the above results, the standard way being via orbit theory. We will be back to this in chapter 13 below, when discussing the orbit theory developed in [44], and its applications.

In connection with our liberation questions now, in the quantum permutation group case, the standard tori parametrized by Fourier matrices play a special role.

This suggests the following definition:

**DEFINITION 11.23.** *Consider a closed subgroup  $G \subset U_N^+$ .*

- (1) *Its standard tori  $T_F$ , with  $F = F_{N_1} \otimes \dots \otimes F_{N_k}$ , and  $N = N_1 + \dots + N_k$  being regarded as a partition, are called Fourier tori.*
- (2) *In the case where we have  $G_N = \langle G_N^c, (T_F)_F \rangle$ , we say that  $G_N$  appears as a Fourier liberation of its classical version  $G_N^c$ .*

The conjecture is that the easy quantum groups should appear as Fourier liberations. With respect to the basic examples, the situation in the free case is as follows:

- (1)  $O_N^+, U_N^+$  are diagonal liberations, so they are Fourier liberations as well.
- (2)  $B_N^+, C_N^+$  are Fourier liberations too, with this being standard.
- (3)  $S_N^+$  is a Fourier liberation too, being generated by its tori [50], [51].
- (4)  $H_N^+, K_N^+$  remain to be investigated, by using the general theory in [83].

As a word of warning here, observe that an arbitrary classical group  $G_N \subset U_N$  is not necessarily generated by its Fourier tori, and nor is an arbitrary discrete group dual, with spinned embedding. Thus, the Fourier tori, and the related notion of Fourier liberation, remain something quite technical, in connection with the easy case.

As an application of all this, let us go back to quantum permutation groups, and more specifically to the quantum symmetry groups of finite graphs, from chapter 9 above. One interesting question is whether  $G^+(X)$  appears as a Fourier liberation of  $G(X)$ . Generally speaking, this is something quite difficult, because for the empty graph itself we are in need of the above-mentioned technical results from [50], [51].

In order to discuss this, let us begin with the following elementary statement:

**THEOREM 11.24.** *In order for a closed subgroup  $G \subset U_K^+$  to appear as  $G = G^+(X)$ , for a certain graph  $X$  having  $N$  vertices, the following must happen:*

- (1) *We must have a representation  $G \subset U_N^+$ .*
- (2) *This representation must be magic,  $G \subset S_N^+$ .*
- (3) *We must have a graph  $X$  having  $N$  vertices, such that  $d \in \text{End}(u)$ .*
- (4)  *$X$  must be in fact such that the Tannakian category of  $G$  is precisely  $\langle d \rangle$ .*

**PROOF.** This is more of an empty statement, coming from the definition of the quantum automorphism group  $G^+(X)$ , as formulated in chapter 9 above. □

In the group dual case, forgetting about Fourier transforms, and imagining that we are at step (1) in the general strategy outlined in Theorem 11.24, we must compute the Tannakian category of  $\widehat{\Gamma} \subset U_N^+$ , diagonally embedded, for the needs of (3,4). We have:

**PROPOSITION 11.25.** *Given a discrete group  $\Gamma = \langle g_1, \dots, g_N \rangle$ , embed diagonally  $\widehat{\Gamma} \subset U_N^+$ , via the unitary matrix  $u = \text{diag}(g_1, \dots, g_N)$ . We have then the formula*

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \left\{ T = (T_{j_1 \dots j_l, i_1 \dots i_k}) \mid g_{i_1} \dots g_{i_k} \neq g_{j_1} \dots g_{j_l} \implies T_{j_1 \dots j_l, i_1 \dots i_k} = 0 \right\}$$

and in particular, with  $k = l = 1$ , we have the formula

$$\text{End}(u) = \left\{ T = (T_{ji}) \mid g_i \neq g_j \implies T_{ji} = 0 \right\}$$

with the linear maps being identified with the corresponding scalar matrices.



PROOF. This is well-known, and elementary, with the first assertion coming from:

$$\begin{aligned}
T \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) &\iff Tu^{\otimes k} = u^{\otimes l}T \\
&\iff (Tu^{\otimes k})_{j_1 \dots j_l, i_1 \dots i_k} = (u^{\otimes l}T)_{j_1 \dots j_l, i_1 \dots i_k} \\
&\iff T_{j_1 \dots j_l, i_1 \dots i_k} g_{i_1} \dots g_{i_k} = g_{j_1} \dots g_{j_l} T_{j_1 \dots j_l, i_1 \dots i_k} \\
&\iff T_{j_1 \dots j_l, i_1 \dots i_k} (g_{i_1} \dots g_{i_k} - g_{j_1} \dots g_{j_l}) = 0
\end{aligned}$$

As for the second assertion, this follows from the first one.  $\square$

Let us go ahead now, with respect to the general strategy outlined in Theorem 11.24, and apply [44] in order to solve (2), and then reformulate (3,4), by using Proposition 11.25, and by choosing to put the multi-Fourier transform on the graph part. We are led in this way into the following refinement of Theorem 11.24, in the group dual setting:

**THEOREM 11.26.** *In order for a group dual  $\widehat{\Gamma}$  to appear as  $G = G^+(X)$ , for a certain graph  $X$  having  $N$  vertices, the following must happen:*

- (1) *First, we need a quotient map  $\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k} \rightarrow \Gamma$ .*
- (2) *Let  $u = \text{diag}(I_1, \dots, I_k)$ , with  $I_r = \text{diag}(\mathbb{Z}_{N_r})$ , for any  $r$ .*
- (3) *Consider also the matrix  $F = \text{diag}(F_{N_1}, \dots, F_{N_k})$ .*
- (4) *We must then have a graph  $X$  having  $N$  vertices.*
- (5) *This graph must be such that  $F^*dF \neq 0 \implies I_i = I_j$ .*
- (6) *In fact,  $\langle F^*dF \rangle$  must be the category in Proposition 11.25.*

PROOF. This is something rather informal, the idea being as follows:

(1) This is what comes out from the classification result in [44], explained above, modulo a unitary base change, as explained before.

(2) This is just a notation, with  $I_r = \text{diag}(\mathbb{Z}_{N_r})$  meaning that  $I_r$  is the diagonal matrix formed by  $1, g, g^2, \dots, g^{N_r-1}$ , with  $g \in \mathbb{Z}_{N_r}$  being the standard generator.

(3) This is another notation, with each Fourier matrix  $F_{N_r}$  being the standard one, namely  $F_{N_r} = \frac{1}{\sqrt{N_r}}(w^{ij})$ , with  $w = e^{2\pi i/N_r}$ , and with indices  $0, 1, \dots, N_r - 1$ .

(4) This is a just a statement, with the precise graph formalism to be clarified later on, in view of the fact that  $X$  will get Fourier-transformed anyway.

(5) This is an actual result, our claim being that the condition  $d \in \text{End}(u)$  from Theorem 11.24 (3) is equivalent to the condition  $F^*dF \neq 0 \implies I_i = I_j$  in the statement. Indeed, we know that with  $F, I$  being as in the statement, we have  $u = FIF^*$ . Now with this formula in hand, we have the following equivalences:

$$\begin{aligned}
\widehat{\Gamma} \curvearrowright X &\iff du = ud \\
&\iff dFIF^* = FIF^*d \\
&\iff [F^*dF, I] = 0
\end{aligned}$$

Also, since the matrix  $I$  is diagonal, with  $M = F^*dF$  have:

$$\begin{aligned} MI = IM &\iff (MI)_{ij} = (IM)_{ij} \\ &\iff M_{ij}I_j = I_iM_{ij} \\ &\iff [M_{ij} \neq 0 \implies I_i = I_j] \end{aligned}$$

We therefore conclude that we have, as desired:

$$\widehat{\Gamma} \curvearrowright X \iff [F^*dF \neq 0 \implies I_i = I_j]$$

(6) This is the Tannakian condition in Theorem 11.24 (4), with reference to the explicit formula for the Tannakian category of  $G = \widehat{\Gamma}$  given in Proposition 11.25.  $\square$

Going ahead now, in connection with the Fourier tori, we have:

PROPOSITION 11.27. *The Fourier tori of  $G^+(X)$  are the biggest quotients*

$$\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k} \rightarrow \Gamma$$

whose duals act on the graph,  $\widehat{\Gamma} \curvearrowright X$ .

PROOF. We have indeed the following computation, at  $F = 1$ :

$$\begin{aligned} C(T_1(G^+(X))) &= C(G^+(X)) / \langle u_{ij} = 0, \forall i \neq j \rangle \\ &= [C(S_N^+) / \langle [d, u] = 0 \rangle] / \langle u_{ij} = 0, \forall i \neq j \rangle \\ &= [C(S_N^+) / \langle u_{ij} = 0, \forall i \neq j \rangle] / \langle [d, u] = 0 \rangle \\ &= C(T_1(S_N^+)) / \langle [d, u] = 0 \rangle \end{aligned}$$

Thus, we obtain the result, with the remark that the quotient that we are interested in appears via relations of type  $d_{ij} = 1 \implies g_i = g_j$ . The proof in general is similar.  $\square$

In connection now with the above-mentioned questions, we have:

THEOREM 11.28. *Consider the following conditions:*

- (1) *We have  $G(X) = G^+(X)$ .*
- (2)  *$G(X) \subset G^+(X)$  is a Fourier liberation.*
- (3)  *$\widehat{\Gamma} \curvearrowright X$  implies that  $\Gamma$  is abelian.*

We have then (1)  $\iff$  (2) + (3).

PROOF. This is something elementary, the proof being as follows:

(1)  $\implies$  (2, 3) Here both the implications are trivial.

(2, 3)  $\implies$  (1) Assuming  $G(X) \neq G^+(X)$ , from (2) we know that  $G^+(X)$  has at least one non-classical Fourier torus, and this contradicts (3).  $\square$

With this in hand, the question is whether (3)  $\implies$  (1) holds. This is a good question, which in practice would make connections between the various conjectures that can be made about a given graph  $X$ , and its quantum symmetry group  $G^+(X)$ .

As an illustration for the potential interest of such considerations, it is known from [75] that the random graphs have no quantum symmetries, with this being something highly non-trivial. Our point now is that, assuming that one day the general compact quantum Lie group theory will solve its Weyl-type questions in relation with the tori, and in particular know, as a theorem, that any  $G^+(X)$  appears as a Fourier liberation of  $G(X)$ , this deep graph result from [75] would become accessible as well via its particular case for the group dual subgroups, which is something elementary, as follows:

**PROPOSITION 11.29.** *For a graph  $X$  having  $N$  vertices, the probability for having an action*

$$\widehat{\Gamma} \curvearrowright X$$

*with  $\Gamma$  being a non-abelian group goes to 0 with  $N \rightarrow \infty$ .*

**PROOF.** This is something quite elementary, the idea being as follows:

(1) First of all, the graphs  $X$  having a fixed number  $N \in \mathbb{N}$  of vertices correspond to the matrices  $d \in M_N(0, 1)$  which are symmetric, and have 0 on the diagonal. The probability mentioned in the statement is the uniform one on such 0-1 matrices.

(2) Regarding now the proof, our claim is that this should come in a quite elementary way, from the  $du = ud$  condition, as reformulated before. Indeed, observe first that in the cyclic case, where  $F = F_N$  is a usual Fourier matrix, associated to a cyclic group  $\mathbb{Z}_N$ , we have the following formula, with  $w = e^{2\pi i/N}$ :

$$\begin{aligned} (F^* dF)_{ij} &= \sum_{kl} (F^*)_{ik} d_{kl} F_{lj} \\ &= \sum_{kl} w^{lj-ik} d_{kl} \\ &= \sum_{k \sim l} w^{lj-ik} \end{aligned}$$

(3) In the general setting now, where we have a quotient map  $\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k} \rightarrow \Gamma$ , with  $N_1 + \dots + N_k = N$ , the computation is similar, as follows, with  $w_i = e^{2\pi i/N_i}$ :

$$\begin{aligned} (F^* dF)_{ij} &= \sum_{kl} (F^*)_{ik} d_{kl} F_{lj} \\ &= \sum_{k \sim l} (F^*)_{ik} F_{lj} \\ &= \sum_{k:i,l:j,k \sim l} (w_{N_i})^{-ik} (w_{N_j})^{lj} \end{aligned}$$

Here the conditions  $k : i$  and  $l : j$  refer to the fact that  $k, l$  must belong respectively to the same matrix blocks as  $i, j$ , with respect to the partition  $N_1 + \dots + N_k = N$ , and  $k \sim l$  means as usual that there is an edge between  $k, l$ , in the graph  $X$ .

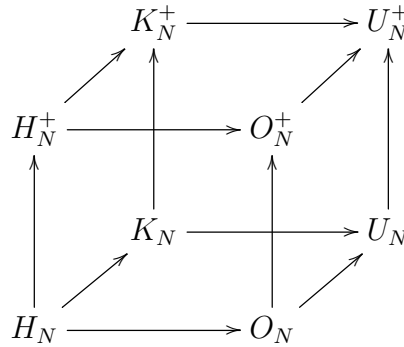
(4) The point now is that with the partition  $N_1 + \dots + N_k = N$  fixed, and with  $d \in M_N(0, 1)$  being random, we have  $(F^*dF)_{ij} \neq 0$  almost everywhere in the  $N \rightarrow \infty$  limit, and so we obtain  $I_i = I_j$  almost everywhere, and so abelianity of  $\Gamma$ , with  $N \rightarrow \infty$ .  $\square$

### 11d. Easy reflections

We discuss here a number of more specialized classification results, for the twistable easy quantum groups, and for more general intermediate quantum groups as follows:

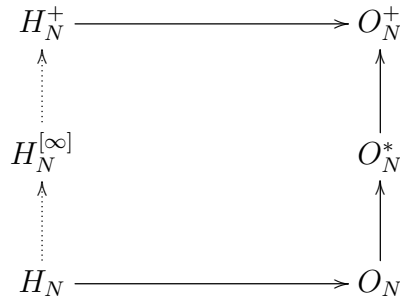
$$H_N \subset G \subset U_N^+$$

The idea will be that of viewing our quantum group as sitting inside the standard cube:



Let us first discuss the classification in the easy case, for the front face of the standard cube, the orthogonal one. We first have the following result:

PROPOSITION 11.30. *The easy quantum groups  $H_N \subset G \subset O_N^+$  are as follows,*



with the dotted arrows indicating that we have intermediate quantum groups there.

PROOF. This is a key result in the classification of easy quantum groups, whose proof is quite technical, the idea being as follows:

(1) We have a first dichotomy concerning the quantum groups in the statement, namely  $H_N \subset G \subset O_N^+$ , which must fall into one of the following two classes:

$$O_N \subset G \subset O_N^+$$

$$H_N \subset G \subset H_N^+$$

This dichotomy comes indeed from the early classification results for the easy quantum groups, from [23], [37], [38], whose proofs are quite elementary.

(2) In addition to this, these early classification results solve as well the first problem, namely  $O_N \subset G \subset O_N^+$ , with  $G = O_N^*$  being the unique non-trivial solution.

(3) We have then a second dichotomy, concerning the quantum groups which are left, namely  $H_N \subset G \subset H_N^+$ , which must fall into one of the following two classes:

$$H_N \subset G \subset H_N^{[\infty]}$$

$$H_N^{[\infty]} \subset G \subset H_N^+$$

This comes indeed from various papers, and more specifically from the final classification paper of Raum and Weber [83], where the quantum groups  $S_N \subset G \subset H_N^+$  with  $G \not\subset H_N^{[\infty]}$  were classified, and shown to contain  $H_N^{[\infty]}$ . For full details, we refer to [83].  $\square$

Summarizing, in order to deal with the front face of the main cube, we are left with classifying the following intermediate easy quantum groups:

$$H_N \subset G \subset H_N^{[\infty]}$$

$$H_N^{[\infty]} \subset G \subset H_N^+$$

Regarding the second case, namely  $H_N^{[\infty]} \subset G \subset H_N^+$ , the result here, from [83], which is quite technical, but has a simple formulation, is as follows:

PROPOSITION 11.31. *Let  $H_N^{[r]} \subset H_N^+$  be the easy quantum group coming from:*

$$\pi_r = \ker \begin{pmatrix} 1 & \dots & r & r & \dots & 1 \\ 1 & \dots & r & r & \dots & 1 \end{pmatrix}$$

*We have then inclusions of quantum groups as follows,*

$$H_N^+ = H_N^{[1]} \supset H_N^{[2]} \supset H_N^{[3]} \supset \dots \supset H_N^{[\infty]}$$

*and we obtain in this way all the intermediate easy quantum groups*

$$H_N^{[\infty]} \subset G \subset H_N^+$$

*satisfying the assumption  $G \neq H_N^{[\infty]}$ .*

PROOF. Once again, this is something technical, and we refer here to [83].  $\square$

It remains to discuss the easy quantum groups  $H_N \subset G \subset H_N^{[\infty]}$ , with the endpoints  $G = H_N, H_N^{[\infty]}$  included. Once again, we follow here [83]. First, we have:

DEFINITION 11.32. *A discrete group generated by real reflections,  $g_i^2 = 1$ ,*

$$\Gamma = \langle g_1, \dots, g_N \rangle$$

*is called uniform if each  $\sigma \in S_N$  produces a group automorphism,  $g_i \rightarrow g_{\sigma(i)}$ .*

Consider now a uniform reflection group, as follows:

$$\mathbb{Z}_2^{*N} \rightarrow \Gamma \rightarrow \mathbb{Z}_2^N$$

We can associate to it a family of subsets  $D(k, l) \subset P(k, l)$ , which form a category of partitions, as follows:

$$D(k, l) = \left\{ \pi \in P(k, l) \mid \ker \binom{i}{j} \leq \pi \implies g_{i_1} \dots g_{i_k} = g_{j_1} \dots g_{j_l} \right\}$$

Observe that we have inclusions of categories as follows, coming respectively from  $\eta \in D$ , and from the quotient map  $\Gamma \rightarrow \mathbb{Z}_2^N$ :

$$P_{even}^{[\infty]} \subset D \subset P_{even}$$

Conversely, consider a category of partitions as follows:

$$P_{even}^{[\infty]} \subset D \subset P_{even}$$

We can associate to it a uniform reflection group  $\mathbb{Z}_2^{*N} \rightarrow \Gamma \rightarrow \mathbb{Z}_2^N$ , as follows:

$$\Gamma = \left\langle g_1, \dots, g_N \mid g_{i_1} \dots g_{i_k} = g_{j_1} \dots g_{j_l}, \forall i, j, k, l, \ker \binom{i}{j} \in D(k, l) \right\rangle$$

As explained in [83], the correspondences  $\Gamma \rightarrow D$  and  $D \rightarrow \Gamma$  constructed above are bijective, and inverse to each other, at  $N = \infty$ .

We have in fact the following result, from [83]:

PROPOSITION 11.33. *We have correspondences between:*

- (1) *Uniform reflection groups  $\mathbb{Z}_2^{* \infty} \rightarrow \Gamma \rightarrow \mathbb{Z}_2^\infty$ .*
- (2) *Categories of partitions  $P_{even}^{[\infty]} \subset D \subset P_{even}$ .*
- (3) *Easy quantum groups  $G = (G_N)$ , with  $H_N^{[\infty]} \supset G_N \supset H_N$ .*

PROOF. This is something quite technical, which follows along the lines of the above discussion. As an illustration, if we denote by  $\mathbb{Z}_2^{\circ N}$  the quotient of  $\mathbb{Z}_2^{*N}$  by the relations

of type  $abc = cba$  between the generators, we have the following correspondences:

$$\begin{array}{ccccc}
 \mathbb{Z}_2^N & \longleftarrow & \mathbb{Z}_2^{\circ N} & \longleftarrow & \mathbb{Z}_2^{*N} \\
 \downarrow & & \downarrow & & \downarrow \\
 H_N & \longrightarrow & H_N^* & \longrightarrow & H_N^{[\infty]}
 \end{array}$$

More generally, for any  $s \in \{2, 4, \dots, \infty\}$ , the quantum groups  $H_N^{(s)} \subset H_N^{[s]}$  constructed in [23] come from the quotients of  $\mathbb{Z}_2^{\circ N} \leftarrow \mathbb{Z}_2^{*N}$  by the relations  $(ab)^s = 1$ . See [83].  $\square$

We can now formulate a final classification result, as follows:

**THEOREM 11.34.** *The easy quantum groups  $H_N \subset G \subset O_N^+$  are as follows,*

$$\begin{array}{ccc}
 H_N^+ & \longrightarrow & O_N^+ \\
 \uparrow & & \uparrow \\
 H_N^{[r]} & & O_N^* \\
 \uparrow & & \uparrow \\
 H_N^{[\infty]} & & O_N \\
 \uparrow & & \uparrow \\
 H_N^\Gamma & & O_N \\
 \uparrow & \longrightarrow & \uparrow \\
 H_N & \longrightarrow & O_N
 \end{array}$$

with the family  $H_N^\Gamma$  covering  $H_N, H_N^{[\infty]}$ , and with the series  $H_N^{[r]}$  covering  $H_N^+$ .

**PROOF.** This follows indeed from the above results. See [83].  $\square$

Beyond easiness, let us start with:

**DEFINITION 11.35.** *Associated to any closed subgroup  $G_N \subset U_N^+$  are its classical, discrete and real versions, given by*

$$\begin{aligned}
 G_N^c &= G_N \cap U_N \\
 G_N^d &= G_N \cap K_N^+ \\
 G_N^r &= G_N \cap O_N^+
 \end{aligned}$$

as well as its free, smooth and unitary versions, given by

$$\begin{aligned}
 G_N^f &= \langle G_N, H_N^+ \rangle \\
 G_N^s &= \langle G_N, O_N \rangle \\
 G_N^u &= \langle G_N, K_N \rangle
 \end{aligned}$$

where  $\langle, \rangle$  is the topological generation operation.

We will need as well a second definition, as follows:

DEFINITION 11.36. *Associated to any closed subgroup  $G_N \subset U_N^+$  are the mixes of its classical, discrete and real versions, given by*

$$G_N^{cd} = G_N \cap K_N$$

$$G_N^{cr} = G_N \cap O_N^+$$

$$G_N^{dr} = G_N \cap H_N^+$$

*as well as the mixes of its free, smooth and unitary versions, given by*

$$G_N^{fs} = \langle G_N, O_N^+ \rangle$$

$$G_N^{fu} = \langle G_N, K_N^+ \rangle$$

$$G_N^{us} = \langle G_N, U_N \rangle$$

*where  $\langle , \rangle$  is the topological generation operation.*

With the above notions in hand, we can formulate:

DEFINITION 11.37. *A closed subgroup  $G_N \subset U_N^+$  is called “oriented” if*

$$G_N = \langle G_N^{cd}, G_N^{cr}, G_N^{dr} \rangle$$

$$G_N = G_N^{fs} \cap G_N^{fu} \cap G_N^{su}$$

*and “weakly oriented” if the following conditions hold,*

$$G_N = \langle G_N^c, G_N^d, G_N^r \rangle$$

$$G_N = G_N^f \cap G_N^s \cap G_N^u$$

*where the various versions are those in Definition 11.35 and Definition 11.36.*

With these notions, our claim is that some classification results are possible:

(1) In the classical case, we believe that the uniform, half-homogeneous, oriented groups are the obvious examples of such groups.

This is of course something quite heavy, well beyond easiness, with the potential tools available for proving such things coming from advanced finite group theory and Lie algebra theory. Our uniformity axiom could play a key role here, when combined with [85], in order to exclude all the exceptional objects which might appear on the way.

(2) In the free case, under similar assumptions, we believe that the solutions should be the obvious examples of such quantum groups.

This is something heavy, too, related to a well-known freeness conjecture, namely  $\langle G_N, S_N^+ \rangle = \{G'_N, S_N^+\}$ . Indeed, assuming that we would have such a formula, and perhaps some more formulae of the same type as well, we can in principle work out our



way inside the cube, from the edge and face projections to  $G_N$  itself, and in this process  $G_N$  would become easy. This would be the straightforward strategy here.

(3) In the group dual case, the orientability axiom simplifies, because the group duals are discrete in our sense. We believe that the uniform, twistable, oriented group duals should appear as combinations of certain abelian groups, which appear in the classical case, with duals of varieties of real reflection groups, which appear in the real case.

This is probably the easiest question in the present series, and the most reasonable one, to start with. However, there are no concrete results so far, in this direction.

### 11e. Exercises

Things have been quite technical in this chapter, and as a unique exercise, summarizing the problems that we have been talking about here, we have:

EXERCISE 11.38. *Find an abstract framework for the “quantum reflection groups”, as intermediate quantum groups of type*

$$A_N \subset G \subset K_N^+$$

*covering at the same time all the intermediate easy quantum groups of type*

$$S_N \subset G \subset K_N^+$$

*and all the classical, non-exceptional complex reflection groups, namely*

$$H_N^{sd} = \left\{ U \in H_N^s \mid (\det U)^d = 1 \right\}$$

*and then start classifying such beasts.*

The point here is that in the classical case, there is a definition and then classification result for the complex reflection groups, with the classification stating that we have as examples the above groups  $H_N^{sd}$ , plus a number of exceptional examples, which can be classified as well. The problem is that of finding the correct quantum extension of this.

## CHAPTER 12

### Twisted reflections

#### 12a. Quantum graphs

We have seen in the previous 3 chapters that some general theory can be developed for the quantum subgroups  $G \subset S_N^+$ , in particular with a notion of “quantum reflection group”. We discuss here the twisted extension of some of these results, and in particular the twisted analogues of the quantum reflection groups, obtained by using generalized quantum permutation groups,  $S_F^+$  with  $F$  being an arbitrary finite quantum space.

Let us first recall from chapter 4 above that we have:

**DEFINITION 12.1.** *A finite quantum space  $F$  is the abstract dual of a finite dimensional  $C^*$ -algebra  $B$ , according to the following formula:*

$$C(F) = B$$

*The number of elements of such a space is by definition the number  $|F| = \dim B$ . By decomposing the algebra  $B$ , we have a formula of the following type:*

$$C(F) = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C})$$

*With  $n_1 = \dots = n_k = 1$  we obtain in this way the space  $F = \{1, \dots, k\}$ . Also, when  $k = 1$  the equation is  $C(F) = M_n(\mathbb{C})$ , and the solution will be denoted  $F = M_n$ .*

As explained in chapter 4, each such finite quantum space  $F$  has a counting measure, corresponding as the algebraic level to the following integration functional, obtained by applying the regular representation, and then the normalized matrix trace:

$$tr : C(F) \rightarrow B(l^2(F)) \rightarrow \mathbb{C}$$

As basic examples, for both  $F = \{1, \dots, N\}$  and  $F = M_N$  we obtain the usual trace. In general, with  $C(F) = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C})$ , the weights of  $tr$  are:

$$c_i = \frac{n_i^2}{\sum_i n_i^2}$$

Let us also mention that the canonical trace is precisely the one making  $\mathbb{C} \subset B$  a Markov inclusion. Equivalently, the counting measure is the one making  $F \rightarrow \{.\}$  a Markov fibration. For a discussion of these facts, see [2], and also [5], [19].

We will also need the definition and main properties of the quantum symmetry groups  $S_F^+$  of such spaces  $F$ . The result here, from chapter 4 as well, is as follows:

**THEOREM 12.2.** *Given a finite quantum space  $F$ , there is a universal compact quantum group  $S_F^+$  acting on  $F$ , leaving the counting measure invariant. We have*

$$C(S_F^+) = C(U_N^+) / \left\langle \mu \in \text{Hom}(u^{\otimes 2}, u), \eta \in \text{Fix}(u) \right\rangle$$

where  $N = |F|$  and where  $\mu, \eta$  are the multiplication and unit maps of  $C(F)$ . For  $F = \{1, \dots, N\}$  we have  $S_F^+ = S_N^+$ . Also, for the space  $F = M_2$  we have  $S_F^+ = SO_3$ .

**PROOF.** This is something that we know as well from chapter 4, with the proof being based on the fact that the coaction axioms for a map  $\Phi : C(F) \rightarrow C(F) \otimes C(G)$ , written as  $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$ , correspond to the fact that  $u = (u_{ij})$  must be a corepresentation, satisfying the conditions  $\mu \in \text{Hom}(u^{\otimes 2}, u)$  and  $\eta \in \text{Fix}(u)$  in the statement.  $\square$

For our purposes here it will be useful to have bases and indices. We will use a single index approach, based on the following formalism:

**DEFINITION 12.3.** *Given a finite quantum space  $F$ , we let  $\{e_i\}$  be the standard multi-matrix basis of the algebra  $B = C(F)$ , so that the multiplication, involution and unit of  $B$  are given by the formulae*

$$\begin{aligned} e_i e_j &= e_{ij} \\ e_i^* &= e_{\bar{i}} \\ 1 &= \sum_{i=\bar{i}} e_i \end{aligned}$$

where  $(i, j) \rightarrow ij$  is the standard partially defined multiplication on the indices, with the convention  $e_\emptyset = 0$ , and where  $i \rightarrow \bar{i}$  is the standard involution on the indices.

To be more precise, let  $\{e_{ab}^r\} \subset B$  be the multimatrix basis. We set then  $i = (abr)$ , and with this convention, the multiplication, coming from  $e_{ab}^r e_{cd}^p = \delta_{rp} \delta_{bc} e_{ad}^r$ , is given by:

$$(abr)(cdp) = \begin{cases} (adr) & \text{if } r = p, b = c \\ \emptyset & \text{otherwise} \end{cases}$$

As for the involution, coming from  $(e_{ab}^r)^* = e_{ba}^r$ , this is given by:

$$\overline{(a, b, r)} = (b, a, r)$$

Finally, the unit formula comes from the following formula for the unit  $1 \in B$ :

$$1 = \sum_{ar} e_{aa}^r$$

Regarding now the generalized quantum permutation groups  $S_F^+$ , the construction in Theorem 12.2 reformulates as follows, by using the single index formalism:

**THEOREM 12.4.** *Given a finite quantum space  $F$ , with basis  $\{e_i\} \subset C(F)$  as above, the algebra  $C(S_F^+)$  is generated by variables  $u_{ij}$  with the following relations,*

$$\begin{aligned} \sum_{ij=p} u_{ik}u_{jl} &= u_{p,kl} \quad , \quad \sum_{kl=p} u_{ik}u_{jl} = u_{ij,p} \\ \sum_{i=\bar{i}} u_{ij} &= \delta_{j\bar{j}} \quad , \quad \sum_{j=\bar{j}} u_{ij} = \delta_{i\bar{i}} \\ u_{ij}^* &= u_{\bar{j}\bar{i}} \end{aligned}$$

with the fundamental corepresentation being the matrix  $u = (u_{ij})$ . We call a matrix  $u = (u_{ij})$  satisfying the above relations “generalized magic”.

**PROOF.** Once again, this is something that we know from chapter 4 above, the idea being that the relations  $\mu \in Hom(u^{\otimes 2}, u)$  and  $\eta \in Fix(u)$  in Theorem 12.2 produce the 1st and 4th relations, then the biunitarity of  $u$  gives the 5th relation, and finally the 2nd and 3rd relations follow from the 1st and 4th relations, by using the antipode.  $\square$

As an illustration, consider the case  $F = \{1, \dots, N\}$ . Here the index multiplication is  $ii = i$  and  $ij = \emptyset$  for  $i \neq j$ , and the involution is  $\bar{i} = i$ . Thus, our relations read:

$$\begin{aligned} u_{ik}u_{il} &= \delta_{kl}u_{ik} \quad , \quad u_{ik}u_{jk} = \delta_{ij}u_{ik} \\ \sum_i u_{ij} &= 1 \quad , \quad \sum_j u_{ij} = 1 \\ u_{ij}^* &= u_{ij} \end{aligned}$$

We recognize here the standard magic conditions on a matrix  $u = (u_{ij})$ .

Getting now to the point where we wanted to get, namely the quantum symmetries of the finite “quantum” graphs, and the generalized quantum reflection groups, let us start with the following straightforward extension of the usual notion of finite graph, from [61], obtained by using a finite quantum space as set of vertices:

**DEFINITION 12.5.** *We call “finite quantum graph” a pair of type*

$$X = (F, d)$$

with  $F$  being a finite quantum space, and with  $d \in M_N(\mathbb{C})$  being a matrix.

Such a quantum graph can be represented as a colored oriented graph on  $\{1, \dots, N\}$ , where  $N = |F|$ , with the vertices being decorated by single indices  $i$  as above, and with the colors being complex numbers, namely the entries of  $d$ . This is quite similar to the formalism from chapter 10, but there is a discussion here in what regards the exact choice of the colors, which are normally irrelevant in connection with our  $G^+(X)$  problematics, and so can be true colors instead of complex numbers. More on this later.

With the above notion in hand, we have the following definition, also from [61]:

DEFINITION 12.6. *The quantum automorphism group of  $X = (F, d)$  is the subgroup*

$$G^+(X) \subset S_F^+$$

*obtained via the relation  $du = ud$ , where  $u = (u_{ij})$  is the fundamental corepresentation.*

We refer to [61] and to [89] for more on this notion, and for a number of advanced computations, in relation with the free wreath products.

At an elementary level, a first problem is that of working out the basics of the correspondence  $X \rightarrow G^+(X)$ , following [6]. There are several things to be done here, namely simplices, complementation, color independence, multi-simplices, and reflections.

Let us start with the simplices. The result here is as follows:

THEOREM 12.7. *Given a finite quantum space  $F$ , we have*

$$G^+(F_{\text{empty}}) = G^+(F_{\text{full}}) = S_F^+$$

*where  $F_{\text{empty}}$  is the empty graph, coming from the matrix  $d = 0$ , and where  $F_{\text{full}}$  is the simplex, coming from the matrix  $d = NP_1 - 1_N$ .*

PROOF. This is something quite tricky, the idea being as follows:

(1) First of all, the formula  $G^+(F_{\text{empty}}) = S_F^+$  is clear from definitions, because the commutation of  $u$  with the matrix  $d = 0$  is automatic.

(2) Regarding  $G^+(F_{\text{full}}) = S_F^+$ , let us first discuss the classical case,  $F = \{1, \dots, N\}$ . Here the simplex  $F_{\text{full}}$  is the graph having having edges between any two vertices, whose adjacency matrix is  $d = \mathbb{1}_N - 1_N$ , where  $\mathbb{1}_N$  is the all-1 matrix. The commutation of  $u$  with  $1_N$  being automatic, and the commutation with  $\mathbb{1}_N$  being automatic too,  $u$  being bistochastic, we have  $[u, d] = 0$ , and so  $G^+(F_{\text{full}}) = S_F^+$  in this case, as stated.

(3) In the general case now, we know from Theorem 12.2 that we have  $\eta \in \text{Fix}(u)$ , with  $\eta : \mathbb{C} \rightarrow C(F)$  being the unit map. Thus we have  $P_1 \in \text{End}(u)$ , and so  $[u, P_1] = 0$  is automatic. Together with the fact that in the classical case we have the formula  $\mathbb{1}_N = NP_1$ , this suggests to define the adjacency matrix of the simplex as being  $d = NP_1 - 1_N$ , and with this definition, we have  $G^+(F_{\text{full}}) = S_F^+$ , as claimed.

(4) Thus, we have the result, and the only piece of discussion still needed concerns the understanding of what the simplex  $F_{\text{full}}$  really is, say pictorially speaking. According to our conventions, the adjacency matrix of the simplex is:

$$\begin{aligned} d_{ij} &= (NP_1 - 1_N)_{ij} \\ &= \bar{1}_i 1_j - \delta_{ij} \\ &= \delta_{i\bar{i}} \delta_{j\bar{j}} - \delta_{ij} \end{aligned}$$

(5) For  $F = \{1, \dots, N\}$ , where the involution on the index set is  $\bar{i} = i$ , we obtain  $d_{ij} = 1 - \delta_{ij}$ , as we should. In the case  $F = M_n$  now, by using double indices we have:

$$\begin{aligned} d_{ab,cd} &= \delta_{ab,ba}\delta_{cd,dc} - \delta_{ab,cd} \\ &= \delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} \end{aligned}$$

Thus, we obtain a matrix  $d \in M_N(-1, 0, 1)$ , which generically has 0 entries. This matrix is symmetric. It has not 0 on the diagonal, the self-edges, worth 1, appearing at the off-diagonal points of  $F$ . The case of edges worth -1 is possible too.  $\square$

With the above result in hand, we can now talk about complementation, as follows:

**THEOREM 12.8.** *For any finite quantum graph  $X$  we have the formula*

$$G^+(X) = G^+(X^c)$$

where  $X \rightarrow X^c$  is the complementation operation, given by  $d_X + d_{X^c} = d_{F_{full}}$ .

**PROOF.** This follows from Theorem 12.7, and more specifically from the fact that the condition  $[u, d_{F_{full}}] = 0$  is automatic, as explained there. There is of course still some discussion here to be done, in what concerns the pictorial representation of  $X^c$ , as a continuation of the discussion in (4) from the proof of Theorem 12.7.  $\square$

As basic examples, with  $F = \{1, \dots, N\}$  we recover the usual  $G^+(X)$  quantum groups. For  $F = M_2$  we have  $S_F^+ = S_F = SO_3$ , and for the few graphs here, having  $|M_2| = 4$  vertices, we recover certain subgroups of  $SO_3$ , which can be computed explicitly.

Following now [6], let us discuss an important point, namely the ‘‘independence on the colors’’ question. The idea indeed is that given a classical graph  $X$  with edges colored by complex numbers, or by other types of colors,  $G(X)$  does not change when changing the colors. This is obvious, and a quantum analogue of this fact, involving  $G^+(X)$ , can be shown to hold as well, as explained in [6], and in chapter 10 above.

In the quantum graph setting things are more complicated, with the independence on the colors not necessarily being true. Let us start with the following definition:

**DEFINITION 12.9.** *We say that a quantum graph  $X = (F, d)$  is washable if, whenever we have another quantum graph  $X' = (F, d')$  with same color scheme, in the sense that*

$$d_{ij} = d_{kl} \iff d'_{ij} = d'_{kl}$$

we have  $G^+(X) = G^+(X')$ .

As already mentioned, it was proved in [6] that in the classical case,  $F = \{1, \dots, N\}$ , all graphs are washable. This is a key result, and this for several reasons: (1) first of all, it gives some intuition on what is going on with respect to colors, in analogy with what happens for  $G(X)$ , which is very intuitive, and trivial, (2) also, it allows the use of true colors (black, blue, red . . .) when drawing colored graphs, instead of complex numbers,

and (3) this can be combined with the fact that  $G^+(X)$  is invariant as well via similar changes in the spectral decomposition of  $d$ , at the level of eigenvalues, with all this leading to some powerful combinatorial methods for the computation of  $G^+(X)$ .

All these things do not necessarily hold in general, and to start with, we have:

PROPOSITION 12.10. *There are quantum graphs, such as the simplex in the homogeneous quantum space case,  $F = M_K \times \{1, \dots, L\}$  with  $K, L \neq 2$ , which are not washable.*

PROOF. We know that the simplex, in the case  $F = M_K \times \{1, \dots, L\}$ , has as adjacency matrix a certain matrix  $d \in M_N(-1, 0, 1)$ , with  $N = K^2L$ . Moreover, assuming  $K, L \geq 2$  as in the statement, entries of all types,  $-1, 0, 1$ , are possible. Now assuming that this simplex is washable, it would follow that we have  $\dim(\text{End}(u)) \geq 3$ , a contradiction.  $\square$

In order to come up with some positive results as well, the idea will be that of using the method in [6]. Let us start with the following statement, coming from there:

PROPOSITION 12.11. *The following matrix belongs to  $\text{End}(u)$ , for any  $n \in \mathbb{N}$ :*

$$d_{ij}^{\times n} = \sum_{i=k_1 \dots k_n} \sum_{j=l_1 \dots l_n} d_{k_1 l_1} \dots d_{k_n l_n}$$

*In particular, in the classical case,  $F = \{1, \dots, N\}$ , all graphs are washable.*

PROOF. We have two assertions here, the idea being as follows:

(1) Consider the multiplication and comultiplication maps of the algebra  $C(F)$ , which in single index notation are given by:

$$\begin{aligned} \mu(e_i \otimes e_j) &= e_{ij} \\ \gamma(e_i) &= \sum_{i=jk} e_j \otimes e_k \end{aligned}$$

Observe that we have  $\mu^* = \gamma$ , with the adjoint taken with respect to the scalar product coming from the canonical trace. We conclude that we have:

$$\begin{aligned} \mu &\in \text{Hom}(u^{\otimes 2}, u) \\ \gamma &\in \text{Hom}(u, u^{\otimes 2}) \end{aligned}$$

The point now is that we can consider the iterations  $\mu^{(n)}, \gamma^{(n)}$  of  $\mu, \gamma$ , constructed in the obvious way, and we have then:

$$\begin{aligned} \mu^{(n)} &\in \text{Hom}(u^{\otimes n}, u) \\ \gamma^{(n)} &\in \text{Hom}(u, u^{\otimes n}) \end{aligned}$$

Now if we assume that we have  $d \in \text{End}(u)$ , we have  $d^{\otimes n} \in \text{End}(u^{\otimes n})$  for any  $n$ , and we conclude that we have:

$$\mu^{(n)} d^{\otimes n} \gamma^{(n)} \in \text{End}(u)$$

In single index notation, we have the following formula:

$$(\mu^{(n)}d^{\otimes n}\gamma^{(n)})_{ij} = \sum_{i=k_1\dots k_n} \sum_{j=l_1\dots l_n} d_{k_1l_1} \dots d_{k_nl_n}$$

Thus, we are led to the conclusion in the statement.

(2) Assuming that we are in the case  $F = \{1, \dots, N\}$ , the matrix  $d^{\times n}$  in the statement is simply the componentwise  $n$ -th power of  $d$ , given by:

$$d_{ij}^{\times n} = d_{ij}^n$$

As explained in [6], a simple analytic argument, using  $n \rightarrow \infty$  and then a recurrence on the number of colors, shows from this that we have washability indeed.  $\square$

In order to exploit now the findings in Proposition 12.11, we will assume that we are in the case  $F = M_K \times \{1, \dots, L\}$ , and we will use an idea which is familiar in random matrices and quantum information, namely assuming that  $d$  is “split”. We have:

**THEOREM 12.12.** *Assuming that we are in the case  $F = M_K \times \{1, \dots, L\}$ , and that the adjacency matrix is split, in the sense that one of the following happens,*

$$d_{ab,cd} = e_{ab}f_{cd}$$

$$d_{ab,cd} = e_{ac}f_{bd}$$

$$d_{ab,cd} = e_{ad}f_{bc}$$

*the quantum graph is washable.*

**PROOF.** The idea here is that of computing the matrix  $d^{\times n}$  from Proposition 12.11, and then adapting the proof from the  $K = 1$  case, from [6], explained above. We know from Proposition 12.11 that we have the following formula, in single index notation:

$$d_{ij}^{\times n} = \sum_{i=k_1\dots k_n} \sum_{j=l_1\dots l_n} d_{k_1l_1} \dots d_{k_nl_n}$$

In double index notation, which is more convenient for our purposes here, we have:

$$d_{ab,cd}^{\times n} = \sum_{x_1\dots x_{n-1}} \sum_{y_1\dots y_{n-1}} d_{ax_1, cy_1} d_{x_1x_2, y_1y_2} d_{x_2x_3, y_2y_3} \dots \dots \dots d_{x_{n-2}x_{n-1}, y_{n-2}y_{n-1}} d_{x_{n-1}b, y_{n-1}d}$$

We have 3 cases to be investigated, and here are the computations of this matrix:



(1) In the case  $d_{ab,cd} = e_{ab}f_{cd}$  we have the following computation:

$$\begin{aligned}
d_{ab,cd}^{\times n} &= \sum_{x_1 \dots x_{n-1}} \sum_{y_1 \dots y_{n-1}} e_{ax_1} f_{cy_1} e_{x_1 x_2} f_{y_1 y_2} e_{x_2 x_3} f_{y_2 y_3} \cdots \cdots \\
&\quad \cdots \cdots e_{x_{n-2} x_{n-1}} f_{y_{n-2} y_{n-1}} e_{x_{n-1} b} f_{y_{n-1} d} \\
&= \sum_{x_1 \dots x_{n-1}} e_{ax_1} e_{x_1 x_2} e_{x_2 x_3} \cdots \cdots e_{x_{n-2} x_{n-1}} e_{x_{n-1} b} \\
&\quad \sum_{y_1 \dots y_{n-1}} f_{cy_1} f_{y_1 y_2} f_{y_2 y_3} \cdots \cdots f_{y_{n-2} y_{n-1}} f_{y_{n-1} d} \\
&= (e^n)_{ab} (f^n)_{cd}
\end{aligned}$$

(2) In the case  $d_{ab,cd} = e_{ac}f_{bd}$  we have the following computation, where the  $\times$  operation at the end is the usual componentwise product of the square matrices, and where  $E$  is the total sum of the entries of a given square matrix:

$$\begin{aligned}
d_{ab,cd}^{\times n} &= \sum_{x_1 \dots x_{n-1}} \sum_{y_1 \dots y_{n-1}} e_{ac} f_{x_1 y_1} e_{x_1 y_1} f_{x_2 y_2} e_{x_2 y_2} f_{x_3 y_3} \cdots \cdots \\
&\quad \cdots \cdots e_{x_{n-2} y_{n-2}} f_{x_{n-1} y_{n-1}} e_{x_{n-1} y_{n-1}} f_{bd} \\
&= e_{ac} f_{bd} \sum_{x_1 \dots x_{n-1}} \sum_{y_1 \dots y_{n-1}} (e \times f)_{x_1 y_1} (e \times f)_{x_2 y_2} \cdots \cdots (e \times f)_{x_{n-1} y_{n-1}} \\
&= e_{ac} e_{bd} E [(e \times f)^{n-1}]
\end{aligned}$$

(3) In the case  $d_{ab,cd} = e_{ad}f_{bc}$  we have the following computation, in a rough form, with the general case depending on the parity of  $n$ :

$$\begin{aligned}
d_{ab,cd}^{\times n} &= \sum_{x_1 \dots x_{n-1}} \sum_{y_1 \dots y_{n-1}} e_{ay_1} f_{x_1 c} e_{x_1 y_2} f_{x_2 y_1} e_{x_2 y_3} f_{x_3 y_2} \cdots \cdots \\
&\quad \cdots \cdots e_{x_{n-2} y_{n-1}} f_{x_{n-1} y_{n-2}} e_{x_{n-1} d} f_{by_{n-1}} \\
&= \sum_{x_1 \dots x_{n-1}} \sum_{y_1 \dots y_{n-1}} e_{ay_1} (f^t)_{y_1 x_2} e_{x_2 y_3} \cdots \cdots (f^t)_{cx_1} e_{x_1 y_2} (f^t)_{x_2 y_3} \cdots \cdots \\
&= [(e f^t)^{n/2}]_{ad} [(f^t e)^{n/2}]_{cb}
\end{aligned}$$

With these formulae in hand, we are led to the conclusion in the statement.  $\square$

As a conclusion, the basic theory of the quantum groups  $G^+(X)$  from chapter 10 extends to the present setting, modulo some subtleties in connection with the colors.

## 12b. Cayley graphs

As a first application of our quantum graph considerations, let us discuss some theoretical questions, in connection with the Cayley theorem. Let us start with:

PROPOSITION 12.13. *For any finite quantum group  $G$ , the counting measure is the Haar measure.*

PROOF. Given an arbitrary finite quantum group  $G$ , we must prove that the counting measure is left and right invariant, in the sense that we have:

$$(tr \otimes id)\Delta = (id \otimes tr)\Delta = tr$$

But this is something well-known, which follows from the definition of the canonical trace, as being the following composition:

$$tr : C(G) \rightarrow B(l^2(G)) \rightarrow \mathbb{C}$$

Indeed, this composition is left and right invariant, as desired.  $\square$

As a consequence, we have a Cayley theorem in the present setting, as follows:

THEOREM 12.14. *For any finite quantum group  $G$ , we have:*

$$G \subset S_G^+$$

*That is, the Cayley theorem holds, in the present setting.*

PROOF. We have an action  $G \curvearrowright G$ , which leaves invariant the Haar measure. Now since by Proposition 12.13 the Haar measure is the counting measure, we conclude that  $G \curvearrowright G$  leaves invariant the counting measure, and so we have  $G \subset S_G^+$ , as claimed.  $\square$

By adding now edges, we are led to the following result:

THEOREM 12.15. *Given a finite quantum group  $G$ , the following happen:*

- (1) *We have  $G \subset G^+(X_G)$  with  $X_G = (G, d)$  when the matrix  $d \in M_G(\mathbb{C})$  belongs to the image of the right regular representation.*
- (2) *In this context, we can always arrange as for the inclusion  $G \subset G^+(X_G)$  to be “optimal”, in the sense that  $\{d\}' = \text{End}(u)$ .*
- (3) *In fact, we can always arrange as for having a formula of type  $G = G^+(X_G)$ , for a certain quantum graph  $X_G$ .*

PROOF. This follows from Theorem 12.14, and from the basic properties of the left and right regular representation, for the finite quantum groups.  $\square$

There are many interesting questions in relation with the above, notably in relation with the problem of realizing a given quantum permutation group  $G \subset S_N^+$ , or more generally  $G \subset S_F^+$ , as the quantum automorphism group of a quantum graph.

### 12c. Twisted reflections

Let us discuss now some basic examples of quantum symmetry groups of quantum graphs, in relation with the reflection problematics. We first have the following result:

**THEOREM 12.16.** *We have the following results:*

- (1)  $G_i \curvearrowright X_i$  implies  $\hat{*}G_i \curvearrowright \sqcup X_i$ .
- (2)  $G \curvearrowright X$  implies  $G \wr_* S_N^+ \curvearrowright NX$ .

**PROOF.** These results are standard, following the proofs from the usual graph case, where  $F = \{1, \dots, N\}$ , and we refer here to [61], and to [89].  $\square$

Let us also mention that under suitable connectedness assumptions on  $X$ , similar to those in the classical case, taken in a functional analytic sense, the action in (2) above can be shown to be universal, when taking  $G = G^+(X)$ , and so we have:

$$G^+(NX) = G^+(X) \wr_* S_N^+$$

For more on this material, we refer to [61], [89] and related papers.

With the above technology in hand, we can talk about multi-simplices, and quantum reflections. The idea is that the quantum automorphism groups  $S_{F \rightarrow E}^+$  of the Markov fibrations  $F \rightarrow E$ , which correspond by definition to the Markov inclusions of finite dimensional  $C^*$ -algebras  $C(E) \subset C(F)$ , were studied in [4].

As explained in [6] and in subsequent papers, in the classical case,  $F = \{1, \dots, N\}$ , the quantum groups  $S_{F \rightarrow E}^+$  are of the form  $G^+(X)$ , with  $X$  being a multi-simplex, obtained as a union of simplices, and correspond to the notion of quantum reflection group.

In the general case, the quantum groups of type  $S_{F \rightarrow E}^+$ , with  $F \rightarrow E$  being a Markov fibration, can be thought of as being “generalized quantum reflection groups”.

In order to discuss this, let us start with the following definition:

**DEFINITION 12.17.** *Let  $B \subset D$  be an inclusion of finite dimensional  $C^*$ -algebras and let  $\varphi$  be a state on  $D$ . We define the universal  $C^*$ -algebra  $A_{aut}(B \subset D)$  generated by the coefficients  $v_{ij}$  of a unitary matrix  $v$  subject to the conditions*

$$m \in \text{Hom}(v^{\otimes 2}, v)$$

$$u \in \text{Hom}(1, v)$$

$$e \in \text{End}(v)$$

where  $m : D \otimes D \rightarrow D$  is the multiplication,  $u : \mathbb{C} \rightarrow D$  is the unit and  $e : D \rightarrow D$  is the projection onto  $B$ , with respect to the scalar product  $\langle x, y \rangle = \varphi(xy^*)$ .

By universality we can construct maps  $\Delta, \varepsilon, S$ , and so we have a Woronowicz algebra in the sense of chapter 1 above. This is something standard, see [4].

Let us discuss now the corepresentation theory of  $A_{aut}(B \subset D)$ . The matrix  $v$  is a corepresentation of  $A_{aut}(B \subset D)$  on the Hilbert space  $D$ , and the three ‘‘Hom’’ conditions translate into the fact that  $v$  corresponds to a coaction of  $A_{aut}(B \subset D)$  on the  $\mathbb{C}^*$ -algebra  $D$ , which leaves  $\varphi$  and  $B$  invariant. Following [4], we will prove that the corresponding Tannakian category is the Fuss-Catalan category, introduced by Bisch-Jones in [47].

The Fuss-Catalan category, as well as other categories to be used in what follows, is a tensor  $\mathbb{C}^*$ -category having  $(\mathbb{N}, +)$  as monoid of objects.

In what follows, we will call such a tensor category a  $\mathbb{N}$ -algebra. If  $C$  is a  $\mathbb{N}$ -algebra we use the following notations:

$$C(m, n) = Hom_C(m, n)$$

$$C(m) = End_C(m)$$

As a first class of examples, which is very wide, associated to any object  $O$  in a tensor  $\mathbb{C}^*$ -category is the  $\mathbb{N}$ -algebra  $NO$  given by the following formula:

$$NO(m, n) = Hom(O^{\otimes m}, O^{\otimes n})$$

Let us first discuss in detail the Temperley-Lieb algebra, as a continuation of the material from chapters 1-4. In the present context, we have the following definition:

DEFINITION 12.18. *The  $\mathbb{N}$ -algebra  $TL^2$  of index  $\delta > 0$  is defined as follows:*

- (1) *The space  $TL^2(m, n)$  consists of linear combinations of noncrossing pairings between  $2m$  points and  $2n$  points:*

$$TL^2(m, n) = \left\{ \sum \alpha \begin{array}{ccc} \cdots \cdots & \leftarrow & 2m \text{ points} \\ \mathfrak{W} & \leftarrow & m + n \text{ strings} \\ \cdots & \leftarrow & 2n \text{ points} \end{array} \right\}$$

- (2) *The operations  $\circ, \otimes, *$  are induced by the vertical and horizontal concatenation and the upside-down turning of diagrams:*

$$A \circ B = \begin{pmatrix} B \\ A \end{pmatrix} \quad , \quad A \otimes B = AB \quad , \quad A^* = \forall$$

- (3) *With the rule  $\bigcirc = \delta$ , erasing a circle is the same as multiplying by  $\delta$ .*

Our first task will be that of finding a suitable presentation for this algebra. Consider the following two elements  $u \in TL^2(0, 1)$  and  $m \in TL^2(2, 1)$ :

$$u = \delta^{-\frac{1}{2}} \cap \quad , \quad m = \delta^{\frac{1}{2}} | \cup |$$

With this convention, we have the following result:

THEOREM 12.19. *The following relations are a presentation of  $TL^2$  by the above rescaled diagrams  $u \in TL^2(0, 1)$  and  $m \in TL^2(2, 1)$ :*

- (1)  $mm^* = \delta^2$ .
- (2)  $u^*u = 1$ .
- (3)  $m(m \otimes 1) = m(1 \otimes m)$ .
- (4)  $m(1 \otimes u) = m(u \otimes 1) = 1$ .
- (5)  $(m \otimes 1)(1 \otimes m^*) = (1 \otimes m)(m^* \otimes 1) = m^*m$ .

PROOF. This is something very standard, well-known, and elementary, which follows by drawing diagrams. □

In more concrete terms, the above result says that  $u, m$  satisfy the above relations, which is something clear, and that if  $C$  is a  $\mathbb{N}$ -algebra and  $v \in C(0, 1)$  and  $n \in C(2, 1)$  satisfy the same relations then there exists a  $\mathbb{N}$ -algebra morphism as follows:

$$TL^2 \rightarrow C$$

$$u \rightarrow v$$

$$m \rightarrow n$$

Now let  $D$  be a finite dimensional  $\mathbb{C}^*$ -algebra with a state  $\varphi$  on it. We have a scalar product  $\langle x, y \rangle = \varphi(xy^*)$  on  $D$ , so  $D$  is an object in the category of finite dimensional Hilbert spaces. Consider the unit  $u$  and the multiplication  $m$  of  $D$ :

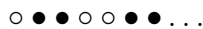
$$u \in ND(0, 1)$$

$$m \in ND(2, 1)$$

The relations in Theorem 12.19 are then satisfied if and only if the first one, namely  $mm^* = \delta^2$ , is satisfied, and this is automatic when  $\varphi$  is the standard trace. One can deduce from Theorem 12.19 that in this case, the category of corepresentations of the Hopf algebra  $A_{aut}(D)$  is the completion of  $TL^2$ , in the sense of [99].

Getting now to Fuss-Catalan algebras, we have here:

DEFINITION 12.20. *A Fuss-Catalan diagram is a planar diagram formed by an upper row of  $4m$  points, a lower row of  $4n$  points, both colored*



*and by  $2m + 2n$  noncrossing strings joining these  $4m + 4n$  points, with the rule that the points which are joined must have the same color.*

Fix  $\beta > 0$  and  $\omega > 0$ . The  $\mathbb{N}$ -algebra  $FC$  is defined as follows. The spaces  $FC(m, n)$  consist of linear combinations of Fuss-Catalan diagrams:

$$FC(m, n) = \left\{ \sum \alpha \begin{array}{c} \circ \bullet \bullet \circ \circ \bullet \bullet \circ \dots \leftarrow 4m \text{ colored points} \\ \mathfrak{W} \leftarrow m + n \text{ black strings} \\ \circ \bullet \bullet \circ \circ \bullet \bullet \circ \dots \leftarrow m + n \text{ white strings} \\ \leftarrow 4n \text{ colored points} \end{array} \right\}$$

As before with the Temperley-Lieb algebra, the operations  $\circ, \otimes, *$  are induced by vertical and horizontal concatenation and upside-down turning of diagrams, but this time with the rule that erasing a black/white circle is the same as multiplying by  $\beta/\omega$ :

$$A \circ B = \begin{pmatrix} B \\ A \end{pmatrix}, \quad A \otimes B = AB, \quad A^* = \nabla$$

$$\text{black} \rightarrow \bigcirc = \beta, \quad \text{white} \rightarrow \bigcirc = \omega$$

Let  $\delta = \beta\omega$ . We have the following bicolored analogues of the elements  $u, m$ :

$$u = \delta^{-\frac{1}{2}} \cap, \quad m = \delta^{\frac{1}{2}} \cup$$

These elements generate in  $FC$  a  $\mathbb{N}$ -subalgebra which is isomorphic to  $TL^2$ .

Consider also the black and white Jones projections, namely:

$$e = \omega^{-1} \left| \begin{array}{c} \cup \\ \cap \end{array} \right|, \quad f = \beta^{-1} \left| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \right|$$

We have  $f = \beta^{-2}(1 \otimes me)m^*$ , so we won't need  $f$  for presenting  $FC$ . For simplifying writing we identify  $x$  and  $x \otimes 1$ , for any  $x$ . We have the following result:

**THEOREM 12.21.** *The following relations, with  $f = \beta^{-2}(1 \otimes me)m^*$ , are a presentation of  $FC$  by  $m \in FC(2, 1)$ ,  $u \in FC(0, 1)$  and  $e \in FC(1)$ :*

- (1) *The relations in Theorem 12.19, with  $\delta = \beta\omega$ .*
- (2)  *$e = e^2 = e^*$ ,  $f = f^*$  and  $(1 \otimes f)f = f(1 \otimes f)$ .*
- (3)  *$eu = u$ .*
- (4)  *$mem^* = m(1 \otimes e)m^* = \beta^2$ .*
- (5)  *$mm(e \otimes e \otimes e) = emm(e \otimes 1 \otimes e)$ .*

**PROOF.** As for any presentation result, we have to prove two assertions:

(I) The elements  $m, u, e$  satisfy the relations (1-5) and generate the  $\mathbb{N}$ -algebra  $FC$ .

(II) If  $M, U$  and  $E$  in a  $\mathbb{N}$ -algebra  $C$  satisfy the relations (1-5), then there exists a morphism of  $\mathbb{N}$ -algebras  $FC \rightarrow C$  sending  $m \rightarrow M, u \rightarrow U, e \rightarrow E$ .

The proof will be based on the results from the paper of Bisch and Jones [47], plus some diagrammatic computations for (I), and some purely algebraic computations for (II).

(I) First, the relations (1-5) are easily verified by drawing pictures.

Let us show now that the  $\mathbb{N}$ -subalgebra  $C = \langle m, u, e \rangle$  of  $FC$  is equal to  $FC$ . First,  $C$  contains the infinite sequence of black and white Jones projections:

$$\begin{aligned}
 p_1 &= e = \omega^{-1} \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \\
 p_2 &= f = \beta^{-1} \left\| \begin{array}{c} \cup \\ \cap \end{array} \right\| \\
 p_3 &= 1 \otimes e = \omega^{-1} \left\| \left\| \begin{array}{c} \cup \\ \cap \end{array} \right\| \right\| \\
 p_4 &= 1 \otimes f = \beta^{-1} \left\| \left\| \left\| \begin{array}{c} \cup \\ \cap \end{array} \right\| \right\| \right\| \\
 &\vdots
 \end{aligned}$$

The algebra  $C$  contains as well the infinite sequence of bicolored Jones projections:

$$\begin{aligned}
 e_1 &= uu^* = \delta^{-1} \begin{array}{c} \cup \\ \cap \end{array} \\
 e_2 &= \delta^{-2} m^* m = \delta^{-1} \left\| \begin{array}{c} \cup \\ \cap \end{array} \right\| \\
 e_3 &= 1 \otimes uu^* = \delta^{-1} \left\| \left\| \begin{array}{c} \cup \\ \cap \end{array} \right\| \right\| \\
 e_4 &= \delta^{-2} (1 \otimes m^* m) = \delta^{-1} \left\| \left\| \left\| \begin{array}{c} \cup \\ \cap \end{array} \right\| \right\| \right\| \\
 &\vdots
 \end{aligned}$$

By the results of Bisch and Jones in [47], these latter projections generate the diagonal  $\mathbb{N}$ -algebra  $\Delta FC$ . Thus we have inclusions as follows:

$$\Delta FC \subset C \subset FC$$

By definition of  $C$ , we have as well the following equality:

$$\Delta FC = \Delta C$$

Also, the existence of semicircles shows that the objects of  $C$  and  $FC$  are self-dual, and by Frobenius reciprocity we obtain that for  $m + n$  even, we have:

$$\begin{aligned} \dim(C(m, n)) &= \dim\left(C\left(\frac{m+n}{2}\right)\right) \\ &= \dim\left(FC\left(\frac{m+n}{2}\right)\right) \\ &= \dim(FC(m, n)) \end{aligned}$$

By tensoring with  $u$  and  $u^*$  we get embeddings as follows:

$$\begin{aligned} C(m, n) &\subset C(m, n+1) \\ FC(m, n) &\subset FC(m, n+1) \end{aligned}$$

But this shows that the above dimension equalities hold for any  $m$  and  $n$ . Together with  $\Delta FC \subset C \subset FC$ , this shows that  $C = FC$ .

(II) Assume that  $M, U, E$  in a  $\mathbb{N}$ -algebra  $C$  satisfy the relations (1-5). We have to construct a morphism  $FC \rightarrow C$  sending:

$$\begin{aligned} m &\rightarrow M \\ u &\rightarrow U \\ e &\rightarrow E \end{aligned}$$

As a first task, we would like to construct a morphism  $\Delta FC \rightarrow \Delta C$  sending:

$$\begin{aligned} m^*m &\rightarrow M^*M \\ uu^* &\rightarrow UU^* \\ e &\rightarrow E \end{aligned}$$

By constructing the corresponding Jones projections  $E_i$  and  $P_i$ , we must send:

$$\begin{aligned} e_i &\rightarrow E_i \\ p_i &\rightarrow P_i \end{aligned}$$

In order to construct these maps, we use now the fact, from [47], that the following relations are a presentation of  $\Delta FC$ :

- (a)  $e_i^2 = e_i$ ,  $e_i e_j = e_j e_i$  if  $|i - j| \geq 2$  and  $e_i e_{i\pm 1} e_i = \delta^{-2} e_i$ .
- (b)  $p_i^2 = p_i$  and  $p_i p_j = p_j p_i$ .
- (c)  $e_i p_i = p_i e_i = e_i$  and  $p_i e_j = e_j p_i$  if  $|i - j| \geq 2$ .
- (d)  $e_{2i\pm 1} p_{2i} e_{2i\pm 1} = \beta^{-2} e_{2i\pm 1}$  and  $e_{2i} p_{2i\pm 1} e_{2i} = \omega^{-2} e_{2i}$ .
- (e)  $p_{2i} e_{2i\pm 1} p_{2i} = \beta^{-2} p_{2i\pm 1} p_{2i}$  and  $p_{2i\pm 1} e_{2i} p_{2i\pm 1} = \omega^{-2} p_{2i} p_{2i\pm 1}$ .



Thus, it remains to verify that we have the following implication, where  $m, u, e$  are now abstract objects, and we are no longer allowed to draw pictures:

$$(1 - 5) \implies (a - e)$$

First, by using  $e_{n+2} = 1 \otimes e_n$  and  $p_{n+2} = 1 \otimes p_n$ , these relations to be checked reduce to the following new collection of relations:

- ( $\alpha$ )  $e_i^2 = e_i$  for  $i = 1, 2$ ,  $e_1 e_2 e_1 = \delta^{-2} e_1$  and  $e_2 e_1 e_2 = \delta^{-2} e_2$ .
- ( $\beta$ )  $p_i^2 = p_i$  for  $i = 1, 2$  and  $[p_1, p_2] = [1 \otimes p_1, p_2] = [1 \otimes p_2, p_2] = 0$ .
- ( $\gamma$ )  $[e_2, 1 \otimes p_2] = [p_2, 1 \otimes e_2] = 0$  and  $e_i p_i = p_i e_i = e_i$  for  $i = 1, 2$ .
- ( $\delta 1$ )  $e_1 p_2 e_1 = \beta^{-2} e_1$  and  $(1 \otimes e_1) p_2 (1 \otimes e_1) = \beta^{-2} (1 \otimes e_1)$ .
- ( $\delta 2$ )  $e_2 p_1 e_2 = e_2 (1 \otimes p_1) e_2 = \omega^{-2} e_2$ .
- ( $\varepsilon 1$ )  $\beta^2 p_2 e_1 p_2 = \omega^2 p_1 e_2 p_1 = p_1 p_2$ .
- ( $\varepsilon 2$ )  $\beta^2 p_2 (1 \otimes e_1) p_2 = \omega^2 (1 \otimes p_1) e_2 (1 \otimes p_1) = (1 \otimes p_1) p_2$ .

With  $e_1 = uu^*$ ,  $e_2 = \delta^{-2} m^* m$ ,  $p_1 = e$  and  $p_2 = f$  we can see that most of these relations are trivial. What is left can be reformulated in the following way:

- (x)  $em^*me = \beta^2 f^*e$ .
- (y)  $(1 \otimes e)m^*m(1 \otimes e) = \beta^2 f^*(1 \otimes e)$ .
- (z)  $f^* = f^*f$ .
- (t)  $[e, f] = [1 \otimes e, f] = [m^*m, 1 \otimes f] = [f, 1 \otimes m^*m] = 0$ .

By multiplying the relation (5) by  $u$  and by  $1 \otimes 1 \otimes u$  to the right we obtain the following useful formula, to be used many times in what follows:

$$m(e \otimes e) = em(1 \otimes e) = eme$$

Let us verify now the above conditions (x-t). First, we have:

$$\beta^2 f^*e = m(e \otimes e)(1 \otimes m^*)$$

By replacing  $m(e \otimes e)$  with  $eme$  we get  $em^*me$ , so (x) is true. Also, we have:

$$(1 \otimes e)m^*m(1 \otimes e) = m(1 \otimes (em(1 \otimes e))^*)$$

By replacing  $em(1 \otimes e)$  with  $eme$  we get  $\beta^2 f^*(1 \otimes e)$ , so (y) is true. We have:

$$f^*f = \beta^{-4} m(1 \otimes em^*me)m^*$$

By replacing  $em^*me$  with  $eme(1 \otimes m^*)$ , then  $eme$  with  $m(e \otimes e)$  we get  $f^*$ , so (z) is true. The first two commutators are zero, because  $fe$  and  $f(1 \otimes e)$  are self-adjoint.

The same happens for the others, because of the following formulae:

$$mm^*(1 \otimes ff^*) = \beta^{-4}(1 \otimes 1 \otimes me)m^*m^*mm(1 \otimes 1 \otimes em^*)$$

$$(1 \otimes m^*m)ff^* = \beta^{-4}(1 \otimes m^*me)m^*m(1 \otimes em^*m)$$

The conclusion is that we constructed a certain  $\mathbb{N}$ -algebra morphism, as follows:

$$\Delta J : \Delta FC \rightarrow \Delta C$$

We have to extend now this morphism into a morphism  $J : FC \rightarrow C$  sending  $u \rightarrow U$  and  $m \rightarrow M$ . We will use a standard argument. For  $w \geq k, l$  we define:

$$\phi : FC(l, k) \rightarrow FC(w)$$

$$x \rightarrow (u^{\otimes(w-k)} \otimes 1_k) x ((u^*)^{\otimes(w-l)} \otimes 1_l)$$

We can define as well a morphism as follows:

$$\theta : FC(w) \rightarrow FC(l, k)$$

$$x \rightarrow ((u^*)^{\otimes(w-k)} \otimes 1_k) x (u^{\otimes(w-l)} \otimes 1_l)$$

Here  $1_k = 1^{\otimes k}$ , and the convention  $x = x \otimes 1$  is no longer used. We define  $\Phi$  and  $\Theta$  in  $C$  by similar formulae. We have  $\theta\phi = \Theta\Phi = Id$ . We define a map  $J$  by:

$$\begin{array}{ccc} FC(l, k) & \xrightarrow{J} & C(l, k) \\ \downarrow \phi & & \uparrow \Theta \\ FC(w) & \xrightarrow{\Delta J} & C(w) \end{array}$$

Since the element  $J(a)$  does now depend on the choice of  $w$ , these  $J$  maps are the components of a global map, as follows:

$$J : FC \rightarrow C$$

This map  $J$  extends  $\Delta J$  and sends  $u \rightarrow U$  and  $m \rightarrow M$ . It remains to prove that  $J$  is a morphism. We have:

$$Im(\phi) = \left\{ x \in FC(w) \mid x = ((uu^*)^{\otimes(w-k)} \otimes 1_k) x ((uu^*)^{\otimes(w-l)} \otimes 1_l) \right\}$$

We have as well as a similar description of  $Im(\Phi)$ , and so  $J$  sends:

$$Im(\phi) \rightarrow Im(\Phi)$$

We have also  $\Theta\Phi = Id$ , so  $\Phi\Theta = Id$  on  $Im(\Phi)$ . Thus the following diagram commutes:

$$\begin{array}{ccc} FC(l, k) & \xrightarrow{J} & C(l, k) \\ \downarrow \phi & & \downarrow \Phi \\ FC(w) & \xrightarrow{\Delta J} & C(w) \end{array}$$

It follows that  $J$  is multiplicative, because we have:

$$\begin{aligned} J(ab) &= \Theta(\Delta J\phi(a)\Delta J\phi(b)) \\ &= \Theta(\Phi J(a)\Phi J(b)) \\ &= \Theta\Phi(J(a)J(b)) \\ &= J(a)J(b) \end{aligned}$$

In order to finish, it remains to prove that we have:

$$J(a \otimes b) = J(a) \otimes J(b)$$

Since we have  $a \otimes b = (a \otimes 1_s)(1_t \otimes b)$  for certain  $s$  and  $t$ , it is enough to prove the above formula for pairs  $(a, b)$  of the form  $(1_t, b)$  or  $(a, 1_s)$ . For  $(a, 1_s)$  this is clear, so it remains to prove that the following set equals  $FC$ :

$$B = \left\{ b \in FC \mid J(1_t \otimes b) = 1_t \otimes J(b), \forall t \in \mathbb{N} \right\}$$

For this purpose, observe first that  $\Delta J$  being a  $\mathbb{N}$ -algebra morphism, we have:

$$\Delta FC \subset B$$

On the other hand, a direct computation gives:

$$J(1_t \otimes u \otimes 1_s) = 1_t \otimes U \otimes 1_s$$

Since  $B$  is stable by involution and multiplication,  $B$  contains the compositions of elements of  $\Delta FC$  with  $1_t \otimes u \otimes 1_s$  and  $1_t \otimes u^* \otimes 1_s$  maps. But any  $b \in FC$  is equal to  $\theta\phi(b)$ , so it is of this form, and we are done.  $\square$

Getting back now to the inclusions  $B \subset D$  of finite dimensional  $\mathbb{C}^*$ -algebras, as in Definition 12.17 above, we have the following result:

**THEOREM 12.22.** *If  $\varphi$  is a  $(\beta, \omega)$ -form on  $B \subset D$  then:*

$$\langle m, u, e \rangle = FC$$

**PROOF.** It is routine to check that the linear maps  $m, u, e$  associated to an inclusion  $B \subset D$  as in the statement satisfy the relations (1-5) in Theorem 12.21. Thus, we obtain a certain  $\mathbb{N}$ -algebra surjective morphism, as follows:

$$J : FC \rightarrow \langle m, u, e \rangle$$

It remains to prove that this morphism  $J$  is faithful. For this purpose, consider the following map, where  $v = m^*u \in FC(0, 2)$ :

$$\begin{aligned} \phi_n &: FC(n) \rightarrow FC(n-1) \\ x &\rightarrow (1^{\otimes(n-1)} \otimes v^*)(x \otimes 1)(1^{\otimes(n-1)} \otimes v) \end{aligned}$$

Consider as well the following map, where  $v = m^*u \in FC(0, 2)$  is as above:

$$\begin{aligned} \psi_n &: C(n) \rightarrow C(n-1) \\ x &\rightarrow (1^{\otimes(n-1)} \otimes J(v)^*)(x \otimes 1)(1^{\otimes(n-1)} \otimes J(v)) \end{aligned}$$

These maps make then the following diagram commutative:

$$\begin{array}{ccc} FC(n) & \xrightarrow{J} & C(n) \\ \downarrow \phi_n & & \downarrow \psi_n \\ FC(n-1) & \xrightarrow{J} & C(n-1) \end{array}$$

By gluing such diagrams we get a factorization by  $J$  of the composition on the left of conditional expectations, which is the Markov trace. By positivity  $J$  is faithful on  $\Delta FC$ , then by Frobenius reciprocity faithfulness has to hold on the whole  $FC$ .  $\square$

Getting back now to quantum groups, we have:

**THEOREM 12.23.** *If  $\varphi$  is a  $(\beta, \omega)$ -form on  $B \subset D$  then the tensor  $\mathbb{C}^*$ -category of finite dimensional corepresentations of  $A_{aut}(B \subset D)$  is the completion of  $FC$ .*

**PROOF.** The algebra  $A_{aut}(B \subset D)$  being by definition presented by the relations corresponding to  $m, u, e$ , its tensor category of corepresentations has to be completion of the tensor category  $\langle m, u, e \rangle$ . On the other hand, the linear form  $\varphi$  being a  $(\beta, \omega)$ -form, Theorem 12.22 applies and gives an isomorphism  $\langle m, u, e \rangle \simeq FC$ .  $\square$

As already mentioned in the above, in terms of quantum spaces and quantum graphs, the conclusion of all this is that the quantum automorphism groups  $S_{F \rightarrow E}^+$  of the Markov fibrations  $F \rightarrow E$ , which can be thought of as being the “twisted versions” of the quantum reflection groups  $H_N^+$  studied in chapter 11 above, correspond to the Fuss-Catalan algebras. We refer to [4] and related papers for more on these topics.

### 12d. General reflections

As a final topic for this chapter, we discuss some further generalizations of all the above. The point indeed is that in the  $F = \{1, \dots, N\}$  case we have seen in chapter 11 that an interesting question is the classification of the general quantum reflection groups, which appear as intermediate quantum groups, as follows:

$$A_N \subset G \subset K_N^+$$

On the other hand, we have seen in this chapter that most of the  $F = \{1, \dots, N\}$  material has extensions to the general case, where  $F$  is an arbitrary finite quantum space. Thus, the story is not over yet, and the most general notion of “quantum reflection group”, using such finite quantum spaces  $F$ , is still to be axiomatized, and studied.

In order to get started, let us go back to the case  $F = \{1, \dots, N\}$ . In the classical case, the classification theorem for the complex reflection groups, a celebrated result by Shephard and Todd [85], from the 50s, is as follows:

**THEOREM 12.24.** *The irreducible complex reflection groups are*

$$H_N^{sd} = \left\{ U \in H_N^s \mid (\det U)^d = 1 \right\}$$

*along with 34 exceptional examples.*

**PROOF.** This is something quite advanced, and we refer here to the paper of Shephard and Todd [85], and to the subsequent literature on the subject.  $\square$

In the general quantum case now, the axiomatization and classification of the quantum reflection groups is a key problem, which is not understood yet. We will be interested in what follows in the “twistable” case, where the theory is more advanced than in the general case. Let us start with the following definition:

**DEFINITION 12.25.** *A closed subgroup  $G \subset U_N^+$  is called:*

- (1) *Half-homogeneous, when it contains the alternating group,  $A_N \subset G$ .*
- (2) *Homogeneous, when it contains the symmetric group,  $S_N \subset G$ .*
- (3) *Twistable, when it contains the hyperoctahedral group,  $H_N \subset G$ .*

These notions are mostly motivated by the easy case. Here we have by definition  $S_N \subset G \subset U_N^+$ , and so our quantum group is automatically homogeneous. The point now is that the twistability assumption corresponds to the following condition, at the level of the associated category of partitions  $D \subset P$ :

$$D \subset P_{\text{even}}$$

We recognize here the condition which is needed for performing the Schur-Weyl twisting operation, explained in chapter 10 above, and based on the signature map:

$$\varepsilon : P_{\text{even}} \rightarrow \{\pm 1\}$$

As a conclusion, in the easy case our notion of twistability is the correct one. In general, there are of course more general twisting methods, usually requiring  $\mathbb{Z}_2^N \subset G$  only. But in the half-homogeneous case, the condition  $\mathbb{Z}_2^N \subset G$  is equivalent to  $H_N \subset G$ .

With this discussion done, let us formulate now the following definition:

DEFINITION 12.26. *A twistable quantum reflection group is an intermediate subgroup*

$$H_N \subset K \subset K_N^+$$

*between the group  $H_N = \mathbb{Z}_2 \wr S_N$ , and the quantum group  $K_N^+ = \mathbb{T} \wr_* S_N^+$ .*

Here is now another definition, which is important for general compact quantum group purposes, and which provides motivations for our formalism from Definition 12.25:

DEFINITION 12.27. *Given a closed subgroup  $G \subset U_N^+$  which is twistable, in the sense that we have  $H_N \subset G$ , we define its associated reflection subgroup to be*

$$K = G \cap K_N^+$$

*with the intersection taken inside  $U_N^+$ . We say that  $G$  appears as a soft liberation of its classical version  $G_{class} = G \cap U_N$  when  $G = \langle G_{class}, K \rangle$ .*

These notions are important in the classification theory of compact quantum groups, and in connection with certain noncommutative geometry questions as well. As a first observation, with  $K$  being as above, we have an intersection diagram, as follows:

$$\begin{array}{ccc} K & \longrightarrow & G \\ \uparrow & & \uparrow \\ K_{class} & \longrightarrow & G_{class} \end{array}$$

The soft liberation condition states that this diagram must be a generation diagram. We will be back to this in a moment, with some further theoretical comments. Let us work out some examples. As a basic result, we have:

THEOREM 12.28. *The reflection subgroups of the basic unitary quantum groups*

$$\begin{array}{ccccc} U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \end{array}$$

*are as follows,*

$$\begin{array}{ccccc} K_N & \longrightarrow & K_N^* & \longrightarrow & K_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ H_N & \longrightarrow & H_N^* & \longrightarrow & H_N^+ \end{array}$$

*and these unitary quantum groups all appear via soft liberation.*

PROOF. The fact that the reflection subgroups of the quantum groups on the left are those on the right is clear in all cases, with the middle objects being by definition:

$$H_N^* = H_N \cap O_N^*$$

$$K_N^* = K_N \cap U_N^*$$

Regarding the second assertion, things are quite tricky here, as follows:

(1) In the classical case there is nothing to prove, because any classical group is by definition a soft liberation of itself.

(2) In the half-classical case the results are non-trivial, but can be proved by using the technology developed by Bichon and Dubois-Violette in [46].

(3) In the free case the results are highly non-trivial, and the only known proof so far uses the recurrence methods developed by Chirvasitu in [51].  $\square$

Summarizing, we are here into recent and interesting quantum group theory. We will discuss a bit later the concrete applications of Theorem 12.28. There is a connection here as well with the notion of diagonal torus, introduced in chapter 1 above. We can indeed refine Definition 12.27, in the following way:

DEFINITION 12.29. *Given  $H_N \subset G \subset U_N^+$ , the diagonal tori  $T = G \cap \mathbb{T}_N^+$  and reflection subgroups  $K = G \cap K_N^+$  for  $G$  and for  $G_{class} = G \cap U_N$  form a diagram as follows:*

$$\begin{array}{ccccc}
 T & \longrightarrow & K & \longrightarrow & G \\
 \uparrow & & \uparrow & & \uparrow \\
 T_{class} & \longrightarrow & K_{class} & \longrightarrow & G_{class}
 \end{array}$$

We say that  $G$  appears as a soft/hard liberation when it is generated by  $G_{class}$  and by  $K/T$ , which means that the right square/whole rectangle should be generation diagrams.

It is in fact possible to further complicate the picture, by adding free versions as well, with these free versions being by definition given by the following formula:

$$G_{free} = \langle G, S_N^+ \rangle$$

Importantly, we can equally add the parameter  $N \in \mathbb{N}$  to the picture, the idea being that we have a kind of “ladder”, whose steps are the diagrams in Definition 12.29, perhaps extended with the free versions too, at fixed values of  $N \in \mathbb{N}$ .

The various generation and intersection properties of this ladder are important properties of  $G = (G_N)$  itself, with subtle relations between them. In fact, as already mentioned in the proof of Theorem 12.28 above, the proof of the soft generation property for  $O_N^+, U_N^+$  uses in fact this ladder, via the recurrence methods developed in [51].

All this is quite technical, so as a concrete result in connection with the above hard liberation notion, we have the following statement, improving Theorem 12.28:

**THEOREM 12.30.** *The diagonal tori of the basic unitary quantum groups*

$$\begin{array}{ccccc}
 U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+
 \end{array}$$

are as follows,

$$\begin{array}{ccccc}
 \mathbb{T}_N & \longrightarrow & \mathbb{T}_N^* & \longrightarrow & \mathbb{T}_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 T_N & \longrightarrow & T_N^* & \longrightarrow & T_N^+
 \end{array}$$

and these unitary quantum groups all appear via hard liberation.

**PROOF.** The first assertion is something that we already know, from section 1 above. As for the second assertion, this can be proved by carefully examining the proof of Theorem 12.28, and performing some suitable modifications, where needed.  $\square$

As an interesting remark, some subtleties appear in the following way:

**PROPOSITION 12.31.** *The diagonal tori of the basic quantum reflection groups*

$$\begin{array}{ccccc}
 K_N & \longrightarrow & K_N^* & \longrightarrow & K_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N & \longrightarrow & H_N^* & \longrightarrow & H_N^+
 \end{array}$$

are as follows,

$$\begin{array}{ccccc}
 \mathbb{T}_N & \longrightarrow & \mathbb{T}_N^* & \longrightarrow & \mathbb{T}_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 T_N & \longrightarrow & T_N^* & \longrightarrow & T_N^+
 \end{array}$$

and these quantum reflection groups do not all appear via hard liberation.



PROOF. The first assertion is clear, as a consequence of Theorem 12.28, because the diagonal torus is the same for a quantum group, and for its reflection subgroup:

$$G \cap \mathbb{T}_N^+ = (G \cap K_N^+) \cap \mathbb{T}_N^+$$

Regarding the second assertion, things are quite tricky here, as follows:

(1) In the classical case the hard liberation property definitely holds, because any classical group is by definition a hard liberation of itself.

(2) In the half-classical case the answer is again positive, and this can be proved by using the technology developed by Bichon and Dubois-Violette in [46].

(3) In the free case the hard liberation property fails, due to some intermediate quantum groups  $H_N^{[\infty]}$ ,  $K_N^{[\infty]}$ , where “hard liberation stops”.  $\square$

Summarizing, we are led into the notion of Fourier liberation from chapter 9, as well into the classification results for the easy quantum reflection groups from [83]. All this is quite technical, and on top of this comes the question of axiomatizing and understanding the structure of the quantum reflection groups, in the case where the base space  $F$  is an arbitrary finite quantum space. There are many interesting questions here.

### 12e. Exercises

This have been quite technical in this chapter, and so will be our exercises here. First we have, in relation with the quantum spaces and graphs in general:

EXERCISE 12.32. *Work out the theory of the discrete group dual actions*

$$\widehat{\Gamma} \curvearrowright X$$

with  $X = (F, D)$  being a finite quantum graph.

To be more precise here, one interesting question is that of understanding when the quantum automorphism groups  $G^+(X) \subset S_F^+$  can be group duals.

In relation now with the quantum reflection groups, we have:

EXERCISE 12.33. *Make the connection between the easiness results from chapter 11, regarding the quantum groups  $H_N^{s+}$ , and the representation theory results from here, regarding the quantum automorphism groups  $S_{F \rightarrow E}^+$  of the Markov fibrations  $F \rightarrow E$ .*

This is something a bit tricky, in the sense that no new computations are needed, with the work instead consisting of making lots of identifications, in order to view the main results from chapter 11 as particular cases of those obtained here.

Finally, we have all the open questions mentioned in the above, in relation with the axiomatization and study of the most general “quantum reflection groups”.

## Part IV

# Matrix models

*And I wait praying to the Northern Star  
I'm afraid it won't lead you anywhere  
He's so cold raining on the world tonight  
All the angels kneeling to the Northern Lights*

## CHAPTER 13

### Orbits, orbitals

#### 13a. Orbit theory

In this fourth part of this book, this chapter and the next 3 ones, we discuss a number of more specialized methods for dealing with the quantum permutation groups  $G \subset S_N^+$ . As before with the third part of the present book, there has been a lot of work here, and our main aim in what follows will be that of surveying this material.

In the classical case,  $G \subset S_N$ , a useful tool for the study of  $G$  are the orbits of the action  $G \curvearrowright \{1, \dots, N\}$ , and more generally the higher orbitals of this action. In the quantum case,  $G \subset S_N^+$ , the theory goes back to Bichon's paper [44]. We first have:

**THEOREM 13.1.** *Given a closed subgroup  $G \subset S_N^+$ , with standard coordinates denoted  $u_{ij} \in C(G)$ , the following defines an equivalence relation on  $\{1, \dots, N\}$ ,*

$$i \sim j \iff u_{ij} \neq 0$$

*that we call orbit decomposition associated to the corresponding action  $G \curvearrowright \{1, \dots, N\}$ . In the classical case,  $G \subset S_N$ , this is the usual orbit equivalence.*

**PROOF.** We first check the fact that we have indeed an equivalence relation. The reflexivity axiom  $i \sim i$  follows by using the counit, as follows:

$$\begin{aligned} \varepsilon(u_{ij}) = \delta_{ij} &\implies \varepsilon(u_{ii}) = 1 \\ &\implies u_{ii} \neq 0 \end{aligned}$$

The symmetry axiom  $i \sim j \implies j \sim i$  follows by using the antipode:

$$S(u_{ij}) = u_{ji} \implies [u_{ij} \neq 0 \implies u_{ji} \neq 0]$$

As for the transitivity axiom  $i \sim j, j \sim k \implies i \sim k$ , this follows by using the comultiplication. Consider indeed the following formula:

$$\Delta(u_{ik}) = \sum_j u_{ij} \otimes u_{jk}$$

On the right we have a sum of projections, and we obtain from this:

$$\begin{aligned} u_{ij} \neq 0, u_{jk} \neq 0 &\implies u_{ij} \otimes u_{jk} > 0 \\ &\implies \Delta(u_{ik}) > 0 \\ &\implies u_{ik} \neq 0 \end{aligned}$$

Finally, in the classical case, where  $G \subset S_N$ , the standard coordinates are:

$$u_{ij} = \chi \left( \sigma \in G \mid \sigma(j) = i \right)$$

Thus  $u_{ij} \neq 0$  is equivalent to the existence of an element  $\sigma \in G$  such that  $\sigma(j) = i$ . But this means that  $i, j$  must be in the same orbit under the action of  $G$ , as claimed.  $\square$

Generally speaking, the theory from the classical case extends well to the quantum group setting, and we have in particular the following result, also from [44]:

**THEOREM 13.2.** *Given a closed subgroup  $G \subset S_N^+$ , with magic matrix denoted  $u = (u_{ij})$ , consider the associated coaction map, on the space  $X = \{1, \dots, N\}$ :*

$$\Phi : C(X) \rightarrow C(X) \otimes C(G) \quad , \quad e_i \rightarrow \sum_j e_j \otimes u_{ji}$$

The following three subalgebras of  $C(X)$  are then equal

$$\begin{aligned} \text{Fix}(u) &= \left\{ \xi \in C(X) \mid u\xi = \xi \right\} \\ \text{Fix}(\Phi) &= \left\{ \xi \in C(X) \mid \Phi(\xi) = \xi \otimes 1 \right\} \\ F &= \left\{ \xi \in C(X) \mid i \sim j \implies \xi(i) = \xi(j) \right\} \end{aligned}$$

where  $\sim$  is the orbit equivalence relation constructed in Theorem 13.1.

**PROOF.** The fact that we have  $\text{Fix}(u) = \text{Fix}(\Phi)$  is standard, with this being valid for any corepresentation  $u = (u_{ij})$ . Regarding now the equality with  $F$ , we know from Theorem 13.1 that the magic unitary  $u = (u_{ij})$  is block-diagonal, with respect to the orbit decomposition there. But this shows that the algebra  $\text{Fix}(u) = \text{Fix}(\Phi)$  decomposes as well with respect to the orbit decomposition, and so in order to prove the result, we are left with a study in the transitive case, where the result is clear. See [44].  $\square$

We have as well a useful analytic result, as follows:

**THEOREM 13.3.** *Given a closed subgroup  $G \subset S_N^+$ , consider the following matrix:*

$$P_{ij} = \int_G u_{ij}$$

Then  $P$  is the orthogonal projection onto the linear space

$$F = \left\{ \xi \in \mathbb{C}^N \mid i \sim j \implies \xi_i = \xi_j \right\}$$

and so the orbits and their sizes can be deduced from the knowledge of  $P$ .

**PROOF.** This follows from the above results, and from the standard fact, coming from the Peter-Weyl theory, that  $P$  is the orthogonal projection onto  $\text{Fix}(u)$ .  $\square$

As a main application of the above orbit theory, let us go back to the group dual results from chapter 2 above. We recall from there that we have:

PROPOSITION 13.4. *Given a quotient group  $\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k} \rightarrow \Gamma$ , we have an embedding  $\widehat{\Gamma} \subset S_N^+$ , with  $N = N_1 + \dots + N_k$ , with magic matrix given by the formula*

$$u = \begin{pmatrix} F_{N_1} I_1 F_{N_1}^* & & \\ & \ddots & \\ & & F_{N_k} I_k F_{N_k}^* \end{pmatrix}$$

where  $F_N = \frac{1}{\sqrt{N}}(w_N^{ij})$  with  $w_N = e^{2\pi i/N}$  are Fourier matrices, and where

$$I_r = \begin{pmatrix} 1 & & & \\ & g_r & & \\ & & \ddots & \\ & & & g_r^{N_r-1} \end{pmatrix}$$

with  $g_1, \dots, g_k$  being the standard generators of  $\Gamma$ .

PROOF. This is something that we already know. To be more precise, given a quotient group  $\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k} \rightarrow \Gamma$  as in the statement, we have an embedding as follows:

$$\begin{aligned} \widehat{\Gamma} &\subset \widehat{\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k}} \\ &= \widehat{\mathbb{Z}_{N_1}} \hat{*} \dots \hat{*} \widehat{\mathbb{Z}_{N_k}} \\ &\simeq \mathbb{Z}_{N_1} \hat{*} \dots \hat{*} \mathbb{Z}_{N_k} \\ &\subset S_{N_1} \hat{*} \dots \hat{*} S_{N_k} \\ &\subset S_{N_1}^+ \hat{*} \dots \hat{*} S_{N_k}^+ \\ &\subset S_N^+ \end{aligned}$$

Here all the embeddings and identifications are standard, with the  $\simeq$  sign standing for a multi-Fourier transform, and when working out what happens at the level of the corresponding magic unitaries, we are led to the formula in the statement.  $\square$

We can now improve the above result, in the following way:

THEOREM 13.5. *Consider a quotient group as follows, with  $N = N_1 + \dots + N_k$ :*

$$\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k} \rightarrow \Gamma$$

We have then  $\widehat{\Gamma} \subset S_N^+$ , and any group dual subgroup of  $S_N^+$  appears in this way.

PROOF. In one sense, this is something that we already know, from Proposition 13.4. Conversely now, assume that we have a group dual subgroup  $\widehat{\Gamma} \subset S_N^+$ . By Peter-Weyl,

the corresponding magic unitary must be of the following form, with  $U \in U_N$ :

$$u = U \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_N \end{pmatrix} U^*$$

Now if we denote by  $N = N_1 + \dots + N_k$  the orbit decomposition for  $\widehat{\Gamma} \subset S_N^+$ , coming from Theorem 13.1, we conclude that  $u$  has a  $N = N_1 + \dots + N_k$  block-diagonal pattern, and so that  $U$  has as well this  $N = N_1 + \dots + N_k$  block-diagonal pattern.

But this discussion reduces our problem to its  $k = 1$  particular case, with the statement here being that the cyclic group  $\mathbb{Z}_N$  is the only transitive group dual  $\widehat{\Gamma} \subset S_N^+$ . The proof of this latter fact being elementary, we obtain the result. See [44].  $\square$

### 13b. Quasi-transitivity

Let us discuss now the notion of transitivity, and its generalizations. We first have:

DEFINITION 13.6. *Associated to a quantum group  $G \subset S_N^+$ , producing the equivalence relation on  $\{1, \dots, N\}$  given by  $i \sim j$  when  $u_{ij} \neq 0$ , are as well:*

- (1) *The partition  $\pi \in P(N)$  having as blocks the equivalence classes under  $\sim$ .*
- (2) *The binary matrix  $\varepsilon \in M_N(0, 1)$  given by  $\varepsilon_{ij} = \delta_{u_{ij}, 0}$ .*

Observe that each of the objects  $\sim, \pi, \varepsilon$  determines the other two ones.

We will often assume, without mentioning it, that the orbits of  $G \subset S_N^+$  come in increasing order, in the sense that the corresponding partition is as follows:

$$\pi = \{1, \dots, K_1\}, \dots, \{K_1 + \dots + K_{M-1} + 1, \dots, K_1 + \dots + K_M\}$$

Indeed, at least for the questions that we are interested in here, we can always assume that it is so, simply by conjugating everything by a suitable permutation  $\sigma \in S_N$ .

In terms of these objects, the notion of transitivity reformulates as follows:

DEFINITION 13.7. *We call  $G \subset S_N^+$  transitive when  $u_{ij} \neq 0$  for any  $i, j$ . Equivalently:*

- (1)  *$\sim$  must be trivial,  $i \sim j$  for any  $i, j$ .*
- (2)  *$\pi$  must be the 1-block partition.*
- (3)  *$\varepsilon$  must be the all-1 matrix.*

Let us discuss now the quantum analogue of the fact that given a subgroup  $G \subset S_N$ , with orbits of lengths  $K_1, \dots, K_M$ , we have an inclusion as follows:

$$G \subset S_{K_1} \times \dots \times S_{K_M}$$

Given two quantum permutation groups  $G \subset S_K^+, H \subset S_L^+$ , with magic corepresentations denoted  $u, v$ , we can consider the following algebra, and matrix:

$$A = C(G) * C(H) \quad , \quad w = \text{diag}(u, v)$$

The pair  $(A, w)$  satisfies Woronowicz’s axioms, and since  $w$  is magic, we therefore obtain a quantum permutation group, denoted  $G \hat{*} H \subset S_{K+L}^+$ . See [94].

With this notion in hand, we have the following result:

PROPOSITION 13.8. *Given a quantum group  $G \subset S_N^+$ , with associated orbit decomposition partition  $\pi \in P(N)$ , having blocks of length  $K_1, \dots, K_M$ , we have an inclusion*

$$G \subset S_{K_1}^+ \hat{*} \dots \hat{*} S_{K_M}^+$$

where the product on the right is constructed with respect to the blocks of  $\pi$ . In the classical case,  $G \subset S_N$ , we obtain in this way the usual inclusion  $G \subset S_{K_1} \times \dots \times S_{K_M}$ .

PROOF. Since the standard coordinates  $u_{ij} \in C(G)$  satisfy  $u_{ij} = 0$  for  $i \not\sim j$ , the algebra  $C(G)$  appears as quotient of the following algebra:

$$\begin{aligned} C(S_N^+) / \langle u_{ij} = 0, \forall i \not\sim j \rangle &= C(S_{K_1}^+) * \dots * C(S_{K_M}^+) \\ &= C(S_{K_1}^+ \hat{*} \dots \hat{*} S_{K_M}^+) \end{aligned}$$

Thus, we have an inclusion of quantum groups, as in the statement. Finally, observe that the classical version of the quantum group  $S_{K_1}^+ \hat{*} \dots \hat{*} S_{K_M}^+$  is given by:

$$\begin{aligned} (S_{K_1}^+ \hat{*} \dots \hat{*} S_{K_M}^+)_{class} &= (S_{K_1} \times \dots \times S_{K_M})_{class} \\ &= S_{K_1} \times \dots \times S_{K_M} \end{aligned}$$

Thus in the classical case we obtain  $G \subset S_{K_1} \times \dots \times S_{K_M}$ , as claimed. □

Let us discuss now an extension of the notion of transitivity, from [32], as follows:

DEFINITION 13.9. *A quantum permutation group  $G \subset S_N^+$  is called quasi-transitive when all its orbits have the same size. Equivalently:*

- (1)  $\sim$  has equivalence classes of same size.
- (2)  $\pi$  has all the blocks of equal length.
- (3)  $\varepsilon$  is block-diagonal with blocks the flat matrix of size  $K$ .

As a first example, if  $G$  is transitive then it is quasi-transitive. In general now, if we denote by  $K \in \mathbb{N}$  the common size of the blocks, and by  $M \in \mathbb{N}$  their multiplicity, then we must have  $N = KM$ . We have the following result:

PROPOSITION 13.10. *Assuming that  $G \subset S_N^+$  is quasi-transitive, we must have*

$$G \subset \underbrace{S_K^+ \hat{*} \dots \hat{*} S_K^+}_{M \text{ terms}}$$

where  $K \in \mathbb{N}$  is the common size of the orbits, and  $M \in \mathbb{N}$  is their number.

PROOF. This follows indeed from definitions. □



Observe that in the classical case, we obtain in this way the usual embedding:

$$G \subset \underbrace{S_K \times \dots \times S_K}_{M \text{ terms}}$$

Let us discuss now the examples. Assume that we are in the following situation:

$$G \subset S_K^+ \hat{*} \dots \hat{*} S_K^+$$

If  $u, v$  are the fundamental corepresentations of  $C(S_N^+), C(S_K^+)$ , consider the quotient map  $\pi_i : C(S_N^+) \rightarrow C(S_K^+)$  constructed as follows:

$$u \rightarrow \text{diag}(1_K, \dots, 1_K, \underbrace{v}_{i\text{-th term}}, 1_K, \dots, 1_K)$$

We can then set  $C(G_i) = \pi_i(C(G))$ , and we have the following result:

**PROPOSITION 13.11.** *If  $G_i$  is transitive for all  $i$ , then  $G$  is quasi-transitive.*

**PROOF.** We know that we have embeddings as follows:

$$G_1 \times \dots \times G_M \subset G \subset \underbrace{S_K^+ \hat{*} \dots \hat{*} S_K^+}_{M \text{ terms}}$$

It follows that the size of any orbit of  $G$  is at least  $K$ , because it contains  $G_1 \times \dots \times G_M$ , and at most  $K$ , because it is contained in  $S_K^+ \hat{*} \dots \hat{*} S_K^+$ . Thus,  $G$  is quasi-transitive.  $\square$

We call the quasi-transitive subgroups appearing as above “of product type”. There are quasi-transitive groups which are not of product type, as for instance:

$$G = S_2 \subset S_2 \times S_2 \subset S_4$$

$$\sigma \rightarrow (\sigma, \sigma)$$

Indeed, the quasi-transitivity is clear, say by letting  $G$  act on the vertices of a square. On the other hand, since we have  $G_1 = G_2 = \{1\}$ , this group is not of product type.

In general, we can construct examples by using various product operations:

**PROPOSITION 13.12.** *Given transitive subgroups  $G_1, \dots, G_M \subset S_K^+$ , the following constructions produce quasi-transitive subgroups as follows, of product type:*

$$G \subset \underbrace{S_K^+ \hat{*} \dots \hat{*} S_K^+}_{M \text{ terms}}$$

- (1) *The usual product:  $G = G_1 \times \dots \times G_M$ .*
- (2) *The dual free product:  $G = G_1 \hat{*} \dots \hat{*} G_M$ .*

**PROOF.** All these assertions are clear from definitions, because in each case, the quantum groups  $G_i \subset S_K^+$  constructed before are those in the statement.  $\square$

In the group dual case, we have the following result:

PROPOSITION 13.13. *The group duals which are of product type*

$$\widehat{\Gamma} \subset \underbrace{S_K^+ \hat{*} \dots \hat{*} S_K^+}_{M \text{ terms}}$$

are precisely those appearing from intermediate groups of the following type:

$$\underbrace{\mathbb{Z}_K * \dots * \mathbb{Z}_K}_{M \text{ terms}} \rightarrow \Gamma \rightarrow \underbrace{\mathbb{Z}_K \times \dots \times \mathbb{Z}_K}_{M \text{ terms}}$$

PROOF. In one sense, this is clear. Conversely, consider a group dual  $\widehat{\Gamma} \subset S_N^+$ , coming from a quotient group  $\mathbb{Z}_K^{*M} \rightarrow \Gamma$ . The subgroups  $G_i \subset \widehat{\Gamma}$  constructed above must be group duals as well,  $G_i = \widehat{\Gamma}_i$ , for certain quotient groups  $\Gamma \rightarrow \Gamma_i$ . Now if  $\widehat{\Gamma}$  is of product type,  $\widehat{\Gamma}_i \subset S_K^+$  must be transitive, and hence equal to  $\widehat{\mathbb{Z}}_K$ . Thus we have  $\Gamma \rightarrow \mathbb{Z}_K^M$ .  $\square$

In order to construct now some other classes of examples, we use the notion of normality for compact quantum groups. This notion, from [57], is introduced as follows:

DEFINITION 13.14. *Given a quantum subgroup  $H \subset G$ , coming from a quotient map  $\pi : C(G) \rightarrow C(H)$ , the following are equivalent:*

- (1) *The following algebra satisfies  $\Delta(A) \subset A \otimes A$ :*

$$A = \left\{ a \in C(G) \mid (id \otimes \pi)\Delta(a) = a \otimes 1 \right\}$$

- (2) *The following algebra satisfies  $\Delta(B) \subset B \otimes B$ :*

$$B = \left\{ a \in C(G) \mid (\pi \otimes id)\Delta(a) = 1 \otimes a \right\}$$

- (3) *We have  $A = B$ , as subalgebras of  $C(G)$ .*

*If these conditions are satisfied, we say that  $H \subset G$  is a normal subgroup.*

Now with this notion in hand, we have, following [32]:

THEOREM 13.15. *Assuming that  $G \subset S_N^+$  is transitive, and that  $H \subset G$  is normal,  $H \subset S_N^+$  follows to be quasi-transitive.*

PROOF. Consider the quotient map  $\pi : C(G) \rightarrow C(H)$ , given at the level of standard coordinates by  $u_{ij} \rightarrow v_{ij}$ . Consider two orbits  $O_1, O_2$  of  $H$  and set:

$$x_i = \sum_{j \in O_1} u_{ij} \quad , \quad y_i = \sum_{j \in O_2} u_{ij}$$

These two elements are orthogonal projections in  $C(G)$  and they are nonzero, because they are sums of nonzero projections by transitivity of  $G$ . We have:

$$\begin{aligned} (id \otimes \pi)\Delta(x_i) &= \sum_k \sum_{j \in O_1} u_{ik} \otimes v_{kj} \\ &= \sum_{k \in O_1} \sum_{j \in O_1} u_{ik} \otimes v_{kj} \\ &= \sum_{k \in O_1} u_{ik} \otimes 1 \\ &= x_i \otimes 1 \end{aligned}$$

Thus by normality of  $H$  we have the following formula:

$$(\pi \otimes id)\Delta(x_i) = 1 \otimes x_i$$

On the other hand, assuming that we have  $i \in O_2$ , we obtain:

$$\begin{aligned} (\pi \otimes id)\Delta(x_i) &= \sum_k \sum_{j \in O_1} v_{ik} \otimes u_{kj} \\ &= \sum_{k \in O_2} v_{ik} \otimes x_k \end{aligned}$$

Multiplying this by  $v_{ik} \otimes 1$  with  $k \in O_2$  yields  $v_{ik} \otimes x_k = v_{ik} \otimes x_i$ , that is to say,  $x_k = x_i$ . In other words,  $x_i$  only depends on the orbit of  $i$ . The same is of course true for  $y_i$ . By using this observation, we can compute the following element:

$$z = \sum_{k \in O_2} \sum_{j \in O_1} u_{kj} = \sum_{k \in O_2} x_k = |O_2|x_i$$

On the other hand, by applying the antipode, we have as well:

$$S(z) = \sum_{k \in O_2} \sum_{j \in O_1} u_{jk} = \sum_{j \in O_1} y_j = |O_1|y_j$$

We therefore obtain the following formula:

$$S(x_i) = \frac{|O_1|}{|O_2|}y_j$$

Now since both  $x_i$  and  $y_j$  have norm one, we conclude that the two orbits have the same size, and this finishes the proof.  $\square$

### 13c. Higher orbitals

Following Lupini, Mančinska, Roberson [75], let us discuss now the higher orbitals. To start with, we have the following standard result, from the classical case:

PROPOSITION 13.16. *Given a subgroup  $G \subset S_N$ , consider its magic unitary:*

$$u_{ij} = \chi \left( \sigma \in G \mid \sigma(j) = i \right)$$

We have then the following equivalence,

$$u_{i_1 j_1} \dots u_{i_k j_k} \neq 0 \iff \exists \sigma \in G, \sigma(i_1) = j_1, \dots, \sigma(i_k) = j_k$$

and these conditions produce an equivalence relation

$$(i_1, \dots, i_k) \sim (j_1, \dots, j_k)$$

whose equivalence classes are the  $k$ -orbitals of  $G$ .

PROOF. The fact that we have indeed an equivalence as in the statement, which produces an equivalence relation, is indeed clear from definitions.  $\square$

In the quantum case, the situation is more complicated. We follow the approach to the orbits and orbitals developed in [75]. We first have:

THEOREM 13.17. *Let  $G \subset S_N^+$  be a closed subgroup, with magic unitary  $u = (u_{ij})$ , and let  $k \in \mathbb{N}$ . The relation*

$$(i_1, \dots, i_k) \sim (j_1, \dots, j_k) \iff u_{i_1 j_1} \dots u_{i_k j_k} \neq 0$$

is then reflexive, symmetric, and transitive at  $k = 1, 2$ .

PROOF. This is known from [75], the proof being as follows:

(1) The reflexivity simply follows by using the counit:

$$\begin{aligned} \varepsilon(u_{i_r i_r}) = 1, \forall r &\implies \varepsilon(u_{i_1 i_1} \dots u_{i_k i_k}) = 1 \\ &\implies u_{i_1 i_1} \dots u_{i_k i_k} \neq 0 \\ &\implies (i_1, \dots, i_k) \sim (i_1, \dots, i_k) \end{aligned}$$

(2) The symmetry follows by applying the antipode, and then the involution:

$$\begin{aligned} (i_1, \dots, i_k) \sim (j_1, \dots, j_k) &\implies u_{i_1 j_1} \dots u_{i_k j_k} \neq 0 \\ &\implies u_{j_k i_k} \dots u_{j_1 i_1} \neq 0 \\ &\implies u_{j_1 i_1} \dots u_{j_k i_k} \neq 0 \\ &\implies (j_1, \dots, j_k) \sim (i_1, \dots, i_k) \end{aligned}$$

(3) The transitivity is something more tricky. We need to prove that we have:

$$u_{i_1 j_1} \dots u_{i_k j_k} \neq 0, u_{j_1 l_1} \dots u_{j_k l_k} \neq 0 \implies u_{i_1 l_1} \dots u_{i_k l_k} \neq 0$$

In order to do so, we use the following formula:

$$\Delta(u_{i_1 l_1} \dots u_{i_k l_k}) = \sum_{s_1 \dots s_k} u_{i_1 s_1} \dots u_{i_k s_k} \otimes u_{s_1 l_1} \dots u_{s_k l_k}$$

At  $k = 1$  the result is clear, because on the right we have a sum of projections, which is therefore strictly positive when one of these projections is nonzero.

At  $k = 2$  now, the result follows from the following trick, from [75]:

$$\begin{aligned} & (u_{i_1 j_1} \otimes u_{j_1 l_1}) \Delta(u_{i_1 l_1} u_{i_2 l_2}) (u_{i_2 j_2} \otimes u_{j_2 l_2}) \\ &= \sum_{s_1 s_2} u_{i_1 j_1} u_{i_1 s_1} u_{i_2 s_2} u_{i_2 j_2} \otimes u_{j_1 l_1} u_{s_1 l_1} u_{s_2 l_2} u_{j_2 l_2} \\ &= u_{i_1 j_1} u_{i_2 j_2} \otimes u_{j_1 l_1} u_{j_2 l_2} \end{aligned}$$

Indeed, we obtain from this that we have  $u_{i_1 l_1} u_{i_2 l_2} \neq 0$ , as desired.  $\square$

In general, the above equivalence relation is not transitive, the basic counterexample at  $k = 3$  being the Kac-Paljutkin quantum group. See [78].

In view of the results that we have so far, we can formulate:

DEFINITION 13.18. *Given a closed subgroup  $G \subset S_N^+$ , consider the relation defined by:*

$$(i_1, \dots, i_k) \sim_k (j_1, \dots, j_k) \iff u_{i_1 j_1} \dots u_{i_k j_k} \neq 0$$

- (1) *The equivalence classes with respect to  $\sim_1$  are called orbits of  $G$ .*
- (2) *The equivalence classes with respect to  $\sim_2$  are called orbitals of  $G$ .*

*In the case where  $\sim_k$  with  $k \geq 3$  happens to be transitive, and so is an equivalence relation, we call its equivalence classes the algebraic  $k$ -orbitals of  $G$ .*

We have as well an analytic approach to this higher orbital problematics, which is particularly useful when  $\sim_k$  is not transitive, that we will explain now.

Let us begin with the following standard result:

PROPOSITION 13.19. *For a subgroup  $G \subset S_N$ , which fundamental corepresentation denoted  $u = (u_{ij})$ , the following numbers are equal:*

- (1) *The number of  $k$ -orbitals.*
- (2) *The dimension of space  $\text{Fix}(u^{\otimes k})$ .*
- (3) *The number  $\int_G \chi^k$ , where  $\chi = \sum_i u_{ii}$ .*

PROOF. This is well-known, the proof being as follows:

(1) = (2) Given  $\sigma \in G$  and vector  $\xi = \sum_{i_1 \dots i_k} \alpha_{i_1 \dots i_k} e_{i_1} \otimes \dots \otimes e_{i_k}$ , we have:

$$\begin{aligned} \sigma^{\otimes k} \xi &= \sum_{i_1 \dots i_k} \alpha_{i_1 \dots i_k} e_{\sigma(i_1)} \otimes \dots \otimes e_{\sigma(i_k)} \\ \xi &= \sum_{i_1 \dots i_k} \alpha_{\sigma(i_1) \dots \sigma(i_k)} e_{\sigma(i_1)} \otimes \dots \otimes e_{\sigma(i_k)} \end{aligned}$$

Thus  $\sigma^{\otimes k} \xi = \xi$  holds for any  $\sigma \in G$  precisely when  $\alpha$  is constant on the  $k$ -orbitals of  $G$ , and this gives the equality between the numbers in (1) and (2).

(2) = (3) This follows from the Peter-Weyl theory, because  $\chi = \sum_i u_{ii}$  is the character of the fundamental corepresentation  $u$ .  $\square$

In the quantum case now,  $G \subset S_N^+$ , by the general Peter-Weyl type results established by Woronowicz in [98], we still have the following formula:

$$\dim \text{Fix}(u^{\otimes k}) = \int_G \chi^k$$

The problem is that of understanding the  $k$ -orbital interpretation of this number. We first have the following result, basically coming from [44], [75]:

PROPOSITION 13.20. *Given a closed subgroup  $G \subset S_N^+$ , and a number  $k \in \mathbb{N}$ , consider the following linear space:*

$$F_k = \left\{ \xi \in (\mathbb{C}^N)^{\otimes k} \mid \xi_{i_1 \dots i_k} = \xi_{j_1 \dots j_k}, \forall (i_1, \dots, i_k) \sim (j_1, \dots, j_k) \right\}$$

- (1) We have  $F_k \subset \text{Fix}(u^{\otimes k})$ .
- (2) At  $k = 1, 2$  we have  $F_k = \text{Fix}(u^{\otimes k})$ .
- (3) In the classical case, we have  $F_k = \text{Fix}(u^{\otimes k})$ .
- (4) For  $G = S_N^+$  with  $N \geq 4$  we have  $F_3 \neq \text{Fix}(u^{\otimes 3})$ .

PROOF. The tensor power  $u^{\otimes k}$  being the corepresentation  $(u_{i_1, \dots, i_k, j_1, \dots, j_k})_{i_1 \dots i_k, j_1 \dots j_k}$ , the corresponding fixed point space  $\text{Fix}(u^{\otimes k})$  consists of the vectors  $\xi$  satisfying:

$$\sum_{j_1 \dots j_k} u_{i_1 j_1} \dots u_{i_k j_k} \xi_{j_1 \dots j_k} = \xi_{i_1 \dots i_k} \quad , \quad \forall i_1, \dots, i_k$$

With this formula in hand, the proof goes as follows:

- (1) Assuming  $\xi \in F_k$ , the above fixed point formula holds indeed, because:

$$\begin{aligned} \sum_{j_1 \dots j_k} u_{i_1 j_1} \dots u_{i_k j_k} \xi_{j_1 \dots j_k} &= \sum_{j_1 \dots j_k} u_{i_1 j_1} \dots u_{i_k j_k} \xi_{i_1 \dots i_k} \\ &= \xi_{i_1 \dots i_k} \end{aligned}$$

- (2) This is something more tricky, coming from the following formulae:

$$u_{ik} \left( \sum_j u_{ij} \xi_j - \xi_i \right) = u_{ik} (\xi_k - \xi_i)$$

$$u_{i_1 k_1} \left( \sum_{j_1 j_2} u_{i_1 j_1} u_{i_2 j_2} \xi_{j_1 j_2} - \xi_{i_1 i_2} \right) u_{i_2 k_2} = u_{i_1 k_1} u_{i_2 k_2} (\xi_{k_1 k_2} - \xi_{i_1 i_2})$$

- (3) This follows indeed from Proposition 13.19 above.

(4) This follows from the representation theory of  $S_N^+$  with  $N \geq 4$ , and from some elementary computations, the dimensions of the two spaces involved being  $4 < 5$ . To be

more precise, let us start with the symmetric group  $S_N$ . It follows from definitions that the  $k$ -orbitals are indexed by the partitions  $\pi \in P(k)$ , as follows:

$$C_\pi = \left\{ (i_1, \dots, i_k) \mid \ker i = \pi \right\}$$

In particular at  $k = 3$  we have 5 such orbitals, corresponding to:

$$\square\square \quad , \quad \square | \quad , \quad | \square \quad , \quad \sqsupset \quad , \quad |||$$

Regarding now  $S_N^+$ , the 3-orbitals are exactly as for  $S_N$ , except for the fact that the  $\sqsupset$  and  $|||$  3-orbitals get merged. Thus, we have 4 such orbitals, corresponding to:

$$\square\square \quad , \quad \square | \quad , \quad | \square \quad , \quad \overline{|||}$$

As for the number of analytic orbitals, this is the same as for  $S_N$ , namely 5. □

The above considerations suggest formulating the following definition:

DEFINITION 13.21. *Given a closed subgroup  $G \subset U_N^+$ , the integer*

$$\dim \text{Fix}(u^{\otimes k}) = \int_G \chi^k$$

*is called number of analytic  $k$ -orbitals.*

To be more precise, in the classical case the situation is of course well understood, and this is the number of  $k$ -orbitals. The same goes for the general case, with  $k = 1, 2$ , where this is the number of  $k$ -orbitals. At  $k = 3$  and higher, however, even in the case where the algebraic 3-orbitals are well-defined, their number is not necessarily the above one.

In the particular case  $k = 3$ , we have as well the following result, which brings some more support for the above definition:

PROPOSITION 13.22. *For a closed subgroup  $G \subset S_N^+$ , and an integer  $k \leq 3$ , the following conditions are equivalent:*

- (1)  *$G$  is  $k$ -transitive, in the sense that  $\text{Fix}(u^{\otimes k})$  has dimension 1, 2, 5.*
- (2) *The  $k$ -th moment of the main character is  $\int_G \chi^k = 1, 2, 5$ .*
- (3) *We have the integration formula*

$$\int_G u_{i_1 j_1} \dots u_{i_k j_k} = \frac{(N - k)!}{N!}$$

*for distinct indices  $i_r$  and distinct indices  $j_r$ .*

- (4) *An arbitrary polynomial integral*

$$\int_G u_{i_1 j_1} \dots u_{i_k j_k}$$

*equals  $\frac{(N - |\ker i|)!}{N!}$  when  $\ker i = \ker j$ , and equals 0, otherwise.*

PROOF. Most of these implications are known, the idea being as follows:

(1)  $\iff$  (2) This follows from the Peter-Weyl type theory from [98], because the  $k$ -th moment of the character counts the number of fixed points of  $u^{\otimes k}$ .

(2)  $\iff$  (3) This follows from the Schur-Weyl duality results for  $S_N, S_N^+$  and from  $P(k) = NC(k)$  at  $k \leq 3$ .

(3)  $\iff$  (4) Once again this follows from  $P(k) = NC(k)$  at  $k \leq 3$ , and from a standard integration result for  $S_N$ .  $\square$

As a conclusion to all these considerations, we have:

**THEOREM 13.23.** *For a closed subgroup  $G \subset S_N^+$ , and an integer  $k \in \mathbb{N}$ , the number  $\dim(\text{Fix}(u^{\otimes k})) = \int_G \chi^k$  of “analytic  $k$ -orbitals” has the following properties:*

- (1) *In the classical case, this is the number of  $k$ -orbitals.*
- (2) *In general, at  $k = 1, 2$ , this is the number of  $k$ -orbitals.*
- (3) *At  $k = 3$ , when this number is minimal,  $G$  is 3-transitive in the above sense.*

PROOF. This follows indeed from the above considerations.  $\square$

### 13d. Finite subgroups

Let us discuss now an alternative take on these questions, in the finite quantum group case. We start with the following standard definition:

**DEFINITION 13.24.** *Associated to any finite quantum group  $F$  is its dual finite quantum group  $G = \widehat{F}$ , given by  $C(G) = C(F)^*$ , with Hopf  $C^*$ -algebra structure as follows:*

- (1) *Multiplication  $(\varphi\psi)a = (\varphi \otimes \psi)\Delta(a)$ .*
- (2) *Unit  $1 = \varepsilon$ .*
- (3) *Involution  $\varphi^*(a) = \overline{\varphi(S(a)^*)}$ .*
- (4) *Comultiplication  $(\Delta\varphi)(a \otimes b) = \varphi(ab)$ .*
- (5) *Counit  $\varepsilon(\varphi) = \varphi(1)$ .*
- (6) *Antipode  $(S\varphi)a = \varphi(S(a))$ .*

Our aim in what follows will be that of reformulating in terms of  $G = \widehat{F}$  the condition  $F \subset S_N^+$ . We will see later how this can potentially help, by dropping the assumption that  $F, G$  are finite, in connection with various quantum permutation group questions.

In order to get started, we have the following well-known fact:

**PROPOSITION 13.25.** *Given  $F$  and  $G = \widehat{F}$  as in Definition 13.24, the formula*

$$\begin{aligned} \pi : C(G) &\rightarrow M_N(\mathbb{C}) \\ \varphi &\rightarrow [\varphi(u_{ij})]_{ij} \end{aligned}$$

*defines a  $*$ -algebra representation precisely when  $u$  is a corepresentation.*



PROOF. In one sense, the fact that  $\pi$  is multiplicative follows from the fact that  $u$  is comultiplicative, in the sense that  $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ , as follows:

$$\begin{aligned} \pi(\varphi\psi) &= [(\varphi\psi)u_{ij}]_{ij} \\ &= [(\varphi \otimes \psi)\Delta(u_{ij})]_{ij} \\ &= \left[ \sum_k \varphi(u_{ik})\psi(u_{kj}) \right]_{ij} \\ &= [\varphi(u_{ij})]_{ij} [\psi(u_{ij})]_{ij} \\ &= \pi(\varphi)\pi(\psi) \end{aligned}$$

The fact that the morphism  $\pi$  constructed above is unital is clear, coming from the fact that  $u$  is counital, in the sense that  $\varepsilon(u_{ij}) = \delta_{ij}$ , as follows:

$$\pi(\varepsilon) = [\varepsilon(u_{ij})]_{ij} = 1$$

Regarding now the fact that  $\pi$  is involutive, observe first that we have:

$$\varphi^*(u_{ij}) = \overline{\varphi(S(u_{ij})^*)} = \overline{\varphi(u_{ji})}$$

Thus, we can prove that  $\pi$  is indeed involutive, as follows, using the fact that  $u$  is coinvolutive, in the sense that  $S(u_{ij}) = u_{ji}^*$ , as follows:

$$\begin{aligned} \pi(\varphi^*) &= [\varphi^*(u_{ij})]_{ij} \\ &= [\overline{\varphi(u_{ji})}]_{ij} \\ &= \left[ [\varphi(u_{ij})]_{ij} \right]^* \\ &= \pi(\varphi)^* \end{aligned}$$

Finally, the proof in the other sense follows from exactly the same computations.  $\square$

In order to reach now to the condition  $F \subset S_N^+$ , we must impose several conditions on the matrix  $u = (u_{ij})$ . Let us start with the bistochasticity condition. We have here:

PROPOSITION 13.26. *Given  $F$  and  $G = \widehat{F}$  as in Proposition 13.25, the matrix  $u = (u_{ij})$  is bistochastic, in the sense that all the row and column sums are 1, precisely when the associated  $*$ -algebra representation  $\pi : C(G) \rightarrow M_N(\mathbb{C})$  satisfies the conditions*

$$\begin{aligned} \pi(\varphi)\xi &= \varphi(1)\xi \\ \pi(\varphi)^t\xi &= \varphi(1)\xi \end{aligned}$$

where  $\xi \in \mathbb{C}^N$  is the all-one vector.

PROOF. We want the following two conditions to be satisfied:

$$\sum_j u_{ij} = 1 \quad , \quad \sum_i u_{ij} = 1$$

In what regards the condition  $\sum_j u_{ij} = 1$ , observe that in terms of  $\pi$ , we have:

$$\begin{aligned} \sum_j \pi(\varphi)_{ij} &= \sum_j \varphi(u_{ij}) \\ &= \varphi\left(\sum_j u_{ij}\right) \end{aligned}$$

Thus, we want this quantity to be  $\varphi(1)$ , for any  $i$ , and this leads to the condition  $\pi(\varphi)\xi = \varphi(1)\xi$  in the statement. As for the second condition, namely  $\sum_i u_{ij} = 1$ , this leads to the second condition in the statement, namely  $\pi(\varphi)^t\xi = \varphi(1)\xi$ .  $\square$

Independently of the above result, we must impose the condition that the coordinates  $u_{ij}$  are self-adjoint. The result here is as follows:

**PROPOSITION 13.27.** *Given  $F$  and  $G = \widehat{F}$  as in Proposition 13.25, we have  $u_{ij} = u_{ij}^*$  precisely when the associated  $*$ -algebra representation  $\pi : C(G) \rightarrow M_N(\mathbb{C})$  satisfies:*

$$\pi S(\varphi) = \pi(\varphi)^t$$

**PROOF.** According to the antipode formula  $(S\varphi)a = \varphi(S(a))$  from Definition 13.24, we have the following computation:

$$\begin{aligned} \pi S(\varphi) &= [S\varphi(u_{ij})]_{ij} \\ &= [\varphi(u_{ji}^*)]_{ij} \end{aligned}$$

With this formula in hand, we see that the condition  $u_{ij} = u_{ij}^*$  means that this latter matrix should be  $[\varphi(u_{ji})]_{ij} = \pi(\varphi)^t$ , as claimed.  $\square$

Let us put now what we have together. We are led to the following statement:

**PROPOSITION 13.28.** *Given  $F$  and  $G = \widehat{F}$  as in Proposition 13.25,  $u = (u_{ij})$  is bistochastic, with self-adjoint entries, precisely when associated  $*$ -algebra representation*

$$\begin{aligned} \pi : C(G) &\rightarrow M_N(\mathbb{C}) \\ \varphi &\rightarrow [\varphi(u_{ij})]_{ij} \end{aligned}$$

*satisfying the following conditions,*

$$\begin{aligned} \pi(\varphi)\xi &= \varphi(1)\xi \\ \pi(\varphi)^t\xi &= \varphi(1)\xi \\ \pi S(\varphi) &= \pi(\varphi)^t \end{aligned}$$

*with  $\xi \in \mathbb{C}^N$  being the all-one vector.*

**PROOF.** This follows indeed from Proposition 13.26 and Proposition 13.27.  $\square$

In order to reach now to  $F \subset S_N^+$ , we must impose one final condition, stating that the entries of  $u = (u_{ij})$  are idempotents,  $u_{ij}^2 = u_{ij}$ . This is something more technical:

PROPOSITION 13.29. *Given  $F$  and  $G = \widehat{F}$  as in Proposition 13.28, we have  $u_{ij}^2 = u_{ij}$  precisely when the associated  $*$ -algebra representation  $\pi : C(G) \rightarrow M_N(\mathbb{C})$  satisfies*

$$m(\pi \otimes \pi)\Delta(\varphi) = \pi(\varphi)m$$

as an equality of maps  $\mathbb{C}^N \otimes \mathbb{C}^N \rightarrow \mathbb{C}^N$ , where  $m$  is the multiplication of  $\mathbb{C}^N$ .

PROOF. This is something which is quite routine. We have indeed the following computation, valid for any indices  $i, j$ , by using the Sweedler notation:

$$\begin{aligned} m(\pi \otimes \pi)\Delta(\varphi)(e_i \otimes e_j) &= m(\pi \otimes \pi) \left( \sum \varphi_1 \otimes \varphi_2 \right) (e_i \otimes e_j) \\ &= m \left( \sum \pi(\varphi_1) \otimes \pi(\varphi_2) \right) (e_i \otimes e_j) \\ &= m \left( \sum \pi(\varphi_1)e_i \otimes \pi(\varphi_2)e_j \right) \\ &= m \left( \sum \sum_{kl} \varphi_1(u_{ki})e_k \otimes \varphi_2(u_{lj})e_l \right) \\ &= \sum \sum_{kl} \varphi_1(u_{ki})e_k \varphi_2(u_{lj})e_l \\ &= \sum \sum_k \varphi_1(u_{ki})\varphi_2(u_{kj})e_k \\ &= \sum \sum_k (\varphi_1 \otimes \varphi_2)(u_{ki} \otimes u_{kj})e_k \\ &= \sum_k \Delta(\varphi)(u_{ki} \otimes u_{kj})e_k \\ &= \sum_k \varphi(u_{ki}u_{kj})e_k \end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned} \pi(\varphi)m(e_i \otimes e_j) &= \pi(\varphi)\delta_{ij}e_i \\ &= [\varphi(u_{ij})]_{ij}\delta_{ij}e_i \\ &= \delta_{ij} \sum_k \varphi(u_{ki})e_k \end{aligned}$$

Thus, the condition in the statement simply reads  $u_{ki}u_{kj} = \delta_{ij}u_{ki}$ , for any  $i, j, k$ . In particular with  $i = j$  we obtain, as desired, the idempotent condition:

$$u_{ki}^2 = u_{ki}$$

Conversely now, if this idempotent condition is satisfied, then  $u = (u_{ij})$  follows to be a matrix of projections, which is bistochastic. Thus this matrix is magic, and so we have  $u_{ki}u_{kj} = \delta_{ij}u_{ki}$  for any  $i, j, k$ , and this leads to the formula in the statement.  $\square$

Let us put now what we have together. We are led to the following statement:

**THEOREM 13.30.** *Given  $F$  and  $G = \widehat{F}$  as in Definition 13.24, we have  $F \subset S_N^+$ , with associated magic matrix  $u = (u_{ij})$ , precisely when we have a  $*$ -algebra representation*

$$\begin{aligned}\pi : C(G) &\rightarrow M_N(\mathbb{C}) \\ \varphi &\rightarrow [\varphi(u_{ij})]_{ij}\end{aligned}$$

satisfying the following conditions,

$$\begin{aligned}\pi(\varphi)\xi &= \varphi(1)\xi \\ \pi(\varphi)^t\xi &= \varphi(1)\xi \\ \pi S(\varphi) &= \pi(\varphi)^t \\ m(\pi \otimes \pi)\Delta(\varphi) &= \pi(\varphi)m\end{aligned}$$

where  $\xi \in \mathbb{C}^N$  is the all-one vector, and  $m$  is the multiplication of  $\mathbb{C}^N$ .

**PROOF.** This follows indeed from Proposition 13.28 and Proposition 13.29, and from the well-known fact, already mentioned in the proof of Proposition 13.29, that a magic matrix  $u = (u_{ij})$  is the same as a matrix of projections which is bistochastic.  $\square$

As a first illustration, in the classical case, we have:

**PROPOSITION 13.31.** *Given a closed subgroup  $F \subset U_N$ , the associated  $*$ -algebra representation constructed in Theorem 13.30 is given by*

$$\begin{aligned}\pi : C^*(F) &\rightarrow M_N(\mathbb{C}) \\ \sum_g \lambda_g g &\rightarrow \sum_g \lambda_g g\end{aligned}$$

and we have  $F \subset S_N$  precisely when the conditions in Theorem 13.30 are satisfied.

**PROOF.** Here the first assertion is clear from definitions. As for the second assertion, this is something that we know from Theorem 13.30, but here is a direct check as well:

(1) For  $\varphi \in C^*(F)$  given by  $\varphi = \sum_g \lambda_g g$  we have  $\pi(\varphi) = \sum_g \lambda_g g$ , and also  $\varphi(1) = \sum_g \lambda_g$  via  $C^*(F) \simeq C(F)^*$  so the bistochasticity condition  $F \subset C_N$  corresponds indeed to the conditions  $\pi(\varphi)\xi = \varphi(1)\xi$  and  $\pi(\varphi)^t\xi = \varphi(1)\xi$  from Theorem 13.30.

(2) Once again with  $\varphi = \sum_g \lambda_g g$ , we have the following formulae:

$$\begin{aligned}\pi S\varphi &= \pi \left( \sum_g \lambda_g g^{-1} \right) = \sum_g \lambda_g g^{-1} \\ \pi(\varphi)^t &= \left( \sum_g \lambda_g g \right)^t = \sum_g \lambda_g g^t\end{aligned}$$

Thus  $F \subset O_N$ , which is the same as saying that  $g^{-1} = g^t$ , for any  $g \in F$ , is indeed equivalent to the condition  $\pi S(\varphi) = \pi(\varphi)^t$  from Theorem 13.30.

(3) As before with  $\varphi = \sum_g \lambda_g g$ , assuming  $F \subset S_N$ , we have the following formula:

$$\begin{aligned} m(\pi \otimes \pi)\Delta(\varphi)(e_i \otimes e_j) &= m\left(\sum_g \lambda_g g \otimes g\right)(e_i \otimes e_j) \\ &= m\left(\sum_g \lambda_g e_{g(i)} \otimes e_{g(j)}\right) \\ &= \delta_{ij} \sum_{g \in G} \lambda_g e_{g(i)} \end{aligned}$$

On the other hand, we have as well the following formula:

$$\begin{aligned} \pi(\varphi)m(e_i \otimes e_j) &= \left(\sum_g \lambda_g g\right)m(e_i \otimes e_j) \\ &= \left(\sum_g \lambda_g g\right)(\delta_{ij} e_i) \\ &= \delta_{ij} \sum_g \delta_g e_{g(i)} \end{aligned}$$

Thus the condition  $m(\pi \otimes \pi)\Delta(\varphi) = \pi(\varphi)m$  in Theorem 13.30 must be indeed satisfied, and the proof of the converse is similar, using the same computations.  $\square$

In the group dual case now, the result is a priori something more subtle, related to Bichon's classification in [44] of the group dual subgroups  $\widehat{\Gamma} \subset S_N^+$ . However, and here comes our point, in the present dual setting everything drastically simplifies, and the complete result, with complete proof, is as follows:

**THEOREM 13.32.** *Given a finite group  $G$ , and setting  $F = \widehat{G}$ , the associated  $*$ -algebra representation constructed in Theorem 13.30 appears as follows, for a certain family of generators  $g_1, \dots, g_N \in H$ , and for a certain unitary  $U \in U_N$ ,*

$$\begin{aligned} \pi : C(G) &\rightarrow M_N(\mathbb{C}) \\ \varphi &\rightarrow U \operatorname{diag}(g_1, \dots, g_N) U^* \end{aligned}$$

and we have  $F \subset S_N^+$  precisely when the conditions in Theorem 13.30 are satisfied, which in turn mean that the representation  $\pi$  appears as in [44].

**PROOF.** Here the first assertion is standard, coming from Woronowicz's Peter-Weyl type theory from [97]. As for the second assertion, this is a priori something which is less obvious, related to Bichon's classification of the group dual subgroups  $\widehat{\Gamma} \subset S_N^+$  in

[44]. However, in our dual formulation this is something clear, because the algebra  $C(G)$  is commutative, so its matrix representation  $\pi$  must appear diagonally, spinned by a unitary. Thus, we obtain the result, without a single computation needed.  $\square$

There are many things that can be done with finite quantum permutation groups, that can be sometimes simpler by using the present dual formalism.

In order to discuss this, let us start with:

PROPOSITION 13.33. *Given  $G = \widehat{F}$  as in Theorem 13.30, the  $*$ -algebra representation*

$$\pi : C(G) \rightarrow M_N(\mathbb{C})$$

*gives rise to a family of  $*$ -algebra representations as follows, for any  $k \in \mathbb{N}$ ,*

$$\begin{aligned} \pi^k : C(G) &\rightarrow M_N(\mathbb{C})^{\otimes k} \\ \pi^k &= \pi^{\otimes k} \Delta^{(k)} \end{aligned}$$

*that we will still denote by  $\pi$ , when there is no confusion, which are given by*

$$\pi_{i_1 \dots i_k, j_1 \dots j_k}(\varphi) = \varphi(u_{i_1 j_1} \dots u_{i_k j_k})$$

*in standard multi-index notation for the elements of  $M_N(\mathbb{C})^{\otimes k}$ .*

PROOF. Let us begin with the following computation, in Sweedler notation:

$$\begin{aligned} &\langle \pi^{\otimes k} \Delta^{(k)}(\varphi)(e_{j_1} \otimes \dots \otimes e_{j_k}), e_{i_1} \otimes \dots \otimes e_{i_k} \rangle \\ &= \left\langle \pi^{\otimes k} \left( \sum \varphi_1 \otimes \dots \otimes \varphi_k \right) (e_{j_1} \otimes \dots \otimes e_{j_k}), e_{i_1} \otimes \dots \otimes e_{i_k} \right\rangle \\ &= \sum \langle (\pi(\varphi_1) \otimes \dots \otimes \pi(\varphi_k))(e_{j_1} \otimes \dots \otimes e_{j_k}), e_{i_1} \otimes \dots \otimes e_{i_k} \rangle \\ &= \sum \langle \pi(\varphi_1)e_{j_1} \otimes \dots \otimes \pi(\varphi_k)e_{j_k}, e_{i_1} \otimes \dots \otimes e_{i_k} \rangle \\ &= \sum \langle \pi(\varphi_1)e_{j_1}, e_{i_1} \rangle \dots \langle \pi(\varphi_k)e_{j_k}, e_{i_k} \rangle \\ &= \sum \varphi_1(u_{i_1 j_1}) \dots \varphi_k(u_{i_k j_k}) \\ &= \sum (\varphi_1 \otimes \dots \otimes \varphi_k)(u_{i_1 j_1} \otimes \dots \otimes u_{i_k j_k}) \\ &= \Delta^{(k)}(\varphi)(u_{i_1 j_1} \dots u_{i_k j_k}) \\ &= \varphi(u_{i_1 j_1} \dots u_{i_k j_k}) \end{aligned}$$

Thus, we have the following formula, valid at any  $k \in \mathbb{N}$ :

$$\pi^{\otimes k} \Delta^{(k)}(\varphi) = [\varphi(u_{i_1 j_1} \dots u_{i_k j_k})]_{i_1 \dots i_k, j_1 \dots j_k}$$

Equivalently, the representation  $\pi = \pi^{\otimes k} \Delta^{(k)}$  is given by the following formula:

$$\pi_{i_1 \dots i_k, j_1 \dots j_k}(\varphi) = \varphi(u_{i_1 j_1} \dots u_{i_k j_k})$$

Thus, we are led to the conclusion in the statement.  $\square$

Following [23], let us discuss now integration results. We have:

**THEOREM 13.34.** *The polynomial integrals over  $G$  are given by*

$$\left[ \int u_{i_1 j_1} \dots u_{i_k j_k} \right]_{i_1 \dots i_k, j_1 \dots j_k} = \pi^{\otimes k} \Delta^{(k)}(f)$$

and the moments of the main character  $\chi = \sum_i u_{ii}$  are given by

$$\int \chi^k = \text{Tr}(\pi^{\otimes k} \Delta^{(k)}(f))$$

where  $\int \in C(G)$  is the Haar integration functional.

**PROOF.** The first formula is clear from Proposition 13.33. Regarding now the moments of the main character, observe first that we have the following general formula:

$$\begin{aligned} \text{Tr}(\pi^{\otimes k} \Delta^{(k)}(\varphi)) &= \sum_{i_1 \dots i_k} \varphi(u_{i_1 i_1} \dots u_{i_k i_k}) \\ &= \varphi \left( \sum_{i_1 \dots i_k} u_{i_1 i_1} \dots u_{i_k i_k} \right) \\ &= \varphi(\chi^k) \end{aligned}$$

In particular, with  $\varphi = \int$ , the Haar integration, we obtain:

$$\text{Tr}(\pi^{\otimes k} \Delta^{(k)}(f)) = \int \chi^k$$

Thus, we are led to the conclusions in the statement.  $\square$

As a second topic, which is of key interest, let us discuss the orbit and orbital theory, following [44], [75]. Regarding the orbits, following [44] we have:

**THEOREM 13.35.** *The orbits of  $F \subset S_N^+$  can be defined dually by  $i \sim j$  when*

$$\pi_{ij}(\varphi) > 0$$

for a certain positive linear form  $\varphi > 0$ .

**PROOF.** We know from [44] that  $i \sim j$  when  $u_{ij} \neq 0$  is an equivalence relation on  $\{1, \dots, N\}$ . Here is a proof of this fact, using our present, dual formalism:

(1) The reflexivity of  $\sim$  as defined in the statement is clear, coming from:

$$\pi(1) = 1 \implies \pi_{ii} \neq 0$$

(2) The symmetry is clear too, coming from  $\pi S(\varphi) = \pi(\varphi)^t$ . Alternatively:

$$\pi(\varphi^*) = \pi(\varphi)^* \implies \pi_{ij}(\varphi^*) = \overline{\pi_{ij}(\varphi)}$$

(3) Regarding now the transitivity, things are a bit more tricky. We have:

$$\pi_{ij}(\varphi\psi) = \sum_k \pi_{ik}(\varphi)\pi_{kj}(\psi)$$

Now since  $\varphi \geq 0$  implies  $\varphi(u_{ij}) \geq 0$  for any  $i, j$ , we obtain the result. □

Regarding the orbitals, following [75], we first have:

PROPOSITION 13.36. *The relation on  $\{1, \dots, N\}^k$  given by  $i \sim j$  when*

$$\pi_{i_1 \dots i_k, j_1 \dots j_k}(\varphi) > 0$$

*for a certain positive linear form  $\varphi > 0$ , is reflexive and symmetric.*

PROOF. The reflexivity is clear exactly as at  $k = 1$ , coming from:

$$\pi(1) = 1 \implies \pi_{i_1 \dots i_k, i_1 \dots i_k} \neq 0$$

The symmetry is clear too, coming from  $\pi S(\varphi) = \pi(\varphi)^t$ . Alternatively:

$$\pi(\varphi^*) = \pi(\varphi)^* \implies \pi_{i_1 \dots i_k, j_1 \dots j_k}(\varphi^*) = \overline{\pi_{i_1 \dots i_k, j_1 \dots j_k}(\varphi)}$$

Thus, we are led to the conclusion in the statement. □

Regarding the transitivity of the relation constructed above, things here are quite tricky, and we do not have something simple, using our dual formalism. Nevertheless, following [75], [78], let us formulate an informal statement, as follows:

THEOREM 13.37. *The relation constructed in Proposition 13.36 is transitive at  $k = 2$ , not necessarily transitive at  $k \geq 3$ , and these results can be both recovered in dual form.*

PROOF. As mentioned, all this is quite complicated, the situation being as follows:

(1) It is known from [75] that we have transitivity at  $k = 2$ , the proof being something tricky, but fairly short, as follows:

$$\begin{aligned} & (u_{i_1 j_1} \otimes u_{j_1 l_1}) \Delta(u_{i_1 l_1} u_{i_2 l_2})(u_{i_2 j_2} \otimes u_{j_2 l_2}) \\ &= \sum_{s_1 s_2} u_{i_1 j_1} u_{i_1 s_1} u_{i_2 s_2} u_{i_2 j_2} \otimes u_{j_1 l_1} u_{s_1 l_1} u_{s_2 l_2} u_{j_2 l_2} \\ &= u_{i_1 j_1} u_{i_2 j_2} \otimes u_{j_1 l_1} u_{j_2 l_2} \end{aligned}$$

Indeed, we obtain from this that we have  $u_{i_1 l_1} u_{i_2 l_2} \neq 0$ , as desired.

(2) The challenge now is to reformulate the above proof from [75] in the dual setting, somehow by applying linear forms on both sides. This is something non-trivial, and a quite technical proof, using a conditioning method, and totalling about 1 page or so, can be found in [78]. It is not clear on how to simplify that proof.

(3) As good news however, the fact that we don't necessarily have transitivity at  $k = 3$ , which was something conjectured in [75] and in follow-up papers, and not accessible with the methods there, was recently done in [78], using dual methods. □



**13e. Exercises**

We have seen in this chapter how to deal with the arbitrary subgroups  $G \subset S_N^+$ , by using orbitals, group duals, and related techniques. As a first exercise, we have:

EXERCISE 13.38. *Formulate and study a notion of transitivity for the subgroups*

$$G \subset S_N^+$$

*by requiring that the corresponding action  $G \curvearrowright \{1, \dots, N\}$  has only 1 orbit.*

There are many things that can be done here, in analogy with the theory of the transitive subgroups  $G \subset S_N$ , and we will actually be back to this later.

Along the same lines, but at a more advanced level, we have:

EXERCISE 13.39. *Formulate and study a notion of  $k$ -transitivity for the subgroups*

$$G \subset S_N^+$$

*by requiring that the action  $G \curvearrowright \{1, \dots, N\}$  has a minimal number of  $k$ -orbitals.*

As before, there are many things that can be done here, in analogy with the theory of the higher transitive subgroups  $G \subset S_N$ , but in the quantum case some of the things will work only at  $k = 1$ , or  $k \leq 2$ , or  $k \leq 3$ . We will be back to all this.

Finally, in relation with the group dual approach, we have:

EXERCISE 13.40. *Extend the theory of the duals  $G = \widehat{F}$  of the finite subgroups*

$$F \subset S_N^+$$

*developed here, notably by solving the orbital problem formulated above.*

As with the previous exercises, this is something quite tricky, partly going into unexplored territory, and the more the results, the better.

## CHAPTER 14

### Transitive groups

#### 14a. Transitivity

We have seen in the previous chapter that a theory of orbits and orbitals can be developed for the closed subgroups  $G \subset S_N^+$ , and that all this is particularly interesting in connection with tori. In this chapter we restrict the attention to the transitive case.

Let us first review the basic theory, that we will need in what follows. The notion of transitivity, which goes back to Bichon's paper [44], can be introduced as follows:

**DEFINITION 14.1.** *Let  $G \subset S_N^+$  be a closed subgroup, with magic unitary  $u = (u_{ij})$ , and consider the equivalence relation on  $\{1, \dots, N\}$  given by  $i \sim j \iff u_{ij} \neq 0$ .*

- (1) *The equivalence classes under  $\sim$  are called orbits of  $G$ .*
- (2)  *$G$  is called transitive when the action has a single orbit.*

*In other words, we call a subgroup  $G \subset S_N^+$  transitive when  $u_{ij} \neq 0$ , for any  $i, j$ .*

This transitivity notion is standard, coming in a straightforward way from the orbit theory. In the classical case, we obtain of course the usual notion of transitivity.

We will need as well the following result, once again coming from [44]:

**THEOREM 14.2.** *For a closed subgroup  $G \subset S_N^+$ , the following are equivalent:*

- (1)  *$G$  is transitive.*
- (2)  *$Fix(u) = \mathbb{C}\xi$ , where  $\xi$  is the all-one vector.*
- (3)  *$\int_G u_{ij} = \frac{1}{N}$ , for any  $i, j$ .*

**PROOF.** This is well-known in the classical case. In general, the proof is as follows:

(1)  $\iff$  (2) We use the standard fact that the fixed point space of a corepresentation coincides with the fixed point space of the associated coaction:

$$Fix(u) = Fix(\Phi)$$

As explained in chapter 13 above, the fixed point space of the magic corepresentation  $u = (u_{ij})$  has the following interpretation, in terms of orbits:

$$Fix(u) = \left\{ \xi \in C(X) \mid i \sim j \implies \xi(i) = \xi(j) \right\}$$

In particular, the transitivity condition corresponds to  $Fix(u) = \mathbb{C}\xi$ , as stated.

(2)  $\iff$  (3) This is clear from the general properties of the Haar integration, and more precisely from the fact that  $(\int_G u_{ij})_{ij}$  is the projection onto  $Fix(u)$ .  $\square$

Here is now a list of basic examples of transitive quantum groups, that we already know, coming from the various constructions from the previous chapters:

**THEOREM 14.3.** *The following are transitive subgroups  $G \subset S_N^+$ :*

- (1) *The quantum permutation group  $S_N^+$  itself.*
- (2) *The transitive subgroups  $G \subset S_N$ . These are the classical examples.*
- (3) *The subgroups  $\widehat{G} \subset S_{|G|}$ , with  $G$  abelian. These are the group dual examples.*
- (4) *The quantum groups  $F \subset S_N^+$  which are finite,  $|F| < \infty$ , and transitive.*
- (5) *The quantum automorphism groups of transitive graphs  $G^+(X)$ , with  $|X| = N$ .*
- (6) *In particular, the hyperoctahedral quantum group  $H_n^+ \subset S_N^+$ , with  $N = 2n$ .*
- (7) *We have as well the twisted orthogonal group  $O_n^{-1} \subset S_N^+$ , with  $N = 2^n$ .*

*In addition, the class of transitive quantum permutation groups  $\{G \subset S_N^+ | N \in \mathbb{N}\}$  is stable under direct products  $\times$ , wreath products  $\wr$  and free wreath products  $\wr_*$ .*

**PROOF.** All these assertions are well-known. In what follows we briefly describe the idea of each proof, and indicate a reference. We will be back to all these examples, gradually, in the context of other questions, to be formulated later on.

(1) This comes from the fact that we have an inclusion  $S_N \subset S_N^+$ . Indeed, since  $S_N$  is transitive, so must be  $S_N^+$ , because its coordinates  $u_{ij}$  map to those of  $S_N$ . See [32].

(2) This is again trivial. Indeed, for a classical group  $G \subset S_N$ , the variables  $u_{ij} = \chi(\sigma \in S_N | \sigma(j) = i)$  are all nonzero precisely when  $G$  is transitive. See [32].

(3) This follows from the general results of Bichon in [44], who classified there all the group dual subgroups  $\widehat{\Gamma} \subset S_N^+$ . For a discussion here, we refer to [32].

(4) Here we use the convention  $|F| = \dim_{\mathbb{C}} C(F)$ , and the statement itself is empty, and is there just for reminding us that these examples are to be investigated.

(5) This is trivial, because  $X$  being transitive means that  $G(X) \curvearrowright X$  is transitive, and by definition of  $G^+(X)$ , we have  $G(X) \subset G^+(X)$ . See [7].

(6) This comes from a result from [18], stating that we have  $H_n^+ = G^+(I_n)$ , where  $I_n$  is the graph formed by  $n$  segments, having  $N = 2n$  vertices.

(7) Once again this comes from a result from [18], stating that we have  $O_n^{-1} = G^+(K_n)$ , where  $K_n$  is the  $n$ -dimensional hypercube, having  $N = 2^n$  vertices.

Finally, the stability assertion is clear from the definition of the various products involved, from [43], [94]. This is well-known, and we will be back later on to this.  $\square$

Summarizing, we have a substantial list of examples. We will see that there are several other interesting examples, coming from the matrix models.

Let us study now the arbitrary transitive subgroups  $G \subset S_N^+$ . As a first result here, in the classical case the situation is very simple, as follows:

PROPOSITION 14.4. *Let  $G$  be a finite group.*

- (1) *Assuming that we have a transitive action  $G \curvearrowright \{1, \dots, N\}$ , by setting*

$$H = \left\{ \sigma \in G \mid \sigma(1) = 1 \right\}$$

*we have an identification as follows:*

$$G/H = \left\{ 1, \dots, N \right\}$$

- (2) *Conversely, any subgroup  $H \subset G$  produces an action as follows*

$$G \curvearrowright G/H$$

$$g(hH) = (gh)H$$

*and so a morphism as follows, with  $N = [G : H]$ :*

$$G \rightarrow S_N$$

- (3) *This latter morphism is injective when the following condition is satisfied:*

$$hgh^{-1} \in H, \forall h \in G \implies g = 1$$

PROOF. All the above assertions are well-known and standard, coming from the definition of the quotient space  $G/H$ , as being the space of cosets  $gH$ .  $\square$

In the quantum case now, it is quite unclear how to generalize the above structure result, to the transitive subgroups  $G \subset S_N^+$ . To be more precise, the various examples from [12], involving for instance group duals, show that we cannot expect to have an elementary generalization of the above  $G/H = \{1, \dots, N\}$  isomorphism.

However, we can at least try to extend the obvious fact that  $G = N|H|$  must be a multiple of  $N$ . And here, we have the following result, from [22]:

THEOREM 14.5. *If  $G \subset S_N^+$  is finite and transitive, then  $N$  divides  $|G|$ . Moreover:*

- (1) *The case  $|G| = N$  comes from the classical finite groups, of order  $N$ , acting on themselves.*
- (2) *The case  $|G| = 2N$  is possible, in the non-classical setting, an example here being the Kac-Paljutkin quantum group, at  $N = 4$ .*

PROOF. In order to prove the first assertion, we use the coaction of  $C(G)$  on the algebra  $\mathbb{C}^N = C(1, \dots, N)$ . In terms of the standard coordinates  $u_{ij}$ , the formula is:

$$\begin{aligned}\Phi : \mathbb{C}^N &\rightarrow \mathbb{C}^N \otimes C(G) \\ e_i &\rightarrow \sum_j e_j \otimes u_{ji}\end{aligned}$$

For  $a \in \{1, \dots, N\}$  consider the evaluation map  $ev_a : \mathbb{C}^N \rightarrow \mathbb{C}$  at  $a$ . By composing  $\Phi$  with  $ev_a \otimes id$  we obtain a  $C(G)$ -comodule map, as follows:

$$\begin{aligned}I_a : \mathbb{C}^N &\rightarrow C(G) \\ e_i &\rightarrow u_{ia}\end{aligned}$$

Our transitivity assumption on  $G$  ensures that this map  $I_a$  is injective. In other words, we have realized  $\mathbb{C}^N$  as a coideal subalgebra of  $C(G)$ .

We recall now that a finite dimensional Hopf algebra is free as a module over a coideal subalgebra  $A$  provided that the latter is Frobenius, in the sense that there exists a non-degenerate bilinear form  $b : A \otimes A \rightarrow \mathbb{C}$  satisfying:

$$b(xy, z) = b(x, yz)$$

We can apply this result to the coideal subalgebra  $I_a(\mathbb{C}^N) \subset C(G)$ , with the remark that  $\mathbb{C}^N$  is indeed Frobenius, with bilinear form as follows:

$$b(fg) = \frac{1}{N} \sum_{i=1}^N f(i)g(i)$$

Thus  $C(G)$  is a free module over the  $N$ -dimensional algebra  $\mathbb{C}^N$ , and this gives the result. Regarding now the remaining assertions, the proof here goes as follows:

(1) Since  $C(G) = \langle u_{ij} \rangle$  is of dimension  $N$ , and its commutative subalgebra  $\langle u_{i1} \rangle$  is of dimension  $N$  already,  $C(G)$  must be commutative. Thus  $G$  must be classical, and by transitivity, the inclusion  $G \subset S_N$  must come from the action of  $G$  on itself.

(2) The closed subgroups  $G \subset S_4^+$  are fully classified, and among them we have indeed the Kac-Paljutkin quantum group, which satisfies  $|G| = 8$ , and is transitive.  $\square$

Summarizing, we have many interesting examples of transitive subgroups  $G \subset S_N^+$ , but at the level of the general theory, many things that we know from the classical case, regarding the structure of the usual transitive subgroups  $G \subset S_N$ , do not have obvious quantum extensions. We will however in what follows that there are some other techniques that can be developed, as for instance certain matrix modelling techniques, in order to deal with the study of the transitive subgroups  $G \subset S_N^+$ , in the non-classical case.

**14b. Higher transitivity**

Let us discuss now the notion of double transitivity. We have here:

DEFINITION 14.6. *Let  $G \subset S_N^+$  be a closed subgroup, with magic unitary  $u = (u_{ij})$ , and consider the equivalence relation on  $\{1, \dots, N\}^2$  given by:*

$$(i, k) \sim (j, l) \iff u_{ij}u_{kl} \neq 0$$

- (1) *The equivalence classes under  $\sim$  are called orbitals of  $G$ .*
- (2)  *$G$  is called doubly transitive when the action has two orbitals.*

*In other words, we call  $G \subset S_N^+$  doubly transitive when  $u_{ij}u_{kl} \neq 0$ , for any  $i \neq k, j \neq l$ .*

To be more precise, it is clear from definitions that the diagonal  $D \subset \{1, \dots, N\}^2$  is an orbital, and that its complement  $D^c$  must be a union of orbitals. With this remark in hand, the meaning of (2) is that the orbitals must be  $D, D^c$ .

Among the results established in [75] is the fact that, with suitable definitions, the space  $Fix(u^{\otimes 2})$  consists of the functions which are constant on the orbitals. We have:

THEOREM 14.7. *For a doubly transitive subgroup  $G \subset S_N^+$ , we have:*

$$\int_G u_{ij}u_{kl} = \begin{cases} \frac{1}{N} & \text{if } i = k, j = l \\ 0 & \text{if } i = k, j \neq l \text{ or } i \neq k, j = l \\ \frac{1}{N(N-1)} & \text{if } i \neq k, j \neq l \end{cases}$$

*Moreover, this formula characterizes the double transitivity.*

PROOF. We use the standard fact, from [98], that the integrals in the statement form the projection onto  $Fix(u^{\otimes 2})$ . Now if we assume that  $G$  is doubly transitive,  $Fix(u^{\otimes 2})$  has dimension 2, and therefore coincides with  $Fix(u^{\otimes 2})$  for the usual symmetric group  $S_N$ . Thus the integrals in the statement coincide with those for the symmetric group  $S_N$ , which are given by the above formula. Finally, the converse is clear as well.  $\square$

Let us discuss the notion of  $k$ -transitivity, at  $k \in \mathbb{N}$ . We begin our study by recalling a few standard facts regarding the symmetric group  $S_N$ , and its subgroups  $G \subset S_N$ , from a representation theory/probabilistic viewpoint. We first have the following result:

PROPOSITION 14.8. *Consider the symmetric group  $S_N$ , together with its standard matrix coordinates  $u_{ij} = \chi(\sigma \in S_N | \sigma(j) = i)$ . We have the formula*

$$\int_{S_N} u_{i_1 j_1} \dots u_{i_k j_k} = \begin{cases} \frac{(N - |\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

*where  $\ker i$  denotes as usual the partition of  $\{1, \dots, k\}$  whose blocks collect the equal indices of  $i$ , and where  $|\cdot|$  denotes the number of blocks.*

PROOF. According to the definition of  $u_{ij}$ , the integrals in the statement are given by:

$$\int_{S_N} u_{i_1 j_1} \cdots u_{i_k j_k} = \frac{1}{N!} \# \left\{ \sigma \in S_N \mid \sigma(j_1) = i_1, \dots, \sigma(j_k) = i_k \right\}$$

Since the existence of  $\sigma \in S_N$  as above requires  $i_m = i_n \iff j_m = j_n$ , this integral vanishes when  $\ker i \neq \ker j$ . As for the case  $\ker i = \ker j$ , if we denote by  $b \in \{1, \dots, k\}$  the number of blocks of this partition, we have  $N - b$  points to be sent bijectively to  $N - b$  points, and so  $(N - b)!$  solutions, and the integral is  $\frac{(N-b)!}{N!}$ , as claimed.  $\square$

We recall now that each action  $G \curvearrowright \{1, \dots, N\}$  produces an action  $G \curvearrowright \{1, \dots, N\}^k$  for any  $k \in \mathbb{N}$ , and by restriction,  $G$  acts on the following set:

$$I_N^k = \left\{ (i_1, \dots, i_k) \in \{1, \dots, N\}^k \mid i_1, \dots, i_k \text{ distinct} \right\}$$

We have the following well-known result:

**THEOREM 14.9.** *Given a subgroup  $G \subset S_N$ , with standard matrix coordinates denoted  $u_{ij} = \chi(\sigma \mid \sigma(j) = i)$ , and a number  $k \leq N$ , the following are equivalent:*

- (1)  $G$  is  $k$ -transitive, in the sense that  $G \curvearrowright I_N^k$  is transitive.
- (2)  $\text{Fix}(u^{\otimes k})$  is minimal, i.e. is the same as for  $G = S_N$ .
- (3)  $\dim \text{Fix}(u^{\otimes k}) = B_k$ , where  $B_k$  is the  $k$ -th Bell number.
- (4)  $\int_G u_{i_1 j_1} \cdots u_{i_k j_k} = \frac{(N-k)!}{N!}$ , for any  $i, j \in I_N^k$ .
- (5)  $\int_G u_{i_1 j_1} \cdots u_{i_k j_k} \neq 0$ , for any  $i, j \in I_N^k$ .
- (6)  $u_{i_1 j_1} \cdots u_{i_k j_k} \neq 0$ , for any  $i, j \in I_N^k$ .

PROOF. All this is well-known, the idea being as follows:

(1)  $\implies$  (2) This follows from the fact that  $u^{\otimes k}$  comes by summing certain actions  $G \curvearrowright I_N^r$  with  $r \leq k$ , and the transitivity at  $k$  implies the transitivity at any  $r \leq k$ .

(2)  $\implies$  (3) This comes from the well-known fact that for the symmetric group  $S_N$ , the multiplicity  $\#(1 \in u^{\otimes k})$  equals the  $k$ -th Bell number  $B_k$ , for any  $k \leq N$ .

(3)  $\implies$  (4) We can use the fact that  $P_{i_1 \dots i_k, j_1 \dots j_k} = \int_G u_{i_1 j_1} \cdots u_{i_k j_k}$  is the orthogonal projection onto  $\text{Fix}(u^{\otimes k})$ . Thus we can assume  $G = S_N$ , and here we have:

$$\int_{S_N} u_{i_1 j_1} \cdots u_{i_k j_k} = \int_{S_N} \chi \left( \sigma \mid \sigma(j_1) = i_1, \dots, \sigma(j_k) = i_k \right) = \frac{(N - k)!}{N!}$$

(4)  $\implies$  (5) This is trivial.

(5)  $\implies$  (6) This is trivial too.

(6)  $\implies$  (1) This is clear, because if  $u_{i_1 j_1} \cdots u_{i_k j_k} = \chi(\sigma \mid \sigma(j_1) = i_1, \dots, \sigma(j_k) = i_k)$  is nonzero, we can find an element  $\sigma \in G$  such that  $\sigma(j_s) = i_s, \forall s$ .  $\square$

Summarizing, we have now a complete picture of the notion of  $k$ -transitivity for the usual permutation groups  $G \subset S_N$ , from a probabilistic viewpoint.

Let us discuss now the quantum permutation group analogue of the above results. Each magic unitary matrix  $u = (u_{ij})$  produces a corepresentation  $u^{\otimes k} = (u_{i_1 j_1} \dots u_{i_k j_k})$ , and so a coaction map, constructed as follows:

$$\begin{aligned} \Phi : (\mathbb{C}^N)^{\otimes k} &\rightarrow (\mathbb{C}^N)^{\otimes k} \otimes C(G) \\ \Phi(e_{i_1 \dots i_k}) &= \sum_{j_1 \dots j_k} e_{j_1 \dots j_k} \otimes u_{j_1 i_1} \dots u_{j_k i_k} \end{aligned}$$

The problem is that  $\text{span}(I_N^k)$  is no longer invariant, due to the fact that the variables  $u_{ij}$  no longer commute. We can only say that  $\text{span}(J_N^k)$  is invariant, where:

$$J_N^k = \left\{ (i_1, \dots, i_k) \in \{1, \dots, N\}^k \mid i_1 \neq i_2 \neq \dots \neq i_k \right\}$$

Indeed, by using the fact, coming from the magic condition on  $u$ , that  $a \neq c, b = d$  implies  $u_{ab}u_{cd} = 0$ , we obtain that for  $i \in J_N^k$  we have, as desired:

$$\Phi(e_{i_1 \dots i_k}) = \sum_{j_1 \neq j_2 \neq \dots \neq j_k} e_{j_1 \dots j_k} \otimes u_{j_1 i_1} \dots u_{j_k i_k}$$

We can study the transitivity properties of this coaction, as follows:

**PROPOSITION 14.10.** *Given a closed subgroup  $G \subset S_N^+$ , consider the associated coaction map, constructed as above:*

$$\Phi : \text{span}(J_N^k) \rightarrow \text{span}(J_N^k) \otimes C(G)$$

The following conditions are then equivalent:

- (1) *The following fixed point space is 1-dimensional:*

$$\text{Fix}(\Phi) = \left\{ \xi \mid \Phi(\xi) = \xi \otimes 1 \right\}$$

- (2) *We have  $\text{Fix}(\Phi) = \mathbb{C}\eta$ , with  $\eta$  being the following vector:*

$$\eta = \sum_{i \in J_N^k} e_{i_1 \dots i_k}$$

- (3) *We have the following formula, valid for any multi-index  $j$ :*

$$\sum_{i \in J_N^k} \int_G u_{i_1 i_1} \dots u_{i_k i_k} = 1$$

If these conditions are satisfied, we say that the coaction  $\Phi$  is transitive.



PROOF. The equivalences are elementary, the idea being as follows:

(1)  $\iff$  (2). Here we just have to check that we have indeed  $\Phi(\eta) = \eta \otimes 1$ , with  $\eta$  being as in the statement. By definition of  $\Phi$ , we have:

$$\Phi(\eta) = \sum_{i_1 \neq i_2 \neq \dots \neq i_k} \sum_{j_1 \neq j_2 \neq \dots \neq j_k} e_{j_1 \dots j_k} \otimes u_{j_1 i_1} \dots u_{j_k i_k}$$

Let us compute the middle sum  $S$ . When summing over indices  $i_1 \neq i_2$  we obtain:

$$(1 - u_{j_2 i_1}) u_{j_2 i_2} \dots u_{j_k i_k} = u_{j_2 i_2} \dots u_{j_k i_k}$$

Then when summing over indices  $i_2 \neq i_3$  we obtain:

$$(1 - u_{j_3 i_2}) u_{j_3 i_3} \dots u_{j_k i_k} = u_{j_3 i_3} \dots u_{j_k i_k}$$

And so on, up to obtaining at the end:

$$\sum_{i_k} u_{i_k i_k} = 1$$

Thus we have  $S = 1$ , and so the condition  $\Phi(\eta) = \eta \otimes 1$  is satisfied indeed.

(1)  $\iff$  (3) This comes from the following general formula, where  $\chi$  is the character of the corepresentation associated to  $\Phi$ :

$$\dim \text{Fix}(\Phi) = \int_G \chi$$

Indeed, in the standard basis  $\{e_i | i \in J_N^k\}$  we have:

$$\chi = \sum_{i \in J_N^k} u_{i_1 i_1} \dots u_{i_k i_k}$$

But this gives the result, by integrating. □

We have the following partial analogue of Theorem 14.9 above:

PROPOSITION 14.11. *Given a closed subgroup  $G \subset S_N^+$ , with  $N \geq 4$ , with matrix coordinates denoted  $u_{ij}$ , and a number  $k \in \mathbb{N}$ , the following conditions are equivalent:*

- (1) *The action  $G \curvearrowright \text{span}(J_N^k)$  is transitive.*
- (2)  *$\text{Fix}(u^{\otimes k})$  is minimal, i.e. is the same as for  $G = S_N^+$ .*
- (3)  *$\dim \text{Fix}(u^{\otimes k}) = C_k$ , where  $C_k$  is the  $k$ -th Catalan number.*
- (4)  *$\int_G u_{i_1 j_1} \dots u_{i_k j_k}$  is the same as for  $G = S_N^+$ , for any  $i, j \in J_N^k$ .*

PROOF. This follows as in the first part of the proof of Theorem 14.9, by performing changes where needed, and by using the general theory from [23], as an input:

(1)  $\implies$  (2) This follows from the fact that  $u^{\otimes k}$  comes by summing certain actions  $G \curvearrowright J_N^r$  with  $r \leq k$ , and the transitivity at  $k$  implies the transitivity at any  $r \leq k$ .

(2)  $\implies$  (3) This comes from the well-known fact that for the quantum group  $S_N^+$  with  $N \geq 4$ , the multiplicity  $\#(1 \in u^{\otimes k})$  equals the  $k$ -th Catalan number  $C_k$ .

(3)  $\implies$  (4) This comes from the well-known fact that  $P_{i_1 \dots i_k, j_1 \dots j_k} = \int_G u_{i_1 j_1} \dots u_{i_k j_k}$  is the orthogonal projection onto  $Fix(u^{\otimes k})$ , coming from [97].

(4)  $\implies$  (1) This follows by taking  $i = j$  and then summing over this index, by using the transitivity criterion for  $G \curvearrowright span(J_N^k)$  from Proposition 14.10 (3).  $\square$

Now let us compare Theorem 14.9 and Proposition 14.11. We conclude that the notion of  $k$ -transitivity for the subgroups  $G \subset S_N$  extends to the quantum group case,  $G \subset S_N^+$ , depending on the value of  $k$ , as follows:

- (1) At  $k = 1, 2$  everything extends well, due to the results in [43], [75].
- (2) At  $k = 3$  we have a good phenomenon,  $P_3 = NC_3$ , and a bad one,  $I_N^3 \neq J_N^3$ .
- (3) At  $k \geq 4$  we have two bad phenomena, namely  $P_k \neq NC_k$  and  $I_N^k \neq J_N^k$ .

Summarizing, our study suggests that things basically stop at  $k = 3$ . So, as a conclusion, let us record the definition and main properties of the 3-transitivity:

**THEOREM 14.12.** *A closed subgroup  $G \subset S_N^+$  is 3-transitive, in the sense that we have  $\dim(Fix(u^{\otimes 3})) = 5$ , if and only if, for any  $i, k, p$  distinct and any  $j, l, q$  distinct:*

$$\int_G u_{ij} u_{kl} u_{pq} = \frac{1}{N(N-1)(N-2)}$$

*In addition, in the classical case, we recover in this way the usual notion of 3-transitivity.*

**PROOF.** We know from Proposition 14.11 that the 3-transitivity condition is equivalent to the fact that the integrals of type  $\int_G u_{ij} u_{kl} u_{pq}$  with  $i \neq k \neq p$  and  $j \neq l \neq q$  have the same values as those for  $S_N^+$ . But these values are computed by Proposition 14.8 and the Weingarten formula, and the 3-transitivity condition follows to be equivalent to:

$$\int_G u_{ij} u_{kl} u_{pq} = \begin{cases} \frac{1}{N(N-1)(N-2)} & \text{if } \ker(ikp) = \ker(jlq) = ||| \\ \frac{1}{N(N-1)} & \text{if } \ker(ikp) = \ker(jlq) = \sqcap \\ 0 & \text{if } \{\ker(ikp), \ker(jlq)\} = \{|||, \sqcap\} \end{cases}$$

Now observe that the last formula is automatic, by using the traciality of the integral and the magic assumption on  $u$ , and that the middle formula follows from the first one, by summing over  $i, j$ . Thus we have are left with the first formula, as stated.

Finally, the last assertion follows from Theorem 14.9, applied at  $k = 3$ .  $\square$

### 14c. Classification results

Let us discuss now classification results at small values of  $N$ .

In order to discuss the case  $N = 4$ , we will need a very precise result, stating that  $S_4^+$  is a twist of  $SO_3$ . Let us start with the following definition, from [12]:

DEFINITION 14.13.  $C(SO_3^{-1})$  is the universal  $C^*$ -algebra generated by the entries of a  $3 \times 3$  orthogonal matrix  $a = (a_{ij})$ , with the following relations:

- (1) Skew-commutation:  $a_{ij}a_{kl} = \pm a_{kl}a_{ij}$ , with sign  $+$  if  $i \neq k, j \neq l$ , and  $-$  otherwise.
- (2) Twisted determinant condition:  $\sum_{\sigma \in S_3} a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} = 1$ .

Normally, our first task would be to prove that  $C(SO_3^{-1})$  is a Woronowicz algebra. This is of course possible, by doing some computations, but we will not need to do these computations, because the result follows from the following result, from [12]:

THEOREM 14.14. We have an isomorphism of compact quantum groups

$$S_4^+ = SO_3^{-1}$$

given by the Fourier transform over the Klein group  $K = \mathbb{Z}_2 \times \mathbb{Z}_2$ .

PROOF. Consider indeed the following matrix, corresponding to the standard vector space action of  $SO_3^{-1}$  on  $\mathbb{C}^4$ :

$$a^+ = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

We apply to this matrix the Fourier transform over the Klein group  $K = \mathbb{Z}_2 \times \mathbb{Z}_2$ :

$$u = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & a_{13} \\ 0 & a_{21} & a_{22} & a_{23} \\ 0 & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

It is routine to check that this matrix is magic, and vice versa, i.e. that the Fourier transform over  $K$  converts the relations in Definition 14.13 into the magic relations. Thus, we obtain the identification from the statement.  $\square$

We have the following classification result, also from [12]:

THEOREM 14.15. The closed subgroups of  $S_4^+ = SO_3^{-1}$  are as follows:

- (1) Infinite quantum groups:  $S_4^+, O_2^{-1}, \widehat{D}_\infty$ .
- (2) Finite groups:  $S_4$ , and its subgroups.
- (3) Finite group twists:  $S_4^{-1}, A_5^{-1}$ .
- (4) Series of twists:  $D_{2n}^{-1}$  ( $n \geq 3$ ),  $DC_{2n}^{-1}$  ( $n \geq 2$ ).
- (5) A group dual series:  $\widehat{D}_n$ , with  $n \geq 3$ .

Moreover, these quantum groups are subject to an ADE classification result.

PROOF. The idea here is that the classification result can be obtained by taking some inspiration from the McKay classification of the subgroups of  $SO_3$ . See [12].

To be more precise, with the convention that prime denotes twists, all unique in the cases below, and double prime denotes pseudo-twists, the ADE classification is:

- (A)  $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, K, \widehat{D}_n (n = 2, 3, \dots, \infty), S_4^+$ .
- (D)  $\mathbb{Z}_4, D'_{2n}, D''_{2n} (n = 2, 3, \dots), H_2^+, D_1, S_3$ .
- (E)  $A_4, S_4, S'_4, A'_5$ .

There are many comments to be made here, regarding our various conventions, and the construction of some of the above quantum groups, as follows:

(1) The 2-element group  $\mathbb{Z}_2 = \{1, \tau\}$  can act in 2 ways on 4 points: either with  $\tau$  acting without fixed point, and we use here the notation  $\mathbb{Z}_2$ , or with  $\tau$  acting with 2 fixed points, and we use here the notation  $D_1$ .

(2) Similarly, the Klein group  $K = \mathbb{Z}_2 \times \mathbb{Z}_2$  can act in 2 ways on 4 points: either with 2 non-trivial elements having 2 fixed points each, and we use here the notation  $K$ , or with all non-trivial elements having no fixed points, and we use here the notation  $D_2 = \widehat{D}_2$ .

(3) We have  $D'_4 = D_4$ , and  $D''_4 = G_0$ , the Kac-Paljutkin quantum group. Besides being a pseudo-twist of  $D_{2n}$ , the quantum group  $D''_{2n}$  with  $n \geq 2$  is known to be as well a pseudo-twist of the dicyclic, or binary cyclic group  $DC_{2n}$ .

(4) As explained below, the definition of  $D'_{2n}, D''_{2n}$  can be extended at  $n = 1, \infty$ , and we formally have  $D'_2 = D''_2 = K$ , and  $D'_\infty = D''_\infty = H_2^+$ , but these conventions are not very useful. Also, as explained in [12], the groups  $D_1, S_3$  are a bit special at (D).

We refer to [12] for the construction and various properties of the various twists and pseudo-twists in the above ADE list. In what follows we will just explain the construction of  $D'_{2n}, D''_{2n}$ , which is of particular interest for us. The idea here is as follows:

(1) Consider the subgroup  $H_2^+ \subset O_2^+$  given by the “cubic” relations, namely:

$$v_{ij}v_{ik} = v_{ji}v_{ki} = 0 \quad \forall j \neq k$$

The following are then projections, producing a  $4 \times 4$  magic matrix of “sudoku” type:

$$u_{ij}^\pm = \frac{v_{ij}^2 \pm v_{ij}}{2}$$

Thus, we obtain an embedding as follows:

$$H_2^+ \subset S_4^+$$

It is known that we have  $H_2^+ = \mathbb{Z}_2 \wr S_2^+$ , that we have  $H_2^+ = G^+(\mid \mid) = G^+(\square)$ , and also that we have  $H_2^+ = O'_2$ .

(2) Given  $k \in \{1, 2, \dots, \infty\}$  we define a subgroup  $G_n^\pm \subset H_2^+$  by imposing the following relations, with everywhere alternating words, of length  $n$ :

$$v_{11}v_{22}v_{11} \dots = v_{22}v_{11}v_{22} \dots$$

$$v_{12}v_{21}v_{12} \dots = \pm v_{21}v_{12}v_{21} \dots$$

This subgroup is then as follows:

(1) At  $n = 1$  we obtain the Klein group,  $G_1^\pm = K$ .

(2) At  $n \geq 2$  we obtain the unique twist  $G_n^+ = D'_{2n}$  and a pseudo-twist  $G_n^- = D''_{2n}$  of the dihedral group  $D_{2n}$ , with the remark that at  $n = 2$  we obtain respectively  $D'_4 = D_4$  and  $D''_4 = G_0$ , and also with the remark that  $D''_{2n}$  is as well a pseudo-twist of the dicyclic, or binary cyclic group  $DC_{2n}$ .

(3) At  $n = \infty$  the above relations disappear by definition, and we have  $G_\infty^\pm = H_2^+$ .  $\square$

We refer to [12] for more on the above result.

We will be back to these subgroups later, with a study of them by using matrix modelling techniques.

By restricting now the attention to the transitive case, we obtain:

**THEOREM 14.16.** *The small order transitive quantum groups are as follows:*

(1) At  $N = 1, 2, 3$  we have  $\{1\}, \mathbb{Z}_2, \mathbb{Z}_3, S_3$ .

(2) At  $N = 4$  we have  $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, D_4, A_4, S_4, O_2^{-1}, S_4^+, S_4^{-1}, A_5^{-1}$ .

**PROOF.** This follows from the above result, the idea being as follows:

(1) This follows from the fact that we have  $S_N = S_N^+$  at  $N \leq 3$ , from [95].

(2) This follows from the ADE classification of the subgroups  $G \subset S_4^+$ , from [11], with all the twists appearing in the statement being standard twists. See [11].  $\square$

As an interesting consequence of the above result, we have:

**PROPOSITION 14.17.** *The inclusion of compact quantum groups*

$$S_4 \subset S_4^+$$

*is maximal, in the sense that there is no quantum group in between.*

**PROOF.** This follows indeed from the above classification result. See [12].  $\square$

It is conjectured in fact that  $S_N \subset S_N^+$  should be maximal, for any  $N \in \mathbb{N}$ . We will be back to this.

Let us study now the quantum subgroups  $G \subset S_5^+$ .

We first have the following elementary observations, regarding such subgroups:

PROPOSITION 14.18. *We have the following examples of subgroups  $G \subset S_5^+$ :*

(1) *The classical subgroups,  $G \subset S_5$ . There are 16 such subgroups, having order:*

$$1, 2, 3, 4, 4, 5, 6, 6, 8, 10, 12, 12, 20, 24, 60, 120$$

(2) *The group duals,  $G = \widehat{\Gamma} \subset S_5^+$ . These appear, via a Fourier transform construction, from the various quotients  $\Gamma$  of the following groups:*

$$\mathbb{Z}_4, \mathbb{Z}_2 * \mathbb{Z}_2, \mathbb{Z}_2 * \mathbb{Z}_3$$

*In addition, we have as well all the ADE quantum groups  $G \subset S_4^+ \subset S_5^+$  from Theorem 14.15 above, embedded via the 5 standard embeddings  $S_4^+ \subset S_5^+$ .*

PROOF. These results are well-known, the proof being as follows:

(1) This is a classical result, with the groups which appear being respectively:

- The cyclic groups  $\{1\}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ .
- The Klein group  $K = \mathbb{Z}_2 \times \mathbb{Z}_2$ .
- The groups  $\mathbb{Z}_5, \mathbb{Z}_6, S_3, D_4, D_5, A_4$ .
- A copy of  $S_3 \times \mathbb{Z}_2$ .
- The general affine group  $GA_1(5) = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$ .
- And finally  $S_4, A_5, S_5$ .

(2) This follows from Bichon’s result in [44], stating that the group dual subgroups  $G = \widehat{\Gamma} \subset S_N^+$  appear from the various quotients  $\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k} \rightarrow \Gamma$ , with  $N_1 + \dots + N_k = N$ . At  $N = 5$  the partitions are  $5 = 1 + 4, 1 + 2 + 2, 2 + 3$ , and this gives the result.  $\square$

Summarizing, the classification of the subgroups  $G \subset S_5^+$  is a particularly difficult task, the situation here being definitely much more complicated than at  $N = 4$ .

Consider now an intermediate compact quantum group, as follows:

$$S_N \subset G \subset S_N^+$$

Then  $G$  must be transitive. Thus, we can restrict the attention to such quantum groups. Regarding such quantum groups, we first have the following elementary result:

PROPOSITION 14.19. *We have the following examples of transitive subgroups  $G \subset S_5^+$ :*

- (1) *The classical transitive subgroups  $G \subset S_5$ . There are only 5 such subgroups, namely  $\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5$ .*
- (2) *The transitive group duals,  $G = \widehat{\Gamma} \subset S_5^+$ . There is only one example here, namely the dual of  $\Gamma = \mathbb{Z}_5$ , which is  $\mathbb{Z}_5$ , already appearing above.*

*In addition, all the ADE quantum groups  $G \subset S_4^+ \subset S_5^+$  are not transitive.*

PROOF. This follows indeed by examining the lists in Proposition 14.18:

(1) The result here is well-known, and elementary. Observe that  $GA_1(5) = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$ , which is by definition the general affine group of  $\mathbb{F}_5$ , is indeed transitive.

(2) This follows from the results in [44], because with  $\mathbb{Z}_{N_1} * \dots * \mathbb{Z}_{N_k} \rightarrow \Gamma$  as in the proof of Proposition 14.18 (2), the orbit decomposition is precisely  $N = N_1 + \dots + N_k$ .

Finally, the last assertion is clear, because the embedding  $S_4^+ \subset S_5^+$  is obtained precisely by fixing a point. Thus  $S_4^+$  and its subgroups are not transitive, as claimed.  $\square$

In order to prove the uniqueness result, we will use the recent progress in subfactor theory [71], concerning the classification of the small index subfactors.

For our purposes, the most convenient formulation of the result in [71] is:

THEOREM 14.20. *The principal graphs of the irreducible index 5 subfactors are:*

- (1)  $A_\infty$ , and a non-extremal perturbation of  $A_\infty^{(1)}$ .
- (2) The McKay graphs of  $\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5$ .
- (3) The twists of the McKay graphs of  $A_5, S_5$ .

PROOF. This is a heavy result, and we refer to [71] for the whole story. The above formulation is the one from [71], with the subgroup subfactors there replaced by fixed point subfactors [3], and with the cyclic groups denoted as usual by  $\mathbb{Z}_N$ .  $\square$

In the quantum permutation group setting, this result becomes:

THEOREM 14.21. *The set of principal graphs of the transitive subgroups  $G \subset S_5^+$  coincide with the set of principal graphs of the following subgroups:*

$$\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5, S_5^+$$

PROOF. We must take the list of graphs in Theorem 14.20, and exclude some of the graphs, on the grounds that the graph cannot be realized by a transitive subgroup  $G \subset S_5^+$ .

We have 3 cases here to be studied, as follows:

(1) The graph  $A_\infty$  corresponds to  $S_5^+$  itself. As for the perturbation of  $A_\infty^{(1)}$ , this disappears, because our notion of transitivity requires the subfactor extremality.

(2) For the McKay graphs of  $\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5$  there is nothing to be done, all these graphs being solutions to our problem.

(3) The possible twists of  $A_5, S_5$ , coming from the graphs in Theorem 14.20 (3) above, cannot contain  $S_5$ , because their cardinalities are smaller or equal than  $|S_5| = 120$ .  $\square$

In connection now with our maximality questions, we have:

THEOREM 14.22. *The inclusion  $S_5 \subset S_5^+$  is maximal.*

PROOF. This follows indeed from Theorem 14.21, with the remark that  $S_5$  being transitive, so must be any intermediate subgroup  $S_5 \subset G \subset S_5^+$ .  $\square$

With a little more work, the above considerations can give the full list of transitive subgroups  $G \subset S_5^+$ . To be more precise, we have here the various subgroups appearing in Theorem 14.21, plus some possible twists of  $A_5, S_5$ , which remain to be investigated.

### 14d. Maximality questions

In general, the maximality of  $S_N \subset S_N^+$  is a difficult question. The only known general result here is in the easy case, as follows:

THEOREM 14.23. *There is no intermediate easy quantum group as follows:*

$$S_N \subset G \subset S_N^+$$

PROOF. This follows by doing some combinatorics. To be more precise, the idea is to show that any  $\pi \in P - NC$  has the following property:

$$\langle \pi \rangle = P$$

And, in order to establish this formula, the idea is to cap  $\pi$  with semicircles, as to preserve one crossing, chosen in advance, and to end up, by a recurrence procedure, with the standard crossing. We refer to [28] for full details here.

We can actually prove this at easiness level 2, as follows. Our first claim is that, assuming that  $G \subset H$  comes from an inclusion of categories of partitions  $D \subset E$ , the maximality at order 2 is equivalent to the following condition, for any  $\pi, \sigma \in E$ , not both in  $D$ , and for any  $\alpha, \beta \neq 0$ :

$$\langle \text{span}(D), \alpha T_\pi + \beta T_\sigma \rangle = \text{span}(E)$$

Consider indeed a category  $\text{span}(D) \subset C \subset \text{span}(E)$ , corresponding to a quantum group  $G \subset K \subset H$  having order 2. The order 2 condition means that we have  $C = \langle C \cap \text{span}_2(P) \rangle$ , where  $\text{span}_2$  denotes the space of linear combinations having 2 components. Since we have  $\text{span}(E) \cap \text{span}_2(P) = \text{span}_2(E)$ , the order 2 formula reads:

$$C = \langle C \cap \text{span}_2(E) \rangle$$

Now observe that the category on the right is generated by the categories  $C_{\pi\sigma}^{\alpha\beta}$  constructed in the statement. Thus, the order 2 condition reads:

$$C = \left\langle C_{\pi\sigma}^{\alpha\beta} \mid \pi, \sigma \in E, \alpha, \beta \in \mathbb{C} \right\rangle$$

Now since the maximality at order 2 of the inclusion  $G \subset H$  means that we have  $C \in \{\text{span}(D), \text{span}(E)\}$ , for any such  $C$ , we are led to the following condition:

$$C_{\pi\sigma}^{\alpha\beta} \in \{\text{span}(D), \text{span}(E)\} \quad , \quad \forall \pi, \sigma \in E, \alpha, \beta \in \mathbb{C}$$



Thus, we have proved our claim. In order to show now that  $S_N \subset S_N^+$  is maximal at order 2, we can use the old “semicircle capping” method. That method shows that any  $\pi \in P - NC$  has the property  $\langle \pi \rangle = P$ , and in order to establish this formula, the idea is to cap  $\pi$  with semicircles, as to preserve one crossing, chosen in advance, and to end up, by a recurrence procedure, with the standard crossing.

In our present case now, at level 2, the statement that we have to prove is as follows: “for  $\pi \in P - NC, \sigma \in P$  and  $\alpha, \beta \neq 0$  we have  $\langle \alpha T_\pi + \beta T_\sigma \rangle = span(P)$ ”.

In order to do this, our claim is that the same method as at level 1 applies, after some suitable modifications. We have indeed two cases, as follows:

(1) Assuming that  $\pi, \sigma$  have at least one different crossing, we can cap the partition  $\pi$  as to end up with the basic crossing, and  $\sigma$  becomes in this way an element of  $P(2, 2)$  different from this basic crossing, and so a noncrossing partition, from  $NC(2, 2)$ . Now by subtracting this noncrossing partition, which belongs to  $C_{S_N^+} = span(NC)$ , we obtain that the standard crossing belongs to  $\langle \alpha T_\pi + \beta T_\sigma \rangle$ , and we are done.

(2) In the case where  $\pi, \sigma$  have exactly the same crossings, we can start our descent procedure by selecting one common crossing, and then two strings of  $\pi, \sigma$  which are different, and then joining the crossing to these two strings. We obtain in this way a certain linear combination  $\alpha' T_{\pi'} + \beta' T_{\sigma'} \in \langle \alpha T_\pi + \beta T_\sigma \rangle$  which satisfies the conditions in (1) above, and we can continue as indicated there. □

The corresponding orthogonal quantum group questions are somehow easier, and our purpose in what follows will be that of discussing all this. We first have:

**THEOREM 14.24.** *There is only one proper intermediate easy quantum group*

$$O_N \subset G \subset O_N^+$$

*namely the half-classical orthogonal group  $O_N^*$ .*

**PROOF.** We must compute here the categories of pairings  $NC_2 \subset D \subset P_2$ , and this can be done via some standard combinatorics, in three steps, as follows:

- (1) Let  $\pi \in P_2 - NC_2$ , having  $s \geq 4$  strings. Our claim is that:
  - If  $\pi \in P_2 - P_2^*$ , there exists a semicircle capping  $\pi' \in P_2 - P_2^*$ .
  - If  $\pi \in P_2^* - NC_2$ , there exists a semicircle capping  $\pi' \in P_2^* - NC_2$ .

Indeed, both these assertions can be easily proved, by drawing pictures.

- (2) Consider now a partition  $\pi \in P_2(k, l) - NC_2(k, l)$ . Our claim is that:
  - If  $\pi \in P_2(k, l) - P_2^*(k, l)$  then  $\langle \pi \rangle = P_2$ .
  - If  $\pi \in P_2^*(k, l) - NC_2(k, l)$  then  $\langle \pi \rangle = P_2^*$ .

This can be indeed proved by recurrence on the number of strings,  $s = (k + l)/2$ , by using (1), which provides us with a descent procedure  $s \rightarrow s - 1$ , at any  $s \geq 4$ .

(3) Finally, assume that we are given an easy quantum group  $O_N \subset G \subset O_N^+$ , coming from certain sets of pairings  $D(k, l) \subset P_2(k, l)$ . We have three cases:

- If  $D \not\subset P_2^*$ , we obtain  $G = O_N$ .
- If  $D \subset P_2, D \not\subset NC_2$ , we obtain  $G = O_N^*$ .
- If  $D \subset NC_2$ , we obtain  $G = O_N^+$ .

Thus, we are led to the conclusion in the statement. □

We have as well the following result, from [18], going beyond easiness:

**THEOREM 14.25.** *The following inclusions are maximal:*

- (1)  $\mathbb{T}O_N \subset U_N$ .
- (2)  $PO_N \subset PU_N$ .
- (3)  $O_N \subset O_N^*$ .

**PROOF.** In order to prove these results, consider as well the group  $\mathbb{T}SO_N$ . Observe that we have  $\mathbb{T}SO_N = \mathbb{T}O_N$  if  $N$  is odd. If  $N$  is even the group  $\mathbb{T}O_N$  has two connected components, with  $\mathbb{T}SO_N$  being the component containing the identity.

Let us denote by  $\mathfrak{so}_N, \mathfrak{u}_N$  the Lie algebras of  $SO_N, U_N$ . It is well-known that  $\mathfrak{u}_N$  consists of the matrices  $M \in M_N(\mathbb{C})$  satisfying  $M^* = -M$ , and that  $\mathfrak{so}_N = \mathfrak{u}_N \cap M_N(\mathbb{R})$ . Also, it is easy to see that the Lie algebra of  $\mathbb{T}SO_N$  is  $\mathfrak{so}_N \oplus i\mathbb{R}$ .

Step 1. Our first claim is that if  $N \geq 2$ , the adjoint representation of  $SO_N$  on the space of real symmetric matrices of trace zero is irreducible.

Let indeed  $X \in M_N(\mathbb{R})$  be symmetric with trace zero. We must prove that the following space consists of all the real symmetric matrices of trace zero:

$$V = \text{span} \left\{ UXU^t \mid U \in SO_N \right\}$$

We first prove that  $V$  contains all the diagonal matrices of trace zero. Since we may diagonalize  $X$  by conjugating with an element of  $SO_N$ , our space  $V$  contains a nonzero diagonal matrix of trace zero. Consider such a matrix:

$$D = \text{diag}(d_1, d_2, \dots, d_N)$$

We can conjugate this matrix by the following matrix:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix} \in SO_N$$

We conclude that our space  $V$  contains as well the following matrix:

$$D' = \text{diag}(d_2, d_1, d_3, \dots, d_N)$$

More generally, we see that for any  $1 \leq i, j \leq N$  the diagonal matrix obtained from  $D$  by interchanging  $d_i$  and  $d_j$  lies in  $V$ . Now since  $S_N$  is generated by transpositions, it

follows that  $V$  contains any diagonal matrix obtained by permuting the entries of  $D$ . But it is well-known that this representation of  $S_N$  on the diagonal matrices of trace zero is irreducible, and hence  $V$  contains all such diagonal matrices, as claimed.

In order to conclude now, assume that  $Y$  is an arbitrary real symmetric matrix of trace zero. We can find then an element  $U \in SO_N$  such that  $UYU^t$  is a diagonal matrix of trace zero. But we then have  $UYU^t \in V$ , and hence also  $Y \in V$ , as desired.

Step 2. Our claim is that the inclusion  $\mathbb{T}SO_N \subset U_N$  is maximal in the category of connected compact groups.

Let indeed  $G$  be a connected compact group satisfying  $\mathbb{T}SO_N \subset G \subset U_N$ . Then  $G$  is a Lie group. Let  $\mathfrak{g}$  denote its Lie algebra, which satisfies:

$$\mathfrak{so}_N \oplus i\mathbb{R} \subset \mathfrak{g} \subset \mathfrak{u}_N$$

Let  $ad_G$  be the action of  $G$  on  $\mathfrak{g}$  obtained by differentiating the adjoint action of  $G$  on itself. This action turns  $\mathfrak{g}$  into a  $G$ -module. Since  $SO_N \subset G$ ,  $\mathfrak{g}$  is also a  $SO_N$ -module.

Now if  $G \neq \mathbb{T}SO_N$ , then since  $G$  is connected we must have  $\mathfrak{so}_N \oplus i\mathbb{R} \neq \mathfrak{g}$ . It follows from the real vector space structure of the Lie algebras  $\mathfrak{u}_N$  and  $\mathfrak{so}_N$  that there exists a nonzero symmetric real matrix of trace zero  $X$  such that:

$$iX \in \mathfrak{g}$$

We know that the space of symmetric real matrices of trace zero is an irreducible representation of  $SO_N$  under the adjoint action. Thus  $\mathfrak{g}$  must contain all such  $X$ , and hence  $\mathfrak{g} = \mathfrak{u}_N$ . But since  $U_N$  is connected, it follows that  $G = U_N$ .

Step 3. Our claim is that the commutant of  $SO_N$  in  $M_N(\mathbb{C})$  is as follows:

- (1)  $SO'_2 = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mid \alpha, \beta \in \mathbb{C} \right\}$ .
- (2) If  $N \geq 3$ ,  $SO'_N = \{\alpha I_N \mid \alpha \in \mathbb{C}\}$ .

Indeed, at  $N = 2$  this is a direct computation. At  $N \geq 3$ , an element in  $X \in SO'_N$  commutes with any diagonal matrix having exactly  $N - 2$  entries equal to 1 and two entries equal to  $-1$ . Hence  $X$  is a diagonal matrix. Now since  $X$  commutes with any even permutation matrix and  $N \geq 3$ , it commutes in particular with the permutation matrix associated with the cycle  $(i, j, k)$  for any  $1 < i < j < k$ , and hence all the entries of  $X$  are the same. We conclude that  $X$  is a scalar matrix, as claimed.

Step 4. Our claim is that the set of matrices with nonzero trace is dense in  $SO_N$ .

At  $N = 2$  this is clear, since the set of elements in  $SO_2$  having a given trace is finite. So assume  $N > 2$ , and let  $T \in SO_N \simeq SO(\mathbb{R}^N)$  with  $Tr(T) = 0$ . Let  $E \subset \mathbb{R}^N$  be a 2-dimensional subspace preserved by  $T$ , such that  $T|_E \in SO(E)$ .

Let  $\varepsilon > 0$  and let  $S_\varepsilon \in SO(E)$  with  $\|T|_E - S_\varepsilon\| < \varepsilon$ , and with  $Tr(T|_E) \neq Tr(S_\varepsilon)$ , in the  $N = 2$  case. Now define  $T_\varepsilon \in SO(\mathbb{R}^N) = SO_N$  by:

$$T_{\varepsilon|E} = S_\varepsilon \quad , \quad T_{\varepsilon|E^\perp} = T|_{E^\perp}$$

It is clear that  $\|T - T_\varepsilon\| \leq \|T|_E - S_\varepsilon\| < \varepsilon$  and that:

$$Tr(T_\varepsilon) = Tr(S_\varepsilon) + Tr(T|_{E^\perp}) \neq 0$$

Thus, we have proved our claim.

Step 5. Our claim is that  $\mathbb{T}O_N$  is the normalizer of  $\mathbb{T}SO_N$  in  $U_N$ , i.e. is the subgroup of  $U_N$  consisting of the unitaries  $U$  for which  $U^{-1}XU \in \mathbb{T}SO_N$  for all  $X \in \mathbb{T}SO_N$ .

It is clear that the group  $\mathbb{T}O_N$  normalizes  $\mathbb{T}SO_N$ , so in order to prove the result, we must show that if  $U \in U_N$  normalizes  $\mathbb{T}SO_N$  then  $U \in \mathbb{T}O_N$ .

First note that  $U$  normalizes  $SO_N$ . Indeed if  $X \in SO_N$  then  $U^{-1}XU \in \mathbb{T}SO_N$ , so  $U^{-1}XU = \lambda Y$  for some  $\lambda \in \mathbb{T}$  and  $Y \in SO_N$ . If  $Tr(X) \neq 0$ , we have  $\lambda \in \mathbb{R}$  and hence:

$$\lambda Y = U^{-1}XU \in SO_N$$

The set of matrices having nonzero trace being dense in  $SO_N$ , we conclude that  $U^{-1}XU \in SO_N$  for all  $X \in SO_N$ . Thus, we have:

$$\begin{aligned} X \in SO_N &\implies (UXU^{-1})^t(UXU^{-1}) = I_N \\ &\implies X^tU^tUX = U^tU \\ &\implies U^tU \in SO'_N \end{aligned}$$

It follows that at  $N \geq 3$  we have  $U^tU = \alpha I_N$ , with  $\alpha \in \mathbb{T}$ , since  $U$  is unitary. Hence we have  $U = \alpha^{1/2}(\alpha^{-1/2}U)$  with  $\alpha^{-1/2}U \in O_N$ , and  $U \in \mathbb{T}O_N$ . If  $N = 2$ ,  $(U^tU)^t = U^tU$  gives again that  $U^tU = \alpha I_2$ , and we conclude as in the previous case.

Step 6. Our claim is that the inclusion  $\mathbb{T}O_N \subset U_N$  is maximal in the category of compact groups.

Suppose indeed that  $\mathbb{T}O_N \subset G \subset U_N$  is a compact group such that  $G \neq U_N$ . It is a well-known fact that the connected component of the identity in  $G$  is a normal subgroup, denoted  $G_0$ . Since we have  $\mathbb{T}SO_N \subset G_0 \subset U_N$ , we must have  $G_0 = \mathbb{T}SO_N$ . But since  $G_0$  is normal in  $G$ , the group  $G$  normalizes  $\mathbb{T}SO_N$ , and hence  $G \subset \mathbb{T}O_N$ .

Step 7. Our claim is that the inclusion  $PO_N \subset PU_N$  is maximal in the category of compact groups.

This follows from the above result. Indeed, if  $PO_N \subset G \subset PU_N$  is a proper intermediate subgroup, then its preimage under the quotient map  $U_N \rightarrow PU_N$  would be a proper intermediate subgroup of  $\mathbb{T}O_N \subset U_N$ , which is a contradiction.

Step 8. Our claim is that the inclusion  $O_N \subset O_n^*$  is maximal in the category of compact groups.

Consider indeed a sequence of surjective Hopf  $*$ -algebra maps as follows, whose composition is the canonical surjection:

$$C(O_N^*) \xrightarrow{f} A \xrightarrow{g} C(O_N)$$

This produces a diagram of Hopf algebra maps with pre-exact rows, as follows:

$$\begin{array}{ccccccccc} \mathbb{C} & \longrightarrow & C(PO_N^*) & \longrightarrow & C(O_N^*) & \longrightarrow & C(\mathbb{Z}_2) & \longrightarrow & \mathbb{C} \\ & & \downarrow f_{\downarrow} & & \downarrow f & & \parallel & & \\ \mathbb{C} & \longrightarrow & PA & \longrightarrow & A & \longrightarrow & C(\mathbb{Z}_2) & \longrightarrow & \mathbb{C} \\ & & \downarrow g_{\downarrow} & & \downarrow g & & \parallel & & \\ \mathbb{C} & \longrightarrow & PC(O_N) & \longrightarrow & C(O_N) & \longrightarrow & C(\mathbb{Z}_2) & \longrightarrow & \mathbb{C} \end{array}$$

Consider now the following composition, with the isomorphism on the left being something well-known, coming from [46], that we will explain in chapter 15 below:

$$C(PU_N) \simeq C(PO_N^*) \xrightarrow{f_{\downarrow}} PA \xrightarrow{g_{\downarrow}} PC(O_N) \simeq C(PO_N)$$

This induces, at the group level, the embedding  $PO_N \subset PU_N$ . Thus  $f_{\downarrow}$  or  $g_{\downarrow}$  is an isomorphism. If  $f_{\downarrow}$  is an isomorphism we get a commutative diagram of Hopf algebra morphisms with pre-exact rows, as follows:

$$\begin{array}{ccccccccc} \mathbb{C} & \longrightarrow & C(PO_N^*) & \longrightarrow & C(O_N^*) & \longrightarrow & C(\mathbb{Z}_2) & \longrightarrow & \mathbb{C} \\ & & \parallel & & \downarrow f & & \parallel & & \\ \mathbb{C} & \longrightarrow & C(PO_N^*) & \longrightarrow & A & \longrightarrow & C(\mathbb{Z}_2) & \longrightarrow & \mathbb{C} \end{array}$$

Then  $f$  is an isomorphism. Similarly if  $g_{\downarrow}$  is an isomorphism, then  $g$  is an isomorphism. For further details on all this, we refer to [18].  $\square$

Given an exotic quantum permutation group  $S_N \subset G \subset S_N^+$ , we can expect all the combinatorics of  $G$  to be invariant under  $S_N$ , due to the embedding  $S_N \subset G$ . In order to establish some concrete results in this direction, we use the following known formula:

**PROPOSITION 14.26.** *Assuming  $S_N \subset G \subset S_N^+$ , consider the quotient map  $\pi : C(G) \rightarrow C(S_N)$ , and set  $ev_{\sigma}(a) = \pi(a)(\sigma)$ , for any  $a \in C(G)$ . Then with*

$$\varphi_{\sigma\tau} = ev_{\sigma^{-1}} * \varphi * ev_{\tau}$$

*we have the following formula,*

$$\varphi_{\sigma\tau}(u_{i_1 j_1} \cdots u_{i_p j_p}) = \varphi(u_{\sigma(i_1)\tau(j_1)} \cdots u_{\sigma(i_p)\tau(j_p)})$$

*valid for any  $p \in \mathbb{N}$ , and any indices  $i_1, \dots, i_p$  and  $j_1, \dots, j_p$ .*

PROOF. We have indeed the following computation:

$$\begin{aligned}
& \varphi_{\sigma\tau}(u_{i_1 j_1} \dots u_{i_p j_p}) \\
&= (ev_{\sigma^{-1}} \otimes \varphi \otimes ev_{\tau}) \Delta^{(2)}(u_{i_1 j_1} \dots u_{i_p j_p}) \\
&= \sum_{k_1 \dots k_p} \sum_{l_1 \dots l_p} ev_{\sigma^{-1}}(u_{i_1 k_1} \dots u_{i_p k_p}) \varphi(u_{k_1 l_1} \dots u_{k_p l_p}) ev_{\tau}(u_{l_1 j_1} \dots u_{l_p j_p}) \\
&= \sum_{k_1 \dots k_p} \sum_{l_1 \dots l_p} \delta_{\sigma(i_1)k_1} \dots \delta_{\sigma(i_p)k_p} \varphi(u_{k_1 l_1} \dots u_{k_p l_p}) \delta_{\tau(j_1)l_1} \dots \delta_{\tau(j_p)l_p} \\
&= \varphi(u_{\sigma(i_1)\tau(j_1)} \dots u_{\sigma(i_p)\tau(j_p)})
\end{aligned}$$

Thus, we obtain the formula in the statement. See [78].  $\square$

As a first application, we can apply this to the existence problem for the algebraic  $k$ -orbitals. We obtain somehow “half” of the proof of that, as follows:

PROPOSITION 14.27. *Assuming  $S_N \subset G \subset S_N^+$ , the relation*

$$(i_1, \dots, i_k) \sim (j_1, \dots, j_k) \iff u_{i_1 j_1} \dots u_{i_k j_k} \neq 0$$

*depends only on  $\ker i, \ker j \in P(k)$ , and not of the particular multi-indices  $i, j$ .*

PROOF. By using Proposition 14.26 and Hahn-Banach we obtain, for any  $\sigma, \tau \in S_N$ :

$$u_{i_1 j_1} \dots u_{i_k j_k} \neq 0 \implies u_{\sigma(i_1)\tau(j_1)} \dots u_{\sigma(i_k)\tau(j_k)} \neq 0$$

Thus, we are led to the conclusion in the statement.  $\square$

In short, the conjectural algebraic  $k$ -orbitals are invariant under  $S_N$ , and the problem now is if the above relation  $\sim$  is transitive on the partitions of  $P(k)$ .

As another application, we have the following result:

PROPOSITION 14.28. *Given an exotic quantum permutation group  $S_N \subset G \subset S_N^+$ , we have*

$$[u_{ij}, u_{kl}] \neq 0$$

*for any indices  $i \neq k$  and  $j \neq l$ . More generally, the following algebra*

$$\langle u_{ij}, u_{kl} \rangle \subset C(G)$$

*does not depend on the choice of  $i \neq k$  and  $j \neq l$ . Even more generally, the algebra*

$$\langle u_{i_1 j_1}, \dots, u_{i_p j_p} \rangle \subset C(G)$$

*depends only on the relative position of the indices  $(i_r, j_r)$  inside the standard square  $\{1, \dots, N\}^2$ , and not on the precise value of these indices.*

PROOF. This is similar to the proof of Proposition 14.27, by using Proposition 14.26 applied to commutators, or more general quantities, and then using Hahn-Banach.  $\square$

As an interesting consequence of Proposition 14.28, we have:

PROPOSITION 14.29. *Given an exotic quantum permutation group  $S_N \subset G \subset S_N^+$ , the corresponding diagonal algebra, which is by definition given by*

$$D(G) = \langle u_{11}, \dots, u_{NN} \rangle \subset C(G)$$

*when regarded as a quotient of  $C^*(\mathbb{Z}_2^{*N})$  is invariant under the action of  $S_N$ .*

PROOF. According to Proposition 14.28, we have an isomorphism as follows, between subalgebras of  $D(G)$ , for any indices  $i \neq j$  and any  $k \neq l$ :

$$\langle u_{ii}, u_{jj} \rangle \simeq \langle u_{kk}, u_{ll} \rangle$$

More generally, still according to Proposition 14.28, we have such isomorphisms in higher length 3, 4, 5, ..., and at length  $N$  we obtain the result.  $\square$

The above result has some potentially interesting consequences. Indeed, since we have  $S_N \subset G \subset S_N^+$ , at the level of diagonal algebras we have quotient maps as follows:

$$D(S_N^+) \rightarrow D(G) \rightarrow D(S_N)$$

Thus, Proposition 14.29 suggests that  $D(G)$  should come from a certain uniform intermediate quotient  $\mathbb{Z}_2^{*N} \rightarrow \Gamma \rightarrow \mathbb{Z}_2^N$ . This would be very interesting, first in order to understand the structure of  $C(G)$  itself, which is not that much bigger than  $D(G)$ , and also for making the link with the quantum reflection groups.

We discuss here a number of freeness questions regarding the algebra  $C(S_N^+)$ , or rather its dense subalgebra  $\mathcal{C}(S_N^+) \subset C(S_N^+)$  generated by the entries of  $u = (u_{ij})$ . The problem is that of understanding which are the polynomial relations relating the variables  $u_{ij}$ :

$$P(\{u_{ij}\}) = 0$$

That is, we would like to understand what is the kernel of the following map:

$$\begin{aligned} \mathbb{C} \langle \{X_{ij}\} \rangle &\rightarrow \mathcal{C}(S_N^+) \\ X_{ij} &\rightarrow u_{ij} \end{aligned}$$

We know that this kernel is the ideal generated by the magic relations, and the conjecture would be that at  $N \geq 4$  the relations  $P(\{u_{ij}\}) = 0$  could only appear for “trivial reasons”. However, it seems difficult to formulate a precise conjecture in this sense.

Let us look instead at the monomial case. Here we would simply like to understand which are the monomials in the variables  $u_{ij}$  which vanish, and we have:

CONJECTURE 14.30. *For the quantum group  $S_N^+$  with  $N \geq 4$ , an equality of type*

$$u_{i_1 j_1} \dots u_{i_k j_k} = 0$$

*can only appear “for trivial reasons”, due to an occurrence inside this relation of a cancellation formula of type  $pq = 0$ , with  $p, q \in \{u_{ij}\}$ .*

Here is an equivalent form of Conjecture 14.30, using the orbital formalism:

CONJECTURE 14.31. *The relation  $i \sim j \iff u_{i_1 j_1} \dots u_{i_k j_k} \neq 0$  is given by:*

$$(i_1, \dots, i_k) \sim (j_1, \dots, j_k) \iff \begin{cases} i_1 = i_2 \iff j_1 = j_2 \\ i_2 = i_3 \iff j_2 = j_3 \\ \dots \\ i_{k-1} = i_k \iff j_{k-1} = j_k \end{cases}$$

*In particular  $\sim$  is transitive, and we have  $2^{k-1}$  equivalence classes, or orbitals.*

Here the first assertion is a reformulation of Conjecture 14.30, and the transitivity assumption and the orbital count are trivial consequences of it.

As before, the cases  $k = 3, 4$  are of particular interest, but this time not in view of their potential doability, but rather in view of their potential applications.

In relation with this latter conjecture, we have the following useful result:

PROPOSITION 14.32. *Given a quantum permutation group  $G \subset S_N^+$ , the relation*

$$(i_1, \dots, i_k) \sim (j_1, \dots, j_k) \iff u_{i_1 j_1} \dots u_{i_k j_k} \neq 0$$

*is in fact a relation between the  $k$ -orbitals of  $G_{class} \subset S_N$ . In particular, with  $G = S_N^+$ , what we have is a relation on the set  $P(k)$  of partitions of  $\{1, \dots, k\}$ .*

PROOF. This is explained above, the idea being that any linear form  $\varphi \in C(G)^*$  can be suitably modified by permutations  $\sigma, \tau \in G_{class}$  as to take the same values on the quantities  $u_{i_1 j_1} \dots u_{i_k j_k}$ , under the action of  $G_{class}$  on the indices. As explained there, together with Hahn-Banach, this gives the result. Finally, in the case of  $G = S_N^+$  we have  $G_{class} = S_N$ , and the orbits here are  $\{1, \dots, N\}^k / \sim = P(k)$ , as claimed.  $\square$

Finally, getting back now to the general freeness questions formulated in the beginning, an interesting question, which is complementary to the above ‘‘orbital’’ ones, regards the diagonal coefficients  $u_{ii}$ . We have here the following conjecture:

CONJECTURE 14.33. *For the quantum group  $S_N^+$  with  $N \geq 4$ , the variables*

$$u_{ii} \in C(S_N^+)$$

*are algebraically free. In particular, the diagonal algebra*

$$D(S_N^+) = \langle u_{11}, \dots, u_{NN} \rangle \subset C(S_N^+)$$

*is isomorphic to the group algebra  $C^*(\mathbb{Z}_2^{*N})$ .*



Observe that this is indeed complementary to the above orbital questions, because we are dealing here with linear products of type  $u_{i_1 i_1} \dots u_{i_k i_k}$ , and their linear combinations, which are “trivial” from the point of view of the orbital theory.

As before, it is very unclear on how to prove such things. In relation now with the “exotic” case,  $S_N \subset G \subset S_N^+$ , we would like such things to be understood, because the orbitals for  $G$ , if well-defined indeed, should lie between those for  $S_N$  and for  $S_N^+$ , and the same should happen for the diagonal algebra. Thus, the exotic quantum permutation groups  $S_N \subset G \subset S_N^+$  would lead to “exotic” relations on  $P(k)$ , and also to “exotic” quotients of  $\mathbb{Z}_2^{*N}$ , and with a bit of luck, such exotic things could not really exist.

### 14e. Exercises

Things have been fairly technical in this chapter, and as a unique exercise on all this, in connection with the questions at the end, we have:

EXERCISE 14.34. *Prove that the diagonal algebra of the free symmetric group*

$$D(S_N^+) = \langle u_{11}, \dots, u_{NN} \rangle \subset C(S_N^+)$$

*is isomorphic to the group algebra  $C^*(\mathbb{Z}_2^{*N})$ .*

This is something that we already discussed in chapter 6 above, in relation with the quantum partial permutations and of their diagonal algebras, and of the related functional analysis problematics of constructing a quantum group of type  $S_\infty^+$ . In view of the above considerations, making the link with the exotic quantum groups  $S_N \subset G \subset S_N^+$ , this appears to be an increasingly interesting problem, related to many things.

## Matrix models

### 15a. Matrix models

One interesting method for the study of the subgroups  $G \subset S_N^+$ , that we have not tried yet, consists in modelling the coordinates  $u_{ij} \in C(G)$  by concrete variables  $U_{ij} \in B$ . Indeed, assuming that the model is faithful in some suitable sense, that the algebra  $B$  is something quite familiar, and that the variables  $U_{ij}$  are not too complicated, all the questions about  $G$  would correspond in this way to routine questions inside  $B$ .

We discuss here these questions, first for the arbitrary quantum groups  $G \subset U_N^+$ , and then for the quantum permutation groups  $G \subset S_N^+$ . Regarding the choice of the target algebra  $B$ , some very convenient algebras are the random matrix ones,  $B = M_K(C(T))$ , with  $K \in \mathbb{N}$ , and  $T$  being a compact space. These algebras generalize indeed the most familiar algebras that we know, namely the matrix ones  $M_K(\mathbb{C})$ , and the commutative ones  $C(T)$ . We are led in this way to the following general definition:

DEFINITION 15.1. *A matrix model for  $G \subset U_N^+$  is a morphism of  $C^*$ -algebras*

$$\pi : C(G) \rightarrow M_K(C(T))$$

where  $T$  is a compact space, and  $K \geq 1$  is an integer.

There are many examples of such models, and will discuss them later on. For the moment, let us develop some general theory. The question to be solved is that of understanding the suitable faithfulness assumptions needed on  $\pi$ , as for the model to “remind” the quantum group. As we will see, this is something quite tricky.

The simplest situation is when  $\pi$  is faithful in the usual sense. This is of course something quite restrictive, because the algebra  $C(G)$  must be of type I in this case. However, there are many interesting examples here, and all this is worth a detailed look. Let us introduce the following notion, which is related to faithfulness:

DEFINITION 15.2. *A matrix model  $\pi : C(G) \rightarrow M_K(C(T))$  is called stationary when*

$$\int_G = \left( \text{tr} \otimes \int_T \right) \pi$$

where  $\int_T$  is the integration with respect to a given probability measure on  $T$ .

Here the term “stationary” comes from a functional analytic interpretation of all this, with a certain Cesàro limit being needed to be stationary, and this will be explained later. Yet another explanation comes from a certain relation with the lattice models, but this relation is rather something folklore, not axiomatized yet. We will be back to this.

As a first result now, which is something which is not exactly trivial, and whose proof requires some functional analysis, the stationarity property implies the faithfulness:

**THEOREM 15.3.** *Assuming that a closed subgroup  $G \subset U_N^+$  has a stationary model,*

$$\pi : C(G) \rightarrow M_K(C(T))$$

*it follows that  $G$  must be coamenable, and that the model is faithful. Moreover,  $\pi$  extends into an embedding of von Neumann algebras, as follows,*

$$L^\infty(G) \subset M_K(L^\infty(T))$$

*which commutes the canonical integration functionals.*

**PROOF.** Assume that we have a stationary model, as in the statement. By performing the GNS construction with respect to  $\int_G$ , we obtain a factorization as follows, which commutes with the respective canonical integration functionals:

$$\pi : C(G) \rightarrow C(G)_{red} \subset M_K(C(T))$$

Thus, in what regards the coamenability question, we can assume that  $\pi$  is faithful. With this assumption made, we have an embedding as follows:

$$C(G) \subset M_K(C(T))$$

Now observe that the GNS construction gives a better embedding, as follows:

$$L^\infty(G) \subset M_K(L^\infty(T))$$

Now since the von Neumann algebra on the right is of type I, so must be its subalgebra  $A = L^\infty(G)$ . This means that, when writing the center of this latter algebra as  $Z(A) = L^\infty(X)$ , the whole algebra decomposes over  $X$ , as an integral of type I factors:

$$L^\infty(G) = \int_X M_{K_x}(\mathbb{C}) dx$$

In particular, we can see from this that  $C(G) \subset L^\infty(G)$  has a unique  $C^*$ -norm, and so  $G$  is coamenable. Thus we have proved our first assertion, and the second assertion follows as well, because our factorization of  $\pi$  consists of the identity, and of an inclusion.  $\square$

We refer to [23], [32] for more on this. Summarizing, what we have is a slight strengthening of the notion of faithfulness. We will see later that there are many interesting examples of such models, while remaining of course in the coamenable and type I setting.

Let us discuss now the general, non-coamenable case, with the aim of finding a weaker notion of faithfulness, which still does the job, of “reminding” the quantum group. The

idea comes by looking at the group duals  $G = \widehat{\Gamma}$ . Consider indeed a general model for the associated algebra, which can be written as follows:

$$\pi : C^*(\Gamma) \rightarrow M_K(C(T))$$

The point now is that such a representation of the group algebra must come by linearization from a unitary group representation, as follows:

$$\rho : \Gamma \rightarrow C(T, U_K)$$

Now observe that when  $\rho$  is faithful, the representation  $\pi$  is in general not faithful, for instance because when  $T = \{.\}$  its target algebra is finite dimensional. On the other hand, this representation “reminds”  $\Gamma$ , so can be used in order to fully understand  $\Gamma$ .

Summarizing, we have an idea here, basically saying that, for practical purposes, the faithfulness property can be replaced with something much weaker. This weaker notion is called “inner faithfulness”, and the general theory here, from [13], is as follows:

DEFINITION 15.4. *Let  $\pi : C(G) \rightarrow M_K(C(T))$  be a matrix model.*

- (1) *The Hopf image of  $\pi$  is the smallest quotient Hopf  $C^*$ -algebra  $C(G) \rightarrow C(H)$  producing a factorization as follows:*

$$\pi : C(G) \rightarrow C(H) \rightarrow M_K(C(T))$$

- (2) *When the inclusion  $H \subset G$  is an isomorphism, i.e. when there is no non-trivial factorization as above, we say that  $\pi$  is inner faithful.*

These constructions work in fact for any  $C^*$ -algebra representation  $\pi : C(G) \rightarrow B$ , but here we will be only interested in the random matrix case,  $B = M_K(C(T))$ . As a first example, motivated by the above discussion, in the case where  $G = \widehat{\Gamma}$  is a group dual,  $\pi$  must come from a group representation, as follows:

$$\rho : \Gamma \rightarrow C(T, U_K)$$

Thus the minimal factorization in (1) is obtained by taking the image:

$$\rho : \Gamma \rightarrow \Lambda \subset C(T, U_K)$$

Thus, as a conclusion,  $\pi$  is inner faithful precisely when:

$$\Gamma \subset C(T, U_K)$$

Dually now, given a compact Lie group  $G$ , and elements  $g_1, \dots, g_K \in G$ , we have a diagonal representation  $\pi : C(G) \rightarrow M_K(\mathbb{C})$ , appearing as follows:

$$f \rightarrow \begin{pmatrix} f(g_1) & & \\ & \ddots & \\ & & f(g_K) \end{pmatrix}$$

The minimal factorization of this representation  $\pi$ , as in (1) above, is then via the algebra  $C(H)$ , with  $H$  being the following closed subgroup of  $G$ :

$$H = \overline{\langle g_1, \dots, g_K \rangle}$$

Thus, as a conclusion,  $\pi$  is inner faithful precisely when we have:

$$G = H$$

In general, the existence and uniqueness of the Hopf image comes from dividing  $C(G)$  by a suitable ideal. In Tannakian terms, as explained in [13], we have:

**THEOREM 15.5.** *Assuming  $G \subset U_N^+$ , with fundamental corepresentation  $u = (u_{ij})$ , the Hopf image of a model  $\pi : C(G) \rightarrow M_K(C(T))$  comes from the Tannakian category*

$$C_{kl} = \text{Hom}(U^{\otimes k}, U^{\otimes l})$$

where  $U_{ij} = \pi(u_{ij})$ , and where the spaces on the right are taken in a formal sense.

**PROOF.** Since the morphisms increase the intertwining spaces, when defined either in a representation theory sense, or just formally, we have inclusions as follows:

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) \subset \text{Hom}(U^{\otimes k}, U^{\otimes l})$$

More generally, we have such inclusions when replacing  $(G, u)$  with any pair producing a factorization of  $\pi$ . Thus, by Tannakian duality, the Hopf image must be given by the fact that the intertwining spaces must be the biggest, subject to the above inclusions.

On the other hand, since  $u$  is biunitary, so is  $U$ , and it follows that the spaces on the right form a Tannakian category. Thus, we have a quantum group  $(H, v)$  given by:

$$\text{Hom}(v^{\otimes k}, v^{\otimes l}) = \text{Hom}(U^{\otimes k}, U^{\otimes l})$$

By the above discussion,  $C(H)$  follows to be the Hopf image of  $\pi$ , as claimed. □

Regarding now the study of the inner faithful models, a key problem is that of computing the Haar integration functional. The result here, from [31], [96], is as follows:

**THEOREM 15.6.** *Given an inner faithful model  $\pi : C(G) \rightarrow M_K(C(T))$ , the Haar integration over  $G$  is given by*

$$\int_G = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{r=1}^k \int_G^r$$

with the truncations of the integration on the right being given by

$$\int_G^r = (\varphi \circ \pi)^{*r}$$

with  $\phi * \psi = (\phi \otimes \psi)\Delta$ , and with  $\varphi = \text{tr} \otimes \int_T$  being the random matrix trace.

PROOF. As a first observation, there is an obvious similarity here with the Woronowicz construction of the Haar measure, explained in chapter 1 above.

In fact, the above result holds more generally for any model  $\pi : C(G) \rightarrow B$ , with  $\varphi \in B^*$  being a faithful trace.

With this picture in hand, the Woronowicz construction simply corresponds to the case  $\pi = id$ , and the result itself is therefore a generalization of Woronowicz's result.

In order to prove now the result, we can proceed as in chapter 1. If we denote by  $\int'_G$  the limit in the statement, we must prove that this limit converges, and that we have:

$$\int'_G = \int_G$$

It is enough to check this on the coefficients of corepresentations, and if we let  $v = u^{\otimes k}$  be one of the Peter-Weyl corepresentations, we must prove that we have:

$$\left( id \otimes \int'_G \right) v = \left( id \otimes \int_G \right) v$$

We already know, from chapter 1 above, that the matrix on the right is the orthogonal projection onto  $Fix(v)$ :

$$\left( id \otimes \int_G \right) v = Proj[Fix(v)]$$

Regarding now the matrix on the left, the trick in [98] applied to the linear form  $\varphi\pi$  tells us that this is the orthogonal projection onto the 1-eigenspace of  $(id \otimes \varphi\pi)v$ :

$$\left( id \otimes \int'_G \right) v = Proj[1 \in (id \otimes \varphi\pi)v]$$

Now observe that, if we set  $V_{ij} = \pi(v_{ij})$ , we have the following formula:

$$(id \otimes \varphi\pi)v = (id \otimes \varphi)V$$

Thus, we can apply the trick in [98], or rather use the same computation as there, which is only based on the biunitarity condition, and we conclude that the 1-eigenspace that we are interested in equals  $Fix(V)$ .

But, according to Theorem 15.5, we have:

$$Fix(V) = Fix(v)$$

Thus, we have proved that we have  $\int'_G = \int_G$ , and this leads to the conclusion in the statement.  $\square$

Regarding now the law of the main character, we have the following result:

PROPOSITION 15.7. *Assume that a model  $\pi : C(G) \rightarrow M_K(C(T))$  is inner faithful, let*

$$\mu = \text{law}(\chi)$$

and let  $\mu^r$  be the law of  $\chi$  with respect to  $\int_G^r = (\varphi \circ \pi)^{*r}$ , where  $\varphi = \text{tr} \otimes \int_T$ .

(1) *We have the following convergence formula, in moments:*

$$\mu = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{r=0}^k \mu^r$$

(2) *The moments of  $\mu^r$  are the numbers  $c_\varepsilon^r = \text{Tr}(T_\varepsilon^r)$ , where:*

$$(T_\varepsilon)_{i_1 \dots i_p, j_1 \dots j_p} = \left( \text{tr} \otimes \int_T \right) (U_{i_1 j_1}^{\varepsilon_1} \dots U_{i_p j_p}^{\varepsilon_p})$$

PROOF. These formulae are both elementary, by using the convergence result established in Theorem 15.6, the proof being as follows:

(1) This follows from the limiting formula in Theorem 15.6.

(2) This follows from the definition of  $T_\varepsilon$ , by summing over equal indices,  $i_r = j_r$ .  $\square$

In order to detect the stationary models, we have the following criterion, from [14]:

THEOREM 15.8. *For a model  $\pi : C(G) \rightarrow M_K(C(T))$ , the following are equivalent:*

(1)  *$\text{Im}(\pi)$  is a Hopf algebra, and the Haar integration on it is:*

$$\psi = \left( \text{tr} \otimes \int_T \right) \pi$$

(2) *The linear form  $\psi = (\text{tr} \otimes \int_T) \pi$  satisfies the idempotent state property:*

$$\psi * \psi = \psi$$

(3) *We have  $T_\varepsilon^2 = T_\varepsilon$ ,  $\forall p \in \mathbb{N}$ ,  $\forall \varepsilon \in \{1, *\}^p$ , where:*

$$(T_\varepsilon)_{i_1 \dots i_p, j_1 \dots j_p} = \left( \text{tr} \otimes \int_T \right) (U_{i_1 j_1}^{\varepsilon_1} \dots U_{i_p j_p}^{\varepsilon_p})$$

*If these conditions are satisfied, we say that  $\pi$  is stationary on its image.*

PROOF. Given a matrix model  $\pi : C(G) \rightarrow M_K(C(T))$  as in the statement, we can factorize it via its Hopf image, as in Definition 15.4 above:

$$\pi : C(G) \rightarrow C(H) \rightarrow M_K(C(T))$$

Now observe that (1,2,3) above depend only on the factorized representation:

$$\nu : C(H) \rightarrow M_K(C(T))$$

Thus, we can assume in practice that we have  $G = H$ , which means that we can assume that  $\pi$  is inner faithful. With this assumption made, the integration formula in Theorem 15.6 applies to our situation, and the proof of the equivalences goes as follows:

(1)  $\implies$  (2) This is clear from definitions, because the Haar integration on any compact quantum group satisfies the idempotent state equation:

$$\psi * \psi = \psi$$

(2)  $\implies$  (1) Assuming  $\psi * \psi = \psi$ , we have  $\psi^{*r} = \psi$  for any  $r \in \mathbb{N}$ , and Theorem 15.6 gives  $\int_G = \psi$ . By using now Theorem 15.3, we obtain the result.

In order to establish now (2)  $\iff$  (3), we use the following elementary formula, which comes from the definition of the convolution operation:

$$\psi^{*r}(u_{i_1 j_1}^{e_1} \dots u_{i_p j_p}^{e_p}) = (T_e^r)_{i_1 \dots i_p, j_1 \dots j_p}$$

(2)  $\implies$  (3) Assuming  $\psi * \psi = \psi$ , by using the above formula at  $r = 1, 2$  we obtain that the matrices  $T_e$  and  $T_e^2$  have the same coefficients, and so they are equal.

(3)  $\implies$  (2) Assuming  $T_e^2 = T_e$ , by using the above formula at  $r = 1, 2$  we obtain that the linear forms  $\psi$  and  $\psi * \psi$  coincide on any product of coefficients  $u_{i_1 j_1}^{e_1} \dots u_{i_p j_p}^{e_p}$ . Now since these coefficients span a dense subalgebra of  $C(G)$ , this gives the result.  $\square$

### 15b. Flat models

As a first illustration, we can apply the above criterion to certain models for  $O_N^*, U_N^*$ . We first have the following result, coming from the work in [15], [29], [46]:

PROPOSITION 15.9. *We have a matrix model as follows,*

$$\begin{aligned} C(O_N^*) &\rightarrow M_2(C(U_N)) \\ u_{ij} &\rightarrow \begin{pmatrix} 0 & v_{ij} \\ \bar{v}_{ij} & 0 \end{pmatrix} \end{aligned}$$

where  $v$  is the fundamental corepresentation of  $C(U_N)$ , as well as a model as follows,

$$\begin{aligned} C(U_N^*) &\rightarrow M_2(C(U_N \times U_N)) \\ u_{ij} &\rightarrow \begin{pmatrix} 0 & v_{ij} \\ w_{ij} & 0 \end{pmatrix} \end{aligned}$$

where  $v, w$  are the fundamental corepresentations of the two copies of  $C(U_N)$ .

PROOF. It is routine to check that the matrices on the right are indeed biunitaries, and since the first matrix is also self-adjoint, we obtain in this way models as follows:

$$\begin{aligned} C(O_N^+) &\rightarrow M_2(C(U_N)) \\ C(U_N^+) &\rightarrow M_2(C(U_N \times U_N)) \end{aligned}$$

Regarding now the half-commutation relations, this comes from something general, regarding the antidiagonal  $2 \times 2$  matrices. Consider indeed matrices as follows:

$$X_i = \begin{pmatrix} 0 & x_i \\ y_i & 0 \end{pmatrix}$$



We have then the following computation:

$$X_i X_j X_k = \begin{pmatrix} 0 & x_i \\ y_i & 0 \end{pmatrix} \begin{pmatrix} 0 & x_j \\ y_j & 0 \end{pmatrix} \begin{pmatrix} 0 & x_k \\ y_k & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_i y_j x_k \\ y_i x_j y_k & 0 \end{pmatrix}$$

Since this quantity is symmetric in  $i, k$ , we obtain  $X_i X_j X_k = X_k X_j X_i$ . Thus, the antidiagonal  $2 \times 2$  matrices half-commute, and so our models factorize as claimed.  $\square$

We can now formulate our first concrete modelling theorem, as follows:

**THEOREM 15.10.** *The above antidiagonal models, namely*

$$C(O_N^*) \rightarrow M_2(C(U_N))$$

$$C(U_N^*) \rightarrow M_2(C(U_N \times U_N))$$

*are both stationary, and in particular they are faithful.*

**PROOF.** Let us first discuss the case of  $O_N^*$ . We will use Theorem 15.8 (3). Since the fundamental representation is self-adjoint, the various matrices  $T_e$  with  $e \in \{1, *\}^p$  are all equal. We denote this common matrix by  $T_p$ . We have, by definition:

$$(T_p)_{i_1 \dots i_p, j_1 \dots j_p} = \left( tr \otimes \int_H \right) \left[ \begin{pmatrix} 0 & v_{i_1 j_1} \\ \bar{v}_{i_1 j_1} & 0 \end{pmatrix} \dots \dots \begin{pmatrix} 0 & v_{i_p j_p} \\ \bar{v}_{i_p j_p} & 0 \end{pmatrix} \right]$$

Since when multiplying an odd number of antidiagonal matrices we obtain an atidiagonal matrix, we have  $T_p = 0$  for  $p$  odd. Also, when  $p$  is even, we have:

$$\begin{aligned} (T_p)_{i_1 \dots i_p, j_1 \dots j_p} &= \left( tr \otimes \int_H \right) \begin{pmatrix} v_{i_1 j_1} \dots \bar{v}_{i_p j_p} & 0 \\ 0 & \bar{v}_{i_1 j_1} \dots v_{i_p j_p} \end{pmatrix} \\ &= \frac{1}{2} \left( \int_H v_{i_1 j_1} \dots \bar{v}_{i_p j_p} + \int_H \bar{v}_{i_1 j_1} \dots v_{i_p j_p} \right) \\ &= \int_H Re(v_{i_1 j_1} \dots \bar{v}_{i_p j_p}) \end{aligned}$$

We have  $T_p^2 = T_p = 0$  when  $p$  is odd, so we are left with proving that for  $p$  even we have  $T_p^2 = T_p$ . For this purpose, we use the following formula:

$$Re(x)Re(y) = \frac{1}{2} (Re(xy) + Re(x\bar{y}))$$

By using this identity for each of the terms which appear in the product, and multi-index notations in order to simplify the writing, we obtain:

$$\begin{aligned}
 & (T_p^2)_{ij} \\
 = & \sum_{k_1 \dots k_p} (T_p)_{i_1 \dots i_p, k_1 \dots k_p} (T_p)_{k_1 \dots k_p, j_1 \dots j_p} \\
 = & \int_H \int_H \sum_{k_1 \dots k_p} \operatorname{Re}(v_{i_1 k_1} \dots \bar{v}_{i_p k_p}) \operatorname{Re}(w_{k_1 j_1} \dots \bar{w}_{k_p j_p}) dv dw \\
 = & \frac{1}{2} \int_H \int_H \sum_{k_1 \dots k_p} \operatorname{Re}(v_{i_1 k_1} w_{k_1 j_1} \dots \bar{v}_{i_p k_p} \bar{w}_{k_p j_p}) + \operatorname{Re}(v_{i_1 k_1} \bar{w}_{k_1 j_1} \dots \bar{v}_{i_p k_p} w_{k_p j_p}) dv dw \\
 = & \frac{1}{2} \int_H \int_H \operatorname{Re}((vw)_{i_1 j_1} \dots (\bar{v}\bar{w})_{i_p j_p}) + \operatorname{Re}((v\bar{w})_{i_1 j_1} \dots (\bar{v}w)_{i_p j_p}) dv dw
 \end{aligned}$$

Now since  $vw \in H$  is uniformly distributed when  $v, w \in H$  are uniformly distributed, the quantity on the left integrates up to  $(T_p)_{ij}$ . Also, since  $H$  is conjugation-stable,  $\bar{w} \in H$  is uniformly distributed when  $w \in H$  is uniformly distributed, so the quantity on the right integrates up to the same quantity, namely  $(T_p)_{ij}$ . Thus, we have:

$$(T_p^2)_{ij} = \frac{1}{2} \left( (T_p)_{ij} + (T_p)_{ij} \right) = (T_p)_{ij}$$

Summarizing, we have obtained that for any  $p$ , we have  $T_p^2 = T_p$ . Thus Theorem 15.8 applies, and shows that our model is stationary, as claimed. As for the proof of the stationarity for the model for  $U_N^*$ , this is similar. See [15].  $\square$

As a second illustration, regarding  $H_N^*, K_N^*$ , we have:

**THEOREM 15.11.** *We have a stationary matrix model as follows,*

$$\begin{aligned}
 C(H_N^*) & \rightarrow M_2(C(K_N)) \\
 u_{ij} & \rightarrow \begin{pmatrix} 0 & v_{ij} \\ \bar{v}_{ij} & 0 \end{pmatrix}
 \end{aligned}$$

where  $v$  is the fundamental corepresentation of  $C(K_N)$ , as well as a stationary model

$$\begin{aligned}
 C(K_N^*) & \rightarrow M_2(C(K_N \times K_N)) \\
 u_{ij} & \rightarrow \begin{pmatrix} 0 & v_{ij} \\ w_{ij} & 0 \end{pmatrix}
 \end{aligned}$$

where  $v, w$  are the fundamental corepresentations of the two copies of  $C(K_N)$ .

**PROOF.** This follows by adapting the proof of Proposition 15.9 and Theorem 15.10 above, by adding there the  $H_N^+, K_N^+$  relations. All this is in fact part of a more general phenomenon, concerning half-liberation in general, and we refer here to [15], [46].  $\square$

Let us go back now to the general problem of modelling a given quantum permutation group  $G \subset S_N^+$ . The “simplest” matrix models that we can use are as follows:

DEFINITION 15.12. *Given a subgroup  $G \subset S_N^+$ , a random matrix model of type*

$$\pi : C(G) \rightarrow M_K(C(T))$$

*is called flat when the fibers  $P_{ij}^x = \pi(u_{ij})(x)$  all have rank 1.*

Observe that we must have  $N = K$  in this case. Also, the quantum permutation group  $G \subset S_N^+$  to be modelled must be transitive, in order for such a model to exist.

Following [32], let us formulate as well a second definition, which is a bit more general, covering many interesting examples of quantum permutation groups  $G \subset S_N^+$  which are not transitive, such as the quasi-transitive ones discussed in the previous section:

DEFINITION 15.13. *Given a subgroup  $G \subset S_N^+$ , a random matrix model of type*

$$\pi : C(G) \rightarrow M_K(C(T))$$

*is called quasi-flat when the fibers  $P_{ij}^x = \pi(u_{ij})(x)$  all have rank  $\leq 1$ .*

Observe that the functions  $x \mapsto r_{ij}^x = \text{rank}(P_{ij}^x)$  are locally constant over  $T$ , so they are constant over the connected components of  $X$ . Thus, when  $T$  is connected, our assumption is that we have  $r_{ij}^x = r_{ij} \in \{0, 1\}$ , for any  $x \in T$ , and any  $i, j$ .

As a first result now, regarding the quasi-flat models, we have:

PROPOSITION 15.14. *Assume that we have a quasi-flat model  $\pi : C(G) \rightarrow M_K(C(T))$ , mapping  $u_{ij} \rightarrow P_{ij}$ , and consider the matrix  $r_{ij} = \text{rank}(P_{ij})$ .*

- (1)  *$r$  is bistochastic, with sums  $K$ .*
- (2) *We have  $r_{ij} \leq \varepsilon_{ij}$ , for any  $i, j$ .*
- (3) *If  $G$  is quasi-transitive, with orbits of size  $K$ , then  $r_{ij} = \varepsilon_{ij}$  for any  $i, j$ .*
- (4) *If  $\pi$  is assumed to be flat, then  $G$  must be transitive.*

PROOF. These results are all elementary, the proof being as follows:

- (1) This is clear from the fact that each  $P^x = (P_{ij}^x)$  is bistochastic, with sums 1.
- (2) This simply comes from  $u_{ij} = 0 \implies P_{ij} = 0$ .
- (3) The matrices  $r = (r_{ij})$  and  $\varepsilon = (\varepsilon_{ij})$  are both bistochastic, with sums  $K$ , and they satisfy  $r_{ij} \leq \varepsilon_{ij}$ , for any  $i, j$ . Thus, these matrices must be equal, as stated.
- (4) This is clear, because  $\text{rank}(P_{ij}) = 1$  implies  $u_{ij} \neq 0$ , for any  $i, j$ . □

In order to construct now universal quasi-flat models, it is convenient to identify the rank one projections in  $M_N(\mathbb{C})$  with the elements of the complex projective space  $P_{\mathbb{C}}^{N-1}$ . We first have the following observation, which goes back to [33]:

PROPOSITION 15.15. *The algebra  $C(S_N^+)$  has a universal flat model, given by*

$$\pi_N : C(S_N^+) \rightarrow M_N(C(X_N))$$

$$\pi_N(u_{ij}) = [P \rightarrow P_{ij}]$$

where  $X_N$  is the set of matrices  $P \in M_N(P_{\mathbb{C}}^{N-1})$  which are bistochastic with sums 1.

PROOF. This is clear from definitions, because any flat model  $C(S_N^+) \rightarrow M_N(\mathbb{C})$  must map the magic corepresentation  $u = (u_{ij})$  into a matrix  $P = (P_{ij})$  belonging to  $X_N$ .  $\square$

Regarding now the general quasi-transitive case, we have here:

THEOREM 15.16. *Given a quasi-transitive subgroup  $G \subset S_N^+$ , with orbits of size  $K$ , we have a universal quasi-flat model  $\pi : C(G) \rightarrow M_K(C(X))$ , constructed as follows:*

(1) For  $G = \underbrace{S_K^+ \hat{*} \dots \hat{*} S_K^+}_{M \text{ terms}}$  with  $N = KM$ , the universal model space is  $X_{N,K} =$

$\underbrace{X_K \times \dots \times X_K}_{M \text{ terms}}$ , and with  $u = \text{diag}(u^1, \dots, u^M)$  the modelling map is:

$$\pi_{N,K}(u_{ij}^r) = [(P^1, \dots, P^M) \rightarrow P_{ij}^r]$$

(2) In general, the model space is the submanifold  $X_G \subset X_{N,K}$  obtained via the Tannakian relations defining  $G$ .

PROOF. This is standard by using Tannakian duality, as follows:

(1) This follows from Tannakian duality, by using Proposition 15.14 (3), which tells us that the 0 entries of the model must appear exactly where  $u = (u_{ij})$  has 0 entries.

(2) Assume that  $G \subset S_N^+$  is quasi-transitive, with orbits of size  $K$ . We have then an inclusion  $G \subset \underbrace{S_K^+ \hat{*} \dots \hat{*} S_K^+}_{M \text{ terms}}$ , and in order to construct the universal quasi-flat model for

$C(G)$ , we need a universal solution to the following factorization problem:

$$C(\underbrace{S_K^+ \hat{*} \dots \hat{*} S_K^+}_{M \text{ terms}}) \rightarrow M_K(C(X_{N,K}))$$

↓

↓

$$C(G) \rightarrow M_K(C(X_G))$$

But, the solution to this latter question is given by the following construction, with the Hom-spaces at left being taken as usual in a formal sense:

$$C(X_G) = C(X_{N,K}) / \left( T \in \text{Hom}(P^{\otimes k}, P^{\otimes l}), \forall k, l \in \mathbb{N}, \forall T \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \right)$$

With this result in hand, the Gelfand spectrum of the algebra on the left is then an algebraic submanifold  $X_G \subset X_{N,K}$ , having the desired universality property.  $\square$

As an illustration, let us discuss now the classical case. With the convention that we identify the rank one projections in  $M_K(\mathbb{C})$  with the corresponding elements of the complex projective space  $P_{\mathbb{C}}^{K-1}$ , we have the following result, from [32]:

**THEOREM 15.17.** *Given a quasi-transitive group  $G \subset S_N$ , with orbits having size  $K$ , the associated universal quasi-flat model space is  $X_G = E_K \times L_{N,K}^G$ , where:*

$$E_K = \left\{ P_1, \dots, P_K \in P_{\mathbb{C}}^{K-1} \mid P_i \perp P_j, \forall i, j \right\}$$

$$L_{N,K}^G = \left\{ \sigma_1, \dots, \sigma_K \in G \mid \sigma_1(i), \dots, \sigma_K(i) \text{ distinct}, \forall i \in \{1, \dots, N\} \right\}$$

*In addition, assuming that we have  $L_{N,K}^G \neq \emptyset$ , the universal quasi-flat model is stationary, with respect to the Haar measure on  $E_K$  times the discrete measure on  $L_{N,K}^G$ .*

**PROOF.** This result is from [32], the idea being as follows:

(1) Let us call “sparse Latin square” any matrix  $L \in M_N(*, 1, \dots, K)$  whose rows and columns consists of a permutation of the numbers  $1, \dots, K$ , completed with  $*$  entries.

(2) Our claim is that the quasi-flat representations  $\pi : C(S_N) \rightarrow M_K(\mathbb{C})$  appear as follows, where  $P_1, \dots, P_K \in M_K(\mathbb{C})$  are rank 1 projections, summing up to 1, and where  $L \in M_N(*, 1, \dots, K)$  is a sparse Latin square, with the convention  $P_* = 0$ :

$$u_{ij} \rightarrow P_{L_{ij}}$$

Indeed, assuming that  $\pi : C(S_N) \rightarrow M_K(\mathbb{C})$  is quasi-flat, the elements  $P_{ij} = \pi(u_{ij})$  are projections of rank  $\leq 1$ , which pairwise commute, and form a magic unitary.

Let  $P_1, \dots, P_K \in M_K(\mathbb{C})$  be the rank one projections appearing in the first row of  $P = (P_{ij})$ . Since these projections form a partition of unity with rank one projections, any rank one projection  $Q \in M_K(\mathbb{C})$  commuting with all of them satisfies:

$$Q \in \{P_1, \dots, P_K\}$$

In particular we have  $P_{ij} \in \{P_1, \dots, P_K\}$  for any  $i, j$  such that  $P_{ij} \neq 0$ . Thus we can write  $u_{ij} \rightarrow P_{L_{ij}}$ , for a certain matrix  $L \in M_N(*, 1, \dots, K)$ , with the convention  $P_* = 0$ .

In order to finish, the remark is that  $u_{ij} \rightarrow P_{L_{ij}}$  defines a representation  $\pi : C(S_N) \rightarrow M_K(\mathbb{C})$  precisely when the matrix  $P = (P_{L_{ij}})_{ij}$  is magic. But this condition tells us precisely that  $L$  must be a sparse Latin square, as desired.

(3) In order to finish, we must compute the Hopf image. Given a sparse Latin square  $L \in M_N(*, 1, \dots, K)$ , consider the permutations  $\sigma_1, \dots, \sigma_K \in S_N$  given by:

$$\sigma_x(j) = i \iff L_{ij} = x$$

Our claim is that the Hopf image associated to a representation  $\pi : C(S_N) \rightarrow M_K(\mathbb{C})$ ,  $u_{ij} \rightarrow P_{L_{ij}}$  as above is then the algebra  $C(G_L)$ , where:

$$G_L = \langle \sigma_1, \dots, \sigma_K \rangle \subset S_N$$

Indeed, the image of  $\pi$  being generated by  $P_1, \dots, P_K$ , we have an isomorphism of algebras  $\alpha : Im(\pi) \simeq C(1, \dots, K)$  given by  $P_i \rightarrow \delta_i$ . Consider the following diagram:

$$\begin{array}{ccccc}
 C(S_N) & \xrightarrow{\pi} & Im(\pi) & \longrightarrow & M_K(\mathbb{C}) \\
 & \searrow \varphi & \downarrow \alpha & & \\
 & & C(1, \dots, K) & & 
 \end{array}$$

Here the map on the right is the canonical inclusion and  $\varphi = \alpha\pi$ . Since the Hopf image of  $\pi$  coincides with the one of  $\varphi$ , it is enough to compute the latter. We know that  $\varphi$  is given by  $\varphi(u_{ij}) = \delta_{L_{ij}}$ , with the convention  $\delta_* = 0$ . By Gelfand duality,  $\varphi$  must come from a certain map  $\sigma : \{1, \dots, K\} \rightarrow S_N$ , via the transposition formula:

$$\varphi(f)(x) = f(\sigma_x)$$

With the choice  $f = u_{ij}$ , we obtain  $\delta_{L_{ij}}(x) = u_{ij}(\sigma_x)$ . Now observe that:

$$\delta_{L_{ij}}(x) = \begin{cases} 1 & \text{if } L_{ij} = x \\ 0 & \text{otherwise} \end{cases}$$

We have as well the following formula:

$$u_{ij}(\sigma_x) = \begin{cases} 1 & \text{if } \sigma_x(j) = i \\ 0 & \text{otherwise} \end{cases}$$

We conclude that  $\sigma_x$  is the permutation in the statement. Summarizing, we have shown that  $\varphi$  comes by transposing the map  $x \rightarrow \sigma_x$ , with  $\sigma_x$  being as in the statement. Thus the Hopf image of  $\varphi$  is the algebra  $C(G_L)$ , with:

$$G_L = \langle \sigma_1, \dots, \sigma_K \rangle$$

Thus, we are led to the conclusion in the statement. □

There are many other explicit computations in the quasi-flat case, especially in the group dual case, and we refer here to [22], [32].

### 15c. Quasi-flatness

As an illustration, let us discuss the following types of models:

$$\pi : C(S_K^+) \rightarrow M_2(\mathbb{C})$$

In order to do this, let us start with a model  $\pi : C(G) \rightarrow M_N(\mathbb{C})$  for a quantum permutation group  $G \subset S_K^+$  is called flat if  $N = K$ , and the projections  $p_{ij} = \pi(u_{ij})$  are all of rank 1, and quasi-flat if the projections  $p_{ij} = \pi(u_{ij})$  are all of rank 0 or 1. Up to a simple manipulation, we are in this latter situation, of quasi-flat models:

PROPOSITION 15.18. *Up to subtracting an identity diagonal block, which won't change the image of the model, any model of type*

$$\pi : C(S_K^+) \rightarrow M_2(\mathbb{C})$$

*can be assumed to be quasi-flat, in the sense that the projections  $p_{ij} = \pi(u_{ij})$ , belonging to the matrix algebra  $M_2(\mathbb{C})$ , are all of rank 0 or 1.*

PROOF. We know that the matrix  $p = (p_{ij})$  formed by the projections  $p_{ij} = \pi(u_{ij})$  is a  $K \times K$  magic matrix having as entries projections  $r \in M_2(\mathbb{C})$ . The point now is that these latter projections  $r \in M_2(\mathbb{C})$  can be of rank 0, 1, 2, and up to a permutation of the indices, we can isolate the rank 2 projections in a matrix block, so that our magic matrix  $p = (p_{ij})$  looks as follows, with only rank 0, 1 projections in the lower block  $q$ :

$$p = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$$

Thus, we are led to the conclusion in the statement. □

In view of the above result, we can assume that our model is quasi-flat. Now in order to classify such models, let us first recall from [32] that we have:

DEFINITION 15.19. *A “sparse Latin square” is a square matrix*

$$L \in M_K(*, 1, \dots, R)$$

*having the property that each of its rows and columns consists of a permutation of the numbers  $1, \dots, R$ , completed with  $*$  entries.*

In the case  $R = N$ , where there are no  $*$  symbols, we recover the usual Latin squares. In general, however, the combinatorics of these matrices is more complicated than that of the usual Latin squares. Here are a few examples of such matrices, at  $R = 2$ :

$$\begin{pmatrix} 1 & 2 & * \\ 2 & * & 1 \\ * & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & * & * \\ 2 & * & 1 & * \\ * & 1 & * & 2 \\ * & * & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & * & * \\ 2 & 1 & * & * \\ * & * & 1 & 2 \\ * & * & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & * & * & * \\ 2 & * & 1 & * & * \\ * & 1 & 2 & * & * \\ * & * & * & 1 & 2 \\ * & * & * & 2 & 1 \end{pmatrix}$$

We will be back to this, with some structure and classification results.

With this notion in hand, the first result that we need, also from [32], is as follows:

PROPOSITION 15.20. *The quasi-flat representations  $\pi : C(S_K) \rightarrow M_R(\mathbb{C})$  appear as*

$$u_{ij} \rightarrow P_{L_{ij}}$$

where  $P_1, \dots, P_R \in M_R(\mathbb{C})$  are rank 1 projections, summing up to 1, and where

$$L \in M_K(*, 1, \dots, R)$$

is a sparse Latin square, with the convention  $P_* = 0$ .

PROOF. Assuming that  $\pi : C(S_K) \rightarrow M_R(\mathbb{C})$  is quasi-flat, the elements  $P_{ij} = \pi(u_{ij})$  are projections of rank  $\leq 1$ , which pairwise commute, and form a magic unitary.

Let  $P_1, \dots, P_R \in M_R(\mathbb{C})$  be the rank one projections appearing in the first row of  $P = (P_{ij})$ . Since these projections form a partition of unity with rank one projections, any rank one projection  $Q \in M_R(\mathbb{C})$  commuting with all of them satisfies:

$$Q \in \{P_1, \dots, P_R\}$$

In particular we have  $P_{ij} \in \{P_1, \dots, P_R\}$  for any  $i, j$  such that  $P_{ij} \neq 0$ . Thus we can write  $u_{ij} \rightarrow P_{L_{ij}}$ , for a certain matrix  $L \in M_N(*, 1, \dots, K)$ , with the convention  $P_* = 0$ .

In order to finish, the remark is that  $u_{ij} \rightarrow P_{L_{ij}}$  defines a representation  $\pi : C(S_K) \rightarrow M_R(\mathbb{C})$  precisely when the matrix  $P = (P_{L_{ij}})_{ij}$  is magic. But this condition tells us precisely that  $L$  must be a sparse Latin square, in the sense of Definition 15.19.  $\square$

In the case  $R = 2$ , that we are interested in, this result reads:

PROPOSITION 15.21. *The quasi-flat representations  $\pi : C(S_K) \rightarrow M_2(\mathbb{C})$  appear as*

$$u_{ij} \rightarrow P_{L_{ij}}$$

where  $P_1, P_2 \in M_2(\mathbb{C})$  are rank 1 projections, satisfying  $P_1 + P_2 = 1$ , and where  $L \in M_K(*, 1, 2)$  is a sparse Latin square, with the convention  $P_* = 0$ .

PROOF. This follows indeed from Proposition 15.20 above.  $\square$

In what regards now the models of  $S_K^+$  that we are interested in, we have:

THEOREM 15.22. *The matrix models of the following type, assumed as in Proposition 15.18 to be quasi-flat, up to subtracting an identity matrix block,*

$$\pi : C(S_K^+) \rightarrow M_2(\mathbb{C})$$

appear, via a permutations of the rows and columns, in the following way,

$$\pi : C(S_K^+) \rightarrow C(S_{K_1} \hat{*} \dots \hat{*} S_{K_n}) \rightarrow M_2(\mathbb{C})$$

by making a direct sum of sparse Latin square models, as in Proposition 15.21.



PROOF. With notations from Proposition 15.18 and its proof, we know that the square matrix  $p = (p_{ij})$  formed by the projections  $p_{ij} = \pi(u_{ij})$  is a  $K \times K$  magic matrix having as entries projections  $r \in M_2(\mathbb{C})$ , having rank 0 or 1. We must prove that, up to a permutation of the rows and columns, this matrix becomes block-diagonal, with commuting entries in each block. But this can be done as follows:

(1) Up to a permutation of the indices, we can assume that  $r = p_{11}$  has rank 1. But then, in order for the sum on the first row to be 1, this first row must consist of  $r$ , or  $s = 1 - r$ , and of 0 projections. The same goes for the first column. Now by permuting the rows and columns, it follows that our matrix must be as follows:

$$p = \begin{pmatrix} r & s & 0 & \dots \\ s & * & & \\ 0 & & & \\ \vdots & & & \end{pmatrix}$$

If the  $*$  entry is  $r$ , we have our first block, and we can proceed by recurrence.

(2) If the  $*$  entry is 0, we need a copy of  $r$  both in the second row and second column, and so by permuting the rows and columns, our matrix must be as follows:

$$p = \begin{pmatrix} r & s & 0 & 0 & \dots \\ s & 0 & r & 0 & \dots \\ 0 & r & * & & \\ 0 & 0 & & & \\ \vdots & \vdots & & & \end{pmatrix}$$

Now if the  $*$  entry is  $s$ , we have our first block, and we can proceed by recurrence.

(3) And so on, and we end up with a block decomposition of our representation, as in the statement. As for the precise quantum group formulation of the factorization that we obtain, as in the statement, this follows from Proposition 15.21.  $\square$

Observe that the various manipulations in the above proof classify as well the possible sparse Latin squares  $L \in M_K(*, 1, 2)$  that we are interested in. To be more precise, there is one such square for each  $K \in \mathbb{N}$ , as follows:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & * \\ 2 & * & 1 \\ * & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & * & * \\ 2 & * & 1 & * \\ * & 1 & * & 2 \\ * & * & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & * & * & * \\ 2 & * & 1 & * & * \\ * & 1 & * & 2 & * \\ * & * & 2 & * & 1 \\ * & * & * & 1 & 2 \end{pmatrix}$$

In order to compute now the image of the representation in Theorem 15.22, we will need the following result, which is once again from [32]:

PROPOSITION 15.23. *Given a sparse Latin square  $L \in M_K(*, 1, \dots, R)$ , consider the permutations  $\sigma_1, \dots, \sigma_R \in S_K$  given by:*

$$\sigma_x(j) = i \iff L_{ij} = x$$

*The Hopf image associated to a representation  $\pi : C(S_K) \rightarrow M_R(\mathbb{C})$ ,  $u_{ij} \mapsto P_{L_{ij}}$  as above is then the algebra  $C(G_L)$ , where:*

$$G_L = \langle \sigma_1, \dots, \sigma_R \rangle \subset S_K$$

PROOF. This follows by using a well-known method, see [32]. □

We are led in this way to the following conclusion:

PROPOSITION 15.24. *The matrix models of the following type, assumed as in Proposition 15.18 to be quasi-flat, up to subtracting an identity matrix block,*

$$\pi : C(S_K^+) \rightarrow M_2(\mathbb{C})$$

*appear, via a permutations of the rows and columns, by making a direct sum of sparse Latin square models, as in Proposition 15.21, and factorize as follows,*

$$\pi : C(S_K^+) \rightarrow C(G_{K_1} \hat{*} \dots \hat{*} G_{K_n}) \rightarrow M_2(\mathbb{C})$$

*with the groups  $G_i \subset S_{K_i}$  being constructed from the sparse Latin squares as in Proposition 15.23, and with the algebra in the middle being the Hopf image.*

PROOF. This follows indeed from the above results. □

Now recall from the comments after the proof of Theorem 15.22 that the sparse Latin squares that we are interested in are classified, with 1 such square, that we will denote  $L_K$ , at any  $K \geq 2$ . With the convention  $L_1 = (1)$ , which is technically not a sparse Latin square, we can convert Proposition 15.24 into a final result, as follows:

THEOREM 15.25. *The matrix models of the following type,*

$$\pi : C(S_K^+) \rightarrow M_2(\mathbb{C})$$

*appear, via a permutations of the rows and columns, by making a direct sum of sparse Latin square models, and factorize as follows,*

$$\pi : C(S_K^+) \rightarrow C(D_{K_1} \hat{*} \dots \hat{*} D_{K_n}) \rightarrow M_2(\mathbb{C})$$

*with  $D_{K_i}$  being dihedral groups, and with the algebra in the middle being the Hopf image.*

PROOF. In view of of the above discussion, it is enough to prove that the group associated to the unique sparse Latin square  $L_K \in M_K(*, 1, 2)$  is the dihedral group

$D_N \subset S_N$ . For this purpose, observe that the permutations  $\sigma = \sigma_1$  and  $\tau = \sigma_2$  constructed in Proposition 15.23 are the following two involutions of  $S_K$ :

$$\begin{aligned} \sigma &= (1)(23)(45)(67) \dots \\ \tau &= (12)(34)(56)(78) \dots \end{aligned}$$

Since  $G_N = \langle \sigma, \tau \rangle$  is generated by two involutions, we have:

$$G_N = \langle 1, \sigma, \tau, \sigma\tau, \tau\sigma, \sigma\tau\sigma, \tau\sigma\tau, \dots \rangle$$

On the other hand since the permutation  $\sigma\tau \in S_K$  is generically given by  $i \rightarrow i \pm 2$ , with the sign depending on the parity of  $i$ , we have:

$$(\sigma\tau)^K = 1$$

Thus, the above sequence defining our group  $G_N$  by enumerating its elements stops after  $2N = |D_N|$  steps. With a bit more care, by using the quotient map  $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2 \rightarrow G_N$ , we conclude that we have  $G_N = D_N$ , as claimed.  $\square$

Summarizing, we have a good understanding of the models  $\pi : C(S_K^+) \rightarrow M_2(\mathbb{C})$ . As a basic example for the above, we have the “sudoku” construction. Given rank 1 projections satisfying  $r + s = 1, r' + s' = 1$ , consider the following matrix, which is sudoku magic:

$$p = \begin{pmatrix} r & 0 & s & 0 \\ 0 & r' & 0 & s' \\ s & 0 & r & 0 \\ 0 & s' & 0 & r' \end{pmatrix}$$

By permuting rows and columns, we obtain the following matrix, which is still magic, and which appears as above, from two  $2 \times 2$  Latin squares:

$$\tilde{p} = \begin{pmatrix} r & s & 0 & 0 \\ s & r & 0 & 0 \\ 0 & 0 & r' & s' \\ 0 & 0 & s' & r' \end{pmatrix}$$

Following now [22], [33] and related papers, we have the following result:

**PROPOSITION 15.26.** *Assuming that a model  $\pi : C(G) \rightarrow M_K(C(X))$  is inner faithful and quasi-flat, mapping  $u_{ij} \rightarrow Proj(\xi_{ij}^x)$ , with  $\|\xi_{ij}^x\| \in \{0, 1\}$ , we have*

$$T_p = \int_X T_p(\xi^x) dx$$

where the matrix  $T_p(\xi) \in M_{N^p}(\mathbb{C})$ , associated to an array  $\xi \in M_N(\mathbb{C}^K)$  is given by

$$T_p(\xi)_{i_1 \dots i_p, j_1 \dots j_p} = \frac{1}{K} \langle \xi_{i_1 j_1}, \xi_{i_2 j_2} \rangle \langle \xi_{i_2 j_2}, \xi_{i_3 j_3} \rangle \dots \langle \xi_{i_p j_p}, \xi_{i_1 j_1} \rangle$$

with the scalar product being the usual one on  $\mathbb{C}^K$ , taken linear at right.

PROOF. We have the following well-known computation, valid for any vectors  $\xi_1, \dots, \xi_p$  having norms  $\|\xi_i\| \in \{0, 1\}$ , with the scalar product being linear at right:

$$\begin{aligned} Proj(\xi_i)x &= \langle \xi_i, x \rangle \xi_i, \forall i \\ \implies Proj(\xi_1) \dots Proj(\xi_p)(x) &= \langle \xi_1, \xi_2 \rangle \dots \dots \langle \xi_{p-1}, \xi_p \rangle \langle \xi_p, x \rangle \xi_1 \\ \implies Tr(Proj(\xi_1) \dots Proj(\xi_p)) &= \langle \xi_1, \xi_2 \rangle \dots \dots \langle \xi_{p-1}, \xi_p \rangle \langle \xi_p, \xi_1 \rangle \end{aligned}$$

Thus, the matrices  $T_p$  can be computed as follows:

$$\begin{aligned} (T_p)_{i_1 \dots i_p, j_1 \dots j_p} &= \int_X tr \left( Proj(\xi_{i_1 j_1}^x) Proj(\xi_{i_2 j_2}^x) \dots Proj(\xi_{i_p j_p}^x) \right) dx \\ &= \frac{1}{K} \int_X \langle \xi_{i_1 j_1}^x, \xi_{i_2 j_2}^x \rangle \langle \xi_{i_2 j_2}^x, \xi_{i_3 j_3}^x \rangle \dots \dots \langle \xi_{i_p j_p}^x, \xi_{i_1 j_1}^x \rangle dx \\ &= \int_X (T_p(\xi^x))_{i_1 \dots i_p, j_1 \dots j_p} dx \end{aligned}$$

We therefore obtain the formula in the statement. See [22], [33]. □

An even more conceptual result, from [14], [22], [33], is as follows:

THEOREM 15.27. *Given an inner faithful quasi-flat model*

$$\begin{aligned} \pi : C(G) &\rightarrow M_K(C(X)) \\ u_{ij} &\rightarrow Proj(\xi_{ij}^x) \end{aligned}$$

with  $\|\xi_{ij}^x\| \in \{0, 1\}$ , the law of the normalized character  $\chi/K$  with respect to the truncated integral  $\int_G^r$  coincides with that of the Gram matrix of the vectors

$$\xi_{i_1 \dots i_r}^x = \frac{1}{\sqrt{K}} \cdot \xi_{i_1 i_2}^{x_1} \otimes \xi_{i_2 i_3}^{x_2} \otimes \dots \otimes \xi_{i_r i_1}^{x_r}$$

with respect to the normalized matrix trace, and to the integration functional on  $X^r$ .

PROOF. The moments  $C_p$  of the measure that we are interested in are given by:

$$\begin{aligned} C_p &= \frac{1}{K^p} \int_G^r \left( \sum_i u_{ii} \right)^p \\ &= \frac{1}{K^p} \sum_{i_1 \dots i_p} (T_p^r)_{i_1 \dots i_p, i_1 \dots i_p} \\ &= \frac{1}{K^p} \cdot Tr(T_p^r) \end{aligned}$$

The trace on the right is given by the following formula:

$$Tr(T_p^r) = \sum_{i_1^1 \dots i_p^r} (T_p)_{i_1^1 \dots i_p^1, i_1^2 \dots i_p^2} \dots \dots (T_p)_{i_1^r \dots i_p^r, i_1^1 \dots i_p^1}$$

In view of the formula in Proposition 15.26, this quantity will expand in terms of the matrices  $T_p(\xi)$  constructed there. To be more precise, we have:

$$Tr(T_p^r) = \int_{X^r} \sum_{i_1^1 \dots i_p^r} T_p(\xi^{x_1})_{i_1^1 \dots i_p^1, i_1^2 \dots i_p^2} \cdots T_p(\xi^{x_r})_{i_1^r \dots i_p^r, i_1^1 \dots i_p^1} dx$$

By using now the explicit formula of each  $T_p(\xi)$ , from Proposition 15.26, we have:

$$\begin{aligned} Tr(T_p^r) &= \frac{1}{K^r} \int_{X^r} \sum_{i_1^1 \dots i_p^r} \langle \xi_{i_1^1 i_1^2}^{x_1}, \xi_{i_2^1 i_2^2}^{x_1} \rangle \cdots \langle \xi_{i_p^1 i_p^2}^{x_1}, \xi_{i_1^1 i_1^2}^{x_1} \rangle \\ &\quad \cdots \\ &\quad \langle \xi_{i_1^r i_1^1}^{x_r}, \xi_{i_2^r i_2^1}^{x_r} \rangle \cdots \langle \xi_{i_p^r i_p^1}^{x_r}, \xi_{i_1^r i_1^1}^{x_r} \rangle dx \end{aligned}$$

By changing the order of the summation, we can write this formula as:

$$\begin{aligned} Tr(T_p^r) &= \frac{1}{K^r} \int_{X^r} \sum_{i_1^1 \dots i_p^r} \langle \xi_{i_1^1 i_1^2}^{x_1}, \xi_{i_2^1 i_2^2}^{x_1} \rangle \cdots \langle \xi_{i_p^1 i_p^2}^{x_1}, \xi_{i_2^1 i_2^2}^{x_1} \rangle \\ &\quad \cdots \\ &\quad \langle \xi_{i_p^1 i_p^2}^{x_1}, \xi_{i_1^1 i_1^2}^{x_1} \rangle \cdots \langle \xi_{i_p^r i_p^1}^{x_r}, \xi_{i_1^r i_1^1}^{x_r} \rangle dx \end{aligned}$$

But this latter formula can be written as follows:

$$\begin{aligned} Tr(T_p^r) &= K^{p-r} \int_{X^r} \sum_{i_1^1 \dots i_p^r} \frac{1}{K} \langle \xi_{i_1^1 i_1^2}^{x_1} \otimes \cdots \otimes \xi_{i_p^1 i_p^2}^{x_1}, \xi_{i_2^1 i_2^2}^{x_1} \otimes \cdots \otimes \xi_{i_2^1 i_2^2}^{x_1} \rangle \\ &\quad \cdots \\ &\quad \frac{1}{K} \langle \xi_{i_p^1 i_p^2}^{x_1} \otimes \cdots \otimes \xi_{i_p^r i_p^1}^{x_r}, \xi_{i_1^1 i_1^2}^{x_1} \otimes \cdots \otimes \xi_{i_1^r i_1^1}^{x_r} \rangle dx \end{aligned}$$

In terms of the vectors in the statement, and of their Gram matrix  $G_r^x$ , we obtain:

$$\begin{aligned} Tr(T_p^r) &= K^{p-r} \int_{X^r} \sum_{i_1^1 \dots i_p^r} \langle \xi_{i_1^1 \dots i_1^r}^x, \xi_{i_2^1 \dots i_2^r}^x \rangle \cdots \langle \xi_{i_p^1 \dots i_p^r}^x, \xi_{i_1^1 \dots i_1^r}^x \rangle dx \\ &= K^{p-r} \int_{X^r} \sum_{i_1^1 \dots i_p^r} (G_r^x)_{i_1^1 \dots i_1^r, i_2^1 \dots i_2^r} \cdots (G_r^x)_{i_p^1 \dots i_p^r, i_1^1 \dots i_1^r} dx \\ &= K^{p-r} \int_{X^r} Tr((G_r^x)^p) dx \end{aligned}$$

Summarizing, the moments of the measure in the statement are given by:

$$\begin{aligned} C_p &= \frac{1}{K^r} \int_{X^r} Tr((G_r^x)^p) dx \\ &= \left( tr \otimes \int_{X^r} \right) (G_r^p) \end{aligned}$$

This gives the formula in the statement of the theorem.  $\square$

**15d. Universal models**

Following [33], let us study now the universal flat model for  $C(S_N^+)$ . Given a flat magic unitary, we can write it, in a non-unique way, as  $u_{ij} = Proj(\xi_{ij})$ . The array  $\xi = (\xi_{ij})$  is then a “magic basis”, in the sense that each of its rows and columns is an orthonormal basis of  $\mathbb{C}^N$ . We are therefore led to two spaces, as follows:

DEFINITION 15.28. *Associated to any  $N \in \mathbb{N}$  are the following spaces:*

- (1)  $X_N$ , the space of all  $N \times N$  flat magic unitaries  $u = (u_{ij})$ .
- (2)  $K_N$ , the space of all  $N \times N$  magic bases  $\xi = (\xi_{ij})$ .

Let us recall now that the rank 1 projections  $p \in M_N(\mathbb{C})$  can be identified with the corresponding 1-dimensional subspaces  $E \subset \mathbb{C}^N$ , which are by definition the elements of the complex projective space  $P_{\mathbb{C}}^{N-1}$ . In addition, if we consider the complex sphere,  $S_{\mathbb{C}}^{N-1} = \{z \in \mathbb{C}^N \mid \sum_i |z_i|^2 = 1\}$ , we have a quotient map  $\pi : S_{\mathbb{C}}^{N-1} \rightarrow P_{\mathbb{C}}^{N-1}$  given by  $z \rightarrow Proj(z)$ . Observe that  $\pi(z) = \pi(z')$  precisely when  $z' = wz$ , for some  $w \in \mathbb{T}$ .

Consider as well the embedding  $U_N \subset (S_{\mathbb{C}}^{N-1})^N$  given by  $x \rightarrow (x_1, \dots, x_N)$ , where  $x_1, \dots, x_N$  are the rows of  $x$ . Finally, let us call an abstract matrix stochastic/bistochastic when the entries on each row/each row and column sum up to 1.

With these notations, the abstract model spaces  $X_N, K_N$  that we are interested in, and some related spaces, are as follows:

PROPOSITION 15.29. *We have inclusions and surjections as follows,*

$$\begin{array}{ccccc}
 K_N & \subset & U_N^N & \subset & M_N(S_{\mathbb{C}}^{N-1}) \\
 \downarrow & & \downarrow & & \downarrow \\
 X_N & \subset & Y_N & \subset & M_N(P_{\mathbb{C}}^{N-1})
 \end{array}$$

where  $X_N, Y_N$  consist of bistochastic/stochastic matrices, and  $K_N$  is the lift of  $X_N$ .

PROOF. This follows from the above discussion. Indeed, the quotient map  $S_{\mathbb{C}}^{N-1} \rightarrow P_{\mathbb{C}}^{N-1}$  induces the quotient map  $M_N(S_{\mathbb{C}}^{N-1}) \rightarrow M_N(P_{\mathbb{C}}^{N-1})$  at right, and the lift of the space of stochastic matrices  $Y_N \subset M_N(P_{\mathbb{C}}^{N-1})$  is then the rescaled group  $U_N^N$ , as claimed.  $\square$

In order to get some insight into the structure of  $X_N, K_N$ , we can use inspiration from the Sinkhorn algorithm. This algorithm starts with a  $N \times N$  matrix having positive entries and produces, via successive averagings over rows/columns, a bistochastic matrix.

In our situation, we would like to have an “averaging” map  $Y_N \rightarrow Y_N$ , whose infinite iteration lands in the model space  $X_N$ . Equivalently, we would like to have an “averaging” map  $U_N^N \rightarrow U_N^N$ , whose infinite iteration lands in  $K_N$ .

In order to construct such averaging maps, we use the orthogonalization procedure coming from the polar decomposition. First, we have the following result:

PROPOSITION 15.30. *We have orthogonalization maps as follows,*

$$\begin{array}{ccc} (S_{\mathbb{C}}^{N-1})^N & \xrightarrow{\alpha} & (S_{\mathbb{C}}^{N-1})^N \\ \downarrow & & \downarrow \\ (P_{\mathbb{C}}^{N-1})^N & \xrightarrow{\beta} & (P_{\mathbb{C}}^{N-1})^N \end{array}$$

where  $\alpha(x)_i = \text{Pol}([(x_i)_j]_{ij})$ , and  $\beta(p) = (P^{-1/2}p_iP^{-1/2})_i$ , with  $P = \sum_i p_i$ .

PROOF. Our first claim is that we have a factorization as in the statement. Indeed, pick  $p_1, \dots, p_N \in P_{\mathbb{C}}^{N-1}$ , and write  $p_i = \text{Proj}(x_i)$ , with  $\|x_i\| = 1$ . We can then apply  $\alpha$ , as to obtain a vector  $\alpha(x) = (x'_i)_i$ , and then set  $\beta(p) = (p'_i)$ , where  $p'_i = \text{Proj}(x'_i)$ .

Our first task is to prove that  $\beta$  is well-defined. Consider indeed vectors  $\tilde{x}_i$ , satisfying  $\text{Proj}(\tilde{x}_i) = \text{Proj}(x_i)$ . We have then  $\tilde{x}_i = \lambda_i x_i$ , for certain scalars  $\lambda_i \in \mathbb{T}$ , and so the matrix formed by these vectors is  $\tilde{M} = \Lambda M$ , with  $\Lambda = \text{diag}(\lambda_i)$ . It follows that  $\text{Pol}(\tilde{M}) = \Lambda \text{Pol}(M)$ , and so  $\tilde{x}'_i = \lambda_i x'_i$ , and finally  $\text{Proj}(\tilde{x}'_i) = \text{Proj}(x'_i)$ , as desired.

It remains to prove that  $\beta$  is given by the formula in the statement. For this purpose, observe first that, given  $x_1, \dots, x_N \in S_{\mathbb{C}}^{N-1}$ , with  $p_i = \text{Proj}(x_i)$  we have:

$$\begin{aligned} \sum_i p_i &= \sum_i [(\bar{x}_i)_k (x_i)_l]_{kl} \\ &= \sum_i (\bar{M}_{ik} M_{il})_{kl} \\ &= ((M^* M)_{kl})_{kl} \\ &= M^* M \end{aligned}$$

We can now compute the projections  $p'_i = \text{Proj}(x'_i)$ . Indeed, the coefficients of these projections are given by  $(p'_i)_{kl} = \bar{U}_{ik} U_{il}$  with  $U = MP^{-1/2}$ , and we obtain, as desired:

$$\begin{aligned} (p'_i)_{kl} &= \sum_{ab} \bar{M}_{ia} P_{ak}^{-1/2} M_{ib} P_{bl}^{-1/2} \\ &= \sum_{ab} P_{ka}^{-1/2} \bar{M}_{ia} M_{ib} P_{bl}^{-1/2} \\ &= \sum_{ab} P_{ka}^{-1/2} (p_i)_{ab} P_{bl}^{-1/2} \\ &= (P^{-1/2} p_i P^{-1/2})_{kl} \end{aligned}$$

An alternative proof uses the fact that the elements  $p'_i = P^{-1/2}p_iP^{-1/2}$  are self-adjoint, and sum up to 1. The fact that these elements are indeed idempotents can be checked directly, via  $p_iP^{-1}p_i = p_i$ , because this equality holds on  $\ker p_i$ , and also on  $x_i$ .  $\square$

As an illustration, here is how the orthogonalization works at  $N = 2$ :

PROPOSITION 15.31. *At  $N = 2$  the orthogonalization procedure for  $(Proj(x), Proj(y))$  amounts in considering the vectors  $(x \pm y)/\sqrt{2}$ , and then rotating by  $45^\circ$ .*

PROOF. By performing a rotation, we can restrict attention to the case  $x = (\cos t, \sin t)$  and  $y = (\cos t, -\sin t)$ , with  $t \in (0, \pi/2)$ . Here the computations are as follows:

$$\begin{aligned} M = \begin{pmatrix} \cos t & \sin t \\ \cos t & -\sin t \end{pmatrix} &\implies P = M^*M = \begin{pmatrix} 2\cos^2 t & 0 \\ 0 & 2\sin^2 t \end{pmatrix} \\ &\implies P^{-1/2} = |M|^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\cos t} & 0 \\ 0 & \frac{1}{\sin t} \end{pmatrix} \\ &\implies U = M|M|^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

Thus the orthogonalization procedure replaces  $(Proj(x), Proj(y))$  by the orthogonal projections on the vectors  $(\frac{1}{\sqrt{2}}(1, 1), \frac{1}{\sqrt{2}}(-1, 1))$ , and this gives the result.  $\square$

With these preliminaries in hand, let us discuss now the version that we need of the Sinkhorn algorithm. The orthogonalization procedure is as follows:

THEOREM 15.32. *The orthogonalization maps  $\alpha, \beta$  induce maps as follows,*

$$\begin{array}{ccc} U_N^N & \xrightarrow{\Phi} & U_N^N \\ \downarrow & & \downarrow \\ Y_N & \xrightarrow{\Psi} & Y_N \end{array}$$

which are the transposition maps on  $K_N, X_N$ , and which are projections at  $N = 2$ .

PROOF. It follows from definitions that  $\Phi(x)$  is obtained by putting the components of  $x = (x_i)$  in a row, then picking the  $j$ -th column vectors of each  $x_i$ , calling  $M_j$  this matrix, then taking the polar part  $x'_j = Pol(M_j)$ , and finally setting  $\Phi(x) = x'$ . Thus:

$$\begin{aligned} \Phi(x) &= Pol((x_{ij})_{ij}) \\ \Psi(u) &= (P_i^{-1/2}u_{ji}P_i^{-1/2})_{ij} \end{aligned}$$

Thus, the first assertion is clear, and the second assertion is clear too.  $\square$

Our claim is that the algorithm converges, as follows:



CONJECTURE 15.33. *The above maps  $\Phi, \Psi$  increase the volume,*

$$\text{vol} : U_N^N \rightarrow Y_N \rightarrow [0, 1], \quad \text{vol}(u) = \prod_j |\det((u_{ij})_i)|$$

*and respectively land, after an infinite number of steps, in  $K_N/X_N$ .*

As a main application of the above conjecture, the infinite iteration  $(\Phi^2)^\infty : U_N^N \rightarrow K_N$  would provide us with an integration on  $K_N$ , and hence on the quotient space  $K_N \rightarrow X_N$  as well, by taking the push-forward measures, coming from the Haar measure on  $U_N^N$ . In relation now with the matrix model problematics, we have:

CONJECTURE 15.34. *The universal  $N \times N$  flat matrix representation*

$$\pi_N : C(S_N^+) \rightarrow M_N(C(X_N)), \quad \pi_N(w_{ij}) = (u \rightarrow u_{ij})$$

*is faithful at  $N = 4$ , and is inner faithful at any  $N \geq 5$ .*

We refer to [33] and related papers for further details regarding this conjecture, and also for other applications of the Sinkhorn algorithm philosophy to modelling questions for the quantum permutation groups.

### 15e. Exercises

The matrix modelling problematics is something quite exciting, and we have several exercises here. In relation with the notion of stationarity, we have:

EXERCISE 15.35. *Work out the details for the fact that the stationarity of a model*

$$\pi : C(G) \rightarrow M_K(C(T))$$

*implies its faithfulness.*

This is something that we already discussed in the above, but with some standard functional analysis details missing. The problem is that of working out these details.

In relation with the notion of inner faithfulness, we have:

EXERCISE 15.36. *Find an example of an inner faithful model*

$$\pi : C(G) \rightarrow M_K(C(T))$$

*which is not faithful, not coming from a classical group, or a group dual.*

This is something quite tricky, and it is of course possible to cheat a bit here, by using product operations. The exercise asks for a high-quality counterexample.

## CHAPTER 16

### Weyl and Fourier

#### 16a. Weyl matrices

Following [25], [33], let us discuss now some more subtle examples of stationary models, related to the Pauli matrices, and Weyl matrices, and physics. We first have:

DEFINITION 16.1. *Given a finite abelian group  $H$ , the associated Weyl matrices are*

$$W_{ia} : e_b \rightarrow \langle i, b \rangle e_{a+b}$$

where  $i \in H$ ,  $a, b \in \widehat{H}$ , and where  $(i, b) \rightarrow \langle i, b \rangle$  is the Fourier coupling  $H \times \widehat{H} \rightarrow \mathbb{T}$ .

As a basic example, consider the cyclic group  $H = \mathbb{Z}_2 = \{0, 1\}$ . Here the Fourier coupling is given by  $\langle i, b \rangle = (-1)^{ib}$ , and so the Weyl matrices act via:

$$W_{00} : e_b \rightarrow e_b \quad , \quad W_{10} : e_b \rightarrow (-1)^b e_b$$

$$W_{11} : e_b \rightarrow (-1)^b e_{b+1} \quad , \quad W_{01} : e_b \rightarrow e_{b+1}$$

Thus, we have the following formulae:

$$W_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad W_{10} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$W_{11} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad W_{01} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We recognize here, up to some multiplicative factors, the four Pauli matrices. Now back to the general case, we have the following well-known result:

PROPOSITION 16.2. *The Weyl matrices are unitaries, and satisfy:*

- (1)  $W_{ia}^* = \langle i, a \rangle W_{-i, -a}$ .
- (2)  $W_{ia} W_{jb} = \langle i, b \rangle W_{i+j, a+b}$ .
- (3)  $W_{ia} W_{jb}^* = \langle j - i, b \rangle W_{i-j, a-b}$ .
- (4)  $W_{ia}^* W_{jb} = \langle i, a - b \rangle W_{j-i, b-a}$ .

PROOF. The unitarity follows from (3,4), and the rest of the proof goes as follows:

(1) We have indeed the following computation:

$$\begin{aligned}
 W_{ia}^* &= \left( \sum_b \langle i, b \rangle E_{a+b, b} \right)^* \\
 &= \sum_b \langle -i, b \rangle E_{b, a+b} \\
 &= \sum_b \langle -i, b-a \rangle E_{b-a, b} \\
 &= \langle i, a \rangle W_{-i, -a}
 \end{aligned}$$

(2) Here the verification goes as follows:

$$\begin{aligned}
 W_{ia}W_{jb} &= \left( \sum_d \langle i, b+d \rangle E_{a+b+d, b+d} \right) \left( \sum_d \langle j, d \rangle E_{b+d, d} \right) \\
 &= \sum_d \langle i, b \rangle \langle i+j, d \rangle E_{a+b+d, d} \\
 &= \langle i, b \rangle W_{i+j, a+b}
 \end{aligned}$$

(3,4) By combining the above two formulae, we obtain:

$$\begin{aligned}
 W_{ia}W_{jb}^* &= \langle j, b \rangle W_{ia}W_{-j, -b} \\
 &= \langle j, b \rangle \langle i, -b \rangle W_{i-j, a-b}
 \end{aligned}$$

We obtain as well the following formula:

$$\begin{aligned}
 W_{ia}^*W_{jb} &= \langle i, a \rangle W_{-i, -a}W_{jb} \\
 &= \langle i, a \rangle \langle -i, b \rangle W_{j-i, b-a}
 \end{aligned}$$

But this gives the formulae in the statement, and we are done.  $\square$

With  $n = |H|$ , we can use an isomorphism  $l^2(\widehat{H}) \simeq \mathbb{C}^n$  as to view each  $W_{ia}$  as a usual matrix,  $W_{ia} \in M_n(\mathbb{C})$ , and hence as a usual unitary,  $W_{ia} \in U_n$ . Also, given a vector  $\xi$ , we denote by  $Proj(\xi)$  the orthogonal projection onto  $\mathbb{C}\xi$ . Following [33], we have:

PROPOSITION 16.3. *Given a closed subgroup  $E \subset U_n$ , we have a representation*

$$\pi_H : C(S_N^+) \rightarrow M_N(C(E))$$

$$w_{ia, jb} \rightarrow [U \rightarrow Proj(W_{ia}UW_{jb}^*)]$$

where  $n = |H|$ ,  $N = n^2$ , and where  $W_{ia}$  are the Weyl matrices associated to  $H$ .

PROOF. The Weyl matrices being given by  $W_{ia} : e_b \rightarrow \langle i, b \rangle e_{a+b}$ , we have:

$$tr(W_{ia}) = \begin{cases} 1 & \text{if } (i, a) = (0, 0) \\ 0 & \text{if } (i, a) \neq (0, 0) \end{cases}$$

Together with the formulae in Proposition 16.2, this shows that the Weyl matrices are pairwise orthogonal with respect to the following scalar product on  $M_n(\mathbb{C})$ :

$$\langle x, y \rangle = \text{tr}(xy^*)$$

Thus, these matrices form an orthogonal basis of  $M_n(\mathbb{C})$ , consisting of unitaries:

$$W = \left\{ W_{ia} \mid i \in H, a \in \widehat{H} \right\}$$

Thus, each row and each column of the matrix  $\xi_{ia,jb} = W_{ia}UW_{jb}^*$  is an orthogonal basis of  $M_n(\mathbb{C})$ , and so the corresponding projections form a magic unitary, as claimed.  $\square$

We will need the following well-known result:

PROPOSITION 16.4. *With  $T = \text{Proj}(x_1) \dots \text{Proj}(x_p)$  and  $\|x_i\| = 1$  we have*

$$\langle T\xi, \eta \rangle = \langle \xi, x_p \rangle \langle x_p, x_{p-1} \rangle \dots \langle x_2, x_1 \rangle \langle x_1, \eta \rangle$$

for any  $\xi, \eta$ . In particular, we have:

$$\text{Tr}(T) = \langle x_1, x_p \rangle \langle x_p, x_{p-1} \rangle \dots \langle x_2, x_1 \rangle$$

PROOF. For  $\|x\| = 1$  we have  $\text{Proj}(x)\xi = \langle \xi, x \rangle x$ . This gives:

$$\begin{aligned} T\xi &= \text{Proj}(x_1) \dots \text{Proj}(x_p)\xi \\ &= \text{Proj}(x_1) \dots \text{Proj}(x_{p-1}) \langle \xi, x_p \rangle x_p \\ &= \text{Proj}(x_1) \dots \text{Proj}(x_{p-2}) \langle \xi, x_p \rangle \langle x_p, x_{p-1} \rangle x_{p-1} \\ &= \dots \\ &= \langle \xi, x_p \rangle \langle x_p, x_{p-1} \rangle \dots \langle x_2, x_1 \rangle x_1 \end{aligned}$$

Now by taking the scalar product with  $\eta$ , this gives the first assertion. As for the second assertion, this follows from the first assertion, by summing over  $\xi = \eta = e_i$ .  $\square$

Now back to the Weyl matrix models, let us first compute  $T_p$ . We have:

PROPOSITION 16.5. *We have the formula*

$$\begin{aligned} &(T_p)_{ia,jb} \\ &= \frac{1}{N} \langle i_1, a_1 - a_p \rangle \dots \langle i_p, a_p - a_{p-1} \rangle \langle j_1, b_1 - b_2 \rangle \dots \langle j_p, b_p - b_1 \rangle \\ &\quad \int_E \text{tr}(W_{i_1-i_2, a_1-a_2} U W_{j_2-j_1, b_2-b_1} U^*) \dots \text{tr}(W_{i_p-i_1, a_p-a_1} U W_{j_1-j_p, b_1-b_p} U^*) dU \end{aligned}$$

with all the indices varying in a cyclic way.

PROOF. By using the trace formula in Proposition 16.4 above, we obtain:

$$\begin{aligned}
& (T_p)_{ia,jb} \\
&= \left( \text{tr} \otimes \int_E \right) \left( \text{Proj}(W_{i_1 a_1} U W_{j_1 b_1}^*) \dots \text{Proj}(W_{i_p a_p} U W_{j_p b_p}^*) \right) \\
&= \frac{1}{N} \int_E \langle W_{i_1 a_1} U W_{j_1 b_1}^*, W_{i_p a_p} U W_{j_p b_p}^* \rangle \dots \langle W_{i_2 a_2} U W_{j_2 b_2}^*, W_{i_1 a_1} U W_{j_1 b_1}^* \rangle dU
\end{aligned}$$

In order to compute now the scalar products, observe that we have:

$$\begin{aligned}
\langle W_{ia} U W_{jb}^*, W_{kc} U W_{ld}^* \rangle &= \text{tr}(W_{jb} U^* W_{ia}^* W_{kc} U W_{ld}^*) \\
&= \text{tr}(W_{ia}^* W_{kc} U W_{ld}^* W_{jb} U^*) \\
&= \langle i, a - c \rangle \langle l, d - b \rangle \text{tr}(W_{k-i, c-a} U W_{j-l, b-d} U^*)
\end{aligned}$$

By plugging these quantities into the formula of  $T_p$ , we obtain the result.  $\square$

Consider now the Weyl group  $W = \{W_{ia}\} \subset U_n$ , that we already met in the proof of Proposition 16.3 above. We have the following result, from [33]:

THEOREM 16.6. *For any compact group  $W \subset E \subset U_n$ , the model*

$$\begin{aligned}
\pi_H : C(S_N^+) &\rightarrow M_N(C(E)) \\
w_{ia,jb} &\rightarrow [U \rightarrow \text{Proj}(W_{ia} U W_{jb}^*)]
\end{aligned}$$

constructed above is stationary on its image.

PROOF. We must prove that we have  $T_p^2 = T_p$ . We have:

$$\begin{aligned}
& (T_p^2)_{ia,jb} \\
&= \sum_{kc} (T_p)_{ia,kc} (T_p)_{kc,jb} \\
&= \frac{1}{N^2} \sum_{kc} \langle i_1, a_1 - a_p \rangle \dots \langle i_p, a_p - a_{p-1} \rangle \langle k_1, c_1 - c_2 \rangle \dots \langle k_p, c_p - c_1 \rangle \\
&\quad \langle k_1, c_1 - c_p \rangle \dots \langle k_p, c_p - c_{p-1} \rangle \langle j_1, b_1 - b_2 \rangle \dots \langle j_p, b_p - b_1 \rangle \\
&\quad \int_E \text{tr}(W_{i_1 - i_2, a_1 - a_2} U W_{k_2 - k_1, c_2 - c_1} U^*) \dots \text{tr}(W_{i_p - i_1, a_p - a_1} U W_{k_1 - k_p, c_1 - c_p} U^*) dU \\
&\quad \int_E \text{tr}(W_{k_1 - k_2, c_1 - c_2} V W_{j_2 - j_1, b_2 - b_1} V^*) \dots \text{tr}(W_{k_p - k_1, c_p - c_1} V W_{j_1 - j_p, b_1 - b_p} V^*) dV
\end{aligned}$$

By rearranging the terms, this formula becomes:

$$\begin{aligned}
& (T_p^2)_{ia,jb} \\
= & \frac{1}{N^2} \langle i_1, a_1 - a_p \rangle \dots \langle i_p, a_p - a_{p-1} \rangle \langle j_1, b_1 - b_2 \rangle \dots \langle j_p, b_p - b_1 \rangle \\
& \int_E \int_E \sum_{kc} \langle k_1 - k_p, c_1 - c_p \rangle \dots \langle k_p - k_{p-1}, c_p - c_{p-1} \rangle \\
& tr(W_{i_1-i_2, a_1-a_2} U W_{k_2-k_1, c_2-c_1} U^*) tr(W_{k_1-k_2, c_1-c_2} V W_{j_2-j_1, b_2-b_1} V^*) \\
& \dots \dots \dots \\
& tr(W_{i_p-i_1, a_p-a_1} U W_{k_1-k_p, c_1-c_p} U^*) tr(W_{k_p-k_1, c_p-c_1} V W_{j_1-j_p, b_1-b_p} V^*) dU dV
\end{aligned}$$

Let us denote by  $I$  the above double integral. By using  $W_{kc}^* = \langle k, c \rangle W_{-k, -c}$  for each of the couplings, and by moving as well all the  $U^*$  variables to the left, we obtain:

$$\begin{aligned}
I = & \int_E \int_E \sum_{kc} tr(U^* W_{i_1-i_2, a_1-a_2} U W_{k_2-k_1, c_2-c_1}) tr(W_{k_2-k_1, c_2-c_1}^* V W_{j_2-j_1, b_2-b_1} V^*) \\
& \dots \dots \dots \\
& tr(U^* W_{i_p-i_1, a_p-a_1} U W_{k_1-k_p, c_1-c_p}) tr(W_{k_1-k_p, c_1-c_p}^* V W_{j_1-j_p, b_1-b_p} V^*) dU dV
\end{aligned}$$

In order to perform now the sums, we use the following formula:

$$\begin{aligned}
tr(AW_{kc})tr(W_{kc}^*B) &= \frac{1}{N} \sum_{qrst} A_{qr}(W_{kc})_{rq}(W_{kc}^*)_{st} B_{ts} \\
&= \frac{1}{N} \sum_{qrst} A_{qr} \langle k, q \rangle \delta_{r-q, c} \langle k, -s \rangle \delta_{t-s, c} B_{ts} \\
&= \frac{1}{N} \sum_{qs} \langle k, q-s \rangle A_{q, q+c} B_{s+c, s}
\end{aligned}$$

If we denote by  $A_x, B_x$  the variables which appear in the formula of  $I$ , we have:

$$\begin{aligned}
I &= \frac{1}{N^p} \int_E \int_E \sum_{kcqs} \langle k_2 - k_1, q_1 - s_1 \rangle \dots \langle k_1 - k_p, q_p - s_p \rangle \\
& (A_1)_{q_1, q_1+c_2-c_1} (B_1)_{s_1+c_2-c_1, s_1} \dots (A_p)_{q_p, q_p+c_1-c_p} (B_p)_{s_p+c_1-c_p, s_p} \\
= & \frac{1}{N^p} \int_E \int_E \sum_{kcqs} \langle k_1, q_p - s_p - q_1 + s_1 \rangle \dots \langle k_p, q_{p-1} - s_{p-1} - q_p + s_p \rangle \\
& (A_1)_{q_1, q_1+c_2-c_1} (B_1)_{s_1+c_2-c_1, s_1} \dots (A_p)_{q_p, q_p+c_1-c_p} (B_p)_{s_p+c_1-c_p, s_p}
\end{aligned}$$

Now observe that we can perform the sums over  $k_1, \dots, k_p$ . We obtain in this way a multiplicative factor  $n^p$ , along with the condition:

$$q_1 - s_1 = \dots = q_p - s_p$$

Thus we must have  $q_x = s_x + a$  for a certain  $a$ , and the above formula becomes:

$$I = \frac{1}{n^p} \int_E \int_E \sum_{csa} (A_1)_{s_1+a, s_1+c_2-c_1+a} (B_1)_{s_1+c_2-c_1, s_1} \cdots (A_p)_{s_p+a, s_p+c_1-c_p+a} (B_p)_{s_p+c_1-c_p, s_p}$$

Consider now the variables  $r_x = c_{x+1} - c_x$ , which altogether range over the set  $Z$  of multi-indices having sum 0. By replacing the sum over  $c_x$  with the sum over  $r_x$ , which creates a multiplicative  $n$  factor, we obtain the following formula:

$$I = \frac{1}{n^{p-1}} \int_E \int_E \sum_{r \in Z} \sum_{sa} (A_1)_{s_1+a, s_1+r_1+a} (B_1)_{s_1+r_1, s_1} \cdots (A_p)_{s_p+a, s_p+r_p+a} (B_p)_{s_p+r_p, s_p}$$

For an arbitrary multi-index  $r$  we have:

$$\delta_{\sum_i r_i, 0} = \frac{1}{n} \sum_i \langle i, r_1 \rangle \cdots \langle i, r_p \rangle$$

Thus, we can replace the sum over  $r \in Z$  by a full sum, as follows:

$$I = \frac{1}{n^p} \int_E \int_E \sum_{rsia} \langle i, r_1 \rangle (A_1)_{s_1+a, s_1+r_1+a} (B_1)_{s_1+r_1, s_1} \cdots \langle i, r_p \rangle (A_p)_{s_p+a, s_p+r_p+a} (B_p)_{s_p+r_p, s_p}$$

In order to “absorb” now the indices  $i, a$ , we can use the following formula:

$$\begin{aligned} & W_{ia}^* A W_{ia} \\ &= \left( \sum_b \langle i, -b \rangle E_{b, a+b} \right) \left( \sum_{bc} E_{a+b, a+c} A_{a+b, a+c} \right) \left( \sum_c \langle i, c \rangle E_{a+c, c} \right) \\ &= \sum_{bc} \langle i, c - b \rangle E_{bc} A_{a+b, a+c} \end{aligned}$$

Thus we have:

$$(W_{ia}^* A W_{ia})_{bc} = \langle i, c - b \rangle A_{a+b, a+c}$$

Our formula becomes:

$$\begin{aligned} I &= \frac{1}{n^p} \int_E \int_E \sum_{rsia} (W_{ia}^* A_1 W_{ia})_{s_1, s_1+r_1} (B_1)_{s_1+r_1, s_1} \cdots (W_{ia}^* A_p W_{ia})_{s_p, s_p+r_p} (B_p)_{s_p+r_p, s_p} \\ &= \int_E \int_E \sum_{ia} tr(W_{ia}^* A_1 W_{ia} B_1) \cdots tr(W_{ia}^* A_p W_{ia} B_p) \end{aligned}$$

Now by replacing  $A_x, B_x$  with their respective values, we obtain:

$$I = \int_E \int_E \sum_{ia} tr(W_{ia}^* U^* W_{i_1-i_2, a_1-a_2} U W_{ia} V W_{j_2-j_1, b_2-b_1} V^*) \dots \dots \dots tr(W_{ia}^* U^* W_{i_p-i_1, a_p-a_1} U W_{ia} V W_{j_1-j_p, b_1-b_p} V^*) dU dV$$

By moving the  $W_{ia}^* U^*$  variables at right, we obtain, with  $S_{ia} = U W_{ia} V$ :

$$I = \sum_{ia} \int_E \int_E tr(W_{i_1-i_2, a_1-a_2} S_{ia} W_{j_2-j_1, b_2-b_1} S_{ia}^*) \dots \dots \dots tr(W_{i_p-i_1, a_p-a_1} S_{ia} W_{j_1-j_p, b_1-b_p} S_{ia}^*) dU dV$$

Now since  $S_{ia}$  is Haar distributed when  $U, V$  are Haar distributed, we obtain:

$$I = N \int_E \int_E tr(W_{i_1-i_2, a_1-a_2} U W_{j_2-j_1, b_2-b_1} U^*) \dots tr(W_{i_p-i_1, a_p-a_1} U W_{j_1-j_p, b_1-b_p} U^*) dU$$

But this is exactly  $N$  times the integral in the formula of  $(T_p)_{ia,jb}$ , from Proposition 16.5 above. Since the  $N$  factor cancels with one of the two  $N$  factors that we found in the beginning of the proof, when first computing  $(T_p^2)_{ia,jb}$ , we are done.  $\square$

As an illustration for the above result, going back to [25], we have:

THEOREM 16.7. *We have a stationary matrix model*

$$\pi : C(S_4^+) \subset M_4(C(SU_2))$$

given on the standard coordinates by the formula

$$\pi(u_{ij}) = [x \rightarrow Proj(c_i x c_j)]$$

where  $x \in SU_2$ , and  $c_1, c_2, c_3, c_4$  are the Pauli matrices.

PROOF. As already explained in the comments following Definition 16.1, the Pauli matrices appear as particular cases of the Weyl matrices:

$$W_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad W_{10} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$W_{11} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad W_{01} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

By working out the details, we conclude that Theorem 16.6 produces in this case the model in the statement.  $\square$

Observe that, since the projection  $Proj(c_i x c_j)$  depends only on the image of  $x$  in the quotient  $SU_2 \rightarrow SO_3$ , we can replace the model space  $SU_2$  by the smaller space  $SO_3$ .



This can be used in conjunction with the isomorphism  $S_4^+ \simeq SO_3^{-1}$ , and as explained in [12], our model becomes in this way something more conceptual, as follows:

$$\pi : C(SO_3^{-1}) \subset M_4(C(SO_3))$$

As a philosophical conclusion, to this and to some previous findings as well, no matter what we do, we always end up getting back to  $SU_2, SO_3$ . Thus, we are probably doing some physics here. This is indeed the case, the above computations being closely related to the standard computations for the Ising and Potts models. The general relation, however, between quantum permutations and lattice models, is not axiomatized yet.

### 16b. Cocyclic models

Let us discuss now some generalizations of the Weyl matrix models.

We will need the following standard definition:

DEFINITION 16.8. A 2-cocycle on a group  $G$  is a function  $\sigma : G \times G \rightarrow \mathbb{T}$  satisfying:

$$\sigma(gh, k)\sigma(g, h) = \sigma(g, hk)\sigma(h, k)$$

$$\sigma(g, 1) = \sigma(1, g) = 1$$

The algebra  $C^*(G)$ , with multiplication given by

$$g \cdot h = \sigma(g, h)gh$$

and with the involution making the standard generators  $g \in C_\sigma^*(G)$  unitaries, is denoted  $C_\sigma^*(G)$ .

As explained in [33], we have the following general construction:

PROPOSITION 16.9. Given a finite group  $G = \{g_1, \dots, g_N\}$  and a 2-cocycle

$$\sigma : G \times G \rightarrow \mathbb{T}$$

we have a matrix model as follows,

$$\pi : C(S_N^+) \rightarrow M_N(C(E))$$

$$w_{ij} \rightarrow [x \rightarrow Proj(g_i x g_j^*)]$$

for any closed subgroup  $E \subset U_A$ , where  $A = C_\sigma^*(G)$ .

PROOF. This is clear from definitions, because the standard generators  $\{g_1, \dots, g_N\}$  are pairwise orthogonal with respect to the canonical trace of  $A$ . See [33].  $\square$

In order to investigate the stationarity of  $\pi$ , we use:

PROPOSITION 16.10. *We have the formula*

$$(T_p)_{i_1 \dots i_p, j_1 \dots j_p} = \overline{\sigma(i_1, i_1^{-1} i_2)} \dots \overline{\sigma(i_p, i_p^{-1} i_1)} \cdot \overline{\sigma(j_2, j_2^{-1} j_1)} \dots \overline{\sigma(j_1, j_1^{-1} j_p)} \\ \frac{1}{N} \int_E \text{tr}(g_{i_1^{-1} i_2} x g_{j_2^{-1} j_1} x^*) \dots \text{tr}(g_{i_p^{-1} i_1} x g_{j_1^{-1} j_p} x^*) dx$$

with all the indices varying in a cyclic way.

PROOF. According to the definition of  $T_p$ , we have the following formula:

$$(T_p)_{i_1 \dots i_p, j_1 \dots j_p} \\ = \left( \text{tr} \otimes \int_E \right) \left( \text{Proj}(g_{i_1} x g_{j_1}^*) \dots \text{Proj}(g_{i_p} x g_{j_p}^*) \right) dx \\ = \frac{1}{N} \int_E \langle g_{i_1} x g_{j_1}^*, g_{i_2} x g_{j_2}^* \rangle \dots \langle g_{i_p} x g_{j_p}^*, g_{i_1} x g_{j_1}^* \rangle dx$$

We have the following formula:

$$g_i g_{i^{-1} k} = \sigma(i, i^{-1} k) g_k$$

Thus, we have the following formula:

$$g_i^* g_k = \overline{\sigma(i, i^{-1} k)} g_{i^{-1} k}$$

We therefore obtain:

$$\langle g_i x g_j^*, g_k x g_l^* \rangle = \text{tr}(g_j x^* g_i^* g_k x g_l^*) \\ = \text{tr}(g_i^* g_k x g_l^* g_j x^*) \\ = \overline{\sigma(i, i^{-1} k)} \cdot \overline{\sigma(l, l^{-1} j)} \cdot \text{tr}(g_{i^{-1} k} x g_{l^{-1} j} x^*)$$

By plugging these quantities into the formula of  $T_p$ , we obtain the result.  $\square$

We have the following result, which generalizes some previous computations:

THEOREM 16.11. *For any intermediate closed subgroup  $G \subset E \subset U_A$ , the matrix model*

$$\pi : C(S_N^+) \rightarrow M_N(C(E))$$

constructed above is stationary on its image.

PROOF. We use the formula in Proposition 16.10. Let us write  $(T_p)_{ij} = \rho(i, j)(T_p^\circ)_{ij}$ , where  $\rho(i, j)$  is the product of  $\sigma$  terms appearing there. We have:

$$(T_p^2)_{ij} = \sum_k (T_p)_{ik} (T_p)_{kj} \\ = \sum_k \rho(i, k) \rho(k, j) (T_p^\circ)_{ik} (T_p^\circ)_{kj}$$

Let us first compute the  $\rho$  term. We have:

$$\begin{aligned} & \rho(i, k)\rho(k, j) \\ &= \frac{\overline{\sigma(i_1, i_1^{-1}i_2)} \dots \overline{\sigma(i_p, i_p^{-1}i_1)} \cdot \overline{\sigma(k_2, k_2^{-1}k_1)} \dots \overline{\sigma(k_1, k_1^{-1}k_p)}}{\overline{\sigma(k_1, k_1^{-1}k_2)} \dots \overline{\sigma(k_p, k_p^{-1}k_1)} \cdot \overline{\sigma(j_2, j_2^{-1}j_1)} \dots \overline{\sigma(j_1, j_1^{-1}j_p)}} \\ &= \sigma(i, j) \cdot \overline{\sigma(k_2, k_2^{-1}k_1)} \cdot \overline{\sigma(k_1, k_1^{-1}k_2)} \cdot \dots \cdot \overline{\sigma(k_1, k_1^{-1}k_p)} \cdot \overline{\sigma(k_p, k_p^{-1}k_1)} \end{aligned}$$

Now observe that by multiplying  $\sigma(i, i^{-1}k)g_k^*g_k = g_{i^{-1}k}$  and  $\sigma(k, k^{-1}i)g_k^*g_i = g_{k^{-1}i}$  we obtain:

$$\sigma(i, i^{-1}k)\sigma(k, k^{-1}i) = \sigma(i^{-1}k, k^{-1}i)$$

Thus, our expression further simplifies:

$$\rho(i, k)\rho(k, j) = \sigma(i, j) \cdot \overline{\sigma(k_2^{-1}k_1, k_1^{-1}k_2)} \cdot \dots \cdot \overline{\sigma(k_1^{-1}k_p, k_p^{-1}k_1)}$$

On the other hand, the  $T^\circ$  term can be written as follows:

$$\begin{aligned} (T_p^\circ)_{ik}(T_p^\circ)_{kj} &= \frac{1}{N^2} \int_E \int_E \text{tr}(g_{i_1^{-1}i_2} x g_{k_2^{-1}k_1} x^*) \text{tr}(g_{k_1^{-1}k_2} y g_{j_2^{-1}j_1} y^*) \\ &\quad \dots \dots \dots \\ &\quad \text{tr}(g_{i_p^{-1}i_1} x g_{k_1^{-1}k_p} x^*) \text{tr}(g_{k_p^{-1}k_1} y g_{j_1^{-1}j_p} y^*) dx dy \end{aligned}$$

We therefore conclude that we have the following formula:

$$\begin{aligned} & (T_p^2)_{ij} \\ &= \frac{\sigma(i, j)}{N^2} \int_E \int_E \sum_{k_1 \dots k_p} \overline{\sigma(k_2^{-1}k_1, k_1^{-1}k_2)} \text{tr}(g_{i_1^{-1}i_2} x g_{k_2^{-1}k_1} x^*) \text{tr}(g_{k_1^{-1}k_2} y g_{j_2^{-1}j_1} y^*) \\ &\quad \dots \dots \dots \\ &\quad \overline{\sigma(k_1^{-1}k_p, k_p^{-1}k_1)} \text{tr}(g_{i_p^{-1}i_1} x g_{k_1^{-1}k_p} x^*) \text{tr}(g_{k_p^{-1}k_1} y g_{j_1^{-1}j_p} y^*) dx dy \end{aligned}$$

By using now  $g_i^* = \overline{\sigma(i, i^{-1})}g_{i^{-1}}$ , and moving as well the  $x^*$  variables at left, we obtain:

$$\begin{aligned} (T_p^2)_{ij} &= \frac{\sigma(i, j)}{N^2} \int_E \int_E \sum_{k_1 \dots k_p} \text{tr}(x^* g_{i_1^{-1}i_2} x g_{k_2^{-1}k_1}) \text{tr}(g_{k_2^{-1}k_1}^* y g_{j_2^{-1}j_1} y^*) \\ &\quad \dots \dots \dots \\ &\quad \text{tr}(x^* g_{i_p^{-1}i_1} x g_{k_1^{-1}k_p}) \text{tr}(g_{k_1^{-1}k_p}^* y g_{j_1^{-1}j_p} y^*) dx dy \end{aligned}$$

We can compute the products of traces by using the following formula:

$$\begin{aligned} \text{tr}(Ag_k)\text{tr}(g_k^*B) &= \sum_{qs} \langle g_q, Ag_k \rangle \langle g_s, g_k^*B \rangle \\ &= \sum_{qs} \text{tr}(g_q^*Ag_k)\text{tr}(g_s^*g_k^*B) \end{aligned}$$

Thus are left with an integral involving the variable  $z = xy$ , which gives  $T_p^\circ$ . □

Let us discuss now the relationship with the Weyl matrices. We have:

PROPOSITION 16.12. *Given a finite abelian group  $H$ , consider the product  $G = H \times \widehat{H}$ , and endow it with its standard Fourier cocycle.*

- (1) *With  $E = U_n$ , where  $n = |H|$ , the model  $\pi : C(S_N^+) \rightarrow M_N(C(U_n))$  constructed above, where  $N = n^2$ , is the Weyl matrix model associated to  $H$ .*
- (2) *When assuming in addition that  $H$  is cyclic,  $H = \mathbb{Z}_n$ , we obtain in this way the matrix model for  $C(S_N^+)$  coming from the usual Weyl matrices.*
- (3) *In the particular case  $H = \mathbb{Z}_2$ , the model  $\pi : C(S_4^+) \rightarrow M_N(C(U_2))$  constructed above is the matrix model for  $C(S_4^+)$  coming from the Pauli matrices.*

PROOF. All this is well-known. The general construction in Proposition 16.9 above came in fact by successively generalizing (3)  $\rightarrow$  (2)  $\rightarrow$  (1), and then by performing one more generalization, with  $G = H \times \widehat{H}$  with its standard Fourier cocycle being replaced by an arbitrary finite group  $G$ , with a 2-cocycle on it. For full details here, see [33].  $\square$

Regarding now the associated quantum permutation groups, in the general context of Proposition 16.9, we have the following result:

THEOREM 16.13. *For a generalized Weyl matrix model, as in Proposition 16.9 above, the moments of the main character of the associated quantum group are*

$$c_p = \frac{1}{N} \sum_{j_1 \dots j_p}^{\circ} \int_E \text{tr}(g_{j_1} x g_{j_1}^* x^*) \dots \text{tr}(g_{j_p} x g_{j_p}^* x^*) dx$$

where  $\circ$  means that the indices are subject to the condition  $j_1 \dots j_p = 1$ .

PROOF. According to Proposition 16.9 and to Proposition 16.10 above, the moments of the main character are the following numbers:

$$c_p = \frac{1}{N} \sum_{i_1 \dots i_p} \overline{\sigma(i_1, i_1^{-1} i_2)} \dots \overline{\sigma(i_p, i_p^{-1} i_1)} \cdot \overline{\sigma(i_2, i_2^{-1} i_1)} \dots \overline{\sigma(i_1, i_1^{-1} i_p)} \\ \int_E \text{tr}(g_{i_1^{-1} i_2} x g_{i_2^{-1} i_1} x^*) \dots \text{tr}(g_{i_p^{-1} i_1} x g_{i_1^{-1} i_p} x^*) dx$$

We can compact the cocycle part by using the following formulae:

$$\begin{aligned} \sigma(i_p, i_p^{-1} i_{p+1}) \sigma(i_{p+1}, i_{p+1}^{-1} i_p) &= \sigma(i_{p+1}, i_{p+1}^{-1} i_p \cdot i_p^{-1} i_{p+1}) \sigma(i_{p+1}^{-1} i_p, i_p^{-1} i_{p+1}) \\ &= \sigma(i_{p+1}, 1) \sigma(i_{p+1}^{-1} i_p, i_p^{-1} i_{p+1}) \\ &= \sigma(i_{p+1}^{-1} i_p, i_p^{-1} i_{p+1}) \end{aligned}$$

Thus, in terms of the indices  $j_1 = i_1^{-1}i_2, \dots, j_p = i_p^{-1}i_1$ , which are subject to the condition  $j_1 \dots j_p = 1$ , we have the following formula:

$$c_p = \frac{1}{N} \sum_{j_1 \dots j_p}^{\circ} \overline{\sigma(j_1^{-1}, j_1)} \dots \overline{\sigma(j_p^{-1}, j_p)} \int_E \text{tr}(g_{j_1} x g_{j_1^{-1}} x^*) \dots \text{tr}(g_{j_p} x g_{j_p^{-1}} x^*) dx$$

Here the  $\circ$  symbol above the sum is there for reminding us that the indices are subject to the condition  $j_1 \dots j_p = 1$ . We can use now:

$$g_j^* = \overline{\sigma(j^{-1}, j)} g_{j^{-1}}$$

We therefore obtain:

$$c_p = \frac{1}{N} \sum_{j_1 \dots j_p}^{\circ} \int_E \text{tr}(g_{j_1} x g_{j_1}^* x^*) \dots \text{tr}(g_{j_p} x g_{j_p}^* x^*) dx$$

Thus, we have obtained the formula in the statement. □

It is quite unclear whether the above formula further simplifies, in general.

In the context of the Fourier cocycles, as in Proposition 16.12, it is possible to pass to a plain sum, by inserting a certain product of multiplicative factors  $c(j_1) \dots c(j_p)$ , which equals 1 when  $j_1 \dots j_p = 1$ , and the computation can be finished as follows:

$$\begin{aligned} c_p &= \frac{1}{N} \int_E \left( \sum_j c(j) \text{tr}(g_j x g_j^* x^*) \right)^p dx \\ &= \frac{1}{N} \int_E \text{tr}(x x^*) dx \end{aligned}$$

Thus, the law of the main character of the corresponding quantum group coincides with the law of the main character of  $PE$ . All this suggests that the quantum group associated to a Weyl matrix model, as above, should appear as a suitable twist of  $PE$ .

In addition, we believe that in the case where  $E$  is easy these examples should be covered by a suitable projective extension of the Schur-Weyl twisting procedure.

### 16c. Hadamard matrices

Following [8], [14], [19], [45], [34] and related papers, let us discuss now the Hadamard matrix models, which are of particular importance as well. Let us start with:

DEFINITION 16.14. *A complex Hadamard matrix is a square matrix*

$$H \in M_N(\mathbb{C})$$

*whose entries are on the unit circle, and whose rows are pairwise orthogonal.*

Observe that the orthogonality condition tells us that the rescaled matrix  $U = H/\sqrt{N}$  must be unitary. Thus, these matrices form a real algebraic manifold, given by:

$$X_N = M_N(\mathbb{T}) \cap \sqrt{N}U_N$$

The basic example is the Fourier matrix,  $F_N = (w^{ij})$  with  $w = e^{2\pi i/N}$ . With indices  $i, j \in \{0, 1, \dots, N - 1\}$ , this matrix is as follows:

$$F_N = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{N-1} \\ 1 & w^2 & w^4 & \dots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \dots & w^{(N-1)^2} \end{pmatrix}$$

More generally, we have as example the Fourier coupling of any finite abelian group  $G$ , regarded via the isomorphism  $G \simeq \widehat{G}$  as a square matrix,  $F_G \in M_G(\mathbb{C})$ :

$$F_G = \langle i, j \rangle_{i \in G, j \in \widehat{G}}$$

Observe that for the cyclic group  $G = \mathbb{Z}_N$  we obtain in this way the above standard Fourier matrix  $F_N$ . In general, we obtain a tensor product of Fourier matrices  $F_N$ .

There are many other examples of Hadamard matrices, some being elementary, some other fairly exotic, appearing in various branches of mathematics and physics. The idea is that the complex Hadamard matrices can be thought of as being “generalized Fourier matrices”, and this is where the interest in these matrices comes from.

For more on all this, we refer here to [88].

In relation with the quantum groups, the starting observation is as follows:

PROPOSITION 16.15. *If  $H \in M_N(\mathbb{C})$  is Hadamard, the rank one projections*

$$P_{ij} = Proj \left( \begin{pmatrix} H_i \\ H_j \end{pmatrix} \right)$$

where  $H_1, \dots, H_N \in \mathbb{T}^N$  are the rows of  $H$ , form a magic unitary.

PROOF. This is clear, the verification for the rows being as follows:

$$\begin{aligned} \left\langle \begin{pmatrix} H_i \\ H_j \end{pmatrix}, \begin{pmatrix} H_i \\ H_k \end{pmatrix} \right\rangle &= \sum_l \frac{H_{il}}{H_{jl}} \cdot \frac{H_{kl}}{H_{il}} \\ &= \sum_l \frac{H_{kl}}{H_{jl}} \\ &= N\delta_{jk} \end{aligned}$$

The verification for the columns is similar, as follows:

$$\begin{aligned} \left\langle \frac{H_i}{H_j}, \frac{H_k}{H_j} \right\rangle &= \sum_l \frac{H_{il}}{H_{jl}} \cdot \frac{H_{jl}}{H_{kl}} \\ &= \sum_l \frac{H_{il}}{H_{kl}} \\ &= N\delta_{ik} \end{aligned}$$

Thus, we have indeed a magic unitary, as claimed. □

We can proceed now in the same way as we did with the Weyl matrices, namely by constructing a model of  $C(S_N^+)$ , and performing the Hopf image construction:

DEFINITION 16.16. *To any Hadamard matrix  $H \in M_N(\mathbb{C})$  we associate the quantum permutation group  $G \subset S_N^+$  given by the fact that  $C(G)$  is the Hopf image of*

$$\begin{aligned} \pi : C(S_N^+) &\rightarrow M_N(\mathbb{C}) \\ u_{ij} &\rightarrow Proj \left( \begin{matrix} H_i \\ H_j \end{matrix} \right) \end{aligned}$$

where  $H_1, \dots, H_N \in \mathbb{T}^N$  are the rows of  $H$ .

Summarizing, we have a construction  $H \rightarrow G$ , and our claim is that this construction is something really useful, with  $G$  encoding the combinatorics of  $H$ . To be more precise, our claim is that “ $H$  can be thought of as being a kind of Fourier matrix for  $G$ ”.

This is of course quite interesting, philosophically speaking. There are several results supporting this, with the main evidence coming from the following result, coming from [19], [34], which collects the basic known results regarding the construction:

THEOREM 16.17. *The construction  $H \rightarrow G$  has the following properties:*

- (1) *For a Fourier matrix  $H = F_G$  we obtain the group  $G$  itself, acting on itself.*
- (2) *For  $H \notin \{F_G\}$ , the quantum group  $G$  is not classical, nor a group dual.*
- (3) *For a tensor product  $H = H' \otimes H''$  we obtain a product,  $G = G' \times G''$ .*

PROOF. All this material is standard, and elementary, as follows:

(1) Let us first discuss the cyclic group case, where our Hadamard matrix is a usual Fourier matrix,  $H = F_N$ . Here the rows of  $H$  are given by  $H_i = \rho^i$ , where:

$$\rho = (1, w, w^2, \dots, w^{N-1})$$

Thus, we have the following formula, for the associated magic basis:

$$\frac{H_i}{H_j} = \rho^{i-j}$$

It follows that the corresponding rank 1 projections  $P_{ij} = Proj(H_i/H_j)$  form a circulant matrix, all whose entries commute. Since the entries commute, the corresponding quantum group must satisfy  $G \subset S_N$ . Now by taking into account the circulant property of  $P = (P_{ij})$  as well, we are led to the conclusion that we have:

$$G = \mathbb{Z}_N$$

In the general case now, where  $H = F_G$ , with  $G$  being an arbitrary finite abelian group, the result can be proved either by extending the above proof, or by decomposing  $G = \mathbb{Z}_{N_1} \times \dots \times \mathbb{Z}_{N_k}$  and using (3) below, whose proof is independent from the rest.

(2) This is something more tricky, needing some general study of the representations whose Hopf images are commutative, or cocommutative. For details here, along with a number of supplementary facts on the construction  $H \rightarrow G$ , we refer to [34].

(3) Assume that we have a tensor product  $H = H' \otimes H''$ , and let  $G, G', G''$  be the associated quantum permutation groups. We have then a diagram as follows:

$$\begin{array}{ccccc}
 C(S_{N'}^+) \otimes C(S_{N''}^+) & \longrightarrow & C(G') \otimes C(G'') & \longrightarrow & M_{N'}(\mathbb{C}) \otimes M_{N''}(\mathbb{C}) \\
 \uparrow & & & & \downarrow \\
 C(S_N^+) & \longrightarrow & C(G) & \longrightarrow & M_N(\mathbb{C})
 \end{array}$$

Here all the maps are the canonical ones, with those on the left and on the right coming from  $N = N'N''$ . At the level of standard generators, the diagram is as follows:

$$\begin{array}{ccccc}
 u'_{ij} \otimes u''_{ab} & \longrightarrow & w'_{ij} \otimes w''_{ab} & \longrightarrow & P'_{ij} \otimes P''_{ab} \\
 \uparrow & & & & \downarrow \\
 u_{ia,jb} & \longrightarrow & w_{ia,jb} & \longrightarrow & P_{ia,jb}
 \end{array}$$

Now observe that this diagram commutes. We conclude that the representation associated to  $H$  factorizes indeed through  $C(G') \otimes C(G'')$ , and this gives the result.  $\square$

Generally speaking, going beyond the above result, with explicit computations of quantum permutation groups associated to explicit complex Hadamard matrices, is a quite difficult task. The main results which are known so far concern the deformations of the Fourier matrices, and we refer here to [8], [14], [45] and related papers.

At a more abstract level, one interesting question is that of abstractly characterizing the magic matrices coming from the complex Hadamard matrices. We have here:



PROPOSITION 16.18. *Given an Hadamard matrix  $H \in M_N(\mathbb{C})$ , the vectors*

$$\xi_{ij} = \frac{H_i}{H_j}$$

*on which the magic unitary entries  $P_{ij}$  project, have the following properties:*

- (1)  $\xi_{ii} = \xi$  is the all-one vector.
- (2)  $\xi_{ij}\xi_{jk} = \xi_{ik}$ , for any  $i, j, k$ .
- (3)  $\xi_{ij}\xi_{kl} = \xi_{il}\xi_{kj}$ , for any  $i, j, k, l$ .

PROOF. All these assertions are trivial, by using the formula  $\xi_{ij} = H_i/H_j$ . □

Let us call now magic basis of a given Hilbert space  $H$  any square array of vectors  $\xi \in M_N(H)$ , all whose rows and columns are orthogonal bases of  $H$ . With this convention, the above observations lead to the following result, at the magic basis level:

THEOREM 16.19. *The magic bases  $\xi \in M_N(S_{\mathbb{C}}^{N-1})$  coming from the complex Hadamard matrices are those having the following properties:*

- (1) We have  $\xi_{ij} \in \mathbb{T}^N$ , after a suitable rescaling.
- (2) The conditions in Proposition 16.18 are satisfied.

PROOF. By using the multiplicativity conditions (1,2,3) in Proposition 16.18, we conclude that, up to a rescaling, we must have  $\xi_{ij} = \xi_i/\xi_j$ , where  $\xi_1, \dots, \xi_N$  is the first row of the magic basis. Together with our assumption  $\xi_{ij} \in \mathbb{T}^N$ , this gives the result. □

At the general level now, we have the following result, from [19]:

THEOREM 16.20. *The Tannakian category of the quantum group  $G \subset S_N^+$  associated to a complex Hadamard matrix  $H \in M_N(\mathbb{C})$  is given by*

$$T \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \iff T^\circ G^{k+2} = G^{l+2} T^\circ$$

where the objects on the right are constructed as follows:

- (1)  $T^\circ = id \otimes T \otimes id$ .
- (2)  $G_{ia}^{jb} = \sum_k H_{ik} \bar{H}_{jk} \bar{H}_{ak} H_{bk}$ .
- (3)  $G_{i_1 \dots i_k, j_1 \dots j_k}^k = G_{i_k i_{k-1}}^{j_k j_{k-1}} \cdots G_{i_2 i_1}^{j_2 j_1}$ .

PROOF. We use the Tannakian result for the Hopf image of a representation, discussed in chapter 15 above. With the notations here, we have the following formula:

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{Hom}(U^{\otimes k}, U^{\otimes l})$$

The vector space on the right consists by definition of the complex  $N^l \times N^k$  matrices  $T$ , satisfying the following relation:

$$TU^{\otimes k} = U^{\otimes l}T$$

If we denote this equality by  $L = R$ , the left term  $L$  is given by:

$$\begin{aligned} L_{ij} &= (TU^{\otimes k})_{ij} \\ &= \sum_a T_{ia} U_{aj}^{\otimes k} \\ &= \sum_a T_{ia} U_{a_1 j_1} \cdots U_{a_k j_k} \end{aligned}$$

As for the right term  $R$ , this is given by:

$$\begin{aligned} R_{ij} &= (U^{\otimes l} T)_{ij} \\ &= \sum_b U_{ib}^{\otimes l} T_{bj} \\ &= \sum_b U_{i_1 b_1} \cdots U_{i_l b_l} T_{bj} \end{aligned}$$

Consider now the vectors  $\xi_{ij} = H_i/H_j$ . Since these vectors span the ambient Hilbert space, the equality  $L = R$  is equivalent to the following equality:

$$\langle L_{ij} \xi_{pq}, \xi_{rs} \rangle = \langle R_{ij} \xi_{pq}, \xi_{rs} \rangle$$

We use now the following well-known formula, expressing a product of rank one projections  $P_1, \dots, P_k$  in terms of the corresponding image vectors  $\xi_1, \dots, \xi_k$ :

$$\langle P_1 \dots P_k x, y \rangle = \langle x, \xi_k \rangle \langle \xi_k, \xi_{k-1} \rangle \cdots \langle \xi_2, \xi_1 \rangle \langle \xi_1, y \rangle$$

This gives the following formula for  $L$ :

$$\begin{aligned} \langle L_{ij} \xi_{pq}, \xi_{rs} \rangle &= \sum_a T_{ia} \langle P_{a_1 j_1} \cdots P_{a_k j_k} \xi_{pq}, \xi_{rs} \rangle \\ &= \sum_a T_{ia} \langle \xi_{pq}, \xi_{a_k j_k} \rangle \cdots \langle \xi_{a_1 j_1}, \xi_{rs} \rangle \\ &= \sum_a T_{ia} G_{pa_k}^{qj_k} G_{a_k a_{k-1}}^{j_k j_{k-1}} \cdots G_{a_2 a_1}^{j_2 j_1} G_{a_1 r}^{j_1 s} \\ &= \sum_a T_{ia} G_{rap, sjq}^{k+2} \\ &= (T^\circ G^{k+2})_{rip, sjq} \end{aligned}$$

As for the right term  $R$ , this is given by:

$$\begin{aligned}
 \langle R_{ij}\xi_{pq}, \xi_{rs} \rangle &= \sum_b \langle P_{i_1 b_1} \dots P_{i_l b_l} \xi_{pq}, \xi_{rs} \rangle T_{bj} \\
 &= \sum_b \langle \xi_{pq}, \xi_{i_1 b_1} \rangle \dots \langle \xi_{i_l b_l}, \xi_{rs} \rangle T_{bj} \\
 &= \sum_b G_{p i_1}^{q b_1} G_{i_1 i_2}^{b_1 b_2} \dots G_{i_{l-1} i_l}^{b_{l-1} b_l} G_{i_l r}^{b_l s} T_{bj} \\
 &= \sum_b G_{rip, sbq}^{l+2} T_{bj} \\
 &= (G^{l+2} T^o)_{rip, sq}
 \end{aligned}$$

Thus, we obtain the formula in the statement. See [19]. □

We will be back to the above result later on, with several interpretations of the formula found there, which reminds computations from statistical mechanics.

### 16d. Spin models

Let us discuss now the relation with subfactor theory, and with planar algebras. As a starting point, we have the following basic observation of Popa [82]:

**PROPOSITION 16.21.** *Up to a conjugation by a unitary, the pairs of orthogonal MASA in the simplest factor, namely the matrix algebra  $M_N(\mathbb{C})$ , are as follows,*

$$A = \Delta$$

$$B = H\Delta H^*$$

with  $\Delta \subset M_N(\mathbb{C})$  being the diagonal matrices, and with  $H \in M_N(\mathbb{C})$  being Hadamard.

**PROOF.** Any maximal abelian subalgebra (MASA) in  $M_N(\mathbb{C})$  being conjugated to  $\Delta$ , we can assume, up to conjugation by a unitary, that we have, with  $U \in U_N$ :

$$A = \Delta$$

$$B = U\Delta U^*$$

Now observe that given two diagonal matrices  $D, E \in \Delta$ , we have:

$$\begin{aligned}
 \text{tr}(D \cdot U E U^*) &= \frac{1}{N} \sum_i (D U E U^*)_{ii} \\
 &= \frac{1}{N} \sum_{ij} D_{ii} U_{ij} E_{jj} \bar{U}_{ij} \\
 &= \frac{1}{N} \sum_{ij} D_{ii} E_{jj} |U_{ij}|^2
 \end{aligned}$$

Thus, the orthogonality condition  $A \perp B$  reformulates as follows:

$$\frac{1}{N} \sum_{ij} D_{ii} E_{jj} |U_{ij}|^2 = \frac{1}{N^2} \sum_{ij} D_{ii} E_{jj}$$

Thus, we must have the following equality, for any  $i, j$ :

$$|U_{ij}| = \frac{1}{\sqrt{N}}$$

But this tells us that the matrix  $H = \sqrt{N}U$  must be Hadamard, as claimed. □

Along the same lines, but at a more advanced level, we have:

**THEOREM 16.22.** *Given a complex Hadamard matrix  $H \in M_N(\mathbb{C})$ , the diagram formed by the associated pair of orthogonal MASA, namely*

$$\begin{array}{ccc} \Delta & \longrightarrow & M_N(\mathbb{C}) \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & H\Delta H^* \end{array}$$

*is a commuting square in the sense of subfactor theory, in the sense that the expectations onto  $\Delta, H\Delta H^*$  commute, and their product is the expectation onto  $\mathbb{C}$ .*

**PROOF.** The expectation  $E_\Delta : M_N(\mathbb{C}) \rightarrow \Delta$  is the operation  $M \rightarrow M_\Delta$  which consists in keeping the diagonal, and erasing the rest. Consider now the other expectation:

$$E_{H\Delta H^*} : M_N(\mathbb{C}) \rightarrow H\Delta H^*$$

It is better to identify this with the following expectation, with  $U = H/\sqrt{N}$ :

$$E_{U\Delta U^*} : M_N(\mathbb{C}) \rightarrow U\Delta U^*$$

This latter expectation must be of the form  $M \rightarrow UX_\Delta U^*$ , with  $X$  satisfying:

$$\langle M, UDU^* \rangle = \langle UX_\Delta U^*, UDU^* \rangle \quad , \quad \forall D \in \Delta$$

The scalar products being given by  $\langle a, b \rangle = \text{tr}(ab^*)$ , this condition reads:

$$\text{tr}(MUD^*U^*) = \text{tr}(X_\Delta D^*) \quad , \quad \forall D \in \Delta$$

Thus  $X = U^*MU$ , and the formulae of our two expectations are as follows:

$$\begin{aligned} E_\Delta(M) &= M_\Delta \\ E_{U\Delta U^*}(M) &= U(U^*MU)_\Delta U^* \end{aligned}$$

With these formulae in hand, an elementary computation gives the result. □

The point now is that any commuting square  $C$  produces a subfactor of the Murray-von Neumann hyperfinite  $\text{II}_1$  factor  $R$ . Consider indeed such a square:

$$\begin{array}{ccc} C_{01} & \longrightarrow & C_{11} \\ \uparrow & & \uparrow \\ C_{00} & \longrightarrow & C_{10} \end{array}$$

Under suitable assumptions on the inclusions  $C_{00} \subset C_{10}, C_{01} \subset C_{11}$ , we can perform the basic construction for them, in finite dimensions, and we obtain in this way a whole array of commuting squares, as follows:

$$\begin{array}{ccccccc} & A_0 & & A_1 & & A_2 & \\ & \uparrow & & \uparrow & & \uparrow & \\ & \cdots & & \cdots & & \cdots & \\ C_{02} & \longrightarrow & C_{12} & \longrightarrow & C_{22} & \cdots & B_2 \\ \uparrow & & \uparrow & & \uparrow & & \\ C_{01} & \longrightarrow & C_{11} & \longrightarrow & C_{21} & \cdots & B_1 \\ \uparrow & & \uparrow & & \uparrow & & \\ C_{00} & \longrightarrow & C_{10} & \longrightarrow & C_{20} & \cdots & B_0 \end{array}$$

Here the various  $A, B$  letters stand for the von Neumann algebras obtained in the limit, which are all isomorphic to the hyperfinite  $\text{II}_1$  factor  $R$ , and we have:

**THEOREM 16.23.** *In the context of the above diagram, the following happen:*

- (1)  $A_0 \subset A_1$  is a subfactor, and  $\{A_i\}$  is the Jones tower for it.
- (2) The corresponding planar algebra is given by  $A'_0 \cap A_k = C'_{01} \cap C_{k0}$ .
- (3) A similar result holds for the "horizontal" subfactor  $B_0 \subset B_1$ .

**PROOF.** This is something very standard, the idea being as follows:

- (1) This is something quite routine.
- (2) This is a subtle result, called Ocneanu compactness theorem.
- (3) This follows from (1,2), by flipping the diagram. □

Getting back now to the Hadamard matrices, we can extend our lineup of results, namely Proposition 16.21 and Theorem 16.22, with an advanced result, as follows:

**THEOREM 16.24.** *Given a complex Hadamard matrix  $H \in M_N(\mathbb{C})$ , the diagram formed by the associated pair of orthogonal MASA, namely*

$$\begin{array}{ccc} \Delta & \longrightarrow & M_N(\mathbb{C}) \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & H\Delta H^* \end{array}$$

*is a commuting square in the sense of subfactor theory, and the associated planar algebra  $P = (P_k)$  is given by the following formula, in terms of  $H$  itself,*

$$T \in P_k \iff T^\circ G^2 = G^{k+2} T^\circ$$

*where the objects on the right are constructed as follows:*

- (1)  $T^\circ = id \otimes T \otimes id.$
- (2)  $G_{ia}^{jb} = \sum_k H_{ik} \bar{H}_{jk} \bar{H}_{ak} H_{bk}.$
- (3)  $G_{i_1 \dots i_k, j_1 \dots j_k}^k = G_{i_k i_{k-1}}^{j_k j_{k-1}} \dots G_{i_2 i_1}^{j_2 j_1}.$

**PROOF.** We have two assertions here, the idea being as follows:

(1) The fact that we have indeed a commuting square is something that we already know, from Theorem 16.22 above.

(2) The computation of the associated planar algebra is possible thanks to the Ocneanu compactness theorem, corresponding to the formula in Theorem 16.23 (2). To be more precise, by doing some direct computations, which are quite similar to those in the proof of Theorem 16.20 above, we obtain the formula in the statement. See [68].  $\square$

The point now is that all the above is very similar to Theorem 16.20. To be more precise, by comparing the above result with the formula obtained in Theorem 16.20, which is identical, we are led to the following result:

**THEOREM 16.25.** *Let  $H \in M_N(\mathbb{C})$  be a complex Hadamard matrix.*

- (1) *The planar algebra associated to  $H$  is given by  $P_k = Fix(u^{\otimes k})$ , where  $G \subset S_N^+$  is the associated quantum permutation group.*
- (2) *The corresponding Poincaré series  $f(z) = \sum_k \dim(P_k) z^k$  equals the Stieltjes transform  $\int_G \frac{1}{1-z\chi}$  of the law of the main character  $\chi = \sum_i u_{ii}$ .*

**PROOF.** This follows by comparing the quantum group and subfactor results:

(1) As already mentioned above, this simply follows by comparing Theorem 16.20 with the subfactor computation in Theorem 16.24. For full details here, we refer to [19].

(2) This is a consequence of (1), and of the Peter-Weyl type results from [97], which tell us that fixed points can be counted by integrating characters.  $\square$

Summarizing, we have now a clarification of the various quantum algebraic objects associated to a complex Hadamard matrix  $H \in M_N(\mathbb{C})$ , the idea being that the central object, which best encodes the “symmetries” of the matrix, and which allows the computation of the other quantum algebraic objects as well, such as the associated planar algebra, is the associated quantum permutation group  $G \subset S_N^+$ .

Regarding now the subfactor itself, the result here is as follows:

**THEOREM 16.26.** *The subfactor associated to  $H \in M_N(\mathbb{C})$  is of the form*

$$A^G \subset (\mathbb{C}^N \otimes A)^G$$

with  $A = R \rtimes \widehat{G}$ , where  $G \subset S_N^+$  is the associated quantum permutation group.

**PROOF.** This is something more technical, the idea being that the basic construction procedure for the commuting squares, explained before Theorem 16.23, can be performed in an “equivariant setting”, for commuting squares having components as follows:

$$D \otimes_G E = (D \otimes (E \rtimes \widehat{G}))^G$$

To be more precise, starting with a commuting square formed by such algebras, we obtain by basic construction a whole array of commuting squares as follows, with  $\{D_i\}, \{E_i\}$  being by definition Jones towers, and with  $D_\infty, E_\infty$  being their inductive limits:

$$\begin{array}{ccccccc}
 & D_0 \otimes_G E_\infty & & D_1 \otimes_G E_\infty & & D_2 \otimes_G E_\infty & & \\
 & \uparrow \cdots & & \uparrow \cdots & & \uparrow \cdots & & \\
 D_0 \otimes_G E_2 & \longrightarrow & D_1 \otimes_G E_2 & \longrightarrow & D_2 \otimes_G E_2 & \cdots \longrightarrow & D_\infty \otimes_G E_2 & \\
 \uparrow & & \uparrow & & \uparrow & & & \\
 D_0 \otimes_G E_1 & \longrightarrow & D_1 \otimes_G E_1 & \longrightarrow & D_2 \otimes_G E_1 & \cdots \longrightarrow & D_\infty \otimes_G E_1 & \\
 \uparrow & & \uparrow & & \uparrow & & & \\
 D_0 \otimes_G E_0 & \longrightarrow & D_1 \otimes_G E_0 & \longrightarrow & D_2 \otimes_G E_0 & \cdots \longrightarrow & D_\infty \otimes_G E_0 & 
 \end{array}$$

The point now is that this quantum group picture works in fact for any commuting square having  $\mathbb{C}$  in the lower left corner. In the Hadamard matrix case, that we are interested in here, the corresponding commuting square is as follows:

$$\begin{array}{ccc}
 \mathbb{C} \otimes_G \mathbb{C}^N & \longrightarrow & \mathbb{C}^N \otimes_G \mathbb{C}^N \\
 \uparrow & & \uparrow \\
 \mathbb{C} \otimes_G \mathbb{C} & \longrightarrow & \mathbb{C}^N \otimes_G \mathbb{C}
 \end{array}$$

Thus, the subfactor obtained by vertical basic construction appears as follows:

$$\mathbb{C} \otimes_G E_\infty \subset \mathbb{C}^N \otimes_G E_\infty$$

But this gives the conclusion in the statement, with the  $\text{II}_1$  factor appearing there being by definition  $A = E_\infty \rtimes \widehat{G}$ , and with the remark that we have  $E_\infty \simeq R$ . See [1].  $\square$

Summarizing, we have some interesting mathematics here. In practice now, a first problem is that of getting beyond Theorem 16.17, with explicit computations, and we refer here to [8], [14], [45] and related papers. Another problem is that of unifying all this with the Weyl matrix models, and we refer here to [25], [33] and related papers. Finally, in relation with the work of Jones [66], [67], [68], [69], and of Connes as well [53], [54], we have the question of understanding the physical meaning of all this.

Getting back now to Theorem 16.17, going beyond it is a quite delicate task. The next simplest models appear by deforming the Fourier matrices, or rather the tensor products of such matrices,  $F_{G \times H} = F_G \otimes F_H$ , via the following construction, due to Diță:

PROPOSITION 16.27. *The matrix  $\mathcal{F}_{G \times H} \in M_{G \times H}(\mathbb{T}^{G \times H})$  given by*

$$(\mathcal{F}_{G \times H})_{ia,jb}(Q) = Q_{ib}(F_G)_{ij}(F_H)_{ab}$$

*is complex Hadamard, and its fiber at  $Q = (1_{ib})$  is the Fourier matrix  $F_{G \times H}$ .*

PROOF. The fact that the rows of  $F_G \otimes_Q F_H = \mathcal{F}_{G \times H}(Q)$  are pairwise orthogonal follows from definitions. With  $1 = (1_{ij})$  we have  $(F_G \otimes_1 F_H)_{ia,jb} = (F_G)_{ij}(F_H)_{ab}$ , and we recognize here the formula of  $F_{G \times H} = F_G \otimes F_H$ , in double index notation.  $\square$

We have the following result, coming from [15]:

THEOREM 16.28. *When the deformation matrix  $Q$  is generic, the minimal factorization for the representation  $\pi_Q$  coming from Proposition 16.27 is*

$$\begin{array}{ccc} C(S_{G \times H}^+) & \xrightarrow{\pi_Q} & M_{G \times H}(\mathbb{C}) \\ & \searrow & \nearrow \pi \\ & C^*(\Gamma_{G,H}) \rtimes C(G) & \end{array}$$

where the discrete group on the bottom is

$$\Gamma_{G,H} \simeq \mathbb{Z}^{(|G|-1)(|H|-1)} \rtimes H$$

with canonical action of  $G$  on it.

PROOF. This is something quite technical, and we refer here to [15].  $\square$

At the level of the spectral measures, the result is as follows:



THEOREM 16.29. *With  $M = |G|, N = |H|$  we have the formula*

$$\text{law}(\chi) = \left(1 - \frac{1}{N}\right) \delta_0 + \frac{1}{N} \text{law}(A)$$

where the matrix

$$A \in C(\mathbb{T}^{MN}, M_M(\mathbb{C}))$$

is given by  $A(q) = \text{Gram matrix of the rows of } q$ .

PROOF. As before, this is something quite technical, and we refer here to [15].  $\square$

When working out the asymptotics, we are led to Poisson laws. We have in fact the following result, coming from [8], [15] and related papers:

THEOREM 16.30. *Given two finite abelian groups  $G, H$ , with  $|G| = M, |H| = N$ , consider the main character  $\chi$  of the quantum group associated to  $\mathcal{F}_{G \times H}$ . We have then*

$$\text{law}\left(\frac{\chi}{N}\right) = \left(1 - \frac{1}{M}\right) \delta_0 + \frac{1}{M} \pi_t$$

in moments, with  $M = tN \rightarrow \infty$ , where  $\pi_t$  is the free Poisson law of parameter  $t > 0$ . In addition, this formula holds for any generic fiber of  $\mathcal{F}_{G \times H}$ .

PROOF. This is something quite heavy, with the algebraic study of the quantum group and corresponding subfactor being done in [15] in the case of the generic fibers, and with the probabilistic aspects being worked out in [15] for the generic fibers, and in [8] for the parametric model, by using a purely probabilistic approach. The proofs use various algebraic and analytic methods, and free probability from [81], [93] in order to reach to the free Poisson law  $\pi_t$ , which is the same as the Marchenko-Pastur law [77].  $\square$

### 16e. Exercises

Things have been quite technical in this chapter, and as an instructive exercise, coming as a complement to the various questions raised above, we have:

EXERCISE 16.31. *Develop of theory of matrix models for the ADE subgroups*

$$G \subset S_4^+$$

*by suitably restricting the parameter space of the Pauli representation.*

To be more precise, assuming that  $G = H'$  comes as a twist of a subgroup  $H \subset SO_3$ , the problem is that of proving that when restricting to  $H$  the model space of the Pauli representation, the Hopf image of the representation that we obtain is  $C(G)$ .

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