



**HAL**  
open science

# INITIAL-BOUNDARY VALUE PROBLEM FOR SECOND ORDER HYPERBOLIC OPERATOR WITH MIXED BOUNDARY CONDITIONS

Djamal Ait-Akli, Abdelkader Merakeb

► **To cite this version:**

Djamal Ait-Akli, Abdelkader Merakeb. INITIAL-BOUNDARY VALUE PROBLEM FOR SECOND ORDER HYPERBOLIC OPERATOR WITH MIXED BOUNDARY CONDITIONS. 2021. hal-03319255v2

**HAL Id: hal-03319255**

**<https://hal.science/hal-03319255v2>**

Preprint submitted on 25 Aug 2021 (v2), last revised 15 Dec 2021 (v7)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# INITIAL-BOUNDARY VALUE PROBLEM FOR SECOND ORDER HYPERBOLIC OPERATOR WITH MIXED BOUNDARY CONDITIONS

DJAMAL AIT-AKLI<sup>1,\*</sup>, ABDELKADER MERAKEB<sup>1</sup>  
AUGUST 25, 2021

ABSTRACT. We deal with a linear hyperbolic differential operator of the second order on a bounded planar domain with a smooth boundary. We establish a well-posedness result in case where a mixed, Dirichlet-Neumann, condition is prescribed on the boundary. We focus on the case of a non-homogeneous Dirichlet data and a homogeneous Neumann one. The presented proof is based on a functional theoretical approach and on an approximation argument.

## 1. INTRODUCTION

Throughout this paper, we let  $\Omega \subset \mathbb{R}^2$  to be a bounded planar domain of class  $C^\infty$ . The boundary  $\partial\Omega := \Gamma_d \cup \Gamma_n$  is formed with two part. We assume, for simplicity, that  $\Gamma_d$  is connected and denote:

$$\{s_i\}_{i=\overline{1,2}} := \overline{\Gamma_d} \cap \overline{\Gamma_n}. \quad (1.1)$$

We are interested in the study of the second order hyperbolic operator:

$$\mathcal{H}u := \partial_t^2 u - \mathcal{L}u. \quad (1.2)$$

where  $\mathcal{L}$  is a second order uniformly and strongly elliptic operator, one can think of the Laplace or the Lamé operator for instance. In the sequel, and only for the sake of clarity of presentation, the operator  $\mathcal{L}$  will be represented by the Laplacian operator  $\Delta$ . We consider the partial differential equation:

$$\mathcal{H}u = f \quad \text{a.e. in } (0, T) \times \Omega. \quad (1.3)$$

where  $T > 0$ . The equation (1.3) is completed with a mixed boundary condition. More precisely, we prescribe a Dirichlet data on  $\Gamma_d$  and a Neumann condition on  $\Gamma_n$  i.e. we have in the sens of trace:

$$u = G \quad \text{on } (0, T) \times \Gamma_d \quad , \quad \frac{\partial u}{\partial \vec{n}} = 0 \quad \text{on } (0, T) \times \Gamma_n, \quad (1.4)$$

where  $\vec{n}$  is the exterior unit normal defined at each point of  $\Gamma_n$  and  $\frac{\partial u}{\partial \vec{n}}$  denotes the normal derivative of  $u$ . Moreover, initial time conditions are also prescribed:

$$u(0, x) = \Psi^0(x), \quad \partial_t u(0, x) = \Psi^1(x) \quad \text{on } \Omega. \quad (1.5)$$

We address in this work the issue of existence and uniqueness of a solution to the problem associated to equation (1.3) when endowed with the conditions (1.4) and (1.5). Moreover, we establish an energy estimate that shows the continuous

---

2010 *Mathematics Subject Classification.* 35L20, 35A01, 35B30.

*Key words and phrases.* Linear hyperbolic operator; Evolution PDE's; Boundary value problems, Energy estimate, Mixed boundary condition.

dependence of the solution of this problem with respect to the data, this estimate expresses the stability of the solution.

Let's place our work in perspective: a well-posedness result, in case of an homogeneous Dirichlet conditions, can be found in [1]. A similar study for the case of a non-homogeneous Dirichlet condition, on the entire boundary, is dealt with in [2]. Regarding other types of boundary condition, the work presented in [3] deals with the case of the non-homogenous Neumann-type boundary condition. The problem with a mixed homogeneous type condition has been dealt with in [4], see also [5].

The problem under study is given in the following system:

$$\begin{cases} \partial_t^2 u - \Delta u = f & \text{in } (0, T) \times \Omega, \\ u = G & \text{on } (0, T) \times \Gamma_d, \\ \frac{\partial u}{\partial \vec{n}} = 0 & \text{on } (0, T) \times \Gamma_n, \\ u(0, x) = \Psi^0, \quad \partial_t u(0, x) = \Psi^1 & \text{in } \Omega. \end{cases} \quad (1.6)$$

Let  $K(\Omega)$  denote the completion of the set of the up to boundary smooth functions in  $\Omega$  vanishing on  $\Gamma_d$  with respect to the Sobolev  $H^1$ -norm i.e.

$$K(\Omega) := \overline{\{v \in C^\infty(\bar{\Omega}), v|_{\Gamma_d} = 0\}}^{\|\cdot\|_{H^1}}. \quad (1.7)$$

The functions  $(u, v)$  are said to satisfy the weak boundary condition, cf.[4, Definition 2, p.170], if:

$$\begin{aligned} (u, v) &\in K(\Omega) \times K(\Omega), \\ \int_{\Omega} \Delta u \varphi &= \int_{\Omega} \nabla u \nabla \varphi, \quad \forall \varphi \in K(\Omega). \end{aligned} \quad (1.8)$$

The main result of this paper is stated in the following theorem:

**Theorem 1.1.** *Let  $f \in C^1([0, T]; L^2(\Omega))$ . Assume that  $(\Psi^0, \Psi^1)$  satisfy the weak boundary conditions (1.8). Let  $G \in H^1((0, T) \times \Gamma_d)$  such that there exists  $\tilde{G} \in H^1((0, T) \times \partial\Omega)$  satisfying:*

$$\tilde{G} \equiv G \quad \text{on } (0, T) \times \Gamma_d, \quad (1.9)$$

$$\tilde{G}(0, x) = 0, \quad \partial_t \tilde{G}(0, x) = 0 \quad \text{on } \partial\Omega. \quad (1.10)$$

Let  $\{s_i\}_i$  be defined as in (1.1). We assume further that  $G$  satisfy point-wisely a domination condition near  $s_i$ , that is:

$$\exists \gamma > 0, \quad |G(t, x)| = \mathcal{O}_{x \rightarrow s_i}(|x - s_i|^{1+\gamma}), \quad \forall t \in (0, T), \quad i = 1, 2. \quad (1.11)$$

Then, there is a unique solution  $u \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$  of system (1.6), moreover this solution satisfies the following energy estimate:

$$\begin{aligned} &\left\| \frac{\partial u}{\partial \vec{n}} \right\|_{L^2((0, T) \times \Gamma_d)} + \|u\|_{C^0([0, T]; H^1(\Omega))} + \|u'\|_{C^0([0, T]; L^2(\Omega))} \\ &\leq C \left( \|G\|_{H^1((0, T) \times \Gamma_d)} + \|f\|_{L^1(0, T; L^2(\Omega))} + \|\Psi^0\|_{H^1(\Omega)} + \|\Psi^1\|_{L^2(\Omega)} \right). \end{aligned} \quad (1.12)$$

**The plan of the paper.** We prove in section 2 the auxiliary Lemma 2.1 which states that the space of solutions of the a problem similar to (1.6) is non-trivial. Moreover, these solutions satisfy an energy estimate. As a second step, we consider in section 3 the solution of an approximating problem that effectively satisfy this energy estimate. We finally pass to the limit and conclude the results of Theorem 1.1.

2. AUXILIARY RESULT

Before going further, we need to set some preliminary remarks:

**Remark 2.1.** Consider the Dirichlet boundary value problem:

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } (0, T) \times \Omega, \\ u = G & \text{on } (0, T) \times \partial\Omega, \\ u(0, x) = 0, \quad \partial_t u(0, x) = 0 & \text{in } \Omega, \end{cases} \quad (2.1)$$

where  $G \in H^1((0, T) \times \partial\Omega)$  satisfy the same compatibility conditions as (1.10). Following [2, Theorem 2.1, p.151], there exists a unique solution  $u \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$  to problem (2.1), moreover  $\frac{\partial u}{\partial \vec{n}} \in L^2((0, T) \times \partial\Omega)$ . Let us denote  $g := \frac{\partial u}{\partial \vec{n}}$  the Neumann data associated to system (2.1), then the weak problem associated to this system writes:

$$\begin{aligned} \text{find } u \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \text{ such that:} \\ \langle \partial_t^2 u, v \rangle_{-1,1} + B[u, v, t] = (g, v), \end{aligned} \quad (2.2)$$

$$\begin{aligned} \text{for all } v \in H^1(\Omega) \text{ and a.e. } 0 \leq t \leq T. \\ u(0, x) = \Psi^0(x), \quad \partial_t u(0, x) = \Psi^1(x). \end{aligned} \quad (2.3)$$

where  $\langle, \rangle_{-1,1}$  denotes the duality pairing  $H^{-1}(\Omega) \times H^1(\Omega)$ . Regarding the structure of the bilinear form  $B$  and the existence of a, unique, solution of this weak problem, the reader may refer to [6, Sec.7.2.1-7.2.2, p.377-380].

An energy estimate can thus be derived, cf. [6, Theorem 5, p. 389]:

$$\sup_{t \in (0, T)} \|u(t)\|_{H^1(\Omega)} + \sup_{t \in (0, T)} \|u'(t)\|_{L^2(\Omega)} \leq C \|g\|_{L^2((0, T) \times \partial\Omega)}. \quad (2.4)$$

Noicte that the weak problem (2.2) in [6] was stated in the case of a right hand side  $f$  such that  $f \in L^2(0, T; H^{-1}(\Omega))$  but this poses no problem given that  $g \in L^2(0, T; H^{-\frac{1}{2}}(\partial\Omega)) \subset L^2(0, T; H^{-1}(\Omega))$ .

We state an auxiliary Lemma needed subsequently for the proof of Theorem 1.1:

**Lemma 2.1.** Consider the mixed boundary value problem given by:

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } (0, T) \times \Omega, \\ u = G & \text{on } (0, T) \times \Gamma_d, \\ \frac{\partial u}{\partial \vec{n}} = 0 & \text{on } (0, T) \times \Gamma_n, \\ u(0, x) = 0, \quad \partial_t u(0, x) = 0 & \text{in } \Omega. \end{cases} \quad (2.5)$$

where we assume that  $G$  meet the compatibility conditions (1.10) on  $\Gamma_d$ . Then the subspace  $\mathcal{G} \subset H^1((0, T) \times \Gamma_d)$ :

$$\mathcal{G} := \left\{ \begin{array}{l} G \in H^1((0, T) \times \Gamma_d) : \text{problem (2.5)} \\ \text{admits a unique solution } u \text{ such that} \\ u \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \end{array} \right\}. \quad (2.6)$$

isn't trivial. Moreover, any solution  $u$  of problem (2.5) satisfy the energy estimate:

$$\begin{aligned} \left\| \frac{\partial u}{\partial \vec{n}} \right\|_{L^2((0, T) \times \Gamma_d)} + \|u\|_{C^0([0, T]; H^1(\Omega))} + \|\partial_t u\|_{C^0([0, T]; L^2(\Omega))} \\ \leq C_1 \|G\|_{H^1((0, T) \times \Gamma_d)}, \end{aligned} \quad (2.7)$$

for all  $G \in \mathcal{G}$ . Where we mean by a trivial subspace: a vector space containing only the zero element  $\mathbf{0}_{H^1}$ .

We present a proof of Lemma 2.1:

*Proof.* Let us introduce the subspace  $\mathcal{D} \subset H^1((0, T) \times \partial\Omega)$  defined by:

$$\mathcal{D} := \left\{ G \in H^1((0, T) \times \partial\Omega) : G \text{ satisfy the } \right. \\ \left. \text{compatibility conditions (1.10) on } \partial\Omega \right\}. \quad (2.8)$$

The subspace  $\mathcal{D}$  is endowed with the norm  $\| \cdot \|_{\mathcal{D}} := \| \cdot \|_{H^1((0, T) \times \partial\Omega)}$ . It can be easily shown that  $\mathcal{D}$  is closed with respect to  $\| \cdot \|_{\mathcal{D}}$ , and thus it a Hilbert space. We set  $\mathcal{D}'$  to be the topological dual space of  $\mathcal{D}$ . The proof is dived into three steps:

**Step 1. Non triviality of  $\mathcal{G}$ .** Denote  $\mathcal{F}$  to be the subspace:

$$\mathcal{F} := \{ \eta \in H^1((0, T) \times \partial\Omega), \eta(0, x) = 0 \text{ on } \partial\Omega \}. \quad (2.9)$$

Consider the non-homogeneous Neumann problem given by the following system:

$$\begin{cases} \partial_t^2 \phi - \Delta u = 0 & \text{in } (0, T) \times \Omega, \\ \frac{\partial \phi}{\partial n} = \eta & \text{on } (0, T) \times \partial\Omega, \\ \phi(0, x) = 0, \partial_t \phi(0, x) = 0 & \text{in } \Omega. \end{cases} \quad (2.10)$$

where  $\eta \in \mathcal{F}$ . Pose:

$$\mathcal{E} := \{ \phi_{|(0, T) \times \partial\Omega} : \phi \text{ solves problem (2.10)} \} \subset \mathcal{D}.$$

Following [3, Theorem A.3, p.117], for every  $\eta \in \mathcal{F}$ , there exists a unique solution  $\phi$  of problem (2.10), such that  $\phi \in \mathcal{E} \subset H^{1+\delta_0}((0, T) \times \partial\Omega)$ , for some  $\delta_0 > 0$ .

One can construct a function  $\eta$  such that:  $\eta \in \mathcal{F}$  and  $\eta \equiv 0$  on  $(0, T) \times \Gamma_n$  which is not identically zero on  $(0, T) \times \partial\Omega$ . Then, there exists a unique, non identically zero,  $h := \phi_{|(0, T) \times \Gamma_d} \in \mathcal{G}$  which is the trace of the solution  $\phi$  of problem (2.10) corresponding to the Neumann  $\eta$ . This ensures the non triviality of the space  $\mathcal{G}$  given by (2.6).

Consider the non-homogeneous Dirichlet problem given by the following system:

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } (0, T) \times \Omega, \\ u = G & \text{on } (0, T) \times \partial\Omega, \\ u(0, x) = 0, \partial_t u(0, x) = 0 & \text{in } \Omega, \end{cases} \quad (2.11)$$

where  $G \in \mathcal{D}$ , and  $\mathcal{D}$  is defined by (2.8). Consider the operator  $T$  defined by:

$$T : \mathcal{D} \subset H^1((0, T) \times \partial\Omega) \rightarrow \mathcal{A} \subset L^2((0, T) \times \partial\Omega) \\ G \mapsto T(G) = g := \frac{\partial u}{\partial \vec{n}},$$

where  $\mathcal{A} := T(\mathcal{D})$ , the operator  $T$  associates to every Dirichlet data  $G \in \mathcal{D}$ , the unique Neumann data  $g \in \mathcal{A}$  on the boundary associated to problem (2.11). Following the well-posedness result recalled in Remark 2.1,  $T$  is well defined. According to [2, Remark 2.2, p.152], and in addition to being linear, the operator  $T$  is bounded i.e. there exists  $C_0 > 0$  such that:

$$\|T(G)\|_{L^2} = \left\| \frac{\partial u}{\partial \vec{n}} \right\|_{L^2((0, T) \times \partial\Omega)} \leq C_0 \|G\|_{H^1((0, T) \times \partial\Omega)}, \quad (2.12)$$

for all  $G \in \mathcal{D}$ . The operator  $T : \mathcal{D} \subset H^1((0, T) \times \partial\Omega) \rightarrow \mathcal{A}$  is clearly linear and invertible, we would like to establish the boundedness of its inverse, denoted by  $T^{-1}$ . For this we use the Banach isomorphism theorem. In order to apply this theorem, we need first to show that the subspace  $\mathcal{A}$  is complete i.e. is a Banach.

**Step 2. Completeness of  $\mathcal{A}$ .** In order to prove the completeness of  $\mathcal{A}$ , it is sufficient to prove that it is a closed subspace. Let  $(g_n)_n$  be a convergent sequence of functions in  $\mathcal{A}$  i.e. there exists  $g \in L^2((0, T) \times \partial\Omega)$  such that  $\|g_n - g\|_{L^2((0, T) \times \partial\Omega)} \rightarrow 0$ , then we need to show that  $g \in \mathcal{A}$  i.e. that there exists  $G \in \mathcal{D}$  such that  $g = T(G)$ . By assumption, there exists  $(G_n)_{n \in \mathbb{N}}$  such that: for all  $n \in \mathbb{N}$ ,  $G_n \in \mathcal{D}$  and  $g_n = T(G_n)$ . Consider the problem, cf. [2, Problem 2.63, p.163]:

$$\begin{cases} \partial_t^2 \phi - \Delta u = 0 & \text{in } (0, T) \times \Omega, \\ \phi = h & \text{on } (0, T) \times \partial\Omega, \\ \phi(T, x) = \partial_t \phi(T, x) = 0 & \text{in } \Omega, \end{cases} \quad (2.13)$$

Pose:  $\eta(t, x) := \frac{\partial \phi}{\partial n}(t, x)$ . According to [2, Estimate (2.65), p.163] we know that  $(G_n)_n$  and  $(g_n)_n$  do satisfy:

$$\int_{(0, T) \times \partial\Omega} G_n \eta = \int_{(0, T) \times \partial\Omega} g_n h \quad (2.14)$$

for all  $\phi$  solution of the problem (2.13) i.e. Estimate (2.14) is true for all  $h \in \mathcal{D}_T$  and all  $\eta \in T(\mathcal{D}_T)$ , where  $\mathcal{D}_T := \{u(T - \cdot, \cdot) : u \in \mathcal{D}\}$ . An important fact to emphasize is: for any  $\eta$  and  $h$  corresponding to the solution of (2.13),  $T$ -shifted Neumann data  $\eta(T - \cdot, \cdot)$  are mapped into  $T$ -shifted Dirichlet data  $h(T - \cdot, \cdot)$ . To see this, one may consider [3, Identity 1.3, 1.6 p.115]. Our objective is to pass to the limit in (2.14) and deduce the existence of a limit,  $\lim G_n$ , that belongs to  $\mathcal{D}$ .

It is easy to see that Estimate (2.14) is true for all  $\eta \in C_0^\infty((0, T) \times \partial\Omega)$ , then by density we infer that it is true for all  $\eta \in L^2((0, T) \times \partial\Omega)$ . Following [3, Theorem A.1, p.117], Estimate (2.14) is true for all  $\eta \in L^2((0, T) \times \partial\Omega) \equiv L^2(0, T; L^2(\partial\Omega))$  and all  $h = T^{-1}(L^2(0, T; L^2(\partial\Omega))) \subset H^\gamma(\partial\Omega)$ , for some  $\gamma > 0$ . On another hand, define for every  $\eta \in \mathcal{D} \subset L^2((0, T) \times \partial\Omega)$  the linear functional  $Z_\eta \in \mathcal{D}'$ :

$$\begin{aligned} Z_\eta : \mathcal{D} &\rightarrow \mathcal{D}' \\ G &\mapsto Z_\eta(G) := \int_{(0, T) \times \partial\Omega} G \eta. \end{aligned} \quad (2.15)$$

Applying the Riesz representation theorem, we have: every  $Z_\eta \in \mathcal{D}'$ , with  $\eta \in \mathcal{D} \subset L^2((0, T) \times \partial\Omega)$ , defines a unique  $\tilde{Z}_\eta \in \mathcal{D}$  and such that:

$$(G, \tilde{Z}_\eta)_{\mathcal{D}} = Z_\eta(G), \quad \forall G \in \mathcal{D} \quad \text{and} \quad \|\tilde{Z}_\eta\|_{\mathcal{D}} = \|Z_\eta\|_{\mathcal{D}'}, \quad (2.16)$$

where  $(\cdot, \cdot)_{\mathcal{D}}$  stands for the scalar product on  $\mathcal{D}$ . Applying the estimate (2.14) and the definition (2.15) we obtain:

$$(G_n, \tilde{Z}_\eta)_{\mathcal{D}} = Z_\eta(G_n) = \int_{(0, T) \times \partial\Omega} G_n \eta = \int_{(0, T) \times \partial\Omega} g_n h. \quad (2.17)$$

for all  $G_n \in \mathcal{D}$  with  $n \in \mathbb{N}$ . One should remark that  $\tilde{Z}_\eta$  traverses all of  $\mathcal{D}$  when  $\eta$  runs over  $\mathcal{D}$  which is equivalent to the fact that  $Z_\eta$  runs over all  $\mathcal{D}'$ . Hence, passing to the limit in (2.17), we deduce that there exists  $G^* \in \mathcal{D}$  such that:

$$(G_n, \tilde{Z}_\eta)_{\mathcal{D}} \xrightarrow{n \rightarrow \infty} (G^*, \tilde{Z}_\eta)$$

for all  $\tilde{Z}_\eta \in \mathcal{D}$ . This means that the sequence  $G_n$  converges weakly in the Hilbert space  $\mathcal{D}$ . Passing to the limit in (2.17), we infer that the limit function  $G^*$  satisfy :

$$\int_{(0,T) \times \partial\Omega} G^* \eta = \int_{(0,T) \times \partial\Omega} gh. \quad (2.18)$$

Thus  $G^* \in \mathcal{D}$  solves problem (2.11). We conclude that there exists  $G \in \mathcal{D}$  such that  $g = T(G)$  and thus  $\mathcal{A}$  is a Banach space.

**Step 3. Proof of Estimate (2.7).** using the completeness of  $\mathcal{A}$  and (2.12) we infer, by virtue of the isomorphism theorem, that the linear operator  $T^{-1} : \mathcal{A} \rightarrow \mathcal{D}$  is bounded. The continuity of  $T^{-1}$  ensures that  $\exists C_{-1} > 0$  such that:

$$\|T^{-1}(g)\|_{H^1((0,T) \times \partial\Omega)} = \|G\|_{H^1((0,T) \times \partial\Omega)} \leq C_{-1} \|g\|_{L^2((0,T) \times \partial\Omega)}, \quad (2.19)$$

for all  $g \in \mathcal{A}$ . Set  $\mathcal{B}$  to be the subspace:

$$\mathcal{B} := \{g \in L^2((0,T) \times \partial\Omega) : g \equiv 0 \text{ on } \Gamma_n\}.$$

The subspace  $(\mathcal{B}, \|\cdot\|_{L^2})$  is closed. We infer that  $\mathcal{A} \cap \mathcal{B}$  is also closed, thus it is a Banach space. Since  $T : \mathcal{D} \rightarrow \mathcal{A}$  is continuous, as well as its inverse, then  $T^{-1}(\mathcal{A} \cap \mathcal{B})$  is a complete subspace of  $\mathcal{D}$ , thus it is a Banach. We remark that:

- The space  $T^{-1}(\mathcal{A} \cap \mathcal{B})$  isn't trivial. Indeed, one can see with ease that the non trivial space  $\mathcal{G}$ , given by (2.6), is nothing but the restriction to  $\Gamma_d$  of functions of the spaces  $T^{-1}(\mathcal{A} \cap \mathcal{B})$ , this is due to the fact that any solution  $u$  of problem (2.5) satisfy the compatibility conditions (1.10) on  $(0,T) \times \partial\Omega$ .

We immediately infer from estimate (2.19) that:

$$\begin{aligned} \|T^{-1}(g)\|_{H^1((0,T) \times \Gamma_d)} &= \|G\|_{H^1((0,T) \times \Gamma_d)} \\ &\leq C_{-1} \|g\|_{L^2((0,T) \times \Gamma_d)}, \end{aligned} \quad (2.20)$$

for all  $g \in \mathcal{A} \cap \mathcal{B}$ . The operator  $T^{-1}$  is defined by:

$$\begin{aligned} T^{-1} : \mathcal{A} \cap \mathcal{B} &\rightarrow T^{-1}(\mathcal{A} \cap \mathcal{B}) \subset \mathcal{D} \\ g &\mapsto T^{-1}(g) = G. \end{aligned}$$

and is continuous. Consider the operator  $\text{Tr}$  defined by:

$$\begin{aligned} \text{Tr}_{|(0,T) \times \Gamma_d} : T^{-1}(\mathcal{A} \cap \mathcal{B}) &\subset H^1 \rightarrow \mathcal{C} \\ G &\mapsto \text{Tr}_{|(0,T) \times \Gamma_d} G, \end{aligned}$$

where  $\mathcal{C} := \text{Tr}_{|(0,T) \times \Gamma_d} \circ T^{-1}(\mathcal{A} \cap \mathcal{B})$ . The space  $\mathcal{C}$  is the set of restrictions to  $(0,T) \times \Gamma_d$  of functions  $G \in T^{-1}(\mathcal{A} \cap \mathcal{B})$  that correspond to Neumann data which are identically zero on  $\Gamma_n$ . Following the above remark, we have:

$$\text{Tr}_{|(0,T) \times \Gamma_d} \circ T^{-1}(\mathcal{A} \cap \mathcal{B}) \equiv \mathcal{G},$$

where  $\mathcal{G}$  is given by (2.8). Estimate (2.20) shows that the operator  $\text{Tr}_{|(0,T) \times \Gamma_d} \circ T^{-1} : \mathcal{A} \cap \mathcal{B} \rightarrow \mathcal{C}$  is continuous, and it is, obviously, one-to-one and onto. One can easily see, using the completeness of  $\mathcal{A}$  and  $\mathcal{B}$  in addition to the estimate (2.12), that the subspace  $\mathcal{C}$  is a Banach space. Then applying again the isomorphism theorem we show that the operator  $M \circ T^{-1}$  has a bounded inverse i.e. there exists  $C_1 > 0$  s.t.:

$$\|g\|_{L^2((0,T) \times \Gamma_d)} = \|[M \circ T^{-1}]^{-1}(G)\|_{L^2((0,T) \times \Gamma_d)} \leq C_1 \|G\|_{H^1((0,T) \times \Gamma_d)}. \quad (2.21)$$

for all  $G \in \mathcal{C} \subset H^1((0,T) \times \Gamma_d)$ . Finally using estimate (2.21), and taking into account remark 2.1, namely estimate (2.4), we conclude immediately estimate (2.7).  $\square$

## 3. PROOF OF THE MAIN RESULT

Presently we establish Theorem 1.1, which is about the well-posedness of system (1.6), using mainly the result of Lemma 2.1:

*Proof.* The proof is divided into two subsections. In the first one we construct, starting from a Dirichlet problem, a sequence of function each of which solves an approximating problem similar to (1.6) and satisfy the estimate (2.7) of Lemma (2.1). In the second subsection we pass to the limit and conclude the results of theorem 1.1.

**Step 1. Construction of the approximating solution.** Let  $\{s_i\}_{i=1,2}$  be as defined by (1.1). Denote by  $D(s_i, \epsilon_0) \subset \mathbb{R}^2$  the disk centered at  $s_i$  with radius  $\epsilon_0$ , where  $\epsilon_0 := \frac{1}{2}|s_1 - s_2|$ . Let  $\epsilon$ ,  $0 < \epsilon < \epsilon_0$ , be a parameter designed to tend towards zero. We introduce a sequence of truncating functions  $\chi_{\epsilon,i} \in C^1(\mathbb{R}^2)$ ,  $i = 1, 2$  defined by:

$$\chi_{\epsilon,i}(x) := \begin{cases} 0, & |x - s_i| \leq \epsilon^2, \\ \exp\left(\frac{\epsilon - |x - s_i|}{\epsilon^2 - |x - s_i|}\right), & \epsilon^2 < |x - s_i| < \epsilon, \\ 1, & \epsilon < |x - s_i|. \end{cases}$$

Define:

$$\chi_\epsilon(t, x) := \begin{cases} \chi_{\epsilon,1}(x), & \forall x, |x - s_1| \leq \epsilon_0, \forall t \in (0, T), \\ \chi_{\epsilon,2}(x), & \forall x, |x - s_2| \leq \epsilon_0, \forall t \in (0, T), \\ 1, & \forall x \in \mathbb{R}^2 - \mathcal{V}_1 \cup \mathcal{V}_2, \forall t \in (0, T). \end{cases} \quad (3.1)$$

where we denote  $\mathcal{V}_i := D(s_i, \epsilon_0)$ . It is easy to see, from the definition (3.1), that the functions  $(\chi_\epsilon)_\epsilon$  are constant with respect to the time variable and satisfy:

$$1) |\chi_\epsilon(t, x)| \leq 1, \quad \forall t \in (0, T), \forall x \in \mathbb{R}^2, \forall \epsilon > 0. \quad (3.2)$$

$$2) \epsilon^{1+\gamma} \|\partial_x \chi_\epsilon\|_{\infty, (0, T) \times \mathcal{V}_i} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0, \text{ with } \gamma > 0. \quad (3.3)$$

$$3) |\chi_\epsilon(t, x)| \rightarrow 1, \quad \text{as } \epsilon \rightarrow 0, \forall (t, x) \in (0, T) \times \Gamma_d. \quad (3.4)$$

Let us cover the set  $\bar{\Omega}$  by open sets  $U_1$  and  $U_2^\epsilon$  such that:

- (i)  $\bar{\Omega} \subset U_1 \cup U_2^\epsilon$ , for all  $\epsilon > 0$ ,
- (ii)  $U_1 \cap \partial\Omega = \bar{\Gamma}_d$ ,
- (iii)  $|\alpha_i^\epsilon - s_i| < \frac{1}{2}\epsilon^2$ , for every  $i = \bar{1}, \bar{2}$ ,

where we denote  $\{\alpha_1^\epsilon, \alpha_2^\epsilon\} := \partial\Omega \cap \partial U_2^\epsilon$  such that  $d(\alpha_1^\epsilon, s_1) < d(\alpha_2^\epsilon, s_1)$ , for all  $\epsilon > 0$ . The condition (ii) implies that  $\partial U_1 \cap \partial\Omega = \{s_1, s_2\}$ . Consider, for every  $\epsilon > 0$ , the partition of the unity  $(\tilde{\psi}_1^\epsilon, \tilde{\psi}_2^\epsilon)$  subordinate to the cover  $(U_1, U_2^\epsilon)$  i.e.

- (a)  $\tilde{\psi}_i^\epsilon \in C_0^\infty(U_i)$ ,  $0 < \tilde{\psi}_i^\epsilon < 1$ , for  $1 \leq i \leq 2$ ,
- (b)  $\text{supp } \tilde{\psi}_1^\epsilon \subset U_1$ ,  $\text{supp } \tilde{\psi}_2^\epsilon \subset U_2^\epsilon$ ,  $\forall \epsilon > 0$ ,
- (c)  $\tilde{\psi}_1^\epsilon(x) + \tilde{\psi}_2^\epsilon(x) = 1$  for every  $x \in \Omega$ .

Define the functions:

$$\psi_1^\epsilon(t, x) := \tilde{\psi}_1^\epsilon(x), \quad \psi_2^\epsilon(t, x) := \tilde{\psi}_2^\epsilon(x), \quad \forall t \in (0, T), \forall x \in \Omega. \quad (3.5)$$

Thus, the functions  $\psi_1^\epsilon, \psi_2^\epsilon$  are constant with respect to the time variable. One immediate consequence is:

$$\|\psi_1^\epsilon - 1\|_{\infty, (0, T) \times \Gamma_d} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (3.6)$$



Let  $G \in H^1((0, T) \times \Gamma_d)$  and  $\tilde{G} \in H^1((0, T) \times \partial\Omega)$  be such that conditions (1.9)-(1.10) are satisfied. Consider the Dirichlet boundary value problem:

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } (0, T) \times \Omega, \\ u = \tilde{G} & \text{on } (0, T) \times \partial\Omega, \\ u(0, x) = 0, \quad \partial_t u(0, x) = 0 & \text{in } \Omega. \end{cases} \quad (3.7)$$

the compatibility conditions are well met by  $\tilde{G}$  on  $\partial\Omega$ . Denote  $u \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$  to be the unique solution of (3.7). It is easy to see that the function  $\chi_\epsilon \psi_1^\epsilon u$  satisfy the following system:

$$\begin{cases} \partial_t^2(\chi_\epsilon \psi_1^\epsilon u) - \Delta(\chi_\epsilon \psi_1^\epsilon u) = f_\epsilon & \text{in } (0, T) \times \Omega_s, \\ \chi_\epsilon \psi_1^\epsilon u = G_\epsilon & \text{on } (0, T) \times \Gamma_d, \\ \frac{\partial(\chi_\epsilon \psi_1^\epsilon u)}{\partial \vec{n}} = 0 & \text{on } (0, T) \times \Gamma_n, \\ (\chi_\epsilon \psi_1^\epsilon u)(0, x) = 0, \quad \partial_t(\chi_\epsilon \psi_1^\epsilon u)(0, x) = 0 & \text{in } \Omega, \end{cases} \quad (3.8)$$

where  $G_\epsilon := \psi_1^\epsilon \chi_\epsilon \tilde{G}$  and  $f_\epsilon \in C^1([0, T]; L^2(\Omega))$ . One should remark that the homogeneous Neumann condition in (3.8) arises from the conditions (ii) and (b). Consider the function  $w_\epsilon$  satisfying the following system:

$$\begin{cases} \partial_t^2 w_\epsilon - \Delta w_\epsilon = f - f_\epsilon & \text{in } (0, T) \times \Omega_s, \\ w_\epsilon = 0 & \text{on } (0, T) \times \Gamma_d, \\ \frac{\partial w_\epsilon}{\partial \vec{n}} = 0 & \text{on } (0, T) \times \Gamma_n, \\ w_\epsilon(0, x) = \Psi^0(x), \quad \partial_t w_\epsilon(0, x) = \Psi^1(x) & \text{in } \Omega, \end{cases} \quad (3.9)$$

where  $(\Psi^0, \Psi^1)$  satisfy the weak boundary conditions (1.8). In virtue of [4, Theorem 1, p. 170] and its Corollary, see also [5], there is a unique  $w_\epsilon \in C^1([0, T]; K(\Omega)) \cap C^2([0, T]; L^2(\Omega))$  satisfying, for all  $\epsilon > 0$ , system (3.9). Then it is obvious to remark that the function  $u_\epsilon := \chi_\epsilon \psi_1^\epsilon u + w_\epsilon$  solves the following system:

$$\begin{cases} \partial_t^2 u_\epsilon - \Delta u_\epsilon = f & \text{in } (0, T) \times \Omega_s, \\ u_\epsilon = G_\epsilon & \text{on } (0, T) \times \Gamma_d, \\ \frac{\partial u_\epsilon}{\partial \vec{n}} = 0 & \text{on } (0, T) \times \Gamma_n, \\ u_\epsilon(0, x) = \Psi^0(x), \quad \partial_t u_\epsilon(0, x) = \Psi^1(x) & \text{in } \Omega. \end{cases}$$

On the other hand, consider the function  $u^2$  that solves the system:

$$\begin{cases} \partial_t^2 u^2 - \Delta u^2 = f & \text{in } (0, T) \times \Omega, \\ u^2 = 0 & \text{on } (0, T) \times \Gamma_d, \\ \frac{\partial u^2}{\partial \vec{n}} = 0 & \text{on } (0, T) \times \Gamma_n, \\ u^2(0, x) = \Psi^0(x), \quad \partial_t u^2(0, x) = \Psi^1(x) & \text{in } \Omega, \end{cases} \quad (3.10)$$

following again [4, Theorem 1, p. 170], see also [5], we deduce that (3.10) admits a unique solution:

$$u^2 \in C^1([0, T]; K(\Omega)) \cap C^2([0, T]; L^2(\Omega)),$$

moreover we have, cf. [4, Estimate 1.9, p. 170], the following estimate:

$$\begin{aligned} & \|u^2\|_{C^0([0,T];H^1(\Omega)) \cap C^1([0,T];L^2(\Omega))} \\ & \leq C(\|f\|_{L^1(0,T;L^2(\Omega))} + \|\Psi^0\|_{H^1(\Omega)} + \|\Psi^1\|_{L^2(\Omega)}). \end{aligned} \quad (3.11)$$

**Step 2. Passing to the limit using Lemma 2.1.** It is immediate to see that  $u_\epsilon^1 := u_\epsilon - u^2 \in C^0([0,T];H^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$  exists and satisfy, for every  $\epsilon > 0$ , the system:

$$\begin{cases} \partial_t^2 u_\epsilon^1 - \Delta u_\epsilon^1 = 0 & \text{in } (0,T) \times \Omega; \\ u_\epsilon^1 = G_\epsilon & \text{on } (0,T) \times \Gamma_d; \\ \frac{\partial u_\epsilon^1}{\partial n} = 0 & \text{on } (0,T) \times \Gamma_n; \\ u_\epsilon^1(0,x) = 0, \quad \partial_t u_\epsilon^1(0,x) = 0 & \text{in } \Omega, \end{cases} \quad (3.12)$$

and thus  $G_\epsilon \in \mathcal{G}$  for every  $\epsilon > 0$ , where  $\mathcal{G}$  is defined by (2.6). Moreover, the associated solution  $u_\epsilon^1$  of system (3.12) satisfy, for every  $\epsilon > 0$ , the assumption of Lemma 2.1. Consequently, if we note  $g_\epsilon := \frac{\partial u_\epsilon^1}{\partial n}$ , we obtain by applying (2.7):

$$\|g_\epsilon\|_{L^2((0,T) \times \Gamma_d)} \leq C_1 \|G_\epsilon \chi_\epsilon \psi_1^\epsilon\|_{H^1((0,T) \times \Gamma_d)}. \quad (3.13)$$

for all  $\epsilon > 0$ . The  $H^1$ -norm of  $G_\epsilon - G = G_\epsilon \chi_\epsilon \psi_1^\epsilon - G$  satisfy:

$$\begin{aligned} & \|G_\epsilon - G\|_{H^1((0,T) \times \Gamma_d)} \\ & \leq \|G_\epsilon - G\|_{L^2((0,T) \times \Gamma_d)}^2 + \|\partial_t G_\epsilon - \partial_t G\|_{L^2((0,T) \times \Gamma_d)}^2 \\ & \quad + \|\nabla_x G_\epsilon - \nabla_x G\|_{L^2((0,T) \times \Gamma_d)}^2. \end{aligned} \quad (3.14)$$

Since  $\psi_1^\epsilon$  and  $\chi_\epsilon$  are constant with respect to the time variable, and considering (3.6), then we can easily see that the limit of the first two terms of the RHS of (3.14) is zero. Let us estimate the limit of the term  $\|\nabla_x G_\epsilon - \nabla_x G\|_{L^2((0,T) \times \Gamma_d)}$ .

This term involves the  $L^2$ -norm of the gradient on the surface  $(0,T) \times \partial\Omega$  and therefore we need to apply Estimate (3.24) of Remark 3.1 that computes  $\nabla_x(\psi_\epsilon \chi_\epsilon G)$ , and thus we have:

$$\begin{aligned} & \|\nabla_x G_\epsilon - \nabla_x G\|_{L^2((0,T) \times \Gamma_d)} \\ & \leq \|\chi_\epsilon \psi_1^\epsilon \nabla_x G - \nabla_x G\|_{L^2((0,T) \times \Gamma_d)} \\ & \quad + \|G_\epsilon \chi_\epsilon \nabla_x \psi_1^\epsilon\|_{L^2((0,T) \times \Gamma_d)} + \|G \psi_1^\epsilon \nabla_x \chi_\epsilon\|_{L^2((0,T) \times \Gamma_d)}. \end{aligned} \quad (3.15)$$

Let us note the following three facts:

(A) Using Estimates: (3.2), (3.4) and (3.6), we can show easily that:

$$\|\chi_\epsilon \psi_1^\epsilon \nabla_x G - \nabla_x G\|_{L^2((0,T) \times \Gamma_d)} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

(B) Given condition (i)-(iii) and (b), the support of  $\nabla_x \psi_1^\epsilon$  is contained in the region  $\cup_i D(s_i, \epsilon^2)$ , for every  $\epsilon > 0$ . Moreover, according to its definition (3.1), the function  $\chi_\epsilon$  is zero on this region, thus:

$$\begin{aligned} & \|G_\epsilon \chi_\epsilon \nabla_x \psi_1^\epsilon\|_{L^2((0,T) \times \Gamma_d)} \\ & = \sum_{i=1}^2 \|G_\epsilon \chi_\epsilon \nabla_x \psi_1^\epsilon\|_{L^2((0,T) \times \{x \in \Gamma_d, |x - s_i| \leq \epsilon^2\})} = 0. \end{aligned} \quad (3.16)$$

(C) Using condition (1.11), the properties (a) and (3.3), we infer:

$$\begin{aligned} \|G\psi_1^\epsilon \nabla_x \chi_\epsilon\|_{L^2((0,T) \times \Gamma_d)} &= \sum_{i=1}^2 \|G\psi_1^\epsilon \nabla_x \chi_\epsilon\|_{L^2((0,T) \times \{x \in \Gamma_d, \epsilon^2 < |x - s_i| \leq \epsilon\})} \\ &\leq 2\|\epsilon^\gamma\|_{L^2((0,T) \times \{x \in \Gamma_d, \epsilon^2 < |x - s_i| \leq \epsilon\})} \rightarrow 0, \end{aligned}$$

we have used the fact that the support of  $\partial_x \chi_\epsilon$  on  $\Gamma_d$  is contained in  $\cup_i \{x \in \Gamma_d, \epsilon^2 < |x - s_i| \leq \epsilon\}$ .

Consequently, using the facts (A)-(C) and applying the dominated convergence theorem, one can easily show that:

$$\|\nabla_x(G\psi_1^\epsilon \chi_\epsilon) - \nabla_x G\|_{L^2((0,T) \times \Gamma_d)} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (3.17)$$

Finally, using (3.6), (3.14) and (3.17) one can see that  $\|G_\epsilon - G\|_{H^1((0,T) \times \Gamma_d)} \rightarrow 0$ . Consequently, we deduce from estimate (3.13) that there exists  $g \in L^2((0,T) \times \Gamma_d)$  such that  $\|g_\epsilon - g\|_{L^2((0,T) \times \Gamma_d)} \rightarrow 0$  and thus, by passing to the limit in estimate (3.13), we obtain:

$$\|g\|_{L^2((0,T) \times \Gamma_d)} \leq C\|G\|_{H^1((0,T) \times \Gamma_d)}. \quad (3.18)$$

Since  $G_\epsilon$  and  $g_\epsilon$  converge strongly in  $H^1$  resp.  $L^2$ , then after writing the weak problem (2.2)-(2.3) associated to the Neumann data  $g_\epsilon$  and then letting  $\epsilon \rightarrow 0$ , one can easily see that: the limit functions  $G$  resp.  $g$  are the Dirichlet resp. the Neumann data on  $\Gamma_d$  of problem (2.5). Considering the claim in remark 2.1, namely estimate (2.4), and then using (3.18) we conclude the estimate:

$$\|u^1\|_{C^0([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))} \leq C\|G\|_{H^1((0,T) \times \Gamma_d)}, \quad (3.19)$$

where  $u^1$  solves (2.5) with the function  $G$  as a Dirichlet data on  $(0, T) \times \Gamma_d$ .

*Conclusion.* It is immediate to see that the function  $u := u^1 + u^2$  solves uniquely system (1.6), where  $u^2$  is the unique solution of problem (3.10). Combining estimates (3.11) and (3.19) we infer using the triangular inequality:

$$\begin{aligned} \|u^1 + u^2\|_{C^0([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))} \\ \leq \|G\|_{H^1((0,T) \times \Gamma_d)} + \|f\|_{L^1(0,T; L^2(\Omega))} + \|\Psi^0\|_{H^1(\Omega)} + \|\Psi^1\|_{L^2(\Omega)}, \end{aligned}$$

which yields estimate (1.12). Finally, it is easy to see how the uniqueness of the solution  $u$  follows from this estimate, and thus conclude the proof of Theorem 1.1.  $\square$

**Remark 3.1.** Since  $\tilde{G} \in H^1((0, T) \times \partial\Omega)$  we know, cf. [7, Theorem 1, p.40], there exists  $\tilde{G}^n \in C^\infty(\partial\Omega)$  such that  $\|\tilde{G}^n - \tilde{G}\|_{H^1((0,T) \times \partial\Omega)} \rightarrow 0$ . On the other hand, given that  $\partial\Omega \in C^\infty$ , we know, cf. [8, Theorem 2, p.12], that there exists an extension  $\tilde{G}_e^n$  having a sufficient regularity, say  $\tilde{G}_e^n \in C^1(\mathbb{R}^2)$  such that:

$$\tilde{G}_e^n = \tilde{G}^n \quad \text{on } (0, T) \times \partial\Omega \quad (3.20)$$

the rule of derivation of the product on  $\Omega$  yields:

$$\nabla_x(\tilde{G}_e^n \chi_\epsilon \psi_1^\epsilon) = \chi_\epsilon \psi_1^\epsilon \nabla_x \tilde{G}_e^n + \tilde{G}_e^n \chi_\epsilon \nabla_x \psi_1^\epsilon + \tilde{G}_e^n \psi_1^\epsilon \nabla_x \chi_\epsilon, \quad \text{a.e. in } (0, T) \times \Omega. \quad (3.21)$$

Given the smoothness of all the involved functions, identity (3.21) holds true when passing to the boundary points  $\Gamma_d$ . Thus, taking into account (3.20), we have:

$$\nabla_x(G^n \chi_\epsilon \psi_1^\epsilon) = \chi_\epsilon \psi_1^\epsilon \nabla_x G^n + G^n \chi_\epsilon \nabla_x \psi_1^\epsilon + G^n \psi_1^\epsilon \nabla_x \chi_\epsilon, \quad \text{a.e. in } (0, T) \times \Gamma_d. \quad (3.22)$$

Using the density assumption, we pass to the limit  $n \rightarrow \infty$  by using the dominated convergence theorem to deduce that:

$$\begin{aligned} & \|\nabla_x(G^n \chi_\epsilon \psi_1^\epsilon) - (\chi_\epsilon \psi_1^\epsilon \nabla_x G + G \chi_\epsilon \nabla_x \psi_1^\epsilon G \psi_1^\epsilon \nabla_x \chi_\epsilon)\|_{H^1((0,T) \times \partial\Omega)} \\ & \leq \|\chi_\epsilon \psi_1^\epsilon \nabla_x G^n - \chi_\epsilon \psi_1^\epsilon \nabla_x G\| + \|G^n \chi_\epsilon \nabla_x \psi_1^\epsilon - G \chi_\epsilon \nabla_x \psi_1^\epsilon\| \\ & + \|G^n \psi_1^\epsilon \nabla_x \chi_\epsilon - G \psi_1^\epsilon \nabla_x \chi_\epsilon\| \\ & \rightarrow 0. \end{aligned} \quad (3.23)$$

On one hand using the density assumption, and on the other hand converting the normed convergences in (3.23) into point wise convergences, we get using (3.22):

$$\begin{aligned} \nabla_x G_\epsilon(t, x) &= \nabla_x(G \chi_\epsilon \psi_1^\epsilon)(t, x) \\ &= \lim_{n \rightarrow \infty} \nabla_x(G^n \chi_\epsilon \psi_1^\epsilon)(t, x) \\ &= \chi_\epsilon \psi_1^\epsilon \nabla_x G + G \chi_\epsilon \nabla_x \psi_1^\epsilon + G \psi_1^\epsilon \nabla_x \chi_\epsilon, \quad \text{a.e. in } (0, T) \times \Gamma_d. \end{aligned} \quad (3.24)$$

**Remark 3.2.** One should notice that the derivative  $\partial_x \psi_1^\epsilon$  can become infinite when  $\epsilon \rightarrow 0$ , the point behind the domination condition (1.11) and condition (iii) is to annihilate the effect of this possible blow-up.

- The identification  $L^2((0, T) \times \partial\Omega) \equiv L^2(0, T; L^2(\partial\Omega))$  in the second step of the proof of Lemma 2.1 relies on the existence of the isometric isomorphism defined by:

$$\begin{aligned} \mathcal{I} : L^2((0, T) \times \partial\Omega) &\rightarrow L^2(0, T; L^2(\partial\Omega)) \\ \eta &\mapsto (t \mapsto \eta(t, x)). \end{aligned}$$

- Finally, we emphasize that the results stated in [2] and which are used in this paper, were established within the framework of the Laplacian operator. That said, they remain valid within the framework of a strongly and uniformly elliptic operator  $\mathcal{L}$  in (1.2).

## REFERENCES

- [1] Wilcox C H, Initial-boundary value problems for linear hyperbolic partial differential equations of the second order, *Arch. Ration. Mech. Anal.* **10-1** (1962) 361-400.
- [2] Lasićka I, Lions J-L and Trigianni R, Non homogeneous boundary value problems for second order hyperbolic operators, *J. Math. pures et appl.* **65** (1985) 149-192.
- [3] Lasićka I and Trigianni R, Regularity Theory of Hyperbolic Equations with Non-homogeneous Neumann Boundary Conditions II. General Boundary Data, *J. Differential Equations.* **94** (1991) 112-164.
- [4] Ibuki K, On the regularity of solutions of a mixed problem for hyperbolic equations of second order in a domain with corners, *J. Math. Kyoto Univ.* **16-1** (1976) 167-183.
- [5] Hayashida K, On a mixed problem for hyperbolic equations with discontinuous boundary conditions, *Publ. RIMS, Kyoto Univ.* **7** (1971/72) 57-67.
- [6] Evans L C, Partial Differential Equations, Graduate Studies in Mathematics V.19 (1998) (American Mathematical Society).
- [7] V. I. Burenkov, The density of infinitely differentiable functions in Sobolev spaces for an arbitrary open set, Investigations in the theory of differentiable functions of many variables and its applications. Part V, *Trudy Mat. Inst. Steklov.*, 131, 1974, 3950; *Proc. Steklov Inst. Math.*, **131** (1974), 3951
- [8] C. Fefferman, Whitney's extension problems and interpolation of data, *Bull. Amer. Math. Soc.* **46** (2009), 207-220.

<sup>1</sup>L2CSP, MOULOU D MAMMERI UNIVERSITY OF TIZI-OUZOU, 15000, TIZI-OUZOU, ALGERIA.

\*CORRESPONDING AUTHOR: djamel.aitakli@ummto.dz