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INITIAL-BOUNDARY VALUE PROBLEM FOR SECOND ORDER HYPERBOLIC OPERATOR WITH MIXED BOUNDARY CONDITIONS

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ABSTRACT. We deal with a linear hyperbolic differential operator of the second order on a bounded planar domain with a smooth boundary. We establish a well-posedness result in case where a mixed, Dirichlet-Neumann, boundary condition is prescribed on the boundary. We focus on the case of a non-homogeneous Dirichlet data and a homogeneous Neumann one.

1. INTRODUCTION

Throughout this paper, we let $\Omega \subset \mathbb{R}^2$ to be a bounded planar domain of class C^∞ . The boundary $\partial\Omega := \Gamma_d \cup \Gamma_n$ is formed by two parts. We assume, for simplicity, that Γ_d is connected and denote:

$$\{s_i\}_{i=1,2} := \overline{\Gamma_d} \cap \overline{\Gamma_n}. \quad (1.1)$$

We are interested in the study of the time-dependent hyperbolic operator, \mathcal{H} , defined by:

$$\mathcal{H}u := \partial_t^2 u - \mathcal{L}u. \quad (1.2)$$

where $T > 0$ and \mathcal{L} is a second order uniformly and strongly elliptic operator, one can think of the Laplace operator or the Lamé system for instance. In the sequel, and only for the sake of clarity of presentation, the operator \mathcal{L} will be represented by the Laplacian operator Δ . We consider the partial differential equation:

$$\mathcal{H}u = f \quad \text{a.e. in } (0, T) \times \Omega. \quad (1.3)$$

The equation (1.3) is completed with a Dirichlet-Neumann mixed boundary condition. More precisely, we prescribe a Dirichlet data on Γ_d and a Neumann condition on Γ_n :

$$u = G \quad \text{on } \Gamma_d \quad , \quad \frac{\partial u}{\partial \vec{n}} = 0 \quad \text{on } \Gamma_n, \quad (1.4)$$

where \vec{n} is the exterior unit normal defined at each point of Γ_n and $\frac{\partial u}{\partial \vec{n}}$ denotes the normal derivative of u . Moreover, initial time conditions are also prescribed:

$$u(0, x) = \Psi^0(x), \quad \partial_t u(0, x) = \Psi^1(x) \quad \text{on } \Omega. \quad (1.5)$$

We address in this work the issue of existence and uniqueness of a solution to the problem associated to equation (1.3) when endowed with the conditions (1.4) and (1.5). Moreover, we establish an energy estimate that shows the continuous

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dependence of the solution of this problem with respect to the data, this estimate expresses the stability of the solution.

Let's place our work in perspective: a well-posedness result, in case of an homogeneous Dirichlet conditions, can be found in [1]. A similar study for the case of a non-homogeneous Dirichlet condition, on the entire boundary, is dealt with in [2]. Regarding other types of boundary condition, the work presented in [3] deals with the case of the non-homogenous Neumann-type boundary condition. The problem with a mixed homogeneous type condition has been dealt with in [4], see also [5].

The problem we want to study is given in the following system:

$$\begin{cases} \partial_t^2 u - \Delta u = f & \text{in } (0, T) \times \Omega, \\ u = G & \text{on } (0, T) \times \Gamma_d, \\ \frac{\partial u}{\partial \vec{n}} = 0 & \text{on } (0, T) \times \Gamma_n, \\ u(0, x) = \Psi^0, \quad \partial_t u(0, x) = \Psi^1 & \text{in } \Omega. \end{cases} \quad (1.6)$$

Let us introduce the space $K(\Omega)$ defined as the completion of the set, $C^\infty(\bar{\Omega})$, of the up to boundary smooth functions in Ω vanishing on Γ_d with respect to the Sobolev H^1 -norm i.e.

$$K(\Omega) := \overline{\{v \in C^\infty(\bar{\Omega}), v_{\Gamma_d} = 0\}}^{\|\cdot\|_{H^1}}. \quad (1.7)$$

The functions (u, v) are said to satisfy the weak boundary condition if:

$$\begin{aligned} (u, v) &\in K(\Omega) \times K(\Omega), \\ \int_{\Omega} \Delta u \varphi &= \int_{\Omega} \nabla u \nabla \varphi, \quad \forall \varphi \in K(\Omega). \end{aligned} \quad (1.8)$$

The main result of this paper is stated in the following theorem:

Theorem 1.1. *Let $f \in C^1([0, T]; L^2(\Omega))$. Assume that (Ψ^0, Ψ^1) satisfy the weak boundary conditions (1.8). Let $G \in H^1((0, T) \times \Gamma_d)$ such that there exists $\tilde{G} \in H^1((0, T) \times \partial\Omega)$ satisfying:*

$$\tilde{G} \equiv G \quad \text{on } (0, T) \times \Gamma_d, \quad (1.9)$$

$$\tilde{G}(0, x) = 0, \quad \partial_t \tilde{G}(0, x) = 0 \quad \text{on } \partial\Omega. \quad (1.10)$$

Let $\{s_i\}_i$ be defined as in (1.1). We assume further that G satisfy point-wisely a domination condition near s_i , that is:

$$\exists \gamma > 0, \quad |G(t, x)| = \mathcal{O}_{x \rightarrow s_i}(|x - s_i|^{1+\gamma}), \quad \forall t \in (0, T), \quad i = 1, 2. \quad (1.11)$$

Then, there is a unique solution $u \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ of system (1.6), moreover this solution satisfies the following energy estimate:

$$\begin{aligned} &\left\| \frac{\partial u}{\partial \vec{n}} \right\|_{L^2((0, T) \times \Gamma_d)} + \|u\|_{C^0([0, T]; H^1(\Omega))} + \|u'\|_{C^0([0, T]; L^2(\Omega))} \\ &\leq C \left(\|G\|_{H^1((0, T) \times \Gamma_d)} + \|f\|_{L^1(0, T; L^2(\Omega))} + \|\Psi^0\|_{H^1(\Omega)} + \|\Psi^1\|_{L^2(\Omega)} \right). \end{aligned} \quad (1.12)$$

The plan of the proof consists, first of all, in establishing, in section 2, an auxiliary lemma which states that the space of solutions of some auxiliary problem is non-trivial. Moreover, these solutions satisfy an energy estimate. In a second step, we consider, in section 3, an approximation problem whose solutions effectively satisfy this energy estimate. This allows us to pass to the limit and to conclude the results of Theorem 1.1.

The presented proof is based on a functional theoretic approach and on an approximation argument. We emphasize that the results stated in [2] and used in this paper, which are proved in the Laplacian framework, remain valid in the case of a strongly and uniformly elliptic general operator \mathcal{L} in (1.2).

2. AUXILIARY RESULT

Before going further, we need to set some preliminary remarks:

Remark 2.1. *Consider the Dirichlet boundary value problem:*

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } (0, T) \times \Omega, \\ u = G & \text{on } (0, T) \times \partial\Omega, \\ u(0, x) = 0, \quad \partial_t u(0, x) = 0 & \text{in } \Omega, \end{cases} \quad (2.1)$$

where $G \in H^1((0, T) \times \partial\Omega)$ satisfy the compatibility conditions (1.10). Following [2, Theorem 2.1, p. 151], there exists a unique solution $u \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ to problem (2.8), moreover $\frac{\partial u}{\partial \vec{n}} \in L^2((0, T) \times \partial\Omega)$. Let us denote $g := \frac{\partial u}{\partial \vec{n}}$ the Neumann data associated to system (2.1), then the weak problem associated to this system writes:

find $u \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ such that:

$$(u'', v) + B[u, v, t] = (g, v), \quad (2.2)$$

for all $v \in H^1(\Omega)$ and a.e. $0 \leq t \leq T$.

$$u(0, x) = \Psi^0(x), \quad \partial_t u(0, x) = \Psi^1(x). \quad (2.3)$$

Regarding the structure of the bilinear form B and the existence of a, unique, solution of this weak problem, the reader can refer to [6, Sec.7.2.1-7.2.2, p.377-380]. An energy estimate can thus be derived, cf. [6, Theorem 5, p. 389]:

$$\sup_{t \in (0, T)} \|u(t)\|_{H^1(\Omega)} + \sup_{t \in (0, T)} \|u'(t)\|_{L^2(\Omega)} \leq C \|g\|_{L^2((0, T) \times \partial\Omega)}. \quad (2.4)$$

We state an auxiliary Lemma needed subsequently for the proof of Theorem 1.1:

Lemma 2.1. *Consider the, formally posed, mixed boundary value problem given by the following system:*

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } (0, T) \times \Omega, \\ u = G & \text{on } (0, T) \times \Gamma_d, \\ \frac{\partial u}{\partial \vec{n}} = 0 & \text{on } (0, T) \times \Gamma_n, \\ u(0, x) = 0, \quad \partial_t u(0, x) = 0 & \text{in } \Omega. \end{cases} \quad (2.5)$$

with the compatibility conditions (1.10) satisfied by G on Γ_d . Then the subspace $\mathcal{G} \subset H^1((0, T) \times \Gamma_d)$:

$$\mathcal{G} := \left\{ \begin{array}{l} G \in H^1((0, T) \times \Gamma_d) : \text{problem (2.5)} \\ \text{admits a unique solution } u \text{ such that} \\ u \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \end{array} \right\}. \quad (2.6)$$

isn't trivial. Moreover, a solution u of problem (2.5) satisfies the energy estimate:

$$\begin{aligned} & \left\| \frac{\partial u}{\partial \vec{n}} \right\|_{L^2((0,T) \times \Gamma_d)} + \|u\|_{C^0([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))} \\ & \leq C \|G\|_{H^1((0,T) \times \Gamma_d)}. \end{aligned} \quad (2.7)$$

Where we meant by a trivial subspace: a vector space containing only the zero element $\mathbf{0}_{H^1}$. We set $[H^1]'((0,T) \times \partial\Omega)$ to be the topological dual space of the Sobolev space $H^1((0,T) \times \partial\Omega)$. We now propose to give a proof of Lemma 2.1:

Proof. Consider the non-homogeneous Dirichlet problem given by the following system:

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } (0, T) \times \Omega, \\ u = G & \text{on } (0, T) \times \partial\Omega, \\ u(0, x) = 0, \partial_t u(0, x) = 0 & \text{in } \Omega, \end{cases} \quad (2.8)$$

where $G \in H^1((0, T) \times \partial\Omega)$ satisfy the compatibility conditions (1.10). Consider the operator T defined by:

$$\begin{aligned} T : H^1((0, T) \times \partial\Omega) &\rightarrow \mathcal{A} \subset L^2((0, T) \times \partial\Omega) \\ G &\mapsto T(G) = g := \frac{\partial u}{\partial \vec{n}}, \end{aligned}$$

where $\mathcal{A} := T(H^1((0, T) \times \partial\Omega))$. According to the well-posedness result recalled in remark 2.1, the operator T associates to every Dirichlet data $G \in H^1((0, T) \times \partial\Omega)$ the unique Neumann data $g \in \mathcal{A} = T(H^1((0, T) \times \partial\Omega)) \subset L^2((0, T) \times \partial\Omega)$ on the boundary associated to problem (2.8), thus T is well defined.

According to [2, Remark 2.2, p. 152], and in addition to being linear, the operator T is bounded i.e. there exists $C_0 > 0$ such that:

$$\|T(G)\|_{L^2} = \left\| \frac{\partial u}{\partial \vec{n}} \right\|_{L^2((0,T) \times \partial\Omega)} \leq C_0 \|G\|_{H^1((0,T) \times \partial\Omega)}, \quad (2.9)$$

for all $G \in H^1((0, T) \times \partial\Omega)$. The operator $T : H^1((0, T) \times \partial\Omega) \rightarrow \mathcal{A}$ is clearly linear and invertible, we would like to establish the boundedness of its inverse, denoted by T^{-1} . For this we use the Banach isomorphism theorem. In order to apply this theorem, we need first to show that the subspace \mathcal{A} is complete i.e. is a Banach space. In order to prove the completeness of \mathcal{A} , it is sufficient to prove that it is a closed subspace. Let $(g_n)_n$ be a convergent sequence of functions in \mathcal{A} i.e. there exists $g \in L^2((0, T) \times \partial\Omega)$ such that $\|g_n - g\|_{L^2((0,T) \times \partial\Omega)} \rightarrow 0$, then we need to show that $g \in \mathcal{A}$ i.e. that there exists $G \in H^1((0, T) \times \partial\Omega)$ such that $g = T(G)$. Let $(G_n)_{n \in \mathbb{N}}$ be such that $G_n \in H^1((0, T) \times \partial\Omega)$ and $g_n = T(G_n)$. According to [2, Estimate (2.65), p. 163] we know that $(g_n)_n$ and $(G_n)_n$ do satisfy:

$$\int_{(0,T) \times \partial\Omega} G_n \frac{\partial \phi}{\partial \vec{n}} = \int_{(0,T) \times \partial\Omega} g_n h \quad (2.10)$$

for all $\phi \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ solving an auxiliary problem, see [2, Problem 2.63, p. 163], and $h = \phi$ on $(0, T) \times \partial\Omega$. Consequently, the functions ϕ and h satisfy: $\frac{\partial \phi}{\partial \vec{n}} = T(h)$. Let $\mathcal{D} := \{h \in H^{1+\delta}((0, T) \times \partial\Omega) : \frac{\partial \phi}{\partial \vec{n}} \in H^1((0, T) \times \partial\Omega)\}$, for some $\delta > 0$. Following [3, Theorem A.3, p.117], we can choose $\delta > 0$ such that the subspace \mathcal{D} exists and $T(\mathcal{D}) = H^1((0, T) \times \partial\Omega)$, under the condition: $\frac{\partial \phi}{\partial \vec{n}}(0) = 0$, which is indeed satisfied since $\phi(0, x) \equiv 0$ on Ω . On the other hand, by

the Riesz representation theorem, $\forall n \in \mathbb{N}$, $G_n \in H^1((0, T) \times \partial\Omega)$ defines a unique $G_n \in [H^1]'((0, T) \times \partial\Omega)$ and such that:

$$\|G_n\|_{H^1} = \|G_n\|_{[H^1]'}. \quad (2.11)$$

Since $(g_n)_n$ is convergent in L^2 then there exists $\beta > 0$ such that $\|g_n\|_{L^2((0, T) \times \partial\Omega)} \leq \beta$ for all $n \in \mathbb{N}$. Estimate (2.10) yields us:

$$\begin{aligned} \left| \left\langle G_n, \frac{\partial\phi}{\partial\vec{n}} \right\rangle_{[H^1]', H^1} \right| &= \left| \int_{(0, T) \times \partial\Omega} g_n h \right| \leq \|g_n\|_{L^2} \|h\|_{H^1} \\ &\leq \beta \|h\|_{H^{1+\delta}} \\ &\leq \beta \left\| \frac{\partial\phi}{\partial\vec{n}} \right\|_{H^1}. \end{aligned} \quad (2.12)$$

where we have used the continuity stated in the beginning of [3, Subsection 1.2, p.116]. Since the Neumann data $\eta := \frac{\partial\phi}{\partial\vec{n}}$ is arbitrary, for $h \in \mathcal{D}$, then estimate (2.12) yields us:

$$\|G_n\|_{[H^1]'} = \sup_{\eta \in H^1((0, T) \times \partial\Omega) - \{0_{H^1}\}} \frac{|\langle G_n, \eta \rangle|}{\|\eta\|_{H^1}} \leq \beta,$$

for all $n \in \mathbb{N}$. But using (2.11) gives us: $\|G_n\|_{H^1} = \|G_n\|_{[H^1]'} \leq \beta$ for all $n \in \mathbb{N}$, which implies that G_n converges weakly in H^1 to some G . Thus, by passing to the limit in (2.10), we obtain:

$$\int_{(0, T) \times \partial\Omega} G \frac{\partial\phi}{\partial\vec{n}} = \int_{(0, T) \times \partial\Omega} gh. \quad (2.13)$$

We conclude that there exists $G \in H^1$ such that $g = T(G)$ and thus \mathcal{A} is a Banach space. Consequently, using (2.9) and by virtue of the Banach isomorphism theorem, the linear operator $T^{-1} : \mathcal{A} \rightarrow H^1$ is bounded. The continuity of T^{-1} ensures that $\exists C_{-1} > 0$ such that

$$\|T^{-1}(g)\|_{H^1((0, T) \times \partial\Omega)} = \|G\|_{H^1((0, T) \times \partial\Omega)} \leq C_{-1} \|g\|_{L^2((0, T) \times \partial\Omega)}. \quad (2.14)$$

for all $g \in \mathcal{A}$. Set \mathcal{B} to be the space:

$$\mathcal{B} = \{g \in L^2((0, T) \times \partial\Omega) : g \equiv 0 \text{ on } \Gamma_n\}.$$

The subspace \mathcal{B} is closed and then is $\mathcal{A} \cap \mathcal{B}$, thus it is a Banach space. Since $T : H^1 \rightarrow \mathcal{A}$ is continuous, as well as its inverse, then $T^{-1}(\mathcal{A} \cap \mathcal{B})$ is complete, thus it is a Banach. One should notice well that the space $T^{-1}(\mathcal{A} \cap \mathcal{B}) \subset H^1((0, T) \times \partial\Omega)$ isn't empty. Indeed, if $g \in H^1((0, T) \times \partial\Omega) \subset L^2((0, T) \times \partial\Omega)$ such that $g \equiv 0$ on Γ_n then following [3, Theorem A.3, p.117] we have that $T^{-1}(g) \in H^{1+\delta} \subset H^1$. Consequently $g \in \mathcal{A} \cap \mathcal{B}$ and $T^{-1}(g)$ doesn't vanish identically on $(0, T) \times \partial\Omega$. This proves the assertion about the non triviality of the subspace \mathcal{G} , cf. (2.6). We immediately infer from estimate (2.14) that:

$$\|T^{-1}(g)\|_{H^1((0, T) \times \Gamma_d)} = \|G\|_{H^1((0, T) \times \Gamma_d)} \leq C_{-1} \|g\|_{L^2((0, T) \times \Gamma_d)}. \quad (2.15)$$

for all $g \in \mathcal{A} \cap \mathcal{B}$. The operator T^{-1} is defined by:

$$\begin{aligned} T^{-1} : \mathcal{A} \cap \mathcal{B} &\rightarrow T^{-1}(\mathcal{A} \cap \mathcal{B}) \subset H^1 \\ g &\mapsto T^{-1}(g) = G. \end{aligned}$$

and is continuous. Consider the operator M defined by

$$\begin{aligned} M : T^{-1}(\mathcal{A} \cap \mathcal{B}) \subset H^1 &\rightarrow \mathcal{C} \\ G &\mapsto G|_{\Gamma_d}, \end{aligned}$$

where \mathcal{C} is the subspace defined by:

$$\mathcal{C} := M \circ T^{-1}(\mathcal{A} \cap \mathcal{B}). \quad (2.16)$$

Estimate (2.15) shows that the operator $M \circ T^{-1} : \mathcal{A} \cap \mathcal{B} \rightarrow \mathcal{C}$ is continuous, and it is one-to-one. On the other hand one can see easily that the subspace \mathcal{C} given by (2.16) is a Banach space. Then applying again the isomorphism theorem we show that the operator $M \circ T^{-1}$ has a bounded inverse i.e. there exists $C_1 > 0$ s.t.:

$$\|g\|_{L^2((0,T) \times \Gamma_d)} = \|[M \circ T^{-1}]^{-1}(G)\|_{L^2((0,T) \times \Gamma_d)} \leq C_1 \|G\|_{H^1((0,T) \times \Gamma_d)}. \quad (2.17)$$

for all $G \in H^1((0,T) \times \Gamma_d)$. Using estimate (2.17), and taking into account remark 2.1 namely estimate (2.4), we conclude immediately estimate (2.7). \square

3. PROOF OF THE MAIN RESULT

We now establish Theorem 1.1 using mainly the result of Lemma 2.1:

Proof. The proof is divided into two main subsections:

Setting of the approximating problem. Let $\{s_i\}_{i=1,2}$ be such as defined by (1.1). We let $D(s_i, \epsilon_0) \subset \mathbb{R}^2$ to be the disk centred at s_i with radius ϵ_0 , where $\epsilon_0 := \frac{1}{2}|s_1 - s_2|$. Let $\epsilon, 0 < \epsilon < \epsilon_0$, be a parameter designed to tend towards zero. We introduce a sequence of truncating functions $\chi_{\epsilon,i} \in C^1(\mathbb{R}^2)$, $i = 1, 2$ defined by:

$$\chi_{\epsilon,i}(x) := \begin{cases} 0, & |x - s_i| \leq \epsilon^2, \\ \exp\left(\frac{\epsilon - |x - s_i|}{\epsilon^2 - |x - s_i|}\right), & \epsilon^2 < |x - s_i| < \epsilon, \\ 1, & \epsilon < |x - s_i| < \epsilon_0. \end{cases}$$

Define:

$$\chi_\epsilon(t, x) := \begin{cases} \chi_{\epsilon,1}(x), & \forall x, |x - s_1| \leq \epsilon_0, \forall t \in (0, T), \\ \chi_{\epsilon,2}(x), & \forall x, |x - s_2| \leq \epsilon_0, \forall t \in (0, T), \\ 1, & \forall x \in \mathbb{R}^2 - \mathcal{V}_1 \cup \mathcal{V}_2, \forall t \in (0, T). \end{cases} \quad (3.1)$$

where we denote $\mathcal{V}_i := D(s_i, \epsilon_0)$. It is easy to see, from the definition (3.1), that the functions χ_ϵ satisfy:

$$1) |\chi_\epsilon(t, x)| \leq 1, \quad \forall t \in (0, T), \forall x \in \mathbb{R}^2, \forall \epsilon > 0. \quad (3.2)$$

$$2) \epsilon^{1+\gamma} \|\partial_x \chi_\epsilon\|_{\infty, (0,T) \times \mathcal{V}_i} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0, \text{ with } \gamma > 0. \quad (3.3)$$

3) The functions χ_ϵ are constant with respect to the time variable.

Let us cover the set $\overline{\Omega}$ by open sets U_1 and U_2^ϵ such that:

- (i) $\overline{\Omega} \subset U_1 \cup U_2^\epsilon$, for all $\epsilon > 0$,
- (ii) $U_1 \cap \partial\Omega = \overline{\Gamma}_d$,
- (iii) $|\alpha_i^\epsilon - s_i| < \frac{1}{2}\epsilon^2$, for every $i = \overline{1, 2}$,

where we denote $\{\alpha_1^\epsilon, \alpha_2^\epsilon\} := \partial\Omega \cap \partial U_2^\epsilon$ such that $d(\alpha_1^\epsilon, s_1) < d(\alpha_2^\epsilon, s_1)$, for all $\epsilon > 0$. The condition (ii) implies that $\partial U_1 \cap \partial\Omega = \{s_1, s_2\}$. Consider, for every $\epsilon > 0$, the partition of the unity $(\tilde{\psi}_1^\epsilon, \tilde{\psi}_2^\epsilon)$ subordinate to the cover (U_1, U_2^ϵ) i.e.

- (a) $\tilde{\psi}_i^\epsilon \in C_0^\infty(U_i)$, $0 < \tilde{\psi}_i^\epsilon < 1$, for $1 \leq i \leq 2$,

- (b) $\text{supp } \tilde{\psi}_1^\epsilon \subset U_1, \text{supp } \tilde{\psi}_2^\epsilon \subset U_2, \forall \epsilon > 0,$
 (c) $\tilde{\psi}_1^\epsilon(x) + \tilde{\psi}_2^\epsilon(x) = 1$ for every $x \in \Omega$.

Define the functions:

$$\psi_1^\epsilon(t, x) = \tilde{\psi}_1^\epsilon(x), \quad \psi_2^\epsilon(t, x) := \tilde{\psi}_2^\epsilon(x), \quad \forall t \in (0, T), \forall x \in \Omega. \quad (3.4)$$

Thus, the functions $\psi_1^\epsilon, \psi_2^\epsilon$ are constant with respect to the time variable. One immediate consequence is:

$$\|\psi_1^\epsilon - 1\|_{\infty, (0, T) \times \Gamma_d} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (3.5)$$

Let $G \in H^1((0, T) \times \Gamma_d)$ such that there exists $\tilde{G} \in H^1((0, T) \times \partial\Omega)$ with $\tilde{G} \equiv G$ on $(0, T) \times \Gamma_d$. Consider the Dirichlet boundary value problem:

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } (0, T) \times \Omega, \\ u = \tilde{G} & \text{on } (0, T) \times \partial\Omega, \\ u(0, x) = 0, \quad \partial_t u(0, x) = 0 & \text{in } \Omega. \end{cases} \quad (3.6)$$

such that the compatibility conditions (1.10) are met by the function \tilde{G} on $\partial\Omega$. Let $u \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ be the unique solution of the system (3.6). It is easy to see that the function $\chi_\epsilon \psi_1^\epsilon u$ satisfy the following system:

$$\begin{cases} \partial_t^2(\chi_\epsilon \psi_1^\epsilon u) - \Delta(\chi_\epsilon \psi_1^\epsilon u) = f_\epsilon & \text{in } (0, T) \times \Omega_s, \\ \chi_\epsilon \psi_1^\epsilon u = G_\epsilon & \text{on } (0, T) \times \Gamma_d, \\ \frac{\partial(\chi_\epsilon \psi_1^\epsilon u)}{\partial \vec{n}} = 0 & \text{on } (0, T) \times \Gamma_n, \\ \chi_\epsilon \psi_1^\epsilon u(0, x) = 0, \quad \partial_t(\chi_\epsilon \psi_1^\epsilon u)(0, x) = 0 & \text{in } \Omega, \end{cases} \quad (3.7)$$

where $G_\epsilon := \psi_1^\epsilon \chi_\epsilon \tilde{G}$ and $f_\epsilon \in C^1([0, T]; L^2(\Omega))$. One should remark that the homogeneous Neumann condition of system (3.7) arises from the conditions (ii) and (b). Consider the function w_ϵ satisfying the following system:

$$\begin{cases} \partial_t^2 w_\epsilon - \Delta w_\epsilon = f - f_\epsilon & \text{in } (0, T) \times \Omega_s, \\ w_\epsilon = 0 & \text{on } (0, T) \times \Gamma_d, \\ \frac{\partial w_\epsilon}{\partial \vec{n}} = 0 & \text{on } (0, T) \times \Gamma_n, \\ w_\epsilon(0, x) = \Psi^0(x), \quad \partial_t w_\epsilon(0, x) = \Psi^1(x) & \text{in } \Omega, \end{cases} \quad (3.8)$$

where (Ψ^0, Ψ^1) satisfy the weak boundary conditions (1.8). In virtue of [4, Theorem 1, p. 170] and its Corollary, see also [5], there is a unique $w_\epsilon \in C^1([0, T]; K(\Omega)) \cap C^2([0, T]; L^2(\Omega))$ satisfying, for all $\epsilon > 0$, system (3.8). Then it is obvious to remark that the function $u_\epsilon := \chi_\epsilon \psi_1^\epsilon u + w_\epsilon$ solves the following system:

$$\begin{cases} \partial_t^2 u_\epsilon - \Delta u_\epsilon = f & \text{in } (0, T) \times \Omega_s, \\ u_\epsilon = G_\epsilon & \text{on } (0, T) \times \Gamma_d, \\ \frac{\partial u_\epsilon}{\partial \vec{n}} = 0 & \text{on } (0, T) \times \Gamma_n, \\ u_\epsilon(0, x) = \Psi^0(x), \quad \partial_t u_\epsilon(0, x) = \Psi^1(x) & \text{in } \Omega. \end{cases}$$

On the other hand, consider the function u^2 that satisfy the system:

$$\begin{cases} \partial_t^2 u^2 - \Delta u^2 = f & \text{in } (0, T) \times \Omega, \\ u^2 = 0 & \text{on } (0, T) \times \Gamma_d, \\ \frac{\partial u^2}{\partial \vec{n}} = 0 & \text{on } (0, T) \times \Gamma_n, \\ u^2(0, x) = \Psi^0(x), \quad \partial_t u^2(0, x) = \Psi^1(x) & \text{in } \Omega, \end{cases} \quad (3.9)$$

following again [4, Theorem 1, p. 170], see also [5], we deduce that problem (3.9) admits a unique solution

$$u^2 \in C^1([0, T]; K(\Omega)) \cap C^2([0, T]; L^2(\Omega))$$

that satisfy, cf. [4, Estimate 1.9, p. 170], the following estimate:

$$\begin{aligned} & \|u^2\|_{C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))} \\ & \leq C(\|f\|_{L^1(0, T; L^2(\Omega))} + \|\Psi^0\|_{H^1(\Omega)} + \|\Psi^1\|_{L^2(\Omega)}). \end{aligned} \quad (3.10)$$

Passing to the limit using lemma 2.1. It is immediate to see that $u_\epsilon^1 := u_\epsilon - u^2 \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ exists and satisfy the system:

$$\begin{cases} \partial_t^2 u_\epsilon^1 - \Delta u_\epsilon^1 = 0 & \text{in } (0, T) \times \Omega; \\ u_\epsilon^1 = G_\epsilon & \text{on } (0, T) \times \Gamma_d; \\ \frac{\partial u_\epsilon^1}{\partial n} = 0 & \text{on } (0, T) \times \Gamma_n; \\ u_\epsilon^1(0, x) = 0, \quad \partial_t u_\epsilon^1(0, x) = 0 & \text{in } \Omega, \end{cases} \quad (3.11)$$

and thus $G_\epsilon \in \mathcal{G}$ for every $\epsilon > 0$, where \mathcal{G} is defined by (2.6). Moreover, the associated solution u_ϵ^1 that solves system (3.11) satisfy, for every $\epsilon > 0$, the assumption of Lemma 2.1. Consequently, if we note $g_\epsilon := \frac{\partial u_\epsilon^1}{\partial n}$, we obtain by applying estimate (2.7):

$$\|g_\epsilon\|_{L^2((0, T) \times \Gamma_d)} \leq C \|G\chi_\epsilon \psi_1^\epsilon\|_{H^1((0, T) \times \Gamma_d)}. \quad (3.12)$$

Recall that the H^1 -norm of $G_\epsilon = G\chi_\epsilon \psi_1^\epsilon$ is given by:

$$\begin{aligned} & \|G_\epsilon\|_{H^1((0, T) \times \Gamma_d)} \\ & = \left(\|G_\epsilon\|_{L^2((0, T) \times \Gamma_d)}^2 + \|\partial_t G_\epsilon\|_{L^2((0, T) \times \Gamma_d)}^2 + \|\partial_x G_\epsilon\|_{L^2((0, T) \times \Gamma_d)}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (3.13)$$

Since ψ_1^ϵ and χ_ϵ are constant in the time variable and taking into consideration (3.5), we need only to care about the term $\|\partial_x G_\epsilon\|_{L^2((0, T) \times \Gamma_d)}$. Indeed, this term involves the generalized derivative which, given the smoothness of $G\chi_\epsilon \psi_1^\epsilon$, coincide with the almost everywhere derivative, thus we have:

$$\partial_x (G\chi_\epsilon \psi_1^\epsilon) = \chi_\epsilon \psi_1^\epsilon \partial_x G + G\chi_\epsilon \partial_x \psi_1^\epsilon + G\psi_1^\epsilon \partial_x \chi_\epsilon, \quad \text{a.e. in } \Gamma_d. \quad (3.14)$$

We note the following three facts:

(A) Using the estimates (3.2) and (3.5) we can show easily that:

$$\|\chi_\epsilon \psi_1^\epsilon \partial_x G - \partial_x G\|_{L^2((0, T) \times \Gamma_d)} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

(B) Given condition (i)-(iii) and (b), the support of $\partial_x \psi_1^\epsilon$ is contained in the region $\cup_i D(s_i, \epsilon^2)$, for every $\epsilon > 0$. Moreover, given the definition (3.1), the function χ_ϵ vanishes on this region, thus:

$$\|G\chi_\epsilon \partial_x \psi_1^\epsilon\|_{L^2((0,T) \times \Gamma_d)} = \sum_{i=1}^2 \|G\chi_\epsilon \partial_x \psi_1^\epsilon\|_{L^2((0,T) \times (\Gamma_d \cap \{x, |x-s_i| \leq \epsilon^2\}))} = 0.$$

(C) Using condition (1.11), the properties (a) and (3.3) we infer:

$$\begin{aligned} \|G\psi_1^\epsilon \partial_x \chi_\epsilon\|_{L^2} &= \sum_{i=1}^2 \|G\psi_1^\epsilon \partial_x \chi_\epsilon\|_{L^2((0,T) \times \{x, \epsilon^2 < |x-s_i| \leq \epsilon\})} \\ &\leq 2\|\epsilon^\gamma\|_{L^2((0,T) \times \{x, \epsilon^2 < |x-s_i| \leq \epsilon\})} \rightarrow 0, \end{aligned}$$

we have used the fact that the support of $\partial_x \chi_\epsilon$ is contained in $\cup_i \{x, \epsilon^2 < |x-s_i| \leq \epsilon\}$.

On one hand using the facts (A)-(C), and on the other hand applying the dominated convergence theorem one can easily show that:

$$\|\partial_x(G\psi_1^\epsilon \chi_\epsilon) - \partial_x G_\epsilon\|_{L^2((0,T) \times \Gamma_d)} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (3.15)$$

Finally, using (3.5), (3.13) and (3.15) one can easily see that $\|G_\epsilon - G\|_{H^1((0,T) \times \Gamma_d)} \rightarrow 0$. Consequently, we deduce from estimate (3.12) that there exists $g \in L^2((0,T) \times \Gamma_d)$ such that $\|g_\epsilon - g\|_{L^2((0,T) \times \Gamma_d)} \rightarrow 0$ and thus passing to the limit in estimate (3.12) we obtain:

$$\|g\|_{L^2((0,T) \times \Gamma_d)} \leq C\|G\|_{H^1((0,T) \times \Gamma_d)}. \quad (3.16)$$

Since G_ϵ and g_ϵ converge strongly in H^1 resp. L^2 , then after writing the weak problem (2.2)-(2.3) associated to the Neumann data g_ϵ and then letting $\epsilon \rightarrow 0$, one can easily see that: the limit functions G resp. g are the Dirichlet resp. the Neumann data on Γ_d of problem (2.5). Considering the claim in remark 2.1, namely estimate (2.4), and then using (3.16) we conclude the estimate:

$$\|u^1\|_{C^0([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))} \leq C\|G\|_{H^1((0,T) \times \Gamma_d)}, \quad (3.17)$$

where u^1 solves the system (2.5) with the function G as a Dirichlet data on $(0, T) \times \Gamma_d$.

Conclusion. It is immediate to see that the function $u := u^1 + u^2$ solves uniquely system (1.6), where u^2 is the unique solution of problem (3.9). Combining estimates (3.10) and (3.17) we infer using the triangular inequality:

$$\begin{aligned} &\|u^1 + u^2\|_{C^0([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))} \\ &\leq \|G\|_{H^1((0,T) \times \Gamma_d)} + \|f\|_{L^1(0,T; L^2(\Omega))} + \|\Psi^0\|_{H^1(\Omega)} + \|\Psi^1\|_{L^2(\Omega)}, \end{aligned}$$

which yield us estimate (1.12). Finally, we show, using this estimate, the uniqueness of the solution u , and thus conclude the proof of Theorem 1.1. \square

Remark 3.1. *One should note that the derivative $\partial_x \psi_1^\epsilon$ can become infinite when $\epsilon \rightarrow 0$, the point behind the domination condition (1.11) and condition (iii) is to annihilate the effect of this blow-up.*

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