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# SHALLOW WATER WAVES GENERATED BY A FLOATING OBJECT: A CONTROL THEORETICAL PERSPECTIVE

PEI SU AND MARIUS TUCSNAK

**ABSTRACT.** We consider a control system describing the interaction of water waves with a partially immersed rigid body constraint to move only in the vertical direction. The fluid is modeled by the shallow water equations. The control signal is a vertical force acting on the floating body. We first derive the full governing equations of the fluid-body system in a water tank and reformulate them as an initial boundary value problem of a first-order evolution system. We then linearize the equations around the equilibrium state and we study its well-posedness. Finally we focus on the reachability and stabilizability of the linear system. Our main result asserts that, provided that the floating body is situated in the middle of the tank, any symmetric waves with appropriate regularity can be obtained from the equilibrium state by an appropriate control force. This implies, in particular, that we can project this system on the subspace of states with appropriate symmetry properties to obtain a reduced system which is approximately controllable and strongly stabilizable. Note that, in general, this system is not controllable (even approximately).

**Key words.** Shallow water equations, fluid-structure interactions, reachability, stabilizability, operator semigroup, infinite dimensional system.

**2020 AMS subject classifications.** 93B03, 93C20, 93D15, 76B15, 35Q35

## 1. INTRODUCTION

In this work we are interested in the following problem: given a rigid body floating in a fluid at rest in a bounded container, determine the control force acting on the body in order to obtain a prescribed wave profile. We assume that the floating object has vertical lateral walls, with a possibly non-flat but symmetric bottom. More precisely, we assume that the rigid body is restricted to the heave motion (move vertically) and that it floats in a rectangular fluid domain which fits in the shallow water regime (for this concept, please refer to Lannes [22, 24] or Whitham [37]). Moreover, the body is actuated by a vertical control force and in the horizontal direction it does not touch the lateral boundaries of the container. The main contribution in this work consists in showing that, within the linearized shallow water regime and in a spatially symmetric geometry, we can find controls steering the system from rest to any symmetric wave profile having an appropriate space regularity. In order to achieve this goal we pass through the following preliminary steps:

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- Deriving the full nonlinear control model and reformulate it as a first-order evolution system;
- Establishing the well-posedness of the linearized control system.

The system we consider is also of interest for modelling and controlling a class of *wave energy converters* (WECs) where all devices are used to capture the variations of the free surface waves and convert them into electricity. The most popular WECs is the so-called *Point Absorber*, which consists of a floater on the sea surface and hydraulic cylinders vertically installed below the floater (for more details, please refer to Li et al. [25] and Cretel et al. [8]). Mathematically speaking, this device acting from the bottom of the floating body produces a vertical force, as a control signal, to synchronize the motion of the body and of incoming waves and so maximize the energy production or generate a desired waves.

There are a number of works which are devoted to the subject of fluid-structure interaction systems. For instance, the case of the body completely immersed in the fluid is studied in Glass et al. [13], Lacave and Takahashi [21] and the corresponding control problem is considered in Roy and Takahashi [29], Glass et al. [12]. The case when the body is floating i.e. only partially immersed in the fluid, is setup studied in John [18, 19] under simplified assumptions. Recently, Lannes gave in [23] a new formulation of the governing equations and proposed a formulation of the problem as a coupling between a standard wave model (in which the surface elevation is free and the pressure is constrained) and a congested model containing an object (where the pressure is free and the surface elevation is constrained); this method can be implemented with various asymptotic models: non-viscous 1D shallow water model in Iguchi and Lannes [17], viscous 1D shallow water model in Maity et al. [27], 2D radial symmetric shallow water equations in Bocchi [5], Boussinesq equations in Bresch et al. [6] and also in Beck and Lannes [3]. We also refer to Godlewski et al. [14] where the constraint for the equations with the object is released, using a typical "low Mach" technique. For other interesting formulations and asymptotic models (depending on the shallowness parameter) for the water waves system we refer to Lannes [22, 24] and references therein. As far as we know, all the references on floating bodies mentioned above are only concerned with the object freely floating in the fluid and there are almost no work on the control issue.

**1.1. Notation.** We introduce here, constantly referring to Figure 1, some notation which is used throughout this paper. We take the coordinate system as in Figure 1, where the ordinate axis passes through the center of the floating object. The set  $\mathcal{I} := [-l, l]$ , called the *interior region* in the remaining part of this work, is the projection of the object on the bottom of the fluid domain  $\Omega(t)$ . The *exterior region* is denoted by  $\mathcal{E} := \mathcal{E}^- \cup \mathcal{E}^+$  with  $\mathcal{E}^- = (-L, -l)$  and  $\mathcal{E}^+ = (l, L')$ . With the above notation, we assume that the object does not touch one of the lateral boundaries of  $\Omega(t)$ , i.e.  $L \neq l$  and  $L' \neq l$ .

Let  $h_0$  denote the water depth when the object is at equilibrium state. In the same situation of equilibrium, let  $(0, y_{G,\text{eq}})$  denote the coordinate of the center of gravity of the object and let  $h_{\text{eq}}(x)$  denote the distance between the point of abscissa  $x$  of the bottom of the object and the bottom of the fluid domain. We assume that the bottom of the object is symmetric with respect to  $x = 0$ , which implies that  $h_{\text{eq}}(x)$  is a positive single-valued even function. We denote by  $m$  the mass of the object, by  $\rho$  the constant density of the fluid. We also denote by  $\zeta(t, x)$  the elevation

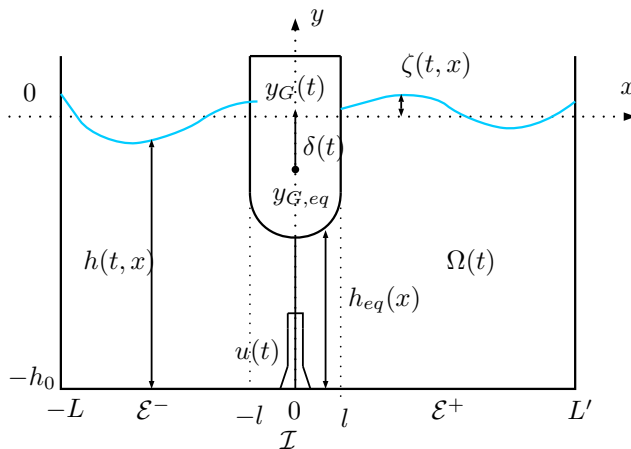


FIGURE 1. Floating body in a tank filled with water

of the water surface with respect to the rest state, by  $h(t, x) = h_0 + \zeta(t, x)$  the total height of the water column. Moreover, we introduce the horizontal discharge, denoted by  $q(t, x)$ , that is the vertical integral of the horizontal velocity of the fluid (in shallow water regime, it is  $h$  times the velocity of the fluid). We use the notation  $\underline{P}$  to represent the pressure on the water surface. When the object moves in the vertical direction, let  $(0, y_G(t))$  be the position of the center of gravity at time  $t$ , and  $\delta(t) = y_G(t) - y_{G,eq}$  be the variation of the position of the center of mass. Furthermore, the vertical control force acting on the object at time  $t$  is denoted by  $u(t)$ .

We define the jump and the average of a function  $f$  defined on  $[-l, l]$  by  $\llbracket f \rrbracket = f(l) - f(-l)$  and  $\langle f \rangle = \frac{1}{2}(f(l) + f(-l))$ , respectively. Moreover,  $f_i = f|_{\mathcal{I}}$  stands for the restriction of  $f$  to the interior domain  $\mathcal{I}$  and  $f_e = f|_{\mathcal{E}}$  denotes the restriction of  $f$  to the exterior domain  $\mathcal{E}$ .

If  $k \in \mathbb{N}$  and  $\mathcal{O} \subset \mathbb{R}$  is an open set, we use the notation  $\mathcal{H}^k(\mathcal{O})$  for the *Sobolev space* formed by the distributions  $f \in \mathcal{D}'(\mathcal{O})$  having the property that  $\partial_x^\alpha f \in L^2(\mathcal{O})$  for every integer  $\alpha \in [0, k]$ . Finally, if a function  $f$  depends on the time  $t$ , we denote by  $\dot{f}$  its derivative with respect to  $t$ . For a matrix  $M$ , we denote by  $M^\top$  the transpose of  $M$ . We use the notation  $X^\perp$  to represent the orthogonal complement of the space  $X$ . For a complex number  $\alpha \in \mathbb{C}$ , we use  $\bar{\alpha}$  to represent the complex conjugate of  $\alpha$ .

**1.2. Main results.** The departure point of our derivation of the control system describing the interaction of the floating body with the fluid is a nonlinear model introduced in Lannes [23], where the fluid fills an infinite strip in the horizontal direction. Taking the control term into account, the governing equations of the floating body system in the fluid domain  $\Omega(t)$  can be obtained from the conservation laws of the total energy and of the volume of the water. In this case, the interior surface pressure  $\underline{P}_i$  is not only determined by the fluid dynamics, but also by the external vertical force below the floater. We show that  $\underline{P}_i$  satisfies a second-order elliptic equation, and its source term and boundary term are given in terms of  $\delta$ ,  $\langle q_i \rangle$  and the exterior functions  $\zeta_e$ ,  $q_e$ . Based on the nonlinear shallow water equations

and Newton's equation, we derive the equations for  $\delta$  and  $\langle q_i \rangle$  and find that their source terms again consist of the exterior functions, respectively. In this way, the whole system is converted to an initial and boundary value problem defined only in the exterior domain  $\mathcal{E}$ . Furthermore, it can be reformulated as a first-order evolution equation with the state  $z$  as

$$z = [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \dot{\delta}]^\top.$$

For the derivation of the fully nonlinear model, please refer to Section 2. Using the notations introduced above, we linearize the nonlinear model around the equilibrium state  $[\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \dot{\delta}]^\top = [0 \quad 0 \quad 0 \quad 0 \quad 0]^\top$  and the resulting linearized fluid-body system, for every  $t \geq 0$  and  $x \in \mathcal{E}$ , reads

$$\begin{cases} \partial_t \zeta = -\partial_x q, \\ \partial_t q = -gh_0 \partial_x \zeta, \\ \frac{d}{dt} \langle q_i \rangle = -\frac{g}{2l\alpha} \llbracket \zeta \rrbracket, \\ \ddot{\delta} = -\frac{2\rho gl}{\underline{M}} \delta + \frac{2\rho gl}{\underline{M}} \langle \zeta \rangle + \frac{1}{\underline{M}} u, \end{cases} \quad (1.1)$$

with the transmission conditions

$$\langle q \rangle = \langle q_i \rangle, \quad \llbracket q \rrbracket = -2l\dot{\delta}, \quad (1.2)$$

and boundary conditions

$$q(t, -L) = 0 = q(t, L'). \quad (1.3)$$

The constants  $\underline{\alpha}$  and  $\underline{M}$  in (1.1) are defined in Section 3. Let the initial data of (1.1) be

$$z_0 = [\zeta_0 \quad q_0 \quad \langle q_i \rangle_0 \quad \delta_0 \quad \delta_1]^\top. \quad (1.4)$$

Our first result is the well-posedness of the linear system (1.1)–(1.4). For the precise definition of the notion of solution of (1.1)–(1.4) we refer to Section 3.

**Theorem 1.1.** *The linearized floating body system (1.1)–(1.4) forms a linear control system with the state space*

$$X = \left\{ [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \eta]^\top \in (L^2(\mathcal{E}))^2 \times \mathbb{C}^3 \mid \int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0 \right\}$$

and the input space  $U = \mathbb{C}$ . For  $u \in L^2_{\text{loc}}([0, \infty); U)$ , the initial data  $z_0 \in X$ , the system (1.1)–(1.4) admits a unique solution  $z \in C([0, \infty); X)$ .

Our main interest is to study the reachable space of the control system (1.1)–(1.4), when the object is put in the middle of the fluid domain in the horizontal direction i.e.  $L = L'$ . This space is formed of all the states that can be reached from equilibrium by means of  $L^2$  controls  $u$ . For every  $\tau > 0$ , the bounded linear map  $\Phi_\tau : L^2([0, \infty); U) \rightarrow X$  is called an *input-to-state map* (briefly, input map) of the system (1.1)–(1.4) with zero initial data (i.e.  $z_0 = 0$ ) defined by

$$\Phi_\tau u = z(\tau) \quad \forall u \in L^2_{\text{loc}}([0, \infty); U). \quad (1.5)$$

Notice that when  $L' = L$  and the initial state is an equilibrium one, the whole floating body-fluid system preserves its symmetry for all  $t \geq 0$ , in the sense that  $\zeta$  and  $q$  satisfy

$$\zeta(t, -x) = \zeta(t, x) \quad q(t, -x) = -q(t, x) \quad \forall x \in \mathcal{E}.$$

We define the symmetry space  $S$  as follows:

$$S = \left\{ \begin{bmatrix} \zeta & q & \langle q_i \rangle & \delta & \eta \end{bmatrix}^\top \in (L^2(\mathcal{E}))^2 \times \mathbb{C}^3 \text{ and } \begin{cases} \zeta(-x) = \zeta(x), & q(-x) = -q(x) \end{cases} \right\}.$$

To state the result, we introduce the Hilbert space  $W$ :

$$W = \left\{ \begin{bmatrix} \zeta & q & \langle q_i \rangle & \delta & \eta \end{bmatrix}^\top \in (\mathcal{H}^1(\mathcal{E}))^2 \times \mathbb{C}^3 \mid \int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0, \right. \\ \left. \begin{cases} \llbracket q \rrbracket = -2l\eta, & \langle q \rangle = \langle q_i \rangle \text{ and } q(-L) = 0 = q(L') \end{cases} \right\}.$$

**Theorem 1.2.** *Assume that the object floats in the middle of the fluid domain in the horizontal direction, i.e.  $L' = L$ . Then for every  $\tau > \frac{2(L-l)}{\sqrt{gh_0}}$ , we have*

$$(W \cap S) \subset \text{Ran } \Phi_\tau \subset (X \cap S), \quad (1.6)$$

where each inclusion is dense and with continuous embedding.

**Remark 1.3.** In the symmetric case described above, the average horizontal discharge  $\langle q_i \rangle$  and the jump of the elevation  $\llbracket \zeta_e \rrbracket$  are both zero, so that the state  $z$  and the linear control system (1.1)–(1.4) can be simplified. We see from the first inclusion in (1.6) that any symmetric state with the regularity as in  $W$  can be reached by the control system (1.1)–(1.4) from the origin. The second inclusion in (1.6) means that the system is not approximately controllable in  $X$ , but in its symmetric subspace  $X \cap S$ . More details on this symmetric case are provided in Section 4.2.

**1.3. Organization of the paper.** In Section 2, we give a detailed derivation for the full governing equations of the floating body system in shallow water, in particular, with a control term. Moreover, we reformulate the equations into a first-order evolution system only defined in the exterior domain with transmission conditions. Then we consider in Section 3 the linearized system and establish its well-posedness by analysing the spectral properties of the evolution operators involved in the control model. Section 4 is devoted to studying the reachability and stabilizability of the linear control system. In the last section, we give some comments for the situation in the general case and introduce some open problems.

## 2. SOME BACKGROUND ON NONLINEAR MODELLING OF FLOATING BODY - SHALLOW WATER INTERACTION

In this section, we derive the nonlinear governing equations describing the motion of the floating object in  $\Omega(t)$ , in the presence of a control applied from the bottom of the object. We follow the approach developed in [23, 3] with modifications to include the external force  $u(t)$  and the presence of the vertical boundaries of the water tank. Here we assume that the fluid fills the domain  $\Omega(t)$ , that it is homogeneous, incompressible, inviscid and irrotational. We also assume that we are here in a configuration where wave motion is correctly described by the nonlinear shallow water equations. We know from [23] that the *nonlinear shallow water equations with a floating structure* are given, for every  $t \geq 0$  and  $x \in \mathbb{R}$ , by

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta = -\frac{h}{\rho} \partial_x \underline{P}, \end{cases} \quad (2.1)$$

where  $g$  is the gravity acceleration. The exterior surface pressure is zero, i.e.  $\underline{P}_e = 0$ , while the interior pressure  $\underline{P}_i$  is determined by the motion of the fluid below the object and also the control signal  $u$ . We denote by  $\zeta_w(t, x)$  the parameterization of the part of the bottom of the object in contact with the fluid (the subscript "w" represents the "wetted" part of the object). Therefore, we have the water surface in the interior domain  $\mathcal{I}$  that matches the bottom of the object, i.e.

$$\zeta_w(t, x) = \zeta_i(t, x) \quad \forall x \in \mathcal{I}. \quad (2.2)$$

It is not difficult to see that we have the relation

$$\zeta_w(t, x) = \delta(t) + h_{\text{eq}}(x) - h_0 \quad \forall x \in \mathcal{I}. \quad (2.3)$$

Moreover, we obtain from (2.3) that  $\partial_t \zeta_w = \dot{\delta}$ , which is the kinematic condition on the water surface. We consider in what follows restricting the model (2.1) to the interval  $[-L, L']$ ,  $-L$  and  $L'$  being the horizontal coordinates of the water tank  $\Omega(t)$ , in particular with the control term. To this end, we observe that the following conditions need to be satisfied:

- *The conservation of the volume of the water.*

We first notice that the two vertical boundary of  $\Omega(t)$  are impermeable, which implies that

$$q(t, -L) = 0 = q(t, L').$$

Therefore, the conservation of the volume of the water implies that

$$q_i(t, \pm l) = q_e(t, \pm l). \quad (2.4)$$

- *The conservation of the total energy of the fluid-structure system.*

We denote by  $E_f$  and  $E_s$  the mechanical energy of the fluid and the mechanical energy of the solid, respectively. Because of the existence of the vertical force  $u$ , the total energy of the floating object system  $E_{\text{tot}}(t) = E_f(t) + E_s(t)$  should satisfy

$$\frac{d}{dt} E_{\text{tot}}(t) = u(t) \dot{\delta}(t). \quad (2.5)$$

Based on the conservation of the energy (2.5), we derive the boundary conditions of the surface pressure  $\underline{P}_i$  at the two contact points  $x = \pm l$ . To do this, we first note that the mechanical energy of the solid  $E_s$  is

$$E_s(t) = mg\delta(t) + \frac{1}{2}m\dot{\delta}^2(t).$$

Recalling the definition of the horizontal discharge  $q$ , the mechanical energy of the fluid  $E_f$  is

$$E_f(t) = \frac{\rho}{2} \int_{\mathcal{E} \cup \mathcal{I}} \left( g\zeta^2(t, x) + \frac{q^2}{h}(t, x) \right) dx.$$

Note that the object at equilibrium satisfies *Archimedes' principle*, we have

$$m = \rho \int_{-l}^l (h_0 - h_{\text{eq}}(x)) dx. \quad (2.6)$$

Newton's law for the motion of the object, together with (2.6), implies that

$$m\ddot{\delta}(t) + 2l\rho g\delta(t) = \int_{-l}^l (\underline{P}_i + \rho g\zeta_i) dx + u(t), \quad (2.7)$$

which means that the motion of the object is determined by its weight, the hydrodynamic force and the external force. Based on the above analysis, we give in the following proposition the boundary condition of  $\underline{P}_i$ .

**Proposition 2.1.** *Assume that the functions  $\zeta$ ,  $q$ ,  $h$ ,  $\delta$  and  $\underline{P}$  are smooth on  $\mathcal{I}$  and  $\mathcal{E}$ . Then the total energy of the floating body system  $E_{\text{tot}}$  is conserved if the interior pressure  $\underline{P}_i$  satisfies*

$$\underline{P}_i(t, \pm l) = \rho g (\zeta_e(t, \pm l) - \zeta_i(t, \pm l)) + \mathfrak{B}_e(t, \pm l) - \mathfrak{B}_i(t, \pm l), \quad (2.8)$$

where  $\mathfrak{B}$  is defined as

$$\mathfrak{B} = \frac{\rho q^2}{2h^2}. \quad (2.9)$$

*Proof.* Taking the derivative of  $E_s$  and  $E_f$  and using (2.7), it is not difficult to obtain that

$$\frac{d}{dt} E_{\text{tot}}(t) = \llbracket \mathfrak{F}_e - \mathfrak{F}_i \rrbracket + u(t) \dot{\delta}(t),$$

where the energy flux  $\mathfrak{F}$  is

$$\mathfrak{F}(\zeta, q) = q(\rho g \zeta + \underline{P} + \mathfrak{B}).$$

Therefore, we conclude that the total energy is conserved in the sense of (2.5) if  $\llbracket \mathfrak{F}_e \rrbracket = \llbracket \mathfrak{F}_i \rrbracket$ , which follows from the boundary values of the interior pressure given in (2.8).  $\square$

It is worthwhile noting that actually  $\mathfrak{B}_i(t, \pm l)$  is fully determined by  $\delta$  and  $\langle q_i \rangle$ . Indeed, we denote by  $h_w(t, x)$  the height of the water column in the interior domain  $\mathcal{I}$ . By the definition of  $h$  and (2.3) we know that

$$h_w(t, x) = h_0 + \zeta_w(t, x) = h_{\text{eq}}(x) + \delta(t) \quad \forall x \in \mathcal{I}. \quad (2.10)$$

Together with (2.2) and the kinematic condition  $\partial_t \zeta_w = \dot{\delta}$ , we obtain that the system (2.1) restricted to the interior domain, for all  $t \geq 0$  and  $x \in \mathcal{I}$ , reads

$$\begin{cases} \partial_x q_i = -\dot{\delta}, \\ \partial_t q_i + \partial_x \left( \frac{q_i^2}{h_w} \right) + g h_w \partial_x \zeta_w = -\frac{h_w}{\rho} \partial_x \underline{P}_i. \end{cases} \quad (2.11)$$

The first equation in (2.11) implies that

$$q_i(t, x) = -x \dot{\delta}(t) + \langle q_i \rangle \quad \forall x \in \mathcal{I}. \quad (2.12)$$

Recalling the definition of  $\mathfrak{B}$  in (2.9) we have

$$\mathfrak{B}_i(t, \pm l) = \frac{\rho}{2} \left( \frac{q_i(t, \pm l)}{h_w(t, \pm l)} \right)^2 = \frac{\rho}{2} \left( \frac{\mp l \dot{\delta}(t) + \langle q_i \rangle}{h_{\text{eq}}(\pm l) + \delta(t)} \right)^2.$$

Up to now, we obtain the governing equations describing the dynamics of the floating object in the bounded domain  $\Omega(t)$  with the control term  $u$ . For the sake



of convenience, we put all the equations together as follows, for all  $t \geq 0$ ,

$$\partial_t \zeta + \partial_x q = 0 \quad x \in \mathcal{I} \cup \mathcal{E}, \quad (2.13a)$$

$$\partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta = -\frac{h}{\rho} \partial_x \underline{P} \quad x \in \mathcal{I} \cup \mathcal{E}, \quad (2.13b)$$

$$\underline{P}_e(t, x) = 0 \quad x \in \mathcal{E}, \quad (2.13c)$$

$$\zeta_i(t, x) = \delta(t) + h_{\text{eq}}(x) - h_0 \quad x \in \mathcal{I}, \quad (2.13d)$$

$$\underline{P}_i(t, \pm l) = \rho g (\zeta_e(t, \pm l) - \zeta_i(t, \pm l)) + \mathfrak{B}_e(t, \pm l) - \mathfrak{B}_i(t, \pm l), \quad (2.13e)$$

$$m \ddot{\delta}(t) = \int_{-l}^l \underline{P}_i(t, x) dx - mg + u(t), \quad (2.13f)$$

$$q_e(t, -L) = 0 = q_e(t, L'), \quad q_i(t, \pm l) = q_e(t, \pm l), \quad (2.13g)$$

with the given initial data

$$\zeta(0, x) = \zeta_0(x), \quad q(0, x) = q_0(x), \quad \delta(0) = \delta_0, \quad \dot{\delta}(0) = \delta_1 \quad \forall x \in \mathcal{I} \cup \mathcal{E}.$$

**Remark 2.2.** There is another interesting formulation for the governing equations (2.13). As in [27], we can define the *Langrangian*  $\mathcal{L}$  and the *action functional*  $\mathcal{S}$  as

$$\mathcal{L}(\zeta, q, \delta) = (K_f + K_s) - (U_f + U_s),$$

$$\mathcal{S}(\zeta, q, \delta) = \int_0^\tau (\mathcal{L}(\zeta, q, \delta) + u \delta) dt \quad \forall \tau > 0,$$

where  $K_f$  and  $U_f$  are the kinetic energy and the potential energy of the fluid, respectively. Similarly,  $K_s$  and  $U_s$  denote the corresponding energies for the solid. The equations (2.13) can be alternatively obtained by using *Hamilton's principle* (see, for instance, [28]) with the equations (2.13a) and (2.13d) as constraints.

We next rephrase the governing equations (2.13) as a first-order evolution system, which will be convenient to study the control problem described in Section 3. To do this, we first show that the pressure term  $\underline{P}_i$  is actually determined by a second-order elliptic equation. Based on the formula for  $q_i$  in (2.12), we shall derive the equations for  $\delta$  and  $\langle q_i \rangle$  by using the interior equations (2.11). We explain in the following theorem that, for given initial data, the average horizontal discharge  $\langle q_i \rangle$  and the displacement  $\delta$  are totally determined by the quantities in the exterior domain  $\mathcal{E}$ .

**Theorem 2.3.** *For smooth solutions, equations (2.13) can be equivalently rewritten as the following system (involving only the exterior domain  $\mathcal{E}$ ):*

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x \left( \frac{q^2}{h} \right) + gh \partial_x \zeta = 0 \end{cases} \quad (t \geq 0, x \in \mathcal{E}), \quad (2.14)$$

with the transmission conditions

$$\langle q \rangle = \langle q_i \rangle, \quad \llbracket q \rrbracket = -2l \dot{\delta}, \quad (2.15)$$

and the boundary conditions

$$q(t, -L) = 0 = q(t, L'). \quad (2.16)$$

Moreover, the discharge  $\langle q_i \rangle$  and the displacement  $\delta$  are determined, for every  $t \geq 0$  and  $x \in \mathcal{E}$ , by

$$\begin{cases} \alpha(\delta) \frac{d}{dt} \langle q_i \rangle + \alpha'(\delta) \dot{\delta} \langle q_i \rangle = -\frac{1}{2\rho l} \llbracket \rho g \zeta + \mathfrak{B} \rrbracket, \\ M(\delta) \ddot{\delta} - 2\rho l \beta(\delta) \dot{\delta}^2 + 2\rho g l \delta - \rho l \alpha'(\delta) \langle q_i \rangle^2 = 2l \langle \rho g \zeta + \mathfrak{B} \rangle + u, \end{cases} \quad (2.17)$$

where  $\mathfrak{B}$  is introduced (2.9) and  $\alpha(\delta)$ ,  $\alpha'(\delta)$ ,  $\beta(\delta)$  and  $M(\delta)$  (with  $h_w$  in (2.10)) are

$$\alpha(\delta) = \frac{1}{2l} \int_{-l}^l \frac{1}{h_w} dx, \quad \alpha'(\delta) = -\frac{1}{2l} \int_{-l}^l \frac{1}{h_w^2} dx, \quad (2.18)$$

$$M(\delta) = m + \int_{-l}^l \frac{\rho x^2}{h_w} dx, \quad \beta(\delta) = \frac{1}{4l} \int_{-l}^l \frac{x^2}{h_w^2} dx. \quad (2.19)$$

*Proof.* According to the conservation of the volume (2.4) and the equation for  $q_i$  in (2.12), we immediately obtain the transmission condition (2.15). We introduce the hydrodynamic pressure  $\Pi_i$  defined by

$$\Pi_i := \underline{P}_i + \rho g \zeta_i.$$

Taking the derivative of the second equation in (2.11) with respect to  $x$  and using the first equation of (2.11), we derive that  $\Pi_i$  satisfies

$$\begin{cases} -\partial_x \left( \frac{h_w}{\rho} \partial_x \Pi_i \right) = -\ddot{\delta} + \partial_x^2 \left( \frac{q_i^2}{h_w} \right), \\ \Pi_i(t, \pm l) = \rho g \zeta_e(t, \pm l) + \mathfrak{B}_e(t, \pm l) - \mathfrak{B}_i(t, \pm l), \end{cases} \quad (2.20)$$

where  $t \geq 0$ ,  $x \in \mathcal{I}$  and  $\mathfrak{B}$  is defined in (2.9). According to what we state around (2.12), the source term and the boundary conditions of (2.20) are determined by  $\delta$ ,  $\langle q_i \rangle$  and the exterior functions  $\zeta_e$ ,  $q_e$ .

Next we derive the equations for  $\delta$  and  $\langle q_i \rangle$ . Recalling that the function  $h_{eq}$  is assumed to be even, we integrate the second equation of (2.11) with respect to  $x$ , which gives

$$2l \alpha \frac{d}{dt} \langle q_i \rangle + \left[ \frac{q_i^2}{h_w^2} \right] + \int_{-l}^l \frac{q_i^2}{h_w^3} \partial_x h_w dx = -\frac{1}{\rho} \llbracket \Pi_i \rrbracket. \quad (2.21)$$

We further derive from (2.21) that

$$\alpha(\delta) \frac{d}{dt} \langle q_i \rangle + \alpha'(\delta) \dot{\delta} \langle q_i \rangle = -\frac{1}{2\rho l} \llbracket \rho g \zeta_e + \mathfrak{B}_e \rrbracket.$$

In the above calculation, we used the formula for  $q_i$  in (2.12) and the integration by parts. To derive the equation for  $\delta$ , we first obtain from Newton's laws presented in (2.7), by doing an integration by parts, that

$$m \ddot{\delta} + 2l \rho g \delta = 2l \langle \Pi_i \rangle - \int_{-l}^l x \partial_x \Pi_i(t, x) dx + u. \quad (2.22)$$

Taking twice integration of the first equation of (2.20) with respect to  $x$  and using the integration by parts, we obtain the expression for the second term on the right side of (2.22). Finally, after doing some trivial derivation we obtain from (2.22) that

$$M(\delta) \ddot{\delta} - 2\rho l \beta(\delta) \dot{\delta}^2 + 2\rho g l \delta = 2l \langle \rho g \zeta_e + \mathfrak{B}_e \rangle + \rho l \alpha'(\delta) \langle q_i \rangle^2 + u. \quad (2.23)$$

In the above calculation, we used a similar technique as in [3], so that we omit the details here. For given initial data of  $\zeta$ ,  $q$ ,  $\langle q_i \rangle$ ,  $\delta$  and  $\dot{\delta}$ , the coupled system (2.14)–(2.17) form a closed initial boundary value problem.  $\square$

It is worthwhile noting that there is an external force  $u$  on the right side of (2.23), which does not appear in the equations obtained in [23, 3]. According to Theorem 2.3, we can further rewrite the system (2.14)–(2.17) into a first-order evolution system in terms of  $\zeta$ ,  $q$ ,  $\langle q_i \rangle$ ,  $\delta$  and  $\dot{\delta}$ . This is straightforward and we omit the details here.

**Remark 2.4.** The well-posedness theory for (2.14)–(2.17) is a delicate question, due to the nonlinear couplings: the boundary conditions (2.15) of the hyperbolic problem (2.14)–(2.16) require the knowledge of  $\langle q_i \rangle$  and  $\delta$ . Conversely, the equations (2.17) require the knowledge of the trace of  $\zeta$  and  $q$  at the contact points  $x = \pm l$ . An interesting question, which lies outside the scope of the present work, is to adapt to our case the local existence theory developed in [17], which tackles the case of an unbounded fluid domain and without control.

### 3. WELL-POSEDNESS AND SPECTRAL ANALYSIS OF THE LINEARIZED MODEL

In this section, we shall work on the linearized version of the first-order evolution system associated with (2.14)–(2.17). Before studying the control problem in Section 4, we first present the linearized model and establish its well-posedness. In the second part of this section, we focus on the spectral analysis of the semigroup generator associated to this linearized equation.

Linearizing the system (2.14)–(2.17) in Theorem 2.3 around the equilibrium state  $[\zeta \ q \ \langle q_i \rangle \ \delta \ \dot{\delta}]^\top = [0 \ 0 \ 0 \ 0 \ 0]^\top$ , we obtain, for all  $t \geq 0$  and  $x \in \mathcal{E}$ ,

$$\begin{cases} \partial_t \zeta = -\partial_x q, \\ \partial_t q = -gh_0 \partial_x \zeta, \\ \frac{d}{dt} \langle q_i \rangle = -\frac{g}{2l\alpha} \llbracket \zeta \rrbracket, \\ \ddot{\delta} = -\frac{2\rho gl}{M} \delta + \frac{2\rho gl}{M} \langle \zeta \rangle + \frac{1}{M} u, \end{cases} \quad (3.1)$$

with transmission conditions

$$\langle q \rangle = \langle q_i \rangle, \quad \llbracket q \rrbracket = -2l\dot{\delta},$$

and boundary conditions

$$q(t, -L) = 0 = q(t, L),$$

and the given initial data

$$\zeta(0, x) = \zeta_0(x), \quad q(0, x) = q_0(x), \quad \langle q_i \rangle(0) = \langle q_i \rangle_0, \quad \delta(0) = \delta_0, \quad \dot{\delta}(0) = \delta_1.$$

The constants  $\underline{\alpha}$  and  $\underline{M}$  in (3.1) are

$$\underline{\alpha} = \alpha(0) \quad \underline{M} = M(0), \quad (3.2)$$

where  $\alpha(\delta)$  and  $M(\delta)$  have been defined in (2.18) and (2.19), respectively.

**3.1. Well-posedness of the linearized system.** Observe that our system has been recast in the exterior domain  $\mathcal{E}$ , so we need to rewrite the energy of the whole system in terms of the exterior functions. Recalling that the mechanical energy for the fluid and for the object are presented in Section 2, we decompose the total energy of the linearized system (3.1) into the interior part  $\underline{E}_{\text{int}}$  and the exterior part  $\underline{E}_{\text{ext}}$  as follows:

$$\begin{aligned}\underline{E}_{\text{ext}}(t) &= \frac{\rho}{2} \int_{\mathcal{E}} \left( \frac{q^2}{h_0}(t, x) + g\zeta^2(t, x) \right) dx, \\ \underline{E}_{\text{int}}(t) &= \frac{\rho}{2} \int_{\mathcal{I}} \left( \frac{q^2}{h_{\text{eq}}}(t, x) + g\zeta^2(t, x) \right) dx + \frac{1}{2}m\dot{\delta}^2 + mg\delta.\end{aligned}$$

The underline in the notation  $E$  represents the corresponding energy for linear system. Using the relation (2.3), (2.10) and (2.12), together with Archimedes' principle (2.6), we obtain that

$$\underline{E}_{\text{int}}(t) = \frac{1}{2}\dot{\delta}^2 \left( m + \int_{\mathcal{I}} \frac{\rho x^2}{h_{\text{eq}}} dx \right) + \langle q_i \rangle^2 \int_{\mathcal{I}} \frac{\rho}{2h_{\text{eq}}} dx + \frac{\rho g}{2} \int_{\mathcal{I}} (h_{\text{eq}}(x) - h_0)^2 dx + \rho g l \delta^2.$$

Therefore, we conclude that the total energy for (3.1), denoted by  $\underline{E}_{\text{tot}}$ , is

$$\begin{aligned}\underline{E}_{\text{tot}}(t) &= \frac{\rho}{2} \int_{\mathcal{E}} \left( \frac{q^2}{h_0}(t, x) + g\zeta^2(t, x) \right) dx + \frac{1}{2}M\dot{\delta}^2 + \langle q_i \rangle^2 \rho l \alpha \\ &\quad + \rho g l \delta^2 + \frac{\rho g}{2} \int_{\mathcal{I}} (h_{\text{eq}}(x) - h_0)^2 dx, \quad (3.3)\end{aligned}$$

where  $\underline{\alpha}$  and  $\underline{M}$  are introduced in (3.2).

Based on the formula of the total energy  $\underline{E}_{\text{tot}}$  in (3.3), we introduce the Hilbert space  $X$  defined by

$$X = \left\{ \left[ \zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \eta \right]^{\top} \in (L^2(\mathcal{E}))^2 \times \mathbb{C}^3 \mid \int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0 \right\}, \quad (3.4)$$

endowed with the inner product

$$\left\langle \begin{bmatrix} \zeta \\ q \\ \langle q_i \rangle \\ \delta \\ \eta \end{bmatrix}, \begin{bmatrix} \tilde{\zeta} \\ \tilde{q} \\ \langle \tilde{q}_i \rangle \\ \tilde{\delta} \\ \tilde{\eta} \end{bmatrix} \right\rangle_X = \frac{\rho g}{2} \langle \zeta, \tilde{\zeta} \rangle_{L^2(\mathcal{E})} + \frac{\rho}{2h_0} \langle q, \tilde{q} \rangle_{L^2(\mathcal{E})} + \rho l \underline{\alpha} \langle q_i \rangle \overline{\langle \tilde{q}_i \rangle} + \rho g l \delta \bar{\tilde{\delta}} + \frac{M}{2} \eta \bar{\tilde{\eta}}. \quad (3.5)$$

**Remark 3.1.** We can see from (3.3) that the total energy only depends on the functions  $\delta$ ,  $\langle q_i \rangle$ ,  $\zeta$  and  $q$  with the space variable  $x \in \mathcal{E}$ . The condition

$$\int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0$$

in the definition of the space  $X$  is motivated by the conservation of the volume.

Equations (3.1) determine a well-posed linear control system (also called *abstract linear control system* in Weiss [36] or Tucsnak and Weiss [35]), with state space  $X$

defined in (3.4) and control space  $U = \mathbb{C}$ , by choosing the appropriate spaces and operators. More precisely, let  $A : \mathcal{D}(A) \rightarrow X$  and  $B \in \mathcal{L}(U, X)$  be defined by

$$A = \begin{bmatrix} 0 & -\frac{d}{dx} & 0 & 0 & 0 \\ -gh_0 \frac{d}{dx} & 0 & 0 & 0 & 0 \\ -\frac{g}{2l\alpha} \llbracket \cdot \rrbracket & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{2}{M} \rho g l \langle \cdot \rangle & 0 & 0 & -\frac{2}{M} \rho g l & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{M} \end{bmatrix}, \quad (3.6)$$

with

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} \zeta & q & \langle q_i \rangle & \delta & \eta \end{bmatrix}^\top \in (\mathcal{H}^1(\mathcal{E}))^2 \times \mathbb{C}^3 \mid \int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0, \right. \\ \left. \llbracket q \rrbracket = -2l\eta, \quad \langle q \rangle = \langle q_i \rangle \quad \text{and} \quad q(-L) = 0 = q(L') \right\}. \quad (3.7)$$

In other words, with the above choice of spaces and operators, the initial boundary value problem of the system (3.1) can be rewritten as

$$\begin{cases} \dot{z} = Az + Bu, \\ z(0) = z_0, \end{cases} \quad (3.8)$$

where  $z$  and  $z_0$  are

$$z = [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \delta]^\top, \quad z_0 = [\zeta_0 \quad q_0 \quad \langle q_i \rangle_0 \quad \delta_0 \quad \delta_1]^\top.$$

The well-posedness of the linearized fluid-structure system (3.8) is a direct consequence of the fact that  $B \in \mathcal{L}(U, X)$  and of following result:

**Proposition 3.2.** *The operator  $A : \mathcal{D}(A) \rightarrow X$  defined in (3.6)–(3.7) is skew-adjoint. Therefore, it generates a group of unitary operators on the Hilbert space  $X$ . Moreover,  $A$  has compact resolvents.*

*Proof.* We first show that  $A$  is skew-symmetric. For the sake of simplicity the computations leading to the property are performed looking to  $X$  as a Hilbert space over  $\mathbb{R}$ . For every  $z = [\zeta \quad q \quad \langle q_i \rangle \quad \delta \quad \eta]^\top \in \mathcal{D}(A)$ , using the inner product defined in (3.5) we have

$$\langle Az, z \rangle_X = -\frac{\rho g}{2} \left( \left\langle \frac{dq}{dx}, \zeta \right\rangle_{L^2(\mathcal{E})} + \left\langle \frac{d\zeta}{dx}, q \right\rangle_{L^2(\mathcal{E})} + \llbracket \zeta_e \rrbracket \langle q_i \rangle - 2l \langle \zeta_e \rangle \eta \right).$$

By using an integration by parts, we get

$$-\left\langle \frac{dq}{dx}, \zeta \right\rangle_{L^2(\mathcal{E})} = \llbracket (\zeta q)_e \rrbracket + \left\langle q, \frac{d\zeta}{dx} \right\rangle_{L^2(\mathcal{E})}.$$

Note that the boundary conditions in  $\mathcal{D}(A)$  implies that

$$q(l) = \langle q_i \rangle - l\eta, \quad q(-l) = \langle q_i \rangle + l\eta,$$

which, by a simple calculation, gives that

$$\langle Az, z \rangle_X = 0.$$

According to [35, Section 3.7], we thus obtain that the operator  $A$  is skew-symmetric.

Secondly, we prove that  $A$  is onto. For every  $f = [f_1 \ f_2 \ f_3 \ f_4 \ f_5]^\top \in X$ , let us solve the equation

$$A \begin{bmatrix} \zeta \\ q \\ \langle q_i \rangle \\ \delta \\ \eta \end{bmatrix} = \begin{bmatrix} -\frac{dq}{dx} \\ -gh_0 \frac{d\zeta}{dx} \\ -\frac{g}{2l\alpha} \llbracket \zeta_e \rrbracket \\ \eta \\ \frac{2}{M} \rho g l \langle \zeta_e \rangle - \frac{2}{M} \rho g l \delta \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} \zeta \\ q \\ \langle q_i \rangle \\ \delta \\ \eta \end{bmatrix} \in \mathcal{D}(A), \quad (3.9)$$

which immediately implies that  $\eta = f_4$ . Solving the equation from the first component of (3.9), i.e.  $-\frac{dq}{dx} = f_1$  with the boundary conditions  $q(-L) = 0$  and  $q(L') = 0$ , we obtain

$$q(x) = \begin{cases} -\int_{-L}^x f_1(\xi) d\xi & \forall x \in (-L, -l), \\ \int_x^{L'} f_1(\xi) d\xi & \forall x \in (l, L'). \end{cases} \quad (3.10)$$

Similarly, from the second equation we get

$$\zeta(x) = \begin{cases} -\frac{1}{gh_0} \int_{-L}^x f_2(\xi) d\xi + c_1 := F(x) + c_1 & \forall x \in (-L, -l), \\ \frac{1}{gh_0} \int_x^{L'} f_2(\xi) d\xi + c_2 := G(x) + c_2 & \forall x \in (l, L'), \end{cases} \quad (3.11)$$

where the constants  $c_1$  and  $c_2$  are to be determined. The above formula, together with the last component of (3.9), gives the expression for  $\delta$ :

$$\delta = \frac{1}{2} (F(-l) + G(l) + c_1 + c_2) - \frac{M}{2\rho g l} f_5.$$

Moreover, we derive from the third equation of (3.9) that

$$-\frac{g}{2l\alpha} (G(l) - F(-l) + c_2 - c_1) = f_3. \quad (3.12)$$

Note that the functions  $\zeta$  and  $\delta$  must satisfy the condition for the conservation of the volume

$$\int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0,$$

which implies that

$$Lc_1 + L'c_2 = \frac{M}{\rho g} f_5 - \int_{-L}^{-l} F(x) dx - \int_l^{L'} G(x) dx - G(l)l - F(-l)l. \quad (3.13)$$

Combining (3.12) and (3.13), we can determine the constants  $c_1$  and  $c_2$  in (3.11). According to the continuity of the discharge (2.4) and (3.10), we have  $\langle q_i \rangle = \langle q \rangle = \frac{1}{2}(q(l) + q(-l))$ . Finally, we still need to verify that  $\llbracket q \rrbracket = -2l\eta$ . Since  $f \in X$ , we have  $\int_{\mathcal{E}} f_1(x) dx + 2lf_4 = 0$ , which, together with (3.10), implies that  $\llbracket q \rrbracket = -2lf_4 = -2l\eta$ . Thus we have found  $z = [\zeta \ q \ \langle q_i \rangle \ \delta \ \eta]^\top \in \mathcal{D}(A)$ , so that (3.9) holds.

According to a classical result [35, Proposition 3.7.2], we conclude that  $A$  is skew-adjoint and  $0 \in \rho(A)$ . By Stone's theorem (see, for instance, [35, Theorem 3.8.6]),  $A$  generates a unitary group on  $X$ . Moreover, it is not difficult to see that

$\mathcal{D}(A)$  is compactly embedded in the state space  $X$ , which implies that the operator  $A$  has compact resolvents.  $\square$

Based on Proposition 3.2, we denote by  $\mathbb{T} = (\mathbb{T}_t)_{t \in \mathbb{R}}$  the *strongly continuous group* (also called  $C_0$ -group) generated by the operator  $A$ . Note that  $B \in \mathcal{L}(\mathbb{C}, X)$ , which is of course an admissible control operator (for this concept, see, for instance, [35, Chapter 4]). Therefore,  $(A, B)$  forms a well-posed linear control system. According to the classical semigroup theory, we have the following conclusion.

**Theorem 3.3.** *For  $u \in L^2_{\text{loc}}[0, \infty)$ , the initial data  $z_0 = [\zeta_0 \quad q_0 \quad \langle q_i \rangle_0 \quad \delta_0 \quad \delta_1]^\top \in X$ , the linear system (3.8) admits a unique solution  $z$ . This solution is given by*

$$z(t) = \mathbb{T}_t z_0 + \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma,$$

and it satisfies

$$z \in C([0, \infty); X).$$

**3.2. Spectral analysis.** In this part, we focus on the study of the spectral structure of the operator  $A$  introduced in (3.6)–(3.7). Note that the operator  $A$  is skew-adjoint, the eigenvalues of  $A$  are purely imaginary, i.e.  $\sigma(A) \subset i\mathbb{R}$ . We give in the following proposition the characteristic equation for the eigenvalues and the formula for the corresponding eigenvectors.

**Proposition 3.4.** *For the operator  $A$  introduced in (3.6)–(3.7),  $i\omega$  with  $\omega \in \mathbb{R}$  is the eigenvalues of  $A$  if and only if  $\omega$  satisfies*

$$\begin{aligned} & -\sqrt{\frac{g}{h_0}} \left[ 2\rho l^2 \omega + (\underline{M}\omega^2 - 2\rho g l) \frac{1}{l\underline{\alpha}\omega} \right] (f_\omega(L)g_\omega(L') + f_\omega(L')g_\omega(L)) \\ & + 2(\underline{M}\omega^2 - 2\rho g l) f_\omega(L)f_\omega(L') + \frac{4\rho g l}{h_0 \underline{\alpha}} g_\omega(L)g_\omega(L') = 0, \end{aligned} \quad (3.14)$$

where  $\underline{\alpha}$  and  $\underline{M}$  are given in (3.2),  $f_\omega$  and  $g_\omega$  are defined as

$$f_\omega(x) = \sin\left(\frac{\omega}{\sqrt{gh_0}}(x-l)\right), \quad g_\omega(x) = \cos\left(\frac{\omega}{\sqrt{gh_0}}(x-l)\right). \quad (3.15)$$

Moreover,  $\phi = [\varphi \quad \psi \quad c \quad a \quad b]^\top$  is an eigenvector corresponding to the eigenvalue  $i\omega$  if and only if

$$\varphi(x) = \begin{cases} \frac{iK_1}{\sqrt{gh_0}} \cos\left(\frac{\omega}{\sqrt{gh_0}}(L+x)\right) & \forall x \in (-L, -l), \\ -\frac{iK_2}{\sqrt{gh_0}} \cos\left(\frac{\omega}{\sqrt{gh_0}}(L'-x)\right) & \forall x \in (l, L'), \end{cases} \quad (3.16)$$

$$\psi(x) = \begin{cases} K_1 \sin\left(\frac{\omega}{\sqrt{gh_0}}(L+x)\right) & \forall x \in (-L, -l), \\ K_2 \sin\left(\frac{\omega}{\sqrt{gh_0}}(L'-x)\right) & \forall x \in (l, L'), \end{cases} \quad (3.17)$$

and

$$\begin{aligned} c &= \frac{1}{2}(\psi(l) + \psi(-l)), \\ a &= \frac{i}{2\omega l}(\psi(l) - \psi(-l)), \quad b = -\frac{1}{2l}(\psi(l) - \psi(-l)), \end{aligned} \quad (3.18)$$

where  $K_1, K_2$  are not simultaneously vanishing real numbers (not necessarily independent).

*Proof.* Let  $\phi = [\varphi \ \psi \ c \ a \ b]^\top \in \mathcal{D}(A)$  be the eigenvector of the operator  $A$  corresponding to the eigenvalue  $i\omega$  with  $\omega \in \mathbb{R}$ . To obtain the formula of  $\phi$ , we solve the equation

$$A \begin{bmatrix} \varphi \\ \psi \\ c \\ a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{d\psi}{dx} \\ -gh_0 \frac{d\varphi}{dx} \\ -\frac{g}{2l\alpha} \llbracket \varphi \rrbracket \\ b \\ \frac{2}{M} \rho gl \langle \varphi \rangle - \frac{2}{M} \rho gla \end{bmatrix} = i\omega \begin{bmatrix} \varphi \\ \psi \\ c \\ a \\ b \end{bmatrix}, \quad (3.19)$$

where  $\underline{\alpha}$  and  $\underline{M}$  are introduced in (3.2). Recalling the definition of  $\mathcal{D}(A)$  in (3.7), we have

$$\psi(-L) = 0 = \psi(L'), \quad \int_{\mathcal{E}} \varphi(x) dx + 2la = 0, \quad (3.20)$$

and

$$\llbracket \psi \rrbracket = -2lb, \quad \langle \psi \rangle = c. \quad (3.21)$$

Combining the first two equations in (3.19), we obtain a second-order differential equation for  $\psi$

$$\frac{d^2\psi}{dx^2} = -\frac{\omega^2}{gh_0} \psi \quad \forall x \in \mathcal{E},$$

which, together with the boundary condition in (3.20), implies that  $\psi$  takes the form (3.17). In (3.17),  $K_1$  and  $K_2$  are not simultaneously zero. Notice that  $-\frac{d\psi}{dx} = i\omega\varphi$ , we further obtain (3.16). Using the relation between  $\varphi$  and  $a$  in (3.20), we derive that

$$a = \frac{i}{2\omega l} (\psi(l) - \psi(-l)),$$

which further, by using the fourth equation of (3.19), implies that

$$b = -\frac{1}{2l} (\psi(l) - \psi(-l)).$$

Taking the conditions (3.21) into account, we have

$$c = \frac{1}{2} (\psi(l) + \psi(-l)).$$

This, together with the third and the last components of (3.19), imply that the imaginary part of the eigenvalue  $\omega$  satisfies

$$\begin{cases} \frac{ig}{l\underline{\alpha}\omega} (\varphi(l) - \varphi(-l)) = \psi(l) + \psi(-l), \\ \frac{\rho gl}{\underline{M}} (\varphi(l) + \varphi(-l)) = \left( \frac{i\rho g}{\underline{M}\omega} - \frac{i\omega}{2l} \right) (\psi(l) - \psi(-l)), \end{cases} \quad (3.22)$$

where  $\underline{\alpha}$  and  $\underline{M}$  are given in (3.2). Using the formula (3.16) and (3.17), the system (3.22) yields that

$$\left[ \sqrt{\frac{g}{h_0}} \frac{1}{l\underline{\alpha}\omega} g_\omega(L) - f_\omega(L) \right] K_1 + \left[ \sqrt{\frac{g}{h_0}} \frac{1}{l\underline{\alpha}\omega} g_\omega(L') - f_\omega(L') \right] K_2 = 0, \quad (3.23)$$



$$\begin{aligned} & \left[ \sqrt{\frac{g}{h_0}} 2\rho l^2 \omega g_\omega(L) - (\underline{M}\omega^2 - 2\rho gl) f_\omega(L) \right] K_1 \\ & + \left[ (\underline{M}\omega^2 - 2\rho gl) f_\omega(L') - \sqrt{\frac{g}{h_0}} 2\rho l^2 \omega g_\omega(L') \right] K_2 = 0, \end{aligned} \quad (3.24)$$

where  $f_\omega$  and  $g_\omega$  are introduced in (3.15). According to the knowledge of linear algebra, the equations (3.23) and (3.24) admit non-trivial solutions  $[K_1 \ K_2]^\top$ , if the determinant of their coefficient matrix is zero. Therefore, we obtain the characteristic equation (3.14).  $\square$

Since  $A$  is skew-adjoint with compact resolvents (see Proposition 3.2), according to a classical result (see, for instance, [35, Chapter 3]), we know that  $A$  is diagonalizable, also called Riesz-spectral operator, for instance, in [9]. We denote by  $(\phi_k)_{k \in \mathbb{Z}^*}$  an orthonormal basis in  $X$  consisting of eigenvectors of  $A$  and by  $(i\omega_k)_{k \in \mathbb{Z}^*}$  the corresponding purely imaginary eigenvalues. Instead of seeing the characteristic equation (3.14), we observe that the coefficient matrix of the system (3.23)–(3.24) can be zero, which implies that the roots of (3.14), i.e. the eigenvalues  $(i\omega_k)_{k \in \mathbb{Z}^*}$ , are not necessarily simple. We specify this situation in what follows.

**Remark 3.5.** Assume that  $\kappa := \underline{M} - 2\rho l^3 \underline{\alpha} > 0$  and that the parameters  $L, L', l$  and  $h_0$  satisfy

$$\sqrt{\frac{2\rho l}{\kappa h_0}} \frac{L' - L}{\pi} \in \mathbb{Z}, \quad (3.25)$$

and

$$\tan \left( \sqrt{\frac{2\rho l}{\kappa h_0}} (L - l) \right) = \frac{1}{l \underline{\alpha}} \sqrt{\frac{\kappa}{2\rho l h_0}}. \quad (3.26)$$

(Recall that the constants  $\underline{M}$  and  $\underline{\alpha}$  have been introduced in (3.2)). Then there exist two double eigenvalues of  $A$ , denoted by  $i\omega^+$  and  $i\omega^-$ , with

$$\omega^\pm = \pm \sqrt{\frac{2\rho gl}{\kappa}}.$$

We are not able to confirm or to inform the existence of  $L, L' > 0, l < \min\{L, L'\}$  and of a function  $h_{\text{eq}}$  to simultaneously satisfying the assumptions at the beginning of this remark. However, it is clear that these conditions are, generically with respect to the parameters listed above, not satisfied, so that the eigenvalues are generically simple. Note that if there is at least one double eigenvalue then the system cannot be controlled (even approximately) by a scalar input. The result below provides a sufficient condition in a special case ensuring that all the eigenvalues of  $A$  are simple.

**Proposition 3.6.** *Assume that the bottom of the floating object is flat. Let  $h_0 > 2\sqrt{\frac{2}{3}}l$  and the function  $h_{\text{eq}}$  satisfies*

$$h_0 > h_{\text{eq}} \geq \frac{1}{2} \left( h_0 + \sqrt{h_0^2 - \frac{8}{3}l^2} \right) \quad \text{or} \quad 0 < h_{\text{eq}} \leq \frac{1}{2} \left( h_0 - \sqrt{h_0^2 - \frac{8}{3}l^2} \right). \quad (3.27)$$

*The all the eigenvalues of  $A$  are simple.*

*Proof.* Recalling the definition of  $h_{\text{eq}}$ , the flat bottom of the object implies that  $h_{\text{eq}}$  is a positive constant function. Using (3.2), (2.18) and (2.19), it is not difficult to

see that if  $h_{\text{eq}}$  satisfies the condition (3.27) then  $\underline{M} - 2\rho l^3 \underline{\alpha} \leq 0$ . This excludes the situation of the double eigenvalues discussed in Remark 3.5.  $\square$

In order to study the reachability and stabilizability properties of the linearized floating-body system in Section 4, it is necessary to make the inner structure of the eigenvalues clear for the explicit decay rate of the solution of the control system (3.8).

**Proposition 3.7.** *Assume that the eigenvalues  $(i\omega_k)_{k \in \mathbb{Z}^*}$  of the operator  $A$  are simple. Then  $(\omega_k)_{k \in \mathbb{Z}^*}$  form a strictly increasing sequence, i.e.  $\lim_{|k| \rightarrow \infty} |\omega_k| = \infty$ .*

*Moreover, we assume that  $\frac{L'-l}{L-l}$  is a real algebraic number of degree  $n$  with  $n \in \mathbb{N}$  (i.e. it is a root of a non-zero polynomial of degree  $n$  in one variable with rational coefficients), then there exists  $C_0 > 0$  such that*

$$\inf_{k \in \mathbb{Z}^*} |\omega_{k+1} - \omega_k| \geq C_0 \quad \text{if } \frac{L'-l}{L-l} \in \mathbb{Q} \quad \text{and} \quad \frac{L'-l}{L-l} \neq \frac{r+1}{r} \quad \forall r \in \mathbb{Z}^*, \quad (3.28)$$

$$\inf_{k \in \mathbb{Z}^*} |k(\omega_{k+1} - \omega_k)| \geq C_0 \quad \text{otherwise.} \quad (3.29)$$

*Proof.* Since  $A$  is skew-adjoint with compact resolvents, according to [35, Proposition 3.2.12], the imaginary part of the eigenvalues  $(\omega_k)_{k \in \mathbb{Z}^*}$  can be ordered to form a strictly increasing sequence such that  $\lim_{|k| \rightarrow \infty} |\omega_k| = \infty$ . Therefore, it suffices to show that (3.28) and (3.29) holds for  $|k|$  large enough. Noting that the functions  $f_{\omega_k}$  and  $g_{\omega_k}$  defined in (3.15) are bounded for large values of  $|k|$ , we rewrite the equation (3.14) as

$$\begin{aligned} \sqrt{\frac{g}{h_0}} \left( \frac{M}{l\underline{\alpha}} + 2\rho l^2 \right) (f_{\omega_k}(L)g_{\omega_k}(L') + f_{\omega_k}(L')g_{\omega_k}(L))\omega_k + r\omega_k \\ = 2\underline{M}f_{\omega_k}(L)f_{\omega_k}(L')\omega_k^2, \end{aligned} \quad (3.30)$$

where  $r\omega_k$  represents the remaining bounded terms. As  $|k|$  approaches to infinity, we observe that the right hand side of (3.30) grows faster than the left side, thus we must have

$$\lim_{|k| \rightarrow \infty} f_{\omega_k}(L)f_{\omega_k}(L') = 0.$$

Based on this observation, the eigenvalues of  $A$  can be split into two subsequences  $(i\omega_{m_k})_{k \in \mathbb{Z}^*}$  and  $(i\omega_{n_k})_{k \in \mathbb{Z}^*}$ , which is induced by  $f_{\omega_k}(L) \rightarrow 0$  and  $f_{\omega_k}(L') \rightarrow 0$  as  $|k| \rightarrow \infty$ , respectively. Therefore, there are two subsequences of  $\mathbb{Z}^*$ :  $(m_k)_{k \in \mathbb{Z}^*}$  and  $(n_k)_{k \in \mathbb{Z}^*}$  such that, for  $|k|$  large enough, we have

$$\begin{aligned} \omega_{m_k} &= \mu m_k \pi + O(\varepsilon_{m_k}) & \text{with } \lim_{|k| \rightarrow \infty} \varepsilon_{m_k} &= 0, \\ \omega_{n_k} &= \nu n_k \pi + O(\tilde{\varepsilon}_{n_k}) & \text{with } \lim_{|k| \rightarrow \infty} \tilde{\varepsilon}_{n_k} &= 0, \end{aligned}$$

where  $\mu = \frac{\sqrt{gh_0}}{L-l}$  and  $\nu = \frac{\sqrt{gh_0}}{L'-l}$ . For large  $|k|$ , substituting the first subsequence  $(\omega_{m_k})_{k \in \mathbb{Z}^*}$  into the equation (3.30), we have

$$\begin{aligned} -\sqrt{\frac{g}{h_0}} \left( \frac{M}{l\underline{\alpha}} + 2\rho l^2 \right) \left[ \varepsilon_{m_k} \cos \left( \frac{L'-l}{L-l} m_k \pi \right) + \sin \left( \frac{L'-l}{L-l} m_k \pi \right) + O(\varepsilon_{m_k}^2) \right] \omega_{m_k} \\ + 2\underline{M} \left[ \varepsilon_{m_k} \sin \left( \frac{L'-l}{L-l} m_k \pi \right) + O(\varepsilon_{m_k}^2) \right] \omega_{m_k}^2 + \text{lower order terms} = 0, \end{aligned}$$

which implies that  $\omega_{m_k} = O(\varepsilon_{m_k}^{-1})$  and thus we derive that  $\varepsilon_{m_k} = O(m_k^{-1})$  for large  $|k|$ . Similarly, we also obtain that  $\tilde{\varepsilon}_{n_k} = O(n_k^{-1})$ . Notice that there is a gap between every two elements both from the sequence  $(\omega_{m_k})_{k \in \mathbb{Z}^*}$  or  $(\omega_{n_k})_{k \in \mathbb{Z}^*}$ . Now we consider the distance between  $(\omega_{m_k})_{k \in \mathbb{Z}^*}$  and  $(\omega_{n_k})_{k \in \mathbb{Z}^*}$ . Since the eigenvalues are strictly increasing, we estimate the difference

$$|\omega_{p+1} - \omega_p| = \left| p\nu\pi \left( \frac{\mu}{\nu} - \frac{p+1}{p} \right) + O\left(\frac{1}{p}\right) \right|, \quad (3.31)$$

where  $\omega_p \in (\omega_{m_k})_{k \in \mathbb{Z}^*}$  and  $\omega_{p+1} \in (\omega_{n_k})_{k \in \mathbb{Z}^*}$  correspond to different type of the eigenvalues. If  $\frac{\mu}{\nu} = \frac{L'-l}{L-l}$  is a rational number but different with  $\frac{k+1}{k}$  for any  $k \in \mathbb{Z}^*$ , we see that there is a uniform gap between the eigenvalues of  $A$ . If  $\frac{\mu}{\nu} = \frac{k_0+1}{k_0}$  for some  $k_0 \in \mathbb{Z}^*$ , we obtain from (3.31) that the distance between the eigenvalues is of order  $\frac{1}{k}$ . If  $\frac{\mu}{\nu}$  is not a rational number, then it is an irrational algebraic number of degree  $n \geq 2$ . According to Liouville's approximation theorem (see, for instance, Stolarsky's book [30, Chapter 3]), there exists a constant  $C > 0$  such that

$$\left| \frac{\mu}{\nu} - \frac{q}{p} \right| \geq \frac{C}{p^n},$$

for all rational numbers  $\frac{q}{p}$ . Hence, we derive from (3.31) that  $|\omega_{p+1} - \omega_p| \geq \frac{c}{p}$ . Putting all the cases together, we finish the proof.  $\square$

**Remark 3.8.** We remark that the set of real algebraic numbers of degree  $n$  with  $n \in \mathbb{N}$  contains all rational numbers and some irrational numbers. All rational numbers form the real algebraic numbers of degree 1, and the other part of the real algebraic numbers are irrational algebraic numbers with  $n \geq 2$ . In particular, the irrational algebraic numbers of degree 2 are called *quadratic irrational numbers*.

**Remark 3.9.** In the proof of Proposition 3.4, we have obtained the specific expression for the eigenvectors  $\phi_k = [\varphi_k \ \psi_k \ c_k \ a_k \ b_k]^\top$ , which is, for every  $k \in \mathbb{Z}^*$ , given by (3.16)–(3.18). Now we normalize  $\phi_k$  in the Hilbert space  $X$  introduced in (3.4). By using (3.16)–(3.18) and after elementary but tedious calculations, we check that for every  $k \in \mathbb{Z}^*$  we have

$$\begin{aligned} \|\phi_k\|_X^2 &= \left( \frac{\rho l \alpha}{4} + \frac{M}{8l^2} + \frac{\rho g}{4\omega_k^2 l} \right) (K_2^2 f_{\omega_k}(L')^2 + K_1^2 f_{\omega_k}(L)^2) \\ &+ \left( \frac{\rho l \alpha}{2} - \frac{M}{4l^2} - \frac{\rho g}{2\omega_k^2 l} \right) K_1 K_2 f_{\omega_k}(L) f_{\omega_k}(L') + \frac{\rho}{2h_0} (K_1^2(L-l) + K_2^2(L'-l)), \end{aligned}$$

where  $\alpha$ ,  $M$  and  $f_{\omega_k}$  are defined in (3.2) and (3.15), respectively. Therefore, we obtain the normalized engenvectors  $\hat{\phi}_k := (\gamma_k \phi_k)_{k \in \mathbb{Z}^*}$  with  $\|\hat{\phi}_k\|_X = 1$ , where  $\gamma_k$  is defined by

$$\begin{aligned} \gamma_k^{-2} &= \left( \frac{\rho l \alpha}{4} + \frac{M}{8l^2} + \frac{\rho g}{4\omega_k^2 l} \right) (K_2^2 f_{\omega_k}(L')^2 + K_1^2 f_{\omega_k}(L)^2) \\ &+ \left( \frac{\rho l \alpha}{2} - \frac{M}{4l^2} - \frac{\rho g}{2\omega_k^2 l} \right) K_1 K_2 f_{\omega_k}(L) f_{\omega_k}(L') \\ &+ \frac{\rho}{2h_0} (K_1^2(L-l) + K_2^2(L'-l)) \quad (k \in \mathbb{Z}^*). \end{aligned}$$

## 4. REACHABILITY AND STABILIZABILITY OF THE LINEARIZED SYSTEM

**4.1. Some background on controllability and reachable spaces.** We begin by recalling some definitions on the controllability of general infinite dimensional systems. We consider the abstract differential equation of the form

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ z(0) = z_0, \end{cases} \quad (4.1)$$

where  $A$  is an infinitesimal generator of a strongly continuous semigroup  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  on a Hilbert space  $X$ , and  $B$  is an admissible control operator of the system (4.1) from the input space  $U$  to the state space  $X$ . This operator is called *bounded* if  $B \in \mathcal{L}(U, X)$ , which is the case of interest in this paper. At a given time  $t$ , the control  $u(t)$  belongs to the input space  $U$ .

Using the semigroup  $\mathbb{T}$  and the control operator  $B$  we can define the input maps  $(\Phi_\tau)_{\tau \geq 0}$  (already appearing in (1.5)) by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-s} B u(s) ds \quad \forall \tau > 0, u \in L_{\text{loc}}^2([0, \infty); U). \quad (4.2)$$

An important role in control theory is played by the range of the operators  $(\Phi_\tau)_{\tau \geq 0}$  defined in (4.2) and denoted, for every  $\tau > 0$ , by  $\text{Ran } \Phi_\tau$ . For each  $\tau > 0$ ,  $\text{Ran } \Phi_\tau$  is called *the reachable space of the system (4.1) in time  $\tau$* . These spaces appear, in particular, in the definition of exact and approximate controllability which are recalled below (see, for instance, [35, Chapter 11] or [9, Chapter 4]).

**Definition 4.1.** Let  $\tau > 0$ .

- (1) The system (4.1) is *exactly controllable* in time  $\tau$  if every element of  $X$  can be reached from the origin at time  $\tau$ , i.e. if

$$\text{Ran } \Phi_\tau = X;$$

- (2) The system (4.1) is *approximately controllable in time  $\tau$*  if

$$\overline{\text{Ran } \Phi_\tau} = X;$$

It is well known, see, for instance, [35, Chapter 6,8], that approximate controllability can be characterized by duality as follows:

**Proposition 4.2.** Let  $\tau > 0$ .

- (1) The system (4.1) is *approximately controllable in time  $\tau$  if and only if*

$$B^* \mathbb{T}_t^* z = 0 \quad \forall t \in [0, \tau] \implies z = 0.$$

- (2) *Assume that  $A$  is skew-adjoint and with compact resolvents, so that there exists an orthonormal basis  $(\phi_k)_{k \in \mathbb{Z}^*}$  in  $X$  consisting of eigenvectors of  $A$  and let  $(i\omega_k)_{k \in \mathbb{Z}^*}$ , with  $\omega_k \in \mathbb{R}$  be the corresponding eigenvalues. Moreover, assume that the eigenvalues of  $A$  are simple and that there exists  $m, \gamma > 0$  such that*

$$|\omega_k - \omega_l| \geq \gamma \quad (k, l \in \mathbb{Z}^*, k \neq l, |k| \geq m, |l| \geq m).$$

*Then the following conditions are equivalent:*

- *The system (4.1) is approximately controllable in any time  $\tau > \frac{2\pi}{\gamma}$ ;*
- *$B^* \phi_k \neq 0$  for every  $k \in \mathbb{Z}^*$ .*

**4.2. Symmetric case.** In this section we come back to the system (3.8), in the particular case of a symmetric geometry and of initial data satisfying appropriate symmetry conditions. We show that in this case the state trajectories of (3.8) coincide with those of a "reduced" system whose state space is a closed subspace of  $X$  defined in (3.4) and we study the reachable spaces of this reduced system.

Let the floating object be in the middle of the fluid domain  $\Omega$  in the horizontal direction, i.e.  $L = L'$ , see Figure 1. We assume that, at the initial state, the floating body system is at equilibrium state, i.e. for every  $x \in \mathcal{E}$ ,

$$z_0 = [\zeta_0 \quad q_0 \quad \langle q_i \rangle_0 \quad \delta_0 \quad \delta_1]^\top = [0 \quad 0 \quad 0 \quad 0 \quad 0]^\top.$$

In this case, when the object moves in the vertical direction, the fluid on two sides of the object goes in opposite directions. To describe this more clearly, we define the Hilbert space  $X_{\text{sym}}$  by

$$X_{\text{sym}} = \left\{ \begin{bmatrix} \zeta & q & 0 & \delta & \eta \end{bmatrix}^\top \in (L^2(\mathcal{E}))^2 \times \mathbb{C}^3 \mid \int_{\mathcal{E}} \zeta(x) dx + 2l\delta = 0, \right. \\ \left. \zeta(-x) = \zeta(x), \quad q(-x) = -q(x) \right\}, \quad (4.3)$$

with the inner product

$$\left\langle \begin{bmatrix} \zeta \\ q \\ 0 \\ \delta \\ \eta \end{bmatrix}, \begin{bmatrix} \tilde{\zeta} \\ \tilde{q} \\ 0 \\ \tilde{\delta} \\ \tilde{\eta} \end{bmatrix} \right\rangle_{X_{\text{sym}}} = \frac{\rho g}{2} \langle \zeta, \tilde{\zeta} \rangle_{L^2(\mathcal{E})} + \frac{\rho}{2h_0} \langle q, \tilde{q} \rangle_{L^2(\mathcal{E})} + \rho g l \delta \tilde{\delta} + \frac{M}{2} \eta \tilde{\eta},$$

where  $\underline{M}$  has been introduced in (3.2).

**Proposition 4.3.** *The Hilbert space  $X_{\text{sym}}$  introduced in (4.3) is  $\mathbb{T}$ -invariant i.e.*

$$\mathbb{T}_t z \in X_{\text{sym}} \quad \forall t \geq 0, z \in X_{\text{sym}},$$

where  $\mathbb{T} = (\mathbb{T}_t)_{t \in \mathbb{R}}$  is the unitary group generated by the operator  $A$  defined in (3.6).

*Proof.* Note that it suffices to show that the system (3.1) preserves the symmetry condition in the Hilbert space  $X_{\text{sym}}$ . Assume that the elevation  $\zeta$  and the horizontal discharge  $q$  satisfy (3.1) and have the following properties

$$\zeta(t, -x) = \zeta(t, x), \quad q(t, -x) = -q(t, x) \quad \forall t \geq 0, x \in \mathcal{E}. \quad (4.4)$$

We define  $\hat{\zeta}$  and  $\hat{q}$  as

$$\hat{\zeta}(t, x) = \zeta(t, -x), \quad \hat{q}(t, x) = -q(t, -x) \quad \forall t \geq 0, x \in \mathcal{E},$$

which implies that

$$\langle q_i \rangle = -\langle \hat{q}_i \rangle, \quad \llbracket \zeta_e \rrbracket = -\llbracket \hat{\zeta}_e \rrbracket, \quad \langle \zeta_e \rangle = \langle \hat{\zeta}_e \rangle.$$

It is not difficult to obtain the corresponding equation for  $\hat{\zeta}$  and  $\hat{q}$ , which implies that  $\hat{\zeta}$  and  $\hat{q}$  also satisfy the system (3.1).  $\square$

Note that the symmetric property (4.4) implies

$$\llbracket \zeta_e \rrbracket = 0 = \langle q_e \rangle = \langle \hat{q}_i \rangle \quad \text{and} \quad \langle \zeta_e \rangle = \zeta_e(t, l),$$

which simplify the linear control system (3.8). Since  $X_{\text{sym}}$  is a closed subspace of  $X$  introduced in (3.4), we have the following decomposition

$$X = X_{\text{sym}} \oplus X_{\text{sym}}^\perp. \quad (4.5)$$

**Remark 4.4.** The word "symmetric" in this section means that not only that the object is in the center of the domain in the horizontal direction ( $L' = L$ ), but also that the functions  $\zeta$  and  $q$  satisfy the symmetry condition (4.4).

We thus obtain a new linear system on the spatial domain  $\mathcal{E}$ . In this symmetric case, the system (3.8) with zero initial data reduces to the following equations defined on  $\mathcal{E}$ , i.e. for all  $t \geq 0$ ,  $x \in \mathcal{E}$ ,

$$\begin{cases} \dot{w} = A_{\text{sym}}w + Bu, \\ w(0) = w_0, \end{cases} \quad (4.6)$$

where  $w$  and  $w_0$  are

$$w = [\zeta \quad q \quad 0 \quad \delta \quad \dot{\delta}]^\top, \quad w_0 = [0 \quad 0 \quad 0 \quad 0 \quad 0]^\top.$$

The operator  $A_{\text{sym}} : \mathcal{D}(A_{\text{sym}}) \rightarrow X_{\text{sym}}$  is densely defined as

$$A_{\text{sym}} = \begin{bmatrix} 0 & -\frac{d}{dx} & 0 & 0 & 0 \\ -gh_0 \frac{d}{dx} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{2}{M}\rho gl \langle \cdot \rangle & 0 & 0 & -\frac{2}{M}\rho gl & 0 \end{bmatrix}, \quad (4.7)$$

with the domain

$$\mathcal{D}(A_{\text{sym}}) = \left\{ \begin{array}{l} [\zeta \quad q \quad 0 \quad \delta \quad \eta]^\top \in (\mathcal{H}^1(\mathcal{E}))^2 \times \mathbb{C}^3 \mid [\zeta \quad q \quad 0 \quad \delta \quad \eta]^\top \in X_{\text{sym}}, \\ \llbracket q \rrbracket = -2l\eta, \quad q(-L) = 0 = q(L) \end{array} \right\}, \quad (4.8)$$

where  $\underline{M}$  is introduced in (3.2). The control operator  $B$  has been defined in (3.6) and we clearly have  $B \in \mathcal{L}(\mathbb{C}, X_{\text{sym}})$ .

Note that  $A_{\text{sym}}$  is the part of  $A$  in the closed subspace  $X_{\text{sym}}$  of  $X$ , so it inherits from  $A$  the properties of being skew-adjoint and has compact resolvents. Therefore, it is diagonalizable and generates a group of unitary operators, denoted by  $\mathbb{T}_{\text{sym}} = (\mathbb{T}_{\text{sym},t})_{t \in \mathbb{R}}$ , on the Hilbert space  $X_{\text{sym}}$  defined in (4.3). Moreover, according to [35, Section 2.4], it is interesting to see from Proposition 4.3 that  $\mathbb{T}_{\text{sym}}$  is the restriction of  $\mathbb{T}$  to  $X_{\text{sym}}$ . Therefore, for  $u \in L_{\text{loc}}^2[0, \infty)$ , the linear system (4.6) is well-posed and the solution  $w \in C([0, \infty); X_{\text{sym}})$ .

**Remark 4.5.** Since  $B \in \mathcal{L}(\mathbb{C}, X_{\text{sym}})$ , it is clear that the input maps of  $(A, B)$  and of  $(A_{\text{sym}}, B)$ , the latter being defined by

$$\Phi_{\text{sym},\tau} u = \int_0^\tau \mathbb{T}_{\text{sym},\tau-s} B u(s) ds \quad \forall u \in L_{\text{loc}}^2([0, \infty); U),$$

have the same range, i.e., that

$$\text{Ran } \Phi_\tau = \text{Ran } \Phi_{\text{sym},\tau} \quad \forall \tau > 0.$$

This means, in particular, that the orthogonal complement space  $X_{\text{sym}}^\perp$  in (4.5) is out of control, justifying the fact that we concentrate on the reachability of the pair  $(A_{\text{sym}}, B)$ .

The spectrum of the operator  $A_{\text{sym}}$  can be obtained, by using the properties (4.4), from the spectrum of  $A$  discussed in Proposition 3.4. More precisely, we have:

**Proposition 4.6.** *Assume that the object is in the middle of the fluid domain which has the symmetry geometry in the sense (4.4). The eigenvalues of the operator  $A_{\text{sym}}$ , denoted by  $i\omega_{\text{sym},k}$ , and the corresponding eigenvectors  $\phi_{\text{sym},k} = [\varphi_{\text{sym},k} \ \psi_{\text{sym},k} \ 0 \ a_{\text{sym},k} \ b_{\text{sym},k}]^T \in \mathcal{D}(A_{\text{sym}})$ , for all  $x \in \mathcal{E}$  and  $k \in \mathbb{Z}^*$ , are*

$$\varphi_{\text{sym},k}(x) = \begin{cases} \frac{iK}{\sqrt{gh_0}} \cos\left(\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L+x)\right) & \forall x \in (-L, -l), \\ \frac{iK}{\sqrt{gh_0}} \cos\left(\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L-x)\right) & \forall x \in (l, L), \end{cases} \quad (4.9)$$

$$\psi_{\text{sym},k}(x) = \begin{cases} K \sin\left(\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L+x)\right) & \forall x \in (-L, -l), \\ -K \sin\left(\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L-x)\right) & \forall x \in (l, L), \end{cases} \quad (4.10)$$

and

$$a_{\text{sym},k} = \frac{i}{\omega_{\text{sym},k} l} \psi_{\text{sym},k}(l), \quad b_{\text{sym},k} = -\frac{1}{l} \psi_{\text{sym},k}(l), \quad (4.11)$$

where  $K$  is an arbitrary constant and the imaginary part of the eigenvalues  $\omega_{\text{sym},k}$  with  $k \in \mathbb{Z}^*$  satisfies

$$(\underline{M}\omega_{\text{sym},k}^2 - 2\rho gl)f_{\omega_{\text{sym},k}}(L) = \sqrt{\frac{g}{h_0}} 2\rho l^2 \omega_{\text{sym},k} g_{\omega_{\text{sym},k}}(L), \quad (4.12)$$

with  $f_{\omega_{\text{sym},k}}$  and  $g_{\omega_{\text{sym},k}}$  introduced in (3.15). Moreover, the eigenvalues  $(i\omega_{\text{sym},k})_{k \in \mathbb{Z}^*}$  are simple and  $(\omega_{\text{sym},k})_{k \in \mathbb{Z}^*}$  form a strictly increasing sequence, with

$$\lim_{k \in \mathbb{Z}^*, |k| \rightarrow \infty} |\omega_{\text{sym},k+1} - \omega_{\text{sym},k}| = \frac{\sqrt{gh_0}}{L-l} \pi.$$

*Proof.* Let  $\phi_{\text{sym}} = [\varphi_{\text{sym}} \ \psi_{\text{sym}} \ 0 \ a_{\text{sym}} \ b_{\text{sym}}]^T \in \mathcal{D}(A_{\text{sym}})$  be an eigenvector of  $A_{\text{sym}}$  corresponding to the eigenvalue  $i\omega_{\text{sym}}$  ( $\omega_{\text{sym}} \in \mathbb{R}$ ), we solve the equation

$$A_{\text{sym}} \phi_{\text{sym}} = i\omega_{\text{sym}} \phi_{\text{sym}}.$$

According to Proposition 3.4, using the symmetry condition (4.4) we obtain that  $\phi_{\text{sym}}$  take the form (4.9)–(4.11), in particular, the third component of  $\phi_{\text{sym}}$  vanishes. In this case, the constants  $K_1$  and  $K_2$  in Proposition 3.4 have the relation  $K_1 = -K_2 = K$ . The equation for  $\omega_{\text{sym}}$  thus becomes

$$\frac{2\rho gl}{\underline{M}} \varphi_{\text{sym}}(l) = i \left( \frac{2\rho g}{\underline{M}\omega_{\text{sym}}} - \frac{\omega_{\text{sym}}}{l} \right) \psi_{\text{sym}}(l),$$

which gives the characteristic equation (4.12). Clearly, the solutions of (4.12), denoted by  $(\omega_{\text{sym},k})_{k \in \mathbb{Z}^*}$ , form a strictly inscreasing sequence. According to the proof of Proposition 3.7, there is one type of the eigenvalues in the symmetric case and for large  $|k|$

$$\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L-l) = k\pi + O\left(\frac{1}{k}\right). \quad (4.13)$$

Moreover, (4.13) implies that there exists  $M > 0$  such that

$$|\omega_{\text{sym},k+1} - \omega_{\text{sym},k}| > \frac{\sqrt{gh_0}}{L-l} \pi \quad \forall k \in \mathbb{Z}^* \text{ and } |k| > M, \quad (4.14)$$

which ends the proof.  $\square$

**Remark 4.7.** Without using Proposition 3.7, the asymptotic behaviour of the eigenvalues in (4.13) can be obtained in an alternative way. By using the characteristic equation (4.12), without loss of generality, we assume that  $\cos\left(\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L-l)\right)$  is non-zero. It follows that

$$\tan\left(\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L-l)\right) = \sqrt{\frac{g}{h_0}} 2\rho l^2 \frac{\omega_{\text{sym},k}}{M\omega_{\text{sym},k}^2 - 2\rho gl} = O\left(\frac{1}{\omega_{\text{sym},k}}\right),$$

for large  $k \in \mathbb{Z}^*$ . Based on the above expression, we assume that

$$\frac{\omega_{\text{sym},k}}{\sqrt{gh_0}}(L-l) = k\pi + \theta_k,$$

with  $\theta_k \rightarrow 0$  as  $k \rightarrow \infty$ . By using the fixed point method introduced in, for instance, the book [11, Chapter 7] or [7, Lemma A.3], we derive that  $\theta_k = O(k^{-1})$ .

By using (4.9)–(4.11), we do some trivial calculations and obtain for every  $k \in \mathbb{Z}^*$  that

$$\|\phi_{\text{sym},k}\|_{X_{\text{sym}}}^2 = \left(\frac{M}{2l^2} + \frac{\rho g}{\omega_{\text{sym},k}^2 l}\right) K^2 f_{\omega_{\text{sym},k}}^2(L) + \frac{\rho}{h_0} K^2(L-l),$$

where  $f_{\text{sym},k}$  and  $M$  are introduced in (3.15) and (3.2) respectively. Now, for every  $k \in \mathbb{Z}^*$ , we define  $\gamma_{\text{sym},k}$  by

$$(\gamma_{\text{sym},k})^{-2} = \left(\frac{M}{2l^2} + \frac{\rho g}{\omega_{\text{sym},k}^2 l}\right) K^2 f_{\omega_{\text{sym},k}}^2(L) + \frac{\rho}{h_0} K^2(L-l). \quad (4.15)$$

We therefore obtain the normalized eigenvectors  $(\widehat{\phi}_{\text{sym},k})_{k \in \mathbb{Z}^*} := (\gamma_{\text{sym},k} \phi_{\text{sym},k})_{k \in \mathbb{Z}^*}$  that form an orthonormal basis in  $X_{\text{sym}}$ .

**Remark 4.8.** As we already realized, the symmetry property (4.4) excludes the case of the double eigenvalues discussed in Section 3.2. Based on the decomposition (4.5), we notice that  $(\widehat{\phi}_{\text{sym},k})_{k \in \mathbb{Z}^*}$  is a proper subset of  $(\widehat{\phi}_k)_{k \in \mathbb{Z}^*}$  introduced in Remark 3.9. Moreover, we have

$$\omega_{\text{sym},k} = \omega_{j(k)} \quad \forall k \in \mathbb{Z}^*,$$

where  $\omega_{j(k)}$  is the eigenvalue of  $A$  and the subscript  $j(k) \in \mathbb{Z}^*$  can be easily found.

**4.3. Proof of the main result.** The adjoint  $B^* \in \mathcal{L}(X_{\text{sym}}, \mathbb{C})$  of the control operator  $B$  defined in (3.6) is

$$B^* = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}. \quad (4.16)$$

We are in a position to prove Theorem 1.2.

*Proof of Theorem 1.2.* According to a classical result (see, for instance, [35, Chapter 4]), we know that for every  $\tau > 0$  and every  $z \in X_{\text{sym}}$ ,

$$(\Phi_{\text{sym},\tau}^* z)(t) = \begin{cases} B^* \mathbb{T}_{\text{sym},\tau-t}^* z & \text{for } t \in [0, \tau], \\ 0 & \text{for } t > \tau, \end{cases}$$

where  $B^*$  is introduced in (4.16) and  $\mathbb{T}_{\text{sym}}$  is the  $C_0$ -group generated by  $A_{\text{sym}}$ . This implies that for every  $\tau > 0$  we have

$$\|(\Phi_{\text{sym},\tau}^* z)\|_{L^2([0,\tau];U)}^2 = \int_0^\tau \|B^* \mathbb{T}_{\text{sym},\tau-t}^* z\|_U^2 dt.$$



Notice that  $0 \in \rho(A_{\text{sym}})$  and the imaginary part of the eigenvalues  $(\omega_{\text{sym},k})_{k \in \mathbb{Z}^*}$  is strictly increasing, there exists  $c > 0$  such that  $|\omega_{\text{sym},k}| \geq c$ , which implies that  $\gamma_{\text{sym},k}$  defined in (4.15) is lower bounded by a positive constant. Combining (4.16) and Proposition 4.6, we have

$$\left| B^*(\widehat{\phi}_{\text{sym},k}) \right| = \left| \frac{\gamma_{\text{sym},k}}{2l} \psi_{\text{sym},k}(l) \right| \geq C \left| \sin \left( \frac{\omega_{\text{sym},k}}{\sqrt{gh_0}} (L-l) \right) \right|, \quad (4.17)$$

for  $k \in \mathbb{Z}^*$ . Putting (4.13) and (4.17) together, we obtain that

$$\left| B^*(\widehat{\phi}_{\text{sym},k}) \right| \geq \frac{C}{k} \quad \forall k \in \mathbb{Z}^*. \quad (4.18)$$

Since the operator  $A_{\text{sym}}$  is diagonalizable and skew-adjoint on  $X_{\text{sym}}$ , we have

$$\mathbb{T}_{\text{sym},t} z = \sum_{k \in \mathbb{Z}^*} e^{i\omega_{\text{sym},k} t} \langle z, \widehat{\phi}_{\text{sym},k} \rangle \widehat{\phi}_{\text{sym},k} \quad \forall z \in X_{\text{sym}},$$

where  $(\widehat{\phi}_{\text{sym},k})_{k \in \mathbb{Z}^*}$ , an orthonormal basis of  $X_{\text{sym}}$ , is introduced around (4.15). Hence, for every  $\tau > 0$  we have

$$\int_0^\tau \|B^* \mathbb{T}_{\text{sym},\tau-t}^* z\|_U^2 dt = \int_0^\tau \left| \sum_{k \in \mathbb{Z}^*} e^{-i\omega_{\text{sym},k} t} \langle z, \widehat{\phi}_{\text{sym},k} \rangle B^* \widehat{\phi}_{\text{sym},k} \right|^2 dt.$$

Recalling (4.14) and using the Ingham theorem (a generalization of Parseval's equality, see, for instance, in [35, Chapter 8] or [20]), there exists  $\tau_0 := \frac{2(L-l)}{\sqrt{gh_0}}$  such that, for every  $\tau > \tau_0$ ,

$$\int_0^\tau \|B^* \mathbb{T}_{\text{sym},\tau-t}^* z\|_U^2 dt \geq C \sum_{k \in \mathbb{Z}^*} |\langle z, \widehat{\phi}_{\text{sym},k} \rangle|^2 |B^* \widehat{\phi}_{\text{sym},k}|^2. \quad (4.19)$$

Therefore, (4.18) and (4.19) imply that, for every  $\tau > \tau_0$ ,

$$\|\Phi_{\text{sym},\tau}^* z\|_{L^2([0,\tau];U)}^2 \geq c \|z\|_{\mathcal{D}(A_{\text{sym}})'}^2 \quad \forall z \in X_{\text{sym}},$$

where  $\mathcal{D}(A_{\text{sym}})'$  is the dual of  $\mathcal{D}(A_{\text{sym}})$  with respect to the pivot space  $X_{\text{sym}}$ . Now we introduce the identity function on  $\mathcal{D}(A_{\text{sym}})$ , denoted by  $\text{id}_{\mathcal{D}(A_{\text{sym}})}$ , then of course we have  $\text{id}_{\mathcal{D}(A_{\text{sym}})} \in \mathcal{L}(\mathcal{D}(A_{\text{sym}}), X_{\text{sym}})$ . Note that, for every  $\tau > 0$ ,  $\Phi_{\text{sym},\tau} \in \mathcal{L}(L^2([0,\tau];U); X_{\text{sym}})$ , we apply next a classical consequence of the closed graph theorem (see, for instance, [35, Proposition 12.1.2]), which follows that

$$\text{Ran } \Phi_{\text{sym},\tau} \supset \mathcal{D}(A_{\text{sym}}).$$

Combined with Remark 4.5, we conclude that  $\text{Ran } \Phi_\tau \supset \mathcal{D}(A_{\text{sym}})$  for every  $\tau > \tau_0$ . Recalling that  $A_{\text{sym}}$  is densely defined, we immediately conclude that (1.6) holds.  $\square$

## 5. CONCLUSIONS, COMMENTS AND OPEN QUESTIONS

In this work, we investigate a coupled PDE-ODE system describing the motion of a floating body in a free boundary ideal fluid, within the linearized shallow water regime. The floating body is constrained to move vertically and it is actuated by a control force applied from the bottom of the object. Our main result asserts that, provided that, in a symmetric geometrical configuration, the system can be steered from rest to any smooth enough symmetric wave profile.

We give below, as a consequence of our main theorem, the following result on the controllability and stabilizability properties of the system (4.6)–(4.8).

**Corollary 5.1.** *Let  $L' = L$  and the initial data  $\zeta_0$  and  $q_0$  satisfy the symmetry condition (4.4). Then the linear system defined by (4.6)–(4.8) on  $X_{\text{sym}}$  (briefly designed by  $(A_{\text{sym}}, B)$ ), has the following properties*

- (1)  $(A_{\text{sym}}, B)$  is not exactly controllable in time  $\tau$  for any finite  $\tau > 0$ ;
- (2)  $(A_{\text{sym}}, B)$  is approximately controllable on  $X_{\text{sym}}$  in time  $\tau$  for any  $\tau > \frac{2(L-l)}{\sqrt{gh_0}}$ ;
- (3)  $(A_{\text{sym}}, B)$  is strongly stabilizable with the feedback operator  $F = -B^*$ . More precisely, there exists  $C > 0$  such that the closed-loop semigroup  $\mathbb{T}_{\text{sym}}^{\text{cl}}$  generated by  $A_{\text{sym}} - BB^*$  satisfies

$$\|\mathbb{T}_{\text{sym},t}^{\text{cl}} w_0\|_{X_{\text{sym}}} \leq \frac{C}{(1+t)^{\frac{1}{2}}} \|w_0\|_{\mathcal{D}(A_{\text{sym}})} \quad \forall w_0 \in \mathcal{D}(A_{\text{sym}}), t \geq 0. \quad (5.1)$$

*Proof.* (1) Note that the operator  $A_{\text{sym}}$  is skew-adjoint and  $B \in \mathcal{L}(\mathbb{C}, X_{\text{sym}})$ , then the first assertion follows directly from Curtain and Zwart [9, Theorem 4.1.5] or [9, Theorem 5.2.6] in the same book, since  $A_{\text{sym}}$  has infinitely many unstable eigenvalues. Equivalently, we know that the system  $(A_{\text{sym}}, B)$  is not exponentially stabilizable (see, for instance, Haraux [16] and Liu [26]). Alternatively, we can apply the main result of Gibson [10] or Guo, Guo and Zhang [15, Theorem 3].

(2) The second assertion is a direct consequence of Theorem 1.2. By duality it suffices to show that there exists  $\tau_0 > 0$ , such that for every  $\tau > \tau_0$ ,

$$B^* \mathbb{T}_{\text{sym},t}^* z = 0 \quad \text{on } [0, \tau] \implies z = 0. \quad (5.2)$$

Let  $B^* \mathbb{T}_{\text{sym},t}^* z = 0$  on  $[0, \tau]$  with  $\tau > \frac{2(L-l)}{\sqrt{gh_0}}$ , we obtain from (4.19) that  $\langle z, \widehat{\phi}_{\text{sym},k} \rangle = 0$  for every  $k \in \mathbb{Z}^*$ , which implies that  $z = 0$ . This, together with Proposition 4.2, gives the result.

(3) The approximate controllability of the system  $(A_{\text{sym}}, B)$  is equivalent to the fact that the semigroup  $\mathbb{T}_{\text{sym}}^{\text{cl}}$  generated by  $A_{\text{sym}} - BB^*$  is strongly stable (for this, please refer to Benchimol [4], Batty and Vu [2]). To obtain the explicit decay rate, we further conclude from (4.18) and (4.19) that

$$\int_0^\tau \|B^* \mathbb{T}_{\text{sym},t} w_0\|_U^2 dt \geq C \|w_0\|_{\mathcal{D}(A_{\text{sym}})'}^2, \quad \forall w_0 \in \mathcal{D}(A_{\text{sym}}).$$

Hence, we have the interpolation

$$[\mathcal{D}(A_{\text{sym}}), \mathcal{D}(A_{\text{sym}})']_\theta = X_{\text{sym}} \quad \text{with } \theta = \frac{1}{2}.$$

We apply Theorem 2.4 in [1] and conclude that the semigroup  $\mathbb{T}_{\text{sym}}^{\text{cl}}$  generated by  $A_{\text{sym}} - BB^*$  satisfies (5.1).  $\square$

The main question left open in our work is the description of the reachable space of the considered system without symmetry conditions. Using the properties of the eigenvalues of the generator (see Subsection 3.2) this could be accomplished provided that one has lower bounds on  $|B^* \widehat{\phi}_k|$ , where  $B^* \in \mathcal{L}(X, \mathbb{C})$  is defined in (4.16), and  $(\widehat{\phi}_k)_{k \in \mathbb{Z}^*}$  is the orthonormal basis introduced in Remark 3.9. Obtaining such lower bounds does not seem an easy task. Indeed, combining (3.16)–(3.18) and (4.16) we obtain that for every  $k \in \mathbb{Z}^*$ ,

$$|B^* \widehat{\phi}_k| = \frac{1}{4l} |\gamma_k (K_2 f_{\omega_k}(L') - K_1 f_{\omega_k}(L))|, \quad (5.3)$$

where  $\widehat{\phi}_k$  and  $\gamma_k$  are introduced in Remark 3.9,  $f_{\omega_k}$  is defined in (3.15); with constant  $K_1$  and  $K_2$  which we are unable to express in a simple manner in terms of  $\omega_k$ . We also recall from Remark 3.5 that we are, in the general case, unable to confirm or to inform the existence of double eigenvalues.

Another open question of interest are the study of the system obtained by adding a viscosity term in the shallow water equations, in the spirit of Maity et al. [27]. This could lead, in particular, to a description of the reachable space for nonlinear systems in which the fluid is modeled by the nonlinear shallow water equations. Finally, let us mention that an interesting question could be to consider the corresponding boundary control problems, in the spirit of [33] (or a short version [34]), [31] and [32].

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