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# ON THE FRACTIONAL LAPLACIAN OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

ARRAN FERNANDEZ, JOEL E. RESTREPO, AND JEAN-DANIEL DJIDA

**ABSTRACT.** The theories of fractional Laplacians and of fractional calculus with respect to functions are combined to produce, for the first time, the concept of a fractional Laplacian with respect to a bijective function. The theory is developed both in the 1-dimensional setting and in the general  $n$ -dimensional setting. Fourier transforms with respect to functions are also defined, and the relationships between Fourier transforms, fractional Laplacians, and Marchaud type derivatives are explored. Function spaces for these operators are carefully defined, including weighted  $L^p$  spaces and a new type of Schwartz space. The theory developed is then applied to construct solutions to some partial differential equations involving both fractional time-derivatives and fractional Laplacians with respect to functions, with illustrative examples.

## 1. INTRODUCTION

Partial differential equations form one of the largest and most important fields of mathematics, with studied problems and topics ranging from abstract mathematical results to direct physical applications. One of the most commonly used operators in partial differential equations, aside from the simple partial derivatives with respect to single independent variables, is the Laplace operator or Laplacian, the study of which is a truly interdisciplinary topic, intersecting with complex analysis, cohomology theory, diffusion modelling, gravitation, and electromagnetics as well as partial differential equations [1, 2].

Fractional calculus is a popular topic of study nowadays in analysis and applied mathematics. The key motivating idea is to take the operators of calculus and extend their definitions to permit fractional orders of differentiation or integration, but the field has expanded to cover different types of operators, with many parameters, or labelled by functions as well as real or complex parameters. A vast number of operators have emerged in the field of fractional calculus, many of which have discovered applications in modelling [3, 4].

Fractional differential equations are frequently used to describe systems with non-local behaviours. Some of the established methods used for classical differential equations can be extended, with modifications, to the fractional setting; detailed studies of fractional differential equations can be found in the textbooks [5, 6]. An important operator for posing fractional partial differential equations (FPDEs) is the fractional Laplacian, which has many equivalent definitions [7] although it is most commonly introduced using Fourier and inverse Fourier transforms. Detailed introductions and studies of the fractional Laplacian can be found in [8, 9] and the references therein, while some related operators and extensions are studied for example in [10, 11].

Attempts to classify and categorise the operators of fractional calculus have shown the breadth of the field while failing to achieve a complete classification. However, there are some broad classes

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of operators which can form a general setting for mathematical studies. One such class consists of the operators of fractional calculus with respect to functions, a fractional analogue of the concept of the Riemann–Stieltjes integral. This was first proposed in the 1970s as a generalisation of the usual fractional integral of a function  $f(x)$  with respect to  $x$ , and it was studied in more detail in the textbooks of the 1990s [5, 12] before rising in popularity in the 2010s due to some researchers defining and promoting the Caputo and Hilfer versions [13, 14]. Fractional differential equations with respect to functions have also discovered applications to diffusion [15], statistics [16], and turbulence [17].

A connection which has not yet been made is between the fractional Laplacian and fractional calculus with respect to functions. The reason for this gap in the field is probably that fractional Laplacians appear mostly in the pure analysis of FPDEs, while fewer advanced mathematical studies exist for fractional differential equations with respect to functions, at least until recently [18, 19, 20, 21]. Extending the notion of fractional Laplacians into the context of fractional calculus with respect to functions, therefore, will represent a useful connection between two different subtopics within the study of fractional differential equations.

In this paper, we investigate for the first time the concept of a fractional Laplacian operator with respect to a function. This has a close connection with the concept of a Fourier transform with respect to a function, which is also defined for the first time in this paper. It turns out that much of the functional analysis of such operators can take place in weighted  $L^p$  spaces. We shall treat the 1-dimensional case separately from the general  $n$ -dimensional case, because the concepts are easier to grasp in 1 dimension, and some of them are not yet extendable to the  $n$ -dimensional setting.

The detailed structure of the paper is as follows. Section 2 introduces some preliminary notions to be used throughout. Section 3 comprises a detailed study of the Fourier transform with respect to a function, both in the 1-dimensional case (§3.1) and the general  $n$ -dimensional case (§3.2). As part of this work, we also establish a new version of the Schwartz space on  $\mathbb{R}$  which is suitable for dealing with operators with respect to functions. Section 4 introduces and analyses the fractional Laplacian with respect to a function, both in the 1-dimensional case (§4.1) and in the general  $n$ -dimensional case (§4.2). This work also involves establishing relationships with Marchaud operators in 1 dimension and with fractional directional derivatives in  $n$  dimensions. Finally, in Section 5 we conduct a lengthy investigation of some fractional space-time Cauchy problems, involving both fractional derivatives with respect to functions in time and fractional Laplacians with respect to functions in space. As special cases of this study, we recover the well-known solutions to the classical heat and wave equations.

## 2. PRELIMINARIES

**2.1. The fractional Laplacian.** In this section, we fix some notations and recall some known results regarding the fractional Laplacian, which will be needed throughout the paper. These results can be found for example in [22, 23, 24] and the references therein.

First of all, we let  $0 < s < 1$  and  $\alpha = 2s$ . Here and in what follows, depending on the context, we may label the order of the fractional Laplacian and related operators as either  $s$  or  $\alpha/2$ , both meaning the same thing, always with the assumption  $\alpha = 2s$ . We also let  $n \in \mathbb{N}$  be the dimension of the domain, so that the setting will be functions on  $\mathbb{R}^n$ . In some of the work below, we will treat the cases  $n = 1$  and  $n > 1$  separately, but the definitions in this section are valid for any  $n \in \mathbb{N}$ .

We recall some standard definitions of function spaces on  $\mathbb{R}^n$ . For any  $p \in [1, \infty)$ , the  $L^p$  space is defined as

$$L^p(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ measurable such that } \int_{\mathbb{R}^n} |f(\mathbf{x})|^p d\mathbf{x} < \infty \right\},$$

which is a Banach space under the standard  $L^p$  norm. For  $p = \infty$ , we have instead the following definition:

$$L^\infty(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ measurable such that } \sup_{\mathbf{x} \in \mathbb{R}^n} |f(\mathbf{x})| < \infty \right\},$$

which is also a Banach space under the standard  $L^\infty$  norm.

Next, weighted  $L^p$  spaces can be defined by altering the measure of integration. Any  $\sigma$ -finite measure gives rise to an  $L^p$  space which is a Banach space. In particular, any measurable function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  gives the following so-called weighted  $L^p$  space:

$$L^p(\mathbb{R}^n, g d\mathbf{x}) := \left\{ f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ measurable such that } \int_{\mathbb{R}^n} |f(\mathbf{x})|^p g(\mathbf{x}) d\mathbf{x} < \infty \right\},$$

endowed with the norm given by the  $(1/p)$ th power of the stated integral. Here  $g$  can be taken as the Radon–Nikodým derivative of one  $\sigma$ -finite measure with respect to another, transforming between two Banach spaces with different weights.

The Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  is the space of smooth and rapidly decreasing functions, namely

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) \text{ such that } \sup_{\mathbf{x} \in \mathbb{R}^n} |\mathbf{x}^{\mathbf{k}} \partial^\ell f(\mathbf{x})| < \infty \text{ for all multi-indices } \mathbf{k}, \ell \in \mathbb{N}^n \right\},$$

and the topology on this space is defined by the following semi-norms:

$$p_N(f) = \sup_{\mathbf{x} \in \mathbb{R}^n} (1 + |\mathbf{x}|)^N \sum_{|\mathbf{k}| \leq N} |\partial^{\mathbf{k}} f(\mathbf{x})|, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where  $N$  can take any value in  $\mathbb{Z}_0^+$ . The  $L^p$  and Schwartz spaces form a natural setting for the fractional Laplacian and related operators.

To provide a rigorous definition of the fractional Laplacian via the principal value integral, we use the following function space for  $0 < s < 1$  and  $n \in \mathbb{N}$ :

$$L_s^1(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ measurable such that } \int_{\mathbb{R}^n} \frac{|f(\mathbf{x})|}{(1 + |\mathbf{x}|)^{n+2s}} d\mathbf{x} < \infty \right\}.$$

For  $f \in L_s^1(\mathbb{R}^n)$  and  $\varepsilon > 0$ , we set

$$(-\Delta)_\varepsilon^s f(\mathbf{x}) := C_{n,s} \int_{\{\mathbf{y} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{y}| > \varepsilon\}} \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n+2s}} d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n,$$

where  $C_{n,s}$  is a normalization constant to be defined explicitly below. The fractional Laplacian  $(-\Delta)^s$  is then defined by the following singular integral:

$$(-\Delta)^s f(\mathbf{x}) := C_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n+2s}} d\mathbf{y} = \lim_{\varepsilon \downarrow 0} (-\Delta)_\varepsilon^s f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \quad (1)$$

provided that the limit exists for a.e.  $\mathbf{x} \in \mathbb{R}^n$ . We refer to [23] and the references therein regarding the class of functions for which the limit in (1) exists for a.e.  $\mathbf{x} \in \mathbb{R}^n$ .

The constant  $C_{n,s}$  is chosen so that the Fourier relation  $\widehat{(-\Delta)^{\alpha/2} f}(\mathbf{k}) = |\mathbf{k}|^\alpha \widehat{f}(\mathbf{k})$  holds for all  $f \in \mathcal{S}(\mathbb{R}^n)$ , where  $\mathbf{k} \in \mathbb{R}^n$  is the variable of the Fourier domain. Explicitly, the normalisation

constant  $C_{n,s}$  is given by

$$C_{n,s} = C_{n,\alpha/2} := \frac{s2^{2s}\Gamma(s + \frac{n}{2})}{\pi^{n/2}\Gamma(1-s)} = \frac{\alpha2^{\alpha-1}\Gamma(\frac{\alpha+n}{2})}{\pi^{n/2}\Gamma(1-\frac{\alpha}{2})}. \quad (2)$$

It is worth mentioning that the fractional Laplacian can be realised as the inverse of the Riesz potential. The Riesz potential of order  $\alpha \in (0, n)$ , which we denote as  $J_\alpha$ , is a pseudo-differential operator with Fourier multiplier  $\mathbf{k} \mapsto |\mathbf{k}|^{-\alpha}$ , i.e. for all  $u \in \mathcal{S}(\mathbb{R}^n)$  we have  $\widehat{J_\alpha u}(\mathbf{k}) = |\mathbf{k}|^{-\alpha}\widehat{u}(\mathbf{k})$ . One can show that for all  $u \in \mathcal{S}(\mathbb{R}^n)$  we have

$$J_\alpha u = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{u(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n-\alpha}} d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n,$$

where the constant  $C_{n,\alpha}$  is given as in (2). Thus, using the Fourier characterization of  $(-\Delta)^{\alpha/2}$  and  $J_\alpha$  for  $\alpha \in (0, 2)$ , we find that they are two-sided inverses to each other:  $(-\Delta)^{\alpha/2} = J_\alpha^{-1}$ .

**2.2. Fractional calculus with respect to functions.** The operators of fractional calculus with respect to functions, sometimes called  $\Psi$ -fractional calculus following the notation of Almeida [13], are a natural extension of the original operators of classical fractional calculus. The concept was first proposed in general by Osler in 1970 [25], although the special case of fractional integrals with respect to power functions was considered by Erdelyi in 1966 [26]. More detailed studies of Riemann–Liouville fractional calculus with respect to functions can be found in the textbooks [5, 12]. The natural extension to Caputo fractional derivatives with respect to functions was formalised by Almeida in 2017 [13]. We present the definitions as follows.

**Definition 2.1** ([13, 25]). Let  $\phi : (c, \infty) \rightarrow \mathbb{R}$  be a function which is differentiable with  $\phi' > 0$  almost everywhere, where  $c$  is a constant of integration. The Riemann–Liouville fractional integral with respect to  $\phi$ , to order  $\alpha > 0$  (or  $\alpha \in \mathbb{C}$  with  $\text{Re}(\alpha) > 0$ ), of a locally  $L^1$  function  $f$ , is defined to be

$${}^{RL}I_{\phi(x)}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (\phi(x) - \phi(y))^{\alpha-1} f(y) \phi'(y) dy, \quad x > 0.$$

The Riemann–Liouville fractional derivative with respect to  $\phi$ , to order  $\alpha > 0$  with  $n - 1 \leq \alpha < n \in \mathbb{N}$  (or  $\alpha \in \mathbb{C}$  with  $\text{Re}(\alpha) \geq 0$  and  $n - 1 \leq \text{Re}(\alpha) < n \in \mathbb{N}$ ), of a  $C^n$  function  $f$ , is defined to be

$${}^{RL}D_{\phi(x)}^\alpha f(x) = \left( \frac{1}{\phi'(x)} \cdot \frac{d}{dx} \right)^n {}^{RL}I_{\phi(x)}^{n-\alpha} f(x).$$

The Caputo fractional derivative with respect to  $\phi$ , to order  $\alpha > 0$  with  $n - 1 \leq \alpha < n \in \mathbb{N}$  (or  $\alpha \in \mathbb{C}$  with  $\text{Re}(\alpha) \geq 0$  and  $n - 1 \leq \text{Re}(\alpha) < n \in \mathbb{N}$ ), of a  $C^n$  function  $f$ , is defined to be

$${}^C D_{\phi(x)}^\alpha f(x) = {}^{RL}I_{\phi(x)}^{n-\alpha} \left( \frac{1}{\phi'(x)} \cdot \frac{d}{dx} \right)^n f(x).$$

One of the most important facts for understanding and studying fractional calculus with respect to functions is the following result expressing these operators as conjugations of the classical Riemann–Liouville and Caputo operators with an invertible operator of composition with  $\phi$ .

**Proposition 2.2** ([5, 12]). *The operators of fractional calculus with respect to functions are related to the classical operators of fractional calculus by the following conjugation relations:*

$$\begin{aligned} {}^{RL}I_{\phi(x)}^\alpha &= Q_\phi \circ {}^{RL}I_{\phi(c)}^\alpha \circ Q_\phi^{-1}, \\ {}^{RL}D_{\phi(x)}^\alpha &= Q_\phi \circ {}^{RL}D_{\phi(c)}^\alpha \circ Q_\phi^{-1}, \\ {}^C D_{\phi(x)}^\alpha &= Q_\phi \circ {}^C D_{\phi(c)}^\alpha \circ Q_\phi^{-1}, \end{aligned}$$

where  $Q_\phi$  is the operator of right composition with  $\phi$ :

$$Q_\phi(f) = f \circ \phi, \quad \text{i.e.} \quad (Q_\phi f)(x) = f(\phi(x)). \quad (3)$$

These facts enable a theory of fractional calculus with respect to functions to be quickly constructed from the existing theory of fractional calculus, simply by applying compositions with  $\phi$  and  $\phi^{-1}$  wherever necessary. For example, the rules for compositions of two Riemann–Liouville or Caputo operators with respect to  $\phi$  are exactly the same as the established rules for compositions of two classical Riemann–Liouville or Caputo operators.

Similarly, the above conjugation relations can also be used for more speedy and efficient solving of fractional differential equations with respect to functions, as in many cases these can be reduced to classical fractional differential equations whose properties may already be known [20]. Let us mention some examples of particular interest, which have their own names and applications.

**Example 2.3.** Three special cases can be mentioned.

- In the case  $\phi(x) = x$ , fractional calculus with respect to this function is exactly the original fractional calculus.
- In the case  $\phi(x) = \log(x)$ , fractional calculus with respect to this function is known as Hadamard fractional calculus. The operators of Hadamard fractional calculus are:

$$\begin{aligned} {}^H I_x^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x \left( \log \frac{x}{y} \right)^{\alpha-1} \frac{f(y)}{y} dy, \\ {}^{HR} D_x^\alpha f(x) &= \left( x \cdot \frac{d}{dx} \right)^n {}^H I_x^{n-\alpha} f(x), \\ {}^{HC} D_x^\alpha f(x) &= {}^H I_x^{n-\alpha} \left( x \cdot \frac{d}{dx} \right)^n f(x). \end{aligned}$$

- In the case  $\phi(x) = x^\rho$ , fractional calculus with respect to this function was proposed by Erdélyi [26] in 1964, although some more recent literature has referred to such operators by the name of Katugampola. The operators here are:

$$\begin{aligned} {}^{RL} I_{x^\rho}^\alpha f(x) &= \frac{\rho}{\Gamma(\alpha)} \int_0^x (x^\rho - y^\rho)^{\alpha-1} y^{\rho-1} f(y) dy, \\ {}^{RL} D_{x^\rho}^\alpha f(x) &= \left( \frac{1}{\rho x^{\rho-1}} \cdot \frac{d}{dx} \right)^n {}^{RL} I_{x^\rho}^{n-\alpha} f(x), \\ {}^C D_{x^\rho}^\alpha f(x) &= {}^{RL} I_{x^\rho}^{n-\alpha} \left( \frac{1}{\rho x^{\rho-1}} \cdot \frac{d}{dx} \right)^n f(x). \end{aligned}$$

Another useful tool in the study of fractional differential equations with respect to functions has been the Laplace transform with respect to a function, or  $\Psi$ -Laplace transform, defined in [27] and further studied from the operational viewpoint in [20].

**Definition 2.4** ([27]). Let  $\phi$  be a function which is differentiable with  $\phi > 0$  and  $\phi' > 0$  on  $\mathbb{R}^+$  and which satisfies  $\phi(0) = 0$ .

The Laplace transform with respect to  $\phi$ , of a function  $f$  such that this integral exists (for example, any  $\phi$ -exponentially bounded function  $f$  [20]), is defined to be

$$[\mathcal{L}_\phi f](s) = \int_0^\infty e^{-s\phi(x)} f(x) \phi'(x) dx,$$

where  $s \in \mathbb{C}$  is the variable of the Laplace domain.

**Proposition 2.5** ([20]). *The Laplace transform with respect to a function is related to the classical Laplace transform by the following relation between operators:*

$$\mathcal{L}_\phi = \mathcal{L} \circ Q_\phi^{-1},$$

where  $Q_\phi$  is the operator of right composition with  $\phi$  as defined in (3).

The above result, establishing the connection between the Laplace transform with respect to  $\phi$  and the classical Laplace transform, leads directly to many important results in the understanding of Laplace transforms with respect to functions. These include the relationship of such transforms with fractional integrals and derivatives with respect to functions [20, §3.1] and with the  $\phi$ -convolution operation defined by

$$(f *_\phi g)(x) = \int_0^x f \circ \phi^{-1}(\phi(x) - \phi(y))g(y)\phi'(y) dy. \quad (4)$$

Further studies of the Laplace transforms with respect to functions, including their applications to solving fractional differential equations with respect to functions, can be found in [27, 20].

### 3. FOURIER TRANSFORM WITH RESPECT TO A FUNCTION

In this section, we introduce the concept of Fourier transforms with respect to functions. These are closely related to the Laplace transforms with respect to functions, as described in [27, 20], and they will be needed later for defining the fractional Laplacian with respect to a function.

#### 3.1. The 1-dimensional case.

**Definition 3.1.** Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a function which is differentiable with  $\phi' > 0$  almost everywhere and  $\phi(x) \rightarrow \pm\infty$  as  $x \rightarrow \pm\infty$  respectively.

The Fourier transform with respect to  $\phi$ , of a function  $f$  such that this integral exists, is defined to be

$$[\mathcal{F}_\phi f](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\phi(x)} f(x)\phi'(x) dx, \quad (5)$$

where  $k \in \mathbb{R}$  is the variable of the Fourier domain.

**Proposition 3.2.** *The Fourier transform with respect to a function is related to the classical Fourier transform by the following relation between operators:*

$$\mathcal{F}_\phi = \mathcal{F} \circ Q_\phi^{-1}, \quad (6)$$

where  $Q_\phi$  is the operator of right composition with  $\phi$  as defined in (3).

*Proof.* This follows directly from substitution in the integral:

$$\mathcal{F} \circ Q_\phi^{-1} f(k) = \mathcal{F} [f \circ \phi^{-1}](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(\phi^{-1}(x)) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\phi(u)} f(u)\phi'(u) du,$$

using the change of variables  $x = \phi(u)$ , as well as the fact that  $\phi(x) \rightarrow \pm\infty$  as  $x \rightarrow \pm\infty$  respectively.  $\square$

**Proposition 3.3.** *The inverse Fourier transform with respect to  $\phi$  is given by*

$$[\mathcal{F}_\phi^{-1}g](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik\phi(x)} g(k) dk;$$

in other words, we have  $g = \mathcal{F}_\phi f \iff f = \mathcal{F}_\phi^{-1}g$ .

*Proof.* This follows from the operational relation in Proposition 3.2: the inverse of  $\mathcal{F} \circ Q_\phi^{-1}$  should be  $Q_\phi \circ \mathcal{F}^{-1}$ .  $\square$

Having defined the operator and its inverse, it is necessary to consider which function spaces this operator can be defined on. The results on  $L^p$  spaces are analogous to those for the classical Fourier transform, but here we must use weighted  $L^p$  spaces instead, due to the  $\phi'$  appearing in the integrand.

**Theorem 3.4.** *The Fourier transform with respect to  $\phi$  is a bounded operator from  $L^1(\mathbb{R}, d\phi)$  to  $L^\infty(\mathbb{R})$ .*

*Proof.* If  $f \in L^1(\mathbb{R}, d\phi)$ , then by definition,

$$\int_{\mathbb{R}} |f(x)| d\phi(x) = \int_{\mathbb{R}} |f(x)| \phi'(x) dx < \infty.$$

We have, for all  $k \in \mathbb{R}$ ,

$$\begin{aligned} \left| [\mathcal{F}_\phi f](k) \right| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\phi(x)} f(x) \phi'(x) dx \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ik\phi(x)} f(x) \phi'(x)| dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \phi'(x) dx, \end{aligned}$$

which means

$$\|\mathcal{F}_\phi f\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^1(\mathbb{R}, d\phi)}.$$

$\square$

The above Theorem makes sense with respect to the compositional relation given by Proposition 3.2, because it is known that the classical Fourier transform is bounded from  $L^1(\mathbb{R})$  to  $L^\infty(\mathbb{R})$ , and the  $Q_\phi$  operator can be described as follows.

**Proposition 3.5.** *Let  $\phi$  be a function as in Definition 3.1, and define  $Q_\phi$  by (3) as the operation of composition with  $\phi$ . For any  $p \in [1, \infty]$ , the map*

$$Q_\phi : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R}, d\phi)$$

*is a linear isometry. For  $p = 2$  specifically, this map is also an isometry between inner product spaces, where the inner product  $\langle \cdot, \cdot \rangle_\phi$  on the weighted  $L^2$  space is defined in the usual way by*

$$\langle f, g \rangle_\phi = \int_{-\infty}^{\infty} f(x) \overline{g(x)} \phi'(x) dx,$$

*and the complex conjugation can be ignored if we assume real-valued functions  $f$  and  $g$ .*

*Proof.* Linearity is clear. To prove it is an isometry, we substitute in the integral as follows:

$$\|Q_\phi f\|_{L^p(\mathbb{R}, d\phi)}^p = \int_{-\infty}^{\infty} |f(\phi(u))|^p \phi'(u) du = \int_{-\infty}^{\infty} |f(x)|^p dx = \|f\|_{L^p(\mathbb{R})}^p,$$

using the change of variables  $x = \phi(u)$ , as well as the fact that  $\phi(x) \rightarrow \pm\infty$  as  $x \rightarrow \pm\infty$  respectively. In the case  $p = 2$ , we can similarly prove that  $Q_\phi$  preserves the inner product as follows:

$$\begin{aligned} \langle Q_\phi f, Q_\phi g \rangle_\phi &= \int_{-\infty}^{\infty} Q_\phi f(u) \overline{Q_\phi g(u)} \phi'(u) du = \int_{-\infty}^{\infty} f(\phi(u)) \overline{g(\phi(u))} \phi'(u) du \\ &= \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx = \langle f, g \rangle. \end{aligned}$$

$\square$



**Theorem 3.6** (Plancherel's theorem with respect to  $\phi$ ). *If  $f \in L^1(\mathbb{R}, d\phi) \cap L^2(\mathbb{R}, d\phi)$ , then  $\mathcal{F}_\phi f \in L^2(\mathbb{R})$  and*

$$\|\mathcal{F}_\phi f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R}, d\phi)}.$$

*More generally, if  $f, g \in L^1(\mathbb{R}, d\phi) \cap L^2(\mathbb{R}, d\phi)$ , then*

$$\langle \mathcal{F}_\phi f, \mathcal{F}_\phi g \rangle = \langle f, g \rangle_\phi.$$

*Proof.* Both results follow from Proposition 3.5 together with the classical Plancherel's theorem:

$$\|f\|_{L^2(\mathbb{R}, d\phi)} = \|Q_\phi^{-1} f\|_{L^2(\mathbb{R})} = \|\mathcal{F} \circ Q_\phi^{-1} f\|_{L^2(\mathbb{R})} = \|\mathcal{F}_\phi f\|_{L^2(\mathbb{R})},$$

and

$$\langle f, g \rangle_\phi = \langle Q_\phi^{-1} f, Q_\phi^{-1} g \rangle = \langle \mathcal{F} \circ Q_\phi^{-1} f, \mathcal{F} \circ Q_\phi^{-1} g \rangle = \langle \mathcal{F}_\phi f, \mathcal{F}_\phi g \rangle,$$

where in both cases we have used the expression (6) for  $\mathcal{F}_\phi$  as a composition of operators.  $\square$

**Corollary 3.7.** *The Fourier transform with respect to  $\phi$  can be extended continuously to an isometry from the space  $L^2(\mathbb{R}, d\phi)$  to the space  $L^2(\mathbb{R})$ .*

*Proof.* Weighted  $L^p$  spaces act the same way as  $L^p$  spaces in that  $L^1(\mathbb{R}, d\phi) \cap L^2(\mathbb{R}, d\phi)$  is dense in  $L^2(\mathbb{R}, d\phi)$ . Then the result follows from Theorem 3.6.  $\square$

Having established how the operator  $Q_\phi$ , and by extension the Fourier transform with respect to  $\phi$ , acts on  $L^p$  space, we can now turn our attention to Schwartz spaces. In order to understand the action of these operators on Schwartz-type spaces, we need to introduce a new space, which we can think of as a  $\phi$ -Schwartz space or a Schwartz space with respect to  $\phi$ .

**Definition 3.8.** Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function with  $\phi' > 0$  almost everywhere and  $\phi(x) \rightarrow \pm\infty$  as  $x \rightarrow \pm\infty$  respectively (i.e., the same assumptions on  $\phi$  as in the work above, with the extra assumption of being a  $C^\infty$  function). The Schwartz space with respect to  $\phi$  is defined as

$$\mathcal{S}_\phi(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}) \text{ such that } \sup_{x \in \mathbb{R}} \left| \phi(x)^k \left( \frac{d}{d\phi(x)} \right)^\ell f(x) \right| < \infty \text{ for all } k, \ell \in \mathbb{N} \right\},$$

where  $\frac{d}{d\phi(x)} = \frac{1}{\phi'(x)} \cdot \frac{d}{dx}$  is the operator of differentiation with respect to  $\phi(x)$ . On this space we can define seminorms by

$$p_{N, \phi}(f) = \sup_{x \in \mathbb{R}} \left( 1 + |\phi(x)| \right)^N \sum_{\ell=0}^N \left| \left( \frac{d}{d\phi(x)} \right)^\ell f(x) \right|, \quad f \in \mathcal{S}_\phi(\mathbb{R}^n),$$

where  $N$  can take any value in  $\mathbb{Z}_0^+$ . This family of seminorms gives rise to a metric and a topology in the same way as the corresponding family of seminorms on the original Schwartz space  $\mathcal{S}(\mathbb{R}^n)$ .

**Theorem 3.9.** *If  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function with  $\phi' > 0$  almost everywhere and  $\phi(x) \rightarrow \pm\infty$  as  $x \rightarrow \pm\infty$  respectively, then the map*

$$Q_\phi : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}_\phi(\mathbb{R})$$

*is a linear isometry.*

*Proof.* Let  $f \in \mathcal{S}(\mathbb{R})$ . This inclusion is equivalent to

$$\sup_{x \in \mathbb{R}} \left| x^k f^{(\ell)}(x) \right| < \infty \quad \forall k, \ell \in \mathbb{N}.$$

Since  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is a bijection, taking the supremum over all  $x \in \mathbb{R}$  is equivalent to taking the supremum over all  $\phi(x) \in \mathbb{R}$ . So the above is equivalent to

$$\sup_{x \in \mathbb{R}} \left| \phi(x)^k f^{(\ell)}(\phi(x)) \right| < \infty \quad \forall k, \ell \in \mathbb{N}.$$

It is known that the operator of differentiation with respect to  $\phi(x)$  can be expressed as a conjugation of the operator of standard differentiation with respect to  $x$ , namely as follows:

$$\frac{d}{d\phi(x)} = Q_\phi \circ \frac{d}{dx} \circ Q_\phi^{-1}.$$

Thus, by composition of operators, we have for any  $\ell \in \mathbb{N}$  that

$$\left( \frac{d}{d\phi(x)} \right)^\ell = Q_\phi \circ \frac{d^\ell}{dx^\ell} \circ Q_\phi^{-1},$$

and therefore

$$f^{(\ell)}(\phi(x)) = Q_\phi \circ \frac{d^\ell}{dx^\ell}(f)(x) = \left( \frac{d}{d\phi(x)} \right)^\ell \circ Q_\phi(f)(x) = \left( \frac{d}{d\phi(x)} \right)^\ell (f \circ \phi)(x).$$

This means the inclusion  $f \in \mathcal{S}(\mathbb{R})$  is equivalent to

$$\sup_{x \in \mathbb{R}} \left| \phi(x)^k \left( \frac{d}{d\phi(x)} \right)^\ell (f \circ \phi)(x) \right| < \infty \quad \forall k, \ell \in \mathbb{N},$$

which, by the definition above, is equivalent to  $f \circ \phi \in \mathcal{S}_\phi(\mathbb{R})$ , i.e.  $Q_\phi f \in \mathcal{S}_\phi(\mathbb{R})$ .

Now we have proved that  $Q_\phi$  is a linear bijection  $\mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}_\phi(\mathbb{R})$ . To show it is an isometry, it suffices to prove that the respective seminorms correspond via the identity  $p_{N,\phi}(Q_\phi f) = p_N(f)$ . This can be done in a straightforward way, as follows:

$$\begin{aligned} p_{N,\phi}(Q_\phi f) &= \sup_{x \in \mathbb{R}} \left( 1 + |\phi(x)| \right)^N \sum_{\ell=0}^N \left| \left( \frac{d}{d\phi(x)} \right)^\ell \circ Q_\phi(f)(x) \right| \\ &= \sup_{x \in \mathbb{R}} \left( 1 + |\phi(x)| \right)^N \sum_{\ell=0}^N \left| Q_\phi \circ \left( \frac{d}{dx} \right)^\ell (f)(x) \right| \\ &= \sup_{x \in \mathbb{R}} \left( 1 + |\phi(x)| \right)^N \sum_{\ell=0}^N \left| f^{(\ell)}(\phi(x)) \right| \\ &= \sup_{x \in \mathbb{R}} (1 + |x|)^N \sum_{\ell=0}^N \left| f^{(\ell)}(x) \right| = p_N(f), \end{aligned}$$

where in the last line we used again the bijectivity of  $\phi$ . □

**Corollary 3.10.** *Let  $\phi$  be a smooth function with  $\phi' > 0$  almost everywhere and  $\phi(x) \rightarrow \pm\infty$  as  $x \rightarrow \pm\infty$  respectively. For any  $p \in [1, \infty)$ , the new Schwartz-type space  $\mathcal{S}_\phi(\mathbb{R})$  is dense in the weighted  $L^p$  space  $L^p(\mathbb{R}, d\phi)$ .*

*Proof.* Since  $Q_\phi$  maps the original Schwartz space and  $L^p$  spaces isometrically onto the new Schwartz space and weighted  $L^p$  spaces, this follows immediately from the well-known fact that  $\mathcal{S}(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$  for  $1 \leq p < \infty$ . □

**Remark 3.11.** It is also clear that the space of test functions (smooth compactly supported functions) on  $\mathbb{R}$  is contained in the Schwartz space  $\mathcal{S}_\phi(\mathbb{R})$  with respect to any smooth bijective function  $\phi$  as above.

**Theorem 3.12.** *Let  $\phi$  be a smooth function with  $\phi' > 0$  almost everywhere and  $\phi(x) \rightarrow \pm\infty$  as  $x \rightarrow \pm\infty$  respectively. The Fourier transform with respect to  $\phi$  is a linear isomorphism from  $\mathcal{S}_\phi(\mathbb{R})$  to  $\mathcal{S}(\mathbb{R})$ .*

*Proof.* We know that

$$\mathcal{F} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R}) \quad \text{and} \quad Q_\phi : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}_\phi(\mathbb{R})$$

are both linear isomorphisms. So, by (6), we have

$$\mathcal{F}_\phi = \mathcal{F} \circ Q_\phi^{-1} : \mathcal{S}_\phi(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R}),$$

the composition of two linear isomorphisms, being another linear isomorphism.  $\square$

Finally, let us define a new type of convolution operator which is well suited for the setting of Fourier transforms with respect to a function. The following definition is by analogy with the Laplace-type convolution operator with respect to a function (4), but here we use an infinite interval of integration, to obtain a Fourier-type convolution operator with respect to a function.

**Definition 3.13.** Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a function which is differentiable with  $\phi' > 0$  almost everywhere and  $\phi(x) \rightarrow \pm\infty$  as  $x \rightarrow \pm\infty$  respectively. The Fourier-type  $\phi$ -convolution of two functions  $f$  and  $g$  in the space  $L^1(\mathbb{R}, d\phi)$  is defined as follows:

$$h(x) = (f *_\phi g)(x) = \int_{-\infty}^{\infty} f(\phi^{-1}(\phi(x) - \phi(y)))g(y)\phi'(y) dy.$$

The Fourier-type  $\phi$ -convolution can be expressed as a conjugation, via  $Q_\phi$ , of the standard Fourier-type convolution, in the same way as was done for Laplace-type convolutions in [20]. We state the result as follows, since it is useful in understanding the interaction of the Fourier-type  $\phi$ -convolution with other operations and function spaces.

**Proposition 3.14.** *For any  $f, g \in L^1(\mathbb{R}, d\phi)$ , we have*

$$f *_\phi g = Q_\phi \left( (Q_\phi^{-1}f) * (Q_\phi^{-1}g) \right),$$

where  $*$  is the usual Fourier-type convolution operator. In other words, treating  $*$  and  $*_\phi$  as binary operations on pairs of functions, we have

$$*_\phi = (Q_\phi) \circ (*) \circ (Q_\phi^{-1}, Q_\phi^{-1}).$$

*Proof.* This follows from simple integral substitution:

$$\left( (Q_\phi^{-1}f) * (Q_\phi^{-1}g) \right)(x) = \int_{-\infty}^{\infty} f(\phi^{-1}(x - y))g(\phi^{-1}(y)) dy,$$

so

$$Q_\phi \left( (Q_\phi^{-1}f) * (Q_\phi^{-1}g) \right)(x) = \int_{-\infty}^{\infty} f(\phi^{-1}(\phi(x) - y))g(\phi^{-1}(y)) dy,$$

which is the same as the formula for  $f *_\phi g(x)$  after substituting  $y = \phi(u)$ .  $\square$

**Proposition 3.15** (Young's inequality for convolutions with respect to a function). *If  $p, q, r \in [1, \infty]$  with  $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ , and  $f \in L^p(\mathbb{R}, d\phi)$  and  $g \in L^q(\mathbb{R}, d\phi)$ , then  $f *_\phi g \in L^r(\mathbb{R}, d\phi)$  with*

$$\|f *_\phi g\|_{L^r(\mathbb{R}, d\phi)} \leq \|f\|_{L^p(\mathbb{R}, d\phi)} \|g\|_{L^q(\mathbb{R}, d\phi)}.$$

*Proof.* This follows from the classical Young's inequality for convolutions in  $L^p$  spaces, together with Proposition 3.5 and Proposition 3.14.  $\square$

**Proposition 3.16** (Fourier convolution theorem with respect to a function). *The Fourier transform with respect to  $\phi$  transforms convolution with respect to  $\phi$  into multiplication:*

$$\mathcal{F}_\phi(f *_\phi g) = (\mathcal{F}_\phi f)(\mathcal{F}_\phi g),$$

for  $f, g$  in  $L^1(\mathbb{R}, d\phi)$  or any other suitable  $L^p, L^q$  spaces.

*Proof.* This follows from the classical Fourier convolution theorem, together with Proposition 3.5, the identity (6), and Proposition 3.14.  $\square$

**Remark 3.17.** In Definition 3.1 and throughout this subsection, we have assumed that  $\phi$  is an increasing function, with positive derivative and appropriate infinite limits.

This assumption is inspired by the corresponding assumption necessary for Laplace transforms with respect to a function, in which case it is usually assumed that  $\phi(0) = 0$  and  $\phi(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . This makes sense for Laplace transforms, where initial values play an important role in the structure and applications of the transform.

For Fourier transforms, however, there is no special finite value in the integral, and the classical Fourier transform can be written as

$$[\mathcal{F}(f)](k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} f(x) dx,$$

integrating over the whole space of real numbers. This way of writing it also extends naturally to the  $n$ -dimensional Fourier transform, where the integration is over the whole space  $\mathbb{R}^n$ .

An alternative assumption we could consider for  $\phi$ , to define Fourier transforms with respect to  $\phi$ , is simply that  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is a bijection which is differentiable almost everywhere. This allows for monotonically decreasing functions as well as monotonically increasing ones, and all the results above hold true, with the caveat that we must use  $|\phi'|$  instead of  $\phi'$  as a factor in the integrals.

Instead of verifying once again the proofs above under this weaker assumption on  $\phi$ , we shall now pass to the  $n$ -dimensional case, in which the results of this section with a general differentiable bijection  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  will appear as special cases.

### 3.2. The $n$ -dimensional case.

**Definition 3.18.** Let  $n \in \mathbb{N}$ , and let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a bijection all of whose 1st-order partial derivatives exist almost everywhere.

The  $n$ -dimensional Fourier transform with respect to  $\phi$ , of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  or  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  such that this integral exists, is defined to be

$$[\mathcal{F}_\phi f](\mathbf{k}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{k} \cdot \phi(\mathbf{x})} f(\mathbf{x}) |J\phi(\mathbf{x})| d\mathbf{x},$$

where  $\mathbf{k} \in \mathbb{R}^n$  is the variable of the Fourier domain, and  $J\phi$  is the Jacobian determinant of the function  $\phi$ .

**Proposition 3.19.** *The  $n$ -dimensional Fourier transform with respect to a function is related to the classical Fourier transform by the following relation between operators:*

$$\mathcal{F}_\phi = \mathcal{F} \circ Q_\phi^{-1}, \tag{7}$$

where  $Q_\phi$  is the operator of right composition with  $\phi$ , as defined in (3), but this time acting on the space of functions defined on  $\mathbb{R}^n$ .

*Proof.* This follows directly from substitution in the integral:

$$\begin{aligned}\mathcal{F} \circ Q_\phi^{-1} f(\mathbf{k}) &= \mathcal{F} [f \circ \phi^{-1}] (\mathbf{k}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{k} \cdot \mathbf{x}} f(\phi^{-1}(\mathbf{x})) \, d\mathbf{x} \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{k} \cdot \phi(\mathbf{u})} f(\mathbf{u}) |J\phi(\mathbf{u})| \, d\mathbf{u},\end{aligned}$$

using the change of variables  $\mathbf{x} = \phi(\mathbf{u})$ . Here we have made use of the fact that  $\phi$  is a bijection from  $\mathbb{R}^n$  to itself, as well as the Jacobian rule for change of variables in an integral over an  $n$ -dimensional region.  $\square$

**Proposition 3.20.** *The  $n$ -dimensional inverse Fourier transform with respect to  $\phi$  is given by*

$$[\mathcal{F}_\phi^{-1} g](\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\mathbf{k} \cdot \phi(\mathbf{x})} g(\mathbf{k}) \, d\mathbf{k};$$

*in other words, we have  $g = \mathcal{F}_\phi f \iff f = \mathcal{F}_\phi^{-1} g$ .*

*Proof.* As before, this follows from the operational relation in Proposition 3.19 as well as the form of the classical  $n$ -dimensional inverse Fourier transform: the inverse of  $\mathcal{F} \circ Q_\phi^{-1}$  should be  $Q_\phi \circ \mathcal{F}^{-1}$ .  $\square$

**Theorem 3.21.** *The  $n$ -dimensional Fourier transform with respect to  $\phi$  is a bounded operator from  $L^1(\mathbb{R}^n, d\phi)$  to  $L^\infty(\mathbb{R}^n)$ .*

*Proof.* Note firstly that  $d\phi$  is a positive measure, since

$$\int_A d\phi = \int_{\phi^{-1}(A)} |J\phi(\mathbf{x})| \, d\mathbf{x}$$

for any measurable set  $A \subseteq \mathbb{R}^n$ , where we take the absolute value of the Jacobian determinant according to the standard result on changing variables in  $n$ -dimensional integrals.

If  $f \in L^1(\mathbb{R}^n, d\phi)$ , then, for all  $\mathbf{k} \in \mathbb{R}^n$ ,

$$\begin{aligned}\left| [\mathcal{F}_\phi f](\mathbf{k}) \right| &= \left| \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{k} \cdot \phi(\mathbf{x})} f(\mathbf{x}) |J\phi(\mathbf{x})| \, d\mathbf{x} \right| \\ &\leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |e^{-i\mathbf{k} \cdot \phi(\mathbf{x})} f(\mathbf{x})| |J\phi(\mathbf{x})| \, d\mathbf{x} \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |f(\mathbf{x})| |J\phi(\mathbf{x})| \, d\mathbf{x} \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |f(\mathbf{x})| \, d\phi(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \|f\|_{L^1(\mathbb{R}^n, d\phi)}.\end{aligned}$$

Taking the supremum over  $\mathbf{k}$ , this means

$$\|\mathcal{F}_\phi f\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(2\pi)^{n/2}} \|f\|_{L^1(\mathbb{R}^n, d\phi)}.$$

$\square$

**Proposition 3.22.** *Let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a function as in Definition 3.18, and define  $Q_\phi$  by (3) as the operation of composition with  $\phi$ . For any  $p \in [1, \infty]$ , the map*

$$Q_\phi : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n, d\phi)$$

is a linear isometry. For  $p = 2$  specifically, this map is also an isometry between inner product spaces, where the inner product  $\langle \cdot, \cdot \rangle_\phi$  on the weighted  $L^2$  space is defined in the natural way by

$$\langle f, g \rangle_\phi = \int_{\mathbb{R}^n} f(\mathbf{x}) \overline{g(\mathbf{x})} |J\phi(\mathbf{x})| \, d\mathbf{x},$$

and the complex conjugation can be ignored if we assume real-valued functions  $f$  and  $g$ .

*Proof.* Linearity is clear. To prove it is an isometry, we substitute in the integral as follows:

$$\|Q_\phi f\|_{L^p(\mathbb{R}^n, d\phi)}^p = \int_{\mathbb{R}^n} |f(\phi(\mathbf{u}))|^p |J\phi(\mathbf{u})| \, d\mathbf{u} = \int_{\mathbb{R}^n} |f(\mathbf{x})|^p \, d\mathbf{x} = \|f\|_{L^p(\mathbb{R}^n)}^p,$$

using the change of variables  $x = \phi(u)$ . Here we have made use of the fact that  $\phi$  is a bijection from  $\mathbb{R}^n$  to itself, as well as the Jacobian rule for change of variables in an integral over an  $n$ -dimensional region. In the case  $p = 2$ , we can similarly prove that  $Q_\phi$  preserves the inner product as follows:

$$\begin{aligned} \langle Q_\phi f, Q_\phi g \rangle_\phi &= \int_{\mathbb{R}^n} Q_\phi f(\mathbf{u}) \overline{Q_\phi g(\mathbf{u})} |J\phi(\mathbf{u})| \, d\mathbf{u} = \int_{\mathbb{R}^n} f(\phi(\mathbf{u})) \overline{g(\phi(\mathbf{u}))} |J\phi(\mathbf{u})| \, d\mathbf{u} \\ &= \int_{\mathbb{R}^n} f(\mathbf{x}) \overline{g(\mathbf{x})} \, d\mathbf{x} = \langle f, g \rangle. \end{aligned}$$

□

**Theorem 3.23** (Plancherel's theorem in  $n$  dimensions with respect to  $\phi$ ). *If  $f \in L^1(\mathbb{R}^n, d\phi) \cap L^2(\mathbb{R}^n, d\phi)$ , then  $\mathcal{F}_\phi f \in L^2(\mathbb{R}^n)$  and*

$$\|\mathcal{F}_\phi f\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n, d\phi)}.$$

More generally, if  $f, g \in L^1(\mathbb{R}^n, d\phi) \cap L^2(\mathbb{R}^n, d\phi)$ , then

$$\langle \mathcal{F}_\phi f, \mathcal{F}_\phi g \rangle = \langle f, g \rangle_\phi.$$

*Proof.* Both results follow from Proposition 3.22 together with the classical  $n$ -dimensional Plancherel's theorem:

$$\|f\|_{L^2(\mathbb{R}^n, d\phi)} = \|Q_\phi^{-1} f\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F} \circ Q_\phi^{-1} f\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F}_\phi f\|_{L^2(\mathbb{R}^n)},$$

and

$$\langle f, g \rangle_\phi = \langle Q_\phi^{-1} f, Q_\phi^{-1} g \rangle = \langle \mathcal{F} \circ Q_\phi^{-1} f, \mathcal{F} \circ Q_\phi^{-1} g \rangle = \langle \mathcal{F}_\phi f, \mathcal{F}_\phi g \rangle,$$

where in both cases we have used the expression (7) for  $\mathcal{F}_\phi$  as a composition of operators. □

**Corollary 3.24.** *The Fourier transform with respect to  $\phi$  can be extended continuously to an isometry from the space  $L^2(\mathbb{R}^n, d\phi)$  to the space  $L^2(\mathbb{R}^n)$ .*

*Proof.* Weighted  $L^p$  spaces act the same way as  $L^p$  spaces in that  $L^1(\mathbb{R}^n, d\phi) \cap L^2(\mathbb{R}^n, d\phi)$  is dense in  $L^2(\mathbb{R}^n, d\phi)$ . Then the result follows from Theorem 3.6. □

Finally, let us define the  $n$ -dimensional Fourier convolution operator with respect to a function  $\phi$  and establish some of its basic properties, similar to what we did in the previous subsection.

**Definition 3.25.** Let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a function as in Definition 3.18. The Fourier-type  $\phi$ -convolution of two functions  $f$  and  $g$  in the space  $L^1(\mathbb{R}^n, d\phi)$  is defined as follows:

$$h(\mathbf{x}) = (f *_\phi g)(\mathbf{x}) = \int_{\mathbb{R}^n} f(\phi^{-1}(\phi(\mathbf{x}) - \phi(\mathbf{y}))) g(\mathbf{y}) |J\phi(\mathbf{y})| \, d\mathbf{y}.$$

**Proposition 3.26.** *For any  $f, g \in L^1(\mathbb{R}^n, d\phi)$ , we have*

$$f *_{\phi} g = Q_{\phi} \left( (Q_{\phi}^{-1} f) * (Q_{\phi}^{-1} g) \right),$$

where  $*$  is the usual Fourier-type convolution operator. In other words, treating  $*$  and  $*_{\phi}$  as binary operations on pairs of functions, we have

$$*_{\phi} = (Q_{\phi}) \circ (*) \circ (Q_{\phi}^{-1}, Q_{\phi}^{-1}).$$

*Proof.* This follows from simple integral substitution:

$$\left( (Q_{\phi}^{-1} f) * (Q_{\phi}^{-1} g) \right) (\mathbf{x}) = \int_{\mathbb{R}^n} f(\phi^{-1}(\mathbf{x} - \mathbf{y})) g(\phi^{-1}(\mathbf{y})) d\mathbf{y},$$

so

$$Q_{\phi} \left( (Q_{\phi}^{-1} f) * (Q_{\phi}^{-1} g) \right) (\mathbf{x}) = \int_{-\infty}^{\infty} f(\phi^{-1}(\phi(\mathbf{x}) - \mathbf{y})) g(\phi^{-1}(\mathbf{y})) d\mathbf{y},$$

which is the same as the formula for  $f *_{\phi} g(\mathbf{x})$  after substituting  $\mathbf{y} = \phi(\mathbf{u})$ .  $\square$

**Proposition 3.27** (Young's inequality for convolutions with respect to a function). *If  $p, q, r \in [1, \infty]$  with  $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ , and  $f \in L^p(\mathbb{R}^n, d\phi)$  and  $g \in L^q(\mathbb{R}^n, d\phi)$ , then  $f *_{\phi} g \in L^r(\mathbb{R}^n, d\phi)$  with*

$$\|f *_{\phi} g\|_{L^r(\mathbb{R}^n, d\phi)} \leq \|f\|_{L^p(\mathbb{R}^n, d\phi)} \|g\|_{L^q(\mathbb{R}^n, d\phi)}.$$

*Proof.* This follows from the classical Young's inequality for convolutions in  $L^p$  spaces, together with Proposition 3.22 and Proposition 3.26.  $\square$

**Proposition 3.28** (Fourier convolution theorem with respect to a function). *The Fourier transform with respect to  $\phi$  transforms convolution with respect to  $\phi$  into multiplication:*

$$\mathcal{F}_{\phi}(f *_{\phi} g) = (\mathcal{F}_{\phi} f)(\mathcal{F}_{\phi} g),$$

for  $f, g$  in  $L^1(\mathbb{R}^n, d\phi)$  or any other suitable  $L^p, L^q$  spaces.

*Proof.* This follows from the classical Fourier convolution theorem, together with Proposition 3.22, the identity (7), and Proposition 3.26.  $\square$

**Remark 3.29.** As a special case of the results in this subsection, putting  $n = 1$ , we obtain all the results of §3.1 in the extended case where  $\phi$  can be any differentiable bijection of  $\mathbb{R}$  to itself, not necessarily an increasing one.

#### 4. FRACTIONAL LAPLACIAN WITH RESPECT TO A FUNCTION

**4.1. Marchaud operators and the 1-dimensional case.** As in the previous section, we begin with the 1-dimensional case, where it is easier to understand what happens, before moving on later to the general case of  $n$ -dimensional functions and operators.

The Marchaud derivatives are known [28] for their connection to the fractional Laplacian. They are defined as follows [12, §5.4]:

$${}^M D_x^{\alpha} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^x \frac{f(x) - f(t)}{(x-t)^{\alpha+1}} dt; \quad (8)$$

$${}^M D_x^{\alpha} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(x) - f(x+t)}{t^{1+\alpha}} dt = \frac{\alpha}{\Gamma(1-\alpha)} \int_x^{\infty} \frac{f(x) - f(t)}{(t-x)^{\alpha+1}} dt, \quad (9)$$

where  $x \in \mathbb{R}$  and  $0 < \alpha < 1$  (similar definitions can also be used for  $\alpha > 1$  [12, §5.6]). Note that the Marchaud derivatives are equivalent to the left and right Riemann–Liouville derivatives,  ${}^{RL} D_x^{\alpha} f(x)$  and  ${}^{RL} D_x^{\alpha} f(x)$  respectively, under sufficiently good behaviour assumptions on the function  $f$ .

A key observation, for our work in this paper, is that the intervals used for the integrals (8)–(9) are chosen precisely so that the base of the fractional power ( $x - t$  in the first integral,  $t - x$  in the second) is always a positive real number. This ensures that, whenever  $f$  is a real function and  $\alpha$  is a real parameter, the Marchaud derivatives are also real, involving fractional powers only of positive real numbers.

The relationship between the Marchaud derivatives and the 1-dimensional fractional Laplacian is given by the following identity:

$${}^M_+D_x^\alpha f(x) + {}^M_-D_x^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{\infty} \frac{f(x) - f(t)}{|x-t|^{\alpha+1}} dt,$$

which is valid because the variables  $x$  and  $t$  are real, and so  $|x - t|$  equals either  $x - t$  or  $t - x$  according to whether  $t \leq x$  or  $t \geq x$ . This new expression is a constant multiple of the 1-dimensional fractional Laplacian, which is defined, using the appropriate normalisation constant according to (2), as follows:

$$(-\Delta)^{\alpha/2} f(x) = \frac{\alpha 2^{\alpha-1} \Gamma(\frac{\alpha+1}{2})}{\sqrt{\pi} \Gamma(1-\frac{\alpha}{2})} \int_{-\infty}^{\infty} \frac{f(x) - f(t)}{|x-t|^{\alpha+1}} dt.$$

Thus, we have

$${}^M_+D_x^\alpha f(x) + {}^M_-D_x^\alpha f(x) = \frac{\sqrt{\pi} \Gamma(1-\frac{\alpha}{2})}{2^{\alpha-1} \Gamma(1-\alpha) \Gamma(\frac{\alpha+1}{2})} (-\Delta)^{\alpha/2} f(x), \quad (10)$$

as discussed in [28]. Making use of the reflection formula and Legendre duplication formula for the gamma function, the normalisation constant here can be greatly simplified so as not to involve gamma functions at all:

$${}^M_+D_x^\alpha f(x) + {}^M_-D_x^\alpha f(x) = 2 \cos\left(\frac{\pi\alpha}{2}\right) \cdot (-\Delta)^{\alpha/2} f(x).$$

The above discussion motivates the following definitions, for the Marchaud derivatives and 1-dimensional fractional Laplacian both taken with respect to a function  $\phi$ .

**Definition 4.1** (Marchaud fractional derivatives with respect to a function). If  $\phi : (-\infty, c) \rightarrow \mathbb{R}$  is a function which is differentiable with  $\phi' > 0$  almost everywhere and  $\phi(x) \rightarrow -\infty$  as  $x \rightarrow -\infty$ , then the lower Marchaud fractional derivative with respect to  $\phi$ , to order  $\alpha \in (0, 1)$ , of a function  $f$  defined on  $(-\infty, c)$ , is defined to be

$${}^M_+D_{\phi(x)}^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^x \frac{f(x) - f(t)}{(\phi(x) - \phi(t))^{\alpha+1}} \phi'(t) dt, \quad (11)$$

where  $x \in (-\infty, c)$ .

If  $\phi : (c, \infty) \rightarrow \mathbb{R}$  is a function which is differentiable with  $\phi' > 0$  almost everywhere and  $\phi(x) \rightarrow \infty$  as  $x \rightarrow \infty$ , then the upper Marchaud fractional derivative with respect to  $\phi$ , to order  $\alpha \in (0, 1)$ , of a function  $f$  defined on  $(c, \infty)$ , is defined to be

$${}^M_-D_{\phi(x)}^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_x^{\infty} \frac{f(x) - f(t)}{(\phi(t) - \phi(x))^{\alpha+1}} \phi'(t) dt, \quad (12)$$

where  $x \in (c, \infty)$ .

**Definition 4.2** (Fractional Laplacian in 1 dimension with respect to a function). If  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is a bijection which is differentiable almost everywhere, then the 1-dimensional fractional Laplacian with respect to  $\phi$ , of a function  $f$  defined on  $\mathbb{R}$ , is

$$(-\Delta)_{\phi(x)}^{\alpha/2} f(x) = \frac{\alpha 2^{\alpha-1} \Gamma(\frac{\alpha+1}{2})}{\sqrt{\pi} \Gamma(1-\frac{\alpha}{2})} \int_{-\infty}^{\infty} \frac{f(x) - f(t)}{|\phi(x) - \phi(t)|^{\alpha+1}} |\phi'(t)| dt, \quad (13)$$



where  $0 < \alpha < 2$  and  $x \in \mathbb{R}$ .

**Remark 4.3.** Note that Definition 4.1 and Definition 4.2 rely on different assumptions on the function  $\phi$ .

This is because the Marchaud formulae (11)–(12) both rely on assumptions about the sign of the difference between  $\phi(x)$  and  $\phi(t)$ , if we want the whole expression to be real when  $f$ ,  $x$ , and  $\alpha$  are real. Indeed, for (11) we have  $-\infty < t \leq x$  and expect  $\phi(x) - \phi(t)$  to be positive real, while for (12) we have  $x \leq t < \infty$  and expect  $\phi(t) - \phi(x)$  to be positive real. In both cases, therefore, it is reasonable to restrict  $\phi$  to be monotonically increasing.

Such a restriction is not necessary for the fractional Laplacian formula (13), since there we only care about the *absolute* difference between  $\phi(x)$  and  $\phi(t)$ , regardless of the values of  $t$  and  $x$  relative to each other. We do still require  $\phi$  to be bijective (therefore either monotonically increasing or monotonically decreasing), because in the work below we shall be transforming integrals over  $\mathbb{R}$  by the substitution  $x = \phi(u)$ .

**Theorem 4.4.** *The Marchaud derivatives with respect to a function, and the fractional Laplacian with respect to a function, are related to the classical Marchaud derivatives and fractional Laplacian, by the following relations between operators:*

$$\begin{aligned} {}_+^M D_{\phi(x)}^\alpha &= Q_\phi \circ {}_+^M D_x^\alpha \circ Q_\phi^{-1}, \\ {}_-^M D_{\phi(x)}^\alpha &= Q_\phi \circ {}_-^M D_x^\alpha \circ Q_\phi^{-1}, \\ (-\Delta)_{\phi(x)}^{\alpha/2} &= Q_\phi \circ (-\Delta)^{\alpha/2} \circ Q_\phi^{-1}, \end{aligned}$$

where in each case the appropriate assumptions on  $\phi$  and  $\alpha$  are imposed, according to Definitions 4.1 and 4.2.

*Proof.* This follows from a simple substitution in the integral formula, but this composition of operators is slightly hard to notate, so we verify it step by step for the case of the lower Marchaud fractional derivatives:

$$\begin{aligned} f &: x \mapsto f(x); \\ Q_\phi^{-1} f &: x \mapsto f(\phi^{-1}(x)); \\ {}_+^M D_x^\alpha \circ Q_\phi^{-1} f &: x \mapsto \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^x \frac{f(\phi^{-1}(x)) - f(\phi^{-1}(t))}{(x-t)^{\alpha+1}} dt; \\ Q_\phi \circ {}_+^M D_x^\alpha \circ Q_\phi^{-1} f &: x \mapsto \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{\phi(x)} \frac{f(x) - f(\phi^{-1}(t))}{(\phi(x)-t)^{\alpha+1}} dt \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^x \frac{f(x) - f(u)}{(\phi(x)-\phi(u))^{\alpha+1}} \phi'(u) du = {}_+^M D_{\phi(x)}^\alpha f(x), \end{aligned}$$

where in the last step we used the change of variables  $t = \phi(u)$ .

Thus, the result is proved for the lower Marchaud fractional derivative; the proofs for the upper Marchaud fractional derivative and for the fractional Laplacian are similar. Note that, in the case of the fractional Laplacian, the multiplier is  $|\phi'|$  instead of  $\phi'$ , because in this case it is no longer assumed that  $\phi$  is strictly increasing; it may also be strictly decreasing, reversing the direction of the interval  $(-\infty, \infty)$  with the negative multiplier  $\phi'$ .  $\square$

The result of Theorem 4.4 follows the same lines as many other theorems expressing fractional-calculus operators with respect to a function  $\phi(x)$  as conjugations, via  $Q_\phi$ , of the corresponding operators with respect to  $x$ . Such results are one of the most important aspects of the whole theory

of fractional calculus with respect to functions, as they enable many results to be proved directly from the corresponding known results for the original operators which are not taken with respect to functions [29, 20]. Theorem 4.4 itself justifies our specific choices of formula (11)–(12) and (13) used in the Definitions above.

**Corollary 4.5.** *Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a bijection which is differentiable almost everywhere, and let  $0 < \alpha < 2$ . The operator  $(-\Delta)_{\phi(x)}^{\alpha/2}$ , the fractional Laplacian with respect to  $\phi$ , is defined on the function space  $L^p(\mathbb{R}, d\phi)$  for any  $p$  with  $1 \leq p \leq 2$ .*

*Proof.* This follows from Proposition 3.5 together with Theorem 4.4: firstly  $Q_\phi^{-1}$  maps the space  $L^p(\mathbb{R}, d\phi)$  isometrically to  $L^p(\mathbb{R})$ , and on this space the fractional Laplacian  $(-\Delta)^{\alpha/2}$  is well-defined.  $\square$

**Corollary 4.6.** *Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a bijection which is smooth almost everywhere, and let  $0 < \alpha < 2$ . The operator  $(-\Delta)_{\phi(x)}^{\alpha/2}$ , the fractional Laplacian with respect to  $\phi$ , is defined on the  $\phi$ -Schwartz space  $\mathcal{S}_\phi(\mathbb{R})$  and therefore on the space of test functions (smooth compactly supported functions) on  $\mathbb{R}$ .*

*Proof.* This can be proved similarly to Corollary 4.5, using Theorem 3.9 together with Theorem 4.4. It can also be seen as a direct consequence of Corollary 4.5, since the space of  $\phi$ -Schwartz functions is dense in the  $\phi$ -weighted  $L^p$  spaces by Corollary 3.10. The space of test functions is contained in  $\mathcal{S}_\phi(\mathbb{R})$  by Remark 3.11.  $\square$

**Proposition 4.7.** *If  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is a function which is differentiable with  $\phi' > 0$  almost everywhere and  $\phi(x) \rightarrow \pm\infty$  as  $x \rightarrow \pm\infty$  respectively, then the Marchaud derivatives with respect to  $\phi$  and the fractional Laplacian with respect to  $\phi$  are related by the following identity:*

$${}_+^M D_{\phi(x)}^\alpha f(x) + {}_-^M D_{\phi(x)}^\alpha f(x) = 2 \cos\left(\frac{\pi\alpha}{2}\right) \cdot (-\Delta)_{\phi(x)}^{\alpha/2} f(x),$$

for all  $\alpha \in (0, 1)$  and  $x \in \mathbb{R}$ .

*Proof.* Since  $\phi$  is a strictly increasing function, we have

$$|\phi(x) - \phi(t)| = \begin{cases} \phi(x) - \phi(t) & \text{if } t \leq x, \\ \phi(t) - \phi(x) & \text{if } t \geq x. \end{cases}$$

So the integral over  $\mathbb{R}$  can be split into two sub-intervals, just like in the previous discussion of Marchaud derivatives and fractional Laplacians without a function  $\phi$ :

$$\begin{aligned} 2 \cos\left(\frac{\pi\alpha}{2}\right) \cdot (-\Delta)_{\phi(x)}^{\alpha/2} f(x) &= \frac{\sqrt{\pi} \Gamma\left(1 - \frac{\alpha}{2}\right)}{2^{\alpha-1} \Gamma(1 - \alpha) \Gamma\left(\frac{\alpha+1}{2}\right)} (-\Delta)_{\phi(x)}^{\alpha/2} f(x) \\ &= \frac{\alpha}{\Gamma(1 - \alpha)} \int_{-\infty}^{\infty} \frac{f(x) - f(t)}{|\phi(x) - \phi(t)|^{\alpha+1}} |\phi'(t)| dt \\ &= \frac{\alpha}{\Gamma(1 - \alpha)} \int_{-\infty}^x \frac{f(x) - f(t)}{(\phi(x) - \phi(t))^{\alpha+1}} \phi'(t) dt + \frac{\alpha}{\Gamma(1 - \alpha)} \int_x^{\infty} \frac{f(x) - f(t)}{(\phi(t) - \phi(x))^{\alpha+1}} \phi'(t) dt \\ &= {}_+^M D_{\phi(x)}^\alpha f(x) + {}_-^M D_{\phi(x)}^\alpha f(x). \end{aligned}$$

Note that the assumption  $\phi' > 0$  was needed in this case for the Marchaud derivatives with respect to  $\phi$  to be well-defined, as well as for the correct splitting of the integral.  $\square$

**Theorem 4.8.** *Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a bijection which is differentiable almost everywhere. The fractional Laplacian with respect to  $\phi$  and the Fourier transform with respect to  $\phi$  are related by the*

following identity:

$$\mathcal{F}_\phi [(-\Delta)_{\phi(x)}^{\alpha/2} f(x)] = |k|^\alpha \mathcal{F}_\phi f(k),$$

for any  $f \in L^p(\mathbb{R}, d\phi)$ ,  $1 \leq p \leq 2$ .

*Proof.* This result is already known for the original fractional Laplacian, not with respect to a function, i.e. the case  $\phi(x) = x$ . We can deduce it in the general case by using Proposition 3.2 for the Fourier transform with respect to  $\phi$  and Theorem 4.4 for the fractional Laplacian with respect to  $\phi$ :

$$\mathcal{F}_\phi \circ (-\Delta)_{\phi(x)}^{\alpha/2} f = [\mathcal{F} \circ Q_\phi^{-1}] \circ [Q_\phi \circ (-\Delta)^{\alpha/2} \circ Q_\phi^{-1}] f = \mathcal{F} \circ (-\Delta)^{\alpha/2} [Q_\phi^{-1} f],$$

then applying the original result to the function  $g = f \circ \phi^{-1} = Q_\phi^{-1} f$ , we have immediately

$$\mathcal{F} \circ (-\Delta)^{\alpha/2} [Q_\phi^{-1} f](k) = |k|^\alpha \mathcal{F} [Q_\phi^{-1} f](k) = |k|^\alpha \mathcal{F}_\phi f(k),$$

which proves the result.  $\square$

**4.2. The  $n$ -dimensional case.** We have seen above that, in the 1-dimensional setting of  $\mathbb{R}$ , the fractional Laplacian can be written as a sum of Marchaud derivatives. In the general  $n$ -dimensional setting, the operators corresponding to Marchaud derivatives are discussed in [12, §24.6] and also in [28], being defined as follows:

$${}_{+;\xi}^M D_{\mathbf{x}}^\alpha f(\mathbf{x}) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(\mathbf{x}) - f(\mathbf{x} - t\xi)}{t^{1+\alpha}} dt; \quad (14)$$

$${}_{-;\xi}^M D_{\mathbf{x}}^\alpha f(\mathbf{x}) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(\mathbf{x}) - f(\mathbf{x} + t\xi)}{t^{1+\alpha}} dt, \quad (15)$$

where  $\mathbf{x} \in \mathbb{R}^n$  and  $0 < \alpha < 1$  (similar definitions can also be used for  $\alpha > 1$  [28]).

Note that, in the general  $n$ -dimensional case with  $n > 1$ , we cannot rewrite these formulae in a way analogous to the second expressions for the Marchaud derivatives in the  $n = 1$  case (8)–(9). The  $n$ -dimensional formulae (14)–(15) are defined by real integrals, and we cannot use a substitution to transform these to integrals over  $\mathbb{R}^n$ .

On the other hand, we do still have a relationship between these  $n$ -dimensional Marchaud analogues and the  $n$ -dimensional fractional Laplacian, analogous to (10). Namely, the following relation proved in [28]:

$$\begin{aligned} \int_{S^{n-1}} {}_{+;\xi}^M D_{\mathbf{x}}^\alpha f(\mathbf{x}) d\xi &= \frac{\pi^{n/2} \Gamma(1 - \frac{\alpha}{2})}{2^{\alpha-1} \Gamma(1-\alpha) \Gamma(\frac{\alpha+n}{2})} (-\Delta)^{\alpha/2} f(\mathbf{x}) \\ &= \frac{2\pi^{\frac{n-1}{2}} \Gamma(\frac{\alpha+1}{2}) \cos(\frac{\pi\alpha}{2})}{\Gamma(\frac{\alpha+n}{2})} (-\Delta)^{\alpha/2} f(\mathbf{x}), \end{aligned} \quad (16)$$

where the constant of integration comes from dividing  $\frac{\alpha}{\Gamma(1-\alpha)}$  by the normalisation constant  $C_{n,\alpha/2}$  defined in (2), and using the reflection formula and Legendre duplication formula for the gamma function to obtain the final expression. Note that using  ${}_{-;\xi}^M D_{\mathbf{x}}^\alpha$  instead of  ${}_{+;\xi}^M D_{\mathbf{x}}^\alpha$  would give the same result, since  ${}_{-;\xi}^M D_{\mathbf{x}}^\alpha = {}_{+;-\xi}^M D_{\mathbf{x}}^\alpha$  for all  $\xi \in \mathbb{R}^n$ , so there is symmetry in the relation.

**Definition 4.9** (Fractional Laplacian in  $n$  dimensions with respect to a function). Let  $n \in \mathbb{N}$ , and let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a bijection all of whose 1st-order partial derivatives exist almost everywhere. The  $n$ -dimensional fractional Laplacian with respect to  $\phi$ , of a function  $f$  defined on  $\mathbb{R}^n$ , is

$$(-\Delta)_{\phi(x)}^{\alpha/2} f(\mathbf{x}) = C_{n,\alpha/2} \text{P.V.} \int_{\mathbb{R}^n} \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\phi(\mathbf{x}) - \phi(\mathbf{y})|^{n+\alpha}} |J\phi(\mathbf{y})| d\mathbf{y}, \quad (17)$$

where  $0 < \alpha < 2$  and  $\mathbf{x} \in \mathbb{R}^n$ .

**Remark 4.10.** The above definition, for the  $n$ -dimensional fractional Laplacian with respect to  $\phi$ , will be justified below, using once again the method of conjugation of operators. It should be noted at this stage that the formula for the  $n$ -dimensional Marchaud-type derivatives (14)–(15) with respect to  $\phi$  is not so elegant as the 1-dimensional case, so this no longer acts as a motivation for the definition of the fractional Laplacian in  $n$  dimensions with respect to  $\phi$ . Indeed, the extensions of (14)–(15) to operators with respect to functions are given by:

$${}_{+;\xi}^M D_{\phi(\mathbf{x})}^{\alpha} f(\mathbf{x}) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(\mathbf{x}) - f(\phi^{-1}(\phi(\mathbf{x}) - t\xi))}{t^{1+\alpha}} dt; \quad (18)$$

$${}_{-;\xi}^M D_{\phi(\mathbf{x})}^{\alpha} f(\mathbf{x}) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(\mathbf{x}) - f(\phi^{-1}(\phi(\mathbf{x}) + t\xi))}{t^{1+\alpha}} dt. \quad (19)$$

We will justify these definitions later, both in terms of conjugation of operators and in terms of their relationship with the  $n$ -dimensional fractional Laplacian with respect to  $\phi$ .

**Theorem 4.11.** *The  $n$ -dimensional fractional Laplacian with respect to a function, and the  $n$ -dimensional Marchaud-type derivatives with respect to a function, are related to the analogous operators with respect to  $\mathbf{x}$  by the following conjugation relations:*

$$\begin{aligned} (-\Delta)_{\phi(\mathbf{x})}^{\alpha/2} &= Q_{\phi} \circ (-\Delta)^{\alpha/2} \circ Q_{\phi}^{-1}, \\ {}_{+;\xi}^M D_{\phi(\mathbf{x})}^{\alpha} &= Q_{\phi} \circ {}_{+;\xi}^M D_{\mathbf{x}}^{\alpha} \circ Q_{\phi}^{-1}, \\ {}_{-;\xi}^M D_{\phi(\mathbf{x})}^{\alpha} &= Q_{\phi} \circ {}_{-;\xi}^M D_{\mathbf{x}}^{\alpha} \circ Q_{\phi}^{-1}, \end{aligned}$$

where  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a bijection all of whose 1st-order partial derivatives exist almost everywhere, and in each case the appropriate assumptions on  $\xi$  and  $\alpha$  are imposed, according to the definitions above (this means  $0 < \alpha < 1$  for the Marchaud derivatives and  $0 < \alpha < 2$  for the fractional Laplacian).

*Proof.* The proof for the fractional Laplacian is similar to the 1-dimensional case seen in Theorem 4.4 above. We verify it step by step as follows:

$$\begin{aligned} f : \mathbf{x} &\mapsto f(\mathbf{x}); \\ Q_{\phi}^{-1} f : \mathbf{x} &\mapsto f(\phi^{-1}(\mathbf{x})); \\ (-\Delta)^{\alpha/2} \circ Q_{\phi}^{-1} f : \mathbf{x} &\mapsto C_{n,\alpha/2} \text{P.V.} \int_{\mathbb{R}^n} \frac{f(\phi^{-1}(\mathbf{x})) - f(\phi^{-1}(\mathbf{y}))}{|\mathbf{x} - \mathbf{y}|^{n+\alpha}} d\mathbf{y}; \\ Q_{\phi} \circ (-\Delta)^{\alpha/2} \circ Q_{\phi}^{-1} f : \mathbf{x} &\mapsto C_{n,\alpha/2} \text{P.V.} \int_{\mathbb{R}^n} \frac{f(\mathbf{x}) - f(\phi^{-1}(\mathbf{y}))}{|\phi(\mathbf{x}) - \mathbf{y}|^{n+\alpha}} d\mathbf{y} \\ &= C_{n,\alpha/2} \text{P.V.} \int_{\mathbb{R}^n} \frac{f(\mathbf{x}) - f(\mathbf{u})}{|\phi(\mathbf{x}) - \phi(\mathbf{u})|^{n+\alpha}} |J\phi(\mathbf{u})| d\mathbf{u} \\ &= (-\Delta)_{\phi(\mathbf{x})}^{\alpha/2} f(\mathbf{x}), \end{aligned}$$

where we used the change of variables  $\mathbf{y} = \phi(\mathbf{u})$  and the fact that  $\phi$  is a bijection from  $\mathbb{R}^n$  to itself, as well as the Jacobian rule for change of variables in an integral over an  $n$ -dimensional region.

For the Marchaud-type derivatives, the proof is different:

$$\begin{aligned}
f &: \mathbf{x} \mapsto f(\mathbf{x}); \\
Q_\phi^{-1} f &: \mathbf{x} \mapsto f(\phi^{-1}(\mathbf{x})); \\
{}_{+\xi}^M D_{\mathbf{x}}^\alpha \circ Q_\phi^{-1} f &: \mathbf{x} \mapsto \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(\phi^{-1}(\mathbf{x})) - f(\phi^{-1}(\mathbf{x} - t\xi))}{t^{1+\alpha}} dt; \\
Q_\phi \circ {}_{+\xi}^M D_{\mathbf{x}}^\alpha \circ Q_\phi^{-1} f &: \mathbf{x} \mapsto \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(\mathbf{x}) - f(\phi^{-1}(\phi(\mathbf{x}) - t\xi))}{t^{1+\alpha}} dt \\
&= {}_{+\xi}^M D_{\phi(\mathbf{x})}^\alpha f(\mathbf{x}),
\end{aligned}$$

directly from the definition (18), and similarly for  ${}_{-\xi}^M D_{\phi(\mathbf{x})}^\alpha$ .  $\square$

The result of Theorem 4.11 partially explains why the formulae (18)–(19) were chosen for defining the operators (14)–(15) with respect to a function  $\phi$ . However, these formulae still seem barely related to the formula (17) for the  $n$ -dimensional fractional Laplacian with respect to  $\phi$ : the latter includes a Jacobian factor and no composite functions inside the integral, while the former have no Jacobian factors and feature a  $\phi$  inside a  $\phi^{-1}$  inside an  $f$  inside the integrals. We now proceed to verify that a relationship analogous to (16) is still valid in the case of operators with respect to  $\phi$ .

**Proposition 4.12.** *If  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a bijection all of whose 1st-order partial derivatives exist almost everywhere, then the  $n$ -dimensional Marchaud-type derivatives with respect to  $\phi$  and the fractional Laplacian with respect to  $\phi$  are related by the following identity:*

$$\begin{aligned}
\int_{S^{n-1}} {}_{+\xi}^M D_{\phi(\mathbf{x})}^\alpha f(\mathbf{x}) d\xi &= \frac{\pi^{n/2} \Gamma(1 - \frac{\alpha}{2})}{2^{\alpha-1} \Gamma(1 - \alpha) \Gamma(\frac{\alpha+n}{2})} (-\Delta)_{\phi(\mathbf{x})}^{\alpha/2} f(\mathbf{x}) \\
&= \frac{2\pi^{\frac{n-1}{2}} \Gamma(\frac{\alpha+1}{2}) \cos(\frac{\pi\alpha}{2})}{\Gamma(\frac{\alpha+n}{2})} (-\Delta)_{\phi(\mathbf{x})}^{\alpha/2} f(\mathbf{x}),
\end{aligned}$$

for all  $\alpha \in (0, 1)$  and  $\mathbf{x} \in \mathbb{R}^n$ .

*Proof.* The proof here is similar to that in [28, §10]. We go through it as follows:

$$\begin{aligned}
\int_{S^{n-1}} {}_{+\xi}^M D_{\phi(\mathbf{x})}^\alpha f(\mathbf{x}) d\xi &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{S^{n-1}} \int_0^\infty \frac{f(\mathbf{x}) - f(\phi^{-1}(\phi(\mathbf{x}) - t\xi))}{t^{1+\alpha}} dt d\xi \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \int_{\partial B_1(\mathbf{0})} \frac{f(\mathbf{x}) - f(\phi^{-1}(\phi(\mathbf{x}) - t\xi))}{t^{1+\alpha}} d\xi dt \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \int_{\partial B_t(\phi(\mathbf{x}))} \frac{f(\mathbf{x}) - f(\phi^{-1}(\mathbf{y}))}{t^{1+\alpha}} \frac{d\mathbf{y}}{t^{n-1}} dt \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \int_{\partial B_t(\phi(\mathbf{x}))} \frac{f(\mathbf{x}) - f(\phi^{-1}(\mathbf{y}))}{|\phi(\mathbf{x}) - \mathbf{y}|^{n+\alpha}} d\mathbf{y} dt \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_{\mathbb{R}^n} \frac{f(\mathbf{x}) - f(\phi^{-1}(\mathbf{y}))}{|\phi(\mathbf{x}) - \mathbf{y}|^{n+\alpha}} d\mathbf{y} \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_{\mathbb{R}^n} \frac{f(\mathbf{x}) - f(\mathbf{u})}{|\phi(\mathbf{x}) - \phi(\mathbf{u})|^{n+\alpha}} |J\phi(\mathbf{u})| d\mathbf{u} \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \cdot \frac{1}{C_{n,\alpha/2}} (-\Delta)_{\phi(\mathbf{x})}^{\alpha/2} f(\mathbf{x}),
\end{aligned}$$

which is the desired result, after some manipulation of constants and usage of the reflection formula and Legendre duplication formula for the gamma function. Here we have used two  $n$ -dimensional substitutions: firstly  $\mathbf{y} = \phi(x) - t\xi$ , which is a linear substitution with a simple Jacobian, and secondly  $\mathbf{y} = \phi(\mathbf{u})$ .  $\square$

**Proposition 4.13.** *Let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a bijection all of whose 1st-order partial derivatives exist almost everywhere, and let  $0 < \alpha < 2$ . The operator  $(-\Delta)_{\phi(\mathbf{x})}^{\alpha/2}$ , the fractional Laplacian with respect to  $\phi$ , is defined on the function space  $L^p(\mathbb{R}^n, d\phi)$  for any  $p$  with  $1 \leq p \leq 2$ .*

*Proof.* This follows from Proposition 3.22 together with Theorem 4.11: firstly  $Q_\phi^{-1}$  maps the space  $L^p(\mathbb{R}^n, d\phi)$  isometrically to  $L^p(\mathbb{R}^n)$ , and on this space the fractional Laplacian  $(-\Delta)^{\alpha/2}$  is well-defined.  $\square$

**Theorem 4.14.** *Let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a bijection all of whose 1st-order partial derivatives exist almost everywhere. The fractional Laplacian with respect to  $\phi$  and the Fourier transform with respect to  $\phi$  are related by the following identity:*

$$\mathcal{F}_\phi [(-\Delta)_{\phi(\mathbf{x})}^{\alpha/2} f(\mathbf{x})] = |\mathbf{k}|^\alpha \mathcal{F}_\phi f(\mathbf{k}),$$

for any  $f \in L^p(\mathbb{R}^n, d\phi)$ ,  $0 < \alpha < 2$ ,  $1 \leq p \leq 2$ .

*Proof.* This result is already known for the original  $n$ -dimensional fractional Laplacian, i.e. for the case  $\phi(\mathbf{x}) = \mathbf{x}$ . We can deduce it in the general case by using Proposition 3.19 for the  $n$ -dimensional Fourier transform with respect to  $\phi$  and Theorem 4.11 for the  $n$ -dimensional fractional Laplacian with respect to  $\phi$ :

$$\mathcal{F}_\phi \circ (-\Delta)_{\phi(\mathbf{x})}^{\alpha/2} f = [\mathcal{F} \circ Q_\phi^{-1}] \circ [Q_\phi \circ (-\Delta)^{\alpha/2} \circ Q_\phi^{-1}] f = \mathcal{F} \circ (-\Delta)^{\alpha/2} [Q_\phi^{-1} f],$$

then applying the original result to the function  $g = f \circ \phi^{-1} = Q_\phi^{-1} f$ , we have immediately

$$\mathcal{F} \circ (-\Delta)^{\alpha/2} [Q_\phi^{-1} f](\mathbf{k}) = |\mathbf{k}|^\alpha \mathcal{F} [Q_\phi^{-1} f](\mathbf{k}) = |\mathbf{k}|^\alpha \mathcal{F}_\phi f(\mathbf{k}),$$

which proves the result.  $\square$

## 5. APPLICATIONS TO DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

In this section, we use the recent results from [19] on fractional differential equations with continuous variable coefficients to show the explicit solution of a general class of space-time fractional Cauchy problems involving time derivatives with respect to a function and also the  $n$ -dimensional fractional Laplacian with respect to  $\phi$ . As a particular case, we shall obtain solutions for fractional wave and heat type equations, and other well-known equations.

We use the following formula for the Caputo derivative of a function with respect to another function in the time domain:

$${}_0^C \partial_{\psi(t)}^\alpha w(\mathbf{x}, t) = {}^{RL} D_{\psi(t)}^\alpha \left[ w(\mathbf{x}, t) - \sum_{j=0}^{n-1} \frac{(\psi(t) - \psi(0))^j}{j!} \cdot \frac{d^j}{d\psi(t)^j} w(\mathbf{x}, 0) \right],$$

where  $\alpha \in \mathbb{C}$ ,  $\text{Re}(\alpha) > 0$ ,  $\mathbf{x} \in \mathbb{R}^n$ ,  $t \in (0, T]$ ,  $n = -\lfloor -\text{Re}(\alpha) \rfloor$  for  $\alpha \notin \mathbb{N}$ ,  $n = \alpha$  for  $\alpha \in \mathbb{N}$ , and  $\psi$  is a differentiable function on  $[0, \infty)$  with  $\psi(0) = 0$  and  $\psi' > 0$  almost everywhere. Defining the Caputo derivatives in this way, instead of by differentiating  $n$  times and then fractionally integrating, leads to a wider range of function spaces, which require weaker conditions for the existence of the fractional integro-differential operators. We use Caputo type derivatives instead of

Riemann–Liouville type because of the variant behaviour in initial conditions which are important in physical interpretations [31, 32].

We consider the following space-time fractional Cauchy problem which involves both fractional time derivatives with respect to  $\psi$  and the fractional Laplacian with respect to  $\phi$ :

$$\left\{ \begin{array}{l} {}^C_0\partial_{\psi(t)}^{\beta_0} w(\mathbf{x}, t) + \sum_{i=1}^{m-1} \varrho_i(t) {}^C_0\partial_{\psi(t)}^{\beta_i} w(\mathbf{x}, t) + \varrho_m(t) (-\Delta)_{\phi(\mathbf{x})}^{\alpha/2} w(\mathbf{x}, t) = h(\mathbf{x}, t), \quad t \in (0, T], \quad \mathbf{x} \in \mathbb{R}^n, \\ w(\mathbf{x}, 0) = w_0(\mathbf{x}), \\ \frac{\partial w}{\partial \psi(t)}(\mathbf{x}, 0) = w_1(\mathbf{x}), \\ \vdots \\ \frac{\partial^{n_0} w}{\partial \psi(t)^{n_0}}(\mathbf{x}, 0) = w_{n_0-1}(\mathbf{x}), \end{array} \right. \quad (20)$$

where  $h(\cdot, t) \in C[0, T]$ ,  $\varrho_i(t) \in C[0, T]$  for  $1 \leq i \leq m$ ,  $0 < \alpha < 2$ ,  $\beta_i \in \mathbb{C}$  with  $\text{Re}(\beta_i) > 0$  for  $0 \leq i \leq m-1$ ,  $\text{Re}(\beta_0) > \text{Re}(\beta_1) > \dots > \text{Re}(\beta_{m-1}) > 0$ , and  $n_i = \lfloor \text{Re}(\beta_i) \rfloor + 1 = -\lfloor -\text{Re}(\beta_i) \rfloor$  for  $0 \leq i \leq m-1$ . We also suppose that  $h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n)$ , and  $w_k \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi w_k \in L^1(\mathbb{R}^n)$ , for  $0 \leq k \leq n_0 - 1$ .

Moreover, we also need to assume that

$$\sum_{k=0}^{\infty} (-1)^k {}^{RL}I_{\psi(t)}^{\beta_0} \left( \sum_{i=1}^{m-1} \varrho_i(t) {}^{RL}I_{\psi(t)}^{\beta_0 - \beta_i} + |\mathbf{k}|^\alpha \varrho_m(t) {}^{RL}I_{\psi(t)}^{\beta_0} \right)^k \mathcal{F}_\phi h(\mathbf{k}, t) \in L^1(\mathbb{R}^n), \quad \mathbf{k} \in \mathbb{R}^n, t > 0.$$

Under the above assumptions, we shall prove existence and uniqueness of a solution of the space-time fractional Cauchy problem (20) for a function  $w(\mathbf{x}, t)$  in the space

$$C^{n_0-1, \beta_0}[0, T] := \left\{ g(t) \in C^{n_0-1}[0, T] : {}^C_0D_{\psi(t)}^{\beta_0} g(t) \in C[0, T] \right\},$$

endowed with the norm

$$\|g\|_{C^{n_0-1, \beta_0}[0, T]} = \sum_{k=0}^{n_0-1} \left\| \frac{d^k g}{d\psi(t)^k} \right\|_{C[0, T]} + \| {}^C_0D_{\psi(t)}^{\beta_0} g \|_{C[0, T]}.$$

**5.1. Solution for the general considered fractional Cauchy problem.** We first apply the  $n$ -dimensional space Fourier transform with respect to  $\phi$  to the fractional Cauchy problem (20), which leads to the following initial-value problem in the Fourier  $\mathbf{k}$ -domain:

$$\left\{ \begin{array}{l} {}^C_0\partial_{\psi(t)}^{\beta_0} (\mathcal{F}_\phi w)(\mathbf{k}, t) + \sum_{i=1}^{m-1} \varrho_i(t) {}^C_0\partial_{\psi(t)}^{\beta_i} (\mathcal{F}_\phi w)(\mathbf{k}, t) + |\mathbf{k}|^\alpha \varrho_m(t) (\mathcal{F}_\phi w)(\mathbf{k}, t) \\ \qquad \qquad \qquad = (\mathcal{F}_\phi h)(\mathbf{k}, t), \quad t \in [0, T], \quad \mathbf{k} \in \mathbb{R}^n, \\ (\mathcal{F}_\phi w)(\mathbf{k}, 0) = (\mathcal{F}_\phi w_0)(\mathbf{k}), \\ \frac{\partial (\mathcal{F}_\phi w)}{\partial \psi(t)}(\mathbf{k}, 0) = (\mathcal{F}_\phi w_1)(\mathbf{k}), \\ \vdots \\ \frac{\partial^{n_0} (\mathcal{F}_\phi w)}{\partial \psi(t)^{n_0}}(\mathbf{k}, 0) = (\mathcal{F}_\phi w_{n_0-1})(\mathbf{k}), \end{array} \right. \quad (21)$$

where Theorem 4.14 is used for the Fourier transform of the fractional Laplacian with respect to  $\phi$ . Below we study three different cases, with respect to the complex orders and the initial conditions of problem (21), which allow us to show the solution of the fractional Cauchy problem (20).

**Theorem 5.1.** *Let  $h(\cdot, t), \varrho_i \in C[0, T]$  ( $1 \leq i \leq m$ ),  $h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n)$ ,  $w_j(\mathbf{x}) = 0$  for  $0 \leq j \leq n_0 - 1$ . Let  $\sum_{i=1}^m \|\varrho_i\|_\infty {}^{RL}I_{\psi(t)}^{\beta_0 - \beta_i} e^{\mu t} \leq C e^{\mu t}$  for some  $\mu > 0$ , where  $0 < C < 1$  is a constant independent of  $t$ . Then the fractional Cauchy problem (20) has a unique solution  $w(\cdot, t) \in C^{n_0-1, \beta_0}[0, T]$  given by*

$$w(\mathbf{x}, t) = \mathcal{F}_\phi^{-1} \left( \sum_{k=0}^{\infty} (-1)^k {}^{RL}I_{\psi(t)}^{\beta_0} \left( \sum_{i=1}^{m-1} \varrho_i(t) {}^{RL}I_{\psi(t)}^{\beta_0 - \beta_i} + |\mathbf{k}|^\alpha \varrho_m(t) {}^{RL}I_{\psi(t)}^{\beta_0} \right)^k (\mathcal{F}_\phi h)(\mathbf{k}, t) \right) (\mathbf{x}) \quad (22)$$

for any  $\mathbf{x} \in \mathbb{R}^n$  and  $t > 0$ .

*Proof.* To solve equation (20), the  $n$ -dimensional space Fourier transform with respect to  $\phi$  can be applied. This leads to the fractional equation (21) with continuous variable coefficients. Using [19, Theorem 3.8] with  $\beta_m = 0$ ,  $d_m(t) = |\mathbf{k}|^\alpha \varrho_m(t)$  and  $d_i(t) = \varrho_i(t)$ ,  $i = 1, \dots, m-1$ , we obtain the unique solution given by

$$\begin{aligned} (\mathcal{F}_\phi w)(\mathbf{k}, t) &= \sum_{k=0}^{+\infty} (-1)^k {}^{RL}I_{\psi(t)}^{\beta_0} \left( \sum_{i=1}^m d_i(t) {}^{RL}I_{\psi(t)}^{\beta_0 - \beta_i} \right)^k (\mathcal{F}_\phi h)(\mathbf{k}, t) \\ &= \sum_{k=0}^{+\infty} (-1)^k {}^{RL}I_{\psi(t)}^{\beta_0} \left( \sum_{i=1}^{m-1} \varrho_i(t) {}^{RL}I_{\psi(t)}^{\beta_0 - \beta_i} + |\mathbf{k}|^\alpha \varrho_m(t) {}^{RL}I_{\psi(t)}^{\beta_0} \right)^k (\mathcal{F}_\phi h)(\mathbf{k}, t). \end{aligned}$$

The explicit solution is found by applying the  $n$ -dimensional inverse Fourier transform with respect to  $\phi$  to both sides of the above formula.  $\square$

Now, for each  $j = 0, \dots, n_0 - 1$ , we define the function

$$\Psi_j(t) = \frac{(\psi(t) - \psi(0))^j}{\Gamma(j+1)}, \quad j \in \mathbb{N} \cup \{0\}, \quad (23)$$

and the set

$$\mathbb{K}_j := \{i : 0 \leq \operatorname{Re}(\beta_i) \leq j, i = 1, \dots, m\}, \quad j = 0, 1, \dots, n_0 - 1,$$

with  $\kappa_j = \min\{\mathbb{K}_j\}$  if  $\mathbb{K}_j \neq \emptyset$ . Note that the inclusion  $s \in \mathbb{K}_j$  implies  $\operatorname{Re}(\beta_s) \leq j$ , while  $\mathbb{K}_{j_1} \subset \mathbb{K}_{j_2}$  for  $j_1 < j_2$ . Also, if  $\beta_m = 0$ , then  $\mathbb{K}_j \neq \emptyset$  for all  $j = 0, 1, \dots, n_0 - 1$ . Following [19], we define

$$\mathcal{K}_j(t, |\mathbf{k}|^\alpha, \varrho_1, \dots, \varrho_m) := \sum_{k=0}^{\infty} (-1)^{k+1} {}^{RL}I_{\psi(t)}^{\beta_0} \left( \sum_{i=1}^m d_i(t) {}^{RL}I_{\psi(t)}^{\beta_0 - \beta_i} \right)^k \sum_{i=1}^m d_i(t) {}^{RL}D_{\psi(t)}^{\beta_i} \Psi_j(t),$$

and

$$\mathcal{K}_j^{\kappa_j}(t, |\mathbf{k}|^\alpha, \varrho_1, \dots, \varrho_m) := \sum_{k=0}^{\infty} (-1)^{k+1} {}^{RL}I_{\psi(t)}^{\beta_0} \left( \sum_{i=1}^m d_i(t) {}^{RL}I_{\psi(t)}^{\beta_0 - \beta_i} \right)^k \sum_{i=\kappa_j}^m d_i(t) {}^{RL}D_{\psi(t)}^{\beta_i} \Psi_j(t),$$

where  $d_m(t) = |\mathbf{k}|^\alpha \varrho_m(t)$  and  $d_i(t) = \varrho_i(t)$  for  $1 \leq i \leq m-1$ . We also define

$$\mathcal{G}((\mathcal{F}_\phi h)(\mathbf{k}, t)) := \sum_{k=1}^{\infty} (-1)^k {}^{RL}I_{\psi(t)}^{\beta_0} \left( \sum_{i=1}^m d_i(t) {}^{RL}I_{\psi(t)}^{\beta_0 - \beta_i} \right)^k (\mathcal{F}_\phi h)(\mathbf{k}, t).$$



We always require that the kernels  $\mathcal{K}_j^{\kappa_j}$  and  $\mathcal{K}_j$  (for  $0 \leq j \leq n_0 - 1$ ) and  $\mathcal{G}((\mathcal{F}_\phi h)(\mathbf{k}, t))$  are in  $L^1(\mathbb{R}^n)$ , which is a natural condition even imposed in the classical case where there are no fractional operators.

**Theorem 5.2.** *Let  $n_0 > n_1$ ,  $\beta_m = 0$ , and  $h(\cdot, t), \varrho_i \in C[0, T]$  for  $1 \leq i \leq m$ . Let  $h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n)$ , and  $w_k \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi w_k \in L^1(\mathbb{R}^n)$  for  $0 \leq k \leq n_0 - 1$ . Let us also assume that  $\sum_{i=1}^m \|\varrho_i\|_\infty {}^{RL}I_{\psi(t)}^{\beta_0 - \beta_i} e^{\mu t} \leq C e^{\mu t}$  for some  $\mu > 0$  and some constant  $0 < C < 1$  which does not depend on  $t$ . Then the initial value problem (20) has a unique solution  $w(\cdot, t) \in C^{n_0-1, \beta_0}[0, T]$  given by*

$$w(\mathbf{x}, t) = \sum_{j=0}^{n_0-1} w_j(\mathbf{x}) \Psi_j(t) + \sum_{j=0}^{n_0-1} \left( w_j *_{\phi} \mathcal{F}_\phi^{-1}(\mathcal{H}_j(t, |\mathbf{k}|^\alpha, \varrho_1, \dots, \varrho_m)) \right) (\mathbf{x}) \\ - {}^{RL}I_{\psi(t)}^{\beta_0} h(\mathbf{x}, t) + (\mathcal{F}_\phi^{-1} \mathcal{G}((\mathcal{F}_\phi h)(\mathbf{k}, t))) (\mathbf{x}),$$

where

$$\mathcal{H}_j(t, |\mathbf{k}|^\alpha, \varrho_1, \dots, \varrho_m) = \begin{cases} \mathcal{K}_j^{\kappa_j}(t, |\mathbf{k}|^\alpha, \varrho_1, \dots, \varrho_m) & \text{if } j = 0, \dots, n_1 - 1, \\ \mathcal{K}_j(t, |\mathbf{k}|^\alpha, \varrho_1, \dots, \varrho_m) & \text{if } j = n_1, \dots, n_0 - 1. \end{cases}$$

*Proof.* From (21), applying [19, Theorem 3.10] with  $\beta_m = 0$ ,  $d_m(t) = |\mathbf{k}|^\alpha \varrho_m(t)$ , and  $d_i(t) = \varrho_i(t)$  for  $1 \leq i \leq m - 1$ , we obtain the unique solution

$$(\mathcal{F}_\phi w)(\mathbf{k}, t) = \sum_{j=0}^{n_0-1} (\mathcal{F}_\phi w_j)(\mathbf{k}) y_j(\mathbf{k}, t) + \sum_{k=0}^{\infty} (-1)^k {}^{RL}I_{\psi(t)}^{\beta_0} \left( \sum_{i=1}^m d_i(t) I_{0+}^{\beta_0 - \beta_i, \psi} \right)^k (\mathcal{F}_\phi h)(\mathbf{k}, t),$$

where  $y_j$  is defined for  $j = 0, \dots, n_1 - 1$  by

$$y_j(\mathbf{k}, t) = \Psi_j(t) + \sum_{k=0}^{\infty} (-1)^{k+1} {}^{RL}I_{\psi(t)}^{\beta_0} \left( \sum_{i=1}^m d_i(t) j = 0, \dots, n_1 - 1 \beta_0 - \beta_i \right)^k \sum_{i=\kappa_j}^m d_i(t) {}^{RL}D_{\psi(t)}^{\beta_i} \Psi_j(t) \\ = \Psi_j(t) + \mathcal{K}_j^{\kappa_j}(t, |\mathbf{k}|^\alpha, \varrho_1, \dots, \varrho_m),$$

and for  $j = n_1, n_1 + 1, \dots, n_0 - 1$  by

$$y_j(\mathbf{k}, t) = \Psi_j(t) + \sum_{k=0}^{\infty} (-1)^{k+1} {}^{RL}I_{\psi(t)}^{\beta_0} \left( \sum_{i=1}^m d_i(t) {}^{RL}I_{\psi(t)}^{\beta_0 - \beta_i} \right)^k \sum_{i=1}^m d_i(t) {}^{RL}D_{\psi(t)}^{\beta_i} \Psi_j(t), \\ = \Psi_j(t) + \mathcal{K}_j(t, |\mathbf{k}|^\alpha, \varrho_1, \dots, \varrho_m),$$

where  $\Psi_j(t)$  is given in (23). Therefore,

$$(\mathcal{F}_\phi w)(\mathbf{k}, t) - \sum_{j=0}^{n_0-1} (\mathcal{F}_\phi w_j)(\mathbf{k}) \Psi_j(t) \\ = \sum_{j=0}^{n_0-1} (\mathcal{F}_\phi w_j)(\mathbf{k}) \mathcal{H}_j(t, |\mathbf{k}|^\alpha, \varrho_1, \dots, \varrho_m) - I_{0+}^{\beta_0, \psi} (\mathcal{F}_\phi h)(\mathbf{k}, t) + \mathcal{G}((\mathcal{F}_\phi h)(\mathbf{k}, t)), \quad (24)$$

with the notations  $\mathcal{H}_j(t, |\mathbf{k}|^\alpha, \varrho_1, \dots, \varrho_m)$  and  $\mathcal{G}((\mathcal{F}_\phi h)(\mathbf{k}, t))$  as defined above.

Applying the  $n$ -dimensional inverse Fourier transform with respect to  $\phi$  in (24), and using the fact that  $\mathcal{K}_j^{\kappa_j}$  and  $\mathcal{K}_j$  (for  $0 \leq j \leq n_0 - 1$ ) and  $\mathcal{G}((\mathcal{F}_\phi h)(\mathbf{k}, t))$  are in  $L^1(\mathbb{R}^n)$ , we get the solution of the fractional Cauchy problem (20), making use of Proposition 3.28 for the Fourier convolution theorem.  $\square$

The following theorem is established in the same way as the above one. Indeed, the proof follows from applying [19, Theorem 3.9] to equation (21) with  $d_m(t) = |\mathbf{k}|^\alpha \varrho_m(t)$  and  $d_i(t) = \varrho_i(t)$  for  $1 \leq i \leq m-1$ .

**Theorem 5.3.** *Let  $n_0 = n_1$ ,  $\beta_m = 0$ , and  $h(\cdot, t), \varrho_i \in C[0, T]$  for  $1 \leq i \leq m$ . Let  $h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n)$ , and  $w_k \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi w_k \in L^1(\mathbb{R}^n)$  for  $0 \leq k \leq n_0 - 1$ . Let us also assume that  $\sum_{i=1}^m \|\varrho_i\|_\infty {}^{RL}I_{\psi(t)}^{\beta_0 - \beta_i} e^{\mu t} \leq C e^{\mu t}$  for some  $\mu > 0$  and some constant  $0 < C < 1$  which does not depend on  $t$ . Then the initial value problem (20) has a unique solution  $w(\cdot, t) \in C^{n_0-1, \beta_0}[0, T]$  given by*

$$w(\mathbf{x}, t) = \sum_{j=0}^{n_0-1} w_j(\mathbf{x}) \Psi_j(t) + \sum_{j=0}^{n_0-1} \left( w_j *_{\phi} \mathcal{F}_\phi^{-1}(\mathcal{K}_j^{\kappa_j}(t, |\mathbf{k}|^\alpha, \varrho_1, \dots, \varrho_m)) \right)(\mathbf{x}) \\ - {}^{RL}I_{\psi(t)}^{\beta_0} h(\mathbf{x}, t) + (\mathcal{F}_\phi^{-1} \mathcal{G}((\mathcal{F}_\phi h)(\mathbf{k}, t)))(\mathbf{x}).$$

**Remark 5.4.** Notice that the above theorems extend some of the obtained results given in [33] to a more general setting. Indeed, the consideration of the  $n$ -dimensional fractional Laplacian in the problems considered here enlarges the scope of the considered fractional Cauchy problem and the allowable function spaces involved.

**5.2. The constant-coefficient case.** We begin this section by recalling from [34] the definition of the multivariate Mittag-Leffler function. This will be used in the explicit representations of solutions of the considered fractional differential equations in the case of constant coefficients. We also refer to the entire book [35] for more discussion on different types of Mittag-Leffler functions.

**Definition 5.5.** The multivariate Mittag-Leffler function  $E_{(a_1, \dots, a_n), b}(z_1, \dots, z_n)$ , of  $n$  variables  $z_1, \dots, z_n \in \mathbb{C}$  and arbitrary parameters  $a_1, \dots, a_n, b \in \mathbb{C}$  with positive real parts, is defined by the following formula:

$$E_{(a_1, \dots, a_n), b}(z_1, \dots, z_n) = \sum_{k=0}^{+\infty} \sum_{l_1 + \dots + l_n = k, l_1, \dots, l_n \geq 0} \frac{k!}{l_1! \times \dots \times l_n!} \cdot \frac{\prod_{i=1}^n z_i^{l_i}}{\Gamma(b + \sum_{i=1}^n a_i l_i)}, \quad (25)$$

where this series is locally uniformly convergent under the given conditions on the parameters.

We consider the following space-time fractional Cauchy problem with constant coefficients and both space and time derivatives taken to be fractional with respect to functions:

$$\left\{ \begin{array}{l} {}^C_0 \partial_{\psi(t)}^{\beta_0} w(\mathbf{x}, t) + \sum_{i=1}^{m-1} \varrho_i {}^C_0 \partial_{\psi(t)}^{\beta_i} w(\mathbf{x}, t) + \varrho_m (-\Delta)_{\phi(\mathbf{x})}^{\alpha/2} w(\mathbf{x}, t) = h(\mathbf{x}, t), \quad t \in (0, T], \quad \mathbf{x} \in \mathbb{R}^n, \\ w(\mathbf{x}, 0) = w_0(\mathbf{x}), \\ \frac{\partial w}{\partial \psi(t)}(\mathbf{x}, 0) = w_1(\mathbf{x}), \\ \vdots \\ \frac{\partial^{n_0} w}{\partial \psi(t)^{n_0}}(\mathbf{x}, 0) = w_{n_0-1}(\mathbf{x}), \end{array} \right. \quad (26)$$

where  $h(\cdot, t) \in C[0, T]$ ,  $0 < \alpha < 2$ ,  $\varrho_i, \beta_i \in \mathbb{C}$  with  $\text{Re}(\beta_i) > 0$  for  $0 \leq i \leq m-1$  and  $\text{Re}(\beta_0) > \text{Re}(\beta_1) > \dots > \text{Re}(\beta_{m-1}) > 0$ , and  $n_i = \lfloor \text{Re}(\beta_i) \rfloor + 1 = -\lfloor -\text{Re}(\beta_i) \rfloor$  for  $0 \leq i \leq m-1$ . We also

suppose that  $h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n)$ , and  $w_k \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi w_k \in L^1(\mathbb{R}^n)$  for  $0 \leq k \leq n_0 - 1$ . Moreover, we also need to assume that

$$\int_0^t E_{(\beta_0-\beta_1, \dots, \beta_0-\beta_m), \beta_0} \left( -\varrho_1(\psi(t) - \psi(u))^{\beta_0-\beta_1}, \dots, -|\mathbf{k}|^\alpha \varrho_m(\psi(t) - \psi(u))^{\beta_0-\beta_m} \right) \\ \times (\psi(t) - \psi(u))^{\beta_0-1} (\mathcal{F}_\phi h(\mathbf{k}, u)) \psi'(u) du \in L^1(\mathbb{R}^n), \quad \mathbf{k} \in \mathbb{R}^n, \quad t > 0.$$

We shall now solve the space-time fractional Cauchy problem (26) by proving existence and uniqueness of a solution in  $C^{n_0-1, \beta_0}[0, T]$ . We follow the same steps as in the previous section: applying the  $n$ -dimensional space Fourier transform with respect to  $\phi$  to the fractional Cauchy problem (26), and then use [19, Theorems 4.2, 4.3 and 4.4] to show the solutions of equation (26) in three different cases. Ultimately we establish the following statements, whose proofs we omit here as they are very similar to those in the previous subsection.

**Theorem 5.6.** *Let  $h(\cdot, t) \in C[0, T]$ ,  $\varrho_i \in \mathbb{C}$  for  $1 \leq i \leq m$ ,  $h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n, d\phi)$  and  $\mathcal{F}_\phi h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n)$ ,  $w_j(\mathbf{x}) = 0$  for  $0 \leq j \leq n_0 - 1$ . Assume that  $\sum_{i=1}^m |\varrho_i| {}^{RL}I_{\psi(t)}^{\beta_0-\beta_i} e^{\mu t} \leq C e^{\mu t}$  for some  $\mu > 0$ , where  $0 < C < 1$  is a constant independent of  $t$ . Then the fractional Cauchy problem (26) has a unique solution  $w(\cdot, t) \in C^{n_0-1, \beta_0}[0, T]$  given by*

$$w(\mathbf{x}, t) = \int_0^t \left( h(\mathbf{x}, u) \right) *_{\phi} \mathcal{F}_\phi^{-1} \left( \psi'(u) (\psi(t) - \psi(u))^{\beta_0-1} \right. \\ \left. \times E_{(\beta_0-\beta_1, \dots, \beta_0-\beta_m), \beta_0} \left( -\varrho_1(\psi(t) - \psi(u))^{\beta_0-\beta_1}, \dots, -|\mathbf{k}|^\alpha \varrho_m(\psi(t) - \psi(u))^{\beta_0-\beta_m} \right) \right) du.$$

**Theorem 5.7.** *Let  $n_0 > n_1$ ,  $\beta_m = 0$ ,  $h(\cdot, t) \in C[0, T]$ , and  $\varrho_i \in \mathbb{C}$  for  $1 \leq i \leq m$ . Let  $h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n)$ , and  $w_k \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi w_k \in L^1(\mathbb{R}^n)$ , for  $0 \leq k \leq n_0 - 1$ . Assume that  $\sum_{i=1}^m |\varrho_i| {}^{RL}I_{\psi(t)}^{\beta_0-\beta_i} e^{\mu t} \leq C e^{\mu t}$  for some  $\mu > 0$  and some constant  $0 < C < 1$  which does not depend on  $t$ . Then the initial value problem (26) has a unique solution  $w(\cdot, t) \in C^{n_0-1, \beta_0}[0, T]$  given by*

$$w(\mathbf{x}, t) = \sum_{j=0}^{n_0-1} w_j(\mathbf{x}) \Psi_j(t) \\ + \sum_{j=0}^{n_1-1} w_j *_{\phi} \mathcal{F}_\phi^{-1} \left( \sum_{i=\kappa_j}^m \varrho_i^* (\psi(t) - \psi(0))^{j+\beta_0-\beta_i} \right. \\ \left. \times E_{(\beta_0-\beta_1, \dots, \beta_0-\beta_m), j+1+\beta_0-\beta_i} \left( \varrho_1^* (\psi(t) - \psi(0))^{\beta_0-\beta_1}, \dots, \varrho_m^* (\psi(t) - \psi(0))^{\beta_0-\beta_m} \right) \right) \\ + \sum_{j=n_1}^{n_0-1} w_j *_{\phi} \mathcal{F}_\phi^{-1} \left( \sum_{i=0}^m \varrho_i^* (\psi(t) - \psi(0))^{j+\beta_0-\beta_i} \right. \\ \left. \times E_{(\beta_0-\beta_1, \dots, \beta_0-\beta_m), j+1+\beta_0-\beta_i} \left( \varrho_1^* (\psi(t) - \psi(0))^{\beta_0-\beta_1}, \dots, \varrho_m^* (\psi(t) - \psi(0))^{\beta_0-\beta_m} \right) \right) \\ + \int_0^t \left( h(\mathbf{x}, u) \right) *_{\phi} \mathcal{F}_\phi^{-1} \left( \psi'(u) (\psi(t) - \psi(u))^{\beta_0-1} \right. \\ \left. \times E_{(\beta_0-\beta_1, \dots, \beta_0-\beta_m), \beta_0} \left( -\varrho_1^* (\psi(t) - \psi(u))^{\beta_0-\beta_1}, \dots, -\varrho_m^* (\psi(t) - \psi(u))^{\beta_0-\beta_m} \right) \right) du,$$

where  $\varrho_i^* = \varrho_i$  for  $0 \leq i \leq m-1$  and  $\varrho_m^* = |\mathbf{k}|^\alpha \varrho_m$ .

**Theorem 5.8.** *Let  $n_0 = n_1$ ,  $\beta_m = 0$ ,  $h(\cdot, t) \in C[0, T]$ , and  $\varrho_i \in \mathbb{C}$  for  $1 \leq i \leq m$ . Let  $h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi h(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^n)$ , and  $w_k \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi w_k \in L^1(\mathbb{R}^n)$ , for  $0 \leq k \leq n_0 - 1$ . Assume that  $\sum_{i=1}^m |\varrho_i| {}^{RL}I_{\psi(t)}^{\beta_0 - \beta_i} e^{\mu t} \leq C e^{\mu t}$  for some  $\mu > 0$  and some constant  $0 < C < 1$  which does not depend on  $t$ . Then the initial value problem (26) has a unique solution  $w(\cdot, t) \in C^{n_0-1, \beta_0}[0, T]$  given by*

$$\begin{aligned} w(\mathbf{x}, t) &= \sum_{j=0}^{n_0-1} w_j(\mathbf{x}) \Psi_j(t) \\ &+ \sum_{j=0}^{n_0-1} w_j *_{\phi} \mathcal{F}_{\phi}^{-1} \left( \sum_{i=\kappa_j}^m \varrho_i^* (\psi(t) - \psi(0))^{j+\beta_0-\beta_i} \right. \\ &\quad \left. \times E_{(\beta_0-\beta_1, \dots, \beta_0-\beta_m), j+1+\beta_0-\beta_i} (\varrho_1^* (\psi(t) - \psi(0))^{\beta_0-\beta_1}, \dots, \varrho_m^* (\psi(t) - \psi(0))^{\beta_0-\beta_m}) \right) \\ &+ \int_0^t \left( h(\mathbf{x}, u) \right) *_{\phi} \mathcal{F}_{\phi}^{-1} \left( \psi'(u) (\psi(t) - \psi(u))^{\beta_0-1} \right. \\ &\quad \left. \times E_{(\beta_0-\beta_1, \dots, \beta_0-\beta_m), \beta_0} (-\varrho_1^* (\psi(t) - \psi(u))^{\beta_0-\beta_1}, \dots, -\varrho_m^* (\psi(t) - \psi(u))^{\beta_0-\beta_m}) \right) du, \end{aligned}$$

where  $\varrho_i^* = \varrho_i$  for  $0 \leq i \leq m-1$  and  $\varrho_m^* = |\mathbf{k}|^{\alpha} \varrho_m$ .

**Remark 5.9.** As a corollary of the results established in this subsection, we can achieve the results previously shown in [36]. Under the choice of functions  $\psi(t) = t$  and  $\phi(\mathbf{x}) = \mathbf{x}$ , the fractional time derivative with respect to the function  $\psi(t)$  becomes the classical one with respect to  $t$ , and the  $n$ -dimensional fractional Laplacian and the  $n$ -dimensional Fourier transform with respect to  $\phi$  become the classical ones in  $\mathbb{R}^n$ . Thus, the following space-time fractional Cauchy problem is obtained:

$$\left\{ \begin{array}{l} C_0 \partial_t^{\beta_0} w(\mathbf{x}, t) + \sum_{i=1}^{m-1} \varrho_i C_0 \partial_t^{\beta_i} w(\mathbf{x}, t) + \varrho_m (-\Delta_{\mathbf{x}})^{\alpha/2} w(\mathbf{x}, t) = h(\mathbf{x}, t), \quad t \in (0, T], \quad \mathbf{x} \in \mathbb{R}^n, \\ w(\mathbf{x}, 0) = w_0(\mathbf{x}), \\ \frac{\partial w}{\partial t}(\mathbf{x}, 0) = w_1(\mathbf{x}), \\ \vdots \\ \frac{\partial^{n_0} w}{\partial t^{n_0}}(\mathbf{x}, 0) = w_{n_0-1}(\mathbf{x}), \end{array} \right. \quad (27)$$

under the same conditions on the functions and parameters as imposed earlier in this subsection. Then, as a special case of Theorems 5.6 to 5.8, we obtain the same results already shown in [36] for variable-coefficient fractional differential equations.

**5.3. The classical heat and wave equations.** In this subsection, we consider the classical heat and wave equations on  $\mathbb{R}^n$  as special cases of the general problems considered above, and compare the solutions obtained using the usual approach in  $\mathbb{R}^n$  with those obtained using our results. As expected, both solutions will coincide, which acts to verify the correctness of our results.

**Example 5.10.** Let us start with the heat equation on  $\mathbb{R}^n$ :

$$\left\{ \begin{array}{l} w_t(\mathbf{x}, t) - \Delta_{\mathbf{x}} w(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathbb{R}^n, \quad t > 0, \\ w(\mathbf{x}, 0) = f(\mathbf{x}), \end{array} \right. \quad (28)$$

where  $f, \mathcal{F}f \in L^1(\mathbb{R}^n)$ . Classically, the solution of equation (28) is found by the application of the  $n$ -dimensional space Fourier transform, which transforms the equation into an ordinary differential equation with respect to the variable  $t$ . The solution thus obtained is:

$$w(\mathbf{x}, t) = \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) \mathcal{F}^{-1} \left( e^{-|\mathbf{k}|^2 t} \right) (\mathbf{y}) \, d\mathbf{y}. \quad (29)$$

Now, using our Theorem 5.2 with  $\psi(t) = t$ ,  $\phi(\mathbf{x}) = \mathbf{x}$ ,  $\beta_0 = 1$ ,  $\alpha = 2$ ,  $\varrho_i(t) \equiv 0$  for  $1 \leq i \leq m-1$ ,  $\varrho_m(t) = 1$ ,  $h(\mathbf{x}, t) \equiv 0$  and  $w_0(\mathbf{x}) = f(\mathbf{x})$ , we obtain the following solution function:

$$w(\mathbf{x}, t) = f(\mathbf{x}) + \left( f * \mathcal{F}^{-1}(\mathcal{K}_0(t, |\mathbf{k}|^2, 1)) \right) (\mathbf{x}),$$

where

$$\mathcal{K}_0(t, |\mathbf{k}|^2, 1) = \sum_{k=0}^{\infty} (-1)^{k+1} \left( {}^{RL}I_t^1 (|\mathbf{k}|^2 {}^{RL}I_t^1)^k \right) (|\mathbf{k}|^2 t^0) = \sum_{k=0}^{\infty} (-|\mathbf{k}|^2)^{k+1} \frac{t^{k+1}}{(k+1)!} = e^{-|\mathbf{k}|^2 t} - 1.$$

Therefore,

$$\begin{aligned} w(\mathbf{x}, t) &= f(\mathbf{x}) + \int_{\mathbb{R}^n} f(\mathbf{y}) \mathcal{F}^{-1} \left( -1 + e^{-|\mathbf{k}|^2 t} \right) (\mathbf{x} - \mathbf{y}) \, d\mathbf{y} \\ &= f(\mathbf{x}) - f(\mathbf{x}) + \int_{\mathbb{R}^n} f(\mathbf{y}) \mathcal{F}^{-1} \left( e^{-|\mathbf{k}|^2 t} \right) (\mathbf{x} - \mathbf{y}) \, d\mathbf{y}, \end{aligned}$$

which is the same as (29).

**Example 5.11.** We now discuss the solution of the wave equation on  $\mathbb{R}^n$  :

$$\begin{cases} \partial_{tt}^2 w(\mathbf{x}, t) - \Delta_{\mathbf{x}} w(\mathbf{x}, t) = 0, & \mathbf{x} \in \mathbb{R}^n, t > 0, \\ w(\mathbf{x}, 0) = f_1(\mathbf{x}), \\ \frac{\partial w}{\partial t}(\mathbf{x}, 0) = f_2(\mathbf{x}), \end{cases} \quad (30)$$

where  $f_1, f_2 \in L^1(\mathbb{R}^n)$  such that  $\mathcal{F}f_1(\mathbf{y}) \cos(|\mathbf{y}|t)$ ,  $\mathcal{F}f_2(\mathbf{y}) \frac{\sin(|\mathbf{y}|t)}{|\mathbf{y}|} \in L^1(\mathbb{R}^n)$  for any  $\mathbf{y} \in \mathbb{R}^n$  and  $t \geq 0$ .

Classically, the solution of (30) is found by the application of the  $n$ -dimensional space Fourier transform, and then solving the transformed equation, which is an ordinary differential equation in the variable  $t$ . Hence,

$$w(\mathbf{x}, t) = \int_{\mathbb{R}^n} \left( f_1(\mathbf{x} - \mathbf{y}) \mathcal{F}^{-1} \left( \cos(|\mathbf{k}|t) \right) (\mathbf{y}) + f_2(\mathbf{x} - \mathbf{y}) \mathcal{F}^{-1} \left( \frac{\sin(|\mathbf{k}|t)}{|\mathbf{k}|} \right) (\mathbf{y}) \right) \, d\mathbf{y}. \quad (31)$$

Now, using our Theorem 5.2 with  $\psi(t) = t$ ,  $\phi(\mathbf{x}) = \mathbf{x}$ ,  $\beta_0 = 2$ ,  $\alpha = 2$ ,  $\varrho_i(t) \equiv 0$  for  $1 \leq i \leq m-1$ ,  $\varrho_m(t) = 1$ ,  $h(\mathbf{x}, t) \equiv 0$ ,  $w_0(\mathbf{x}) = f_1(\mathbf{x})$ , and  $w_1(\mathbf{x}) = f_2(\mathbf{x})$ , we have

$$w(\mathbf{x}, t) = f_1(\mathbf{x}) + f_2(\mathbf{x})t + \left( f_1 * \mathcal{F}^{-1}(\mathcal{K}_0(t, |\mathbf{k}|^2, 1)) \right) (\mathbf{x}) + \left( f_2 * \mathcal{F}^{-1}(\mathcal{K}_1(t, |\mathbf{k}|^2, 1)) \right) (\mathbf{x}),$$

where

$$\begin{aligned} \mathcal{K}_0(t, |\mathbf{k}|^2, 1) &= \sum_{k=0}^{+\infty} (-1)^{k+1} \left( {}^{RL}I_t^2 (|\mathbf{k}|^2 {}^{RL}I_t^2)^k \right) (|\mathbf{k}|^2 t^0) \\ &= \sum_{k=0}^{+\infty} (-|\mathbf{k}|^2)^{k+1} \frac{t^{2(k+1)}}{(2(k+1))!} = -1 + \cos(|\mathbf{k}|t), \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}_1(t, |\mathbf{k}|^2, 1) &= \sum_{k=0}^{+\infty} (-1)^{k+1} \left( {}^{RL}I_t^2 (|\mathbf{k}|^2 {}^{RL}I_t^2)^k \right) (|\mathbf{k}|^2 t^1) \\ &= \sum_{k=0}^{+\infty} (-|\mathbf{k}|^2)^{k+1} \frac{t^{2(k+1)+1}}{(2(k+1)+1)!} = -t + \frac{\sin(|\mathbf{k}|t)}{|\mathbf{k}|}. \end{aligned}$$

We then arrive at

$$\begin{aligned} w(\mathbf{x}, t) &= f_1(\mathbf{x}) + f_2(\mathbf{x})t - f_1(\mathbf{x}) + \int_{\mathbb{R}^n} f_1(\mathbf{y}) \mathcal{F}^{-1}(\cos(|\mathbf{k}|t)) (\mathbf{x} - \mathbf{y}) d\mathbf{y} \\ &\quad - t f_2(\mathbf{x}) + \int_{\mathbb{R}^n} f_2(\mathbf{y}) \mathcal{F}^{-1}\left(\frac{\sin(|\mathbf{y}|t)}{|\mathbf{y}|}\right) (\mathbf{x} - \mathbf{y}) d\mathbf{y}, \end{aligned}$$

which is the same as (31).

**5.4. Further examples.** In this subsection, we give the solution of some fractional Cauchy problems of heat and wave type by using the fractional Laplacian with respect to  $\phi$ . Both of the differential equations considered below, in (32) and (33), look the same, but with different ranges of values for the time-fractional order  $\beta_0$ , which affect the form of the required initial conditions.

**Example 5.12.** Let us consider the fractional initial value problem of wave type with a power time-function:

$$\begin{cases} {}^C_0\partial_t^{\beta_0} w(\mathbf{x}, t) + t^{\beta_0} (-\Delta)_{\phi(\mathbf{x})}^{\alpha/2} w(\mathbf{x}, t) = 0, \\ w(\mathbf{x}, 0) = w_0(\mathbf{x}), \\ \frac{\partial w}{\partial t}(\mathbf{x}, 0) = w_1(\mathbf{x}), \end{cases} \quad (32)$$

where  $w_0, w_1 \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_\phi w_0, \mathcal{F}_\phi w_1 \in L^1(\mathbb{R}^n)$ ,  $1 < \beta_0 \leq 2$ , and  $0 < \alpha < 2$ . By Theorem 5.2 and [19, Remark 3.2], the solution of equation (32) is given by the formula

$$w(\mathbf{x}, t) = w_0(\mathbf{x}) + w_1(\mathbf{x})t + \left( w_0 *_{\phi} \mathcal{F}_\phi^{-1}(\mathcal{K}_0(t, |\mathbf{k}|^\alpha, t^{\beta_0})) \right)(\mathbf{x}) + \left( w_1 *_{\phi} \mathcal{F}_\phi^{-1}(\mathcal{K}_1(t, |\mathbf{k}|^\alpha, t^{\beta_0})) \right)(\mathbf{x}),$$

where

$$\begin{aligned} \mathcal{K}_0(t, |\mathbf{k}|^\alpha, t^{\beta_0}) &= \sum_{k=0}^{\infty} (-|\mathbf{k}|^\alpha)^{k+1} {}^{RL}I_t^{\beta_0} \left( t^{\beta_0} {}^{RL}I_t^{\beta_0} \right)^k t^{\beta_0}, \\ \mathcal{K}_1(t, |\mathbf{k}|^\alpha, t^{\beta_0}) &= \sum_{k=0}^{\infty} (-|\mathbf{k}|^\alpha)^{k+1} {}^{RL}I_t^{\beta_0} \left( t^{\beta_0} {}^{RL}I_t^{\beta_0} \right)^k t^{\beta_0+1}. \end{aligned}$$

From the rules for Riemann–Liouville fractional integrals of power functions, it is clear (cf. [19, Section 3.3]) that

$${}^{RL}I_t^\beta \left( t^\beta {}^{RL}I_t^\beta \right)^k t^\gamma = \prod_{j=0}^k \frac{\Gamma(2j\beta + \gamma + 1)}{\Gamma((2j+1)\beta + \gamma + 1)} t^{(2k+1)\beta + \gamma}, \quad \operatorname{Re}(\beta) > 0, \operatorname{Re}(\gamma) > -1.$$

Therefore,

$$\begin{aligned}
w(\mathbf{x}, t) &= w_0(\mathbf{x}) + w_1(\mathbf{x})t + w_0 *_{\phi} \mathcal{F}_{\phi}^{-1} \left( -1 + E_{1,2\beta_0,\beta_0}^{\beta_0}(-|\mathbf{k}|^{\alpha} t^{2\beta_0}) \right) (\mathbf{x}) \\
&\quad + t w_1 *_{\phi} \mathcal{F}_{\phi}^{-1} \left( -1 + E_{1,2\beta_0,\beta_0+1}^{\beta_0}(-|\mathbf{k}|^{\alpha} t^{2\beta_0}) \right) (\mathbf{x}) \\
&= \int_{\mathbb{R}^n} w_0(\phi^{-1}(\phi(\mathbf{x}) - \phi(\mathbf{y}))) \left( \mathcal{F}_{\phi}^{-1} E_{1,2\beta_0,\beta_0}^{\beta_0}(-|\mathbf{k}|^{\alpha} t^{2\beta_0}) \right) (\mathbf{y}) |J\phi(\mathbf{y})| d\mathbf{y} \\
&\quad + t \int_{\mathbb{R}^n} w_1(\phi^{-1}(\phi(\mathbf{x}) - \phi(\mathbf{y}))) \left( \mathcal{F}_{\phi}^{-1} E_{1,2\beta_0,\beta_0+1}^{\beta_0}(-|\mathbf{k}|^{\alpha} t^{2\beta_0}) \right) (\mathbf{y}) |J\phi(\mathbf{y})| d\mathbf{y}
\end{aligned}$$

where  $E_{\alpha,\beta,\gamma}^{\lambda} = \sum_{k=0}^{+\infty} c_k z^k$ ,  $z \in \mathbb{C}$ , with

$$c_0 = 1, \quad c_k = \prod_{j=0}^{k-1} \frac{\Gamma(\alpha[j\beta + \gamma] + 1)}{\Gamma(\alpha[j\beta + \gamma] + \lambda + 1)}, \quad k = 1, 2, \dots,$$

is a generalised special function of Fox–Wright type. Note that in the case  $\lambda = \alpha$ , the function  $E_{\alpha,\beta,\gamma}^{\alpha}$  becomes the generalized (Kilbas–Saigo) Mittag-Leffler type function [35, Chapter 5].

**Example 5.13.** Let us consider another simpler example of heat type equation, which only needs one initial condition:

$$\begin{cases} {}_0^C \partial_t^{\beta_0} w(\mathbf{x}, t) + t^{\beta_0} (-\Delta)_{\phi(\mathbf{x})}^{\alpha/2} w(\mathbf{x}, t) = 0, \\ w(\mathbf{x}, 0) = w_0(\mathbf{x}), \end{cases} \quad (33)$$

where  $w_0 \in L^1(\mathbb{R}^n, d\phi)$ ,  $\mathcal{F}_{\phi} w_0 \in L^1(\mathbb{R}^n)$ ,  $0 < \beta_0 \leq 1$ , and  $0 < \alpha < 2$ . Again, by Theorem 5.2 and [19, Remark 3.2], and using the same function  $E_{\alpha,\beta,\gamma}^{\lambda}$  as in the previous example, the solution of the problem (33) is given by:

$$\begin{aligned}
w(\mathbf{x}, t) &= w_0(\mathbf{x}) + \left( w_0 *_{\phi} \mathcal{F}_{\phi}^{-1} (\mathcal{K}_0(t, |\mathbf{k}|^{\alpha}, t^{\beta_0})) \right) (\mathbf{x}) \\
&= \int_{\mathbb{R}^n} w_0(\phi^{-1}(\phi(\mathbf{x}) - \phi(\mathbf{y}))) \left( \mathcal{F}_{\phi}^{-1} E_{1,2\beta_0,\beta_0}^{\beta_0}(-|\mathbf{k}|^{\alpha} t^{2\beta_0}) \right) (\mathbf{y}) |J\phi(\mathbf{y})| d\mathbf{y}.
\end{aligned}$$

For the particular case  $\beta_0 = 1$ , we have by [35, Corollary 5.16] that the Fox–Wright type function becomes simply an exponential function:

$$E_{1,2,1}^1(-|\mathbf{k}|^{\alpha} t^2) = {}_1F_1 \left( 1; 1; \frac{-|\mathbf{k}|^{\alpha} t^2}{2} \right) = E_{1,1} \left( \frac{-|\mathbf{k}|^{\alpha} t^2}{2} \right) = \exp \left( \frac{-|\mathbf{k}|^{\alpha} t^2}{2} \right),$$

so in this case we have

$$w(\mathbf{x}, t) = \int_{\mathbb{R}^n} w_0(\phi^{-1}(\phi(\mathbf{x}) - \phi(\mathbf{y}))) \mathcal{F}_{\phi}^{-1} \left( \exp \left( \frac{-|\mathbf{k}|^{\alpha} t^2}{2} \right) \right) (\mathbf{y}) |J\phi(\mathbf{y})| d\mathbf{y}.$$

If we further let  $\alpha \rightarrow 2$  to obtain the non-fractional special case  $\beta_0 = 1$ ,  $\alpha = 2$ , then the solution can be found more explicitly. Indeed, we have

$$\mathcal{F}_{\phi}^{-1} \left( \exp \left( \frac{-|\mathbf{k}|^2 t^2}{2} \right) \right) = \frac{t^{-n}}{(2\pi)^{n/2}} \exp \left( \frac{-|\phi(\mathbf{x})|^2}{2t^2} \right),$$

by using a classical result on Fourier transforms of Gaussian distributions together with the fact that  $\mathcal{F}_\phi^{-1} = Q_\phi \circ \mathcal{F}^{-1}$ . Then,

$$\begin{aligned} w(\mathbf{x}, t) &= \frac{t^{-n}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} w_0(\phi^{-1}(\phi(\mathbf{x}) - \phi(\mathbf{y}))) \exp\left(\frac{-|\phi(\mathbf{y})|^2}{2t^2}\right) |J\phi(\mathbf{y})| d\mathbf{y} \\ &= \frac{t^{-n}}{(2\pi)^{n/2}} \left( w_0 *_\phi \exp\left(\frac{-|\phi(\cdot)|^2}{2t^2}\right) \right) (\mathbf{x}) \\ &= \frac{t^{-n}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left(\frac{-|\phi(\mathbf{x}) - \phi(\mathbf{y})|^2}{2t^2}\right) w_0(\mathbf{y}) |J\phi(\mathbf{y})| d\mathbf{y}, \end{aligned}$$

and this is the explicit solution for the non-fractional version of the modified heat equation (33) with respect to the  $n$ -dimensional function  $\phi$ . We note that the initial condition  $w(\mathbf{x}, 0) = w_0(\mathbf{x})$  is straightforward to verify using the Dirac delta function as a distributional limit of heat kernels.

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