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► **To cite this version:**

Michel de Lara, Jean-Philippe Chancelier, Benjamin Heymann. Topological Conditional Separation. 2021. hal-03315811

HAL Id: hal-03315811

<https://hal.science/hal-03315811>

Preprint submitted on 6 Aug 2021

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Topological Conditional Separation

Michel De Lara[†], Jean-Philippe Chancelier^{*}, Benjamin Heymann[†]

August 6, 2021

Abstract

Pearl's d-separation is a foundational notion to study conditional independence between random variables. We define the topological conditional separation and we show that it is equivalent to the d-separation, extended beyond acyclic graphs, be they finite or infinite.

1 Introduction

As the world shifts toward more and more data-driven decision-making, causal inference is taking more space in applied sciences, statistics and machine learning. This is because it allows for better, more robust decision-making, and provides a way to interpret the data that goes beyond correlation [7]. In his seminal work [6], Pearl builds on graphical models [3] to propose the so-called do-calculus, and he notably introduces the notion of d-separation on a Directed Acyclic Graph (DAG).

This paper has two companion papers [2, 5], Altogether, the three of them aim at providing another perspective on conditional independence and do-calculus. In this paper, we consider directed graphs (DGs), not necessarily acyclic, and we introduce a suitable topology on the set of vertices. Then, we define the new notion of topological conditional separation on DGs, and we prove its equivalence with an extension of Pearl's d-separation on DGs. The topological separation is practical because it just requires to check that two sets are disjoint. By contrast, the d-separation requires to check that *all* the paths that connect two variables are blocked. Moreover, as its name suggests, the topological separation has a theoretical interpretation which motivates a detour by the theory of Alexandrov topologies.

The paper is organized as follows. In Sect. 2, we provide background on binary relations and graphs, and then we present Alexandrov topologies induced by binary relations. In Sect. 3, we recall the definition of d-separation, then introduce a suitable topology on the set of vertices, and define a new notion of conditional topological separation (t-separation). Then, we show that d-separation and t-separation are equivalent, and we put forward a

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practical characterization of t-separation between subsets of vertices. We provide additional material on Alexandrov topologies in Appendix A, and we relegate technical lemmas in Appendix B.

2 Alexandrov topology on a graph

In §2.1, we provide background on binary relations and graphs. In §2.2, we present Alexandrov topologies induced by binary relations.

2.1 Background on binary relations, graphs and topologies

We use the notation $\llbracket r, s \rrbracket = \{r, r+1, \dots, s-1, s\}$ for any two integers r, s such that $r \leq s$.

2.1.1 Binary relations

Let \mathcal{V} be a nonempty set (finite or not). We recall that a *(binary) relation* \mathcal{R} on \mathcal{V} is a subset $\mathcal{R} \subset \mathcal{V} \times \mathcal{V}$ and that $\gamma \mathcal{R} \lambda$ means $(\gamma, \lambda) \in \mathcal{R}$. For any subset $\Gamma \subset \mathcal{V}$, the *(sub)diagonal relation* is $\Delta_\Gamma = \{(\gamma, \lambda) \in \mathcal{V} \times \mathcal{V} \mid \gamma = \lambda \in \Gamma\}$ and the *diagonal relation* is $\Delta = \Delta_\mathcal{V}$. A *foreset* of a relation \mathcal{R} is any set of the form $\mathcal{R}\lambda = \{\gamma \in \mathcal{V} \mid \gamma \mathcal{R} \lambda\}$, where $\lambda \in \mathcal{V}$, or, by extension, of the form $\mathcal{R}\Lambda = \{\gamma \in \mathcal{V} \mid \exists \lambda \in \Lambda, \gamma \mathcal{R} \lambda\}$, where $\Lambda \subset \mathcal{V}$. An *afterset* of a relation \mathcal{R} is any set of the form $\gamma \mathcal{R} = \{\lambda \in \mathcal{V} \mid \gamma \mathcal{R} \lambda\}$, where $\gamma \in \mathcal{V}$, or, by extension, of the form $\Gamma \mathcal{R} = \{\lambda \in \mathcal{V} \mid \exists \gamma \in \Gamma, \gamma \mathcal{R} \lambda\}$, where $\Gamma \subset \mathcal{V}$. The *opposite* or *complementary* \mathcal{R}^c of a binary relation \mathcal{R} is the relation $\mathcal{R}^c = \mathcal{V} \times \mathcal{V} \setminus \mathcal{R}$, that is, defined by $\gamma \mathcal{R}^c \lambda \iff \neg(\gamma \mathcal{R} \lambda)$. The *converse* \mathcal{R}^{-1} of a binary relation \mathcal{R} is defined by $\gamma \mathcal{R}^{-1} \lambda \iff \lambda \mathcal{R} \gamma$. A relation \mathcal{R} is *symmetric* if $\mathcal{R}^{-1} = \mathcal{R}$, and is *anti-symmetric* if $\mathcal{R}^{-1} \cap \mathcal{R} \subset \Delta$.

The *composition* $\mathcal{R}\mathcal{R}'$ of two binary relations $\mathcal{R}, \mathcal{R}'$ on \mathcal{V} is defined by $\gamma(\mathcal{R}\mathcal{R}')\lambda \iff \exists \delta \in \mathcal{V}, \gamma \mathcal{R} \delta$ and $\delta \mathcal{R}' \lambda$; then, by induction we define $\mathcal{R}^{n+1} = \mathcal{R}\mathcal{R}^n$ for $n \in \mathbb{N}^*$. The *transitive closure* of a binary relation \mathcal{R} is $\mathcal{R}^+ = \bigcup_{k=1}^{\infty} \mathcal{R}^k$ (and \mathcal{R} is *transitive* if $\mathcal{R}^+ = \mathcal{R}$) and the *reflexive and transitive closure* is $\mathcal{R}^* = \mathcal{R}^+ \cup \Delta = \bigcup_{k=0}^{\infty} \mathcal{R}^k$ with the convention $\mathcal{R}^0 = \Delta$. A *partial equivalence relation* is a symmetric and transitive binary relation (generally denoted by \sim or \equiv). An *equivalence relation* is a reflexive, symmetric and transitive binary relation.

2.1.2 Preorders

A *preorder* (or “quasi-ordering”) on \mathcal{V} is a reflexive and transitive binary relation (generally denoted by \preceq), whereas an *order* is an anti-symmetric preorder (generally denoted by \leq). For a preorder, the forset (resp. afterset) of a subset $\Gamma \subset \mathcal{V}$ is called the *downset* (resp. *upset*) of Γ and is denoted by $\downarrow \Gamma$ (resp. by $\uparrow \Gamma$):

$$\downarrow \Gamma = \{\alpha \in \mathcal{V} \mid \exists \gamma \in \Gamma, \alpha \preceq \gamma\}, \quad \uparrow \Gamma = \{\alpha \in \mathcal{V} \mid \exists \gamma \in \Gamma, \gamma \preceq \alpha\}.$$

Then, a subset $\Gamma \subset \mathcal{V}$ is called an *upper set* (resp. a *lower set*) — or also an *upward closed set* (resp. *downward closed set*) — with respect to the preorder \preceq if $\downarrow V \subset V$ (resp. $\uparrow V \subset V$) or, equivalently, if $\downarrow V = V$ (resp. $\uparrow V = V$).

2.1.3 Graphs

Let \mathcal{V} be a nonempty set (finite or not), whose elements are called *vertices*. Let $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ be a relation on \mathcal{V} , whose elements are ordered pairs (that is, couples) of vertices called *edges*. The first element of an edge is the *tail of the edge*, whereas the second one is the *head of the edge*. Both tail and head are called *endpoints* of the edge, and we say that the edge connects its endpoints. We define a *loop* as an element of $\Delta \cap \mathcal{E}$, that is, a loop is an edge that connects a vertex to itself.

A *graph*, as we use it throughout this paper, is a couple $(\mathcal{V}, \mathcal{E})$ where $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. This definition is very basic and we now stress proximities and differences with classic notions in graph theory. As we define a graph, it may hold a finite or infinite number of vertices; there is at most one edge that has a couple of ordered vertices as single endpoints, hence a graph (in our sense) is not a multigraph (in graph theory); loops are not excluded (since we do not impose $\Delta \cap \mathcal{E} = \emptyset$). Hence, what we call a graph would be called a directed simple graph permitting loops in graph theory.

2.1.4 Topologies

We refer the reader to [4, Chapter 4] for notions in topology. Let \mathcal{V} be a nonempty set. The set $\mathcal{T} \subset 2^{\mathcal{V}}$ is said to be a *topology* on \mathcal{V} if \mathcal{T} contains both \emptyset, \mathcal{V} and is stable under the union and finite intersection operations. The space $(\mathcal{V}, \mathcal{T})$ is called *topological space*. Any element $O \in \mathcal{T}$ is called an *open set* (more precisely a \mathcal{T} -open set), and any element in

$$\mathcal{T}' = \{C \subset \mathcal{V} \mid C^c \in \mathcal{T}\} \quad (1)$$

is called a *closed set* (more precisely a \mathcal{T} -closed set). For any subset $V \subset \mathcal{V}$, the intersection of all the closed sets that contain V is a closed set called *topological closure* and denoted by \overline{V} (or, when needed, $\overline{V}^{\mathcal{T}}$).

A *clopen set* (more precisely a \mathcal{T} -clopen set) is a subset of \mathcal{V} which is both closed and open, that is, an element of $\mathcal{T} \cap \mathcal{T}'$. A topological space $(\mathcal{V}, \mathcal{T})$ is said to be disconnected if it is the union of two disjoint nonempty open sets; otherwise, it is said to be connected. A subset $V \subset \mathcal{V}$ of \mathcal{V} is said to be *connected* (more precisely \mathcal{T} -connected) if it is connected under its *subspace topology* $\mathcal{T} \cap V = \{O \cap V \in \mathcal{T} \mid O \in \mathcal{T}\}$ (also called *trace topology* or *relative topology*). A *connected component* of the topological space $(\mathcal{V}, \mathcal{T})$ (also called a \mathcal{T} -connected component) is a maximal (for the inclusion order) connected subset. A connected component is necessarily closed and the connected components of $(\mathcal{V}, \mathcal{T})$ form a partition of \mathcal{V} [4, Exercise 4.11.13]. Any clopen set is a union of (possibly infinitely many) connected components.

Let $(\mathcal{V}_i, \mathcal{T}_i)$, $i = 1, 2$ be two topological spaces. The *product topology* $\mathcal{T}_1 \otimes \mathcal{T}_2$ is the smallest subset $\mathcal{T} \subset 2^{\mathcal{V}_1 \times \mathcal{V}_2}$ which is a topology on the product set $\mathcal{V}_1 \times \mathcal{V}_2$ and which contains all the finite rectangles $\{O_1 \times O_2 \mid O_i \in \mathcal{T}_i, i = 1, 2\}$.

Specialization preorder. With any topology \mathcal{T} on \mathcal{V} , one associates the so-called *specialization* (or canonical) *preorder* as the binary relation $\preceq_{\mathcal{T}}$ on \mathcal{V} defined by [4, § 4.2.1,

Lemma 4.2.7]

$$\gamma \preceq_{\mathcal{T}} \lambda \iff \gamma \in \overline{\lambda}^{\mathcal{T}} \quad (\forall \gamma, \lambda \in \mathcal{V}). \quad (2)$$

The relation $\preceq_{\mathcal{T}}$ is reflexive and transitive, hence is a preorder (hence the notation). Following the notation in §2.1.1 — with the notation $\downarrow_{\mathcal{T}}$ for a downset and $\uparrow_{\mathcal{T}}$ for an upset — we have that

$$\downarrow_{\mathcal{T}} \lambda = \overline{\lambda}^{\mathcal{T}}, \quad \forall \lambda \in \mathcal{V}, \quad (3)$$

it is readily shown (and well-known [4, Lemmas 4.2.6 and 4.2.7]) that every open set is an upper set and every closed set is a lower set.

Preorder topology. It can be shown that, for any preorder \preceq on \mathcal{V} , the set

$$\mathcal{T}_{\preceq} = \{O \subset \mathcal{V} \mid \uparrow O \subset O\} \quad (4)$$

is a topology and that it is the finest topology \mathcal{T} that has \preceq as specialization order (that is, such that $\preceq_{\mathcal{T}} = \preceq$) [4, Proposition 4.2.11]. The topology \mathcal{T}_{\preceq} is an Alexandrov topology as follows.

Alexandrov topology. The set $\mathcal{T} \subset 2^{\mathcal{V}}$ is said to be an *Alexandrov topology* on \mathcal{V} if \mathcal{T} contains both \emptyset, \mathcal{V} and is stable under the union and (not necessarily finite) intersection operations. If \mathcal{T} is an Alexandrov topology, then \mathcal{T}' in (1) also is an Alexandrov topology, that we call the *dual (Alexandrov) topology* [1].

It is established that a topology \mathcal{T} is an Alexandrov topology if and only if $\mathcal{T} = \mathcal{T}_{\preceq_{\mathcal{T}}}$, where $\preceq_{\mathcal{T}}$ is the specialization preorder of \mathcal{T} , that is, if and only if the open sets (in \mathcal{T}) are exactly the upper sets (with respect to $\preceq_{\mathcal{T}}$) — or, equivalently, the closed sets (in \mathcal{T}') are exactly the lower sets (with respect to $\preceq_{\mathcal{T}}$) [4, Proposition 4.2.11, Exercise 4.2.13]. Thus, if \mathcal{T} is an Alexandrov topology, we have that

$$\left(O \in \mathcal{T} \iff \downarrow_{\mathcal{T}} O \subset O \right) \text{ and } \left(C \in \mathcal{T}' \iff \uparrow_{\mathcal{T}} C \subset C \right). \quad (5)$$

In an Alexandrov topology, it can be shown that, for any family $\{\Gamma_s\}_{s \in \mathbb{S}}$ of subsets $\Gamma_s \subset \mathcal{V}$, the topological closure satisfies

$$\overline{\bigcup_{s \in \mathbb{S}} \Gamma_s} = \bigcup_{s \in \mathbb{S}} \overline{\Gamma_s}, \quad \forall \Gamma_s \subset \mathcal{V}, \quad s \in \mathbb{S}. \quad (6)$$

Indeed, as $\Gamma_s \subset \overline{\bigcup_{s' \in \mathbb{S}} \Gamma_{s'}}$, we get that $\bigcup_{s \in \mathbb{S}} \Gamma_s \subset \bigcup_{s \in \mathbb{S}} \overline{\Gamma_s} \subset \overline{\bigcup_{s \in \mathbb{S}} \Gamma_s}$. As the set $\bigcup_{s \in \mathbb{S}} \overline{\Gamma_s}$ is closed, by definition of Alexandrov topology, we conclude. By (6), it is readily deduced that

$$\overline{\Gamma} = \downarrow_{\mathcal{T}} \Gamma, \quad \forall \Gamma \subset \mathcal{V}. \quad (7)$$

2.2 Alexandrov topology induced by a binary relation

As recalled, a topology is an Alexandrov topology if and only if it is the topology of a preorder [4, Exercise 4.2.13]. In fact, one can associate a topology with any binary relation (see (8) below) and prove that a topology is an Alexandrov topology if and only if it is the topology of a binary relation [1, Théorème 1.2]. In Proposition 1, we analyze the Alexandrov topology induced by a binary relation; we recover known results [1, Théorème 1.2] and we add some new results. Additional results are provided in Appendix A.

Proposition 1 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, that is, \mathcal{V} is a set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$.*

The following set

$$\mathcal{T}_{\mathcal{E}} = \{O \subset \mathcal{V} \mid O\mathcal{E} \subset O\} \quad (8)$$

is an Alexandrov topology on \mathcal{V} with the property that open subsets are characterized by

$$O \in \mathcal{T}_{\mathcal{E}} \iff O\mathcal{E} \subset O \iff O\mathcal{E}^+ \subset O \iff O\mathcal{E}^* \subset O \iff O\mathcal{E}^* = O. \quad (9)$$

In the Alexandrov topology $\mathcal{T}_{\mathcal{E}}$, the topological closure¹ $\overline{\Gamma}^{\mathcal{E}}$ of a subset $\Gamma \subset \mathcal{V}$ is given by

$$\overline{\Gamma}^{\mathcal{E}} = \mathcal{E}^*\Gamma, \quad \forall \Gamma \subset \mathcal{V}, \quad (10)$$

that is, is the \mathcal{E}^ -foreset; the closed subsets are characterized by*

$$C \in \mathcal{T}'_{\mathcal{E}} \iff \mathcal{E}C \subset C \iff \mathcal{E}^+C \subset C \iff \mathcal{E}^*C \subset C \iff \mathcal{E}^*C = C \iff \overline{C}^{\mathcal{E}} = C. \quad (11)$$

The Alexandrov topology $\mathcal{T}_{\mathcal{E}}$ satisfies

$$\mathcal{T}_{\mathcal{E}} = \mathcal{T}_{\mathcal{E}^+} = \mathcal{T}_{\mathcal{E}^*}, \quad (12a)$$

and the dual Alexandrov topology $\mathcal{T}'_{\mathcal{E}}$ satisfies

$$\mathcal{T}'_{\mathcal{E}} = \mathcal{T}_{\mathcal{E}^{-1}} = \mathcal{T}_{(\mathcal{E}^{-1})^+} = \mathcal{T}_{(\mathcal{E}^{-1})^*}. \quad (12b)$$

Regarding the specialization preorder $\preceq_{\mathcal{T}}$, it is well-known that, for any preorder \preceq on \mathcal{V} , we have that $\preceq_{\mathcal{T}_{\preceq}} = \preceq$ [4, Proposition 4.2.11]. More generally, it holds that

$$\preceq_{\mathcal{T}_{\mathcal{E}}} = \mathcal{E}^*, \quad \forall \mathcal{E} \subset \mathcal{V}^2. \quad (13)$$

Proof. We prove that the set $\mathcal{T}_{\mathcal{E}}$ in (8) contains both \emptyset, \mathcal{V} and is stable under the union and intersection operations, be they finite or infinite, which is what is required for an Alexandrov topology. Indeed, both $\emptyset, \mathcal{V} \in \mathcal{T}_{\mathcal{E}}$ as $\emptyset\mathcal{E} = \emptyset$ and $\mathcal{V}\mathcal{E} \subset \mathcal{V}$. Let $\{O_s\}_{s \in \mathbb{S}}$ be a family in $\mathcal{T}_{\mathcal{E}}$, that is, $O_s\mathcal{E} \subset O_s$ for all $s \in \mathbb{S}$. We deduce that $(\cup_{s \in \mathbb{S}} O_s)\mathcal{E} = \cup_{s \in \mathbb{S}} O_s\mathcal{E} \subset \cup_{s \in \mathbb{S}} O_s$, hence stability by union, and also that $(\cap_{s \in \mathbb{S}} O_s)\mathcal{E} \subset \cap_{s \in \mathbb{S}} O_s\mathcal{E} \subset \cap_{s \in \mathbb{S}} O_s$, hence stability by intersection.

¹To alleviate the notation, we have denoted the topological closure by $\overline{\Gamma}^{\mathcal{E}}$ instead of $\overline{\Gamma}^{\mathcal{T}_{\mathcal{E}}}$.

We establish the useful equivalences:

$$\begin{aligned}
O\mathcal{E}^* = O &\iff O\mathcal{E}^* \subset O && \text{(because } O \subset O\mathcal{E}^* \text{ since } \Delta \subset \mathcal{E}^* = \mathcal{E}^+ \cup \Delta) \\
&\iff O\mathcal{E}^+ \subset O && \text{(because } \mathcal{E}^* = \mathcal{E}^+ \cup \Delta) \\
&\iff O\mathcal{E} \subset O && \text{(because } \mathcal{E}^+ = \bigcup_{k=1}^{\infty} \mathcal{E}^k \text{ and then by induction)} \\
&\iff \mathcal{E}O^c \subset O^c
\end{aligned}$$

indeed, suppose by contradiction that $O\mathcal{E} \subset O$ but that there exists $\alpha \in \mathcal{E}O^c$ such that $\alpha \notin O^c$, that is, $\alpha \in O$; as a consequence, there exists $\gamma \in O^c$ such that $\alpha\mathcal{E}\gamma$, hence that $\gamma \in \alpha\mathcal{E}$; now, as $\alpha \in O$, we get that $\gamma \in \alpha\mathcal{E} \subset O\mathcal{E} \subset O$ by assumption; but this contradicts that $\gamma \in O^c$; the reverse implication is proved in the same way ; the rest of the equivalences below are proved as above

$$\begin{aligned}
&\iff \mathcal{E}^+O^c \subset O^c \\
&\iff \mathcal{E}^*O^c \subset O^c \\
&\iff \mathcal{E}^*O^c = O^c .
\end{aligned}$$

We deduce that (9) holds true, hence also that $\mathcal{T}_{\mathcal{E}} = \mathcal{T}_{\mathcal{E}^+} = \mathcal{T}_{\mathcal{E}^*}$ by (8), and that (11) holds true, hence also that $\mathcal{T}'_{\mathcal{E}} = \mathcal{T}_{\mathcal{E}^{-1}} = \mathcal{T}_{(\mathcal{E}^{-1})^+} = \mathcal{T}_{(\mathcal{E}^{-1})^*}$ by (8) and by (1).

Finally, we consider a subset $\Gamma \subset \mathcal{V}$ and we characterize its topological closure $\overline{\Gamma}^{\mathcal{E}}$, the smallest closed subset that contains Γ . On the one hand, we have that $\Gamma \subset \mathcal{E}^*\Gamma$ since $\mathcal{E}^* = \mathcal{E}^+ \cup \Delta$. On the other hand, the set $\mathcal{E}^*\Gamma$ is closed since $\mathcal{E}^*(\mathcal{E}^*\Gamma) = (\mathcal{E}^*)^2\Gamma = \mathcal{E}^*\Gamma$, because the relation \mathcal{E}^* is transitive. By definition of the topological closure $\overline{\Gamma}^{\mathcal{E}}$, we deduce that $\overline{\Gamma}^{\mathcal{E}} \subset \mathcal{E}^*\Gamma$. Now, let $\Lambda \subset \mathcal{V}$ be a closed subset such that $\Gamma \subset \Lambda$. We necessarily have that $\mathcal{E}^*\Gamma \subset \mathcal{E}^*\Lambda = \Lambda$, where the last equality is by (11) as Λ is closed.

As a consequence, the topological closure $\overline{\Gamma}^{\mathcal{E}}$ will always contain the closed set $\mathcal{E}^*\Lambda$, from which we get that $\mathcal{E}^*\Lambda \subset \overline{\Gamma}^{\mathcal{E}}$. We conclude that $\overline{\Gamma}^{\mathcal{E}} = \mathcal{E}^*\Gamma$.

This ends the proof. □

3 Equivalence between d-separation and t-separation

In §3.1, we recall the (extended) definition of d-separation, then introduce a suitable topology on the set of vertices, and define a new notion of conditional topological separation (t-separation). Then, we show that d-separation and t-separation between vertices (and between subsets of vertices) are equivalent. In §3.2, we put forward a practical characterization of t-separation between subsets of vertices.

3.1 d- and t-separation between vertices

We first recall the (extended) definition of d-separation, second define a new notion of conditional topological separation (t-separation) and third prove their equivalence.

3.1.1 d-separation between vertices

In the companion paper [2] we generalize Pearl's d-separation beyond acyclic graphs as follows.

Definition 2 ([2, Definition 3]) *Let $(\mathcal{V}, \mathcal{E})$ be a graph, that is, \mathcal{V} is a set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and let $W \subset \mathcal{V}$ be a subset of vertices. We define the conditional parental relation \mathcal{E}^W as*

$$\mathcal{E}^W = \Delta_{W^c} \mathcal{E} \quad \text{that is, } \gamma \mathcal{E}^W \lambda \iff \gamma \in W^c \text{ and } \gamma \mathcal{E} \lambda \quad (\forall \gamma, \lambda \in \mathcal{V}), \quad (15a)$$

the conditional ascendent relation \mathcal{B}^W as

$$\mathcal{B}^W = \mathcal{E}(\Delta_{W^c} \mathcal{E})^* = \mathcal{E} \mathcal{E}^{W*} \text{ where } \mathcal{E}^{W*} = (\mathcal{E}^W)^* \quad (15b)$$

which relates a descendent with an ascendent by means of elements in W^c . We define their converses \mathcal{E}^{-W} and \mathcal{B}^{-W} as

$$\mathcal{E}^{-W} = (\mathcal{E}^W)^{-1} = \mathcal{E}^{-1} \Delta_{W^c}, \quad (15c)$$

$$\mathcal{B}^{-W} = (\mathcal{B}^W)^{-1} = (\mathcal{E}^{-1} \Delta_{W^c})^* \mathcal{E}^{-1} = \mathcal{E}^{-W*} \mathcal{E}^{-1} \text{ where } \mathcal{E}^{-W*} = (\mathcal{E}^{-W})^*. \quad (15d)$$

With these elementary binary relations, we define the conditional common cause relation \mathcal{K}^W as the symmetric relation

$$\mathcal{K}^W = \mathcal{B}^{-W} \Delta_{W^c} \mathcal{B}^W = \mathcal{E}^{-W+} \mathcal{E}^{W+}, \quad (15e)$$

the conditional cousinhood relation \mathcal{C}^W as the partial equivalence relation

$$\mathcal{C}^W = (\Delta_W \mathcal{K}^W \Delta_W)^+ \cup \Delta_W, \quad (15f)$$

and the conditional active relation \mathcal{A}^W as the symmetric relation

$$\mathcal{A}^W = \Delta \cup \mathcal{B}^W \cup \mathcal{B}^{-W} \cup \mathcal{K}^W \cup (\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W). \quad (15g)$$

With the conditional active relation \mathcal{A}^W , we can now define the notion of d-separation between vertices (which can readily be extended to d-separation between subsets of vertices).

Definition 3 (*d-separation between vertices, [2, Definition 2]*) *Let $(\mathcal{V}, \mathcal{E})$ be a graph, that is, \mathcal{V} is a set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and let $W \subset \mathcal{V}$ be a subset of vertices. Let $\gamma, \lambda \in \mathcal{V}$ be two vertices. We denote*

$$\gamma \perp\!\!\!\perp_d \lambda \mid W \iff \neg(\gamma \mathcal{A}^W \lambda), \quad (16)$$

and we say that the vertices γ and λ are d-separated (w.r.t. W).

3.1.2 t-separation between vertices

We introduce a suitable topology on the set of vertices, and we define a new notion of conditional topological separation.

Let $(\mathcal{V}, \mathcal{E})$ be a graph, $W \subset \mathcal{V}$ be a subset of vertices, and \mathcal{E}^W in (15a) be the corresponding conditional parental relation. To alleviate the notation, in the Alexandrov topology $\mathcal{T}_{\mathcal{E}^W}$ in (8), we use the following. For any subset $\Gamma \subset \mathcal{V}$, the topological closure is denoted² by $\overline{\Gamma}^W$, and the downset is denoted³ by $\downarrow_W \Gamma$. By (7) and (10), we get that

$$\overline{\Gamma}^W = \mathcal{E}^{W*} \Gamma = \downarrow_W \Gamma. \quad (17)$$

Notice that the subset W is $\mathcal{T}_{\mathcal{E}^W}$ -open, that is, $W \in \mathcal{T}_{\mathcal{E}^W}$. Indeed, the complementary set W^c is closed as it satisfies $\mathcal{E}^{W*} W^c = (\mathcal{E}^W)^+ W^c \cup W^c \subset W^c$, as $\mathcal{E}^W \mathcal{V} \subset W^c$ because $\mathcal{E}^W = \Delta_{W^c} \mathcal{E}$ by (15a) and by definition of the subdiagonal relation Δ_{W^c} .

With the conditional ascendent relation \mathcal{B}^W , the conditional common cause relation \mathcal{K}^W the conditional cousinhood relation \mathcal{C}^W and the $\mathcal{T}_{\mathcal{E}^W}$ -topological closure, we can now define the notion of t-separation between vertices (which can readily be extended to t-separation between subsets of vertices).

Definition 4 (Conditional topological separation between vertices, t-separation)

Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices. We set

$$\mathfrak{S}^W = \Delta \cup \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W). \quad (18)$$

Let $\gamma, \lambda \in \mathcal{V}$ be two vertices. We denote

$$\gamma \underset{t}{\parallel} \lambda \mid W \iff \overline{\mathfrak{S}^W \gamma}^W \cap \overline{\mathfrak{S}^W \lambda}^W = \emptyset, \quad (19)$$

and we say that the vertices γ and λ are conditionally topologically separated (w.r.t. W) or, shortly, t-separated.

With the above definitions, we now show that the notions of d- and t-separation are equivalent on the complementary set W^c .

Theorem 5 Let $(\mathcal{V}, \mathcal{E})$ be a graph, that is, \mathcal{V} is a set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and let $W \subset \mathcal{V}$ be a subset of vertices. We have the equivalence

$$\gamma \underset{t}{\parallel} \lambda \mid W \iff \gamma \underset{d}{\parallel} \lambda \mid W \quad (\forall \gamma, \lambda \in W^c). \quad (20)$$

Proof. To prove (20), it is equivalent, by Definition 3, to prove the equivalence

$$\gamma \underset{t}{\parallel} \lambda \mid W \iff \neg(\gamma \mathcal{A}^W \lambda) \quad (\forall \gamma, \lambda \in W^c). \quad (21)$$

²Instead of $\overline{\Gamma}^{\mathcal{E}^W}$ or even of $\overline{\Gamma}^{\mathcal{T}_{\mathcal{E}^W}}$.

³Instead of $\downarrow_{\mathcal{T}_{\mathcal{E}^W}}$.

For this purpose, we set

$$\mathfrak{S}^{-W} = \Delta \cup (\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W = (\mathfrak{S}^W)^{-1}. \quad (22)$$

Let $\gamma, \lambda \in \mathcal{V}$ be two vertices such that $\gamma, \lambda \in W^c$. We have

$$\overline{\mathfrak{S}^W \gamma}^W \cap \overline{\mathfrak{S}^W \lambda}^W \neq \emptyset \iff (\mathcal{E}^{W*} \mathfrak{S}^W \gamma) \cap (\mathcal{E}^{W*} \mathfrak{S}^W \lambda) \neq \emptyset$$

because the topological closure of a subset $\Gamma \subset \mathcal{V}$ is given by $\overline{\Gamma}^W = \mathcal{E}^{W*} \Gamma$ by (17)

$$\begin{aligned} &\iff (\gamma \mathfrak{S}^{-W} (\mathcal{E}^{W*})^{-1}) \cap (\mathcal{E}^{W*} \mathfrak{S}^W \lambda) \neq \emptyset \\ &\hspace{15em} \text{(by definition of the converse relation)} \\ &\iff \gamma \mathfrak{S}^{-W} \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathfrak{S}^W \lambda \end{aligned}$$

by definition of relation composition and by $(\mathcal{E}^{W*})^{-1} = \mathcal{E}^{-W*}$

$$\begin{aligned} &\iff \gamma \Delta_{W^c} \mathfrak{S}^{-W} \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathfrak{S}^W \Delta_{W^c} \lambda \quad \text{(because } \gamma, \lambda \in W^c \text{ by assumption)} \\ &\iff \gamma \Delta_{W^c} \mathcal{A}^W \Delta_{W^c} \lambda \quad \text{(by (36) in Appendix B)} \\ &\iff \gamma \mathcal{A}^W \lambda. \quad \text{(because } \gamma, \lambda \in W^c \text{ by assumption)} \end{aligned}$$

Thus, by taking the negation, we have obtained (21), hence (20) by Definition 3. \square

3.2 Characterization of t-separation between subsets

We put forward a practical characterization of t-separation between subsets of vertices. For this purpose, we introduce the notion of splitting, which slightly generalizes the notion of partition.

For any subset $\Gamma \subset \mathcal{V}$ and for any family $\{\Gamma_s\}_{s \in \mathbb{S}}$ of subsets $\Gamma_s \subset \mathcal{V}$, we write $\sqcup_{s \in \mathbb{S}} \Gamma_s = \Gamma$ when we have, on the one hand, $(s \neq s' \implies \Gamma_s \cap \Gamma_{s'} = \emptyset)$ and, on the other hand, $\bigcup_{s \in \mathbb{S}} \Gamma_s = \Gamma$. We will also say that $\{\Gamma_s\}_{s \in \mathbb{S}}$ is a *splitting* of Γ (we do not use the vocable of partition because it is not required that the subsets Γ_s be nonempty).

Proposition 6 (Topological separation between subsets) *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices. Let $\Gamma, \Lambda \subset \mathcal{V}$ be two subsets of vertices such that*

$$\Gamma \cap \Lambda = \emptyset, \quad \Gamma \cap W = \emptyset, \quad \Lambda \cap W = \emptyset. \quad (23)$$

The following statements are equivalent:

1. *For any $\gamma \in \Gamma, \lambda \in \Lambda$, we have that $\gamma \perp\!\!\!\perp_t \lambda \mid W$, as in Definition 4,*
2. *There exists a splitting W_Γ, W_Λ of W such that*

$$W_\Gamma \sqcup W_\Lambda = W \text{ and } \overline{\Gamma \cup W_\Gamma}^W \cap \overline{\Lambda \cup W_\Lambda}^W = \emptyset. \quad (24)$$

Proof.

• (Item 1 \implies Item 2). We consider two subsets $\Gamma, \Lambda \subset \mathcal{V}$ such that (23) holds true, and we prove the existence of a splitting W_Γ, W_Λ of W satisfying (23) in two steps.

First, we set $W'_\Gamma = \mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)\Gamma \subset W$ by definition (15f) of \mathcal{C}^W and $W'_\Lambda = \mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)\Lambda \subset W$, and we prove that

$$\overline{\Gamma \cup W'_\Gamma}^W \cap \overline{\Lambda \cup W'_\Lambda}^W = \emptyset \quad (25)$$

We have that

$$\begin{aligned} \overline{\Gamma \cup W'_\Gamma}^W \cap \overline{\Lambda \cup W'_\Lambda}^W &= \overline{\Gamma \cup \mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)\Gamma}^W \cap \overline{\Lambda \cup (\mathcal{B}^{-W} \cup \mathcal{K}^W)\Lambda}^W \\ &= \overline{\mathfrak{S}^W \Gamma}^W \cap \overline{\mathfrak{S}^W \Lambda}^W \quad (\text{by definition (18) of } \mathfrak{S}^W) \\ &= \left(\bigcup_{\gamma \in \Gamma} \overline{\mathfrak{S}^W \gamma}^W \right) \cap \left(\bigcup_{\lambda \in \Lambda} \overline{\mathfrak{S}^W \lambda}^W \right) \end{aligned}$$

by property (6) of the topological closure in an Alexandrov topology

$$= \bigcup_{\gamma \in \Gamma, \lambda \in \Lambda} \underbrace{\left(\overline{\mathfrak{S}^W \gamma}^W \cap \overline{\mathfrak{S}^W \lambda}^W \right)}_{=\emptyset}$$

by (19) as $\gamma \not\perp \lambda \mid W$ by assumption, with $\gamma \in \Gamma \subset W^c$ and $\lambda \in \Lambda \subset W^c$ by (23).

Thus, we have proven (25).

Second, we are now going to prove that we can enlarge the subsets W'_Γ and W'_Λ to obtain a splitting of W satisfying Equation (24).

For this purpose, we set $\widetilde{W} = W \setminus (W'_\Gamma \cup W'_\Lambda) \subset W$. The cousinhood relation \mathcal{C}^W in (15f) is a partial equivalence relation on \mathcal{V} , and it is easily seen to be an equivalence relation on W . This is why we consider the partition $\widetilde{W} = \sqcup_{i \in I} \widetilde{W}_i$ of \widetilde{W} , where, for each $i \in I$, the elements of the subset \widetilde{W}_i belong to the same equivalence class of the equivalence relation $\mathcal{C}^W \subset W^2$ (understood as the restriction of the cousinhood relation \mathcal{C}^W to W). We are now going to prove that

$$\forall i \in I, \text{ either } \overline{\widetilde{W}_i}^W \cap \overline{\Gamma \cup W'_\Gamma}^W = \emptyset \quad \text{or} \quad \overline{\widetilde{W}_i}^W \cap \overline{\Lambda \cup W'_\Lambda}^W = \emptyset. \quad (26)$$

The proof is by contradiction. Let $i \in I$ be fixed and suppose that both $\overline{\widetilde{W}_i}^W \cap \overline{\Gamma \cup W'_\Gamma}^W \neq \emptyset$ and $\overline{\widetilde{W}_i}^W \cap \overline{\Lambda \cup W'_\Lambda}^W \neq \emptyset$. First, as $\overline{\widetilde{W}_i}^W \cap \overline{\Gamma \cup W'_\Gamma}^W \neq \emptyset$, there would exist $\lambda \in \overline{\widetilde{W}_i}^W \cap \overline{\Gamma \cup W'_\Gamma}^W$, hence there would exist $w_i^{(1)} \in \widetilde{W}_i$ such that $\lambda \mathcal{E}^{W^*} w_i^{(1)}$ (as $\overline{\widetilde{W}_i}^W = \mathcal{E}^{W^*} \widetilde{W}_i$ by (10)), and there would exist $\gamma \in \Gamma$ such that $\lambda \mathcal{E}^{W^*} (\Delta \cup \mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)) \gamma$ (as $\overline{\Gamma \cup W'_\Gamma}^W = \mathcal{E}^{W^*} (\Gamma \cup W'_\Gamma) = \mathcal{E}^{W^*} (\Delta \cup \mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)) \Gamma$ by (10) and by definition of W'_Γ). Thus, we would have that $w_i^{(1)} \mathcal{E}^{-W^*} \mathcal{E}^{W^*} (\Delta \cup \mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)) \gamma$. Second, as $\overline{\widetilde{W}_i}^W \cap \overline{\Lambda \cup W'_\Lambda}^W \neq \emptyset$ and proceeding in the same way, there would exist $w_i^{(2)} \in \widetilde{W}_i$ and $\lambda \in \Lambda$ such that $w_i^{(2)} \mathcal{E}^{-W^*} \mathcal{E}^{W^*} (\Delta \cup \mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)) \lambda$. Now, as $w_i^{(1)}$ and $w_i^{(2)}$ would be both in \widetilde{W}_i they would be in the same equivalence class for the equivalence relation \mathcal{C}^W , giving thus $w_i^{(1)} \mathcal{C}^W w_i^{(2)}$. Finally, we would obtain that

$$\gamma \Delta_{W^c} (\Delta \cup (\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W) \mathcal{E}^{-W^*} \mathcal{E}^{W^*} \mathcal{C}^W \mathcal{E}^{-W^*} \mathcal{E}^{W^*} (\Delta \cup \mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)) \Delta_{W^c} \lambda. \quad (27)$$

Combining (27), Equation (35) and the definition (15g) of \mathcal{A}^W , we would obtain that $\gamma\mathcal{A}^W\lambda$. Thus, we would arrive at a contradiction as we have, by assumption, $\gamma \perp\!\!\!\perp_t \lambda \mid W$, hence $\neg(\gamma\mathcal{A}^W\lambda)$ by (21).

Thus, we have proven that the disjunction (26) holds true, from which we obtain a splitting $I = I_\Gamma \sqcup I_\Lambda$ and a splitting $\widetilde{W} = \widetilde{W}_\Gamma \sqcup \widetilde{W}_\Lambda$ defined by

$$\begin{aligned} I_\Gamma &= \{i \in I \mid \widetilde{W}_i^W \cap \overline{\Lambda \cup W'_\Lambda}^W = \emptyset\} & \text{and} & & I_\Lambda &= I \setminus I_\Gamma, \\ \widetilde{W}_\Gamma &= \bigcup_{i \in I_\Gamma} \widetilde{W}_i^W & \text{and} & & \widetilde{W}_\Lambda &= \bigcup_{i \in I_\Lambda} \widetilde{W}_i^W, \end{aligned}$$

which, by construction, satisfies

$$\widetilde{W}_\Gamma^W \cap \overline{\Lambda \cup W'_\Lambda}^W = \emptyset \quad \text{and} \quad \widetilde{W}_\Lambda^W \cap \overline{\Gamma \cup W'_\Gamma}^W = \emptyset. \quad (28)$$

Now, we define $W_\Gamma = W'_\Gamma \cup \widetilde{W}_\Gamma$ and $W_\Lambda = W'_\Lambda \cup \widetilde{W}_\Lambda$, and we are going to prove that (24) holds true.

To check the second part of (24), we calculate

$$\begin{aligned} \overline{\Gamma \cup W'_\Gamma}^W \cap \overline{\Lambda \cup W'_\Lambda}^W &= \overline{\Gamma \cup W'_\Gamma \cup \widetilde{W}_\Gamma}^W \cap \overline{\Lambda \cup W'_\Lambda \cup \widetilde{W}_\Lambda}^W \\ &= (\overline{\Gamma \cup W'_\Gamma}^W \cup \widetilde{W}_\Gamma^W) \cap (\overline{\Lambda \cup W'_\Lambda}^W \cup \widetilde{W}_\Lambda^W) & (\text{by (6)}) \\ &= \underbrace{(\overline{\Gamma \cup W'_\Gamma}^W \cap \overline{\Lambda \cup W'_\Lambda}^W)}_{=\emptyset \text{ by (25)}} \cup \underbrace{(\overline{\Gamma \cup W'_\Gamma}^W \cap \widetilde{W}_\Lambda^W)}_{=\emptyset \text{ by (28)}} \\ &\quad \cup \underbrace{(\widetilde{W}_\Gamma^W \cap \overline{\Lambda \cup W'_\Lambda}^W)}_{=\emptyset \text{ by (28)}} \cup (\widetilde{W}_\Gamma^W \cap \widetilde{W}_\Lambda^W) \\ &= \left(\bigcup_{i \in I_\Gamma} \widetilde{W}_i^W \right) \cap \left(\bigcup_{j \in I_\Lambda} \widetilde{W}_j^W \right) \end{aligned}$$

by (6) and by definition of \widetilde{W}_Γ and \widetilde{W}_Λ

$$= \bigcup_{i \in I_\Gamma} \bigcup_{j \in I_\Lambda} \underbrace{(\widetilde{W}_i^W \cap \widetilde{W}_j^W)}_{=\emptyset} = \emptyset$$

as $I_\Gamma \cap I_\Lambda = \emptyset$, using the postponed Lemma 8 which gives $\widetilde{W}_i^W \cap \widetilde{W}_j^W = \emptyset$, for any $i, j \in I$ with $i \neq j$. From the just proven equality $\overline{\Gamma \cup W'_\Gamma}^W \cap \overline{\Lambda \cup W'_\Lambda}^W = \emptyset$, we readily get that $W_\Gamma \cap W_\Lambda = \emptyset$. Therefore, to check the first part of (24), it remains to calculate

$$W_\Gamma \cup W_\Lambda = (W'_\Gamma \cup W'_\Lambda) \cup \underbrace{\bigcup_{i \in I_\Gamma} \widetilde{W}_i^W \cup \bigcup_{i \in I_\Lambda} \widetilde{W}_i^W}_{=\widetilde{W} = W \setminus (W'_\Gamma \cup W'_\Lambda)} = (W'_\Gamma \cup W'_\Lambda) \cup (W \setminus (W'_\Gamma \cup W'_\Lambda)) = W.$$

• (Item 2 \implies Item 1). The proof is by contradiction. For this purpose, we suppose we suppose that there exists a splitting $W = W_\Gamma \sqcup W_\Lambda$ such that $\overline{\Gamma \cup W_\Gamma}^W \cap \overline{\Lambda \cup W_\Lambda}^W = \emptyset$ (Item 2) and that there exists $\gamma \in \Gamma$ and $\lambda \in \Lambda$ such that $\gamma \mathcal{A}^W \lambda$ (\neg Item 1). We show that we arrive at a contradiction.

Using the fact that $\gamma \mathcal{A}^W \lambda$ and that $\overline{\{\gamma\}}^W \cap \overline{\{\lambda\}}^W \subset \overline{\Gamma \cup W_\Gamma}^W \cap \overline{\Lambda \cup W_\Lambda}^W = \emptyset$, we would obtain, by the postponed Lemma 7, that there would exist a nonempty subset $W_{\gamma,\lambda} \subset W$ such that all the elements of $W_{\gamma,\lambda}$ would be in the same class for the \mathcal{C}^W partial equivalence relation, and such that

$$\overline{\{\gamma\}}^W \cap \overline{W_{\gamma,\lambda}}^W \neq \emptyset \quad \text{and} \quad \overline{\{\lambda\}}^W \cap \overline{W_{\gamma,\lambda}}^W \neq \emptyset. \quad (29)$$

Now, using the fact that $\overline{W_\Gamma}^W \cap \overline{W_\Lambda}^W \subset \overline{\Gamma \cup W_\Gamma}^W \cap \overline{\Lambda \cup W_\Lambda}^W = \emptyset$, we would obtain that $\overline{W_\Gamma}^W \cap \overline{W_\Lambda}^W = \emptyset$, which, would imply that $W_{\gamma,\lambda}$ would necessarily be included either in W_Γ or in W_Λ , by using the second part of Lemma 8. Therefore, Assuming that $W_{\gamma,\lambda} \subset W_\Gamma$, we obtain a contradiction using Equation (29), as we would have that

$$\emptyset \neq \overline{\{\lambda\}}^W \cap \overline{W_{\gamma,\lambda}}^W \subset \overline{\{\lambda\}}^W \cap \overline{W_\Gamma}^W \subset \overline{\Lambda \cup W_\Lambda}^W \cap \overline{\Gamma \cup W_\Gamma}^W = \emptyset.$$

Proceeding in a similar way in the case $W_{\gamma,\lambda} \subset W_\Lambda$ leads to a similar contradiction. \square

We end with Lemma 7 and Lemma 8 which are instrumental in the proof of Proposition 6.

Lemma 7 *Suppose that the assumptions of Proposition 6 are satisfied. Let $\gamma \in \Gamma$ and $\lambda \in \Lambda$ be given such that $\gamma \mathcal{A}^W \lambda$ and $\overline{\{\gamma\}}^W \cap \overline{\{\lambda\}}^W = \emptyset$. Then, there exists $w_\gamma, w_\lambda \in W$ such that w_γ and w_λ are in the same equivalence class of the partial equivalence relation \mathcal{C}^W (that is, $w_\gamma \mathcal{C}^W w_\lambda$) and such that $\overline{\{\gamma\}}^W \cap \overline{\{w_\gamma\}}^W \neq \emptyset$ and $\overline{\{\lambda\}}^W \cap \overline{\{w_\lambda\}}^W \neq \emptyset$.*

Proof. As a preliminary result, we prove that $\gamma(\Delta \cup \mathcal{B}^W \cup \mathcal{B}^{-W} \cup \mathcal{K}^W)\lambda$ contradicts the assumptions of Lemma 7. First, $\gamma \Delta \lambda$ contradicts the assumption $\Gamma \cap \Lambda = \emptyset$ in (23). Second, $\gamma \mathcal{B}^W \lambda$ contradicts the assumption $\overline{\{\gamma\}}^W \cap \overline{\{\lambda\}}^W = \emptyset$. Indeed, using the definition (15b) of the conditional ascendent relation \mathcal{B}^W , and using the fact that $\gamma \in \Gamma \subset W^c$, we have that $\gamma \mathcal{B}^W \lambda = \gamma \mathcal{E} \mathcal{E}^{W^*} \lambda = \gamma \Delta_{W^c} \mathcal{E} \mathcal{E}^{W^*} \lambda = \gamma \mathcal{E}^{W^*} \lambda$. Thus, $\gamma \mathcal{B}^W \lambda$ implies that $\gamma \in \mathcal{E}^{W^*} \lambda = \overline{\{\lambda\}}^W$ and thus a contradiction as $\emptyset = \overline{\{\gamma\}}^W \cap \overline{\{\lambda\}}^W \supset \gamma \cap \overline{\{\lambda\}}^W$. Third, following the same lines, $\gamma \mathcal{B}^{-W} \lambda$ implies that $\lambda \in \overline{\{\gamma\}}^W$ and contradicts $\overline{\{\gamma\}}^W \cap \overline{\{\lambda\}}^W = \emptyset$. Fourth, $\gamma \mathcal{K}^W \lambda = \gamma \mathcal{E}^{-W^+} \mathcal{E}^{W^+} \lambda$ and thus $\gamma \mathcal{K}^W \lambda$ implies that $\mathcal{E}^{W^+} \gamma \cap \mathcal{E}^{W^+} \lambda \neq \emptyset$, hence that $\mathcal{E}^{W^*} \gamma \cap \mathcal{E}^{W^*} \lambda \neq \emptyset$, that is, $\overline{\{\gamma\}}^W \cap \overline{\{\lambda\}}^W \neq \emptyset$ by (17). This again leads to a contradiction.

Therefore, we get that $\neg(\gamma(\Delta \cup \mathcal{B}^W \cup \mathcal{B}^{-W} \cup \mathcal{K}^W)\lambda)$ and $\gamma \mathcal{A}^W \lambda$ imply that $\gamma(\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \lambda$, using the definition (15g) of the conditional active relation \mathcal{A}^W . Thus, there exist $w_\gamma, w_\lambda \in W$ such that

$$\gamma(\mathcal{B}^W \cup \mathcal{K}^W)w_\gamma \text{ and } w_\lambda(\mathcal{B}^{-W} \cup \mathcal{K}^W)\lambda \text{ and } w_\gamma \mathcal{C}^W w_\lambda.$$

Now, we prove that $\gamma(\mathcal{B}^W \cup \mathcal{K}^W)w_\gamma$ implies that we have $\overline{\{\gamma\}}^W \cap \overline{\{w_\gamma\}}^W \neq \emptyset$. Indeed, as $\gamma(\mathcal{B}^W \cup \mathcal{K}^W)w_\gamma$, we have two possibilities. First, suppose that $\gamma \mathcal{B}^W w_\gamma$. Then, as $\gamma \in W^c$, this implies that $\gamma \Delta_{W^c} \mathcal{B}^W w_\gamma = \gamma \mathcal{E}^{W^*} w_\gamma$ which implies that $\gamma \in \mathcal{E}^{W^*} w_\gamma = \overline{\{w_\gamma\}}^W$ by (17). Therefore, we get that $\overline{\{\gamma\}}^W \cap \overline{\{w_\gamma\}}^W \neq \emptyset$. Second, if $\gamma \mathcal{K}^W w_\gamma$, then, as already seen at the beginning of the proof, we obtain that $\overline{\{\gamma\}}^W \cap \overline{\{w_\gamma\}}^W \neq \emptyset$.

Then, following similar arguments, we prove that $w_\lambda(\mathcal{B}^{-W} \cup \mathcal{K}^W)\lambda$ implies that we have $\overline{\{\lambda\}}^W \cap \overline{\{w_\lambda\}}^W \neq \emptyset$.

Finally, we have obtained that $\overline{\{\gamma\}}^W \cap \overline{\{w_\gamma\}}^W \neq \emptyset$ and $\overline{\{\lambda\}}^W \cap \overline{\{w_\lambda\}}^W \neq \emptyset$. Moreover w_γ and w_λ are in the same equivalence class of the partial equivalence relation \mathcal{C}^W as $w_\gamma \mathcal{C}^W w_\lambda$. Thus, we have found two elements $w_\gamma, w_\lambda \in W$ satisfying the conclusion of Lemma 7. This concludes the proof. \square

Lemma 8 *Let W' and W'' be two subsets of W which are included in two distinct equivalence classes of the partial equivalence relation \mathcal{C}^W . Then, we have that $\overline{W'}^W \cap \overline{W''}^W = \emptyset$.*

Conversely, assume given a splitting $W = W' \sqcup W''$ such that $\overline{W'}^W \cap \overline{W''}^W = \emptyset$. Then, there does not exist $w' \in W'$ and $w'' \in W''$ such that w' and w'' are in the same equivalence classes of \mathcal{C}^W

Proof. For the first assertion, we make a proof by contradiction. For this purpose, we consider W' and W'' , two subsets of W which are included in two distinct equivalence classes of the partial equivalence relation \mathcal{C}^W , and we suppose that $\overline{W'}^W \cap \overline{W''}^W \neq \emptyset$. Then, by (10), there exists $w' \in W'$ and $w'' \in W''$ and $\gamma \in \mathcal{V}$ such that $\gamma \in \mathcal{E}^{W^*} w'$ and $\gamma \in \mathcal{E}^{W^*} w''$. Therefore we have that $w' \mathcal{E}^{-W^*} \mathcal{E}^{W^*} w''$, from which we deduce that $w'(\Delta \cup \Delta_{W^c} \mathcal{B}^W \cup \mathcal{B}^{-W} \Delta_{W^c} \cup \mathcal{K}^W)w''$, using Equation (34a). Moreover, as by assumption, W' and W'' are two subsets of W which are included in two distinct equivalence classes of \mathcal{C}^W , we have that $w' \in W, w'' \in W$ and $w' \neq w''$. Hence, among the four possible cases corresponding to the union of four terms, only the last one is possible: we must necessarily have that $w' \Delta_W \mathcal{K}^W \Delta_W w''$, which implies that $w' \mathcal{C}^W w''$ by definition (15f) of \mathcal{C}^W . Thus, w' and w'' belong to the same equivalence class of \mathcal{C}^W , but this contradicts that $w' \in W'$ and $w'' \in W''$ where W' and W'' are included in two distinct equivalence classes of \mathcal{C}^W .

Now, we prove the converse assertion again by contradiction. For this purpose, consider a splitting $W = W' \sqcup W''$ such that $\overline{W'}^W \cap \overline{W''}^W = \emptyset$, and suppose that there exists $w' \in W'$ and $w'' \in W''$ such that w' and w'' are in the same equivalence classes of \mathcal{C}^W or otherwise said, such that $w' \mathcal{C}^W w''$. Using Equation (15f) and the fact that $w' \neq w''$, as $W' \cap W'' = \emptyset$, we deduce that $w'(\Delta_W \mathcal{K}^W \Delta_W)^+ w''$. Hence, there exists $k \in \mathbb{N}, k \geq 1$, and a sequence $\{w_i\}_{i \in [1, k]}$ in W such that, for all $i \in [1, k-1]$, we have $w_i \mathcal{K}^W w_{i+1}$ and $w' \mathcal{K}^W w_1$ and $w_k \mathcal{K}^W w''$. Setting $w_0 = w'$ and $w_{k+1} = w''$ and using the property that $\gamma \mathcal{K}^W \lambda \implies \overline{\{\gamma\}}^W \cap \overline{\{\lambda\}}^W \neq \emptyset$ (shown at the beginning of the proof of Lemma 7), we get that $\overline{\{w_i\}}^W \cap \overline{\{w_{i+1}\}}^W \neq \emptyset$ for $i \in [0, k]$. Now, the sequence $\{w_i\}_{i \in [0, k+1]}$ is in W , with the first element, w' , in W' and the last one, w'' , in W'' . As $W = W' \sqcup W''$, we can find two consecutive elements in the sequence such that one is in W' and the other one is in W'' and which are such that the intersection of their topological closure is not empty. Thus, we have obtained that $\overline{W'}^W \cap \overline{W''}^W \neq \emptyset$, which gives a contradiction.

This ends the proof. \square

4 Conclusion

Together with its two companion papers [2, 5], this paper is a contribution to providing another perspective on conditional independence and do-calculus. In this paper, we consider

directed graphs (DGs), not necessarily acyclic, and we introduce a suitable topology on the set of vertices and the new notion of topological conditional separation on DGs. Then, we prove its equivalence with an extension of Pearl’s d-separation on DGs. What is more, we put forward a practical characterization of t-separation between subsets of vertices. The proofs partially rely on results proven in [2].

Checking topological separation is a two steps process. The first one, which is combinatorial, consists in exploring the possible splitting of the conditioning set W and the second one consists in checking that the two closures induced by the splitting do not intersect. It should be noted that, once given the splitting, the second step is computationally easy and thus the splitting appears as a “certificate” of conditional independence. By contrast, checking d-separation is a one step combinatorial process as it requires to check that *all* the paths that connect two variables are blocked.

A Additional material on Alexandrov topology

We use the material introduced in §2.2, and we provide additional results on Alexandrov topologies.

Proposition 9 *Let $\mathcal{E}_1, \mathcal{E}_2$ be two binary relations on the set \mathcal{V} . We have that*

$$\mathcal{E}_1 \subset \mathcal{E}_2 \implies \mathcal{T}_{\mathcal{E}_2} \subset \mathcal{T}_{\mathcal{E}_1}, \quad (30)$$

$$\mathcal{T}_{\mathcal{E}_1 \cup \mathcal{E}_2} = \mathcal{T}_{\mathcal{E}_1} \cap \mathcal{T}_{\mathcal{E}_2}. \quad (31)$$

The product topology $\mathcal{T}_{\mathcal{E}_1} \otimes \mathcal{T}_{\mathcal{E}_2}$ on the product set \mathcal{V}^2 coincides with the topology $\mathcal{T}_{\mathcal{E}_1 \times \mathcal{E}_2}$ in (8), where $\mathcal{E}_1 \times \mathcal{E}_2 \subset \mathcal{V}^2 \times \mathcal{V}^2$ is the product binary relation on the product set \mathcal{V}^2 :

$$\mathcal{T}_{\mathcal{E}_1} \otimes \mathcal{T}_{\mathcal{E}_2} = \mathcal{T}_{\mathcal{E}_1 \times \mathcal{E}_2}. \quad (32)$$

The topological closure of a subset $\mathcal{R} \subset \mathcal{V}^2$ w.r.t. the topology $\mathcal{T}_{\mathcal{E}_1} \otimes \mathcal{T}_{\mathcal{E}_2} = \mathcal{T}_{\mathcal{E}_1 \times \mathcal{E}_2}$ is given by

$$\overline{\mathcal{R}}^{\mathcal{E}_1 \times \mathcal{E}_2} = \mathcal{E}_1^* \mathcal{R} \mathcal{E}_2^{-*}. \quad (33)$$

Proof. The proof of (32) relies on the following identity between open rectangles:

$$\begin{aligned} \Gamma_1 \times \Gamma_2 \in \mathcal{T}_{\mathcal{E}_1} \otimes \mathcal{T}_{\mathcal{E}_2} &\iff \Gamma_1 \in \mathcal{T}_{\mathcal{E}_1} \text{ and } \Gamma_2 \in \mathcal{T}_{\mathcal{E}_2} \\ &\quad \text{(by definition of the product topology } \mathcal{T}_{\mathcal{E}_1} \otimes \mathcal{T}_{\mathcal{E}_2}\text{)} \\ &\iff \mathcal{E}_1 \Gamma_1 \subset \Gamma_1 \text{ and } \mathcal{E}_2 \Gamma_2 \subset \Gamma_2 \quad \text{(by (9))} \\ &\iff (\mathcal{E}_1 \times \mathcal{E}_2)(\Gamma_1 \times \Gamma_2) \subset \Gamma_1 \times \Gamma_2. \end{aligned}$$

Regarding (33), by property (6) of an Alexandrov topology, we have that

$$\overline{\mathcal{R}}^{\mathcal{E}_1 \times \mathcal{E}_2} = \bigcup_{(\gamma', \lambda') \in \mathcal{R}} \overline{\{(\gamma', \lambda')\}}^{\mathcal{E}_1 \times \mathcal{E}_2}$$

so that, for any $\gamma, \lambda \in \mathcal{V}$, we have

$$\begin{aligned} (\gamma, \lambda) \in \overline{\mathcal{R}}^{\mathcal{E}_1 \times \mathcal{E}_2} &\iff \exists(\gamma', \lambda') \in \mathcal{R}, (\gamma, \lambda) \in \overline{\{(\gamma', \lambda')\}}^{\mathcal{E}_1 \times \mathcal{E}_2} \\ &\iff \exists(\gamma', \lambda') \in \mathcal{R}, (\gamma, \lambda) \in \overline{\{\gamma'\}}^{\mathcal{E}_1} \times \overline{\{\lambda'\}}^{\mathcal{E}_2} \end{aligned}$$

by property of the topological closure of rectangles in the product topology

$$\begin{aligned} &\iff \exists(\gamma', \lambda') \in \mathcal{R}, \gamma \in \mathcal{E}_1^* \gamma', \lambda \in \mathcal{E}_2^* \lambda' && \text{(by (10))} \\ &\iff \gamma \mathcal{E}_1^* \mathcal{R} \mathcal{E}_2^{-*} \lambda \end{aligned}$$

This ends the proof. □

Proposition 10 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $V \subset \mathcal{V}$ a subset. The following statements are equivalent:*

1. *The subset V is a connected component of the topological space $(\mathcal{V}, \mathcal{T}_{\mathcal{E}})$,*
2. *The subset V is a connected component of the topological space $(\mathcal{V}, \mathcal{T}'_{\mathcal{E}}) = (\mathcal{V}, \mathcal{T}_{\mathcal{E}^{-1}})$,*
3. *The subset V is a connected component of the topological space $(\mathcal{V}, \mathcal{T}_{\mathcal{E} \cup \mathcal{E}^{-1}}) = (\mathcal{V}, \mathcal{T}_{\mathcal{E}} \cap \mathcal{T}_{\mathcal{E}^{-1}}) = (\mathcal{V}, \mathcal{T}_{\mathcal{E}} \cap \mathcal{T}'_{\mathcal{E}})$,*
4. *The subset V is an equivalence class of the equivalence relation $(\mathcal{E} \cup \mathcal{E}^{-1})^*$.*

Proof. For any $v \in \mathcal{V}$, we denote by $\hat{v} \subset \mathcal{V}$ the $\mathcal{T}_{\mathcal{E}}$ -connected component of the topological space $(\mathcal{V}, \mathcal{T}_{\mathcal{E}})$ that contains v , that is, the union of all $\mathcal{T}_{\mathcal{E}}$ -connected subsets of \mathcal{V} that contain v , and we prove that $\hat{v} = (\mathcal{E} \cup \mathcal{E}^{-1})^* v$. For this purpose, we notice that, by (10), we have that $(\mathcal{E} \cup \mathcal{E}^{-1})^* v = \overline{v}^{\mathcal{E} \cup \mathcal{E}^{-1}}$, which is also the smallest $\mathcal{T}_{\mathcal{E}}$ -clopen set containing v by (31). On the one hand, it is known, and can readily be shown, that $\hat{v} \subset \overline{v}^{\mathcal{E} \cup \mathcal{E}^{-1}}$. Indeed, if F is any $\mathcal{T}_{\mathcal{E}}$ -clopen set containing v , then $\hat{v} \cap F$ and $\hat{v} \cap F^c$ are two disjoint $\mathcal{T}_{\mathcal{E}}$ -open sets whose union equal \hat{v} . As this latter set is $\mathcal{T}_{\mathcal{E}}$ -connected, and as $v \in \hat{v} \cap F$, we deduce that $\hat{v} \cap F = \hat{v}$, hence that $\hat{v} \subset F$. On the other hand, it is well-known that the $\mathcal{T}_{\mathcal{E}}$ -connected component \hat{v} is closed. In an Alexandrov topology, \hat{v} is also $\mathcal{T}_{\mathcal{E}}$ -open since the connected components form a partition of \mathcal{V} , so that \hat{v}^c is a union of $\mathcal{T}_{\mathcal{E}}$ -closed sets, hence is closed. Therefore, \hat{v} is a $\mathcal{T}_{\mathcal{E}}$ -clopen set, and we deduce that $\overline{v}^{\mathcal{E} \cup \mathcal{E}^{-1}} \subset \hat{v}$. Therefore, we have obtained that $\hat{v} = \overline{v}^{\mathcal{E} \cup \mathcal{E}^{-1}} = (\mathcal{E} \cup \mathcal{E}^{-1})^* v$.

Then, we easily deduce the equivalence between the four assertions. This ends the proof. □

B Technical lemmas

Here below, the relations \mathfrak{S}^W and \mathfrak{S}^{-W} have been introduced in (18) and (22). The following lemma is proved in [2].

Lemma 11 ([2]) *We have that*

$$\mathcal{E}^{-W*}\mathcal{E}^{W*} = \Delta \cup \Delta_{W^c}\mathcal{B}^W \cup \mathcal{B}^{-W}\Delta_{W^c} \cup \mathcal{K}^W, \quad (34a)$$

$$\mathcal{C}^W\mathcal{E}^{-W*}\mathcal{E}^{W*} = \mathcal{C}^W(\Delta \cup \mathcal{B}^{-W}\Delta_{W^c} \cup \mathcal{K}^W), \quad (34b)$$

$$\mathcal{C}^W\mathcal{E}^{-W*}\mathcal{E}^{W*}\mathcal{C}^W = \mathcal{C}^W, \quad (34c)$$

$$\mathcal{C}^W\mathcal{E}^{-W*}\mathcal{E}^{W*}\Delta_{W^c} = \mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)\Delta_{W^c}, \quad (34d)$$

$$\Delta_{W^c}\mathcal{E}^{-W*}\mathcal{E}^{W*}\mathcal{C}^W = \Delta_{W^c}(\mathcal{B}^W \cup \mathcal{K}^W)\mathcal{C}^W. \quad (34e)$$

Lemma 12 *We have that*

$$\begin{aligned} \Delta_{W^c}(\Delta \cup (\mathcal{B}^W \cup \mathcal{K}^W)\mathcal{C}^W)\mathcal{E}^{-W*}\mathcal{E}^{W*}\mathcal{C}^W\mathcal{E}^{-W*}\mathcal{E}^{W*}(\Delta \cup \mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W))\Delta_{W^c} \\ = \Delta_{W^c}(\mathcal{B}^W \cup \mathcal{K}^W)\mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)\Delta_{W^c} \end{aligned} \quad (35)$$

Proof. We have that

$$\begin{aligned} & \Delta_{W^c}(\Delta \cup (\mathcal{B}^W \cup \mathcal{K}^W)\mathcal{C}^W)\mathcal{E}^{-W*}\mathcal{E}^{W*}\mathcal{C}^W\mathcal{E}^{-W*}\mathcal{E}^{W*}(\Delta \cup \mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W))\Delta_{W^c} \\ &= \Delta_{W^c}\mathcal{E}^{-W*}\mathcal{E}^{W*}\mathcal{C}^W\mathcal{E}^{-W*}\mathcal{E}^{W*}\Delta_{W^c} && \text{(by developing)} \\ & \quad \cup \Delta_{W^c}\mathcal{E}^{-W*}\mathcal{E}^{W*}\mathcal{C}^W\mathcal{E}^{-W*}\mathcal{E}^{W*}(\mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W))\Delta_{W^c} \\ & \quad \cup \Delta_{W^c}((\mathcal{B}^W \cup \mathcal{K}^W)\mathcal{C}^W)\mathcal{E}^{-W*}\mathcal{E}^{W*}\mathcal{C}^W\mathcal{E}^{-W*}\mathcal{E}^{W*}\Delta_{W^c} \\ & \quad \cup \Delta_{W^c}((\mathcal{B}^W \cup \mathcal{K}^W)\mathcal{C}^W)\mathcal{E}^{-W*}\mathcal{E}^{W*}\mathcal{C}^W\mathcal{E}^{-W*}\mathcal{E}^{W*}(\mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W))\Delta_{W^c} \\ &= \Delta_{W^c}\mathcal{E}^{-W*}\mathcal{E}^{W*}\mathcal{C}^W\mathcal{E}^{-W*}\mathcal{E}^{W*}\Delta_{W^c} \\ & \quad \cup \Delta_{W^c}\mathcal{E}^{-W*}\mathcal{E}^{W*}\mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)\Delta_{W^c} && \text{(as } \mathcal{C}^W\mathcal{E}^{-W*}\mathcal{E}^{W*}\mathcal{C}^W = \mathcal{C}^W \text{ by (34c))} \\ & \quad \cup \Delta_{W^c}(\mathcal{B}^W \cup \mathcal{K}^W)\mathcal{C}^W\mathcal{E}^{-W*}\mathcal{E}^{W*}\Delta_{W^c} && \text{(also by (34c))} \\ & \quad \cup \Delta_{W^c}(\mathcal{B}^W \cup \mathcal{K}^W)\mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)\Delta_{W^c} && \text{(also by (34c) applied twice)} \\ &= \Delta_{W^c}(\mathcal{B}^W \cup \mathcal{K}^W)\mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)\Delta_{W^c} && \text{(by (34d) and (34e))} \\ & \quad \cup \Delta_{W^c}(\mathcal{B}^W \cup \mathcal{K}^W)(\mathcal{B}^{-W} \cup \mathcal{K}^W)\Delta_{W^c} && \text{(by (34e))} \\ & \quad \cup \Delta_{W^c}(\mathcal{B}^W \cup \mathcal{K}^W)\mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)\Delta_{W^c} && \text{(by (34d))} \\ & \quad \cup \Delta_{W^c}(\mathcal{B}^W \cup \mathcal{K}^W)\mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)\Delta_{W^c} \\ &= \Delta_{W^c}(\mathcal{B}^W \cup \mathcal{K}^W)\mathcal{C}^W(\mathcal{B}^{-W} \cup \mathcal{K}^W)\Delta_{W^c}. \end{aligned}$$

This ends the proof. □

Lemma 13 *We have that*

$$\Delta_{W^c}\mathfrak{S}^{-W}\mathcal{E}^{-W*}\mathcal{E}^{W*}\mathfrak{S}^W\Delta_{W^c} = \Delta_{W^c}\mathcal{A}^W\Delta_{W^c}. \quad (36)$$

Proof. First, we write

$$\begin{aligned}
& \mathfrak{S}^{-W} \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathfrak{S}^W \\
&= \underbrace{\left(\Delta \cup (\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W \right)}_{\mathfrak{S}^{-W} \text{ by (22)}} \mathcal{E}^{-W*} \mathcal{E}^{W*} \underbrace{\left(\Delta \cup \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \right)}_{\mathfrak{S}^W \text{ by (22)}} \\
&= (\mathcal{E}^{-W*} \mathcal{E}^{W*}) \cup \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W \mathcal{E}^{-W*} \mathcal{E}^{W*} \right) \cup \left(\mathcal{E}^{-W*} \mathcal{E}^{W*} \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \right) \\
&\quad \cup \left((\mathcal{B}^W \cup \mathcal{K}^W) \underbrace{\mathcal{C}^W \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathcal{C}^W}_{=\mathcal{C}^W \text{ by (34c)}} (\mathcal{B}^{-W} \cup \mathcal{K}^W) \right) \quad (\text{by developing}) \\
&= (\mathcal{E}^{-W*} \mathcal{E}^{W*}) \cup \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W \mathcal{E}^{-W*} \mathcal{E}^{W*} \right) \cup \left(\mathcal{E}^{-W*} \mathcal{E}^{W*} \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \right) \\
&\quad \cup \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \right).
\end{aligned}$$

Second, we obtain that

$$\begin{aligned}
& \Delta_{W^c} \mathfrak{S}^{-W} \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathfrak{S}^W \Delta_{W^c} \\
&= \left(\Delta_{W^c} (\mathcal{E}^{-W*} \mathcal{E}^{W*}) \Delta_{W^c} \right) \cup \left(\Delta_{W^c} (\mathcal{B}^W \cup \mathcal{K}^W) \underbrace{\mathcal{C}^W \mathcal{E}^{-W*} \mathcal{E}^{W*} \Delta_{W^c}}_{=\mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \Delta_{W^c} \text{ by (34d)}} \right) \\
&\quad \cup \left(\underbrace{\Delta_{W^c} \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathcal{C}^W}_{=\Delta_{W^c} (\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W \text{ by (34e)}} (\mathcal{B}^{-W} \cup \mathcal{K}^W) \Delta_{W^c} \right) \\
&\quad \cup \left(\Delta_{W^c} \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \right) \Delta_{W^c} \right) \\
&= \left(\Delta_{W^c} (\mathcal{E}^{-W*} \mathcal{E}^{W*}) \Delta_{W^c} \right) \cup \left(\Delta_{W^c} \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \right) \Delta_{W^c} \right)
\end{aligned}$$

because the three last terms in the union are all equal

$$\begin{aligned}
&= \left(\Delta_{W^c} (\Delta \cup \Delta_{W^c} \mathcal{B}^W \cup \mathcal{B}^{-W} \Delta_{W^c} \cup \mathcal{K}^W) \Delta_{W^c} \right) \\
&\quad \cup \left(\Delta_{W^c} \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \right) \Delta_{W^c} \right) \quad (\text{by (34a)}) \\
&= \Delta_{W^c} \left(\Delta \cup \mathcal{B}^W \cup \mathcal{B}^{-W} \cup \mathcal{K}^W \cup \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \right) \right) \Delta_{W^c} \\
&= \Delta_{W^c} \mathcal{A}^W \Delta_{W^c}. \quad (\text{by definition of } \mathcal{A}^W \text{ in (15g)})
\end{aligned}$$

This ends the proof. \square

In Lemma 13, it was proved that the two relations $\mathfrak{S}^{-W} \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathfrak{S}^W$ and \mathcal{A}^W coincide when restricted to the subset W^c . More generally, we give in this last lemma the relationship between these two relations.

Lemma 14 *We have that*

$$\left(\mathfrak{S}^{-W} \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathfrak{S}^W \right) \cup \mathfrak{S}^{-W} \mathcal{B}^{-W} \Delta_W \cup \Delta_W \mathcal{B}^W \mathfrak{S}^W = \mathcal{A}^W \cup \mathfrak{S}^{-W} \cup \mathfrak{S}^W. \quad (37)$$

Proof. We use the notation $\Theta^W = \mathcal{B}^W \cup \mathcal{K}^W$ and $\Theta^{-W} = \mathcal{B}^{-W} \cup \mathcal{K}^W = (\Theta^W)^{-1}$ to simplify the reading of the proof, so that

$$\begin{aligned}
& \mathfrak{S}^{-W} \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathfrak{S}^W \\
&= (\Delta \cup (\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W) \mathcal{E}^{-W*} \mathcal{E}^{W*} (\Delta \cup \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W)) \\
&\quad \text{(by definition of the relations } \mathfrak{S}^W \text{ and } \mathfrak{S}^{-W} \text{ in (18) and (22))} \\
&= (\Delta \cup \Theta^W \mathcal{C}^W) \mathcal{E}^{-W*} \mathcal{E}^{W*} (\Delta \cup \mathcal{C}^W \Theta^{-W}) \\
&\quad \text{(using the just defined } \Theta^W = \mathcal{B}^W \cup \mathcal{K}^W \text{ and } \Theta^{-W} = \mathcal{B}^{-W} \cup \mathcal{K}^W) \\
&= (\mathcal{E}^{-W*} \mathcal{E}^{W*}) \cup (\Theta^W \mathcal{C}^W \mathcal{E}^{-W*} \mathcal{E}^{W*}) \cup (\Theta^W \mathcal{C}^W \Theta^{-W}) \cup (\mathcal{E}^{-W*} \mathcal{E}^{W*} \mathcal{C}^W \Theta^{-W}) \\
&\quad \text{(by developing and by (34c) giving } \mathcal{C}^W \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathcal{C}^W = \mathcal{C}^W) \\
&= (\mathcal{E}^{-W*} \mathcal{E}^{W*}) \cup \Theta^W \mathcal{C}^W (\Delta \cup \mathcal{B}^{-W} \Delta_{W^c} \cup \mathcal{K}^W) \cup (\Theta^W \mathcal{C}^W \Theta^{-W}) \cup (\Delta \cup \Delta_{W^c} \mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W \Theta^{-W}
\end{aligned}$$

as by (34b) $\Theta^W \mathcal{C}^W \mathcal{E}^{-W*} \mathcal{E}^{W*} = \Theta^W \mathcal{C}^W (\Delta \cup \mathcal{B}^{-W} \Delta_{W^c} \cup \mathcal{K}^W)$ and by symmetry for the last term. Thus, using the last equality and performing a union with $\Theta^W \mathcal{C}^W \mathcal{B}^{-W} \Delta_W \cup \Delta_W \mathcal{B}^W \mathcal{C}^W \Theta^{-W}$ on both sides of the equality, we obtain

$$\begin{aligned}
& (\mathfrak{S}^{-W} \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathfrak{S}^W) \cup \Theta^W \mathcal{C}^W \mathcal{B}^{-W} \Delta_W \cup \Delta_W \mathcal{B}^W \mathcal{C}^W \Theta^{-W} \\
&= (\mathcal{E}^{-W*} \mathcal{E}^{W*}) \cup \Theta^W \mathcal{C}^W (\Delta \cup \mathcal{B}^{-W} \cup \mathcal{K}^W) \cup (\Theta^W \mathcal{C}^W \Theta^{-W}) \cup (\Delta \cup \mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W \Theta^{-W} \\
&= (\mathcal{E}^{-W*} \mathcal{E}^{W*}) \cup \Theta^W \mathcal{C}^W (\Delta \cup \Theta^{-W}) \cup (\Theta^W \mathcal{C}^W \Theta^{-W}) \cup (\Delta \cup \Theta^W) \mathcal{C}^W \Theta^{-W} \\
&= (\mathcal{E}^{-W*} \mathcal{E}^{W*}) \cup \Theta^W \mathcal{C}^W \cup (\Theta^W \mathcal{C}^W \Theta^{-W}) \cup \mathcal{C}^W \Theta^{-W} \\
&= (\Delta \cup \Delta_{W^c} \mathcal{B}^W \cup \mathcal{B}^{-W} \Delta_{W^c} \cup \mathcal{K}^W) \cup \Theta^W \mathcal{C}^W \cup (\Theta^W \mathcal{C}^W \Theta^{-W}) \cup \mathcal{C}^W \Theta^{-W} \quad \text{(by (34a))} \\
&= (\Delta \cup \Delta_{W^c} \mathcal{B}^W \cup \mathcal{B}^{-W} \Delta_{W^c} \cup \mathcal{K}^W) \cup \underbrace{(\Delta \cup \Theta^W \mathcal{C}^W)}_{=\mathfrak{S}^{-W}} \cup (\Theta^W \mathcal{C}^W \Theta^{-W}) \cup \underbrace{(\Delta \cup \mathcal{C}^W \Theta^{-W})}_{=\mathfrak{S}^W} \\
&= \Delta \cup \Delta_{W^c} \mathcal{B}^W \cup \mathcal{B}^{-W} \Delta_{W^c} \cup \mathcal{K}^W \cup (\Theta^W \mathcal{C}^W \Theta^{-W}) \cup \mathfrak{S}^{-W} \cup \mathfrak{S}^W .
\end{aligned}$$

Finally, using the last equality and performing a union with $\Delta_W \mathcal{B}^W \cup \mathcal{B}^{-W} \Delta_W$ on both sides of the equality, we obtain

$$\begin{aligned}
& (\mathfrak{S}^{-W} \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathfrak{S}^W) \cup \Theta^W \mathcal{C}^W \mathcal{B}^{-W} \Delta_W \cup \Delta_W \mathcal{B}^W \mathcal{C}^W \Theta^{-W} \cup \Delta_W \mathcal{B}^W \cup \mathcal{B}^{-W} \Delta_W \\
&= \Delta \cup \mathcal{B}^W \cup \mathcal{B}^{-W} \cup \mathcal{K}^W \cup (\Theta^W \mathcal{C}^W \Theta^{-W}) \cup \mathfrak{S}^{-W} \cup \mathfrak{S}^W \\
&= \Delta \cup \mathcal{B}^W \cup \mathcal{B}^{-W} \cup \mathcal{K}^W \cup (\Theta^W \mathcal{C}^W \Theta^{-W}) \cup \mathfrak{S}^{-W} \cup \mathfrak{S}^W \\
&= \mathcal{A}^W \cup \mathfrak{S}^{-W} \cup \mathfrak{S}^W . \quad \text{(by definition of } \mathcal{A}^W \text{ in (15g))}
\end{aligned}$$

This ends the proof. □

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