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Conditional Separation as a Binary Relation

Jean-Philippe Chancelier[†], Michel De Lara*, Benjamin Heymann[†]

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Abstract

Pearl's d-separation is a foundational notion to study conditional independence between random variables on directed acyclic graphs. Pearl defined the d-separation of two subsets conditionally on a third one. We show how the d-separation can be extended beyond acyclic graphs and can be expressed and characterized as a binary relation between vertices.

1 Introduction

As the world shifts toward more and more data-driven decision-making, causal inference is taking more space in applied sciences, statistics and machine learning. This is because it allows for better, more robust decision-making, and provides a way to interpret the data that goes beyond correlation [8]. In his seminal work [7], Pearl builds on graphical models [1] to introduce the so-called do-calculus and the notion of d-separation on a directed acyclic graph (DAG).

This paper was written in parallel to two other papers [2, 4], all of which aim at providing another perspective on conditional independence and do-calculus. Here we show how the d-separation can be extended beyond DAGs and can be expressed and characterized as a binary relation between vertices. We think this work will allow for new proof strategies in the developpement of the theory of causal inference. In particular, the main result is an intermediate step that we reuse in the two other above mentionned papers. Last, we think the developpement of a theory based on binary relations makes it amenable to computer aided proof [9].

In Sect. 2, we present our extended definition of d-separation and then characterize it as the complementary of the conditional active binary relation between vertices in Sect. 3, we state and sketch the proof of our main result, the characterization of the d-separation relation as the complementary of the conditional active relation. Part of the proof is relegated in Appendices A and B, as well as a treatment of the moral relation [5] with binary relations in Appendices C.

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2 A formal Pearl's d -separation definition

In §2.1, we deal with graphs but using the concepts of binary relations; we formally define what we call extended-oriented paths in a graph. Thus equipped, in §2.2 we formally adapt Pearl's definition of active (and blocked) extended-oriented paths in a graph, from which we deduce the (conditional) d -separation binary relation. Then, we also introduce elementary binary relations that are the bricks to define a new binary relation that we call conditional active relation.

2.1 Extended-oriented paths in a graph

We employ the vocabulary and concepts both of binary relations and of graph theory. We take inspiration from Pearl in [6] to define the notion of blocked paths on a graph, not necessarily finite nor acyclic. We use the notation $\llbracket r, s \rrbracket = \{r, r+1, \dots, s-1, s\}$ for two integers $r \leq s$.

2.1.1 Background on binary relations

Let \mathcal{V} be a nonempty set (finite or not). We recall that a (*binary*) *relation* \mathcal{R} on \mathcal{V} is a subset $\mathcal{R} \subset \mathcal{V} \times \mathcal{V}$ and that $\gamma \mathcal{R} \lambda$ means $(\gamma, \lambda) \in \mathcal{R}$. For any subset $\Gamma \subset \mathcal{V}$, the (*sub*)*diagonal relation* is $\Delta_\Gamma = \{(\gamma, \lambda) \in \mathcal{V} \times \mathcal{V} \mid \gamma = \lambda \in \Gamma\}$ and the *diagonal relation* is $\Delta = \Delta_\mathcal{V}$. A *foreset* of a relation \mathcal{R} is any set of the form $\mathcal{R}\lambda = \{\gamma \in \mathcal{V} \mid \gamma \mathcal{R} \lambda\}$, where $\lambda \in \mathcal{V}$, or, by extension, of the form $\mathcal{R}\Lambda = \{\gamma \in \mathcal{V} \mid \exists \lambda \in \Lambda, \gamma \mathcal{R} \lambda\}$, where $\Lambda \subset \mathcal{V}$. An *afterset* of a relation \mathcal{R} is any set of the form $\gamma \mathcal{R} = \{\lambda \in \mathcal{V} \mid \gamma \mathcal{R} \lambda\}$, where $\gamma \in \mathcal{V}$, or, by extension, of the form $\Gamma \mathcal{R} = \{\lambda \in \mathcal{V} \mid \exists \gamma \in \Gamma, \gamma \mathcal{R} \lambda\}$, where $\Gamma \subset \mathcal{V}$. The *opposite* or *complementary* \mathcal{R}^c of a binary relation \mathcal{R} is the relation $\mathcal{R}^c = \mathcal{V} \times \mathcal{V} \setminus \mathcal{R}$, that is, defined by $\gamma \mathcal{R}^c \lambda \iff \neg(\gamma \mathcal{R} \lambda)$. The *converse* \mathcal{R}^{-1} of a binary relation \mathcal{R} is defined by $\gamma \mathcal{R}^{-1} \lambda \iff \lambda \mathcal{R} \gamma$ (and \mathcal{R} is *symmetric* if $\mathcal{R}^{-1} = \mathcal{R}$). The *composition* $\mathcal{R}\mathcal{R}'$ of two binary relations $\mathcal{R}, \mathcal{R}'$ on \mathcal{V} is defined by $\gamma(\mathcal{R}\mathcal{R}')\lambda \iff \exists \delta \in \mathcal{V}, \gamma \mathcal{R} \delta$ and $\delta \mathcal{R}' \lambda$; then, by induction we define¹ $\mathcal{R}^{n+1} = \mathcal{R}\mathcal{R}^n$ for $n \in \mathbb{N}^*$. The *transitive closure* of a binary relation \mathcal{R} is $\mathcal{R}^+ = \bigcup_{k=1}^{\infty} \mathcal{R}^k$ (and \mathcal{R} is *transitive* if $\mathcal{R}^+ = \mathcal{R}$) and the *reflexive and transitive closure* is $\mathcal{R}^* = \mathcal{R}^+ \cup \Delta = \bigcup_{k=0}^{\infty} \mathcal{R}^k$ with the convention $\mathcal{R}^0 = \Delta$. A *partial equivalence relation* is a symmetric and transitive binary relation (generally denoted by \sim or \equiv). An *equivalence relation* is a reflexive, symmetric and transitive binary relation.

2.1.2 Extended-oriented paths in a (directed simple) graph (permitting loops)

Let \mathcal{V} be a nonempty set (finite or not), whose elements are called *vertices*. Let $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ be a relation on \mathcal{V} , whose elements are ordered pairs (that is, couples) of vertices called *edges*. The first element of an edge is the *tail of the edge*, whereas the second one is the *head*

¹In what follows, when we consider a binary relation as a subset $\mathcal{R} \subset \mathcal{V} \times \mathcal{V}$, we will use the notation $\prod_{i=1}^n \mathcal{R} \subset \prod_{i=1}^n \mathcal{V} \times \mathcal{V}$, where n is a positive integer, to denote a product subset of the product set \mathcal{V}^{2n} , thus making the distinction with the binary relation $\mathcal{R}^n \subset \mathcal{V} \times \mathcal{V}$ obtained by n compositions.

of the edge. Both tail and head are called *endpoints* of the edge, and we say that the edge connects its endpoints. We define a *loop* as an element of $\Delta \cap \mathcal{E}$, that is, a loop is an edge that connects a vertex to itself.

A *graph*, as we use it throughout this paper, is a couple $(\mathcal{V}, \mathcal{E})$. This definition is very basic and we now stress proximities and differences with classic notions in graph theory. As we define a graph, it may hold a finite or infinite number of vertices; there is at most one edge that has a couple of ordered vertices as single endpoints, hence a graph (in our sense) is not a multigraph (in graph theory); loops are not excluded (since we do not impose $\Delta \cap \mathcal{E} = \emptyset$). Hence, what we call a graph would be called a directed simple graph permitting loops in graph theory.

To define blocked and active extended-oriented paths relative to the graph $(\mathcal{V}, \mathcal{E})$, we need to fix additional vocabulary and notation. In the graph $(\mathcal{V}, \mathcal{E})$, the *undirected edges* are the elements of $\mathcal{E} \cap \mathcal{E}^{-1}$ — that is, edges with both $(\lambda, \gamma) \in \mathcal{E}$ and $(\gamma, \lambda) \in \mathcal{E}$ (hence, including loops). Then, the graph $(\mathcal{V}, \mathcal{E})$ is said to be *undirected* if all edges are undirected edges, or, equivalently, if $\mathcal{E} = \mathcal{E} \cap \mathcal{E}^{-1}$ or if $\mathcal{E}^{-1} = \mathcal{E}$. The *undirected extension* of a graph $(\mathcal{V}, \mathcal{E})$ is the graph $(\mathcal{V}, \mathcal{E} \cup \mathcal{E}^{-1})$.

In the graph $(\mathcal{V}, \mathcal{E})$, the *directed edges* are the elements of $\mathcal{E} \cap (\mathcal{E}^{-1})^c$ — that is, edges with $(\lambda, \gamma) \in \mathcal{E}$ such that $(\gamma, \lambda) \notin \mathcal{E}$ (recall that we do not assume that $\mathcal{E} \cap \mathcal{E}^{-1} = \emptyset$). Then, the graph $(\mathcal{V}, \mathcal{E})$ is said to be *directed* if all edges are directed edges, or, equivalently, if $\mathcal{E} \cap \mathcal{E}^{-1} = \emptyset$, that is, when no two edges have the same endpoints.

In graph theory, one finds the notions of path, chain and walk. To avoid ambiguities, we formally define an (edge) path in $(\mathcal{V}, \mathcal{E})$ (in our sense) as the classical notion of path in a graph [3]

We also define an extended-oriented path in $(\mathcal{V}, \mathcal{E})$ (in our sense) as what corresponds to chain path in [5].

• An (*edge*) *path* of length n ($n \geq 1$) relative to the graph $(\mathcal{V}, \mathcal{E})$ is a sequence $\varrho = \{(v_i^b, v_i^\#)\}_{i \in \llbracket 1, n \rrbracket} \in \prod_{i=1}^n \mathcal{E}$ of n edges such that $v_i^\# = v_{i+1}^b$, $i \in \llbracket 1, n-1 \rrbracket$ or, equivalently, is a sequence $\varrho = \{(v_i, v_{i+1})\}_{i \in \llbracket 1, n \rrbracket} \in \prod_{i=1}^n \mathcal{E}$. The first element v_1^b of an (edge) path $\varrho = \{(v_i^b, v_i^\#)\}_{i \in \llbracket 1, n \rrbracket}$ is the *tail of the (edge) path*, whereas the last one $v_n^\#$ is the *head of the (edge) path*. Both tail and head are called *endpoints* of the (edge) path. We denote by

$$P^n(\mathcal{V}, \mathcal{E}) = \left\{ \{(v_i^b, v_i^\#)\}_{i \in \llbracket 1, n \rrbracket} \in \prod_{i=1}^n \mathcal{E} \mid v_i^\# = v_{i+1}^b \text{ for } i \in \llbracket 1, n-1 \rrbracket \right\} \subset \prod_{i=1}^n \mathcal{E} \quad (1a)$$

the set of (edge) paths of length n ($n \geq 1$) relative to the graph $(\mathcal{V}, \mathcal{E})$, and by

$$P(\mathcal{V}, \mathcal{E}) = \bigcup_{n \geq 0} P^n(\mathcal{V}, \mathcal{E}) \subset \mathcal{V}^2 \cup \bigcup_{n \geq 1} \prod_{i=1}^n \mathcal{E} \quad (1b)$$

the set of finite (edge) paths relative to the graph $(\mathcal{V}, \mathcal{E})$, where $P^0(\mathcal{V}, \mathcal{E})$, made of what we call *paths of length 0*, is defined by

$$P^0(\mathcal{V}, \mathcal{E}) = \{(v, v) \mid v \in \mathcal{V} \text{ and } (v, v) \notin \mathcal{E}\} = \Delta \cap \mathcal{E}^c \subset \mathcal{V}^2. \quad (1c)$$

Thus, by definition, we have that

$$P^0(\mathcal{V}, \mathcal{E}) \cup P^1(\mathcal{V}, \mathcal{E}) \supset (\Delta \cap \mathcal{E}^c) \cup (\Delta \cap \mathcal{E}) = \Delta, \quad (1d)$$

so that (edge) paths of length 0 or 1 contain the diagonal Δ . We denote by $|\varrho|$ the length of a path $\varrho \in P(\mathcal{V}, \mathcal{E})$. An *(edge) subpath* of the (edge) path ϱ is an (edge) path obtained by a subsequence of consecutive indices.

• An *extended-oriented path* of length n ($n \geq 1$) relative to the graph $(\mathcal{V}, \mathcal{E})$ is a couple $\rho = (\varrho, o) \in \prod_{i=1}^n (\mathcal{E} \cup \mathcal{E}^{-1}) \times \{-1, +1\}^n$, where $\varrho = \{(v_i^b, v_i^\#)\}_{i \in \llbracket 1, n \rrbracket}$ and $o = \{o_i\}_{i \in \llbracket 1, n \rrbracket} \in \{-1, +1\}^n$ are such that, for all $i \in \llbracket 1, n-1 \rrbracket$, we have $v_i^\# = v_{i+1}^b$ and either $(v_i^b, v_i^\#) \in \mathcal{E}$ if $o_i = +1$ or $(v_i^b, v_i^\#) \in \mathcal{E}^{-1}$ if $o_i = -1$. We denote

$$\mathcal{E}^{(+1)} = \mathcal{E}, \quad \mathcal{E}^{(-1)} = \mathcal{E}^{-1} \quad (2a)$$

and we introduce (for $n \geq 1$)

$$\begin{aligned} U^n(\mathcal{V}, \mathcal{E}) &= \left\{ (\varrho, o) \in \prod_{i=1}^n (\mathcal{E} \cup \mathcal{E}^{-1}) \times \{-1, +1\}^n \mid \right. \\ &\quad \left. \varrho = \{\varrho_i\}_{i \in \llbracket 1, n \rrbracket} \in P^n(\mathcal{V}, \mathcal{E} \cup \mathcal{E}^{-1}) \text{ and } \varrho_i \in \mathcal{E}^{(o_i)} \text{ for } i \in \llbracket 1, n \rrbracket \right\} \quad (2b) \\ &\subset \prod_{i=1}^n (\mathcal{E} \cup \mathcal{E}^{-1}) \times \{-1, +1\}^n \end{aligned}$$

the set of extended-oriented paths of length n ($n \geq 1$) relative to the graph $(\mathcal{V}, \mathcal{E})$, and by

$$U(\mathcal{V}, \mathcal{E}) = \bigcup_{n \geq 0} U^n(\mathcal{V}, \mathcal{E}) \subset (\mathcal{V}^2 \times \{+1\}) \cup \bigcup_{n \geq 1} \left(\prod_{i=1}^n (\mathcal{E} \cup \mathcal{E}^{-1}) \times \{-1, +1\}^n \right) \quad (2c)$$

the set of finite extended-oriented paths² relative to the graph $(\mathcal{V}, \mathcal{E})$, where $U^n(\mathcal{V}, \mathcal{E})$ for $n \geq 1$ is defined by (2b) and $U^0(\mathcal{V}, \mathcal{E})$, the set of extended-oriented paths of length 0, is defined by

$$U^0(\mathcal{V}, \mathcal{E}) = \left\{ ((v, v), +1) \in (\mathcal{V} \times \mathcal{V}) \times \{+1\} \mid (v, v) \notin \mathcal{E} \right\} = (\Delta \cap \mathcal{E}^c) \times \{+1\}. \quad (2d)$$

Thus, by definition, we have that

$$U^0(\mathcal{V}, \mathcal{E}) \cup U^1(\mathcal{V}, \mathcal{E}) \supset (\Delta \cap \mathcal{E}^c) \times \{+1\} \cup (\Delta \cap \mathcal{E}) \times \{+1\} = \Delta \times \{+1\}, \quad (2e)$$

²It is to be noted that an extended-oriented path is *not a path in the graph* $(\mathcal{V}, \mathcal{E})$, neither in the undirected graph $(\mathcal{V}, \mathcal{E} \cup \mathcal{E}^{-1})$. However, considering a couple $(\varrho, o) \in U^n(\mathcal{V}, \mathcal{E})$, we obtain that $\varrho \in P^n(\mathcal{V}, \mathcal{E} \cup \mathcal{E}^{-1})$, that is, ϱ is an (edge) path in the unoriented graph $(\mathcal{V}, \mathcal{E} \cup \mathcal{E}^{-1})$. We thus obtain a natural surjection $(\varrho, o) \mapsto \varrho$ from $U(\mathcal{V}, \mathcal{E})$ to $P(\mathcal{V}, \mathcal{E} \cup \mathcal{E}^{-1})$. This canonical surjection is not necessary injective because a path in $P(\mathcal{V}, \mathcal{E} \cup \mathcal{E}^{-1})$ that has an edge in $\mathcal{E} \cap \mathcal{E}^{-1}$ is the image of two distinct extended-oriented paths. The surjection $(\varrho, o) \mapsto \varrho$ is a bijection in the special case when the graph $(\mathcal{V}, \mathcal{E})$ is directed, that is, when $\mathcal{E} \cap \mathcal{E}^{-1} = \emptyset$, that is, when no two edges have the same endpoints.

so that extended-oriented paths of length 0 or 1 contain $\Delta \times \{+1\}$. An *extended-oriented subpath* of the extended-oriented path $\rho \in U(\mathcal{V}, \mathcal{E})$ is an extended-oriented path obtained by a subsequence of consecutive indices.

- We denote by $\varpi^n : P^n(\mathcal{V}, \mathcal{E}) \rightarrow \mathcal{V} \times \mathcal{V}$ the *projection on the tail and head endpoints* of an (edge) path of length n . It is defined, for $\varrho \in P^n(\mathcal{V}, \mathcal{E})$, by $\varpi^n(\varrho) = \varpi^n((v_i^b, v_i^\#)_{i \in \llbracket 1, n \rrbracket}) = (v_1^b, v_n^\#)$. We denote by $\varpi : P(\mathcal{V}, \mathcal{E}) \rightarrow \mathcal{V} \times \mathcal{V}$ the projection mapping on the tail and head endpoints defined by

$$\forall \varrho \in P(\mathcal{V}, \mathcal{E}), \quad \varpi(\varrho) = \varpi^{|\varrho|}(\varrho) \in \mathcal{V} \times \mathcal{V}. \quad (3a)$$

We also distinguish the tail and the head endpoints projection mappings by

$$\varpi = (\varpi^b, \varpi^\#) \text{ where } \varpi^b : P(\mathcal{V}, \mathcal{E}) \rightarrow \mathcal{V} \text{ and } \varpi^\# : P(\mathcal{V}, \mathcal{E}) \rightarrow \mathcal{V}. \quad (3b)$$

We use the notation $\varpi_U : U(\mathcal{V}, \mathcal{E}) \rightarrow \mathcal{V} \times \mathcal{V}$ to denote the projection mappings on the tail and head endpoints of an extended-oriented path. It is defined, for $(\varrho, o) \in U(\mathcal{V}, \mathcal{E})$, by

$$\varpi_U((\varrho, o)) = \varpi(\varrho), \quad \forall (\varrho, o) \in U(\mathcal{V}, \mathcal{E}). \quad (3c)$$

- We denote by $\iota^n : P^n(\mathcal{V}, \mathcal{E}) \rightrightarrows \mathcal{V}$ the set-valued mapping giving the *intermediate vertices* of an (edge) path of length n ($n \geq 0$). It is defined, for $\varrho = \{v_i, v_{i+1}\}_{i \in \llbracket 1, n \rrbracket} \in P^n(\mathcal{V}, \mathcal{E})$ in (1b), by

$$\iota^n(\varrho) = \iota^n(\{v_i, v_{i+1}\}_{i \in \llbracket 1, n \rrbracket}) = \{v_2, \dots, v_n\} \quad (4a)$$

(which reduces to the empty set for $n = 0$ or 1). Notice that

$$\{v_1, v_2, \dots, v_n, v_{n+1}\} = \{v_1\} \cup \{v_2, \dots, v_n\} \cup \{v_{n+1}\} = \varpi^b(\varrho) \cup \iota^n(\varrho) \cup \varpi^\#(\varrho).$$

We denote by $\iota : P(\mathcal{V}, \mathcal{E}) \rightrightarrows \mathcal{V}$ the set-valued mapping giving the intermediate vertices of an (edge) path by

$$\forall \varrho \in P(\mathcal{V}, \mathcal{E}), \quad \iota(\varrho) = \iota^{|\varrho|}(\varrho). \quad (4b)$$

We use the notation $\iota_U : U(\mathcal{V}, \mathcal{E}) \rightrightarrows \mathcal{V}$ to denote the set-valued mapping giving the intermediate vertices of an extended-oriented path. It is defined, for $(\varrho, o) \in U(\mathcal{V}, \mathcal{E})$ in (2c), by

$$\forall (\varrho, o) \in U(\mathcal{V}, \mathcal{E}), \quad \iota_U((\varrho, o)) = \iota(\varrho). \quad (4c)$$

- To define blocked extended-oriented paths, the following notions prove useful. With any binary relation $\mathcal{R} \subset \mathcal{V} \times \mathcal{V}$, we associate the subset $D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}]$ of $U(\mathcal{V}, \mathcal{E})$ in (2c), that we call the *deployment in extended-oriented paths*, defined by

$$\forall \mathcal{R} \subset \mathcal{V} \times \mathcal{V}, \quad D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}] = \varpi_U^{-1}(\mathcal{R}) \subset U(\mathcal{V}, \mathcal{E}), \quad (5)$$

where the projection ϖ_U has been defined in (3c). The deployment $D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}]$ is made of the extended-oriented paths whose endpoints satisfy the binary relation \mathcal{R} . We also associate with \mathcal{R} the subset $D_P[\mathcal{R} \mid \mathcal{V}, \mathcal{E}]$ of $U(\mathcal{V}, \mathcal{E})$, that we call the *deployment in paths*, defined by

$$\forall \mathcal{R} \subset \mathcal{V} \times \mathcal{V}, \quad D_P[\mathcal{R} \mid \mathcal{V}, \mathcal{E}] = \varpi^{-1}(\mathcal{R}) \subset P(\mathcal{V}, \mathcal{E}), \quad (6)$$

where the projection ϖ has been defined in (3a). The deployment in paths $D_P[\mathcal{R} \mid \mathcal{V}, \mathcal{E}]$ is made of the (edge) paths whose endpoints satisfy the binary relation \mathcal{R} .

2.2 Definitions of active extended-oriented paths, of conditional directional separation and of conditional active binary relation

We define active extended-oriented paths relative to the graph $(\mathcal{V}, \mathcal{E})$ in Definition 1. Then, we obtain the definition of blocked extended-oriented paths relative to the graph $(\mathcal{V}, \mathcal{E})$, as defined by Pearl in [6], by switching to the complementary set.

Definition 1 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices. We say that an extended-oriented path $\rho = (\varrho, o) \in U(\mathcal{V}, \mathcal{E})$ in (2c) relative to the graph $(\mathcal{V}, \mathcal{E})$ is an active extended-oriented path (w.r.t.³ the subset W) if either its length is 0 or 1, or (when its length is at least 2) $\rho = (\varrho, o) = (((v_i^b, v_i^\sharp))_{i \in \llbracket 1, |\varrho| \rrbracket}, (o_i)_{i \in \llbracket 1, |\varrho| \rrbracket})$ is such that, for all $i \in \llbracket 1, |\varrho| - 1 \rrbracket$, the consecutive extended-oriented subpaths $((v_i^b, v_i^\sharp), o_i)$ and $((v_{i+1}^b, v_{i+1}^\sharp), o_{i+1})$ satisfy one of the four following cases*

1. $o_i = +1, o_{i+1} = +1$ and $v_i^\sharp = v_{i+1}^b \in W^c$,
2. $o_i = -1, o_{i+1} = -1$ and $v_i^\sharp = v_{i+1}^b \in W^c$,
3. $o_i = -1, o_{i+1} = +1$ and $v_i^\sharp = v_{i+1}^b \in W^c$,
4. $o_i = +1, o_{i+1} = -1$ and $v_i^\sharp = v_{i+1}^b \in W^*$,

where $\mathcal{E}^* = \mathcal{E}^+ \cup \Delta$ is the reflexive and transitive closure of the relation \mathcal{E} , and where

$$W^* = \mathcal{E}^* W. \quad (7)$$

We denote by $U_a^W(\mathcal{V}, \mathcal{E}) \subset U(\mathcal{V}, \mathcal{E})$ the subset of all active extended-oriented paths (w.r.t. the subset W). We say that an extended-oriented path is blocked if it is not active and we denote by $U_b^W(\mathcal{V}, \mathcal{E}) = (U_a^W(\mathcal{V}, \mathcal{E}))^c$ the subset of all blocked extended-oriented paths (w.r.t. the subset W).

Notice that any extended-oriented path of length 0 or 1 is active by definition.

We introduce in Definition 2 a new binary relation between vertices: we say that two vertices are (conditionally) directionally separated if and only if all the extended-oriented paths, having them as endpoints, are blocked. This definition mimics Pearl's d-separation, but with two differences: the graph is not supposed to be acyclic, and the separation is between vertices and not between disjoint subsets.

³With respect to

Definition 2 Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices. We denote

$$\gamma \underset{d}{\parallel} \lambda \mid W \iff D_U[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{E}] \subset U_b^W(\mathcal{V}, \mathcal{E}) \quad (\forall \gamma, \lambda \in \mathcal{V}), \quad (8)$$

and we say that the vertices γ and λ are (conditionally) directionally separated (w.r.t. the subset W).

3 Characterization of d -separation by means of binary relations

Our main result is the characterization of the conditional directional separation relation (extension of the d -separation) as the complementary of the conditional active relation defined below in Equation (9g).

For the purpose of characterizing the conditional directional separation relation $\underset{d}{\parallel}$, we introduce the following binary relations on \mathcal{V} .

Definition 3 Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices. We define the conditional parental relation \mathcal{E}^W as

$$\mathcal{E}^W = \Delta_{W^c} \mathcal{E} \quad \text{that is, } \gamma \mathcal{E}^W \lambda \iff \gamma \in W^c \text{ and } \gamma \mathcal{E} \lambda \quad (\forall \gamma, \lambda \in \mathcal{V}), \quad (9a)$$

the conditional ascendent relation \mathcal{B}^W as

$$\mathcal{B}^W = \mathcal{E}(\Delta_{W^c} \mathcal{E})^* = \mathcal{E} \mathcal{E}^{W*} \quad (9b)$$

which relates a descendent with an ascendent by means of elements in W^c . We define their converses \mathcal{E}^{-W} and \mathcal{B}^{-W} as

$$\mathcal{E}^{-W} = (\mathcal{E}^W)^{-1} = \mathcal{E}^{-1} \Delta_{W^c}, \quad (9c)$$

$$\mathcal{B}^{-W} = (\mathcal{B}^W)^{-1} = (\mathcal{E}^{-1} \Delta_{W^c})^* \mathcal{E}^{-1} = \mathcal{E}^{-W*} \mathcal{E}^{-1}. \quad (9d)$$

With these elementary binary relations, we define the conditional common cause relation \mathcal{K}^W as the symmetric relation

$$\mathcal{K}^W = \mathcal{B}^{-W} \Delta_{W^c} \mathcal{B}^W = \mathcal{E}^{-W+} \mathcal{E}^{W+}, \quad (9e)$$

the conditional cousinhood relation \mathcal{C}^W as the partial equivalence relation

$$\mathcal{C}^W = (\Delta_W \mathcal{K}^W \Delta_W)^+ \cup \Delta_W, \quad (9f)$$

and the conditional active relation \mathcal{A}^W as the symmetric relation

$$\mathcal{A}^W = \Delta \cup \mathcal{B}^W \cup \mathcal{B}^{-W} \cup \mathcal{K}^W \cup (\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W). \quad (9g)$$

Theorem 4 Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices. The conditional directional separation relation $\perp\!\!\!\perp_d$ (Definition 2) is the complementary $(\mathcal{A}^W)^c$ of the conditional active relation \mathcal{A}^W (Equation (9g) in Definition 3):

$$\left(\perp\!\!\!\perp_d \mid W\right) = (\mathcal{A}^W)^c \quad \text{that is, } \gamma \perp\!\!\!\perp_d \lambda \mid W \iff \neg(\gamma \mathcal{A}^W \lambda) \quad (\forall \gamma, \lambda \in \mathcal{V}). \quad (10)$$

In other words, we have that

$$\{(\gamma, \lambda) \in \mathcal{V} \times \mathcal{V} \mid D_U[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{E}] \subset U_b^W(\mathcal{V}, \mathcal{E})\} = (\mathcal{A}^W)^c. \quad (11)$$

Proof. Let $\gamma, \lambda \in \mathcal{V}$ be two vertices. In Proposition 12, we show that $\gamma \mathcal{A}^W \lambda \implies \neg(\gamma \perp\!\!\!\perp_d \lambda \mid W)$ or, equivalently (see (8) in Definition 2), that $\gamma \mathcal{A}^W \lambda \implies D_U[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E}) \neq \emptyset$. We simply give a sketch of proof here, as details are to be found in the Appendix B. As the binary relation \mathcal{A}^W is given by the union of five relations in (9g), the proof examines five cases and exhibits an active path (one in $U_a^W(\mathcal{V}, \mathcal{E})$, see Definition 1) that joins the vertices γ and λ in the five cases when $\gamma \mathcal{A}^W \lambda$.

Now, we show that $\neg(\gamma \perp\!\!\!\perp_d \lambda \mid W) \implies \gamma \mathcal{A}^W \lambda$ or, equivalently (see (8) in Definition 2), that $D_U[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E}) \neq \emptyset \implies \gamma \mathcal{A}^W \lambda$. For this purpose, we assume that there exists an extended-oriented path $\rho \in U^n(\mathcal{V}, \mathcal{E})$ joining the two vertices γ and λ and such that ρ is active. We are going to prove that $\gamma \mathcal{A}^W \lambda$.

If the path length of ρ is smaller or equal to one, then, by definition (2b) of $U^1(\mathcal{V}, \mathcal{E})$, we necessarily have that either $\gamma \mathcal{E} \lambda$, or $\gamma \mathcal{E}^{-1} \lambda$, or $\gamma \Delta \lambda$. Now, as $\mathcal{E} \subset \mathcal{B}^W \subset \mathcal{A}^W$ by (9b) and (9g), as $\mathcal{E}^{-1} \subset \mathcal{B}^{-W} \subset \mathcal{A}^W$ by (9d) and (9g), and as $\Delta \subset \mathcal{A}^W$ by (9b) and (9g), we conclude that $\gamma \mathcal{A}^W \lambda$.

If the path length of ρ is ≥ 2 , we use Lemma 13 (proof by induction on the path length) and we obtain that $\gamma \mathcal{A}_{*+}^W \lambda$ or $\gamma \mathcal{A}_{*-}^W \lambda$. We therefore conclude that $\gamma \mathcal{A}^W \lambda$ since $\mathcal{A}^W = \Delta \cup \mathcal{A}_{*+}^W \cup \mathcal{A}_{*-}^W$ by (27a).

This ends the proof. □

4 Conclusion

Together with its two companion papers [2, 4], this paper is a contribution to providing another perspective on conditional independence and do-calculus. In this paper, we have considered directed graphs (DGs) not necessarily acyclic, and we have shown how the d-separation can be extended beyond acyclic graphs and can be expressed and characterized as a binary relation between vertices. The results in this paper are instrumental in proving those in [2] on topological conditional separation (t-separation), hence in the use of t-separation to establish conditional independence in [4].

Moreover, there are other perspectives. First, such developpement of a theory based on binary relations is interesting in itself as it makes it amenable to computer aided proof. Second, there are other notions of separation (between subsets) in graph theory that can also be expressed by means of binary relations (between vertices). We illustrate how. Let

$(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices, and $\gamma, \lambda \in \mathcal{V}$ be two vertices. Then, the three following statements are equivalent: any path from γ to λ passes through W ; there does not exist a path from γ to λ which passes through W^c ; $\neg(\gamma(\mathcal{E}\Delta_{W^c}\mathcal{E})^+\lambda)$.

A Additional material

A.1 Material on extended-oriented paths in a graph

In §A.1.1, we introduce additional definitions related to extended-oriented paths in a graph. In §A.1.2, we show Lemmas that will be instrumental in the proof of Theorem 4.

A.1.1 Definitions

We refer the reader to §2.1 for the basic definitions.

Concatenation of extended-oriented paths in a graph. We recall that $P(\mathcal{V}, \mathcal{E})$ in (1b) is the set of finite (edge) paths in the graph $(\mathcal{V}, \mathcal{E})$, and that $U(\mathcal{V}, \mathcal{E})$ in (2c) is the set of finite extended-oriented paths relative to the graph $(\mathcal{V}, \mathcal{E})$.

• The (edge) paths $\varrho' \in P(\mathcal{V}, \mathcal{E})$ and $\varrho'' \in P(\mathcal{V}, \mathcal{E})$ can be *concatenated* when $\varpi^\sharp(\varrho') = \varpi^b(\varrho'')$ (see Equation (3b)) to give the new (edge) path

$$\varrho = \varrho' \times \varrho'' = (\varrho', \varrho'') \in P(\mathcal{V}, \mathcal{E}), \quad (12a)$$

called *concatenation* of ϱ' and ϱ'' . It is easily seen that the operation \times is associative.

The extended-oriented paths $\rho' = (\varrho', \sigma') \in U(\mathcal{V}, \mathcal{E})$ and $\rho'' = (\varrho'', \sigma'') \in U(\mathcal{V}, \mathcal{E})$ can be *concatenated* when $\varpi^\sharp(\rho') = \varpi^b(\rho'')$ to give the new extended-oriented path

$$\rho = \rho' \times \rho'' = ((\varrho', \varrho''), (\sigma', \sigma'')) = (\varrho' \times \varrho'', (\sigma', \sigma'')) \in U(\mathcal{V}, \mathcal{E}), \quad (12b)$$

called *concatenation* of ρ' and ρ'' . It is easily seen that the operation \times is associative. For any subsets $S' \subset U(\mathcal{V}, \mathcal{E})$ and $S'' \subset U(\mathcal{V}, \mathcal{E})$ of extended-oriented paths, we define

$$S' \times S'' = \{\rho' \times \rho'' \mid \rho' \in S', \rho'' \in S''\} \subset U(\mathcal{V}, \mathcal{E}), \quad (12c)$$

with $S' \times S'' = \emptyset$ when $\varpi^\sharp(S') \cap \varpi^b(S'') = \emptyset$. It is easily obtained that the intermediate vertices (4b) of a concatenation are given by

$$\iota_W(\rho) = \iota_W(\rho' \times \rho'') = \iota_W(\rho') \cup \{\varpi^\sharp(\rho')\} \cup \iota_W(\rho'') = \iota_W(\rho') \cup \{\varpi^b(\rho'')\} \cup \iota_W(\rho''). \quad (13)$$

More on deployment in paths.

• For any binary relation $\mathcal{R} \subset \mathcal{V} \times \mathcal{V}$, the subset $D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}] \subset U(\mathcal{V}, \mathcal{E})$ is made of extended-oriented paths (relative to the graph $(\mathcal{V}, \mathcal{E})$) whose ordered endpoints (head and tail) satisfy

the relation \mathcal{R} . The mapping $\mathcal{R} \mapsto D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}]$ is nondecreasing w.r.t. the inclusion order, as it satisfies

$$\forall \mathcal{R} \subset \mathcal{V} \times \mathcal{V}, \forall \mathcal{S} \subset \mathcal{V} \times \mathcal{V}, D_U[\mathcal{R} \cup \mathcal{S} \mid \mathcal{V}, \mathcal{E}] = D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}] \cup D_U[\mathcal{S} \mid \mathcal{V}, \mathcal{E}]. \quad (14)$$

• By the very definition (5) of the deployment in extended-oriented paths, and by definition (12c) of the concatenation between subsets, we have the inclusion

$$\forall \mathcal{R} \subset \mathcal{V} \times \mathcal{V}, \forall \mathcal{S} \subset \mathcal{V} \times \mathcal{V}, D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}] \times D_U[\mathcal{S} \mid \mathcal{V}, \mathcal{E}] \subset D_U[\mathcal{RS} \mid \mathcal{V}, \mathcal{E}]. \quad (15)$$

We use the following notations

$$D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}]^n = \overbrace{D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}] \times \cdots \times D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}]}^{n \text{ times}}, \quad (16a)$$

$$D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}]^+ = \bigcup_{n \geq 1} D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}]^n. \quad (16b)$$

Endpoint relation in extended-oriented paths.

• With any subset $S \subset U(\mathcal{V}, \mathcal{E})$ of extended-oriented paths, we associate the following binary relation $\mathfrak{X}_U[S \mid \mathcal{V}, \mathcal{E}]$ on \mathcal{V} , that we call *endpoint relation*, defined by

$$\forall S \subset U(\mathcal{V}, \mathcal{E}), \mathfrak{X}_U[S \mid \mathcal{V}, \mathcal{E}] = \varpi(S) = \{\varpi(\rho) \mid \rho \in S\} \subset \mathcal{V} \times \mathcal{V}, \quad (17)$$

where the projection ϖ has been defined in (3c).

The subset $\mathfrak{X}_U[S \mid \mathcal{V}, \mathcal{E}] \subset \mathcal{V} \times \mathcal{V}$ is the subset of couples of vertices which are ordered endpoints (head and tail) of an extended-oriented path contained in S . By construction, the mapping $S \mapsto \mathfrak{X}_U[S \mid \mathcal{V}, \mathcal{E}]$ is nondecreasing w.r.t. the inclusion order, and takes values that are binary relations included in the binary relation $(\mathcal{E} \cup \mathcal{E}^{-1})^+$, that is,

$$\forall S \subset U(\mathcal{V}, \mathcal{E}), \mathfrak{X}_U[S \mid \mathcal{V}, \mathcal{E}] \subset (\mathcal{E} \cup \mathcal{E}^{-1})^+. \quad (18)$$

• Joint properties between boundary relation $\mathfrak{X}_U[\cdot \mid \mathcal{V}, \mathcal{E}]$ and deployment in extended-oriented paths $D_U[\cdot \mid \mathcal{V}, \mathcal{E}]$ are as follows:

$$\mathfrak{X}_U[D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}] \mid \mathcal{V}, \mathcal{E}] \subset \mathcal{R}, \quad \forall \mathcal{R} \subset \mathcal{V} \times \mathcal{V} \quad (19a)$$

$$D_U[\mathfrak{X}_U[S \mid \mathcal{V}, \mathcal{E}] \mid \mathcal{V}, \mathcal{E}] \supset S, \quad \forall S \subset U(\mathcal{V}, \mathcal{E}). \quad (19b)$$

A.1.2 Lemmas

We develop useful links between active extended-oriented paths (see Definition 1) and binary relations between ordered endpoints (head and tail) of the path. In the next lemma, we give a first insight in that direction for extended-oriented paths of length 2.

Lemma 5 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices. Let $\rho = (\varrho, o) \in U_a^W(\mathcal{V}, \mathcal{E})$ be a given active extended-oriented path of length 2 (see Definition 1) with γ as head endpoint and λ as tail endpoint, that is, there exists a vertex $\delta \in \mathcal{V}$ such that*

$$\rho = \left(((\gamma, \delta), (\delta, \lambda)), (o_1, o_2) \right) \text{ with } (o_1, o_2) \in \{-1, +1\}^2. \quad (20)$$

Then, one of the following two possibilities holds true:

1. *the extended-oriented path ρ ends with $o_2 = +1$ orientation, and then either $\gamma\mathcal{E}\Delta_{W^c}\mathcal{E}\lambda$ or $\gamma\mathcal{E}^{-1}\Delta_{W^c}\mathcal{E}\lambda$,*
2. *the extended-oriented path ρ ends with $o_2 = -1$ orientation, and then either $\gamma\mathcal{E}\Delta_{W^*}\mathcal{E}^{-1}\lambda$ or $\gamma\mathcal{E}^{-1}\Delta_{W^c}\mathcal{E}^{-1}\lambda$.*

Proof. As $\rho = (\varrho, o)$ in (20) belongs to $U_a^W(\mathcal{V}, \mathcal{E})$ — hence to $U(\mathcal{V}, \mathcal{E})$, the set of finite extended-oriented paths in (2c) — we have that $(\gamma, \delta) \in \mathcal{E}$ if $o_1 = +1$, $(\gamma, \delta) \in \mathcal{E}^{-1}$ if $o_1 = -1$, $(\delta, \lambda) \in \mathcal{E}$ if $o_2 = +1$, and $(\delta, \lambda) \in \mathcal{E}^{-1}$ if $o_2 = -1$. Now, we consider the four conditions enumerated in Definition 1 which must be satisfied for the extended-oriented path ρ to be active and which impose constraints on the vertex δ according to the possible orientations.

1. First, we consider the case when the extended-oriented path ρ ends with +1 orientation, that is, when $o_2 = +1$.
 - Item 1 in Definition 1 corresponds to $o_1 = +1$, $o_2 = +1$ and $\delta \in W^c$, which gives that $\gamma\mathcal{E}^{(+1)}\Delta_{W^c}\delta$ and $\delta\Delta_{W^c}\mathcal{E}^{(+1)}\lambda$. Hence, by composition of binary relations, we get that $\gamma\mathcal{E}\Delta_{W^c}\mathcal{E}\lambda$, using the property $\Delta_{W^c}\Delta_{W^c} = \Delta_{W^c}$.
 - Item 3 in Definition 1 corresponds to $o_1 = -1$, $o_2 = +1$ and $\delta \in W^c$, which gives that $\gamma\mathcal{E}^{(-1)}\Delta_{W^c}\delta$ and $\delta\Delta_{W^c}\mathcal{E}^{(+1)}\lambda$. Hence, we get that $\gamma\mathcal{E}^{-1}\Delta_{W^c}\mathcal{E}\lambda$.
2. Second, we consider the case when the extended-oriented path ρ ends with -1 orientation, that is, when $o_2 = -1$:
 - Item 2 in Definition 1 corresponds to $o_1 = -1$, $o_2 = -1$ and $\delta \in W^c$, which gives that $\gamma\mathcal{E}^{(-1)}\Delta_{W^c}\delta$ and $\delta\Delta_{W^c}\mathcal{E}^{(-1)}\lambda$. Hence, we get that $\gamma\mathcal{E}^{-1}\Delta_{W^c}\mathcal{E}^{-1}\lambda$,
 - Item 4 in Definition 1 corresponds to $o_1 = +1$, $o_2 = -1$ and $\delta \in W^*$, which gives that $\gamma\mathcal{E}^{(+1)}\Delta_{W^*}\delta$ and $\delta\Delta_{W^*}\mathcal{E}^{(-1)}\lambda$. Hence, we get that $\gamma\mathcal{E}\Delta_{W^*}\mathcal{E}^{-1}\lambda$, using the property $\Delta_{W^*}\Delta_{W^*} = \Delta_{W^*}$.

This ends the proof. □

Now, we develop the machinery to analyze active extended-oriented paths by considering decomposition into subpaths and junctions when reconcatenating. For this purpose, we need notation.

We denote by $\Omega^b : U(\mathcal{V}, \mathcal{E}) \rightarrow U^1(\mathcal{V}, \mathcal{E})$ (resp. $\Omega^\sharp : U(\mathcal{V}, \mathcal{E}) \rightarrow U^1(\mathcal{V}, \mathcal{E})$) the projection on the tail (resp. head) endpoint of an extended-oriented path, defined, for $(\varrho, o) \in U^n(\mathcal{V}, \mathcal{E})$ and $n \geq 1$, by

$$\Omega^b((\varrho, o)) = \Omega^b(\{v_i^b, v_i^\sharp\}_{i \in \llbracket 1, n \rrbracket}, \{o_i\}_{i \in \llbracket 1, n \rrbracket}) = ((v_1^b, v_1^\sharp), o_1), \quad (21a)$$

$$\Omega^\sharp((\varrho, o)) = \Omega^\sharp(\{v_i^b, v_i^\sharp\}_{i \in \llbracket 1, n \rrbracket}, \{o_i\}_{i \in \llbracket 1, n \rrbracket}) = ((v_n^b, v_n^\sharp), o_n). \quad (21b)$$

The following Lemma is a straightforward consequence of the definitions of active extended-oriented paths in $U_a^W(\mathcal{V}, \mathcal{E})$ (see Definition 1), of concatenation \times in (12b) and of the projection mappings Ω^b and Ω^\sharp in (21). The proof is left to the reader.

Lemma 6 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices. Let $\rho \in U(\mathcal{V}, \mathcal{E})$ be an extended-oriented path of length $n \geq 2$. We have that*

$$\rho \in U_a^W(\mathcal{V}, \mathcal{E}) \iff \exists \rho', \rho'' \in U_a^W(\mathcal{V}, \mathcal{E}), \quad \rho = \rho' \times \rho'', \quad \Omega^\sharp(\rho') \times \Omega^b(\rho'') \in U_a^W(\mathcal{V}, \mathcal{E}). \quad (22)$$

A.2 Material on binary relations

In §A.2.1, we prove Lemmas on the binary relations of Definition 3. In §A.2.2, we define new binary relations and we establish properties.

A.2.1 Lemmas on the binary relations of Definition 3

We refer the reader to Definition 3 for the definitions of basic binary relations.

Lemma 7 *We have that*

$$\mathcal{E}^{-W*} \mathcal{E}^{W*} = \Delta \cup \Delta_{W^c} \mathcal{B}^W \cup \mathcal{B}^{-W} \Delta_{W^c} \cup \mathcal{K}^W, \quad (23a)$$

$$\mathcal{C}^W \mathcal{E}^{-W*} \mathcal{E}^{W*} = \mathcal{C}^W (\Delta \cup \mathcal{B}^{-W} \Delta_{W^c} \cup \mathcal{K}^W), \quad (23b)$$

$$\mathcal{C}^W \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathcal{C}^W = \mathcal{C}^W, \quad (23c)$$

$$\mathcal{C}^W \mathcal{E}^{-W*} \mathcal{E}^{W*} \Delta_{W^c} = \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \Delta_{W^c}, \quad (23d)$$

$$\Delta_{W^c} \mathcal{E}^{-W*} \mathcal{E}^{W*} \mathcal{C}^W = \Delta_{W^c} (\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}^W. \quad (23e)$$

Proof.

- We prove Equation (23a) as follows:

$$\mathcal{E}^{-W*} \mathcal{E}^{W*} = (\Delta \cup \mathcal{E}^{-W+}) (\Delta \cup \mathcal{E}^{W+})$$

as $\mathcal{E}^{W*} = \Delta \cup \mathcal{E}^{W+}$ by definition of the reflexive and transitive closure of a binary relation

$$= \Delta \cup \mathcal{E}^{W+} \cup \mathcal{E}^{-W+} \cup \mathcal{E}^{-W+} \mathcal{E}^{W+} \quad (\text{by developing})$$

$$= \Delta \cup \mathcal{E}^{W+} \cup \mathcal{E}^{-W+} \cup \mathcal{K}^W \quad (\text{by definition of } \mathcal{K}^W \text{ in (9e)})$$

$$= \Delta \cup (\Delta_{W^c} \mathcal{E} (\Delta_{W^c} \mathcal{E})^*) \cup ((\mathcal{E}^{-1} \Delta_{W^c})^* \mathcal{E}^{-1} \Delta_{W^c}) \cup \mathcal{K}^W$$

as $\mathcal{R}^+ = \mathcal{R}\mathcal{R}^* = \mathcal{R}^*\mathcal{R}$ for any binary relation \mathcal{R} , used here with $\mathcal{R} = \Delta_{W^c}\mathcal{E}$ and with $\mathcal{R} = \mathcal{E}^{-1}\Delta_{W^c}$

$$= \Delta \cup \Delta_{W^c}\mathcal{B}^W \cup \mathcal{B}^{-W}\Delta_{W^c} \cup \mathcal{K}^W . \quad (\text{by definition of } \mathcal{B}^W \text{ in 9c})$$

- We prove Equation (23b) as follows:

$$\mathcal{C}^W \mathcal{E}^{-W^*} \mathcal{E}^{W^*} = \mathcal{C}^W \Delta_W \mathcal{E}^{-W^*} \mathcal{E}^{W^*}$$

as \mathcal{C}^W ends with Δ_W in (9f), hence $\mathcal{C}^W = \mathcal{C}^W \Delta_W$

$$\begin{aligned} &= \mathcal{C}^W \Delta_W (\Delta \cup \Delta_{W^c}\mathcal{B}^W \cup \mathcal{B}^{-W}\Delta_{W^c} \cup \mathcal{K}^W) && (\text{by (23a)}) \\ &= \mathcal{C}^W (\Delta_W \Delta \cup \Delta_W \Delta_{W^c}\mathcal{B}^W \cup \Delta_W \mathcal{B}^{-W}\Delta_{W^c} \cup \Delta_W \mathcal{K}^W) && (\text{by developing}) \\ &= \mathcal{C}^W \Delta_W (\Delta \cup \mathcal{B}^{-W}\Delta_{W^c} \cup \mathcal{K}^W) && (\text{as } \Delta_W \Delta_{W^c} = \emptyset) \\ &= \mathcal{C}^W (\Delta \cup \mathcal{B}^{-W}\Delta_{W^c} \cup \mathcal{K}^W) . && (\text{as } \mathcal{C}^W \Delta_W = \mathcal{C}^W) \end{aligned}$$

- We prove Equation (23c) as follows:

$$\begin{aligned} \mathcal{C}^W \mathcal{E}^{-W^*} \mathcal{E}^{W^*} \mathcal{C}^W &= \mathcal{C}^W (\Delta \cup \mathcal{B}^{-W}\Delta_{W^c} \cup \mathcal{K}^W) \mathcal{C}^W && (\text{by (23b)}) \\ &= \mathcal{C}^W \Delta_W (\Delta \cup \mathcal{B}^{-W}\Delta_{W^c} \cup \mathcal{K}^W) \Delta_W \mathcal{C}^W \end{aligned}$$

as \mathcal{C}^W begins and ends with Δ_W in (9f), hence $\mathcal{C}^W = \mathcal{C}^W \Delta_W = \Delta_W \mathcal{C}^W$

$$\begin{aligned} &= \mathcal{C}^W \Delta_W (\Delta \cup \mathcal{K}^W) \Delta_W \mathcal{C}^W && (\text{as } \Delta_{W^c}\Delta_W = \emptyset) \\ &= \mathcal{C}^W (\Delta_W \cup \Delta_W \mathcal{K}^W \Delta_W) \mathcal{C}^W && (\text{by developing}) \\ &= (\Delta_W \cup (\Delta_W \mathcal{K}^W \Delta_W)^+) (\Delta_W \cup \Delta_W \mathcal{K}^W \Delta_W) (\Delta_W \cup (\Delta_W \mathcal{K}^W \Delta_W)^+) && (\text{by definition (9f) of } \mathcal{C}^W) \\ &= (\Delta_W \cup (\Delta_W \mathcal{K}^W \Delta_W)^+) (\Delta_W \cup (\Delta_W \mathcal{K}^W \Delta_W)^+) \end{aligned}$$

as $(\Delta_W \cup \mathcal{R}^+) (\Delta_W \cup \mathcal{R}) = \Delta_W \cup \mathcal{R}^+ \Delta_W \cup \Delta_W \mathcal{R} \cup \mathcal{R}^+ \mathcal{R} = \Delta_W \cup \mathcal{R}^+ \cup \mathcal{R}^+ \mathcal{R} = \Delta_W \cup \mathcal{R}^+$ whenever $\mathcal{R}\Delta_W = \Delta_W \mathcal{R} = \mathcal{R}$, which is the case for $\mathcal{R} = \Delta_W \mathcal{K}^W \Delta_W$

$$\begin{aligned} &= \Delta_W \cup (\Delta_W \mathcal{K}^W \Delta_W)^+ && (\text{by developing}) \\ &= \mathcal{C}^W . && (\text{by definition (9f) of } \mathcal{C}^W) \end{aligned}$$

- We prove Equation (23d) as follows:

$$\begin{aligned} \mathcal{C}^W \mathcal{E}^{-W^*} \mathcal{E}^{W^*} \Delta_{W^c} &= \mathcal{C}^W (\Delta \cup \mathcal{B}^{-W}\Delta_{W^c} \cup \mathcal{K}^W) \Delta_{W^c} && (\text{by (23b)}) \\ &= \mathcal{C}^W \Delta_W (\Delta \cup \mathcal{B}^{-W}\Delta_{W^c} \cup \mathcal{K}^W) \Delta_{W^c} \end{aligned}$$

as \mathcal{C}^W ends with Δ_W in (9f), hence $\mathcal{C}^W = \mathcal{C}^W \Delta_W$

$$\begin{aligned} &= \mathcal{C}^W \Delta_W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \Delta_{W^c} && (\text{as } \Delta_W \Delta_{W^c} = \emptyset) \\ &= \mathcal{C}^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \Delta_{W^c} . && (\text{as } \mathcal{C}^W = \mathcal{C}^W \Delta_W) \end{aligned}$$

- Equation (23e) is obtained from Equation (23d) by switching to converse relation.

This ends the proof. □

A.2.2 Definitions and properties of new binary relations

We refer the reader to Definition 3 for the definitions of basic binary relations. We add two new ones, where we recall that $W^* = \mathcal{E}^*W$ by (7):

$$\mathcal{C}_*^W = (\Delta_{W^*} \mathcal{K}^W \Delta_{W^*})^+ \cup \Delta_{W^*}, \quad (24)$$

$$\mathcal{A}_*^W = \Delta \cup \mathcal{B}^W \cup \mathcal{B}^{-W} \cup \mathcal{K}^W \cup (\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}_*^W (\mathcal{B}^{-W} \cup \mathcal{K}^W). \quad (25)$$

Notice that the *star conditional cousinhood relation* \mathcal{C}_*^W in (24) is the relation \mathcal{C}^W in (9f) with W replaced by W^* , and the *star conditional active relation* \mathcal{A}_*^W in (25) is the relation \mathcal{A}^W in (9g) with \mathcal{C}^W replaced by \mathcal{C}_*^W .

Lemma 8 *The two following relations*

$$\mathcal{A}_{*+}^W = \mathcal{B}^W \cup \mathcal{K}^W \cup \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}_*^W \mathcal{K}^W \right), \quad (26a)$$

$$\mathcal{A}_{*-}^W = \mathcal{B}^{-W} \cup \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}_*^W \mathcal{B}^{-W} \right), \quad (26b)$$

satisfy the following properties

$$\mathcal{A}_*^W = \Delta \cup \mathcal{A}_{*+}^W \cup \mathcal{A}_{*-}^W, \quad (27a)$$

$$\mathcal{A}_{*+}^W \supset \mathcal{A}_{*-}^W \Delta_{W^c} \mathcal{E}, \quad (27b)$$

$$\mathcal{A}_{*-}^W \supset \mathcal{A}_{*+}^W \Delta_{W^*} \mathcal{E}^{-1}. \quad (27c)$$

Proof.

• We prove (27a) as follows:

$$\begin{aligned} \mathcal{A}_*^W &= \Delta \cup \mathcal{B}^W \cup \mathcal{B}^{-W} \cup \mathcal{K}^W \cup (\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}_*^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \quad (\text{by definition (25) of } \mathcal{A}_*^W) \\ &= \Delta \cup \underbrace{\mathcal{B}^W \cup \mathcal{K}^W \cup (\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}_*^W \mathcal{K}^W}_{=\mathcal{A}_{*+}^W \text{ by (26a)}} \cup \underbrace{\mathcal{B}^{-W} \cup (\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}_*^W \mathcal{B}^{-W}}_{=\mathcal{A}_{*-}^W \text{ by (26b)}}. \end{aligned}$$

• We prove (27b) as follows:

$$\begin{aligned} \mathcal{A}_{*-}^W \Delta_{W^c} \mathcal{E} &= \left(\mathcal{B}^{-W} \cup \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}_*^W \mathcal{B}^{-W} \right) \right) \Delta_{W^c} \mathcal{E} \quad (\text{by definition (26b) of } \mathcal{A}_{*-}^W) \\ &= \left(\Delta \cup \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}_*^W \right) \right) \mathcal{B}^{-W} \Delta_{W^c} \mathcal{E} \quad (\text{by factorizing } \mathcal{B}^{-W}) \\ &\subset \left(\Delta \cup \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}_*^W \right) \right) \mathcal{B}^{-W} \Delta_{W^c} \mathcal{B}^W \quad (\text{as } \mathcal{E} \subset \mathcal{B}^W \text{ by (9b)}) \\ &= \left(\Delta \cup \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}_*^W \right) \right) \mathcal{K}^W \quad (\text{by definition (9e) of } \mathcal{K}^W) \\ &= \mathcal{K}^W \cup (\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}_*^W \mathcal{K}^W \quad (\text{by developing}) \\ &\subset \mathcal{A}_{*+}^W. \quad (\text{by definition (26a) of } \mathcal{A}_{*+}^W) \end{aligned}$$

- We prove (27c) as follows:

$$\begin{aligned} \mathcal{A}_{*+}^w \Delta_{W^*} \mathcal{E}^{-1} &= \left(\mathcal{B}^w \cup \mathcal{K}^w \cup \left((\mathcal{B}^w \cup \mathcal{K}^w) \mathcal{C}_*^w \mathcal{K}^w \right) \right) \Delta_{W^*} \mathcal{E}^{-1} && \text{(by definition (26a) of } \mathcal{A}_{*+}^w \text{)} \\ &= \left((\mathcal{B}^w \cup \mathcal{K}^w) \Delta_{W^*} \mathcal{E}^{-1} \right) \cup \left((\mathcal{B}^w \cup \mathcal{K}^w) \mathcal{C}_*^w \mathcal{K}^w \Delta_{W^*} \mathcal{E}^{-1} \right). && \text{(by developing)} \end{aligned}$$

We treat each of the two terms in the union separately. We are going to show that each term is included in \mathcal{A}_{*-}^w .

For the first term, we have that

$$(\mathcal{B}^w \cup \mathcal{K}^w) \Delta_{W^*} \mathcal{E}^{-1} \subset (\mathcal{B}^w \cup \mathcal{K}^w) \mathcal{C}_*^w \mathcal{B}^{-w}$$

as $\Delta_{W^*} \subset \mathcal{C}_*^w$ by (24), and as $\mathcal{E}^{-1} \subset \mathcal{B}^{-w}$ by (9d)

$$\subset \mathcal{A}_{*-}^w. \quad \text{(as } \mathcal{A}_{*-}^w = \mathcal{B}^{-w} \cup ((\mathcal{B}^w \cup \mathcal{K}^w) \mathcal{C}_*^w \mathcal{B}^{-w}) \text{ by definition (26b))}$$

For the second term, we have that

$$\begin{aligned} &(\mathcal{B}^w \cup \mathcal{K}^w) \mathcal{C}_*^w \mathcal{K}^w \Delta_{W^*} \mathcal{E}^{-1} \\ &= (\mathcal{B}^w \cup \mathcal{K}^w) \left((\Delta_{W^*} \mathcal{K}^w \Delta_{W^*})^+ \cup \Delta_{W^*} \right) \mathcal{K}^w \Delta_{W^*} \mathcal{E}^{-1} && \text{(by definition (24) of } \mathcal{C}_*^w \text{)} \\ &= (\mathcal{B}^w \cup \mathcal{K}^w) \left((\Delta_{W^*} \mathcal{K}^w \Delta_{W^*})^+ \cup \Delta_{W^*} \right) (\Delta_{W^*} \mathcal{K}^w \Delta_{W^*}) \mathcal{E}^{-1} && \text{(by inserting } \Delta_{W^*} \text{)} \\ &\subset (\mathcal{B}^w \cup \mathcal{K}^w) \left((\Delta_{W^*} \mathcal{K}^w \Delta_{W^*})^+ \cup \Delta_{W^*} \right) \mathcal{E}^{-1} \end{aligned}$$

as $(\mathcal{R}^+ \cup \Delta_{W^*}) \mathcal{R} \subset \mathcal{R}^+$ for any relation \mathcal{R} such that $\mathcal{R} \Delta_{W^*} = \Delta_{W^*} \mathcal{R} = \mathcal{R}$, which is the case for $\mathcal{R} = \Delta_{W^*} \mathcal{K}^w \Delta_{W^*}$

$$\begin{aligned} &= (\mathcal{B}^w \cup \mathcal{K}^w) \mathcal{C}_*^w \mathcal{E}^{-1} && \text{(by definition (24) of } \mathcal{C}_*^w \text{)} \\ &\subset (\mathcal{B}^w \cup \mathcal{K}^w) \mathcal{C}_*^w \mathcal{B}^{-w} && \text{(as } \mathcal{E}^{-1} \subset \mathcal{B}^{-w} \text{ by (9d))} \\ &\subset \mathcal{A}_{*-}^w. && \text{(as } \mathcal{A}_{*-}^w = \mathcal{B}^{-w} \cup ((\mathcal{B}^w \cup \mathcal{K}^w) \mathcal{C}_*^w \mathcal{B}^{-w}) \text{ by definition (26b))} \end{aligned}$$

We conclude that $\mathcal{A}_{*+}^w \Delta_{W^*} \mathcal{E}^{-1} \subset \mathcal{A}_{*-}^w$.

This ends the proof. □

Lemma 9 *We have that*

$$\mathcal{A}^w = \mathcal{A}_*^w. \quad (28)$$

Proof. The proofs is in three steps.

- We prove that

$$\forall \mathcal{R} \subset \mathcal{V} \times \mathcal{V}, \quad \forall \Gamma \subset \mathcal{V}, \quad \mathcal{R}^* \Gamma = (\Delta_{\Gamma^c} \mathcal{R})^* \Gamma. \quad (29)$$

For this purpose, we prove the following induction assumption \mathcal{H}_n : for any $n \geq 1$, we have that $(\cup_{k=0}^n \mathcal{R}^k) \Gamma = (\cup_{k=0}^n (\Delta_{\Gamma^c} \mathcal{R})^k) \Gamma$, where we recall the convention $\mathcal{R}^0 = \Delta$.

As a preliminary result, for any $\Gamma, \Lambda \subset \mathcal{V}$, from the sequence of equalities $(\Delta\Gamma) \cup (\mathcal{R}\Lambda) = \Gamma \cup ((\mathcal{R}\Lambda) \setminus \Gamma) = \Gamma \cup (\Delta_{\Gamma^c} \mathcal{R}\Lambda) = (\Delta\Gamma) \cup (\Delta_{\Gamma^c} \mathcal{R}\Lambda)$, we deduce that

$$(\Delta\Gamma) \cup (\mathcal{R}\Lambda) = (\Delta\Gamma) \cup (\Delta_{\Gamma^c} \mathcal{R}\Lambda). \quad (30)$$

Thus, with $\Lambda = \Gamma$, we obtain that $(\cup_{k=0}^1 \mathcal{R}^k)\Gamma = (\cup_{k=0}^1 (\Delta_{\Gamma^c} \mathcal{R})^k)\Gamma$, that is, assumption \mathcal{H}_1 holds true.

Now, we suppose that, for a given $n \geq 1$, the induction assumption \mathcal{H}_n holds true. Then, we have that

$$\begin{aligned} (\cup_{k=0}^{n+1} \mathcal{R}^k)\Gamma &= (\Delta\Gamma) \cup \left(\mathcal{R}((\cup_{k=0}^n \mathcal{R}^k))\Gamma \right) && \text{(using the convention } \mathcal{R}^0 = \Delta) \\ &= (\Delta\Gamma) \cup \left(\Delta_{\Gamma^c} \mathcal{R}((\cup_{k=0}^n \mathcal{R}^k))\Gamma \right) \end{aligned}$$

using the preliminary result (30) but with the binary relation $\mathcal{R}((\cup_{k=0}^n \mathcal{R}^k))$

$$\begin{aligned} &= (\Delta\Gamma) \cup \left(\Delta_{\Gamma^c} \mathcal{R}(\cup_{k=0}^n (\Delta_{\Gamma^c} \mathcal{R})^k)\Gamma \right) && \text{(using the induction assumption } \mathcal{H}_n) \\ &= (\Delta\Gamma) \cup \left((\cup_{k=1}^{n+1} (\Delta_{\Gamma^c} \mathcal{R})^k)\Gamma \right) \\ &= (\cup_{k=0}^{n+1} (\Delta_{\Gamma^c} \mathcal{R})^k)\Gamma. && \text{(as } (\Delta_{\Gamma^c} \mathcal{R})^0 = \Delta) \end{aligned}$$

Thus, we have proven the induction assumption \mathcal{H}_{n+1} .

Now, let us suppose that $\gamma \in \mathcal{R}^* \Gamma$. Then, there exists an integer $n \geq 1$ such that $\gamma \in (\cup_{k=0}^n \mathcal{R}^k)\Gamma$; using the just proven property \mathcal{H}_n , we get that $\gamma \in (\cup_{k=0}^n (\Delta_{\Gamma^c} \mathcal{R})^k)\Gamma$ and, therefore, $\gamma \in (\Delta_{\Gamma^c} \mathcal{R})^* \Gamma$. Thus, we have shown that $\mathcal{R}^* \Gamma \subset (\Delta_{\Gamma^c} \mathcal{R})^* \Gamma$. The converse inclusion is easier to prove as $\Delta_{\Gamma^c} \mathcal{R} \subset \mathcal{R}$. Finally, we have shown the equality $\mathcal{R}^* \Gamma = (\Delta_{\Gamma^c} \mathcal{R})^* \Gamma$, which is (29).

- The following inclusion is easy to prove:

$$\forall \mathcal{R} \subset \mathcal{V} \times \mathcal{V}, \forall \Gamma \subset \mathcal{V}, \Delta_{\mathcal{R}\Gamma} \subset \mathcal{R}\Delta_{\Gamma}\mathcal{R}^{-1}. \quad (31)$$

- We prove that

$$\mathcal{E}^{W^*} \Delta_W \mathcal{E}^{-W^*} = \mathcal{E}^{W^*} \Delta_{(\mathcal{E}^* W)} \mathcal{E}^{-W^*}. \quad (32)$$

Using Equation (29) with $\mathcal{R}=\mathcal{E}$ and $\Gamma=W$ gives $\mathcal{E}^* W = (\Delta_W \mathcal{E})^* W = \mathcal{E}^{W^*} W$. Combined with the Inclusion (31), we get $\Delta_{\mathcal{E}^* W} = \Delta_{\mathcal{E}^* W} \subset \mathcal{E}^{W^*} \Delta_W \mathcal{E}^{-W^*}$. Thus, we obtain that

$$\mathcal{E}^{W^*} \Delta_{(\mathcal{E}^* W)} \mathcal{E}^{-W^*} \subset \mathcal{E}^{W^*} (\mathcal{E}^{W^*} \Delta_W \mathcal{E}^{-W^*}) \mathcal{E}^{-W^*} = \mathcal{E}^{W^*} \Delta_W \mathcal{E}^{-W^*}.$$

Thus, we have obtained the inclusion $\mathcal{E}^{W^*} \Delta_W \mathcal{E}^{-W^*} \supset \mathcal{E}^{W^*} \Delta_{(\mathcal{E}^* W)} \mathcal{E}^{-W^*}$. The reverse inclusion follows from the fact that $W \subset W^* = \mathcal{E}^* W$ by (7), which gives $\mathcal{E}^{W^*} \Delta_W \mathcal{E}^{-W^*} \subset \mathcal{E}^{W^*} \Delta_{(\mathcal{E}^* W)} \mathcal{E}^{-W^*}$.

- Finally, we prove that $\mathcal{A}^W = \mathcal{A}_*^W$. For that purpose, it suffices to show that replacing the subexpressions Δ_W by $\Delta_{(\mathcal{E}^* W)}$ in the expression (9g) of \mathcal{A}^W does not change the relation. Using the definition of \mathcal{A}^W in Equation (9g), we obtain that Δ_W appears only in subexpressions of the form $\mathcal{B}^W \Delta_W \mathcal{B}^{-W}$ or $\mathcal{K}^W \Delta_W \mathcal{B}^{-W}$ or $\mathcal{B}^W \Delta_W \mathcal{K}^W$ or $\mathcal{K}^W \Delta_W \mathcal{K}^W$. Now, using the fact that the two relations \mathcal{B}^W and \mathcal{K}^W always end with \mathcal{E}^{W^*} and the two relation \mathcal{B}^{-W} and \mathcal{K}^W always start with \mathcal{E}^{-W^*} we obtain that Δ_W appears only in subexpressions of the form $\mathcal{E}^{W^*} \Delta_W \mathcal{E}^{-W^*}$. We conclude, using Equation (32), that Δ_W can be replaced by $\Delta_{(\mathcal{E}^* W)}$ in \mathcal{A}^W without changing the relation.

This ends the proof. \square

B Proof of Theorem 4

The proof of Theorem 4 relies on the following Proposition 12 (itself based on Lemmas 10 and 11) and on Lemma 13, that we are going to prove.

The following Lemma 10 displays elementary relational patterns (composition of relations) whose deployment in paths contain active extended-oriented paths. The binary relations below have been introduced in Definition 3, except for the two additional ones defined in (24) and in (25). For any integer $n \geq 1$, we denote by $\mathbf{1}_n = (+1, \dots, +1)$ (resp. $-\mathbf{1}_n = (-1, \dots, -1)$) the vector of length n made of $+1$ (resp. of -1). We recall that the deployment $D_U[\mathcal{R} \mid \mathcal{V}, \mathcal{E}]$ in extended-oriented paths of a binary relation \mathcal{R} has been defined in (5), and the intermediate vertices ι_U of an extended-oriented path in (4c).

Lemma 10 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices. For any two vertices $\gamma, \lambda \in \mathcal{V}$, we have*

$$\begin{aligned} \gamma \mathcal{B}^W \lambda &\implies \text{there exists } \rho \in U(\mathcal{V}, \mathcal{E}) \text{ such that} \\ &\quad (\varrho, \mathbf{1}_{|\varrho|}) \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E}) \\ &\quad \text{and } \iota_U(\rho) \subset \mathcal{E}^{W+\lambda}, \end{aligned} \tag{33a}$$

$$\begin{aligned} \gamma \mathcal{B}^{-W} \lambda &\implies \text{there exists } \rho \in U(\mathcal{V}, \mathcal{E}) \text{ such that} \\ &\quad (\varrho, -\mathbf{1}_{|\varrho|}) \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E}) \\ &\quad \text{and } \iota_U(\rho) \subset \mathcal{E}^{W+\gamma}, \end{aligned} \tag{33b}$$

$$\begin{aligned} \gamma \mathcal{K}^W \lambda &\implies \text{there exists } \rho = (\varrho, o) \in U(\mathcal{V}, \mathcal{E}), \text{ with } |\rho| \geq 2 \\ &\quad \text{and } o = (-1, \dots, +1) \in \{-1\} \times \{-1, +1\}^{|\rho|-2} \times \{+1\}, \text{ such that} \\ &\quad \rho = (\varrho, o) = (\varrho, (-1, \dots, +1)) \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E}) \\ &\quad \text{and } \iota_U(\rho) \subset \mathcal{E}^{W+\{\gamma, \lambda\}}, \end{aligned} \tag{33c}$$

$$\begin{aligned} \gamma \mathcal{C}_*^W \lambda &\implies \gamma \in W^*, \lambda \in W^* \text{ and} \\ &\quad \text{there exists } \rho = (\varrho, o) \in U(\mathcal{V}, \mathcal{E}), \text{ with } |\rho| \geq 2 \\ &\quad \text{and } o = (-1, \dots, +1) \in \{-1\} \times \{-1, +1\}^{|\rho|-2} \times \{+1\}, \text{ such that} \\ &\quad \rho = (\varrho, o) = (\varrho, (-1, \dots, +1)) \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E}) \\ &\quad \text{and } \iota_U(\rho) \subset \mathcal{E}^{W^*W}. \end{aligned} \tag{33d}$$

Moreover, in Implication (33d), if a vertice $\delta \in \iota_U(\rho)$ belongs to W , it necessarily appears in a subpath of ρ of the form $\left((v^\flat, \delta), (\delta, v^\sharp), (+1, -1) \right)$.

Proof.

- We prove the implication (33a). Let $\gamma, \lambda \in \mathcal{V}$ be such that $\gamma \mathcal{B}^W \lambda$. By (9b), we have that $\mathcal{B}^W \lambda = \mathcal{E}(\Delta_{W^c} \mathcal{E})^*$, hence that $\gamma \mathcal{E}(\Delta_{W^c} \mathcal{E})^* \lambda$. As $(\Delta_{W^c} \mathcal{E})^* = \Delta \cup \bigcup_{n=1}^{\infty} (\Delta_{W^c} \mathcal{E})^n$ by definition, if $\gamma \mathcal{E}(\Delta_{W^c} \mathcal{E})^* \lambda$, then either $\gamma \mathcal{E} \lambda$ or there exists $n \geq 1$ such that $\gamma \mathcal{E}(\Delta_{W^c} \mathcal{E})^n \lambda$. Thus, we consider two cases.

If $\gamma \mathcal{E} \lambda$, the extended-oriented path $((\gamma, \lambda), +1)$ is both in $D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}]$, by definition (5), and belongs to $U_a^W(\mathcal{V}, \mathcal{E})$, as it is of length 1 hence is active (see Definition 1).

If $\gamma\mathcal{E}(\Delta_{W^c}\mathcal{E})^n\lambda$, with $n \geq 1$, then there exists a sequence $\{v_i\}_{i \in \llbracket 0, n+1 \rrbracket}$ in \mathcal{V} such that $v_0 = \gamma$, $v_{n+1} = \lambda$, and $v_{i-1}\mathcal{E}v_i$, $v_i \in W^c$, $v_i\mathcal{E}v_{i+1}$, for $i \in \llbracket 1, n \rrbracket$. The following $\rho = (\{(v_i, v_{i+1})\}_{i \in \llbracket 0, n \rrbracket}, \mathbf{1}_{n+1})$ is an extended-oriented path that belongs to $D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}]$, as $\varpi(\rho) = (\gamma, \lambda)$, and also to $U_a^W(\mathcal{V}, \mathcal{E})$. Indeed, all the extended-oriented subpaths

$\left(((v_{i-1}, v_i), (v_i, v_{i+1})), (+1, +1) \right)$ for $i \in \llbracket 1, n \rrbracket$, satisfy Item 1 in Definition 1 because $v_i \in W^c$. It remains to show that $\iota_U(\rho) \subset \mathcal{E}^{W+\lambda}$. This inclusion easily follows from the fact that, by definition (4b) of the intermediate vertices, we have that $\iota_U(\rho) = \{v_2, \dots, v_{n-1}\}$ where $v_{n-1} \in W^c \cap \mathcal{E}\lambda = \Delta_{W^c}\mathcal{E}\lambda = \mathcal{E}^W\lambda \subset \mathcal{E}^{W+\lambda}$, \dots , $v_{n-k} \in (\mathcal{E}^W)^k\lambda \subset \mathcal{E}^{W+\lambda}$, \dots , $v_2 \in (\mathcal{E}^W)^{n-2}\lambda \subset \mathcal{E}^{W+\lambda}$.

- We prove the implication (33b) in the same way (here, all the extended-oriented subpaths satisfy Item 2 in Definition 1).

- We prove the implication (33c). Let $\gamma, \lambda \in \mathcal{V}$ be such that $\gamma\mathcal{K}^W\lambda$. By definition (9e) of \mathcal{K}^W , we obtain that $\gamma\mathcal{B}^{-W}\Delta_{W^c}\mathcal{B}^W\lambda$. As a consequence of the definition of the composition of relations, there exists $\delta \in W^c$ such that $\gamma\mathcal{B}^{-W}\delta$ and $\delta\mathcal{B}^W\lambda$. Thus, by (33b), there exists $\rho' = (\varrho', -\mathbf{1}_{|\varrho'|}) \in D_U[(\gamma, \delta) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$ and by (33a), there exists $\rho'' = (\varrho'', \mathbf{1}_{|\varrho''|}) \in D_U[(\delta, \lambda) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$. We consider the extended-oriented path $\rho = \rho' \times \rho'' \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}]$ obtained by concatenation as in (12b), and which is such that $|\rho| \geq 2$. We claim that $\rho \in U_a^W(\mathcal{V}, \mathcal{E})$. Indeed, $\rho' \in U_a^W(\mathcal{V}, \mathcal{E})$ and $\rho'' \in U_a^W(\mathcal{V}, \mathcal{E})$ by assumption, so that, by Equation (22) in Lemma 6, it only remains to show that

$$\left(((v', \delta), (\delta, v'')), (-1, +1) \right) = \Omega^\sharp(\rho') \times \Omega^\flat(\rho'') \in U_a^W(\mathcal{V}, \mathcal{E}),$$

where (v', δ) is the first edge of the extended-oriented path $\rho' = (\varrho', -\mathbf{1}_{|\varrho'|})$ and (δ, v'') is the last edge of the extended-oriented path $\rho'' = (\varrho'', \mathbf{1}_{|\varrho''|})$. Now, the above subpath satisfies Item 3 in Definition 1 because $v_i \in W^c$. We deduce that $\rho = \rho' \times \rho'' = \left((\varrho', \varrho''), (-\mathbf{1}_{|\varrho'|}, \mathbf{1}_{|\varrho''|}) \right) \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$. It remains to show that $\iota_U(\rho) \subset \mathcal{E}^{W+\{\gamma, \lambda\}}$ but this comes from (13) which gives

$$\iota_U(\rho) = \underbrace{\iota_U(\rho')}_{\subset \mathcal{E}^{W+\gamma}} \cup \underbrace{\{\delta\}}_{\subset \mathcal{E}^{W+\gamma} \cap \mathcal{E}^{W+\lambda}} \cup \underbrace{\iota_U(\rho'')}_{\subset \mathcal{E}^{W+\lambda}} \subset \mathcal{E}^{W+\gamma} \cup (\mathcal{E}^{W+\gamma} \cap \mathcal{E}^{W+\lambda}) \cup \mathcal{E}^{W+\lambda} = \mathcal{E}^{W+\{\gamma, \lambda\}}.$$

- We prove the implication (33d). We suppose that $\gamma\mathcal{C}_*^W\lambda$. As $\mathcal{C}_*^W = \Delta_{W^*} \cup (\Delta_{W^*}\mathcal{K}^W\Delta_{W^*})^+$ by (24), we consider three cases: either $\gamma\Delta_{W^*}\lambda$, or $\gamma\Delta_{W^*}\mathcal{K}^W\Delta_{W^*}\lambda$ or there exists $n \geq 1$ such that $\gamma(\Delta_{W^*}\mathcal{K}^W\Delta_{W^*})^{n+1}\lambda$.

Suppose that $\gamma\Delta_{W^*}\lambda$. Then $\gamma = \lambda$, and thus the extended-oriented path $\rho = ((\gamma, \lambda), +1) \in U^0(\mathcal{V}, \mathcal{E})$ is active as any extended-oriented path of length 0 or 1 is active by Definition 1 and by (2e).

Suppose that $\gamma\Delta_{W^*}\mathcal{K}^W\Delta_{W^*}\lambda$. Then, $\gamma \in W^*$, $\lambda \in W^*$ and $\gamma\mathcal{K}^W\lambda$. Therefore, by (33c) there exists $\rho \in U(\mathcal{V}, \mathcal{E})$, with $|\rho| \geq 2$, and $o \in \{-1\} \times \{-1, +1\}^{|\rho|-2} \times \{+1\}$ such that $(\varrho, o) \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$ and that $\iota_U(\rho) \subset \mathcal{E}^{W+\{\gamma, \lambda\}}$. As $\{\gamma, \lambda\} \subset W^*$, we get that $\iota_U(\rho) \subset \mathcal{E}^{W+W^*} = \mathcal{E}^{W+\mathcal{E}^{W^*}W} = \mathcal{E}^{W+W}$, since $W^* = \mathcal{E}^*W$ by (7) and $\mathcal{E}^{W+\mathcal{E}^{W^*}} = \mathcal{E}^{W+}$.

Suppose that $\gamma(\Delta_{W^*}\mathcal{K}^W\Delta_{W^*})^{n+1}\lambda$. Then, there exists a sequence $\{\delta_i\}_{i \in \llbracket 0, n+1 \rrbracket}$ in W^* such that $\delta_0 = \gamma$, $\delta_{n+1} = \lambda$ and $\delta_i\mathcal{K}^W\delta_{i+1}$ for $i \in \llbracket 0, n \rrbracket$. Therefore, by (33c), there exists a sequence $\{\rho_i\}_{i \in \llbracket 0, n \rrbracket}$ in $U(\mathcal{V}, \mathcal{E})$ of extended-oriented paths such that $|\rho_i| \geq 2$ for $i \in \llbracket 0, n \rrbracket$, and that $\rho_i =$

$(\varrho_i, o_i) \in D_U[(\delta_i, \delta_{i+1}) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$ for $i \in \llbracket 0, n \rrbracket$, where o_i is of the form $(-1, \dots, +1)$ and such that $\iota_U(\rho_i) \subset \mathcal{E}^{W+W}$ for $i \in \llbracket 0, n \rrbracket$. We define the extended-oriented path $\rho = \rho_0 \times \dots \times \rho_n \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}]$, by iterated (associative) concatenation as in (12b), which is such that $|\rho| \geq 2$ and that

$$\rho = (\varrho_0 \times \dots \times \varrho_n, (o_0, \dots, o_n)) \text{ where } o_i \in \{-1\} \times \{-1, +1\}^{|\varrho_i|-2} \times \{+1\}, \forall i \in \llbracket 0, n \rrbracket.$$

We claim that $\rho \in U_a^W(\mathcal{V}, \mathcal{E})$. Indeed, $\rho_i \in U_a^W(\mathcal{V}, \mathcal{E})$ for $i \in \llbracket 0, n \rrbracket$, so that, by Equation (22) in Lemma 6, it only remains to show that, for $i \in \llbracket 0, n-1 \rrbracket$,

$$\left((v_i^\flat, \delta_{i+1}), (\delta_{i+1}, v_{i+1}^\sharp), (+1, -1) \right) = \Omega^\sharp(\rho_i) \times \Omega^\flat(\rho_{i+1}) \in U_a^W(\mathcal{V}, \mathcal{E}), \quad (34)$$

where $(v_i^\flat, \delta_{i+1})$ is the last edge of the extended-oriented path $\rho_i = (\varrho_i, o_i)$ and $(\delta_{i+1}, v_{i+1}^\sharp)$ is the first edge of the extended-oriented path $\rho_{i+1} = (\varrho_{i+1}, o_{i+1})$. As $\delta_{i+1} \in W^*$, for $i \in \llbracket 0, n-1 \rrbracket$, and because of the orientation $(-1, +1)$, all the above subpaths satisfy Item 4 in Definition 1. We conclude that $\rho = \rho_0 \times \dots \times \rho_n \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$. Finally, from (13), we get that

$$\iota_U(\rho) = \underbrace{\left(\bigcup_{i=0}^n \underbrace{\mathcal{E}^{W+W}}_{\subset \mathcal{E}^{W+W} \subset W^c} \right)}_{\subset \mathcal{E}^{W+W} \cup \mathcal{E}^{W^*W}} \cup \underbrace{\left\{ \delta_0, \dots, \delta_n \right\}}_{\subset \mathcal{E}^{W^*W} = \mathcal{E}^{W+W} \cup W} \subset \mathcal{E}^{W+W} \cup \mathcal{E}^{W^*W} \subset \mathcal{E}^{W^*W} \quad (35)$$

since $\iota_U(\rho_i) \subset \mathcal{E}^{W+W}$ and $\delta_i \in \mathcal{E}^{W^*W}$ for $i \in \llbracket 0, n \rrbracket$.

• Moreover, in the implication (33d), if a vertex $\delta \in \iota_U(\rho)$ belongs to W , it necessarily appears in a subpath of ρ of the form $\left((v^\flat, \delta), (\delta, v^\sharp), (+1, -1) \right)$. Indeed, using Equation (35), we obtain that a vertex $\delta \in \iota_U(\rho)$ possibly belong to W only when $\delta \in \{\delta_1, \dots, \delta_n\}$ which gives the result.

This ends the proof. \square

Lemma 11 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices. Let $\gamma, \lambda \in \mathcal{V}$ be two vertices and assume that $\gamma \mathcal{A}^W \lambda$ where \mathcal{A}^W is the conditional active relation (9g). Then, there exists an active extended-oriented path ρ , joining the two vertices γ and λ , whose intermediate vertices belong to $\mathcal{E}^{W^+} \cup \mathcal{E}^{W^*W}$, that is,*

$$\begin{aligned} \gamma \mathcal{A}^W \lambda &\implies \exists \rho \in D_U[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E}) \\ &\text{and } \iota_U(\rho) \subset \mathcal{E}^{W^+} \cup \mathcal{E}^{W^*W}. \end{aligned} \quad (36)$$

Moreover, an intermediate vertex v is in W only if it appears in a 2-length subpath of ρ of the form $\left((v_{i-1}, v), (v, v_{i+1}), (+1, -1) \right)$.

Proof. As the relation \mathcal{A}_*^W defined in (25) equals the relation \mathcal{A}^W in (9g) (as proved in Lemma 9), we suppose that we are given two vertices $\gamma, \lambda \in \mathcal{V}$ such that $\gamma \mathcal{A}_*^W \lambda$ and we prove the existence of $\rho \in D_U[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$ that joins γ and λ and such that $\iota(\rho) \subset \mathcal{E}^{W^+} \cup \mathcal{E}^{W^*W}$. By (25), giving \mathcal{A}_*^W , we have that

$$\gamma \left(\Delta \cup \mathcal{B}^W \cup \mathcal{B}^{-W} \cup \mathcal{K}^W \cup \left((\mathcal{B}^W \cup \mathcal{K}^W) \mathcal{C}_*^W (\mathcal{B}^{-W} \cup \mathcal{K}^W) \right) \right) \lambda.$$

We consider the five cases, one by one. More precisely, for each case, we are going to show that there exists $\rho \in D_U[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$ such that $\iota_U(\rho) \subset \mathcal{E}^{w+}\{\gamma, \lambda\} \cup \mathcal{E}^{w*}W$.

The first case is $\gamma\Delta\lambda$, that is, $\gamma = \lambda$. As any extended-oriented path of length 0 or 1 is active by definition (see the comment following Definition 1), we conclude that $((\gamma, \lambda), +1) \in D_U[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$ and, as $\iota_U(((\gamma, \lambda), +1)) = \emptyset$, the second assertion is satisfied.

The second case is $\gamma\mathcal{B}^w\lambda$. We conclude that there exists $\rho \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$ such that $\iota_U(\rho) \subset \mathcal{E}^{w+}\lambda$ thanks to (33a) in Lemma 10.

The third case is $\gamma\mathcal{B}^{-w}\lambda$. We conclude that there exists $\rho \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$ such that $\iota_U(\rho) \subset \mathcal{E}^{w+}\gamma$ thanks to (33b) in Lemma 10.

The fourth case is $\gamma\mathcal{K}^w\lambda$. We conclude that there exists $\rho \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$ such that $\iota_U(\rho) \subset \mathcal{E}^{w+}\{\gamma, \lambda\}$ thanks to (33c) in Lemma 10.

It remains to tackle the fifth case. Suppose that $\gamma\left(\left(\mathcal{B}^w \cup \mathcal{K}^w\right)\mathcal{C}_*^w\left(\mathcal{B}^{-w} \cup \mathcal{K}^w\right)\right)\lambda$. Therefore, there exist δ_1 and δ_2 in \mathcal{V} such that

$$\gamma\left(\mathcal{B}^w \cup \mathcal{K}^w\right)\delta_1 \text{ and } \delta_1\mathcal{C}_*^w\delta_2 \text{ and } \delta_2\left(\mathcal{B}^{-w} \cup \mathcal{K}^w\right)\lambda.$$

We are going to display an extended-oriented path $\rho \in D_U[(\delta, \gamma) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$.

- Considering the left hand side $\gamma\left(\mathcal{B}^w \cup \mathcal{K}^w\right)\delta_1$ and using Lemma 10, we obtain — either by (33a) applied to $\gamma\mathcal{B}^w\delta_1$, or by (33c) applied to $\gamma\mathcal{K}^w\delta_1$ — that there exists

$$\rho_1 = (\varrho_1, o_1) = (\varrho_1, (\dots, +1)) \in D_U[(\gamma, \delta_1) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E}),$$

and we have $\iota_U(\rho_1) \subset \mathcal{E}^{w+}\{\gamma, \delta_1\}$.

- In the same way, considering the right hand side $\delta_2\left(\mathcal{B}^{-w} \cup \mathcal{K}^w\right)\lambda$ and using Lemma 10, we obtain — either by (33b) applied to $\delta_2\mathcal{B}^{-w}\lambda$, or by (33c) applied to $\delta_2\mathcal{K}^w\lambda$ — that there exists

$$\rho_2 = (\varrho_2, o_2) = (\varrho_2, (-1, \dots)) \in D_U[(\delta_2, \gamma) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E}),$$

and we have $\iota_U(\rho_2) \subset \mathcal{E}^{w+}\{\delta_2, \lambda\}$.

- Considering the middle expression $\delta_1\mathcal{C}_*^w\delta_2$ and using Lemma 10, we obtain by (33d) that $\delta_1 \in W^*$, $\delta_2 \in W^*$, and that there exists

$$\rho_{1,2} = (\varrho_{1,2}, o_{1,2}) = (\varrho_{1,2}, (-1, \dots, +1)) \in D_U[(\delta_1, \delta_2) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E}),$$

and we have $\iota_U(\rho_{1,2}) \subset \mathcal{E}^{w*}W$.

We consider the extended-oriented path $\rho = \rho_1 \times \rho_{1,2} \times \rho_2$ obtained by concatenation as in (12b). By construction, we have that $\rho \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}]$. We claim that $\rho \in U_a^W(\mathcal{V}, \mathcal{E})$. Indeed, $\rho_1, \rho_{1,2}, \rho_2 \in U_a^W(\mathcal{V}, \mathcal{E})$ by assumption, so that, by Equation (22) in Lemma 6, it only remains to show that

$$\left(\left((v_1^b, \delta_1), (\delta_1, v_{1,2}^\sharp)\right), (+1, -1)\right) = \Omega^\sharp(\rho_1) \times \Omega^b(\rho_{1,2}) \in U_a^W(\mathcal{V}, \mathcal{E}), \quad (37)$$

where (v_1^b, δ_1) is the last edge of the extended-oriented path ρ_1 and $(\delta_1, v_{1,2}^\sharp)$ is the first edge of the extended-oriented path $\rho_{1,2}$, and that

$$\left(\left((v_{1,2}^b, \delta_2), (\delta_2, v_2^\sharp)\right), (+1, -1)\right) = \Omega^\sharp(\rho_{1,2}) \times \Omega^b(\rho_2) \in U_a^W(\mathcal{V}, \mathcal{E}), \quad (38)$$

where $(v_{1,2}^b, \delta_2)$ is the last edge of the extended-oriented path $\rho_{1,2}$ and (δ_2, v_2^\sharp) is the first edge of the extended-oriented path ρ_2 . As $\delta_1, \delta_2 \in W^*$ and because of the orientation $(-1, +1)$, the two subpaths hereabove satisfy Item 4 in Definition 1. We conclude that $\rho \in D_U[(\gamma, \lambda) \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$. Now, we have that

$$\begin{aligned} \iota_W(\rho) &= \iota_W(\rho_1) \cup \{\delta_1\} \cup \iota_W(\rho_{1,2}) \cup \{\delta_2\} \cup \iota_W(\rho_2) \\ &\subset \mathcal{E}^{W^+} \{\gamma, \delta_1\} \cup \{\delta_1\} \cup \mathcal{E}^{W^*} W \cup \{\delta_2\} \cup \mathcal{E}^{W^+} \{\delta_2, \lambda\} \end{aligned}$$

as $\iota_W(\rho_1) \subset \mathcal{E}^{W^+} \{\gamma, \delta_1\}$, $\iota_W(\rho_{1,2}) \subset \mathcal{E}^{W^*} W$ and $\iota_W(\rho_2) \subset \mathcal{E}^{W^+} \{\delta_2, \lambda\}$

$$\begin{aligned} &= \mathcal{E}^{W^+} \gamma \cup \mathcal{E}^{W^+} \delta_1 \cup \{\delta_1\} \cup \mathcal{E}^{W^*} W \cup \{\delta_2\} \cup \mathcal{E}^{W^+} \delta_2 \cup \mathcal{E}^{W^+} \lambda \\ &= \mathcal{E}^{W^+} \gamma \cup \mathcal{E}^{W^*} \delta_1 \cup \mathcal{E}^{W^*} W \cup \mathcal{E}^{W^*} \delta_2 \cup \mathcal{E}^{W^+} \lambda \\ &\subset \mathcal{E}^{W^+} \gamma \cup \mathcal{E}^{W^*} W \cup \mathcal{E}^{W^+} \lambda && \text{(as } \delta_1 \in W^*, \delta_2 \in W^*) \\ &= \mathcal{E}^{W^+} \{\gamma, \lambda\} \cup \mathcal{E}^{W^*} W . \end{aligned}$$

The last assertion of the Proposition — namely, that an intermediate vertice v is in W only if it appears in a 2-length subpath of ρ of the form $\left((v_{i-1}, v), (v, v_{i+1}), (+1, -1) \right)$ — follows from the last assertion of Lemma 10 and from Equations (37) and (38).

This ends the proof. □

The following Proposition 12 is half of the proof of Theorem 4.

Proposition 12 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices. Let $\gamma, \lambda \in \mathcal{V}$ be two vertices. We have the implication*

$$\gamma \mathcal{A}^W \lambda \implies \neg(\gamma \underset{d}{\parallel} \lambda \mid W) \tag{39}$$

where \mathcal{A}^W is the conditional active relation (9g) and $\underset{d}{\parallel}$ is the conditional directional separation relation (8).

Proof. The proof is an easy consequence of Lemma 11. Let $\gamma, \lambda \in \mathcal{V}$ be two vertices. We have that

$$\begin{aligned} \gamma \mathcal{A}^W \lambda &\implies \exists \rho \in D_U[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E}) && \text{(by Lemma 11)} \\ &\implies \neg\left(D_U[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{E}] \subset U_b^W(\mathcal{V}, \mathcal{E})\right) && \text{(by definition of } U_b^W(\mathcal{V}, \mathcal{E})) \\ &\implies \neg(\gamma \underset{d}{\parallel} \lambda \mid W) . && \text{(by (8) in Definition 2)} \end{aligned}$$

This ends the proof. □

The following Lemma 13 is instrumental in the second half of the proof of Theorem 4.

Lemma 13 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices. For any $n \geq 2$ and $\gamma, \lambda \in \mathcal{V}$, the following statement holds true. For any extended-oriented path $\rho = (\varrho, o) \in U^n(\mathcal{V}, \mathcal{E})$ of length n joining two vertices γ and λ (that is, $\mathfrak{X}_U[\{\rho\} \mid \mathcal{V}, \mathcal{E}] = \{(\gamma, \lambda)\}$ as in (17)), and such that ρ is active (that is, $\rho \in U_a^W(\mathcal{V}, \mathcal{E})$ as in Definition 1), one of the two following properties is fulfilled:*

1. *Either $\gamma \mathcal{A}_{*+}^W \lambda$ and the last orientation of ρ is $o_n = +1$,*
2. *Or $\gamma \mathcal{A}_{*-}^W \lambda$ and the last orientation of ρ is $o_n = -1$.*

Proof. We call \mathcal{H}_n the statement in Lemma 13 and we prove by induction that it is satisfied for all $n \geq 2$.

We prove \mathcal{H}_2 . For this purpose, we consider an extended-oriented path ρ of length 2, joining two vertices γ and λ in the graph $(\mathcal{V}, \mathcal{E})$, and which is active, that is, by definition (17) of the endpoint relation $\mathfrak{X}_U[S \mid \mathcal{V}, \mathcal{E}]$ on \mathcal{V} ,

$$\rho \in U^2(\mathcal{V}, \mathcal{E}) \text{ and } \mathfrak{X}_U[\{\rho\} \mid \mathcal{V}, \mathcal{E}] = \{(\gamma, \lambda)\} \text{ and } \rho \in U_a^W(\mathcal{V}, \mathcal{E}).$$

Using Lemma 5, there are two cases to consider. In the first case, the extended-oriented path ends with orientation $+1$ and is such that either $\gamma \mathcal{E} \Delta_{W^c} \mathcal{E} \lambda$ or $\gamma \mathcal{E}^{-1} \Delta_{W^c} \mathcal{E} \lambda$. Using the properties that $\mathcal{E} \Delta_{W^c} \mathcal{E} \subset \mathcal{B}^W \subset \mathcal{A}_{*+}^W$ (by (9b) and (26a)) and that $\mathcal{E}^{-1} \Delta_{W^c} \mathcal{E} \subset \mathcal{K}^W \subset \mathcal{A}_{*+}^W$ (by (9e) and (26a)), we obtain that the case 2 of \mathcal{H}_2 is satisfied. In the second case, the extended-oriented path path ends with orientation -1 and is such that either $\gamma \mathcal{E} \Delta_{W^*} \mathcal{E}^{-1} \lambda$ or $\gamma \mathcal{E}^{-1} \Delta_{W^*} \mathcal{E}^{-1} \lambda$. Using the properties that $\mathcal{E} \Delta_{W^*} \mathcal{E}^{-1} \subset \mathcal{B}^W \Delta_{W^*} \mathcal{B}^{-W} \subset \mathcal{A}_{*-}^W$ (by (9b) and (26b)) and that $\mathcal{E}^{-1} \Delta_{W^*} \mathcal{E}^{-1} \subset \mathcal{B}^{-W} \subset \mathcal{A}_{*-}^W$ (by (9d) and (26b)), we obtain that the case 2 of \mathcal{H}_2 is satisfied.

We suppose that the induction assumption \mathcal{H}_{n-1} holds true, where $n-1 \geq 2$, and we are going to show that \mathcal{H}_n holds true. For this purpose, we consider an extended-oriented path ρ of length n ($n \geq 2$), joining two vertices γ and λ in the graph $(\mathcal{V}, \mathcal{E})$, and which is active, that is,

$$\rho \in U^n(\mathcal{V}, \mathcal{E}) \text{ and } \mathfrak{X}_U[\{\rho\} \mid \mathcal{V}, \mathcal{E}] = \{(\gamma, \lambda)\} \text{ and } \rho \in U_a^W(\mathcal{V}, \mathcal{E}).$$

We decompose the extended-oriented path ρ as

$$\rho = (\varrho, o) = (((v_i^b, v_i^\sharp))_{i \in \llbracket 1, n \rrbracket}, (o_i)_{i \in \llbracket 1, n \rrbracket}) = \rho_{n-1} \times \bar{\rho},$$

where $\rho_{n-1} = (((v_i^b, v_i^\sharp))_{i \in \llbracket 1, n-1 \rrbracket}, (o_i)_{i \in \llbracket 1, n-1 \rrbracket}) \in U^{n-1}(\mathcal{V}, \mathcal{E})$ is an extended-oriented path of length $n-1$, and where $\bar{\rho} = ((v_n^b, v_n^\sharp), o_n) \in U^1(\mathcal{V}, \mathcal{E})$ is an extended-oriented path of length 1. We have that $v_1^b = \gamma$ and $v_n^\sharp = \lambda$. It is clear that the extended-oriented path ρ_{n-1} is active, that is, $\rho_{n-1} \in U_a^W(\mathcal{V}, \mathcal{E})$. Indeed, otherwise, the extended-oriented path ρ_{n-1} would be in one of the four cases listed in Definition 2, hence so would be the extended-oriented path ρ . But this would contradict the assumption that $\rho \in U_a^W(\mathcal{V}, \mathcal{E})$. As the extended-oriented path ρ_{n-1} is active and of length $n-1$, it satisfies the induction assumption \mathcal{H}_{n-1} . We deduce that either $\gamma \mathcal{A}_{*+}^W v_{n-1}^\sharp$ and the last orientation of ρ_{n-1} is $o_{n-1} = +1$, or $\gamma \mathcal{A}_{*-}^W v_{n-1}^\sharp$ and the last orientation of ρ_{n-1} is $o_{n-1} = -1$. We analyze the two cases separately.

- Assume that we have $\gamma \mathcal{A}_{*+}^W v_{n-1}^\sharp$ and that the last orientation of ρ_{n-1} is $o_{n-1} = +1$, that is, ρ_{n-1} ends with $((v_{n-1}^b, v_{n-1}^\sharp), +1)$. There are two possibilities for the extended-oriented path $\bar{\rho} = ((v_n^b, v_n^\sharp), o_n)$.

- Suppose that $\bar{\rho} = ((v_n^b, v_n^\sharp), +1) = ((v_n^b, \lambda), +1)$, that is, $(v_n^b, \lambda) \in \mathcal{E}$ by (2b). As the path ρ is active by assumption, the pattern $\left(((v_{n-1}^b, v_{n-1}^\sharp), +1), ((v_n^b, v_n^\sharp), +1) \right)$ must satisfy Item 1 in Definition 1. We deduce that $v_{n-1}^\sharp = v_n^b \in W^c$. Now, we wrap up the results obtained so far. On the one hand, from $\gamma \mathcal{A}_{*+}^W v_{n-1}^\sharp$, $v_{n-1}^\sharp = v_n^b \in W^c$ and $(v_n^b, \lambda) \in \mathcal{E}$, we get that $\gamma \mathcal{A}_{*+}^W \Delta_{W^c} \mathcal{E} \lambda$, hence that $\gamma \mathcal{A}_{*+}^W \lambda$ because the relation \mathcal{A}_{*+}^W in (26a) ends with the relation \mathcal{B}^W and as $\mathcal{B}^W \Delta_{W^c} \mathcal{E} = \mathcal{E} (\Delta_{W^c} \mathcal{E})^* (\Delta_{W^c} \mathcal{E}) \subset \mathcal{B}^W$ by (9b). On the other hand, the extended-oriented path ρ ends with $+1$, as it is the case for $\bar{\rho}$. We conclude that the extended-oriented path ρ of length n satisfies the case 1 of \mathcal{H}_n , since it ends with $+1$ and its endpoints are such that $\gamma \mathcal{A}_{*-}^W \lambda$. Therefore, we have proven the case 1 of the induction assumption \mathcal{H}_n for the extended-oriented paths of length n .
- Suppose that $\bar{\rho} = ((v_n^b, v_n^\sharp), -1) = ((v_n^b, \lambda), -1)$, that is, $(v_n^b, \lambda) \in \mathcal{E}^{-1}$ by (2b). As the path ρ is active, the pattern $\left(((v_{n-1}^b, v_{n-1}^\sharp), +1), ((v_n^b, v_n^\sharp), -1) \right)$ must satisfy Item 4 in Definition 1. We deduce that $v_{n-1}^\sharp = v_n^b \in W^*$. Now, we wrap up the results obtained so far. On the one hand, from $\gamma \mathcal{A}_{*+}^W v_{n-1}^\sharp$, $v_{n-1}^\sharp = v_n^b \in W^*$ and $(v_n^b, \lambda) \in \mathcal{E}^{-1}$, we get that $\gamma \mathcal{A}_{*+}^W \Delta_{W^*} \mathcal{E}^{-1} \lambda$, hence that $\gamma \mathcal{A}_{*-}^W \lambda$ by (27c). On the other hand, the extended-oriented path ρ ends with -1 , as it is the case for $\bar{\rho}$. We conclude that the extended-oriented path ρ of length n satisfies the case 2 of \mathcal{H}_n , since it ends with -1 and its endpoints are such that $\gamma \mathcal{A}_{*-}^W \lambda$. Therefore, we have proven the case 2 of the induction assumption \mathcal{H}_n for the extended-oriented paths of length n .
- Assume that we have $\gamma \mathcal{A}_{*-}^W v_{n-1}^\sharp$ and that the last orientation of ρ_{n-1} is $o_{n-1} = -1$, that is, $(v_n^b, \lambda) \in \mathcal{E}^{-1}$. There are two possibilities for the extended-oriented path $\bar{\rho} = ((v_n^b, v_n^\sharp), o_n)$.
 - Suppose that $\bar{\rho} = ((v_n^b, v_n^\sharp), +1) = ((v_n^b, \lambda), +1)$, that is, $(v_n^b, \lambda) \in \mathcal{E}$ by (2b). As the path ρ is active, the pattern $\left(((v_{n-1}^b, v_{n-1}^\sharp), -1), ((v_n^b, v_n^\sharp), +1) \right)$ must satisfy Item 3 in Definition 1. We deduce that $v_{n-1}^\sharp = v_n^b \in W^c$. Now, we wrap up the results obtained so far. On the one hand, from $\gamma \mathcal{A}_{*-}^W v_{n-1}^\sharp$, $v_{n-1}^\sharp = v_n^b \in W^c$ and $(v_n^b, \lambda) \in \mathcal{E}$, we get that $\gamma \mathcal{A}_{*-}^W \Delta_{W^c} \mathcal{E}^{-1} \lambda$, hence that $\gamma \mathcal{A}_{*+}^W \lambda$ by (27c). On the other hand, the extended-oriented path ρ ends with $+1$, as it is the case for $\bar{\rho}$. We conclude that the extended-oriented path ρ of length n satisfies the case 1 of \mathcal{H}_n , since it ends with $+1$ and its endpoints are such that $\gamma \mathcal{A}_{*+}^W \lambda$. Therefore, we have proven the case 1 of the induction assumption \mathcal{H}_n for the extended-oriented paths of length n .
 - Suppose that $\bar{\rho} = ((v_n^b, v_n^\sharp), -1) = ((v_n^b, \lambda), -1)$, that is, $(v_n^b, \lambda) \in \mathcal{E}^{-1}$ by (2b). As the path ρ is active, the pattern $\left(((v_{n-1}^b, v_{n-1}^\sharp), -1), ((v_n^b, v_n^\sharp), -1) \right)$ must satisfy Item 2 in Definition 1. We deduce that $v_{n-1}^\sharp = v_n^b \in W^c$. Now, we wrap up the results obtained so far. On the one hand, from $\gamma \mathcal{A}_{*-}^W v_{n-1}^\sharp$, $v_{n-1}^\sharp = v_n^b \in W^c$ and $(v_n^b, \lambda) \in \mathcal{E}^{-1}$, we get that $\gamma \mathcal{A}_{*-}^W \Delta_{W^c} \mathcal{E}^{-1} \lambda$, hence that $\gamma \mathcal{A}_{*-}^W \lambda$ because the relation \mathcal{A}_{*-}^W ends with the relation \mathcal{B}^{-W} and we have that $\mathcal{B}^{-W} \Delta_{W^c} \mathcal{E}^{-1} = (\mathcal{E}^{-1} \Delta_{W^c})^* \mathcal{E}^{-1} \Delta_{W^c} \mathcal{E}^{-1} \subset \mathcal{B}^{-W}$ which implies that $\mathcal{A}_{*-}^W \Delta_{W^c} \mathcal{E}^{-1} \subset \mathcal{A}_{*-}^W$. On the other hand, the extended-oriented path ρ ends with -1 , as it is the case for $\bar{\rho}$. We conclude that the extended-oriented path ρ of length n satisfies the case 2 of \mathcal{H}_n , since it ends with -1

and its endpoints are such that $\gamma \mathcal{A}_{*-}^W \lambda$. Therefore, we have proven the case 2 of the induction assumption \mathcal{H}_n for the extended-oriented paths of length n .

This ends the proof. \square

C Moral relation

Let $(\mathcal{V}, \mathcal{E})$ be a graph. The *moral relation* \mathcal{M} on $(\mathcal{V}, \mathcal{E})$ is defined by [5]

$$\mathcal{M} = (\mathcal{E} \cup \mathcal{E}\mathcal{E}^{-1}) \cup (\mathcal{E}^{-1} \cup \mathcal{E}\mathcal{E}^{-1}) \subset \mathcal{E} \times \mathcal{E} . \quad (40)$$

In Proposition 14, which is a consequence of Lemma 11, we prove that, when two vertices γ and λ are in relation by the conditional active relation (9g), then there exists an (edge) path in the graph $(\mathcal{V}, \mathcal{M})$ joining the two vertices γ and λ with intermediate vertices in $\mathcal{E}^{W+}(\{\gamma, \lambda\} \cup W)$.

Proposition 14 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and $W \subset \mathcal{V}$ be a subset of vertices. Let \mathcal{A}^W be the conditional active relation (9g) and \mathcal{M} be the relation (40). Then, for all $\gamma, \lambda \in \mathcal{V}$, we have that*

$$\gamma \mathcal{A}^W \lambda \implies \exists \varrho \in D_P[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{M}] \text{ and } \iota(\varrho) \subset \mathcal{E}^{W+}(\{\gamma, \lambda\} \cup W) . \quad (41)$$

Proof. Let $\gamma, \lambda \in \mathcal{V}$. Using Lemma 11, there exists an active extended-oriented path ρ joining the two vertices γ and λ and such that $\iota_U(\rho) \subset \mathcal{E}^{W+}\{\gamma, \lambda\} \cup \mathcal{E}^{W*}W$. We denote by $\{v_i\}_{i=0, \dots, |\rho|}$ the sequence of intermediate vertices of ρ . For $i = 0, \dots, |\rho| - 1$, we have that $v_i \mathcal{M} v_{i+1}$ as, by definition (2c) of an extended-oriented path, we have that $v_i \mathcal{E} v_{i+1}$ or $v_i \mathcal{E}^{-1} v_{i+1}$. In the same way, we have that $\gamma \mathcal{M} v_0$ and $v_{|\rho|} \mathcal{M} \lambda$. Thus, we build an (edge) path $\varrho' = \{(v_i, v_{i+1})\}_{i \in [1, |\rho|-1]}$ which belongs to $D_P[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{M}]$ as defined in Equation (6). However, the (edge) path ϱ' does not satisfy the assumption $\iota(\varrho') \subset \mathcal{E}^{W+}(\{\gamma, \lambda\} \cup W)$ as some intermediate vertex of ϱ' may belong to W . But using the last statement of Lemma 11, we have that, given an intermediate vertex of ϱ' , $v_i \in W$ is only possible when v_i is a “ δ ” vertex, that is, v_i appears in a subpath of ρ of the form $\left((v_{i-1}, v_i), (v_i, v_{i+1}), (+1, -1) \right)$ which implies that $v_{i-1} \mathcal{E} v_i$ and $v_i \mathcal{E}^{-1} v_{i+1}$ hence that $v_{i-1} \mathcal{M} v_{i+1}$ by definition of the moral relation in (40). We define the (edge) path ϱ'' as obtained from the (edge) path ϱ' by removing all the “ δ ” vertices. We readily get that ϱ'' belongs, as does ϱ' , to $D_P[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{M}]$ and has the same internal vertices in W^c as does ϱ' . We conclude that

$$\iota(\varrho'') \subset \iota(\varrho') \cap W^c \subset (\mathcal{E}^{W+}\{\gamma, \lambda\} \cup \mathcal{E}^{W*}W) \cap W^c \subset \mathcal{E}^{W+}\{\gamma, \lambda\} \cup \mathcal{E}^{W+}W = \mathcal{E}^{W+}(\{\gamma, \lambda\} \cup W) .$$

This ends the proof. \square

Let $W, \Gamma, \Lambda \subset \mathcal{V}$ be three subsets of vertices, that are pairwise disjoint. We define the subset $S = \mathcal{E}^*(\Gamma \cup \Lambda \cup W) \subset \mathcal{V}$ of vertices, and the binary relation $\mathcal{E}_S = \mathcal{E} \cap (S \times S)$ on S . The subgraph (S, \mathcal{E}_S) is composed of the set of vertices S and of the set \mathcal{E}_S of edges, given

by the restriction of the relation \mathcal{E} to the subset S . The *moral relation* \mathcal{M}_S on the subgraph (S, \mathcal{E}_S) is the subset of $S \times S$ defined by

$$\mathcal{M}_S = (\mathcal{E}_S \cup \mathcal{E}_S \mathcal{E}_S^{-1}) \cup (\mathcal{E}_S \cup \mathcal{E}_S \mathcal{E}_S^{-1})^{-1} \subset S \times S. \quad (42)$$

Now, under the assumptions of Proposition 14 and the additional assumption that $\gamma \in \Gamma$ and $\lambda \in \Lambda$, we obtain a stronger result.

Corollary 15 *Let $W, \Gamma, \Lambda \subset \mathcal{V}$ be three subsets of vertices, that are pairwise disjoint. Let $\gamma \in \Gamma$ and $\lambda \in \Lambda$ be given. Then, we have that*

$$\gamma \mathcal{A}^W \lambda \implies \exists \varrho \in D_P[\{(\gamma, \lambda)\} \mid S, \mathcal{M}_S] \text{ and } \iota(\varrho) \subset W^c. \quad (43)$$

Proof. The proof is a simple consequence of Proposition 14 as the path obtained in Equation (41) is a path in the subgraph (S, \mathcal{E}_S) and its intermediate vertices satisfy $\iota(\varrho) \subset \mathcal{E}^{W^+}(\{\gamma, \lambda\} \cup W) \subset W^c$. \square

Proposition 16 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and let $W, \Gamma, \Lambda \subset \mathcal{V}$ be three subsets of vertices, that are pairwise disjoint. We have the implication:*

$$\begin{aligned} \exists \varrho \in D_P[\Gamma \times \Lambda \mid (S, \mathcal{M}_S)] \text{ such that } \iota(\varrho) \subset W^c \\ \implies \exists \rho \in D_U[\Gamma \times \Lambda \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E}). \end{aligned} \quad (44)$$

Proof. Let $\varrho \in D_P[\Gamma \times \Lambda \mid (S, \mathcal{M}_S)]$ such that $\iota(\varrho) \subset W^c$. Then, there exists a sequence $\{v_i\}_{i \in [1, |\varrho|+1]}$ of vertices in S such that $\varrho = \{(v_i, v_{i+1})\}_{i \in [1, |\varrho|]}$ with $v_1 \in \Gamma$, $v_{|\varrho|+1} \in \Lambda$ and such that $v_i \mathcal{M}_S v_{i+1}$ and $v_i \in W^c$ for all $i = 2, \dots, |\varrho|$. Using the definition of \mathcal{M}_S , there are two cases: either for all $i = 2, \dots, |\varrho|$ there exists $o_i \in \{-1, +1\}$ such that $v_i \mathcal{E}^{(o_i)} v_{i+1}$ and, in that case, we build the extended-oriented path $\rho_i = ((v_i, v_{i+1}), o_i)$; or there exists $\delta_i \in S$ such that $v_i \mathcal{E}^{(1)} \delta_i \mathcal{E}^{(-1)} v_{i+1}$ and, in that case, we build the extended-oriented path $\rho_i = (((v_i, \delta_i), (\delta_i, v_{i+1})), \{1, -1\})$ of length 2. By concatenation, we obtain the extended-oriented path $\rho = \rho_1 \times \dots \times \rho_{|\varrho|}$.

It remains to show that the extended-oriented path ρ is active. As, by assumption, the vertices $\{v_i\}_{i \in [1, |\varrho|+1]}$ are in W^c , then Definition 1 implies that ρ can only be blocked at the vertices appearing in 2-length extended-oriented subpath of the form $\left(((v, \delta), (\delta, v')), \{1, -1\} \right)$ with $\delta \in S$. If $\delta \in \mathcal{E}^* W$, then the subpath is active. If $\delta \in S$ and $\delta \notin \mathcal{E}^* W$, then it easily follows that $\delta \in \mathcal{E}^{W^+}(\Gamma \cup \Lambda) = \mathcal{E}^{W^+} \Gamma \cup \mathcal{E}^{W^+} \Lambda$ and the subpath is blocked. We denote by n the number of blocked subpaths of length 2 of ρ for which $\delta \in \mathcal{E}^{W^+} \Gamma \cup \mathcal{E}^{W^+} \Lambda$ and we prove that, when $n > 0$, we can build a new ρ which has a strictly smaller number of blocked subpaths. We consider only the case where the vertex at the center of the blocked subpath of length 2 is such that $\delta \in \mathcal{E}^{W^+} \Gamma$ as the case $\delta \in \mathcal{E}^{W^+} \Lambda$ can be treated in a similar way. Denote by ρ' the subpath of ρ starting at δ and ending in Λ . The subpath ρ contains a number of blocked subpaths which is strictly smaller than n . As $\delta \in \mathcal{E}^{W^+} \Gamma$, there exists an active extended-oriented subpath ρ'' starting in Γ and ending at δ . Then, by concatenation, we obtain that $\rho'' \times \rho' \in D_U[\Gamma \times \Lambda \mid \mathcal{V}, \mathcal{E}]$ for which the number of blocked subpaths is strictly smaller than n , since it is the case for ρ' , and since ρ'' does not contain blocked

subpaths and the connection at node δ is active in the concatenated path $\rho'' \times \rho'$. We can iterate the process a finite number of times (at most n) to obtain at the end an active extended-oriented subpath. This ends the proof. \square

We relate in Proposition 17 the conditional active relation \mathcal{A}^W to the moral relation \mathcal{M}_S defined on the subset S in Equation (42).

Proposition 17 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and let $W, \Gamma, \Lambda \subset \mathcal{V}$ be three subsets of vertices, that are pairwise disjoint. We have the equivalence:*

$$\Delta_\Gamma \mathcal{A}^W \Delta_\Lambda \neq \emptyset \iff \Delta_\Gamma \mathcal{M}_S (\Delta_{W^c} \mathcal{M}_S)^* \Delta_\Lambda \neq \emptyset. \quad (45)$$

Proof.

(\implies) Let $\gamma \in \Gamma$ and $\lambda \in \Lambda$ be two given vertices such that $\gamma \mathcal{A}^W \lambda$. Then, using Equation (41), there exists $\varrho \in D_P[\{(\gamma, \lambda)\} \mid \mathcal{V}, \mathcal{M}]$ such that $\iota(\varrho) \subset \mathcal{E}^{W^+}(\{\gamma, \lambda\} \cup W)$. As $\gamma \in \Gamma$ and $\lambda \in \Lambda$, we get that $\iota(\varrho) \subset S \cap W^c$ and, thus, that ϱ is a path in the subgraph (S, \mathcal{M}_S) with intermediate vertices in W^c . Hence, we obtain that $\varrho \in D_P[\{(\gamma, \lambda)\} \mid S, \mathcal{M}_S]$ and $\iota(\varrho) \subset W^c$. Then, we easily get that $\gamma \mathcal{M}_S (\Delta_{W^c} \mathcal{M}_S)^* \lambda$.

(\impliedby) Let $\gamma \in \Gamma$ and $\lambda \in \Lambda$ be two vertices such that $\gamma \mathcal{M}_S (\Delta_{W^c} \mathcal{M}_S)^* \lambda$. Then, there exists $\varrho \in D_P[\{(\gamma, \lambda)\} \mid S, \mathcal{M}_S]$ with $\iota(\varrho) \subset W^c$. Thus we get, by Proposition 16, that $\rho \in D_U[\Gamma \times \Lambda \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E})$ and, as proved in Theorem 4, it gives that $\gamma \mathcal{A}^W \lambda$. \square

Definition 18 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and let $W, \Gamma, \Lambda \subset \mathcal{V}$ be three subsets of vertices, that are pairwise disjoint. The two subsets $\Gamma, \Lambda \subset \mathcal{V}$ of vertices are said to be conditionally morally active (w.r.t the subset W) if there exists an (edge) path in the moral graph (S, \mathcal{M}_S) with endpoints in Γ and Λ and such that all the intermediate vertices are in W^c .*

The two subsets $\Gamma, \Lambda \subset \mathcal{V}$ of vertices are said to be conditionally morally blocked (w.r.t the subset W), denoted by $\Gamma \underset{m}{\parallel} \Lambda \mid W$ when they are not conditionally morally active (w.r.t the subset W), that is,

$$\Gamma \underset{m}{\parallel} \Lambda \mid W \iff \{\varrho \in D_P[\Gamma \times \Lambda \mid S, \mathcal{M}_S] \mid \iota(\varrho) \subset W^c\} = \emptyset. \quad (46)$$

Theorem 19 *Let $(\mathcal{V}, \mathcal{E})$ be a graph, and let $W, \Gamma, \Lambda \subset \mathcal{V}$ be three subsets of vertices, that are pairwise disjoint. We have the equivalence:*

$$\Gamma \underset{m}{\parallel} \Lambda \mid W \iff \Gamma \underset{d}{\parallel} \Lambda \mid W \quad (\forall \Gamma, \Lambda \subset \mathcal{V}). \quad (47)$$

Proof. We have

$$\begin{aligned} \neg(\Gamma \underset{m}{\parallel} \Lambda \mid W) &\iff \exists \varrho \in D_P[\Gamma \times \Lambda \mid (S, \mathcal{M}_S)] \text{ such that } \iota(\varrho) \subset W^c && \text{(by (46))} \\ &\implies \exists \rho \in D_U[\Gamma \times \Lambda \mid \mathcal{V}, \mathcal{E}] \cap U_a^W(\mathcal{V}, \mathcal{E}) && \text{(by (44))} \\ &\iff \neg(\Gamma \underset{d}{\parallel} \Lambda \mid W). && \text{(by Theorem 4)} \end{aligned}$$

and

$$\begin{aligned}
\neg(\Gamma \underset{d}{\parallel} \Lambda \mid W) &\iff \Delta_\Gamma \mathcal{A}^W \Delta_\Lambda \neq \emptyset \\
&\implies \exists \varrho \in D_P[\{(\gamma, \lambda)\} \mid S, \mathcal{M}_S] \text{ such that } \iota(\varrho) \subset W^c && \text{(by (43))} \\
&\iff \neg(\Gamma \underset{m}{\parallel} \Lambda \mid W) . && \text{(by (46))}
\end{aligned}$$

This ends the proof. □

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