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Trajectory under harmonic potential and magnetic force

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Abstract: We study the trajectories of a point particle which is subjected to the simultaneous effect of a restoring force and a force perpendicular to its speed, which can thus be a magnetic force. Solving the equations of motion, we show these trajectories are, when the force is attractive, a part of the centered trochoïde curves, to which we give a physical signification. Several possible trajectories are detailed with geometrical properties and some characteristics of the motion. A second family of trajectories, this time original and corresponding on a repulsive restoring force, is detailed. We discuss the results with references to works previously published, especially about the normal Zeeman effect.

Keywords: Central force; magnetic field; harmonic oscillator;

Introduction

The study of the trajectories of particles which move under the influence of restoring forces is an important topic in Physics, considering the many physical applications in almost all the areas. This is particularly true when these forces are attractive, i.e. when the mechanical system is referred to a harmonic motion, and when the particle is simultaneously submitted to the influence of a magnetic field. Naturally, this kind of motion has already been studied in the history of classical Physics, especially when an atom was modelled as a simple harmonic oscillator, in order to explain the Zeeman effect ([1], [2]).

This paper presents results which seems represent a significant progress in this topic. Indeed, studying in our turn this problem, we have obtained others and original solutions, which contain the solutions already known as limiting cases. In particular, we show that the trajectories of a charged bi-dimensional harmonic oscillator, under a constant and perpendicular magnetic field, is represented by a part of the centered trochoïde curves. It is well known that this important family of curves, which have been studied since several centuries by generations of mathematicians, are classically build from geometrical considerations, using the motion of two circles. Paper provides thus, that seems for the first time, physical significations to these trajectories. Complete mathematical demonstrations of this result are detailed, as several fundamental characteristics of the motion, such angular momentum and energy. Moreover, limiting cases are presented, naturally ellipse of Lissajous-Bowdith (if magnetic field is null), but also circular trajectories (if restoring fore is null) and rose trajectories (if initial harmonic oscillator is mono dimensional).

In its second part the paper presents another contribution to the topic. Indeed, we study the case where the restoring force is repulsive. Using the same approach, we show that trajectories are described by a

family of spiral curves, which seems, this time, entirely original, except a limiting case, corresponding on the rotating rod spiral. Paper presents some of these curves, but doesn't provide a complete geometrical study, which should be the object of later mathematical works.

In the last part of the paper, we discuss the results, especially a comparison with the model historically suggested by Lorentz. We should remind the reader that we have only studied cases where the magnetic field is perpendicular to the plane where particle moves. Moreover, we have not taken into consideration relativistic or quantum effects.

1. Attractive restoring force : Dynamics

Note : In all the paper the mass of the point particle is considered equal to 1, such forces and accelerations are confused.

We consider thus the motion of a charged point particle which is simultaneously submitted to an attractive restoring force and a magnetic force due to a constant magnetic field B perpendicular to the plane on which a charged particle moves. The force at distance we propose to study can be written

$$\vec{F} = -k\vec{r} + q\vec{V} * \vec{B} \quad (1)$$

Where

k is the spring constant

q the charge of the particle

V its speed.

A limiting case is obtained when the magnetic field is null: this case is corresponding on the harmonic oscillator and the point particle describes an ellipse of Lissajous – Bowditch.

1.1. Solving

We begin introducing the cartesian system of coordinate $(O; \vec{X}; \vec{Y}; \vec{Z})$ where O is the center of the restoring force. We can rewrite (1) as

$$\vec{F} = -k(X\vec{x} + Y\vec{y}) + qB(\dot{X}\vec{y} - \dot{Y}\vec{x})$$

For convenience for the calculation we introduce the pulsation of the initial harmonic oscillator and the Larmor frequency (per mass unity) [3]

$$\left\{ \begin{array}{l} w_0 = \sqrt{k} \\ \alpha = \frac{qB}{2} \end{array} \right\}$$

System becomes using the Newton law of motion

$$\left\{ \begin{array}{l} \ddot{X} = -w_0^2 X - 2\alpha\dot{Y} \\ \ddot{Y} = -w_0^2 Y + 2\alpha\dot{X} \end{array} \right\}$$

A classical method to solve this kind of system is to introduce the complex number (comparable solving can be found for example in references [1], [5])

$$u = X + jY$$

We obtain the differential equation

$$\frac{d^2u}{dt^2} + w_0^2u - 2j\alpha \frac{du}{dt} = 0$$

To solve it we consider a limiting case whose we know an exact solution, i.e. the ellipse of Lissajous-Bowditch we have evoked, corresponding on $\alpha = 0$. We name a and b respectively the semi major and minor axis of this conic. By choosing initial conditions such

$$\begin{cases} t = 0 \\ X = a \\ Y = 0 \end{cases}$$

The solution is given by the relations

$$\begin{cases} X = a \cos wt \\ Y = \pm b \sin wt \end{cases}$$

(Dependent on the direction of rotation). In this case the complex number u is

$$u = a \cos wt \pm jb \sin wt$$

Or,

$$\begin{cases} u = \frac{a+b}{2} e^{j\omega t} + \frac{a-b}{2} e^{-j\omega t} \\ u = \frac{a-b}{2} e^{j\omega t} + \frac{a+b}{2} e^{-j\omega t} \end{cases}$$

We add now a magnetic fields perpendicular to the plane of this ellipse. We choose to investigate solutions of the form

$$u = a_0 e^{j(w+\alpha)t}$$

Introducing it in the differential equation we obtain

$$-(w + \alpha)^2 + w_0^2 + 2\alpha(w + \alpha) = 0$$

Consequently

$$w_0^2 = w^2 - \alpha^2$$

Rewriting now our equation

$$\frac{d^2u}{dt^2} + (w^2 - \alpha^2)u - 2i\alpha \frac{du}{dt} = 0$$

We notice that a second solution exists

$$u = b_0 e^{j(-w+\alpha)t}$$

Where b_0 is our second constant length. Whereas these results we investigate thus a general solution such

$$u = a_0 e^{j(w+\alpha)t} + b_0 e^{j(-w+\alpha)t}$$

To determine the constants we study the initial conditions : considering the initial ellipse, i.e. for $\alpha = 0$, and considering the origin of the time we obtain

for $wt = 0$

$$a = a_0 + b_0$$

And for $wt = \frac{\pi}{2}$, we obtain two possibilities depending on the direction of rotation

$$\begin{cases} b = a_0 - b_0 \\ -b = a_0 - b_0 \end{cases}$$

Where a and b are successively the semi major and minor axis of the conic. We deduce that

$$\begin{cases} a_0 = \frac{a+b}{2} \\ b_0 = \frac{a-b}{2} \end{cases}$$

Or

$$\begin{cases} a_0 = \frac{a-b}{2} \\ b_0 = \frac{a+b}{2} \end{cases}$$

We obtain thus two general solutions

$$\begin{cases} u = \frac{a+b}{2} e^{j(+w+\alpha)t} + \frac{a-b}{2} e^{j(-w+\alpha)t} \\ u = \frac{a-b}{2} e^{j(+w+\alpha)t} + \frac{a+b}{2} e^{j(-w+\alpha)t} \end{cases}$$

We recognize the expressions of the centered trochoïde curves. as presented in reference [4] under the mathematical form

$$u = r_1 e^{jw_1 t} + r_2 e^{jw_2 t}$$

And, by identifying the real and imaginary parts

$$\left\{ \begin{array}{l} X = \frac{a+b}{2} \cos(w+\alpha)t + \frac{a-b}{2} \cos(w-\alpha)t \\ Y = \frac{a+b}{2} \sin(w+\alpha)t - \frac{a-b}{2} \sin(w-\alpha)t \end{array} \right\} \quad (2.1)$$

Or

$$\left\{ \begin{array}{l} X = \frac{a-b}{2} \cos(w+\alpha)t + \frac{a+b}{2} \cos(w-\alpha)t \\ Y = \frac{a-b}{2} \sin(w+\alpha)t - \frac{a+b}{2} \sin(w-\alpha)t \end{array} \right\} \quad (2.2)$$

Under another form

$$\left\{ \begin{array}{l} X = a \cos wt \cos at - b \sin wt \sin at \\ Y = a \cos wt \sin at + b \sin wt \cos at \end{array} \right\}$$

And

$$\left\{ \begin{array}{l} X = a \cos wt \cos at + b \sin wt \sin at \\ Y = a \cos wt \sin at - b \sin wt \cos at \end{array} \right\}$$

i.e. two possible parametrized curves dependent on the initial state of the harmonic oscillator . Note these trajectories are solutions of the problem if

$$w_0^2 = k = w^2 - \alpha^2 > 0$$

Or, introducing the ratio of pulsations

$$n = \frac{\alpha}{w}$$

If

$$-1 < n < 1$$

1.2. Verification

At this step we verify the solutions by direct checking: considering for example the first of them and the successive derivatives with respect to the time

$$\begin{cases} \dot{X} = -a(w \sin wt \cos at + \alpha \cos wt \sin at) - b(w \cos wt \sin at + \alpha \sin wt \cos at) \\ \dot{Y} = -a(w \sin wt \sin at - \alpha \cos wt \cos at) + b(w \cos wt \cos at - \alpha \sin wt \sin at) \end{cases}$$

And

$$\begin{cases} \ddot{X} = a(-w^2 \cos wt \cos at + 2w\alpha \sin wt \sin at - \alpha^2 \cos wt \cos at) \\ \quad + b(w^2 \sin wt \sin at - 2w\alpha \cos wt \cos at + \alpha^2 \sin wt \sin at) \\ \ddot{Y} = -a(w^2 \cos wt \sin at + 2w\alpha \sin wt \cos at + \alpha^2 \cos wt \sin at) \\ \quad - b(w^2 \sin wt \cos at + 2b\alpha \cos wt \sin at + \alpha^2 \sin wt \cos at) \end{cases}$$

A check by direct proof leads well to

$$\begin{cases} \ddot{X} = -(w^2 - \alpha^2)X - 2\alpha\dot{Y} \\ \ddot{Y} = -(w^2 - \alpha^2)Y + 2\alpha\dot{X} \end{cases}$$

Where

$$k = w^2 - \alpha^2$$

2. Presentation of the curves

2.1. Definition

Curves are thus definite with

$$\begin{cases} u_1 = \frac{a+b}{2} e^{j(w+\alpha)t} + \frac{a-b}{2} e^{j(-w+\alpha)t} \\ u_2 = \frac{a-b}{2} e^{j(w+\alpha)t} + \frac{a+b}{2} e^{j(-w+\alpha)t} \end{cases}$$

Rewriting them

$$\begin{cases} u_1 = r_1 e^{jw_1 t} + r_2 e^{jw_2 t} \\ u_2 = r_2 e^{jw_1 t} + r_1 e^{jw_2 t} \end{cases}$$

Geometrical properties of these centered trochoïde curves has already been studied by generations of Mathematicians and Physicians. It appears the construction of the curves is classically done using geometrical considerations : it is a roulette traced by a point attached to a circle rolling around a second fixed circle. We have provided a physical signification, valuable for $-1 < n < 1$, consequently when

$$w_2 < 0$$

This indicates that trajectories are corresponding on the hypotrochoïde curves [6]. The second part of the family (epitrochoïdes) are not corresponding on a real solution (w_0 should be imaginary).

2.1. Several possible trajectories

To do it we, choose the solution (2.1). Using the ratio n they are given by

$$\begin{cases} X = a \cos wt \cos nwt - b \sin wt \sin nwt \\ Y = a \cos wt \sin nwt + b \sin wt \cos nwt \end{cases}$$

These equations are corresponding on a precession of the initial ellipse of Lissajous-Bowditch with an angle of precession equal to α . This precession is visible when n is small (see for example Figure 1). Hypotrochoïdes curves can thus be obtained making a precession of Larmor on an ellipse of Hooke. It is illustrated in Figure 1

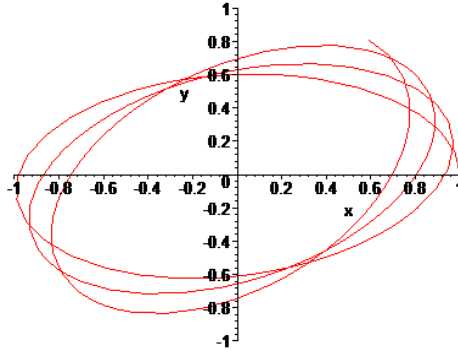
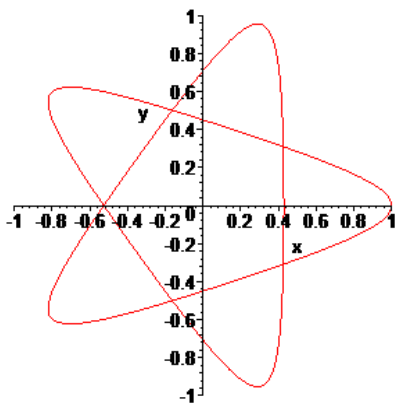


Figure 1 : $n = 0.05$

For a revolution angle of precession is simply given by

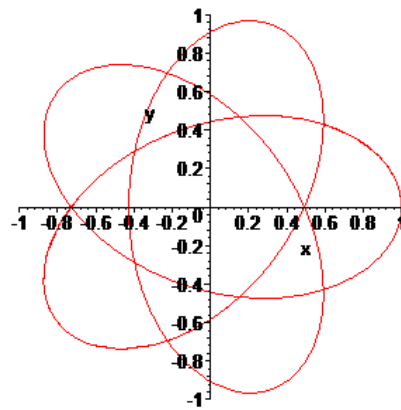
$$\hat{p} = 2\pi n$$

Trajectories are closed if n is a rational number. When this ratio varies we obtain a large variety of curves. To illustrate the paper we build certain of them considering the semi major axis a of the initial ellipse equal to 1. See for example the following hypotrochoïdes curves, with corresponding value of initial semi minor axis and ratio n



$$b = 3/7$$

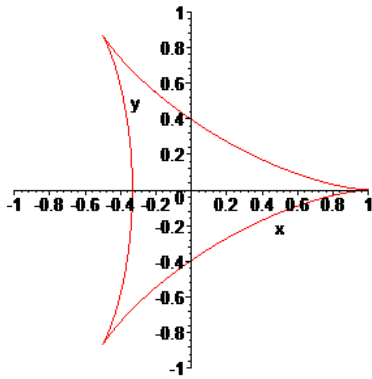
$$n = -1/5$$



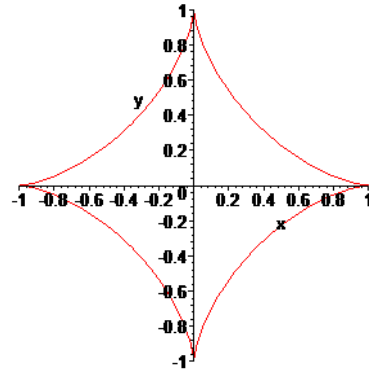
$$n = 1/5$$

The famous hypocycloïde, which are definite with $r_1 w_1 = r_2 |w_2|$, can also be drawn, for example

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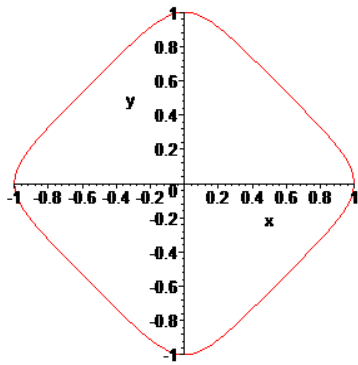


$b = 1/3$ $n = -1/3$ Deltoïde

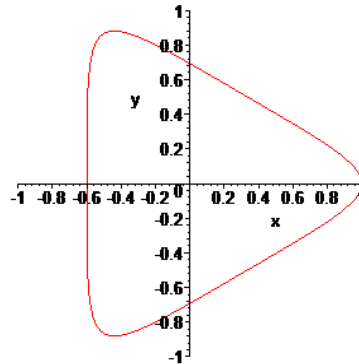


$b = 0.5$ $n = -0.5$ Astroïde

And « flat » Trochoïdes, definite with $r_1 w_1^2 = r_2 w_2^2$

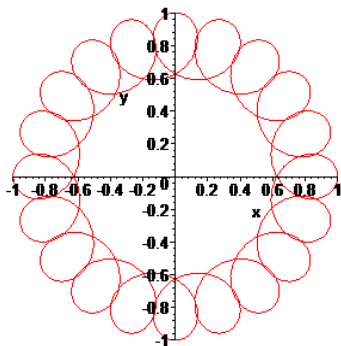


$b = 0.8$ $n = -0.5$

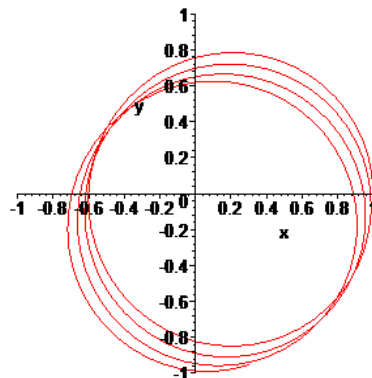


$b = 3/5$ $n = -1/3$

Corresponding now the cases where the ratio n is proximate to 1.



$b = 0.8$ $n = -0.9$



$n = 0.9$

Trajectories becomes gradually circular, as we detail it later.

2.3. Energy and angular momentum

For the reason that magnetic force doesn't work Mechanical Energy E (sum of the kinetic energy T and the harmonic potential V) is conserved on all the trajectories

$$E = T + V = \frac{1}{2}(\dot{X}^2 + \dot{Y}^2 + k(X^2 + Y^2)) = \frac{1}{2}w(wb^2 + 2ab\alpha + a^2w)$$

Angular momentum is given at O by the relation

$$\vec{p} = \vec{r} * \vec{V}$$

And can be written in our system of coordinate

$$\vec{p} = (X\vec{x} + Y\vec{y}) * (\dot{X}\vec{x} + \dot{Y}\vec{y}) = (X\dot{Y} - Y\dot{X})\vec{z}$$

Using previously relations we obtain after simplifications

$$\vec{p} = (abw + \alpha a^2[1 - e^2 \sin^2 wt])\vec{z}$$

2.4. Radius of curvature

Radius of curvature is naturally an important property of the curves. This one, for plane parametrized curves depending on the time is given by

$$R(t) = \frac{[\dot{x}^2 + \dot{y}^2]^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

Where the time derivatives are written by

$$\left\{ \begin{array}{l} \dot{x} = \frac{dx}{dt} \\ \ddot{x} = \frac{d^2x}{dt^2} \end{array} \right.$$

After calculations we have obtained

$$R(t) = \frac{A^{3/2}}{abw^3 + \alpha[A + w(b^2w + ab\alpha + wa^2)]}$$

where

$$A = (aw + b\alpha)^2 - (w^2 - \alpha^2)(a^2 - b^2)\cos^2 wt$$

It extends along an axis (IM) where M is a point of the curve

$$M(X, Y, 0)$$

And I is point located on the X axis such

$$I(I_x, 0, 0)$$

To obtain it we write the condition

$$\overrightarrow{IM} \cdot \vec{V} = 0$$

Where \vec{V} is the speed (naturally tangential to the curve)

$$\vec{V}(\dot{X}; \dot{Y}; 0)$$

Thus

$$(I_x - X). \dot{X} - Y. \dot{Y} = 0$$

And point I is defined by

$$I_x = \frac{w(a^2 - b^2)(\cos wt \sin wt)}{(aw + b\alpha)(\sin wt \cos at) + (a\alpha + bw)(\cos wt \sin at)}$$

Two interesting limiting cases can be noted : indeed, when $n = \pm 1$, radius of curvature R are constant. It is also the case of the speed V (tab. 1)

$n = 1$	$R = \frac{1}{2}(a + b)$	$I_x = \frac{1}{2}(a - b)$	$V = \alpha(a + b)$
$n = -1$	$R = \frac{1}{2}(a - b)$	$I_x = \frac{1}{2}(a + b)$	$V = \alpha(a - b)$

Tab 1 : particular cases : $\alpha = \pm w$

We obtain thus two circles

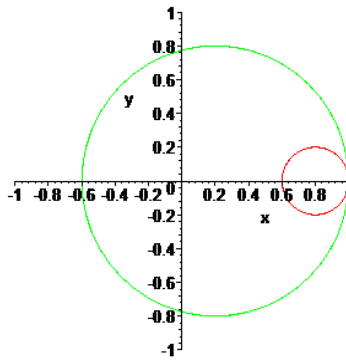


Figure 5 : $n = \pm 1$

In fact it corresponds to a well-known situation. Indeed, in these cases the coefficient k given by

$$k = (w^2 - \alpha^2)$$

Becomes equal to zero : This indicates that the harmonic force is null and, consequently, point particle is only submitted to the constant magnetic field. Trajectories are thus circular and we re-obtain for each cases the classical relation

$$V = qBR$$

Moreover these two circles are the circles used to draw the curves.

2.5. Polar equation

It is often difficult to write a system of parametric equations in a polar system of coordinate $(O, \vec{e}_r, \vec{e}_\theta)$. In our case we managed to do it only for certain limiting cases, writing the classical equations

$$\begin{cases} X = R \cos \theta \\ Y = R \sin \theta \end{cases}$$

Where R is given by

$$R = \sqrt{X^2 + Y^2} = \sqrt{a^2 \cos^2 wt + b^2 \sin^2 wt}$$

Indeed, we can obtain the angular speed using the relation

$$\dot{\theta} = -\frac{\left(\frac{\dot{X}}{R}\right)}{\frac{Y}{R}}$$

Which leads to

$$\dot{\theta} = \frac{(a^2 \cos^2 wt + b^2 \sin^2 wt)\alpha + wab}{a^2 \cos^2 wt + b^2 \sin^2 wt}$$

We have noted note that the angular speed is constant and given by $\dot{\theta} = \alpha$ if $w = 0$, $a = 0$ or $b = 0$

Which allows to obtain in these cases the polar angle

$$\theta = \alpha t$$

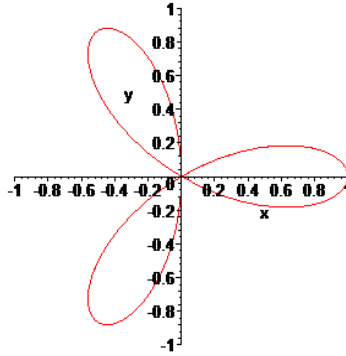
Considering for example the case where $b = 0$. The curves becomes

$$\begin{cases} X = a \cos wt \cos nwt \\ Y = a \cos wt \sin nwt \end{cases}$$

And can be written in the polar system of coordinate

$$\begin{cases} X = a \cos \frac{\theta}{n} \cos \theta \\ Y = a \cos \frac{\theta}{n} \sin \theta \end{cases}$$

Which is corresponding on a well known family of curves (limiting cases of the centered trochoïde), called in the history of mathematics as « Rose Curves » since they have been studied by the mathematician Guido Grandi ([7], [8]). To illustrate the paper we draw the case $n = 1/3$ (« regular trifolium »)



$$n = 1/3$$

This indicates that when the initial oscillator is moved in a straight (i.e. when it is a one-dimensional oscillator) the point particle (after addition of the magnetic field) describes one of these curves, where the angular speed is constant and simply given by $\dot{\theta} = \alpha$. The speed of the point particle along these lines are thus

$$V = \frac{a\alpha}{n} \sqrt{1 + (n^2 - 1)\cos^2 \frac{\theta}{n}}$$

We list the extrema in the following table:

$\theta = 2kn\pi$	$V_{min} = a\alpha$
$\theta = n\pi \left(\frac{2k+1}{2}\right)$	$V_{max} = \frac{a\alpha}{n}$

Tab 2. Speed on the rose curves

Speed is well at the most when the point particle reaches the origin of the restoring force.

3. Repulsive restoring force

We consider in this second part the case where the restoring force is repulsive. In this case the differential equation we have to solve is

$$\vec{F} = k\vec{r} + q\vec{V} * \vec{B}$$

As previously we rewrite the force

$$\vec{F} = k(X\vec{x} + Y\vec{y}) + 2\alpha(\dot{X}\vec{y} - \dot{Y}\vec{x})$$

Finally we obtain the system of differential equations

$$\begin{cases} \ddot{X} = kX - 2\alpha\dot{Y} \\ \ddot{Y} = kY + 2\alpha\dot{X} \end{cases}$$

Introducing again the complex number

$$u = X + jY$$

We obtain this time the differential equation

$$\frac{d^2u}{dt^2} - ku - 2j\alpha \frac{du}{dt} = 0$$

We investigate a form such

$$u = a_0 e^{(w+j\alpha)t}$$

Introducing it in the differential equation we obtain

$$(w + j\alpha)^2 - k - 2j\alpha(w + j\alpha) = 0$$

So

$$w^2 - \alpha^2 + 2jw\alpha - k - 2jw\alpha + 2\alpha^2 = 0$$

Consequently relation is this time

$$k = w^2 + \alpha^2$$

Rewriting our initial equation

$$\frac{d^2u}{dt^2} - (w^2 + \alpha^2)u - 2j\alpha \frac{du}{dt} = 0$$

We note that a second solution is possible

$$u = b_0 e^{(-w+j\alpha)t}$$

Indeed we obtain well

$$w^2 - \alpha^2 - 2jw\alpha - k + 2jw\alpha + 2\alpha^2 = 0$$

Consequently we suggest a general solution on the form

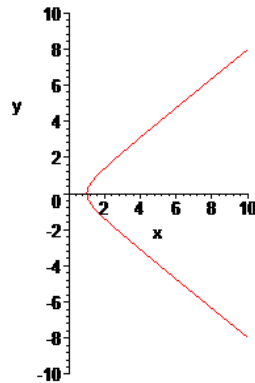
$$u = \left(\frac{a + jb}{2}\right)e^{(w+j\alpha)t} + \left(\frac{a - jb}{2}\right)e^{(-w+j\alpha)t}$$

Such the initial condition (if $\alpha = 0$) is verified. This general solution, identifying real and imaginary parts, leads to the relations

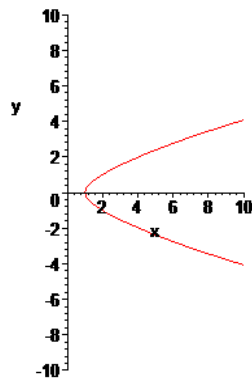
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$$\begin{cases} X = a \cosh wt \cos at - b \sinh wt \sin at \\ Y = a \cosh wt \sin at + b \sinh wt \cos at \end{cases}$$

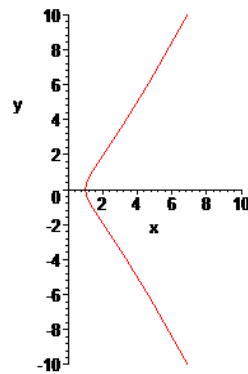
This family of curves seems this time original. We present the simulation, for values of α variant de $\alpha \in [0,2]$, and for $a = 1, b = 0.8, k = 1$. Considering an initial trajectory of the point particle



We add a magnetic field perpendicular to its osculating plane. When magnitude is weak we note an inflexion of the trajectory, depending on the direction of the magnetic field

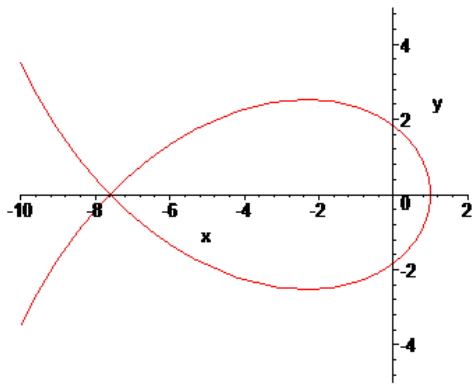


$n = -0.1$

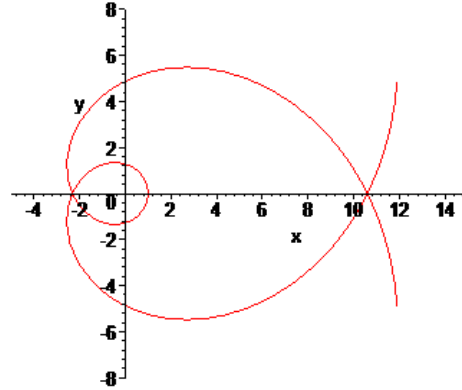


$n = +0.1$

When magnitude of magnetic field increases the spiral form of the curves appears



$n = 1$



$n = 2$

3.1 – Polar equation and limiting cases : rotating rod spirals

Using the same method than in 2.5., with

$$R = \sqrt{X^2 + Y^2} = \sqrt{a^2 \cosh^2 wt - b^2(1 - \cosh^2 wt)}$$

We obtain this time

$$\dot{\theta} = \frac{(a^2 \cosh^2 wt - b^2(1 - \cosh^2 wt))\alpha + wab}{a^2 \cosh^2 wt - b^2(1 - \cosh^2 wt)}$$

We note that $\dot{\theta}$ is constant and equal to α for the three cases : if $w = 0$, $a = 0$ or $b = 0$. This last case leads to

$$\begin{cases} X = a \cosh wt \cos at \\ Y = a \cosh wt \sin at \end{cases} \quad \text{or} \quad \begin{cases} X = -b \sinh wt \sin at \\ Y = b \sin wt \cos at \end{cases}$$

We can thus rewrite these equations

$$\begin{cases} X = a \cosh wt \cos \theta \\ Y = a \cosh wt \sin \theta \end{cases} \quad \text{or} \quad \begin{cases} X = b \sinh wt \cos \theta \\ Y = b \sinh wt \sin \theta \end{cases}$$

And introducing the ratio

$$n = \frac{w}{\alpha} = \frac{\sqrt{k - \alpha^2}}{\alpha}$$

We obtain

$$\begin{cases} X = a \cosh n\theta \cos \theta \\ Y = a \cosh n\theta \sin \theta \end{cases} \quad \text{or} \quad \begin{cases} X = b \sinh n\theta \cos \theta \\ Y = b \sinh n\theta \sin \theta \end{cases}$$

Consequently it exists two possible trajectories which can be written in the polar system of coordinate

$$\begin{cases} r_1 = a \cosh n\theta \\ r_2 = b \sinh n\theta \end{cases}$$

It appears these curves are already known under the name of « rotating rod spirals». Classically they describes the « trajectory of a massive ring sliding without friction on a horizontal rod in rotation around a center » [9]. They constitute thus a limiting case of the solutions.

4. Discussion

4.1. Central force is attractive

The solutions of the problem, as centered trochoïde curves, seem haven't been noticed until today. Indeed, the solutions historically due to Lorentz, to explain the Normal Zeeman effect, are different, as we present it. Demonstrations of solving equation (1) can be found in reference [1].

It appears these solutions have been obtained considering that the angle between the direction of the constant magnetic field and the plane of the initial trajectory is random, which is not the case of our work. But fundamental difference is they have been obtained considering approximations, which isn't the case here, for example assuming that $\alpha \ll \omega_0$ [1] or that the point particle describes initially a circular orbit around the center of harmonic force [5].

The solutions presented in the paper are different because we have chosen different conditions:

- Initial trajectory of the point particle (before the addition of the magnetic field) is a conic and not necessary a circle.
- Angle between the magnetic field and the osculating plane of the trajectory is right.
- We didn't do approximation on the ratio α/ω_0

Consequently, the family of curve is different but can be compared when initial trajectory is circular, i.e. when the two lengths a and b are equals, and when the orientation of the magnetic field is perpendicular. Indeed, our two possible trajectories are in this case given by

$$\begin{cases} X = a \cos(\omega + \alpha)t \\ Y = a \sin(\omega + \alpha)t \end{cases}$$

and

$$\begin{cases} X = a \cos(\omega - \alpha)t \\ Y = a \sin(\omega - \alpha)t \end{cases}$$

Can be compared with the solutions presented in reference [5], for these conditions, which are respectively

$$\begin{cases} X = a \cos(\omega_0 + \alpha)t \\ Y = a \sin(\omega_0 + \alpha)t \end{cases}$$

And

$$\begin{cases} X = a \cos(w_0 - \alpha)t \\ Y = a \sin(w_0 - \alpha)t \end{cases}$$

We see that the two expressions are in correct agreement as much the magnetic field is weak. Indeed classical explanation of the effect provides that “The circular motion of the oscillating electron in the xy-plane at angular frequencies $\omega_0 + \Omega_L$ and $\omega_0 - \Omega_L$ produces radiation at these frequencies » [1] (Ω_L is the frequency of Larmor). We note we obtain well the same gap between the two frequencies. However we also note a difference about the central position, which isn't in our solution w_0 but w . This difference is thus

$$\Delta = w_0 - \sqrt{w_0^2 + \alpha^2}$$

Naturally this difference is small in the case of the Normal Zeeman effect, because the magnetic fields is weak. But, when it increases (case of the Paschen-Back effect) the difference could also increase, which constitutes a prediction for the model.

4.2. Central force is repulsive

The trajectories we have obtained seems this time entirely original (except the limiting case) and can be considered as a contribution to classical electrodynamics, but not also. Indeed, it appears the study of this kind of spirals trajectories present a modern interest in other fields of the modern Physics, for example to study the motion of spacecraft. For this reason, furthers works should be devoted to a mathematical study of them. Moreover, we attract the attention of the reader on the fact that the solution we have obtained shouldn't be confused with the Poincot-Cotes spirals [9].

4. Conclusion

We have presented two families of plane parametrized curves which are obtained by performing a precession around the center of an initial ellipse or hyperbola. We have showed that a part of these curves, known as hypotrochoïde curves, can describe the trajectory of a point-particle simultaneously submitted to a harmonic force and a force perpendicular to its speed. Besides their mathematical and physical interest, these results could have applications in some areas such, for example, the classical electrodynamic, or others. We suggest a prediction about the normal Zeeman and the Paschen-Back effects. Research perspectives could be to verify this prediction and to extend the study, for example to a three-dimensional harmonic oscillator. Another point should to consider the consequences of quantum effects.

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