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# Trajectory under harmonic potential and magnetic force

Eric Guiot

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**Abstract:** We study a family of plane curves which accepts as a limiting case a Lissajous – Bowditch curve. Several closed curves are detailed and we give some geometrical properties such the radius of curvature. Moreover, we show that this family describes the trajectories of a point particle which is subjected to the simultaneous effect of a harmonic force and a force perpendicular to its speed, which can thus be a magnetic force. We present some characteristics of the motion, such angular momentum and energy, and we discuss our results with references to works previously published, especially about the normal effect of Zeeman.

*Keywords:* Central force; magnetic field; harmonic oscillator;

## 1. Introduction

We study in the paper a family of plane parametrized curves given by

$$\begin{cases} X = a \cos wt \cos at - b \sin wt \sin at \\ Y = a \cos wt \sin at + b \sin wt \cos at \end{cases} \quad (1)$$

Where : cartesian coordinate system is

$$(O; \vec{X}; \vec{Y}; \vec{Z})$$

$a$  and  $b$  are two constant lengths

$w$  and  $\alpha$  are two constant pulsations

$t$  is the time.

One of these lines is thus a part of the Lissajous – Bowditch curve, in the limiting case where

$$\alpha = 0$$

And corresponding curve is the ellipse, with

$$b = a\sqrt{1 - e^2}$$

$a$  is its semi major axis,  $e$  its eccentricity and  $O$  its center.

The paper begins to present several curves which are drawn. We detail some geometrical properties, such radius of curvature. But the principal interest of the paper seems to be the study of the dynamics of curves: indeed, we show that a part of these trajectories describe the motion of a point particle, which is subjected to the simultaneous effect of two classical forces: a harmonic force to which is added a force perpendicular to its speed. We have considered the case where this force is due to a constant magnetic field  $B$  perpendicular to the plane on which a charged particle moves. The force at distance we have obtained can thus be written

$$\vec{F} = -k\vec{r} + q\vec{V} * \vec{B} \quad (2)$$

Where

$k$  is the spring constant

$q$  the charge of the particle  
 $V$  its speed.

Naturally this kind of motion has already been studied in the history of the classical Physics, in particular when an atom was modelled as a simple harmonic oscillator [1]. Several solving of the equation (2) can thus be found (see for example [2], [3]) but our family of curves seems not have been noticed except when  $a = b$  (initial trajectory in this case is a circle).

## 2. A brief presentation of the curves

### 2.1 Curves

To simplify their presentation we introduce the ratio of pulsations

$$n = \frac{\alpha}{\omega}$$

Consequently (1) becomes

$$\begin{cases} X = a \cos \omega t \cos n\omega t - b \sin \omega t \sin n\omega t \\ Y = a \cos \omega t \sin n\omega t + b \sin \omega t \cos n\omega t \end{cases}$$

We present quickly some of them. First of all, the precession is visible when  $n$  is small (see for example Figure 1). (Note the curves corresponding on figures 1, 2, 3, 4) are drawn for values  $a = 1$ ,  $e = 0.8$  in our system of coordinates) ( $e$  is the eccentricity of the conic).

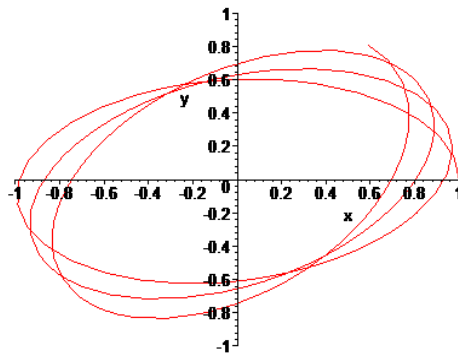
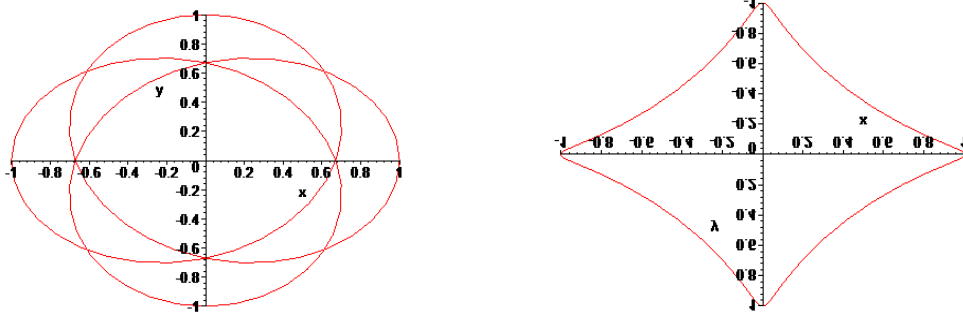
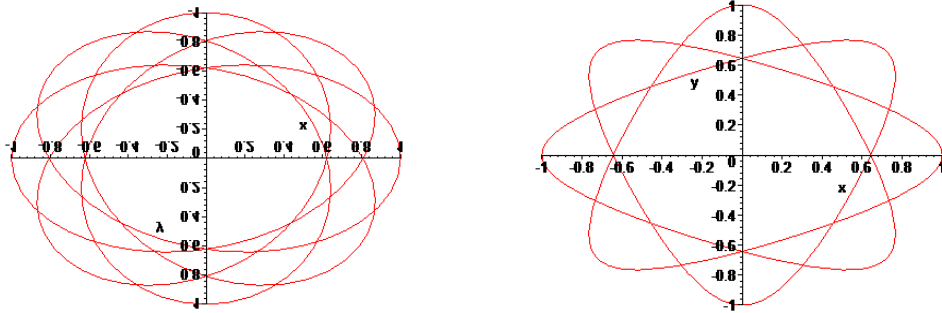
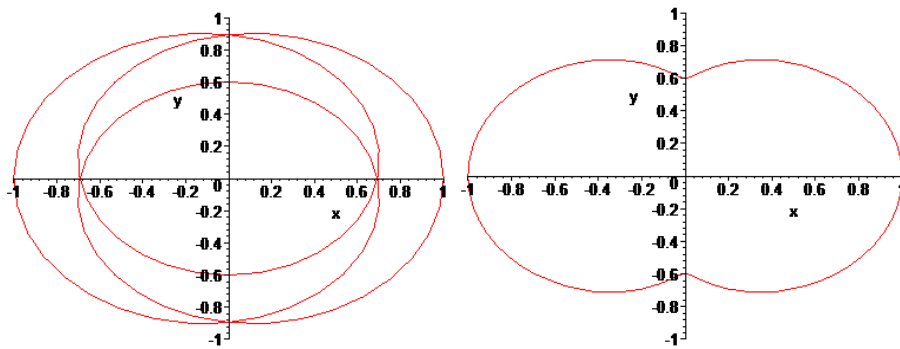


Figure 1 :  $n = 0.05$

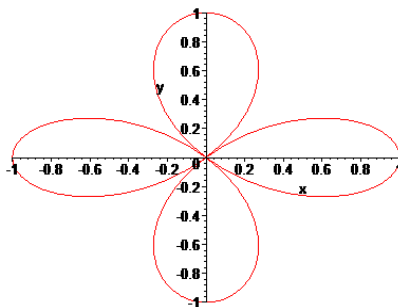
For a revolution angle of precession is simply given by

$$\hat{p} = 2\pi n$$

Such trajectories are closed if  $n$  is a rational number. When this ratio varies we obtain a large variety of curves, See for example

Figure 2 :  $n = \pm 0.5$ Figure 3 :  $n = \pm 0.25$ Figure 4 :  $n = \pm 2$ 

But the curves are also depending on the value of the eccentricity. If  $e = 0$  trajectory is naturally always circular. An interesting case is related to  $e = 1$  : if  $\alpha = 0$  trajectory is a straight but for others values we obtain a remarkable under family of curves : see for example for

Figure 5 :  $n = 0.5$

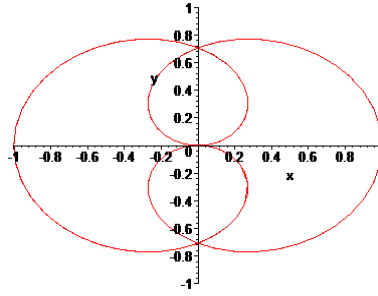


Figure 6 :  $n = 2$

### 2.2 Radius of curvature

This one, for plane parametrized curves depending on the time is given by

$$R(t) = \frac{[\dot{x}^2 + \dot{y}^2]^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

Where the time derivatives are written by

$$\begin{cases} \dot{x} = \frac{dx}{dt} \\ \ddot{x} = \frac{d^2x}{dt^2} \end{cases}$$

After calculations we have obtained

$$R(t) = \frac{A^{3/2}}{abw^3 + \alpha[A + w(b^2w + ab\alpha + wa^2)]}$$

where

$$A = (aw + b\alpha)^2 - (w^2 - \alpha^2)(a^2 - b^2)\cos^2wt$$

It extends along an axis ( $IM$ ) where  $M$  is a point of the curve

$$M(X, Y, 0)$$

And  $I$  is point located on the  $X$  axis such

$$I(I_x, 0, 0)$$

To obtain it we write the condition

$$\overrightarrow{IM} \cdot \vec{V} = 0$$

Where  $\vec{V}$  is the speed (naturally tangential to the curve)

$$\vec{V}(\dot{X}; \dot{Y}; 0)$$

Thus

$$(I_x - X) \cdot \dot{X} - Y \cdot \dot{Y} = 0$$

And point  $I$  is defined by

$$I_x = \frac{w(a^2 - b^2)(\cos wt \sin wt)}{(aw + b\alpha)(\sin wt \cos at) + (a\alpha + bw)(\cos wt \sin at)}$$

An interesting limiting case can be noticed : indeed, when  $\alpha = \pm w$ , radius of curvature are constant and given by

$n = 1$	$R = \frac{1}{2}(a + b)$	$I_x = \frac{1}{2}(a - b)$
$n = -1$	$R = \frac{1}{2}(a - b)$	$I_x = \frac{1}{2}(a + b)$

We obtain thus two circles

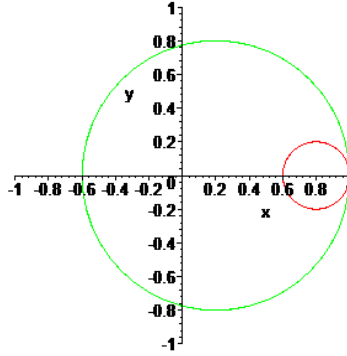


Figure 7 :  $n = \pm 1$

### 3 – Dynamics

#### 3.1 Force at distance

*Note : In all this part the mass of the point particle is considered equal to 1, such forces and accelerations are confused.*

As presented in the introduction we investigate the force given by

$$\vec{F} = -k\vec{r} + q\vec{v} * \vec{B}$$

Considering that the magnetic field  $B$  is perpendicular to the plane containing the trajectory we can write in our system of coordinate

$$\vec{F} = -k(X\vec{x} + Y\vec{y}) + qB(\dot{X}\vec{y} - \dot{Y}\vec{x})$$

Using the Principle of Dynamics we can write the system of differential equations

$$\begin{cases} \ddot{X} = -kX - qB\dot{Y} \\ \ddot{Y} = -kY + qB\dot{X} \end{cases}$$

We consider now the limiting case whose we know the exact solution, i.e. the Lissajous-Bodwitch curve, obtained for  $n = 0$ . The point particle is then submitted to a simple restoring force. This one is thus

$$F = -kr$$

And corresponding pulsation  $w_0$  is simply linked to the spring constant with

$$w_0 = \sqrt{k}$$

To simplify the calculation we introduce the frequency of the initial harmonic oscillator and the Larmor frequency (per mass unity) [4]

$$\left\{ \begin{array}{l} w_0 = \sqrt{k} \\ \alpha = \frac{qB}{2} \end{array} \right\}$$

And system of equations becomes

$$\left\{ \begin{array}{l} \ddot{X} = -w_0^2 X - 2\alpha \dot{Y} \\ \ddot{Y} = -w_0^2 Y + 2\alpha \dot{X} \end{array} \right\}$$

A classical method to solve this kind of system is to introduce the complex number ([1], [2])

$$u = X + jY$$

To obtain the differential equation

$$\frac{d^2 u}{dt^2} + w_0^2 u - 2j\alpha \frac{du}{dt} = 0$$

Generally solutions are obtained investigating a solution such

$$u = a_0 e^{j\omega t}$$

Where  $a_0$  is a constant length. But in our case we need two lengths to describe the initial ellipse. Consequently we are looking for another form. Moreover, we want a periodic solution. We consider a solution function of two pulsations given by

$$u = a_0 e^{j(\omega + \alpha)t}$$

Introducing it in the differential equation we obtain

$$-(\omega + \alpha)^2 + w_0^2 + 2\alpha(\omega + \alpha) = 0$$

Consequently

$$w_0^2 = \omega^2 - \alpha^2$$

Rewriting now our equation

$$\frac{d^2 u}{dt^2} + (\omega^2 - \alpha^2)u - 2i\alpha \frac{du}{dt} = 0$$

We notice that a second solution exists

$$u = b_0 e^{j(-w+\alpha)}$$

Where  $b_0$  is our second constant length. Whereas these results we investigate thus a general solution such

$$u = a_0 e^{j(w+\alpha)t} + b_0 e^{j(-w+\alpha)t}$$

To determine the constants we study the initial conditions : considering this ellipse, i.e. for  $\alpha = 0$  , and considering the origin of the time when the point particle is located on the semi major axis of the ellipse we obtain

for  $wt = 0$

$$a = a_0 + b_0$$

And for  $wt = \frac{\pi}{2}$ , we obtain two possibilities depending on the direction of rotation

$$\begin{cases} b = a_0 - b_0 \\ -b = a_0 - b_0 \end{cases}$$

Where  $a$  and  $b$  are successively the semi major and minor axis of the conic. We deduce that

$$\begin{cases} a_0 = \frac{a+b}{2} \\ b_0 = \frac{a-b}{2} \end{cases}$$

Or

$$\begin{cases} a_0 = \frac{a-b}{2} \\ b_0 = \frac{a+b}{2} \end{cases}$$

We obtain thus two general solutions

$$\begin{cases} u = \frac{a+b}{2} e^{j(+w+\alpha)} + \frac{a-b}{2} e^{j(-w+\alpha)} \\ u = \frac{a-b}{2} e^{j(+w+\alpha)} + \frac{a+b}{2} e^{j(-w+\alpha)} \end{cases}$$

And, by identifying the real and imaginary parts

$$\begin{cases} X = \frac{a+b}{2} \cos(w+\alpha) + \frac{a-b}{2} \cos(w-\alpha) \\ Y = \frac{a+b}{2} \sin(w+\alpha) - \frac{a-b}{2} \sin(w-\alpha) \end{cases}$$

Or

$$\begin{cases} X = \frac{a-b}{2} \cos(w + \alpha) + \frac{a+b}{2} \cos(w - \alpha) \\ Y = \frac{a-b}{2} \sin(w + \alpha) - \frac{a+b}{2} \sin(w - \alpha) \end{cases}$$

Finally

$$\begin{cases} X = a \cos wt \cos \alpha t - b \sin wt \sin \alpha t \\ Y = a \cos wt \sin \alpha t + b \sin wt \cos \alpha t \end{cases}$$

And

$$\begin{cases} X = a \cos wt \cos \alpha t + b \sin wt \sin \alpha t \\ Y = a \cos wt \sin \alpha t - b \sin wt \cos \alpha t \end{cases}$$

At this step we note that our family of curves is well one of the two solutions of the problem when

$$w^2 - \alpha^2 > 0$$

Thus when

$$-1 < n < 1$$

### 3.2 Proof by direct check

We can verify our solution by direct check. Indeed with the solution

$$\begin{cases} X = a \cos wt \cos \alpha t - b \sin wt \sin \alpha t \\ Y = a \cos wt \sin \alpha t + b \sin wt \cos \alpha t \end{cases}$$

We obtain the successive derivatives

$$\begin{cases} \dot{X} = -a(w \sin wt \cos \alpha t + \alpha \cos wt \sin \alpha t) - b(w \cos wt \sin \alpha t + \alpha \sin wt \cos \alpha t) \\ \dot{Y} = -a(w \sin wt \sin \alpha t - \alpha \cos wt \cos \alpha t) + b(w \cos wt \cos \alpha t - \alpha \sin wt \sin \alpha t) \end{cases}$$

And

$$\begin{cases} \ddot{X} = a(-w^2 \cos wt \cos \alpha t + 2w\alpha \sin wt \sin \alpha t - \alpha^2 \cos wt \cos \alpha t) \\ \quad + b(w^2 \sin wt \sin \alpha t - 2w\alpha \cos wt \cos \alpha t + \alpha^2 \sin wt \sin \alpha t) \\ \ddot{Y} = -a(w^2 \cos wt \sin \alpha t + 2w\alpha \sin wt \cos \alpha t + \alpha^2 \cos wt \sin \alpha t) \\ \quad - b(w^2 \sin wt \cos \alpha t + 2w\alpha \cos wt \sin \alpha t + \alpha^2 \sin wt \cos \alpha t) \end{cases}$$

And we obtain well

$$\begin{cases} \ddot{X} = -(w^2 - \alpha^2)X - 2\alpha\dot{Y} \\ \ddot{Y} = -(w^2 - \alpha^2)Y + 2\alpha\dot{X} \end{cases}$$

Where

$$k = w^2 - \alpha^2$$

### 3-3 Repulsive restoring force

In this case we obtain a similar result. Force becomes now

$$\vec{F} = k\vec{r} + q\vec{V} * \vec{B}$$

And system of differential equation

$$\begin{cases} \ddot{X} = w_0^2 X - 2\alpha \dot{Y} \\ \ddot{Y} = w_0^2 Y + 2\alpha \dot{X} \end{cases}$$

With the same methode we investigate this time

$$u = a_0 e^{(w+j\alpha)t} + b_0 e^{(-w+j\alpha)t}$$

One of the solution is given by

$$\begin{cases} X = a \cosh wt \cos at - b \sinh wt \sin at \\ Y = a \cosh wt \sin at + b \sinh wt \cos at \end{cases}$$

With, this time

$$k = w^2 + \alpha^2$$

Wich leads to a family of spiral curves, such presented in Figure 8, which won't be studied in this paper.

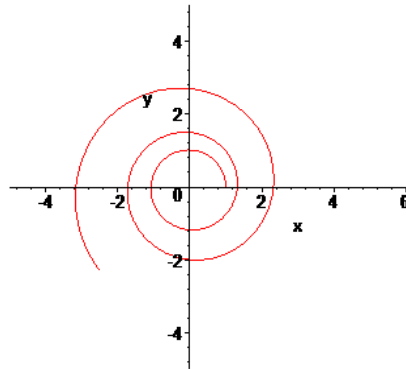


Figure 8 repulsive restoring force with magnetic field  $a = b = 1$  ,  $n = 10$

### 4. Limiting cases

Our family of curve accepts thus two limiting cases : first is obtained for  $n = 0$  and is simply an ellipse of Lissajoux- Bowditch.

Second limiting case, which is obtained when  $n = \pm 1$ , also corresponds to a wellknown situation. Indeed, as we presented it in figure 7, curves in these cases are two circles. Moreover, the coefficient  $k$  given by

$$k = (w^2 - \alpha^2)$$

Becomes here equal to zero : This indicates that the harmonic force is null and, consequently, point particle is only submitted to the constant magnetic field. Trajectories are thus circular and we re-obtain for their two cases the wellknown relation

$$V = qBR$$

But, naturally, considering the case of our point particle these cases aren't possible, because  $k$  isn't null. But we can consider the cases where

$$k \ll \alpha^2$$

Such

$$\alpha \approx w$$

The two trajectories are not stable anymore (Figure 9)

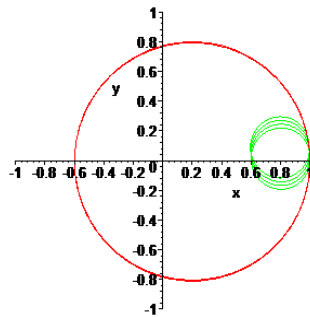


Figure 9.  $n = \pm 0.99$

Indeed it is interesting to note that the “small” circle (corresponding on  $\alpha = -1$ ) doesn't contain the origin of the system of coordinate. But the restoring force leads to a motion of rotation around this point (Figure 10)

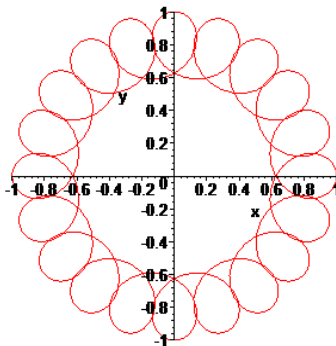


Figure 10.  $n = -0.9$  for  $e = 0.8$

## 5. Discussion

As we introduced the results presented in the paper have to be understood as general results of classical mechanics, that is without specifics applications. But if we consider that an atom can be

modelled as a harmonic oscillator, we can compare our work with important results of the contemporary Physics: we think in particular to the Normal Zeeman Effect (if the magnetic field is weak) and the Paschen-Back effect (if it is strong) [5]. Indeed, these effects describes the behavior of an atom submitted to a magnetic field. The classical theoretical explanation of the Normal Zeeman Effect is historically due to Lorentz, solving equation (2), as it can be found in reference [1].

Our own results can thus be comparable with these works if the spring coefficient given by

$$k = w_0^2 = w^2 - \alpha^2$$

Is positive, thus when

$$-1 < n < 1$$

And, for the Normal Zeeman effect, if

$$\alpha \ll w$$

Or, identically,

$$w_0 \approx w$$

To compare results we need to consider that our initial harmonic oscillator is isotropic (i.e  $a = b$ ). In this case our solutions are

$$\begin{cases} X = a[\cos wt \cos \alpha t - \sin wt \sin \alpha t] \\ Y = a[\cos wt \sin \alpha t + \sin wt \cos \alpha t] \end{cases}$$

and

$$\begin{cases} X = a[\cos wt \cos \alpha t + \sin wt \sin \alpha t] \\ Y = a[\cos wt \sin \alpha t - \sin wt \cos \alpha t] \end{cases}$$

Or, simpler

$$\begin{cases} X = a \cos(w + \alpha)t \\ Y = a \sin(w + \alpha)t \end{cases}$$

and

$$\begin{cases} X = a \cos(w - \alpha)t \\ Y = a \sin(w - \alpha)t \end{cases}$$

We can now compare them with the solutions presented in reference [2], for these conditions, which are respectively

$$\begin{cases} X = a \cos(w_0 + \alpha)t \\ Y = a \sin(w_0 + \alpha)t \end{cases}$$

And

$$\begin{cases} X = a \cos(w_0 - \alpha)t \\ Y = a \sin(w_0 - \alpha)t \end{cases}$$

We see that the two expressions are in correct agreement as much the magnetic field is weak. We obtain the same gap between the two frequencies. However we note a difference about the central position, which isn't in our solution  $w_0$  but  $w$ . This difference is thus

$$\Delta = w_0 - \sqrt{w_0^2 + \alpha^2}$$

Naturally this difference is small in the case of the Normal Zeeman effect, because the magnetic fields is weak. But, when it increases (case of the Paschen Back effect) the difference could also increase. This point can thus be considered as a prediction of the model, to the extent that, naturally, classical mechanics can describe this effect.

## 5 – Conclusion

We have presented a family of plane curves, which can describe the trajectory of a point-particle simultaneous submitted to a harmonic force and a force perpendicular to its speed. Besides their mathematical and physical interest, these results could perhaps have applications in some areas such, for example, the classical electrodynamic, or others. Research perspectives could be to extend the study to a three-dimensions harmonic oscillator and to consider the consequences of quantum effects.

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