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Representations of Deligne-Mostow lattices into $\mathrm{PGL}(3, \mathbb{C})$

E. Falbel, I. Pasquinelli and A. Ucan-Puc

Abstract

We classify representations of a class of Deligne-Mostow lattices into $\mathrm{PGL}(3, \mathbb{C})$. In particular, we show local rigidity for the representations (of Deligne-Mostow lattices with 3-fold symmetry and of type one) where the generators we chose are in the same conjugacy class as the generators of Deligne-Mostow lattices. We use formal computations in SAGE to obtain the results. The code files are available on GitHub ([FPUP21]).

1 Introduction

Lattices in semi-simple Lie groups have strong rigidity properties. When the Lie group is a connected component of the isometry group G of an irreducible hermitian space, Mostow's rigidity theorem states that, in dimension greater than two, any cocompact (or finite covolume by Prasad) lattice $\Gamma \subset G$ is rigid in the sense that an isomorphism between lattices extends to a homomorphism between isometry groups. For hermitian symmetric spaces of higher rank, Margulis' super-rigidity theorem implies that any homomorphism $\phi : \Gamma \to G'$ (with G', connected with trivial centre and no compact factors) with Zariski dense image extends to a homomorphism $\hat{\phi} : G \to G'$.

In rank one, it is well known that super rigidity fails. In particular, there might exist representations of a given lattice in $\mathrm{PU}(2, 1)$ into other Lie groups which do not extend to homomorphisms of the group $\mathrm{PU}(2, 1)$. A very active study of representations of lattices into hermitian Lie groups originated with Toledo's ([Tol89]) description of maximal representations of lattices in $\mathrm{PU}(1, 1)$ into $\mathrm{PU}(2, 1)$ (see [Poz15] for a recent overview).

In this paper we are interested in a target Lie group which is not hermitian. We classify representations (up to conjugation) of a class of Deligne-Mostow lattices into $\mathrm{PGL}(3, \mathbb{C})$.

An important motivation for our results is the study of complex projective structures. Indeed, given a complex projective structure on a complex surface M, one can define a developing map $D : \tilde{M} \to \mathbb{C}P^2$ and a holonomy representation $\rho : \pi_1(M) \to \mathrm{PGL}(3, \mathbb{C})$. Complex hyperbolic manifolds are complete complex projective structures. It is an open question whether a complex hyperbolic manifold has a different complex projective structure. On the other hand, fixing the complex structure of a finite volume complex hyperbolic surface, its complex projective structure is unique by a theorem of Mok-Yeung [MY93] (see also [Kli98]). It is possible that some of the representations in our list correspond (taking torsion free subgroups) to complex projective structures on complex manifolds diffeomorphic but not biholomorphic to complex hyperbolic manifolds. One should be aware that in

the compact case, Siu's rigidity theorem (valid for any compact locally hermitian symmetric space) implies that any other Kähler complex structure on the manifold is biholomorphic or conjugate-biholomorphic to the original complex hyperbolic structure.

In section 2 we give the classification of all representations where the generators we chose are of the same type as the generators of the Deligne-Mostow lattice.

In Sections 3 and 4 we review rigidity results and the definition of the Deligne-Mostow lattices in $PU(2,1)$. The main result which allows computations is the explicit presentation of the Deligne-Mostow lattices. It was obtained in several steps culminating in [Pas16], where a unified treatment of all lattices is given. We will use a presentation of each Deligne-Mostow group using two generators easily obtained from the presentations in [Pas16].

In section 5 we give the detailed proofs for one case. The other cases are calculated in the same way. The main strategy is to use Gröbner basis methods to solve the equations imposed by the relators in the presentation. The computations show that the representations obtained are all locally rigid. With the exception of the Deligne-Mostow lattice itself (because of the local rigidity theorem for complex hyperbolic lattices), we do not know if this local rigidity phenomenon is specific to the lattices we studied or has a more general scope. With an appropriate conjugation, the representations have values in an algebraic field extension of $\mathbb{Q}$ and we studied the orbits by the Galois group.

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2 The results

We refer to section 4 for the definition of Deligne-Mostow lattices. For the statement of the results we simply recall that 3-fold Deligne-Mostow lattices of type one are parametrised by a couple of integers (p,k) and are generated by two elements J and R_1 which are, respectively, a regular elliptic element and a complex reflection.

Theorem 2.1. *Representations of 3-fold type one Deligne-Mostow lattices into* $\mathrm{PGL}(3,\mathbb{C})$ *(with image of generators conjugated to those of the lattice) are locally rigid and are classified in the following tables.*

(p,k)	Total Number	Galois Orbits	Irreducible	$\mathbb{Q}-$extension
(3,4)	18	2	2	$\mathbb{Q}(1-x^6+x^{12})$
(3,5)	24	1	1	$\mathbb{Q}(1-x^3+x^9-x^{12}+x^{15}-x^{21}+x^{24})$
(4,3)	22	5	3	$\mathbb{Q}(1-x^6+x^{12})$
(5,2)	12	2	2	$\mathbb{Q}(1-x+x^3-x^4+x^5-x^7+x^8)$
(5,3)	24	1	1	$\mathbb{Q}(1+x^3-x^9-x^{12}-x^{15}+x^{21}+x^{24})$

Table 1: Compact 3-fold Deligne-Mostow Lattices

(p,k)	Total	Galois Orbits	$\mathbb{Q}-$extension	Irreducible	Reducible	Hermitian Form	Factors
(3,6)	34	11	$\mathbb{Q}(x^6+x^3+1)$	1	0	(2,1)	0
				2	0	(3,0)	2
			$\mathbb{Q}(x^2+x+1)$	0	(6,2)	Degenerate	8
(4,4)	31	11	$\mathbb{Q}(x^4-x^2+1)$	1	0	(2,1)	0
				1	0	(3,0)	1
				0	(2,2)	Degenerate	4
			$\mathbb{Q}(x^2-x+1)$	1	0	(3,0)	1
			$\mathbb{Q}(x^2+1)$	1	0	(2,1)	0
				1	0	(3,0)	1
				0	(2,0)	Degenerate	2
			$\mathbb{Q}$	1	0	(3,0)	1
(6,2)	12	4	$\mathbb{Q}(x^6-x^3+1)$	1	0	(2,1)	0
				1	0	(3,0)	1
			$\mathbb{Q}(x^2-x+1)$	1	0	(3,0)	1
			$\mathbb{Q}$	1	0	(3,0)	1
(6,3)	30	13	$\mathbb{Q}(x^6+x^3+1)$	1	0	(2,1)	0
				2	0	(3,0)	2
			$\mathbb{Q}(x^2+x+1)$	0	(6,4)	Degenerate	9

Table 2: Non-compact 3-fold Deligne-Mostow Lattices

In Table 1 are the information for each compact 3-fold type one Deligne-Mostow lattice. The representations are defined with coefficients in a cyclotomic extension of $\mathbb{Q}$, which depends on (p,k) and for which we give the corresponding minimal polynomial. We also compute the orbits of the Galois group action on the space of representations. For the compact Deligne-Mostow lattices, there exits an invariant Hermitian form which is always non-degenerate.

In Table 2 are the information on non-compact 3-fold type one lattices. Each representation has coefficients in a subfield of a $\mathbb{Q}-$extension which depends on (p,k). For each representation there exists an invariant Hermitian form that can be non-degenerate with signatures (2,1) or (3,0), or degenerate. While the irreducible representations can only come from non-degenerate configurations, the reducible ones can come from either degenerate or non-degenerate configurations (see Section 5). In the Reducible column the numbers (m,n) mean that m reducible representations come from non-degenerate configurations and n come from degenerate ones. The last column corresponds to the number of representations which contain in their kernel an element of the centraliser of the cusp group. One may interpret those representations as representations which factor through the orbifold fundamental group of a compactification of the complex hyperbolic orbifold defined by the Deligne-Mostow lattice.

We also decide which representations may be lifted to $\mathrm{GL}(3,\mathbb{C})$ representations (see [FPUP21]). The representations with generators $\rho(J)$ and $\rho(R_1)$ of other types are not

tabulated. The computations show that they are all locally rigid except for one dimensional branches for Deligne-Mostow lattices $(4,4),(6,2)$ and $(4,3)$.

In the following sections, we give details for the calculations in the case of the Deligne-Mostow lattice $(3,6)$. We also compute the reducible representations of this lattice when the generators are *not* of the same type as the ones of the Deligne-Mostow lattice. The same calculations for the other Deligne-Mostow lattices in the table can be found on GitHub ([FPUP21]).

3 Rigidity of lattices and projective structures

In this section we recall some rigidity theorems in the two dimensional complex hyperbolic context. The first one is Mostow's (Prasad's in the non-compact quotient case) rigidity for lattices in $PU(2,1)$:

Theorem 3.1. *If $\rho : \Gamma_1 \to \Gamma_2$ is an isomorphism between two lattices in $PU(2,1)$ then it extends to an automorphism of $PU(2,1)$.*

Therefore Γ_2 is conjugated to Γ_1 or to its complex conjugate by an element of $PU(2,1)$.

The representations we find in this paper have images in $\mathrm{PGL}(3,\mathbb{C})$ and are counter-examples to Margulis' super-rigidity theorem in rank one, which states as follows. Let G and G' be connected Lie groups, with trivial centre and no compact factors. Suppose that G has rank higher than one and that it is not isogenous to a group of the form $SO(1,n)\times K$ or $SU(n,1)\times K$ where K is compact. Let $\Gamma \subset G$ be a lattice and $\rho : \Gamma \to G'$ a homomorphism with Zariski dense image. Margulis' super-rigidity theorem states that ρ extends to a continuous homomorphism of G into G'.

In this paper we construct a large class of homomorphisms of Deligne-Mostow lattices in $PU(2,1)$ into $PGL(3,\mathbb{C})$. One could think that these representations (taking torsion free subgroups of the lattice) might be holonomy representations of different complex projective structures on the quotient of complex hyperbolic space by the lattice. But that is not the case, at least if the complex structure of the quotient is fixed:

Theorem 3.2 (Mok-Yeung [MY93])**.** *Let $\Gamma\backslash H^2_{\mathbb{C}}$ be a complex hyperbolic manifold of finite volume. Then its complex projective structure induced by the embedding $H^2_{\mathbb{C}} \subset \mathbb{C}P^2$ is the unique projective structure compatible with its complex structure.*

The theorem is also true for complex hyperbolic manifolds of dimension greater than two (see [MY93]) and for higher rank hermitian symmetric domains (see [Kli98]). On the other hand, the existence of different complex structures on the quotient $\Gamma\backslash H^2_{\mathbb{C}}$ (for a torsion free discrete group Γ) is an open problem except for the local rigidity result of Calabi-Vesentini ([CV60]). This states that, in the two dimensional complex hyperbolic setting, the complex structure of $\Gamma\backslash H^2_{\mathbb{C}}$ is locally rigid. One should also observe that local rigidity (up to conjugation) in $\mathrm{PGL}(3,\mathbb{C})$ of the holonomy representation of the complex hyperbolic structure, that is $H^1(\Gamma,\mathfrak{sl}(3,\mathbb{C})) = 0$, follows easily from the local rigidity of hyperbolic structures (see [Kli98] prop. 3.2).

4 Deligne-Mostow lattices

The Deligne-Mostow lattices were introduced and studied by Mostow and by Deligne and Mostow in various works, including [Mos80] and [DM86]. They have a long history dating back to by Picard, Le Vavasseur and Lauricella. Deligne and Mostow start with a *ball N-tuple* $\mu = (\mu_1, \ldots, \mu_N)$ which is a set of N real numbers in $(0, 1)$ whose sum is 2. Then they look at a *hypergeometric function* defined using these parameters. Finally, they use the *monodromy action* to build some lattices in $PU(N-3, 1)$. This list of lattices contains the first known examples on non-arithmetic complex hyperbolic lattices in dimensions 2 and 3.

Here we will concentrate on a very special class of the Deligne-Mostow lattices in $PU(2, 1)$, called the lattices with 3-fold symmetry and of type one. More precisely, first we restrict to the case of dimension 2. This means that we are looking at the group $PU(2, 1)$ acting on 2-dimensional complex hyperbolic space and hence that we are looking at ball 5-tuples. In this dimension, the finite list of lattices that Deligne and Mostow found contains only lattices with some special symmetry. These are called the 2-fold and 3-fold symmetry lattices. A lattice has m-fold symmetry is m of the N elements of the ball N-tuple are equal.

In this work we will only look at some of the lattices with 3-fold symmetry. For all of those, one can find a construction for a fundamental domain and a presentation in [Pas16]. In principle the 2-dimensional Deligne-Mostow lattices depend on the 5 elements of the ball 5-tuple. The condition on the sum of the elements of the ball 5-tuple reduces the parameters to 4. Of these 4, 3 are equal, so the lattices only depend on 2 parameters. One common choice for the parameters is to choose the orders of some of the generators. We will hence identify the lattices using a pair (p, k) which corresponds to the ball 5-tuple $(1/2 - 1/p, 1/2 - 1/p, 1/2 - 1/p, 1/2 + 1/p - 1/k, 2/p + 1/k)$.

The Deligne-Mostow lattices in $PU(2, 1)$ with 3-fold symmetry are divided in 4 *types* according to the ranges of p and k. The type determines the presentation, the volume formula and the combinatorics of a fundamental domain. Here we will look at the lattices *of type one*, which were first looked at in [BP15]. We are choosing these because they have minimal complexity (minimum number of facets in a fundamental domain, shortest presentation). In terms of p and k they are characterised by

$$0 < p \leq 6, \qquad\qquad k \leq \frac{2p}{p-2}.$$

Then a presentation for them is given by

$$\Gamma = \left\langle J, P, R_1, R_2 \colon \begin{array}{l} J^3 = R_1^p = R_2^p = (P^{-1}J)^k = I, \\ R_2 = PR_1P^{-1} = JR_1J^{-1},\ P = R_1R_2 \end{array} \right\rangle, \tag{1}$$

We can rewrite this presentation in terms of the two generators J and R_1 in the following way:

$$\Gamma = \left\langle J, R_1 \colon \ J^3 = R_1^p = (R_1J)^{2k} = R_1JR_1J^2R_1JR_1^{p-1}J^2R_1^{p-1}JR_1^{p-1}J^2 = Id \right\rangle, \tag{2}$$

One can go back to the previous presentation writing $R_2 = JR_1J^{-1}$ and $P = R_1R_2$.

5 The group (3,6)

In this section we look in details at the case $(p, k) = (3, 6)$. The presentation for this case is:

$$\Gamma = \left\langle J, R_1 : \ J^3 = R_1^3 = (R_1 J)^{12} = R_1 J R_1 J^2 R_1 J R_1^2 J^2 R_1^2 J R_1^2 J^2 = Id \ \right\rangle, \tag{3}$$

We first note that for the Deligne-Mostow lattices, one has that J is a regular elliptic element, while R_1 is a complex reflection. In general, while the presentation guarantees that $\rho(J)$ and $\rho(R_1)$ are both elliptic, we cannot say more about their type. We will hence look at all possible combinations of types:

- when $\rho(J)$ is regular elliptic and $\rho(R_1)$ is a complex reflection,
- when $\rho(J)$ is a complex reflection and $\rho(R_1)$ is regular elliptic,
- when $\rho(J)$ and $\rho(R_1)$ are both regular elliptic,
- when $\rho(J)$ and $\rho(R_1)$ are both complex reflections.

Note that if $\rho(J)$ and $\rho(R_1)$ are both complex reflections, then the group preserves the intersection of two complex lines in $\mathbb{C}P^2$. The group is therefore reducible and we will hence ignore this last case.

5.0.1 $\rho(J)$ is regular elliptic and $\rho(R_1)$ is a complex reflection

First, let us assume that the fixed points of $\rho(J)$ do not intersect the fixed points of $\rho(R_1)$. We will call these the *non-degenerate configurations*. We remark that this case might give rise to reducible representations when there exists a degenerate Hermitian form preserved by the generators. We note in the following $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ and ζ_9 a primitive 9th-root of unity.

Proposition 5.1. *A representation $\rho : \Gamma \to PGL(3, \mathbb{C})$ such that $\rho(R_1)$ is a complex reflection and $\rho(J)$ has three distinct eigenvectors which do not intersect the fixed line of $\rho(R_1)$ are given, up to conjugation, by the matrices*

$$J = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{4}$$

and

$$\rho(R_1) = \begin{bmatrix} 1 & -r_1 & r_1 \\ 0 & 1 - r_2 & r_2 \\ 0 & 1 - r_2 - \omega & r_2 + \omega \end{bmatrix}, \tag{5}$$

with (r_1, r_2) in the following list

$$
\begin{aligned}
\alpha_1 &= (0,1),\\
\alpha_2 &= (\omega-1,1),\\
\alpha_3 &= (\frac{1}{2}(\omega-1),\frac{1}{2}(\omega+3)),\\
\alpha_4 &= (\frac{1}{2}(\omega-1),\frac{-1}{2}(3\omega+1)),\\
\alpha_5 &= (0,-\omega),\\
\alpha_6 &= (\omega-1,-\omega),\\
\alpha_7 &= \left(\frac{1}{2}(\zeta_9^3+\zeta_9^2-\zeta_9-1),\frac{1}{2}(-\zeta_9^3+\zeta_9^2+\zeta_9+1)\right),\\
\alpha_8 &= \left(\frac{1}{2}(\zeta_9^5+\zeta_9^4+\zeta_9^3+\zeta_9-1),\frac{1}{2}(\zeta_9^5-\zeta_9^4-\zeta_9^3-\zeta_9+1)\right),\\
\alpha_9 &= \left(\frac{1}{2}(-\zeta_9^5-\zeta_9^4+\zeta_9^3-\zeta_9^2-1),\frac{1}{2}(-\zeta_9^5+\zeta_9^4-\zeta_9^3-\zeta_9^2+1)\right),\\
\alpha_{10} &= \left(\frac{1}{2}(\zeta_9^3+\zeta_9^2-\zeta_9-1),\frac{1}{2}(-\zeta_9^3-\zeta_9^2-\zeta_9+1)\right),\\
\alpha_{11} &= \left(\frac{1}{2}(-\zeta_9^5-\zeta_9^4+\zeta_9^3-\zeta_9^2-1),\frac{1}{2}(\zeta_9^5-\zeta_9^4-\zeta_9^3+\zeta_9^2+1)\right),\\
\alpha_{12} &= \left(\frac{1}{2}(\zeta_9^5+\zeta_9^4+\zeta_9^3+\zeta_9-1),\frac{1}{2}(-\zeta_9^5+\zeta_9^4-\zeta_9^3+\zeta_9+1)\right),\\
\alpha_{13} &= \left(\frac{1}{2}(\zeta_9^3-\zeta_9^2+\zeta_9-1),\frac{1}{2}(-\zeta_9^3-\zeta_9^2-\zeta_9+1)\right),\\
\alpha_{14} &= \left(\frac{1}{2}(\zeta_9^5+\zeta_9^4+\zeta_9^3+\zeta_9^2-1),\frac{1}{2}(\zeta_9^5-\zeta_9^4-\zeta_9^3+\zeta_9^2+1)\right),\\
\alpha_{15} &= \left(\frac{1}{2}(-\zeta_9^5-\zeta_9^4+\zeta_9^3-\zeta_9-1),\frac{1}{2}(-\zeta_9^5+\zeta_9^4-\zeta_9^3+\zeta_9+1)\right).
\end{aligned}
$$

It is interesting to note that only the first six representations can be lifted to $\mathrm{GL}(3,\mathbb{C})$ representations (see [FPUP21]).

Proof. The projective group acts transitively on ordered sets of three distinct points and a line not containing them in $\mathbb{C}P^2$. One observes that the fixed points of J are $[1,1,-1]$, $[\omega,\bar{\omega},1]$, $[\bar{\omega},\omega,1]$ and the line fixed by R_1 (generated by $[1,0,0]$ and $[1,1,1]$) does not meet the fixed points.

Then the matrix of $\rho(J)$ may be fixed to be the one in the statement of the lemma and the matrix of $\rho(R_1)$ will be of the form

$$
\rho(R_1)=\begin{bmatrix}1 & -r_1 & r_1\\ 0 & 1-r_2 & r_2\\ 0 & 1-r_3 & r_3\end{bmatrix}. \tag{6}
$$

By a computation, $\rho(R_1)^3=Id$ if and only if $r_3=r_2+x$ with $x^2+x+1=0$. The proof now is a computation using Gröbner basis package in SAGE. In fact, the first six solutions

appear from a simple factorisation of the equations. The others come from the following system:

$$(8r_2^3 + 6(i\sqrt{3} - 3)r_2^2 - 12(i\sqrt{3} - 1)r_2 + 6i\sqrt{3} - 1)(8r_2^3 + 6(i\sqrt{3} - 3)r_2^2 - 12(i\sqrt{3} - 1)r_2 + 6i\sqrt{3} + 1) = 0,$$

$$(32ir_2^4\sqrt{3} - 96(i+1)r_2^3\sqrt{3} + (92i + 276)\sqrt{3}r_2^2 - 25i\sqrt{3} - 12r_1 - 264r_2 + 75)(8r_2^3 + 6(i\sqrt{3} - 3)r_2^2 - 12(i\sqrt{3} - 1)r_2 + 6i\sqrt{3} - 1) = 0,$$

$$(3 - i\sqrt{3})r_1 + (3 - i\sqrt{3})r_2 + 2r_1^2 - 2r_2^2 - 1 + \sqrt{3} = 0.$$

Writing

$$\begin{aligned} p_1(r_2) &= 8r_2^3 + 6(i\sqrt{3} - 3)r_2^2 - 12(i\sqrt{3} - 1)r_2 + 6i\sqrt{3} - 1, \\ p_2(r_2) &= 8r_2^3 + 6(i\sqrt{3} - 3)r_2^2 - 12(i\sqrt{3} - 1)r_2 + 6i\sqrt{3} + 1, \end{aligned}$$

both polynomials factor over $\mathbb{Q}(\zeta_9)$. For each root of p_1, the corresponding values of r_1 satisfy a quadratic polynomial equation which factors over $\mathbb{Q}(\zeta_9)$. In a similar way, for each root of p_2, the corresponding values of r_1 satisfy a linear polynomial equation whose root belong to $\mathbb{Q}(\zeta_9)$. Therefore the whole set of solutions belongs to the field $\mathbb{Q}(\zeta_9)$. The solutions are then written in the basis of $\mathbb{Q}(\zeta_9)$.

□

We now look at the *degenerate configurations*, where one eigenvector of $\rho(J)$ intersects the fixed complex line of $\rho(R_1)$. This gives rise to reducible representations:

Proposition 5.2. *The representation $\rho : \Gamma \to PGL(3, \mathbb{C})$ such that $\rho(R_1)$ is a complex reflection and $\rho(J)$ has three distinct eigenvectors with one intersecting the fixed line of $\rho(R_1)$ are given, up to conjugation, by the matrices*

$$\rho(J) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{2i\pi/3} & 0 \\ 0 & 0 & e^{4i\pi/3} \end{bmatrix} \tag{7}$$

and

$$\rho(R_1) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 - r_2 & r_2 \\ 0 & 1 - r_2 - x & r_2 + x \end{bmatrix}, \tag{8}$$

where $x^2 + x + 1 = 0$ and with r_2 determined as follows:

1. *for $x = e^{2i\pi/3}$, $r_2 = 1 - \sqrt{3}i/3$ or $r_2 = 1/2 - \sqrt{3}i/6$,*
2. *for $x = e^{4i\pi/3}$, $r_2 = 1 + \sqrt{3}i/3$ or $r_2 = 1/2 + \sqrt{3}i/6$.*

Proof. We choose a diagonal form for $\rho(J)$ and impose that $\rho(R_1)$ has the same eigenvector e_1. One can then impose that the fixed line in projective space passes through $[1, 0, 0]$ and $[1, 1, 1]$. By conjugating by a diagonal matrix, one can suppose that $\rho(R_1)$ is as above with r_2 to be determined. As in the previous proposition a computation using Gröbner basis package in SAGE gives the result. Note that the two sets of solutions are complex conjugates.

□

5.0.2 $\rho(J)$ is a complex reflection and $\rho(R_1)$ is regular elliptic

Interchanging the type of $\rho(J)$ and $\rho(R_1)$, a computation with Gröbner basis proves the following:

Proposition 5.3. *There are no representations $\rho : \Gamma \to PGL(3, \mathbb{C})$ such that $\rho(J)$ is a complex reflection and $\rho(R_1)$ has three distinct eigenvectors which do not intersect the fixed line of $\rho(R_1)$.*

5.0.3 $\rho(J)$ and $\rho(R_1)$ are regular elliptic

The final case to consider occurs when both $\rho(J)$ and $\rho(R_1)$ are regular elliptic. A simple computation shows that the two regular elliptic elements cannot have the same eigenspaces.

Proposition 5.4. *The representations $\rho : \Gamma \to PGL(3, \mathbb{C})$ such that $\rho(R_1)$ and $\rho(J)$ are regular elliptic with at least one distinct eigenspace are given, up to conjugation, by the matrices*

$$J = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{9}$$

and

$$\rho(R_1) = \begin{bmatrix} 1 & s_1 & r_1 \\ 0 & s_2 & r_2 \\ 0 & s_3 & -1 - s_2 \end{bmatrix}, \tag{10}$$

with r_1, r_2, s_1, s_2, s_3 in the following list (up to complex conjugation)

1. $r_1 = 0$, $r_2 = -1$, $s_1 = -1$, $s_2 = -1$, $s_3 = 1$,
2. $r_1 = 1$, $r_2 = 1$, $s_1 = 0$, $s_2 = 0$, $s_3 = -1$,
3. $r_1 = 0$, $r_2 = -\omega$, $s_1 = \omega$, $s_2 = -1$, $s_3 = \bar{\omega}$,
4. $r_1 = 0$, $r_2 = 0, s_1 = 0$, $s_2 = \omega$, $s_3 = 0$,
5. $r_1 = 6/7 + 4/7i\sqrt{3}$, $r_2 = -4/7 + 2/7i\sqrt{3}$, $s_1 = -4/7 + 2/7i\sqrt{(}3)$, $s_2 = -11/14 - 5/14i\sqrt{3}$, $s_3 = 1/7 + 3/7i\sqrt{3}$,
6. $r_1 = 4/7 - 2/7i\sqrt{3}$, $r_2 = 1/7 + 3/7i\sqrt{3}$, $s_1 = -6/7 - 4/7i\sqrt{(}3)$, $s_2 = -3/14 + 5/14i\sqrt{3}$, $s_3 = -4/7 + 2/7i\sqrt{3}$,
7. $r_1 = \omega$, $r_2 = \bar{\omega}$, $s_1 = 0$, $s_2 = 0$, $s_3 = -\omega$,
8. $r_1 = 3/7 - 5/7i\sqrt{3}$, $r_2 = -1/7 - 3/7i\sqrt{3}$, $s_1 = -1/7 - 3/7i\sqrt{(}3)$, $s_2 = -11/14 - 5/14i\sqrt{3}$, $s_3 = 4/7 - 2/7i\sqrt{3}$,
9. $r_1 = -9/7 + 1/7i\sqrt{3}$, $r_2 = 5/7 + 1/7i\sqrt{3}$, $s_1 = 5/7 + 1/7i\sqrt{(}3)$, $s_2 = -11/14 - 5/14i\sqrt{3}$, $s_3 = -5/7 - 1/7i\sqrt{3}$,
10. $r_1 = 1/7 + 3/7i\sqrt{3}$, $r_2 = 4/7 - 2/7i\sqrt{3}$, $s_1 = -3/7 + 5/7i\sqrt{(}3)$, $s_2 = -3/14 + 5/14i\sqrt{3}$, $s_3 = -1/7 - 3/7i\sqrt{3}$,

11. $r_1 = -5/7 + 1/7i\sqrt{3}$, $r_2 = -5/7 + 1/7i\sqrt{3}$, $s_1 = 9/7 + 1/7i\sqrt(3)$, $s_2 = -3/14 - 5/14i\sqrt{3}$, $s_3 = 5/7 - 1/7i\sqrt{3}$,

Proof. We can again fix J to be of the form above and one eigenvector for R_1 to be the first column vector. Observe that a regular elliptic element of order three has null trace and, therefore, we obtain the form of the matrix for R_1. We obtain the result finding a Gröbner basis for the system of equations as in the previous proposition. □

5.1 The Galois Conjugation

Consider the $\mathbb{Q}-$extension by the cyclotomic polynomial $x^6 + x^3 + 1$, that we will denote $\mathbb{Q}(\zeta_9)$ where ζ_9 is again the 9-th root of unity. This extension has a subfield isomorphic to the $\mathbb{Q}-$extension by $x^2 + x + 1$, that we will denote by $\mathbb{Q}(\omega)$ where we recall that $\omega = \zeta_9^3$. Recall also that the Galois group of $\mathbb{Q}(\zeta_9)$, $\mathrm{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q})$, is isomorphic $\mathbb{Z}_6$.

Remark 5.5. *For the first six solutions in Proposition 5.1, the group image belongs to* $\mathrm{PGL}(3, \mathcal{O}_3)$ *up to a projective representative, therefore the corresponding groups have discrete image.*

The group image for the other solutions, up to projective representative, belongs to $\mathrm{PGL}(3, \mathcal{O}_9)$ *where* $\mathcal{O}_9$ *denotes the ring of integers of* $\mathbb{Q}(\zeta_9)$*. We do not know which of these representations are discrete.*

The action of the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q})$ over the field $\mathbb{Q}(\zeta_9)$ induces an action over the set of solutions, therefore on the set of representations. Recall that $\mathrm{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q})$ is a cyclic group, and we will denote its generator by g. In particular, $g(\zeta_9) = \zeta_9^2$. Therefore the element g^2 sends ζ_9 to ζ_9^4, and one can verify that

$$g^2(\omega) = g^2(\zeta_9^3) = \zeta_9^3,$$

and so the group generated by g^2 is the unique subgroup of the Galois group that fixes ω. Therefore, the subgroup $\langle g^2 \rangle$ fixes the subfield $\mathbb{Q}(\omega)$ and in particular, all solutions $\{\alpha_j\}_{j=1}^6$ are fixed points under the Galois group action.

By a direct computation, one can check that the action of $\langle g^2 \rangle$ on the solutions $\{\alpha_j\}_{j=7}^1 5$ has three closed orbits, namely $\{\alpha_7, \alpha_8, \alpha_9\}$, $\{\alpha_{10}, \alpha_{11}, \alpha_{12}\}$ and $\{\alpha_{13}, \alpha_{14}, \alpha_{15}\}$, and these are the only closed orbits in the set of solutions under the Galois group action.

5.2 The Hermitian form

In what follows, we will find the Hermitian form that is preserved by each representation and it will sometimes be degenerate. Assume that H is an Hermitian form. Now if Γ preserve H, then the generators in the presentation (3) satisfy

$$J^* H J = H, \qquad R_1^* H R_1 = H.$$

From the first equation, we can assure that H is of the form

$$\begin{bmatrix} a & -\overline{c} & c \\ -c & a & \overline{c} \\ \overline{c} & c & a \end{bmatrix},$$

where $a \in \mathbb{R}$, and we can assume that $c \in \mathbb{Q}(\zeta_9)$. Let $c = b_0 + \Sigma_{j=1}^{5} b_j \zeta_9^j$. Then the second equation provides a polynomial system on $\mathbb{Q}(\zeta_9)[a, b_0, b_1, b_2, b_3, b_4, b_5]$. For each solution α_i, we use Gröbner basis to reduce the system into two linear equations.

Once we solved the equation, we have that for the Galois orbits $\{\alpha_7, \alpha_8, \alpha_9\}$ and $\{\alpha_{10}, \alpha_{11}, \alpha_{12}\}$, the Hermitian form is non-degenerate and of signature (3,0). For the orbit $\{\alpha_{13}, \alpha_{14}, \alpha_{15}\}$ the Hermitian form is non-degenerate of signature (2,1). In the case of the Galois fixed points $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, all Hermitian forms are degenerate with two zero eigenvalues.

5.3 The cusp group

For a non-compact lattice, one defines a cusp holonomy as a conjugacy class of maximal subgroups (not containing loxodromic elements) fixing a point in the boundary of complex hyperbolic space. The following proposition was proved using a fundamental domain for the lattice (see [BP15] and [Pas16]).

Proposition 5.6. *The Deligne-Mostow (3,6)-lattice $\Gamma_{(3,6)} \subset \mathrm{PU}(2,1)$ has only one cusp. The cusp holonomy is the class determined by $\Gamma_{cusp} = \langle R_2 = JR_1J^2,\ A_1 = JR_1^2J^2R_1^2J \rangle$.*

The cusp holonomy might contain elliptic elements. The purely parabolic cusp holonomy is the maximal subgroup of the cusp holonomy with no elliptics.

Proposition 5.7. *The purely parabolic cusp holonomy of $\Gamma_{(3,6)}$ is $\langle [A_1, R_2], [A_1, R_2^2], (R_2A_1)^2 \rangle$. The centre of this group is the cyclic group generated by $(R_2A_1)^2$.*

A straightforward computation with the list of representations gives the following description of cusp groups. The representations for which the generator of the centraliser is elliptic, may be factorised (up to a finite index subgroup) through a representation of the fundamental group of the Satake-Baily-Borel compactification. Indeed, in the compactification, the centraliser of the cusp holonomy disappears.

Proposition 5.8. *The generator of the centraliser, $(R_2A_1)^2$, is elliptic of order at most three for all representations except for three (up to conjugation) where it can be chosen to be unipotent of the form*

$$\rho((R_2A_1)^2) = \begin{bmatrix} 1 & \xi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

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