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Affine noncommutative geometry

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ABSTRACT. The space \mathbb{C}^N has no free analogue, but we can talk instead about the free sphere $S_{\mathbb{C},+}^{N-1}$, as the manifold defined by the equations $\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$. We discuss here the structure and hierarchy of the submanifolds $X \subset S_{\mathbb{C},+}^{N-1}$, with particular attention to the manifolds having an integration functional $tr : C(X) \rightarrow \mathbb{C}$.

Preface

Classical geometry has its origins in classical mechanics, with some of its most fundamental objects, such as the conics, coming from the trajectories of planets and other celestial objects around the Sun. The whole process of creation of abstract modern geometry, in interaction with classical mechanics, has taken around 2000 years. Similarly, quantum mechanics, which is however a far newer discipline, has inspired several theories of quantum geometry, more commonly called “noncommutative geometry”.

Generally speaking, a quantum space is an abstract space X , described by an algebra of functions of type $A = C(X)$, which can be noncommutative. However, at the technical level, the questions abound. Shall X be finite, discrete, compact, or locally compact? Shall X be an algebraic manifold, and if so, shall it be real or complex, and affine or projective? What about X being smooth, or even Riemannian? Shall the scalars for A be the complex numbers, $k = \mathbb{C}$, or something else? Shall the integration over X exist, and by which mechanism, and if so, shall that be a trace or not? And so on.

No one really knows the answer to these questions, and several noncommutative geometry theories have been built, by using various combinations of answers to the above questions, depending on knowledge and taste. The problem comes of course from our current poor understanding of quantum mechanics, which is no easy business. But this is how life is, and people since Einstein have been struggling with this problem.

The present book is an introduction to noncommutative geometry, using an intuitive and elementary approach. Our philosophy will be very close to that of Connes, with substantial input from Jones, Voiculescu and Woronowicz. To be more precise, we will be interested in quantum spaces X which are compact, are real algebraic manifolds, either affine or projective, are a bit Riemannian too, without however being smooth, and with the corresponding algebra $A = C(X)$ being an operator algebra, and so with $k = \mathbb{C}$, and coming with an integration functional $tr : A \rightarrow \mathbb{C}$, which must be a trace, $tr(ab) = tr(ba)$, and which, importantly, must appear by theorem, rather than by axiom.

All this might seem a bit random, at a first glance, but all these choices will be explained, carefully and slowly, throughout the book. The book is organized in four parts, with Part I dealing with generalities and axiomatics, Part II dealing with the case

of “free manifolds”, which is of particular interest, Part III dealing with more general manifolds, and with Part IV going back to the free manifolds, with further results.

Getting back to our axioms and philosophical choices, a first observation, which makes a good point for us, mathematically speaking, is that what you’re holding in hands, before even reading it, suspiciously looks like an elementary graduate textbook. And this is indeed the case, the present book being indeed a graduate textbook, corresponding to what can be taught during a 1-year graduate course. The math will be not that complicated, without of course being trivial either, with everything being accessible with just some basic knowledge of abstract algebra, functional analysis, and probability.

Thus, good point, our math will be quite simple. But whether this simple math can correspond indeed to some simple physics, of the type that can be met in the usual life, remains a difficult question. Getting back to our axioms, our choices there are particularly motivated by nuclear physics, with for instance our assumption that X is compact being inspired by the fact that the strong force is “confined”. Our belief, or rather dream, is that the strong force could one day be understood as being some kind of “free electricity”. But for the moment, we do not have much concrete evidence for this.

Getting back to Earth now, as already mentioned, the present book and the math inside correspond to what can be taught during a 1-year pleasant graduate course, on the theme “Introduction to noncommutative geometry”. There are also two 1-semester graduate course options, based on this book. The first option consists in doing the same thing, but just using Parts I-II, and with a quick look into Parts III-IV as well. As for the second option, this can be a course of type “Introduction to free geometry”, based on Parts II and IV, and with a quick look into Parts I and III as well.

This book is partly based on a number of recent joint papers on quantum groups and noncommutative geometry, and I am particularly grateful to Julien Bichon, for his heavy involvement in the subject. Many thanks go as well to my cats. Their timeless views and opinions, on everyone and everything, have always been of great help.

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Part I

Abstract geometries

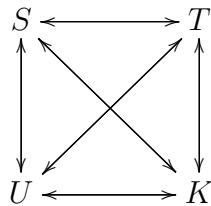
*Hot chili peppers in the blistering sun
Dust on my face and my cape
Me and Magdalena on the run
I think this time we shall escape*

CHAPTER 1

Spheres and tori

1a. Classical geometries

What is geometry? A naive approach to this question suggests that we should have at least a sphere S , a torus T , a unitary group U , and a reflection group K , as starting objects. These basic objects should have relations between them, as follows:



Our idea here will be that of axiomatizing such quadruplets (S, T, U, K) . With this axiomatization in hand, and some classification results as well, we will discuss then the development of each of the geometries that we found. This will be our plan.

Let us first discuss the case of the usual geometry, in \mathbb{R}^N . Basic common sense would suggest to add \mathbb{R}^N itself to our list of objects, and with this addition done, why not erasing afterwards all the other objects, which can be reconstructed anyway from \mathbb{R}^N .

Unfortunately, this is something that we cannot do, in view of our noncommutative geometry goals and motivations. To be more precise, it is well-known that \mathbb{R}^N has no interesting noncommutative analogues. Technically speaking, the problem comes from the fact that \mathbb{R}^N is not compact. We will be back later to this issue.

So, let us go ahead, and construct our quadruplet (S, T, U, K) . We have:

DEFINITION 1.1. *The real sphere, torus, unitary group and reflection group are:*

$$\begin{aligned} S_{\mathbb{R}}^{N-1} &= \left\{ x \in \mathbb{R}^N \mid \sum_i x_i^2 = 1 \right\} \\ T_N &= \left\{ x \in \mathbb{R}^N \mid x_i = \pm \frac{1}{\sqrt{N}} \right\} \\ O_N &= \left\{ U \in M_N(\mathbb{R}) \mid U^t = U^{-1} \right\} \\ H_N &= \left\{ U \in M_N(-1, 0, 1) \mid U^t = U^{-1} \right\} \end{aligned}$$

These are the usual sphere, cube, orthogonal group, and hyperoctahedral group.

Here the superscript $N - 1$ for the sphere, which is very standard, stands for the real dimension as manifold, which is $N - 1$. Also, the $1/\sqrt{N}$ normalization for the cube/torus is there in order to have an embedding $T_N \subset S_{\mathbb{R}}^{N-1}$, this being convenient for us.

Regarding the correspondences between our objects, there are many ways of establishing them, depending on knowledge and taste, but this is not crucial for us. We just need a statement here, in order to get started, so let us formulate things as follows:

THEOREM 1.2. *We have a full set of correspondences, as follows,*

$$\begin{array}{ccc} S_{\mathbb{R}}^{N-1} & \longleftrightarrow & T_N \\ \updownarrow & \swarrow \quad \searrow & \updownarrow \\ O_N & \longleftrightarrow & H_N \end{array}$$

obtained via various results from basic geometry and group theory.

PROOF. As already mentioned, there are several possible solutions to the problem, and all this is not crucial for us. Here is a way of constructing these correspondences:

(1) $S_{\mathbb{R}}^{N-1} \leftrightarrow T_N$. Here T_N comes from $S_{\mathbb{R}}^{N-1}$ via $|x_1| = \dots = |x_N|$, while $S_{\mathbb{R}}^{N-1}$ appears from $T_N \subset \mathbb{R}^N$ by “deleting” this relation, while still keeping $\sum_i x_i^2 = 1$.

(2) $S_{\mathbb{R}}^{N-1} \leftrightarrow O_N$. This comes from the fact that O_N is the isometry group of $S_{\mathbb{R}}^{N-1}$, and that, conversely, $S_{\mathbb{R}}^{N-1}$ appears as $\{Ux \mid U \in O_N\}$, where $x = (1, 0, \dots, 0)$.

(3) $S_{\mathbb{R}}^{N-1} \leftrightarrow H_N$. This is something trickier, but the passage can definitely be obtained, for instance via T_N , by using the constructions in (1) above and (5) below.

(4) $T_N \leftrightarrow O_N$. Here $T_N \simeq \mathbb{Z}_2^N$ is a maximal torus of O_N , and the group O_N itself can be reconstructed from this maximal torus, by using various methods.

(5) $T_N \leftrightarrow H_N$. Here, similarly, $T_N \simeq \mathbb{Z}_2^N$ is a maximal torus of H_N , and the group H_N itself can be reconstructed from this torus as a wreath product, $H_N = T_N \wr S_N$.

(6) $O_N \leftrightarrow H_N$. This is once again something trickier, but the passage can definitely be obtained, for instance via T_N , by using the constructions in (4) and (5) above. \square

The above result is of course something quite non-trivial, and having it understood properly would take some time. However, as already said, we will technically not need all this. Our purpose for the moment is just to explain our (S, T, U, K) philosophy.

As a second basic example of geometry, we have the usual geometry of \mathbb{C}^N . Here, as before, we cannot include the space \mathbb{C}^N itself in our formalism, because this space is not compact, and as already said, we would like to deal with compact spaces only. The corresponding quadruplet (S, T, U, K) can be constructed as follows:

DEFINITION 1.3. *The complex sphere, torus, unitary group and reflection group are:*

$$\begin{aligned} S_{\mathbb{C}}^{N-1} &= \left\{ x \in \mathbb{C}^N \mid \sum_i |x_i|^2 = 1 \right\} \\ \mathbb{T}_N &= \left\{ x \in \mathbb{C}^N \mid |x_i| = \frac{1}{\sqrt{N}} \right\} \\ U_N &= \left\{ U \in M_N(\mathbb{C}) \mid U^* = U^{-1} \right\} \\ K_N &= \left\{ U \in M_N(\mathbb{T} \cup \{0\}) \mid U^* = U^{-1} \right\} \end{aligned}$$

These are the usual complex sphere, torus, unitary group, and complex reflection group.

As before, the superscript $N - 1$ for the sphere does not fit with the rest, but is quite standard, somewhat coming from dimension considerations. We will use it as such. Also, the $1/\sqrt{N}$ factor is there in order to have an embedding $\mathbb{T}_N \subset S_{\mathbb{C}}^{N-1}$.

Also as before, in what regards the correspondences between our objects, there are many ways of establishing them, will all this being not crucial for us. In analogy with Theorem 1.2, let us formulate a second informal statement, as follows:

THEOREM 1.4. *We have a full set of correspondences, as follows,*

$$\begin{array}{ccc} S_{\mathbb{C}}^{N-1} & \longleftrightarrow & \mathbb{T}_N \\ \uparrow & \swarrow & \searrow \\ & & \\ \downarrow & \swarrow & \searrow \\ U_N & \longleftrightarrow & K_N \end{array}$$

obtained via various results from basic geometry and group theory.

PROOF. We follow the proof in the real case, by making adjustments where needed, and with of course the reiterated comment that all this is not crucial for us:

- (1) $S_{\mathbb{C}}^{N-1} \leftrightarrow \mathbb{T}_N$. Same proof as before, using $|x_1| = \dots = |x_N|$.
- (2) $S_{\mathbb{C}}^{N-1} \leftrightarrow U_N$. Here “isometry” must be taken in an affine complex sense.
- (3) $S_{\mathbb{C}}^{N-1} \leftrightarrow K_N$. Trickier as before, best viewed by passing via \mathbb{T}_N .
- (4) $\mathbb{T}_N \leftrightarrow U_N$. Coming from the fact that $\mathbb{T}_N \simeq \mathbb{T}^N$ is a maximal torus of U_N .
- (5) $\mathbb{T}_N \leftrightarrow K_N$. Once again, maximal torus argument, and $K_N = \mathbb{T}_N \wr S_N$.
- (6) $U_N \leftrightarrow K_N$. Trickier as before, best viewed by passing via \mathbb{T}_N . □

As a conclusion, our (S, T, U, K) philosophy seems to work, in the sense that these 4 objects, and the relations between them, encode interesting facts about $\mathbb{R}^N, \mathbb{C}^N$. Our plan in what follows will be that of leaving aside the complete understanding of what has been said above, and going directly for the noncommutative case. We will see that in the noncommutative setting things are more rigid, and therefore, simpler.

1b. Quantum spaces

In order to talk about noncommutative geometry, the idea will be that of defining our quantum spaces X as being abstract manifolds, whose coordinates x_1, \dots, x_N do not necessarily commute. Thus, we are in need of some good algebraic geometry correspondence, between such abstract spaces X , and the corresponding algebras of coordinates A . Following Heisenberg, von Neumann and many others, we will use here the correspondence coming from operator algebras. A first idea is that of using “continuous coordinates”, with each quantum space X corresponding to a certain C^* -algebra, via:

$$A = C(X)$$

With this idea in mind, getting back to our (S, T, U, K) philosophy, we would like to have C^* -algebras with correspondences between them, as follows:

$$\begin{array}{ccc}
 C(S) & \longleftrightarrow & C(T) \\
 \uparrow & \swarrow \quad \searrow & \uparrow \\
 C(U) & \longleftrightarrow & C(K)
 \end{array}$$

A second idea, which is viable as well, and is probably more far-reaching, in view of the loads of uncertainty and probability theory involved with quantum mechanics, but which is technically more complicated to develop, is that of using L^∞ coordinates for our manifolds, according to a formula of the following type:

$$A'' = L^\infty(X)$$

With this second idea in mind, in connection with our (S, T, U, K) program, we would like to have von Neumann algebras with correspondences between them, as follows:

$$\begin{array}{ccc}
 L^\infty(S) & \longleftrightarrow & L^\infty(T) \\
 \updownarrow & \swarrow \quad \searrow & \updownarrow \\
 L^\infty(U) & \longleftrightarrow & L^\infty(K)
 \end{array}$$

In what follows we will use both the above ideas, which are both fruitful. To be more precise, our plan will be that of developing first the continuous theory, and leaving the more advanced aspects, involving von Neumann algebras and probability, for later.

In order to get started now, we will need a number of preliminaries on operators and operator algebras. Let us begin with the following standard definition:

DEFINITION 1.5. *A Hilbert space is a complex vector space H , given with a scalar product $\langle x, y \rangle$, satisfying the following conditions:*

- (1) $\langle x, y \rangle$ is linear in x , and antilinear in y .
- (2) $\overline{\langle x, y \rangle} = \langle y, x \rangle$, for any x, y .
- (3) $\langle x, x \rangle \geq 0$, for any $x \neq 0$.
- (4) H is complete with respect to the norm $\|x\| = \sqrt{\langle x, x \rangle}$.

Here the fact that $\|\cdot\|$ is indeed a norm comes from the Cauchy-Schwarz inequality, $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$, which comes from the fact that the following degree 2 polynomial, with $t \in \mathbb{R}$ and $w \in \mathbb{T}$, being positive, its discriminant must be negative:

$$f(t) = \|x + wty\|^2$$

In finite dimensions, any algebraic basis $\{f_1, \dots, f_N\}$ can be turned into an orthonormal basis $\{e_1, \dots, e_N\}$, by using the Gram-Schmidt procedure. Thus, we have $H \simeq \mathbb{C}^N$, with this latter space being endowed with its usual scalar product:

$$\langle x, y \rangle = \sum_i x_i \bar{y}_i$$

The same happens in infinite dimensions, once again by Gram-Schmidt, coupled if needed with the Zorn lemma, in case our space is really very big. In other words, any Hilbert space has an orthonormal basis $\{e_i\}_{i \in I}$, and we have:

$$H \simeq l^2(I)$$

Of particular interest is the “separable” case, where I is countable. According to the above, there is up to isomorphism only one Hilbert space here, namely:

$$H = l^2(\mathbb{N})$$

All this is, however, quite tricky, and can be a bit misleading. Consider for instance the space $H = L^2[0, 1]$ of square-summable functions $f : [0, 1] \rightarrow \mathbb{C}$, with:

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx$$

This space is of course separable, because we can use the basis $f_n = x^n$ with $n \in \mathbb{N}$, orthogonalized by Gram-Schmidt. However, the orthogonalization procedure is something non-trivial, and so the isomorphism $H \simeq l^2(\mathbb{N})$ that we obtain is something non-trivial as well. Doing some computations here is actually a very good exercise.

In what follows we will be interested in the linear operators $T : H \rightarrow H$ which are bounded. Regarding such operators, we have the following result:

THEOREM 1.6. *Given a Hilbert space H , the linear operators $T : H \rightarrow H$ which are bounded, in the sense that*

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$$

is finite, form a complex algebra $B(H)$, having the following properties:

- (1) $B(H)$ is complete with respect to $\|\cdot\|$, so we have a Banach algebra.
- (2) $B(H)$ has an involution $T \rightarrow T^*$, given by $\langle Tx, y \rangle = \langle x, T^*y \rangle$.

In addition, the norm and involution are related by the formula $\|TT^\| = \|T\|^2$.*

PROOF. The fact that we have indeed an algebra follows from:

$$\|S + T\| \leq \|S\| + \|T\|$$

$$\|\lambda T\| = |\lambda| \cdot \|T\|$$

$$\|ST\| \leq \|S\| \cdot \|T\|$$

(1) Assuming that $\{T_n\} \subset B(H)$ is Cauchy then $\{T_n x\}$ is Cauchy for any $x \in H$, so we can define indeed the limit $T = \lim_{n \rightarrow \infty} T_n$ by setting:

$$Tx = \lim_{n \rightarrow \infty} T_n x$$

(2) Here the existence of T^* comes from the fact that $\varphi(x) = \langle Tx, y \rangle$ being a linear form $H \rightarrow \mathbb{C}$, we must have $\varphi(x) = \langle x, T^*y \rangle$, for a certain vector $T^*y \in H$. Moreover, since this vector is unique, T^* is unique too, and we have as well:

$$(S + T)^* = S^* + T^*$$

$$(\lambda T)^* = \bar{\lambda} T^*$$

$$(ST)^* = T^* S^*$$

$$(T^*)^* = T$$

Observe also that we have indeed $T^* \in B(H)$, because:

$$\begin{aligned} \|T\| &= \sup_{\|x\|=1} \sup_{\|y\|=1} \langle Tx, y \rangle \\ &= \sup_{\|y\|=1} \sup_{\|x\|=1} \langle x, T^*y \rangle \\ &= \|T^*\| \end{aligned}$$

Regarding now the last assertion, we have:

$$\|TT^*\| \leq \|T\| \cdot \|T^*\| = \|T\|^2$$

We have as well the following estimate:

$$\begin{aligned} \|T\|^2 &= \sup_{\|x\|=1} | \langle Tx, Tx \rangle | \\ &= \sup_{\|x\|=1} | \langle x, T^*Tx \rangle | \\ &\leq \|T^*T\| \end{aligned}$$

By replacing $T \rightarrow T^*$ we obtain from this:

$$\|T\|^2 \leq \|TT^*\|$$

Thus, we have proved the last equality, and we are done. \square

Observe that when H comes with an orthonormal basis $\{e_i\}_{i \in I}$, the linear map $T \rightarrow M$ given by $M_{ij} = \langle Te_j, e_i \rangle$ produces an embedding as follows:

$$B(H) \subset M_I(\mathbb{C})$$

Moreover, in this picture the operation $T \rightarrow T^*$ takes a very simple form, namely:

$$(M^*)_{ij} = \overline{M_{ji}}$$

However, as explained before Theorem 1.6, it is better in general not to use bases, and this because very simple spaces like $L^2[0, 1]$ do not have simple bases.

The conditions found in Theorem 1.6 suggest the following definition:

DEFINITION 1.7. *A C^* -algebra is a complex algebra A , having:*

- (1) *A norm $a \rightarrow \|a\|$, making it a Banach algebra.*
- (2) *An involution $a \rightarrow a^*$, satisfying $\|aa^*\| = \|a\|^2$.*

Generally speaking, the elements $a \in A$ are best thought of as being some kind of “generalized operators”, on some Hilbert space which is not present. By using this idea, one can emulate spectral theory in this setting, as follows:

PROPOSITION 1.8. *Given $a \in A$, define its spectrum as being the set*

$$\sigma(a) = \left\{ \lambda \in \mathbb{C} \mid a - \lambda \notin A^{-1} \right\}$$

and its spectral radius $\rho(a)$ as the radius of the smallest centered disk containing $\sigma(a)$.

- (1) *The spectrum of a norm one element is in the unit disk.*
- (2) *The spectrum of a unitary element ($a^* = a^{-1}$) is on the unit circle.*
- (3) *The spectrum of a self-adjoint element ($a = a^*$) consists of real numbers.*
- (4) *The spectral radius of a normal element ($aa^* = a^*a$) is equal to its norm.*

PROOF. Our first claim is that for any polynomial $f \in \mathbb{C}[X]$, and more generally for any rational function $f \in \mathbb{C}(X)$ having poles outside $\sigma(a)$, we have:

$$\sigma(f(a)) = f(\sigma(a))$$

This indeed something well-known for the usual matrices. In the general case, assume first that we have a polynomial, $f \in \mathbb{C}[X]$. If we pick an arbitrary number $\lambda \in \mathbb{C}$, and write $f(X) - \lambda = c(X - r_1) \dots (X - r_k)$, we have then, as desired:

$$\begin{aligned} \lambda \notin \sigma(f(a)) &\iff f(a) - \lambda \in A^{-1} \\ &\iff c(a - r_1) \dots (a - r_k) \in A^{-1} \\ &\iff a - r_1, \dots, a - r_k \in A^{-1} \\ &\iff r_1, \dots, r_k \notin \sigma(a) \\ &\iff \lambda \notin f(\sigma(a)) \end{aligned}$$

Assume now that we are in the general case, $f \in \mathbb{C}(X)$. We pick $\lambda \in \mathbb{C}$, we write $f = P/Q$, and we set $F = P - \lambda Q$. By using the above finding, we obtain, as desired:

$$\begin{aligned} \lambda \in \sigma(f(a)) &\iff F(a) \notin A^{-1} \\ &\iff 0 \in \sigma(F(a)) \\ &\iff 0 \in F(\sigma(a)) \\ &\iff \exists \mu \in \sigma(a), F(\mu) = 0 \\ &\iff \lambda \in f(\sigma(a)) \end{aligned}$$

Regarding now the assertions in the statement, these basically follows from this:

- (1) This comes from the following formula, valid when $\|a\| < 1$:

$$\frac{1}{1 - a} = 1 + a + a^2 + \dots$$

- (2) Assuming $a^* = a^{-1}$, we have the following norm computations:

$$\begin{aligned} \|a\| &= \sqrt{\|aa^*\|} = \sqrt{1} = 1 \\ \|a^{-1}\| &= \|a^*\| = \|a\| = 1 \end{aligned}$$

If we denote by D the unit disk, we obtain from this, by using (1):

$$\|a\| = 1 \implies \sigma(a) \subset D$$

$$\|a^{-1}\| = 1 \implies \sigma(a^{-1}) \subset D$$

On the other hand, by using the rational function $f(z) = z^{-1}$, we have:

$$\sigma(a^{-1}) \subset D \implies \sigma(a) \subset D^{-1}$$

Now by putting everything together we obtain, as desired:

$$\sigma(a) \subset D \cap D^{-1} = \mathbb{T}$$

(3) This follows by using (2), and the following rational function, depending on a parameter $t \in \mathbb{R}$:

$$f(z) = \frac{z + it}{z - it}$$

Indeed, for $t \gg 0$ the element $f(a)$ is well-defined, and we have:

$$\left(\frac{a + it}{a - it}\right)^* = \frac{a - it}{a + it} = \left(\frac{a + it}{a - it}\right)^{-1}$$

Thus the element $f(a)$ is a unitary, and by using (1) its spectrum is contained in \mathbb{T} . We conclude from this that we have:

$$f(\sigma(a)) = \sigma(f(a)) \subset \mathbb{T}$$

But this shows that we have $\sigma(a) \subset f^{-1}(\mathbb{T}) = \mathbb{R}$, as desired.

(4) We already know that we have $\rho(a) \leq \|a\|$, for any $a \in A$. For the reverse inequality, when a is normal, we fix a number $\rho > \rho(a)$. We have then:

$$\begin{aligned} \int_{|z|=\rho} \frac{z^n}{z - a} dz &= \int_{|z|=\rho} \sum_{k=0}^{\infty} z^{n-k-1} a^k dz \\ &= \sum_{k=0}^{\infty} \left(\int_{|z|=\rho} z^{n-k-1} dz \right) a^k \\ &= \sum_{k=0}^{\infty} \delta_{n,k+1} a^k \\ &= a^{n-1} \end{aligned}$$

By applying the norm and taking n -th roots we obtain from this formula, modulo some elementary manipulations, the following estimate:

$$\rho \geq \lim_{n \rightarrow \infty} \|a^n\|^{1/n}$$

Now recall that ρ was by definition an arbitrary number satisfying $\rho > \rho(a)$. Thus, we have obtained the following estimate, valid for any $a \in A$:

$$\rho(a) \geq \lim_{n \rightarrow \infty} \|a^n\|^{1/n}$$

In order to finish, we must prove that when a is normal, this estimate implies the missing estimate, namely $\rho(a) \geq \|a\|$. We can proceed in two steps, as follows:

Step 1. In the case $a = a^*$ we have $\|a^n\| = \|a\|^n$ for any exponent of the form $n = 2^k$, by using the C^* -algebra condition $\|aa^*\| = \|a\|^2$, and by taking n -th roots we get:

$$\rho(a) \geq \|a\|$$

Thus, we are done with the self-adjoint case, with the result $\rho(a) = \|a\|$.

Step 2. In the general normal case $aa^* = a^*a$ we have $a^n(a^n)^* = (aa^*)^n$, and by using this, along with the result from Step 1, applied to aa^* , we obtain:

$$\begin{aligned} \rho(a) &\geq \lim_{n \rightarrow \infty} \|a^n\|^{1/n} \\ &= \sqrt{\lim_{n \rightarrow \infty} \|a^n(a^n)^*\|^{1/n}} \\ &= \sqrt{\lim_{n \rightarrow \infty} \|(aa^*)^n\|^{1/n}} \\ &= \sqrt{\rho(aa^*)} \\ &= \sqrt{\|a\|^2} \\ &= \|a\| \end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

We can now formulate a key theorem, from [62], as follows:

THEOREM 1.9 (Gelfand). *If X is a compact space, the algebra $C(X)$ of continuous functions $f : X \rightarrow \mathbb{C}$ is a commutative C^* -algebra, with structure as follows:*

(1) *The norm is the usual sup norm:*

$$\|f\| = \sup_{x \in X} |f(x)|$$

(2) *The involution is the usual involution:*

$$f^*(x) = \overline{f(x)}$$

Conversely, any commutative C^ -algebra is of the form $C(X)$, with its “spectrum” $X = \text{Spec}(A)$ appearing as the space of characters $\chi : A \rightarrow \mathbb{C}$.*

PROOF. In what regards the first assertion, almost everything here is trivial. We have indeed a commutative algebra, with norm and involution, the Cauchy sequences inside are well-known to converge, and the condition $\|ff^*\| = \|f\|^2$ is satisfied. Conversely, given a

commutative C^* -algebra A , we can define X to be the set of characters $\chi : A \rightarrow \mathbb{C}$, with the topology making continuous all the evaluation maps $ev_a : \chi \rightarrow \chi(a)$. Then X is a compact space, and $a \rightarrow ev_a$ is a morphism of algebras:

$$ev : A \rightarrow C(X)$$

We first prove that ev is involutive. We use the following formula:

$$a = \frac{a + a^*}{2} - i \cdot \frac{i(a - a^*)}{2}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements a . But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in \mathbb{R} .

Since A is commutative, each element is normal, so ev is isometric:

$$\|ev_a\| = \rho(a) = \|a\|$$

It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. \square

The Gelfand theorem suggests formulating the following definition:

DEFINITION 1.10. *Given a C^* -algebra A , not necessarily commutative, we write*

$$A = C(X)$$

and call the abstract object X a “compact quantum space”.

In other words, the category of the compact quantum spaces will be by definition the category of the C^* -algebras, with the arrows reversed. We will be back to this, with examples, and with some technical comments as well.

Let us discuss now the other basic result regarding the C^* -algebras, namely the GNS representation theorem. We will need some more spectral theory, as follows:

PROPOSITION 1.11. *For a normal element $a \in A$, the following are equivalent:*

- (1) *a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.*
- (2) *$a = b^2$, for some $b \in A$ satisfying $b = b^*$.*
- (3) *$a = cc^*$, for some $c \in A$.*

PROOF. This is something very standard, as follows:

(1) \implies (2) Since our element a is normal the algebra $\langle a \rangle$ that is generated is commutative, and by using the Gelfand theorem, we can set $b = \sqrt{a}$.

(2) \implies (3) This is trivial, because we can set $c = b$.

(3) \implies (1) We proceed by contradiction. By multiplying c by a suitable element of $\langle cc^* \rangle$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$. By writing now $d = x + iy$ with $x = x^*, y = y^*$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$. But this contradicts the elementary fact that $\sigma(dd^*), \sigma(d^*d)$ must coincide outside $\{0\}$, which can be checked by explicit inversion. \square

Here is now the representation theorem from [63], along with the idea of the proof:

THEOREM 1.12 (GNS theorem). *Let A be a C^* -algebra.*

- (1) *A appears as a closed $*$ -subalgebra $A \subset B(H)$, for some Hilbert space H .*
- (2) *When A is separable (usually the case), H can be chosen to be separable.*
- (3) *When A is finite dimensional, H can be chosen to be finite dimensional.*

PROOF. Let us first discuss the commutative case, $A = C(X)$. Our claim here is that if we pick a probability measure on X , we have an embedding as follows:

$$C(X) \subset B(L^2(X)) \quad , \quad f \rightarrow (g \rightarrow fg)$$

Indeed, given a function $f \in C(X)$, consider the operator $T_f(g) = fg$, acting on $H = L^2(X)$. Observe that T_f is indeed well-defined, and bounded as well, because:

$$\|fg\|_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 dx} \leq \|f\|_\infty \|g\|_2$$

The application $f \rightarrow T_f$ being linear, involutive, continuous, and injective as well, we obtain in this way a C^* -algebra embedding $C(X) \subset B(H)$, as claimed.

In general, we can use a similar idea, with the algebraic aspects being fine, and with the positivity issues being taken care of by Proposition 1.8 and Proposition 1.11.

Indeed, assuming that a linear form $\varphi : A \rightarrow \mathbb{C}$ has some suitable positivity properties, making it analogous to the integration functionals $\int_X : A \rightarrow \mathbb{C}$ from the commutative case, we can define a scalar product on A , by the following formula:

$$\langle a, b \rangle = \varphi(ab^*)$$

By completing we obtain a Hilbert space H , and we have an embedding as follows:

$$A \subset B(H) \quad , \quad a \rightarrow (b \rightarrow ab)$$

Thus we obtain the assertion (1), and a careful examination of the construction $A \rightarrow H$, outlined above, shows that the assertions (2,3) are in fact proved as well. \square

1c. Free spheres

With the above formalism is hand, we can go ahead, and construct two free quadruplets (S, T, U, K) , in analogy with those corresponding to the classical real and complex geometries. Let us begin with the spheres. Following [4], [28], we have:

DEFINITION 1.13. *We have free real and complex spheres, defined via*

$$C(S_{\mathbb{R},+}^{N-1}) = C^* \left(x_1, \dots, x_N \mid x_i = x_i^*, \sum_i x_i^2 = 1 \right)$$

$$C(S_{\mathbb{C},+}^{N-1}) = C^* \left(x_1, \dots, x_N \mid \sum_i x_i x_i^* = \sum_i x_i^* x_i = 1 \right)$$

where the symbol C^* stands for universal enveloping C^* -algebra.

All this deserves some explanations. Given an integer $N \in \mathbb{N}$, consider the free complex algebra on $2N$ variables, denoted x_1, \dots, x_N and x_1^*, \dots, x_N^* :

$$A = \langle x_1, \dots, x_N, x_1^*, \dots, x_N^* \rangle$$

In other words, the elements of A are the formal linear combinations, with complex coefficients, of products between our variables x_i, x_i^* , and of the unit 1.

This algebra has an involution $*$: $A \rightarrow A$, given by:

$$x_i \leftrightarrow x_i^*$$

Now let us consider the following $*$ -algebra quotients of our $*$ -algebra A :

$$A_R = A / \langle x_i = x_i^*, \sum_i x_i^2 = 1 \rangle$$

$$A_C = A / \langle \sum_i x_i x_i^* = \sum_i x_i^* x_i = 1 \rangle$$

Since the first relations imply the second ones, we have quotient maps as follows:

$$A \rightarrow A_C \rightarrow A_R$$

Our claim now is both A_C, A_R admit enveloping C^* -algebras, in the sense that the biggest C^* -norms on these $*$ -algebras are bounded. We only have to check this for the bigger algebra A_C . But here, our claim follows from the following estimate:

$$\|x_i\|^2 = \|x_i x_i^*\| \leq \left\| \sum_i x_i x_i^* \right\| = 1$$

Summarizing, our claim is proved, so we can define $C(S_{\mathbb{R},+}^{N-1}), C(S_{\mathbb{C},+}^{N-1})$ as being the enveloping C^* -algebras of A_R, A_C , and so Definition 1.13 makes sense.

In order to formulate some results, let us introduce as well:

DEFINITION 1.14. *Given a compact quantum space X , its classical version is the usual compact space $X_{class} \subset X$ obtained by dividing $C(X)$ by its commutator ideal:*

$$C(X_{class}) = C(X)/I \quad , \quad I = \langle [a, b] \rangle$$

In this situation, we also say that X appears as a “liberation” of X .

In other words, the space X_{class} appears as the Gelfand spectrum of the commutative C^* -algebra $C(X)/I$. Observe in particular that X_{class} is indeed a classical space. As a first result now, regarding the above free spheres, we have:

THEOREM 1.15. *We have embeddings of compact quantum spaces, as follows,*

$$\begin{array}{ccc} S_{\mathbb{R},+}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\ \uparrow & & \uparrow \\ S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{C}}^{N-1} \end{array}$$

and the spaces on top appear as liberations of the spaces on the bottom.

PROOF. The first assertion, regarding the inclusions, comes from the fact that at the level of the associated C^* -algebras, we have surjective maps, as follows:

$$\begin{array}{ccc} C(S_{\mathbb{R},+}^{N-1}) & \longleftarrow & C(S_{\mathbb{C},+}^{N-1}) \\ \downarrow & & \downarrow \\ C(S_{\mathbb{R}}^{N-1}) & \longleftarrow & C(S_{\mathbb{C}}^{N-1}) \end{array}$$

For the second assertion, we must establish the following isomorphisms, where the symbol C_{comm}^* stands for “universal commutative C^* -algebra generated by”:

$$\begin{aligned} C(S_{\mathbb{R}}^{N-1}) &= C_{comm}^* \left(x_1, \dots, x_N \mid x_i = x_i^*, \sum_i x_i^2 = 1 \right) \\ C(S_{\mathbb{C}}^{N-1}) &= C_{comm}^* \left(x_1, \dots, x_N \mid \sum_i x_i x_i^* = \sum_i x_i^* x_i = 1 \right) \end{aligned}$$

As a first observation, it is enough to establish the second isomorphism, because the first one will follow from it, by dividing by the relations $x_i = x_i^*$.

So, consider the second universal commutative C^* -algebra A constructed above. Since the standard coordinates on $S_{\mathbb{C}}^{N-1}$ satisfy the defining relations for A , we have a quotient map of as follows, mapping standard coordinates to standard coordinates:

$$A \rightarrow C(S_{\mathbb{C}}^{N-1})$$

Conversely, let us write $A = C(S)$, by using the Gelfand theorem. The variables x_1, \dots, x_N become in this way true coordinates, providing us with an embedding $S \subset \mathbb{C}^N$. Also, the quadratic relations become $\sum_i |x_i|^2 = 1$, so we have $S \subset S_{\mathbb{C}}^{N-1}$. Thus, we have a quotient map $C(S_{\mathbb{C}}^{N-1}) \rightarrow A$, as desired, and this gives all the results. \square

Summarizing, we are done with the spheres. Before getting into tori, let us talk about algebraic manifolds. By using the free spheres constructed above, we can formulate:

DEFINITION 1.16. *A real algebraic manifold $X \subset S_{\mathbb{C},+}^{N-1}$ is a closed quantum subspace defined, at the level of the corresponding C^* -algebra, by a formula of type*

$$C(X) = C(S_{\mathbb{C},+}^{N-1}) / \langle f_i(x_1, \dots, x_N) = 0 \rangle$$

for certain family of noncommutative polynomials, as follows:

$$f_i \in \mathbb{C} \langle x_1, \dots, x_N \rangle$$

We denote by $\mathcal{C}(X)$ the $*$ -subalgebra of $C(X)$ generated by the coordinates x_1, \dots, x_N .

As a basic example of such a manifold, we have the free real sphere $S_{\mathbb{R},+}^{N-1}$. The classical spheres $S_{\mathbb{C}}^{N-1}$, $S_{\mathbb{R}}^{N-1}$, and their real submanifolds, are covered as well by this formalism. At the level of the general theory, we have the following version of the Gelfand theorem, which is something very useful, and that we will use many times in what follows:

THEOREM 1.17. *If $X \subset S_{\mathbb{C},+}^{N-1}$ is an algebraic manifold, as above, we have*

$$X_{class} = \left\{ x \in S_{\mathbb{C}}^{N-1} \mid f_i(x_1, \dots, x_N) = 0 \right\}$$

and X appears as a liberation of X_{class} .

PROOF. This is something that we already met, in the context of the free spheres. In general, the proof is similar, by using the Gelfand theorem. Indeed, if we denote by X'_{class} the manifold constructed in the statement, then we have a quotient map of C^* -algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class}) \rightarrow C(X'_{class})$$

Conversely now, from $X \subset S_{\mathbb{C},+}^{N-1}$ we obtain $X_{class} \subset S_{\mathbb{C}}^{N-1}$. Now since the relations defining X'_{class} are satisfied by X_{class} , we obtain an inclusion $X_{class} \subset X'_{class}$. Thus, at the level of algebras of continuous functions, we have a quotient map of C^* -algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X'_{class}) \rightarrow C(X_{class})$$

Thus, we have constructed a pair of inverse morphisms, and we are done. \square

Finally, once again at the level of the general theory, we have:

DEFINITION 1.18. *We agree to identify two real algebraic submanifolds $X, Y \subset S_{\mathbb{C},+}^{N-1}$ when we have a $*$ -algebra isomorphism between $*$ -algebras of coordinates*

$$f : \mathcal{C}(Y) \rightarrow \mathcal{C}(X)$$

mapping standard coordinates to standard coordinates.

We will see later on some precise reasons for making this convention, in connection with the group duals, in the non-amenable case.

1d. Free tori

Let us go back now to our general (S, T, U, K) program. Now that we are done with the free spheres, we can introduce as well free tori, as follows:

DEFINITION 1.19. *We have free real and complex tori, defined via*

$$C(T_N^+) = C^* \left(x_1, \dots, x_N \mid x_i = x_i^*, x_i^2 = \frac{1}{N} \right)$$

$$C(\mathbb{T}_N^+) = C^* \left(x_1, \dots, x_N \mid x_i x_i^* = x_i^* x_i = \frac{1}{N} \right)$$

where the symbol C^ stands for universal enveloping C^* -algebra.*

The fact that these tori are indeed well-defined comes from the fact that they are algebraic manifolds, in the sense of Definition 1.16 above. In fact, we have:

PROPOSITION 1.20. *We have inclusions of algebraic manifolds, as follows:*

$$\begin{array}{ccc} S_{\mathbb{R},+}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\ \uparrow & & \uparrow \\ T_N^+ & \longrightarrow & \mathbb{T}_N^+ \end{array}$$

In addition, this is an intersection diagram, in the sense that $T_N^+ = \mathbb{T}_N^+ \cap S_{\mathbb{R},+}^{N-1}$.

PROOF. All this is clear indeed, by using the equivalence relation in Definition 1.18, in order to get rid of functional analytic issues at the C^* -algebra level. \square

In analogy with Theorem 1.15, we have the following result:

THEOREM 1.21. *We have inclusions of algebraic manifolds, as follows,*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

and the manifolds on top appear as liberations of those of the bottom.

PROOF. This follows exactly as Theorem 1.15, and best here is to invoke Theorem 1.17 above, which is there precisely for dealing with such situations. \square

Summarizing, we have free spheres and tori, having quite similar properties.

Let us further study the tori. Up to a rescaling, these are given by algebras generated by unitaries, so studying the algebras generated by unitaries will be our next task. The point is that we have many such algebras, coming from the following construction:

THEOREM 1.22. *Let Γ be a discrete group, and consider the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by the fact that all group elements are unitaries, $g^* = g^{-1}$.*

- (1) *The maximal C^* -seminorm on $\mathbb{C}[\Gamma]$ is a C^* -norm, and the closure of $\mathbb{C}[\Gamma]$ with respect to this norm is a C^* -algebra, denoted $C^*(\Gamma)$.*
- (2) *When Γ is abelian, we have an isomorphism $C^*(\Gamma) \simeq C(G)$, where $G = \widehat{\Gamma}$ is its Pontrjagin dual, formed by the characters $\chi : \Gamma \rightarrow \mathbb{T}$.*

PROOF. All this is very standard, the idea being as follows:

(1) In order to prove the result, we must find a $*$ -algebra embedding $\mathbb{C}[\Gamma] \subset B(H)$, with H being a Hilbert space. For this purpose, consider the space $H = \ell^2(\Gamma)$, having $\{h\}_{h \in \Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi : \mathbb{C}[\Gamma] \subset B(H) \quad , \quad \pi(g)(h) = gh$$

Indeed, since $\pi(g)$ maps the basis $\{h\}_{h \in \Gamma}$ into itself, this operator is well-defined, bounded, and is an isometry. It is also clear from the formula $\pi(g)(h) = gh$ that $g \rightarrow \pi(g)$ is a morphism of algebras, and since this morphism maps the unitaries $g \in \Gamma$ into isometries, this is a morphism of $*$ -algebras. Finally, the faithfulness of π is clear.

(2) Since Γ is abelian, the corresponding group algebra $A = C^*(\Gamma)$ is commutative. Thus, we can apply the Gelfand theorem, and we obtain $A = C(X)$, with:

$$X = \text{Spec}(A)$$

But the spectrum $X = \text{Spec}(A)$, consisting of the characters $\chi : C^*(\Gamma) \rightarrow \mathbb{C}$, can be identified with the Pontrjagin dual $G = \widehat{\Gamma}$, and this gives the result. \square

The above result suggests the following definition:

DEFINITION 1.23. *Given a discrete group Γ , the compact quantum space G given by*

$$C(G) = C^*(\Gamma)$$

is called abstract dual of Γ , and is denoted $G = \widehat{\Gamma}$.

This is in fact something which is not very satisfactory, in general, due to amenability issues. However, in the case of the finitely generated discrete groups $\Gamma = \langle g_1, \dots, g_N \rangle$, which is the one that we are interested in here, the corresponding duals appear as algebraic submanifolds $\widehat{\Gamma} \subset S_{\mathbb{C},+}^{N-1}$, and the notion of equivalence from Definition 1.18 is precisely the one that we need, identifying full and reduced group algebras.

We can now refine our findings about tori, as follows:

THEOREM 1.24. *The basic tori are all group duals, as follows,*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array} = \begin{array}{ccc} \widehat{\mathbb{Z}_2^{*N}} & \longrightarrow & \widehat{F}_N \\ \uparrow & & \uparrow \\ \mathbb{Z}_2^N & \longrightarrow & \mathbb{T}^N \end{array}$$

where F_N is the free group on N generators, and $*$ is a group-theoretical free product.

PROOF. The basic tori appear indeed as group duals, and together with the Fourier transform identifications from Theorem 1.22 (2), this gives the result. \square

Following [17], let us try now to understand the correspondence between the spheres S and tori T . We first have the following result, summarizing our knowledge so far:

THEOREM 1.25. *The four main quantum spheres produce the main quantum tori*

$$\begin{array}{ccc} S_{\mathbb{R},+}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\ \uparrow & & \uparrow \\ S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{C}}^{N-1} \end{array} \rightarrow \begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

via the formula $T = S \cap \mathbb{T}_N^+$, with the intersection being taken inside $S_{\mathbb{C},+}^{N-1}$.

PROOF. This comes from the above results, the situation being as follows:

(1) Free complex case. Here the formula in the statement reads $\mathbb{T}_N^+ = S_{\mathbb{C},+}^{N-1} \cap \mathbb{T}_N^+$. But this is something trivial, because we have $\mathbb{T}_N^+ \subset S_{\mathbb{C},+}^{N-1}$.

(2) Free real case. Here the formula in the statement reads $T_N^+ = S_{\mathbb{R},+}^{N-1} \cap \mathbb{T}_N^+$. But this is something that we already know, from Proposition 1.20 above.

(3) Classical complex case. Here the formula in the statement reads $\mathbb{T}_N = S_{\mathbb{C}}^{N-1} \cap \mathbb{T}_N^+$. But this is clear as well, the classical version of \mathbb{T}_N^+ being \mathbb{T}_N .

(4) Classical real case. Here the formula in the statement reads $T_N = S_{\mathbb{R}}^{N-1} \cap \mathbb{T}_N^+$. But this follows by intersecting the formulae from the proof of (2) and (3). \square

The correspondence $S \rightarrow T$ found above is not the only one. In order to discuss this, let us start with a general result, as follows:

THEOREM 1.26. *Given an algebraic manifold $X \subset S_{\mathbb{C},+}^{N-1}$, the category of toral subgroups $T \subset \mathbb{T}_N^+$ acting affinely on X , in the sense that $\Phi(x_i) = x_i \otimes g_i$ defines a morphism*

$$\Phi : C(X) \rightarrow C(X) \otimes C(T)$$

has a universal object, denoted $T^+(X)$, and called toral isometry group of X .

PROOF. In order to prove the result, assume that $X \subset S_{\mathbb{C},+}^{N-1}$ comes as follows:

$$C(X) = C(S_{\mathbb{C},+}^{N-1}) / \left\langle f_{\alpha}(x_1, \dots, x_N) = 0 \right\rangle$$

Consider now the following variables:

$$X_i = x_i \otimes g_i \in C(X) \otimes C(\mathbb{T}_N^+)$$

Our claim is that the torus $T = T^+(X)$ in the statement appears as follows:

$$C(T) = C(\mathbb{T}_N^+) / \left\langle f_{\alpha}(X_1, \dots, X_N) = 0 \right\rangle$$

In order to prove this claim, we have to clarify how the relations $f_{\alpha}(X_1, \dots, X_N) = 0$ are interpreted inside $C(\mathbb{T}_N^+)$, and then show that T is indeed a toral subgroup.

So, pick one of the defining polynomials, $f = f_{\alpha}$, and write it as follows:

$$f(x_1, \dots, x_N) = \sum_r \sum_{i_1^r \dots i_{s_r}^r} \lambda_r \cdot x_{i_1^r} \dots x_{i_{s_r}^r}$$

With $X_i = x_i \otimes g_i$ as above, we have the following formula:

$$f(X_1, \dots, X_N) = \sum_r \sum_{i_1^r \dots i_{s_r}^r} \lambda_r x_{i_1^r} \dots x_{i_{s_r}^r} \otimes g_{i_1^r} \dots g_{i_{s_r}^r}$$

Since the variables on the right span a certain finite dimensional space, the relations $f(X_1, \dots, X_N) = 0$ correspond to certain relations between the variables g_i . Thus, we have indeed a subspace $T \subset \mathbb{T}_N^+$, with a universal map, as follows:

$$\Phi : C(X) \rightarrow C(X) \otimes C(T)$$

In order to show now that T is a group dual, consider the following elements:

$$g'_i = g_i \otimes g_i \quad , \quad X'_i = x_i \otimes g'_i$$

Then from $f(X_1, \dots, X_N) = 0$ we deduce that, with $\Delta(g) = g \otimes g$, we have:

$$f(X'_1, \dots, X'_N) = (id \otimes \Delta)f(X_1, \dots, X_N) = 0$$

Thus we can map $g_i \rightarrow g'_i$, and it follows that T is a group dual, as desired. \square

We will be back to this in chapter 3 below, with a full discussion of the various types of quantum isometries an algebraic manifold $X \subset S_{\mathbb{C},+}^{N-1}$ can have. Now with the above toral isometry group formalism in hand, we can formulate a second result regarding the spheres and tori, which is complementary to Theorem 1.25, as follows:

THEOREM 1.27. *The four main quantum spheres produce via*

$$T = T^+(S)$$

the corresponding four main quantum tori.

PROOF. This is something elementary, which can be established as follows:

(1) Free complex case. Here there is nothing to be proved, because we obviously have an action $\mathbb{T}_N^+ \curvearrowright S_{\mathbb{C},+}^{N-1}$, and this action can only be universal.

(2) Free real case. Here the situation is similar, because we have an obvious action $T_N^+ \curvearrowright S_{\mathbb{R},+}^{N-1}$, and it is clear that this action can only be universal.

(3) Classical complex case. Once again, we have a similar situation here, with the obvious action, namely $\mathbb{T}_N \curvearrowright S_{\mathbb{C}}^{N-1}$, being easily seen to be universal.

(4) Classical real case. Here the obvious action, namely $T_N \curvearrowright S_{\mathbb{R}}^{N-1}$, is universal as well, the reasons for this coming from (2) and (3) above. \square

As a conclusion now, following [17], we can formulate:

DEFINITION 1.28. *A “baby noncommutative geometry” consists of a quantum sphere S and a quantum torus T , which are by definition algebraic manifolds as follows,*

$$S_{\mathbb{R}}^{N-1} \subset S \subset S_{\mathbb{C},+}^{N-1}$$

$$T_N \subset T \subset \mathbb{T}_N^+$$

which must be subject to the following compatibility conditions,

$$T = S \cap \mathbb{T}_N^+ = T^+(S)$$

with the intersection being taken inside $S_{\mathbb{C},+}^{N-1}$, and T^+ being the toral isometry group.

With this notion in hand, our main results so far can be summarized as follows:

THEOREM 1.29. *We have 4 baby noncommutative geometries, as follows,*

$$\begin{array}{ccc} \mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\ \uparrow & & \uparrow \\ \mathbb{R}^N & \longrightarrow & \mathbb{C}^N \end{array}$$

with each symbol \mathbb{K}_\times^N standing for the corresponding pair (S, T) .

PROOF. This follows indeed from Theorem 1.25 and Theorem 1.27. \square

In what follows we will extend our baby theory, with pairs of type (U, K) , consisting of unitary and reflection groups. This will lead to a theory which is more advanced.

1e. Exercises

As a first exercise, which is an obvious one in view of the above, we have:

EXERCISE 1.30. *Establish correspondences as follows*

$$\begin{array}{ccc} S_{\mathbb{R}}^{N-1} & \longleftrightarrow & T_N \\ \uparrow & \swarrow & \searrow \\ & & \\ \downarrow & \swarrow & \searrow \\ O_N & \longleftrightarrow & H_N \end{array}$$

by using results from basic geometry and group theory.

We have already talked about all this, in the beginning of the present chapter, and the problem now is to have the thing done.

As a second exercise, which is obvious as well in view of the above, we have:

EXERCISE 1.31. *Establish correspondences as follows*

$$\begin{array}{ccc} S_{\mathbb{C}}^{N-1} & \longleftrightarrow & \mathbb{T}_N \\ \uparrow & \swarrow & \searrow \\ & & \\ \downarrow & \swarrow & \searrow \\ U_N & \longleftrightarrow & K_N \end{array}$$

by using results from basic geometry and group theory.

As before, this is something that we discussed, and the problem is to have it done.

Regarding now linear algebra, operator theory, spectral theory and operator algebras, which are key for the considerations to follow, the best homework is to pick up some books, and read more. Here is however an instructive exercise, that we have here:

EXERCISE 1.32. *Prove that given two complex matrices $A, B \in M_N(\mathbb{C})$ we have:*

$$\sigma(AB) = \sigma(BA)$$

More generally, with σ^+ being the set of eigenvalues, taken with multiplicities, we have:

$$\sigma^+(AB) = \sigma^+(BA)$$

Prove also that given two arbitrary operators $S, T \in B(H)$ we have

$$\sigma(ST) \cup \{0\} = \sigma(TS) \cup \{0\}$$

and that the spectra of ST, TS do not necessarily coincide on the point 0.

This is something quite long, but very instructive, and having all this done, in detail, will certainly bring you to a good level in linear algebra and spectral theory.

In relation now with the free spheres, tori and other algebraic manifolds, we have:

EXERCISE 1.33. *Make a list of interesting examples of real algebraic manifolds*

$$X \subset S_{\mathbb{C}}^{N-1}$$

based on your classical geometry knowledge, and then a list of examples of manifolds

$$X \subset S_{\mathbb{C},+}^{N-1}$$

based on your noncommutative geometry knowledge, or intuition.

As a hint, many interesting answers to the first question can be constructed by taking $N = n^2$, or more generally N to be a composite number. As for the second question, things here are a bit up to you, depending on whether you have or not some previous knowledge here. At worst, just try “liberating” the answers to the first question.

Finally, here is a more difficult exercise, in relation with the tori:

EXERCISE 1.34. *Prove that the quotient map*

$$C^*(F_N) \rightarrow C_{red}^*(F_N)$$

is not an isomorphism.

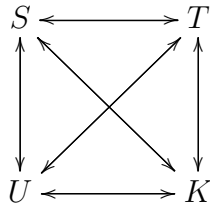
This is something quite difficult, so the exercise here is rather that of finding a proof for this in the literature, and then writing down a brief account of that proof, with the main ideas explained. All this is important, in connection with Definition 1.18 above.

CHAPTER 2

Quantum groups

2a. Quantum groups

We have seen so far that the pairs sphere/torus (S, T) corresponding to the real and complex geometries, of $\mathbb{R}^N, \mathbb{C}^N$, have some natural free analogues. Our objective now will be that of adding to the picture a pair of quantum groups (U, K) , as to reach to a quadruplet of objects (S, T, U, K) , with relations between them, as follows:



The quantum group axioms that we need, coming from [99], are as follows:

DEFINITION 2.1. *A Woronowicz algebra is a C^* -algebra A , given with a unitary matrix $u \in M_N(A)$ whose coefficients generate A , such that the formulae*

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$$

$$\varepsilon(u_{ij}) = \delta_{ij}$$

$$S(u_{ij}) = u_{ji}^*$$

define morphisms of C^ -algebras as follows,*

$$\Delta : A \rightarrow A \otimes A$$

$$\varepsilon : A \rightarrow \mathbb{C}$$

$$S : A \rightarrow A^{opp}$$

called comultiplication, counit and antipode.

In this definition $A \otimes A$ is the universal C^* -algebraic completion of the usual algebraic tensor product of A with itself, and A^{opp} is the opposite C^* -algebra, with multiplication $a \cdot b = ba$. The reasons for using A^{opp} instead of A itself will become clear later on.

We say that a Woronowicz algebra A is cocommutative when we have $\Sigma\Delta = \Delta$, where $\Sigma(a \otimes b) = b \otimes a$ is the flip operator. With this convention, we have the following result, which justifies the terminology and axioms:

THEOREM 2.2. *The following are Woronowicz algebras:*

- (1) $C(G)$, with $G \subset U_N$ compact Lie group. Here the structural maps are:

$$\Delta(\varphi) = (g, h) \rightarrow \varphi(gh)$$

$$\varepsilon(\varphi) = \varphi(1)$$

$$S(\varphi) = g \rightarrow \varphi(g^{-1})$$

- (2) $C^*(\Gamma)$, with $F_N \rightarrow \Gamma$ finitely generated group. Here the structural maps are:

$$\Delta(g) = g \otimes g$$

$$\varepsilon(g) = 1$$

$$S(g) = g^{-1}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

PROOF. In both cases, we have to exhibit a certain matrix u :

- (1) Here we can use the matrix $u = (u_{ij})$ formed by matrix coordinates of G :

$$g = \begin{pmatrix} u_{11}(g) & \dots & u_{1N}(g) \\ \vdots & & \vdots \\ u_{N1}(g) & \dots & u_{NN}(g) \end{pmatrix}$$

- (2) Here we can use the diagonal matrix formed by generators of Γ :

$$u = \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_N \end{pmatrix}$$

Finally, the last assertion follows from the Gelfand theorem, in the commutative case. In the cocommutative case, this is something more technical, to be discussed later. \square

In general now, the structural maps Δ, ε, S have the following properties:

PROPOSITION 2.3. *Let (A, u) be a Woronowicz algebra.*

- (1) Δ, ε satisfy the usual axioms for a comultiplication and a counit, namely:

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$$

- (2) S satisfies the antipode axiom, on the $*$ -subalgebra generated by entries of u :

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(\cdot)1$$

- (3) In addition, the square of the antipode is the identity, $S^2 = id$.

PROOF. Observe first that the result holds in the case where A is commutative. Indeed, by using Theorem 2.2 (1) we can write:

$$\begin{aligned}\Delta &= m^T \\ \varepsilon &= u^T \\ S &= i^T\end{aligned}$$

The 3 conditions in the statement come then by transposition from the basic 3 group theory conditions satisfied by m, u, i , namely:

$$\begin{aligned}m(m \times id) &= m(id \times m) \\ m(id \times u) &= m(u \times id) = id \\ m(id \times i)\delta &= m(i \times id)\delta = 1\end{aligned}$$

Here $\delta(g) = (g, g)$. Observe also that the last condition, $S^2 = id$, is satisfied as well, coming from the identity $i^2 = id$, which is a consequence of the group axioms.

Observe also that the result holds as well in the case where A is cocommutative, by using Theorem 2.2 (1). Indeed, the 3 formulae in the statement are all trivial, and the condition $S^2 = id$ follows once again from the group theory formula $(g^{-1})^{-1} = g$.

In the general case now, the proof goes as follows:

(1) We have the following computation:

$$(\Delta \otimes id)\Delta(u_{ij}) = \sum_l \Delta(u_{il}) \otimes u_{lj} = \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj}$$

We have as well the following computation, which gives the first formula:

$$(id \otimes \Delta)\Delta(u_{ij}) = \sum_k u_{ik} \otimes \Delta(u_{kj}) = \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj}$$

On the other hand, we have the following computation:

$$(id \otimes \varepsilon)\Delta(u_{ij}) = \sum_k u_{ik} \otimes \varepsilon(u_{kj}) = u_{ij}$$

We have as well the following computation, which gives the second formula:

$$(\varepsilon \otimes id)\Delta(u_{ij}) = \sum_k \varepsilon(u_{ik}) \otimes u_{kj} = u_{ij}$$

(2) By using the fact that the matrix $u = (u_{ij})$ is unitary, we obtain:

$$\begin{aligned} m(id \otimes S)\Delta(u_{ij}) &= \sum_k u_{ik}S(u_{kj}) \\ &= \sum_k u_{ik}u_{jk}^* \\ &= (uu^*)_{ij} \\ &= \delta_{ij} \end{aligned}$$

We have as well the following computation, which gives the result:

$$\begin{aligned} m(S \otimes id)\Delta(u_{ij}) &= \sum_k S(u_{ik})u_{kj} \\ &= \sum_k u_{ki}^*u_{kj} \\ &= (u^*u)_{ij} \\ &= \delta_{ij} \end{aligned}$$

(3) Finally, the formula $S^2 = id$ holds as well on the generators, and we are done. \square

Let us record as well the following technical result:

PROPOSITION 2.4. *Given a Woronowicz algebra (A, u) , we have $u^t = \bar{u}^{-1}$, so u is biunitary, in the sense that it is unitary, with unitary transpose.*

PROOF. We have the following computation, based on the fact that u is unitary:

$$\begin{aligned} (uu^*)_{ij} = \delta_{ij} &\implies \sum_k S(u_{ik}u_{jk}^*) = \delta_{ij} \\ &\implies \sum_k u_{kj}u_{ki}^* = \delta_{ij} \\ &\implies (u^t\bar{u})_{ji} = \delta_{ij} \end{aligned}$$

Similarly, we have the following computation, once again using the unitarity of u :

$$\begin{aligned} (u^*u)_{ij} = \delta_{ij} &\implies \sum_k S(u_{ki}^*u_{kj}) = \delta_{ij} \\ &\implies \sum_k u_{jk}^*u_{ik} = \delta_{ij} \\ &\implies (\bar{u}u^t)_{ji} = \delta_{ij} \end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

Summarizing, the Woronowicz algebras appear to have nice properties. In view of Theorem 2.2 and of Proposition 2.3, we can formulate the following definition:

DEFINITION 2.5. *Given a Woronowicz algebra A , we formally write*

$$A = C(G) = C^*(\Gamma)$$

and call G compact quantum group, and Γ discrete quantum group.

When A is commutative and cocommutative, G and Γ are usual abelian groups, dual to each other. In general, we still agree to write $G = \widehat{\Gamma}$, $\Gamma = \widehat{G}$, but in a formal sense.

With this picture in mind, let us call now corepresentation of A any unitary matrix $v \in M_n(\mathcal{A})$, where $\mathcal{A} \subset A$ is the $*$ -algebra generated by the standard coordinates u_{ij} , satisfying the same conditions are those satisfied by u , namely:

$$\begin{aligned} \Delta(v_{ij}) &= \sum_k v_{ik} \otimes v_{kj} \\ \varepsilon(v_{ij}) &= \delta_{ij} \\ S(v_{ij}) &= v_{ji}^* \end{aligned}$$

These corepresentations can be thought of as corresponding to the finite dimensional unitary smooth representations of the underlying compact quantum group G .

Following Woronowicz [99], we have the following key result:

THEOREM 2.6. *Any Woronowicz algebra $A = C(G)$ has a Haar integration functional,*

$$\left(\int_G \otimes id \right) \Delta = \left(id \otimes \int_G \right) \Delta = \int_G (\cdot) 1$$

which can be constructed by starting with any faithful positive form $\varphi \in A^$, and setting*

$$\int_G = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}$$

*where $\phi * \psi = (\phi \otimes \psi) \Delta$. Moreover, for any corepresentation $v \in M_n(\mathbb{C}) \otimes A$ we have*

$$\left(id \otimes \int_G \right) v = P$$

where P is the orthogonal projection onto the corresponding fixed point space:

$$Fix(v) = \left\{ \xi \in \mathbb{C}^n \mid v\xi = \xi \right\}$$

PROOF. Following [99], this can be done in 3 steps, as follows:

(1) Given $\varphi \in A^*$, our claim is that the following limit converges, for any $a \in A$:

$$\int_\varphi a = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}(a)$$

Indeed, we can assume, by linearity, that a is the coefficient of a corepresentation:

$$a = (\tau \otimes id)v$$

But in this case, an elementary computation shows that we have the following formula, where P_φ is the orthogonal projection onto the 1-eigenspace of $(id \otimes \varphi)v$:

$$\left(id \otimes \int_\varphi \right) v = P_\varphi$$

(2) Since $v\xi = \xi$ implies $[(id \otimes \varphi)v]\xi = \xi$, we have $P_\varphi \geq P$, where P is the orthogonal projection onto the following fixed point space:

$$Fix(v) = \left\{ \xi \in \mathbb{C}^n \mid v\xi = \xi \right\}$$

The point now is that when $\varphi \in A^*$ is faithful, by using a standard positivity trick, one can prove that we have $P_\varphi = P$. Assume indeed $P_\varphi \xi = \xi$, and let us set:

$$a = \sum_i \left(\sum_j v_{ij} \xi_j - \xi_i \right) \left(\sum_k v_{ik} \xi_k - \xi_i \right)^*$$

We must prove that we have $a = 0$. Since v is biunitary, we have:

$$\begin{aligned} a &= \sum_i \left(\sum_j \left(v_{ij} \xi_j - \frac{1}{N} \xi_i \right) \right) \left(\sum_k \left(v_{ik}^* \bar{\xi}_k - \frac{1}{N} \bar{\xi}_i \right) \right) \\ &= \sum_{ijk} v_{ij} v_{ik}^* \xi_j \bar{\xi}_k - \frac{1}{N} v_{ij} \xi_j \bar{\xi}_i - \frac{1}{N} v_{ik}^* \xi_i \bar{\xi}_k + \frac{1}{N^2} \xi_i \bar{\xi}_i \\ &= \sum_j |\xi_j|^2 - \sum_{ij} v_{ij} \xi_j \bar{\xi}_i - \sum_{ik} v_{ik}^* \xi_i \bar{\xi}_k + \sum_i |\xi_i|^2 \\ &= \|\xi\|^2 - \langle v\xi, \xi \rangle - \overline{\langle v\xi, \xi \rangle} + \|\xi\|^2 \\ &= 2(\|\xi\|^2 - \operatorname{Re}(\langle v\xi, \xi \rangle)) \end{aligned}$$

By using now our assumption $P_\varphi \xi = \xi$, we obtain from this:

$$\begin{aligned} \varphi(a) &= 2\varphi(\|\xi\|^2 - \operatorname{Re}(\langle v\xi, \xi \rangle)) \\ &= 2(\|\xi\|^2 - \operatorname{Re}(\langle P_\varphi \xi, \xi \rangle)) \\ &= 2(\|\xi\|^2 - \|\xi\|^2) \\ &= 0 \end{aligned}$$

Now since φ is faithful, this gives $a = 0$, and so $v\xi = \xi$. Thus \int_φ is independent of φ , and is given on coefficients $a = (\tau \otimes id)v$ by the following formula:

$$\left(id \otimes \int_\varphi \right) v = P$$

(3) With the above formula in hand, the left and right invariance of $\int_G = \int_\varphi$ is clear on coefficients, and so in general, and this gives all the assertions. See [99]. \square

Consider the dense $*$ -subalgebra $\mathcal{A} \subset A$ generated by the coefficients of the fundamental corepresentation u , and endow it with the following scalar product:

$$\langle a, b \rangle = \int_G ab^*$$

Once again following [99], we have the following result:

THEOREM 2.7. *We have the following Peter-Weyl type results:*

- (1) *Any corepresentation decomposes as a sum of irreducible corepresentations.*
- (2) *Each irreducible corepresentation appears inside a certain $u^{\otimes k}$.*
- (3) $\mathcal{A} = \bigoplus_{v \in \text{Irr}(A)} M_{\dim(v)}(\mathbb{C})$, *the summands being pairwise orthogonal.*
- (4) *The characters of irreducible corepresentations form an orthonormal system.*

PROOF. All these results are from [99], the idea being as follows:

- (1) Given a corepresentation $v \in M_n(A)$, consider its interwiner algebra:

$$\text{End}(v) = \left\{ T \in M_n(\mathbb{C}) \mid Tv = vT \right\}$$

It is elementary to see that this is a finite dimensional C^* -algebra, and we conclude from this that we have a decomposition as follows:

$$\text{End}(v) = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C})$$

To be more precise, such a decomposition appears by writing the unit of our algebra as a sum of minimal projections, as follows, and then working out the details:

$$1 = p_1 + \dots + p_r$$

But this decomposition allows us to define subcorepresentations $v_i \subset v$, which are irreducible, so we obtain, as desired, a decomposition as follows:

$$v = v_1 + \dots + v_r$$

- (2) To any corepresentation $v \in M_n(A)$ we associate its space of coefficients, given by $C(v) = \text{span}(v_{ij})$. The construction $v \rightarrow C(v)$ is then functorial, in the sense that it maps subcorepresentations into subspaces. Observe also that we have:

$$\mathcal{A} = \sum_{k \in \mathbb{N} * \mathbb{N}} C(u^{\otimes k})$$

Now given an arbitrary corepresentation $v \in M_n(A)$, the corresponding coefficient space is a finite dimensional subspace $C(v) \subset \mathcal{A}$, and so we must have, for certain positive integers k_1, \dots, k_p , an inclusion of vector spaces, as follows:

$$C(v) \subset C(u^{\otimes k_1} \oplus \dots \oplus u^{\otimes k_p})$$

We deduce from this that we have an inclusion of corepresentations, as follows:

$$v \subset u^{\otimes k_1} \oplus \dots \oplus u^{\otimes k_p}$$

Thus, by using (1), we are led to the conclusion in the statement.

(3) By using (1) and (2), we obtain a linear space decomposition as follows:

$$\mathcal{A} = \sum_{v \in \text{Irr}(A)} C(v) = \sum_{v \in \text{Irr}(A)} M_{\dim(v)}(\mathbb{C})$$

In order to conclude, it is enough to prove that for any two irreducible corepresentations $v, w \in \text{Irr}(A)$, the corresponding spaces of coefficients are orthogonal:

$$v \not\sim w \implies C(v) \perp C(w)$$

As a first observation, which follows from an elementary computation, for any two corepresentations v, w we have a Frobenius type isomorphism, as follows:

$$\text{Hom}(v, w) \simeq \text{Fix}(\bar{v} \otimes w)$$

Now let us set $P_{ia,jb} = \int_G v_{ij} w_{ab}^*$. According to Theorem 2.6, the matrix P is the orthogonal projection onto the following vector space:

$$\text{Fix}(v \otimes \bar{w}) \simeq \text{Hom}(\bar{v}, \bar{w}) = \{0\}$$

Thus we have $P = 0$, and so $C(v) \perp C(w)$, which gives the result.

(4) The algebra $\mathcal{A}_{\text{central}}$ contains indeed all the characters, because we have:

$$\Sigma \Delta(\chi_v) = \sum_{ij} v_{ji} \otimes v_{ij} = \Delta(\chi_v)$$

The fact that the characters span $\mathcal{A}_{\text{central}}$, and form an orthogonal basis of it, follow from (3). Finally, regarding the norm 1 assertion, consider the following integrals:

$$P_{ik,jl} = \int_G v_{ij} v_{kl}^*$$

We know from Theorem 2.6 that these integrals form the orthogonal projection onto $\text{Fix}(v \otimes \bar{v}) \simeq \text{End}(\bar{v}) = \mathbb{C}1$. By using this fact, we obtain the following formula:

$$\begin{aligned} \int_G \chi_v \chi_v^* &= \sum_{ij} \int_G v_{ii} v_{jj}^* \\ &= \sum_i \frac{1}{N} \\ &= 1 \end{aligned}$$

Thus the characters have indeed norm 1, and we are done. □

We refer to [99] for full details on all the above, and for some applications as well. Let us just record here the fact that in the cocommutative case, we obtain from (4) that the irreducible corepresentations must be all 1-dimensional, and so that we must have $A = C^*(\Gamma)$ for some discrete group Γ , as mentioned in Theorem 2.2 above.

At a more technical level now, following [42], we have the following result:

THEOREM 2.8. *Let A_{full} be the enveloping C^* -algebra of \mathcal{A} , and let A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*

- (1) *The Haar functional of A_{full} is faithful.*
- (2) *The projection map $A_{full} \rightarrow A_{red}$ is an isomorphism.*
- (3) *The counit map $\varepsilon : A_{full} \rightarrow \mathbb{C}$ factorizes through A_{red} .*
- (4) *We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside A_{red} .*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

PROOF. This is well-known in the group dual case, $A = C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) \iff (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red} .

(2) \iff (3) Here \implies is trivial, and conversely, a counit map $\varepsilon : A_{red} \rightarrow \mathbb{C}$ produces an isomorphism $A_{red} \rightarrow A_{full}$, via a formula of type $(\varepsilon \otimes id)\Phi$.

(3) \iff (4) Here \implies is clear, coming from $\varepsilon(N - Re(\chi(u))) = 0$, and the converse can be proved by doing some standard functional analysis. \square

Yet another important technical result is Tannakian duality, as follows:

THEOREM 2.9. *The following operations are inverse to each other:*

- (1) *The construction $A \rightarrow C$, which associates to any Woronowicz algebra A the tensor category formed by the intertwiner spaces $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.*
- (2) *The construction $C \rightarrow A$, which associates to any tensor category C the Woronowicz algebra A presented by the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C_{kl}$.*

PROOF. This is something quite deep, going back to [100] in a slightly different form, and to [73] in the simplified form presented above. The idea is as follows:

(1) We have indeed a construction $A \rightarrow C$ as above, whose output is a tensor C^* -subcategory with duals of the tensor C^* -category of Hilbert spaces.

(2) We have as well a construction $C \rightarrow A$ as above, simply by dividing the free $*$ -algebra on N^2 variables by the relations in the statement.

Regarding now the bijection claim, some elementary algebra shows that $C = C_{A_C}$ implies $A = A_{C_A}$, and also that $C \subset C_{A_C}$ is automatic. Thus we are left with proving

$C_{AC} \subset C$. But this latter inclusion can be proved indeed, by doing some algebra, and using von Neumann's bicommutant theorem, in finite dimensions. See [73]. \square

As a concrete consequence of the above result, we have:

THEOREM 2.10. *We have an embedding as follows, using double indices,*

$$G \subset S_{\mathbb{C},+}^{N^2-1} \quad , \quad x_{ij} = \frac{u_{ij}}{\sqrt{N}}$$

making G an algebraic submanifold of the free sphere.

PROOF. The fact that we have an embedding as above follows from the fact that $u = (u_{ij})$ is biunitary, that we know from Proposition 2.4. As for the algebraicity claim, this follows from Theorem 2.9. Indeed, assuming that $A = C(G)$ is of the form $A = A_C$, it follows that G is algebraic. But this is always the case, because we can take $C = C_A$. \square

Observe that the embedding constructed above makes the link between our isomorphism conventions for quantum groups and for algebraic manifolds.

2b. Free rotations

Let us get back now to our original objective, namely constructing pairs of quantum unitary and reflection groups (O_N^+, H_N^+) and (U_N^+, K_N^+) , as to complete the pairs $(S_{\mathbb{R},+}^{N-1}, T_N^+)$ and $(S_{\mathbb{C},+}^{N-1}, \mathbb{T}_N^+)$ that we already have. Following Wang [93], we have:

THEOREM 2.11. *The following constructions produce compact quantum groups,*

$$\begin{aligned} C(O_N^+) &= C^* \left((u_{ij})_{i,j=1,\dots,N} \mid u = \bar{u}, u^t = u^{-1} \right) \\ C(U_N^+) &= C^* \left((u_{ij})_{i,j=1,\dots,N} \mid u^* = u^{-1}, u^t = \bar{u}^{-1} \right) \end{aligned}$$

which appear respectively as liberations of the groups O_N and U_N .

PROOF. This first assertion follows from the elementary fact that if a matrix $u = (u_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$\begin{aligned} u_{ij}^\Delta &= \sum_k u_{ik} \otimes u_{kj} \\ u_{ij}^\varepsilon &= \delta_{ij} \\ u_{ij}^S &= u_{ji}^* \end{aligned}$$

Regarding now the matrix $u^\varepsilon = 1_N$, this is clearly biunitary. Finally, regarding the matrix u^S , there is nothing to prove here either, because its unitarity is clear too. Finally, observe that if u is real, then so are the above matrices $u^\Delta, u^\varepsilon, u^S$.

Thus, we can define morphisms Δ, ε, S as in Definition 2.1, by using the universal properties of $C(O_N^+)$, $C(U_N^+)$. As for the second assertion, this follows exactly as for the free spheres, by adapting the sphere proof from chapter 1 above. \square

The basic properties of O_N^+, U_N^+ can be summarized as follows:

THEOREM 2.12. *The quantum groups O_N^+, U_N^+ have the following properties:*

- (1) *The closed subgroups $G \subset U_N^+$ are exactly the $N \times N$ compact quantum groups. As for the closed subgroups $G \subset O_N^+$, these are those satisfying $u = \bar{u}$.*
- (2) *We have liberation embeddings $O_N \subset O_N^+$ and $U_N \subset U_N^+$, obtained by dividing the algebras $C(O_N^+), C(U_N^+)$ by their respective commutator ideals.*
- (3) *We have as well embeddings $\widehat{L}_N \subset O_N^+$ and $\widehat{F}_N \subset U_N^+$, where L_N is the free product of N copies of \mathbb{Z}_2 , and where F_N is the free group on N generators.*

PROOF. All these assertions are elementary, as follows:

(1) This is clear from definitions, with the remark that, in the context of Definition 2.1 above, the formula $S(u_{ij}) = u_{ji}^*$ shows that the matrix \bar{u} must be unitary too.

(2) This follows from the Gelfand theorem. To be more precise, this shows that we have presentation results for $C(O_N), C(U_N)$, similar to those in Theorem 2.11, but with the commutativity between the standard coordinates and their adjoints added:

$$\begin{aligned} C(O_N) &= C_{comm}^* \left((u_{ij})_{i,j=1,\dots,N} \mid u = \bar{u}, u^t = u^{-1} \right) \\ C(U_N) &= C_{comm}^* \left((u_{ij})_{i,j=1,\dots,N} \mid u^* = u^{-1}, u^t = \bar{u}^{-1} \right) \end{aligned}$$

Thus, we are led to the conclusion in the statement.

(3) This follows from (1) and from Theorem 2.2 above, with the remark that with $u = \text{diag}(g_1, \dots, g_N)$, the condition $u = \bar{u}$ is equivalent to $g_i^2 = 1$, for any i . \square

The last assertion in Theorem 2.12 suggests the following construction:

PROPOSITION 2.13. *Given a closed subgroup $G \subset U_N^+$, consider its “diagonal torus”, which is the closed subgroup $T \subset G$ constructed as follows:*

$$C(T) = C(G) / \left\langle u_{ij} = 0 \mid \forall i \neq j \right\rangle$$

This torus is then a group dual, $T = \widehat{\Lambda}$, where $\Lambda = \langle g_1, \dots, g_N \rangle$ is the discrete group generated by the elements $g_i = u_{ii}$, which are unitaries inside $C(T)$.

PROOF. Since u is unitary, its diagonal entries $g_i = u_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have $C(T) = C^*(\Lambda)$, modulo identifying as usual the C^* -completions of the various group algebras, and so that we have $T = \widehat{\Lambda}$, as claimed. \square

With this notion in hand, Theorem 2.12 (3) reformulates as follows:

THEOREM 2.14. *The diagonal tori of the basic unitary groups are the basic tori:*

$$\begin{array}{ccc}
 O_N^+ & \longrightarrow & U_N^+ \\
 \uparrow & & \uparrow \\
 O_N & \longrightarrow & U_N
 \end{array}
 \quad \longrightarrow \quad
 \begin{array}{ccc}
 T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\
 \uparrow & & \uparrow \\
 T_N & \longrightarrow & \mathbb{T}_N
 \end{array}$$

In particular, the basic unitary groups are all distinct.

PROOF. This is something clear and well-known in the classical case, and in the free case, this is a reformulation of Theorem 2.12 (3) above, which tells us that the diagonal tori of O_N^+, U_N^+ , in the sense of Proposition 2.13, are the group duals $\widehat{L}_N, \widehat{F}_N$. \square

There is an obvious relation here with the considerations from chapter 1 above, that we will analyse later on. As a second result now regarding our free quantum groups, relating them this time to the free spheres constructed in chapter 1, we have:

PROPOSITION 2.15. *We have embeddings of algebraic manifolds as follows, obtained in double indices by rescaling the coordinates, $x_{ij} = u_{ij}/\sqrt{N}$:*

$$\begin{array}{ccc}
 O_N^+ & \longrightarrow & U_N^+ \\
 \uparrow & & \uparrow \\
 O_N & \longrightarrow & U_N
 \end{array}
 \quad \longrightarrow \quad
 \begin{array}{ccc}
 S_{\mathbb{R},+}^{N^2-1} & \longrightarrow & S_{\mathbb{C},+}^{N^2-1} \\
 \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N^2-1} & \longrightarrow & S_{\mathbb{C}}^{N^2-1}
 \end{array}$$

Moreover, the quantum groups appear from the quantum spheres via

$$G = S \cap U_N^+$$

with the intersection being computed inside the free sphere $S_{\mathbb{C},+}^{N^2-1}$.

PROOF. As explained in Theorem 2.10 above, the biunitarity of the matrix $u = (u_{ij})$ gives an embedding of algebraic manifolds, as follows:

$$U_N^+ \subset S_{\mathbb{C},+}^{N^2-1}$$

Now since the relations defining $O_N, O_N^+, U_N \subset U_N^+$ are the same as those defining $S_{\mathbb{R}}^{N^2-1}, S_{\mathbb{R},+}^{N^2-1}, S_{\mathbb{C}}^{N^2-1} \subset S_{\mathbb{C},+}^{N^2-1}$, this gives the result. \square

2c. Free reflections

Summarizing, in connection with our (S, T, U, K) program, we have so far triples of type (S, T, U) , along with some correspondences between S, T, U . In order to introduce now the reflection groups K , things are more tricky, involving quantum permutation groups. Following Wang [94], these quantum groups are introduced as follows:

THEOREM 2.16. *The following construction, where “magic” means formed of projections, which sum up to 1 on each row and column,*

$$C(S_N^+) = C^* \left((u_{ij})_{i,j=1,\dots,N} \mid u = \text{magic} \right)$$

produces a quantum group liberation of S_N . Moreover, the inclusion

$$S_N \subset S_N^+$$

is an isomorphism at $N \leq 3$, but not at $N \geq 4$, where S_N^+ is not classical, nor finite.

PROOF. The quantum group assertion follows by using the same arguments as those in the proof of Theorem 2.11. Consider indeed the following matrix:

$$U_{ij} = \sum_k u_{ik} \otimes u_{kj}$$

As a first observation, the entries of this matrix are self-adjoint:

$$U_{ij} = U_{ij}^*$$

In fact the entries U_{ij} are orthogonal projections, because we have as well:

$$\begin{aligned} U_{ij}^2 &= \sum_{kl} u_{ik} u_{il} \otimes u_{kj} u_{lj} \\ &= \sum_k u_{ik} \otimes u_{kj} \\ &= U_{ij} \end{aligned}$$

In order to prove now that the matrix $U = (U_{ij})$ is magic, it remains to verify that the sums on the rows and columns are 1. For the rows, this can be checked as follows:

$$\begin{aligned} \sum_j U_{ij} &= \sum_{jk} u_{ik} \otimes u_{kj} \\ &= \sum_k u_{ik} \otimes 1 \\ &= 1 \otimes 1 \end{aligned}$$

For the columns the computation is similar, as follows:

$$\begin{aligned} \sum_i U_{ij} &= \sum_{ik} u_{ik} \otimes u_{kj} \\ &= \sum_k 1 \otimes u_{kj} \\ &= 1 \otimes 1 \end{aligned}$$

Thus the $U = (U_{ij})$ is magic, and so we can define a comultiplication map by using the universality property of $C(S_N^+)$, by setting $\Delta(u_{ij}) = U_{ij}$. By using a similar reasoning, we can define as well a counit map by $\varepsilon(u_{ij}) = \delta_{ij}$, and an antipode map by $S(u_{ij}) = u_{ji}$. Thus the Woronowicz algebra axioms from Definition 2.1 are satisfied, and this finishes the proof of the first assertion, stating that S_N^+ is indeed a compact quantum group.

Observe now that we have an embedding of compact quantum groups $S_N \subset S_N^+$, obtained by using the standard coordinates of S_N , viewed as an algebraic group:

$$u_{ij} = \chi \left(\sigma \in S_N \mid \sigma(j) = i \right)$$

By using the Gelfand theorem and working out the details, as we did with the free spheres are free unitary groups, the embedding $S_N \subset S_N^+$ is indeed a liberation.

Finally, regarding the last assertion, the study here is as follows:

Case $N = 2$. The result here is trivial, the 2×2 magic matrices being by definition as follows, with p being a projection:

$$U = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

Indeed, this shows that the entries of a 2×2 magic matrix must pairwise commute, and so the algebra $C(S_2^+)$ follows to be commutative, which gives the result.

Case $N = 3$. This is more tricky, and we present here a short proof from [72]. By using the same abstract argument as in the $N = 2$ case, and by permuting rows and columns, it is enough to check that u_{11}, u_{22} commute. But this follows from:

$$\begin{aligned} u_{11}u_{22} &= u_{11}u_{22}(u_{11} + u_{12} + u_{13}) \\ &= u_{11}u_{22}u_{11} + u_{11}u_{22}u_{13} \\ &= u_{11}u_{22}u_{11} + u_{11}(1 - u_{21} - u_{23})u_{13} \\ &= u_{11}u_{22}u_{11} \end{aligned}$$

Indeed, by applying the involution to this formula, we obtain from this:

$$u_{22}u_{11} = u_{11}u_{22}u_{11}$$

Thus we obtain $u_{11}u_{22} = u_{22}u_{11}$, as desired.

Case $N = 4$. In order to prove our various claims about S_4^+ , consider the following matrix, with p, q being projections, on some infinite dimensional Hilbert space:

$$U = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

This matrix is magic, and if we choose p, q as for the algebra $\langle p, q \rangle$ to be not commutative, and infinite dimensional, we conclude that $C(S_4^+)$ is not commutative and infinite dimensional as well, and in particular is not isomorphic to $C(S_4)$.

Case $N \geq 5$. Here we can use the standard embedding $S_4^+ \subset S_N^+$, obtained at the level of the corresponding magic matrices in the following way:

$$u \rightarrow \begin{pmatrix} u & 0 \\ 0 & 1_{N-4} \end{pmatrix}$$

Indeed, with this embedding in hand, the fact that S_4^+ is a non-classical, infinite compact quantum group implies that S_N^+ with $N \geq 5$ has these two properties as well. \square

The above result came as a surprise at the time of [94], and there has been a lot of work since then, in order to understand what the quantum permutations really are, at $N \geq 4$. We will be back to this, with further details, on several occasions.

For the moment, let us just record the following alternative approach to S_N^+ , also from [94], which shows that we are not wrong with our formalism:

PROPOSITION 2.17. *The quantum group S_N^+ acts on the set $X = \{1, \dots, N\}$, the corresponding coaction map $\Phi : C(X) \rightarrow C(X) \otimes C(S_N^+)$ being given by:*

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

In fact, S_N^+ is the biggest compact quantum group acting on X , by leaving the counting measure invariant, in the sense that $(\text{tr} \otimes \text{id})\Phi = \text{tr}(\cdot)1$, where $\text{tr}(e_i) = \frac{1}{N}, \forall i$.

PROOF. Our claim is that given a compact matrix quantum group G , the following formula defines a morphism of algebras, which is a coaction map, leaving the trace invariant, precisely when the matrix $u = (u_{ij})$ is a magic corepresentation of $C(G)$:

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

Indeed, let us first determine when Φ is multiplicative. We have:

$$\begin{aligned}\Phi(e_i)\Phi(e_k) &= \sum_{jl} e_j e_l \otimes u_{ji} u_{lk} \\ &= \sum_j e_j \otimes u_{ji} u_{jk}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}\Phi(e_i e_k) &= \delta_{ik} \Phi(e_i) \\ &= \delta_{ik} \sum_j e_j \otimes u_{ji}\end{aligned}$$

We conclude that the multiplicativity of Φ is equivalent to the following conditions:

$$u_{ji} u_{jk} = \delta_{ik} u_{ji} \quad , \quad \forall i, j, k$$

Regarding now the unitality of Φ , we have the following formula:

$$\begin{aligned}\Phi(1) &= \sum_i \Phi(e_i) \\ &= \sum_{ij} e_j \otimes u_{ji} \\ &= \sum_j e_j \otimes \left(\sum_i u_{ji} \right)\end{aligned}$$

Thus Φ is unital when $\sum_i u_{ji} = 1$, $\forall j$. Finally, the fact that Φ is a $*$ -morphism translates into $u_{ij} = u_{ij}^*$, $\forall i, j$. Summing up, in order for $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$ to be a morphism of C^* -algebras, the elements u_{ij} must be projections, summing up to 1 on each row of u . Regarding now the preservation of the trace condition, observe that we have:

$$(tr \otimes id)\Phi(e_i) = \frac{1}{N} \sum_j u_{ji}$$

Thus the trace is preserved precisely when the elements u_{ij} sum up to 1 on each of the columns of u . We conclude from this that $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$ is a morphism of C^* -algebras preserving the trace precisely when u is magic, and since the coaction conditions on Φ are equivalent to the fact that u must be a corepresentation, this finishes the proof of our claim. But this claim proves all the assertions in the statement. \square

With the above results in hand, we can now introduce the quantum reflections:

THEOREM 2.18. *The following constructions produce compact quantum groups,*

$$\begin{aligned} C(H_N^+) &= C^* \left((u_{ij})_{i,j=1,\dots,N} \mid u_{ij} = u_{ij}^*, (u_{ij}^2) = \text{magic} \right) \\ C(K_N^+) &= C^* \left((u_{ij})_{i,j=1,\dots,N} \mid [u_{ij}, u_{ij}^*] = 0, (u_{ij}u_{ij}^*) = \text{magic} \right) \end{aligned}$$

which appear respectively as liberations of the reflection groups H_N and K_N .

PROOF. This can be proved in the usual way, with the first assertion coming from the fact that if u satisfies the relations in the statement, then so do the matrices $u^\Delta, u^\varepsilon, u^S$, and with the second assertion coming as in the sphere case. See [13], [18]. \square

Summarizing, we are done with our construction task for the quadruplets (S, T, U, K) , in the free real and complex cases, and we can now formulate:

PROPOSITION 2.19. *We have a quadruplet as follows, called free real,*

$$\begin{array}{ccc} S_{\mathbb{R},+}^{N-1} & \text{---} & T_N^+ \\ | & \diagdown & | \\ O_N^+ & & H_N^+ \\ | & \diagup & | \\ S_{\mathbb{R},+}^{N-1} & \text{---} & T_N^+ \end{array}$$

and a quadruplet as follows, called free complex:

$$\begin{array}{ccc} S_{\mathbb{C},+}^{N-1} & \text{---} & \mathbb{T}_N^+ \\ | & \diagdown & | \\ U_N^+ & & K_N^+ \\ | & \diagup & | \\ S_{\mathbb{C},+}^{N-1} & \text{---} & \mathbb{T}_N^+ \end{array}$$

PROOF. This is more of an empty statement, coming from the various constructions above, from chapter 1, and from the present chapter. \square

Going ahead now, we must construct correspondences between our objects (S, T, U, K) , completing the work for the pairs (S, T) started in chapter 1 above. This will take some time, and we will need some preliminaries. To start with, let us record the following result, which refines the various liberation statements formulated above:

THEOREM 2.20. *The quantum unitary and reflection groups are as follows,*

$$\begin{array}{ccccc}
 & & K_N^+ & \longrightarrow & U_N^+ \\
 & \nearrow & \uparrow & & \nearrow \\
 H_N^+ & \longrightarrow & O_N^+ & & \\
 \uparrow & & \uparrow & & \uparrow \\
 & & K_N & \longrightarrow & U_N \\
 & \nearrow & \uparrow & & \nearrow \\
 H_N & \longrightarrow & O_N & &
 \end{array}$$

and in this diagram, any face $P \subset Q, R \subset S$ has the property $P = Q \cap R$.

PROOF. The fact that we have inclusions as in the statement follows from the definition of the various quantum groups involved. As for the various intersection claims, these follow as well from definitions. For some further details on all this, we refer to [12]. \square

2d. Diagrams, easiness

In order to efficiently deal with the quantum groups introduced above, and to eventually construct the correspondences that we want to construct, we will need some specialized Tannakian duality results, in the spirit of the Brauer theorem [44].

Following [33], let us start with the following definition:

DEFINITION 2.21. *Associated to any partition $\pi \in P(k, l)$ between an upper row of k points and a lower row of l points is the linear map $T_\pi : (\mathbb{C}^N)^{\otimes k} \rightarrow (\mathbb{C}^N)^{\otimes l}$ given by*

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l}$$

with the Kronecker type symbols $\delta_\pi \in \{0, 1\}$ depending on whether the indices fit or not.

To be more precise, we agree to put the two multi-indices on the two rows of points, in the obvious way. The Kronecker symbols are then defined by $\delta_\pi = 1$ when all the strings of π join equal indices, and by $\delta_\pi = 0$ otherwise. This construction is motivated by:

PROPOSITION 2.22. *The assignment $\pi \rightarrow T_\pi$ is categorical, in the sense that we have*

$$\begin{aligned}
 T_\pi \otimes T_\sigma &= T_{[\pi\sigma]} \\
 T_\pi T_\sigma &= N^{c(\pi, \sigma)} T_{[\frac{\pi}{\sigma}]} \\
 T_\pi^* &= T_{\pi^*}
 \end{aligned}$$

where $c(\pi, \sigma)$ are certain integers, coming from the erased components in the middle.

PROOF. This follows from some routine computations, as follows:

(1) The concatenation axiom follows from the following computation:

$$\begin{aligned}
& (T_\pi \otimes T_\sigma)(e_{i_1} \otimes \dots \otimes e_{i_p} \otimes e_{k_1} \otimes \dots \otimes e_{k_r}) \\
&= \sum_{j_1 \dots j_q} \sum_{l_1 \dots l_s} \delta_\pi \begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_q \end{pmatrix} \delta_\sigma \begin{pmatrix} k_1 & \dots & k_r \\ l_1 & \dots & l_s \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_q} \otimes e_{l_1} \otimes \dots \otimes e_{l_s} \\
&= \sum_{j_1 \dots j_q} \sum_{l_1 \dots l_s} \delta_{[\pi\sigma]} \begin{pmatrix} i_1 & \dots & i_p & k_1 & \dots & k_r \\ j_1 & \dots & j_q & l_1 & \dots & l_s \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_q} \otimes e_{l_1} \otimes \dots \otimes e_{l_s} \\
&= T_{[\pi\sigma]}(e_{i_1} \otimes \dots \otimes e_{i_p} \otimes e_{k_1} \otimes \dots \otimes e_{k_r})
\end{aligned}$$

(2) The composition axiom follows from the following computation:

$$\begin{aligned}
& T_\pi T_\sigma(e_{i_1} \otimes \dots \otimes e_{i_p}) \\
&= \sum_{j_1 \dots j_q} \delta_\sigma \begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_q \end{pmatrix} \sum_{k_1 \dots k_r} \delta_\pi \begin{pmatrix} j_1 & \dots & j_q \\ k_1 & \dots & k_r \end{pmatrix} e_{k_1} \otimes \dots \otimes e_{k_r} \\
&= \sum_{k_1 \dots k_r} N^{c(\pi, \sigma)} \delta_{[\sigma]} \begin{pmatrix} i_1 & \dots & i_p \\ k_1 & \dots & k_r \end{pmatrix} e_{k_1} \otimes \dots \otimes e_{k_r} \\
&= N^{c(\pi, \sigma)} T_{[\sigma]}(e_{i_1} \otimes \dots \otimes e_{i_p})
\end{aligned}$$

(3) Finally, the involution axiom follows from the following computation:

$$\begin{aligned}
& T_\pi^*(e_{j_1} \otimes \dots \otimes e_{j_q}) \\
&= \sum_{i_1 \dots i_p} \langle T_\pi^*(e_{j_1} \otimes \dots \otimes e_{j_q}), e_{i_1} \otimes \dots \otimes e_{i_p} \rangle e_{i_1} \otimes \dots \otimes e_{i_p} \\
&= \sum_{i_1 \dots i_p} \delta_\pi \begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_q \end{pmatrix} e_{i_1} \otimes \dots \otimes e_{i_p} \\
&= T_{\pi^*}(e_{j_1} \otimes \dots \otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical. See [33]. \square

We have the following notion, from [33], [85]:

DEFINITION 2.23. A collection of sets $D = \bigsqcup_{k,l} D(k, l)$ with $D(k, l) \subset P(k, l)$ is called a category of partitions when it has the following properties:

- (1) Stability under the horizontal concatenation, $(\pi, \sigma) \rightarrow [\pi\sigma]$.
- (2) Stability under vertical concatenation $(\pi, \sigma) \rightarrow [\sigma]$, with matching middle symbols.
- (3) Stability under the upside-down turning $*$, with switching of colors, $\circ \leftrightarrow \bullet$.
- (4) Each set $P(k, k)$ contains the identity partition $|| \dots ||$.
- (5) The sets $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ both contain the semicircle \cap .

As a basic example, the set $D = P$ itself, formed by all partitions, is a category of partitions. The same goes for the category of pairings $P_2 \subset P$. There are many other examples, and we will gradually explore them, in what follows.

Generally speaking, the axioms in Definition 2.23 can be thought of as being a “de-linearized version” of the axioms for the Tannakian categories.

We can now formulate a key result, from [33], as follows:

THEOREM 2.24. *Each category of partitions $D = (D(k, l))$ produces a family of compact quantum groups $G = (G_N)$, one for each $N \in \mathbb{N}$, via the formula*

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{span} \left(T_\pi \mid \pi \in D(k, l) \right)$$

which produces a Tannakian category, and therefore a closed subgroup $G_N \subset U_N^+$. The quantum groups which appear in this way are called “easy”.

PROOF. This follows indeed from Woronowicz’s Tannakian duality, in its “soft” form from [73], as explained in Theorem 2.9 above. Indeed, let us set:

$$C(k, l) = \text{span} \left(T_\pi \mid \pi \in D(k, l) \right)$$

By using the axioms in Definition 2.23, and the categorical properties of the operation $\pi \rightarrow T_\pi$, from Proposition 2.22, we deduce that $C = (C(k, l))$ is a Tannakian category. Thus the Tannakian duality result applies, and gives the result. \square

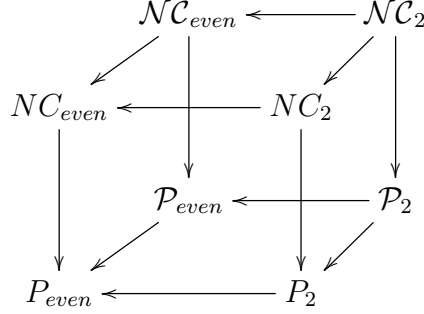
We can now formulate a general Brauer theorem, as follows:

THEOREM 2.25. *The basic quantum unitary and quantum reflection groups, namely*

$$\begin{array}{ccccc}
 & & K_N^+ & \longrightarrow & U_N^+ \\
 & & \uparrow & & \uparrow \\
 H_N^+ & \longrightarrow & O_N^+ & \longrightarrow & U_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 & & K_N & \longrightarrow & U_N \\
 & & \uparrow & & \uparrow \\
 H_N & \longrightarrow & O_N & \longrightarrow & U_N
 \end{array}$$

are all easy. The corresponding categories of partitions form an intersection diagram.

PROOF. This is well-known, the corresponding categories being as follows, with P_{even} being the category of partitions having even blocks, and with $\mathcal{P}_{even}(k, l) \subset P_{even}(k, l)$ consisting of the partitions satisfying $\# \circ = \# \bullet$ in each block, when flattening the partition:



To be more precise, the proof goes as follows:

(1) The quantum group U_N^+ is defined via the following relations:

$$\begin{aligned}
 u^* &= u^{-1} \\
 u^t &= \bar{u}^{-1}
 \end{aligned}$$

But these relations tell us precisely that the following two operators must be in the associated Tannakian category C :

$$\begin{aligned}
 T_\pi &, \quad \pi = \begin{array}{c} \cap \\ \circ \bullet \end{array} \\
 T_\pi &, \quad \pi = \begin{array}{c} \cap \\ \bullet \circ \end{array}
 \end{aligned}$$

Thus the associated Tannakian category is $C = span(T_\pi | \pi \in D)$, with:

$$D = \langle \begin{array}{c} \cap \\ \circ \bullet \end{array}, \begin{array}{c} \cap \\ \bullet \circ \end{array} \rangle = \mathcal{NC}_2$$

Thus, we are led to the conclusion in the statement.

(2) The quantum group $O_N^+ \subset U_N^+$ is defined by imposing the following relations:

$$u_{ij} = \bar{u}_{ij}$$

But these relations tell us that the following operators must be in the associated Tannakian category C :

$$\begin{aligned}
 T_\pi &, \quad \pi = \begin{array}{c} \circ \\ \updownarrow \end{array} \\
 T_\pi &, \quad \pi = \begin{array}{c} \bullet \\ \updownarrow \end{array}
 \end{aligned}$$

Thus the associated Tannakian category is $C = span(T_\pi | \pi \in D)$, with:

$$D = \langle \mathcal{NC}_2, \begin{array}{c} \circ \\ \updownarrow \end{array}, \begin{array}{c} \bullet \\ \updownarrow \end{array} \rangle = \mathcal{NC}_2$$

Thus, we are led to the conclusion in the statement.

(3) The group $U_N \subset U_N^+$ is defined via the following relations:

$$[u_{ij}, u_{kl}] = 0$$

$$[u_{ij}, \bar{u}_{kl}] = 0$$

But these relations tell us that the following operators must be in the associated Tannakian category C :

$$\begin{aligned} T_\pi & \quad , \quad \pi = \text{crossing} \\ T_\pi & \quad , \quad \pi = \text{cup} \end{aligned}$$

Thus the associated Tannakian category is $C = \text{span}(T_\pi | \pi \in D)$, with:

$$D = \langle \mathcal{NC}_2, \text{crossing}, \text{cup} \rangle = \mathcal{P}_2$$

Thus, we are led to the conclusion in the statement.

(4) In order to deal now with O_N , we can simply use the following formula:

$$O_N = O_N^+ \cap U_N$$

At the categorical level, this tells us indeed that the associated Tannakian category is given by $C = \text{span}(T_\pi | \pi \in D)$, with:

$$D = \langle \mathcal{NC}_2, \mathcal{P}_2 \rangle = \mathcal{P}_2$$

Thus, we are led to the conclusion in the statement.

(5) The proof for the reflection groups is similar, by first proving that S_N is easy, corresponding to the category of all partitions P , and then by adding and suitably interpreting the reflection relations. We refer here to [18] and [13], for full details.

(6) The proof for the quantum reflection groups is similar, by first proving that the quantum permutation group S_N^+ is easy, corresponding to the category of all noncrossing partitions NC , and then by adding and suitably interpreting the quantum reflection relations. As before, we refer here to [18] and [13], for full details.

(7) As for the last assertion, which will be of use later on, this is something well-known and standard too. We refer here to [13], [18], [21], and to [12], [33] as well. \square

Getting back now to our axiomatization questions, we must establish correspondences between our objects (S, T, U, K) , as a continuation of the work started in chapter 1, for the pairs (S, T) . And there is quite some work to be done here.

Let us start by discussing the following correspondences:

$$U \rightarrow K \rightarrow T$$

We know from Theorem 2.14 that the correspondences $U \rightarrow T$ appear by taking the diagonal tori. In fact, the correspondences $K \rightarrow T$ appear by taking the diagonal tori as well, and the correspondences $U \rightarrow K$ are something elementary too, obtained by taking the “reflection subgroup”. The complete statement here is as follows:

THEOREM 2.26. For the basic quadruplets (S, T, U, K) , the correspondences

$$\begin{array}{ccc}
 O_N^+ \longrightarrow U_N^+ & & H_N^+ \longrightarrow K_N^+ & & T_N^+ \longrightarrow \mathbb{T}_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 O_N \longrightarrow U_N & \longrightarrow & H_N \longrightarrow K_N & \longrightarrow & T_N \longrightarrow \mathbb{T}_N
 \end{array}$$

appear in the following way:

- (1) $U \rightarrow K$ appears by taking the reflection subgroup, $K = U \cap K_N^+$.
- (2) $U \rightarrow T$ appears by taking the diagonal torus, $T = U \cap \mathbb{T}_N^+$.
- (3) $K \rightarrow T$ appears as well by taking the diagonal torus, $T = K \cap \mathbb{T}_N^+$.

PROOF. This follows from the results that we already have, as follows:

- (1) This follows from Theorem 2.25, because the left face of the cube diagram there appears by intersecting the right face with the quantum group K_N^+ .
- (2) This is something that we already know, from Theorem 2.14 above.
- (3) This follows exactly as in the unitary case, via the proof of Theorem 2.14. □

As a conclusion now, with respect to the “baby theory” developed in section 1 above, concerning the pairs (S, T) , we have some advances. First, we have completed the pairs (S, T) there into quadruplets (S, T, U, K) . And second, we have established some correspondences between our objects, the situation here being as follows:

$$\begin{array}{ccc}
 S & \longrightarrow & T \\
 & \nearrow & \uparrow \\
 U & \longrightarrow & K
 \end{array}$$

There is still a long way to go, in order to establish a full set of correspondences, and to reach to an axiomatization, the idea being that the correspondences $S \leftrightarrow U$ can be established by using quantum isometries, and that the correspondences $T \rightarrow K \rightarrow U$ can be established by using advanced quantum group theory, and with all this heavily relying on the easiness theory developed above. We will discuss this in chapters 3-4 below.

2e. Exercises

As before with the first chapter, the theory explained in the above will be of key importance in what follows, and as best homework we can only recommend downloading and reading a good quantum group book. Here is however a first exercise:

EXERCISE 2.27. *Develop a theory of finite quantum groups, in analogy with the usual group theory, by assuming that the Woronowicz algebras A are finite dimensional.*

This is a bit vague, but as a bottom line, such a theory would need clear axioms, the Pontrjagin type duality worked out in detail, a few theorems, and examples.

In relation now with Haar integration and Peter-Weyl theory, we have:

EXERCISE 2.28. *Work out what happens with the Haar integration and Peter-Weyl theory, theorems and proofs, in the classical group case, and in the group dual case.*

This is something instructive, leading to a good functional analysis knowledge.

Along the same lines, we have, at a more advanced level:

EXERCISE 2.29. *Work out what happens with the amenability and Tannakian duality, theorems and proofs, in the classical group case, and in the group dual case.*

As before with the previous exercise, this is something quite instructive.

All the above is a bit abstract, and as a concrete exercise now, we have:

EXERCISE 2.30. *Prove that the dual of S_5^+ is not amenable.*

As a hint here, try finding a subgroup which is not amenable. Also, as a bonus exercise, you can try as well to prove that the dual of S_4^+ is amenable, although this is quite tricky, and reputed to be undoable with bare hands. But who knows.

Finally, in connection with easiness and quantum reflections, we have:

EXERCISE 2.31. *Develop a full theory for the main quantum reflection groups*

$$\begin{array}{ccc}
 H_N^+ & \longrightarrow & K_N^+ \\
 \uparrow & & \uparrow \\
 H_N & \longrightarrow & K_N
 \end{array}$$

including defining relations, worked out in detail, and easiness property.

This is something that we talked about in the above, and the problem now is that of filling all the details, in order to have all this well understood.

CHAPTER 3

Affine isometries

3a. Quantum isometries

We have seen so far that we have quadruplets (S, T, U, K) consisting of a sphere S , a torus T , a unitary group U and a reflection group K , corresponding to the four main geometries, namely real and complex, classical and free, which are as follows:

$$\begin{array}{ccc}
 \mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\
 \uparrow & & \uparrow \\
 \mathbb{R}^N & \longrightarrow & \mathbb{C}^N
 \end{array}$$

We have to work out now the various correspondences between our objects (S, T, U, K) , as to reach to a full set of correspondences, in each of the above 4 cases, as follows:

$$\begin{array}{ccc}
 S & \longleftrightarrow & T \\
 \uparrow & \swarrow & \uparrow \\
 & & \\
 \downarrow & \searrow & \downarrow \\
 U & \longleftrightarrow & K
 \end{array}$$

We know from chapters 1-2 that we already have 4 such correspondences, namely $S \rightarrow T$ and $U \rightarrow K \rightarrow T$. In this chapter we discuss 3 more correspondences, namely $S \leftrightarrow U$ and $T \rightarrow K$, as to reach to a total of 7 correspondences, as follows:

$$\begin{array}{ccc}
 S & \longrightarrow & T \\
 \uparrow & & \uparrow \\
 & \nearrow & \\
 \downarrow & & \downarrow \\
 U & \longrightarrow & K
 \end{array}$$

In order to connect the spheres and tori (S, T) to the quantum groups (U, K) , the idea will be that of using quantum isometry groups. Let us start with:

PROPOSITION 3.1. *Given an algebraic manifold $X \subset S_{\mathbb{C}}^{N-1}$, the formula*

$$G(X) = \left\{ U \in U_N \mid U(X) = X \right\}$$

defines a compact group of unitary matrices, or isometries, called affine isometry group of X . For the spheres $S_{\mathbb{R}}^{N-1}, S_{\mathbb{C}}^{N-1}$ we obtain in this way the groups O_N, U_N .

PROOF. The fact that $G(X)$ as defined above is indeed a group is clear, its compactness is clear as well, and finally the last assertion is clear as well. In fact, all this works for any closed subset $X \subset \mathbb{C}^N$, but we are not interested here in such general spaces. \square

Observe that in the case of the real and complex spheres, the affine isometry group $G(X)$ leaves invariant the Riemannian metric, because this metric is equivalent to the one inherited from \mathbb{C}^N , which is preserved by our isometries $U \in U_N$. Thus, we could have constructed as well $G(X)$ as being the group of metric isometries of X , with of course some extra care in relation with the complex structure, as for the complex sphere $X = S_{\mathbb{C}}^{N-1}$ to produce $G(X) = U_N$ instead of $G(X) = O_{2N}$.

However, in the noncommutative setting, all this becomes considerably more complicated, and we prefer to use the above construction, as such. We will be back later to metric aspects, at the end of the present chapter.

We have the following quantum analogue of Proposition 3.1:

THEOREM 3.2. *Given an algebraic manifold $X \subset S_{\mathbb{C},+}^{N-1}$, the category of the closed subgroups $G \subset U_N^+$ acting affinely on X , in the sense that the formula*

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

defines a morphism of C^ -algebras*

$$\Phi : C(X) \rightarrow C(X) \otimes C(G)$$

has a universal object, denoted $G^+(X)$, and called affine quantum isometry group of X .

PROOF. Observe first that in the case where the above morphism Φ exists, this morphism is automatically a coaction, in the sense that it satisfies the following conditions:

$$(\Phi \otimes id)\Phi = (id \otimes \Delta)\Phi$$

$$(id \otimes \varepsilon)\Phi = id$$

In order to prove now the result, assume that $X \subset S_{\mathbb{C},+}^{N-1}$ comes as follows:

$$C(X) = C(S_{\mathbb{C},+}^{N-1}) / \left\langle f_{\alpha}(x_1, \dots, x_N) = 0 \right\rangle$$

Consider now the following variables:

$$X_i = \sum_j x_j \otimes u_{ji} \in C(X) \otimes C(U_N^+)$$

Our claim is that the quantum group in the statement $G = G^+(X)$ appears as:

$$C(G) = C(U_N^+) / \left\langle f_\alpha(X_1, \dots, X_N) = 0 \right\rangle$$

In order to prove this claim, we have to clarify how the relations $f_\alpha(X_1, \dots, X_N) = 0$ are interpreted inside $C(U_N^+)$, and then show that G is indeed a quantum group.

So, pick one of the defining polynomials, $f = f_\alpha$, and write it as follows:

$$f(x_1, \dots, x_N) = \sum_r \sum_{i_1^r \dots i_{s_r}^r} \lambda_r \cdot x_{i_1^r} \dots x_{i_{s_r}^r}$$

With $X_i = \sum_j x_j \otimes u_{ji}$ as above, we have the following formula:

$$f(X_1, \dots, X_N) = \sum_r \sum_{i_1^r \dots i_{s_r}^r} \lambda_r \sum_{j_1^r \dots j_{s_r}^r} x_{j_1^r} \dots x_{j_{s_r}^r} \otimes u_{j_1^r i_1^r} \dots u_{j_{s_r}^r i_{s_r}^r}$$

Since the variables on the right span a certain finite dimensional space, the relations $f(X_1, \dots, X_N) = 0$ correspond to certain relations between the variables u_{ij} . Thus, we have indeed a subspace $G \subset U_N^+$, with a universal map:

$$\Phi : C(X) \rightarrow C(X) \otimes C(G)$$

In order to show now that G is a quantum group, consider the following elements:

$$u_{ij}^\Delta = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u_{ij}^\varepsilon = \delta_{ij} \quad , \quad u_{ij}^S = u_{ji}^*$$

Consider as well the following elements, with $\gamma \in \{\Delta, \varepsilon, S\}$:

$$X_i^\gamma = \sum_j x_j \otimes u_{ji}^\gamma$$

From the relations $f(X_1, \dots, X_N) = 0$ we deduce that we have:

$$f(X_1^\gamma, \dots, X_N^\gamma) = (id \otimes \gamma) f(X_1, \dots, X_N) = 0$$

Thus we can map $u_{ij} \rightarrow u_{ij}^\gamma$ for any $\gamma \in \{\Delta, \varepsilon, S\}$, and we are done. \square

Before getting further, we should clarify the relation between Proposition 3.1, Theorem 3.2, and the ‘‘toral isometry’’ constructions from chapter 1 above. We have:

THEOREM 3.3. *Given an algebraic manifold $X \subset S_{\mathbb{C},+}^{N-1}$, the category of the closed subgroups $G \subset \mathcal{G}$ acting affinely on X , with \mathcal{G} being one of the following quantum groups,*

$$\begin{array}{ccccc} \mathbb{T}_N^+ & \longrightarrow & K_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{T}_N & \longrightarrow & K_N & \longrightarrow & U_N \end{array}$$

has a universal object, denoted respectively as follows,

$$\begin{array}{ccccc} T^+(X) & \longrightarrow & K^+(X) & \longrightarrow & G^+(X) \\ \uparrow & & \uparrow & & \uparrow \\ T(X) & \longrightarrow & K(X) & \longrightarrow & G(X) \end{array}$$

which appears by intersecting $G^+(X)$ and \mathcal{G} , inside U_N^+ .

PROOF. Here the assertion regarding $G^+(X)$ is something that we know, from Theorem 3.2, and all the other assertions follow from this, by intersecting with \mathcal{G} . \square

3b. Spheres and rotations

In connection with our axiomatization questions for the quadruplets (S, T, U, K) , we can construct now the correspondences $S \rightarrow U$, in the following way:

THEOREM 3.4. *The quantum isometry groups of the basic spheres,*

$$\begin{array}{ccc} S_{\mathbb{R},+}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\ \uparrow & & \uparrow \\ S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{C}}^{N-1} \end{array}$$

are respectively the basic unitary quantum groups, namely

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array}$$

modulo identifying, as usual, the various C^ -algebraic completions.*

PROOF. We have 4 results to be proved, and we can proceed as follows:

$S_{\mathbb{C},+}^{N-1}$. Let us first construct an action $U_N^+ \curvearrowright S_{\mathbb{C},+}^{N-1}$. We must prove here that the variables $X_i = \sum_j x_j \otimes u_{ji}$ satisfy the defining relations for $S_{\mathbb{C},+}^{N-1}$, namely:

$$\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$$

By using the biunitarity of u , we have the following computation:

$$\begin{aligned} \sum_i X_i X_i^* &= \sum_{ijk} x_j x_k^* \otimes u_{ji} u_{ki}^* \\ &= \sum_j x_j x_j^* \otimes 1 \\ &= 1 \otimes 1 \end{aligned}$$

Once again by using the biunitarity of u , we have as well:

$$\begin{aligned} \sum_i X_i^* X_i &= \sum_{ijk} x_j^* x_k \otimes u_{ji}^* u_{ki} \\ &= \sum_j x_j^* x_j \otimes 1 \\ &= 1 \otimes 1 \end{aligned}$$

Thus we have an action $U_N^+ \curvearrowright S_{\mathbb{C},+}^{N-1}$, which gives $G^+(S_{\mathbb{C},+}^{N-1}) = U_N^+$, as desired.

$S_{\mathbb{R},+}^{N-1}$. Let us first construct an action $O_N^+ \curvearrowright S_{\mathbb{R},+}^{N-1}$. We already know that the variables $X_i = \sum_j x_j \otimes u_{ji}$ satisfy the defining relations for $S_{\mathbb{C},+}^{N-1}$, so we just have to check that these variables are self-adjoint. But this follows from $u = \bar{u}$, as follows:

$$\begin{aligned} X_i^* &= \sum_j x_j^* \otimes u_{ji}^* \\ &= \sum_j x_j \otimes u_{ji} \\ &= X_i \end{aligned}$$

Conversely, assume that we have an action $G \curvearrowright S_{\mathbb{R},+}^{N-1}$, with $G \subset U_N^+$. The variables $X_i = \sum_j x_j \otimes u_{ji}$ must be then self-adjoint, and the above computation shows that we must have $u = \bar{u}$. Thus our quantum group must satisfy $G \subset O_N^+$, as desired.

$S_{\mathbb{C}}^{N-1}$. The fact that we have an action $U_N \curvearrowright S_{\mathbb{C}}^{N-1}$ is clear. Conversely, assume that we have an action $G \curvearrowright S_{\mathbb{C}}^{N-1}$, with $G \subset U_N^+$. We must prove that this implies $G \subset U_N$,

and we will use a standard trick from [38]. We have:

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

By multiplying this formula with itself we obtain:

$$\Phi(x_i x_k) = \sum_{jl} x_j x_l \otimes u_{ji} u_{lk}$$

$$\Phi(x_k x_i) = \sum_{jl} x_l x_j \otimes u_{lk} u_{ji}$$

Since the variables x_i commute, these formulae can be written as:

$$\Phi(x_i x_k) = \sum_{j < l} x_j x_l \otimes (u_{ji} u_{lk} + u_{li} u_{jk}) + \sum_j x_j^2 \otimes u_{ji} u_{jk}$$

$$\Phi(x_i x_k) = \sum_{j < l} x_j x_l \otimes (u_{lk} u_{ji} + u_{jk} u_{li}) + \sum_j x_j^2 \otimes u_{jk} u_{ji}$$

Since the tensors at left are linearly independent, we must have:

$$u_{ji} u_{lk} + u_{li} u_{jk} = u_{lk} u_{ji} + u_{jk} u_{li}$$

By applying the antipode to this formula, then applying the involution, and then relabelling the indices, we successively obtain:

$$u_{kl}^* u_{ij}^* + u_{kj}^* u_{il}^* = u_{ij}^* u_{kl}^* + u_{il}^* u_{kj}^*$$

$$u_{ij} u_{kl} + u_{il} u_{kj} = u_{kl} u_{ij} + u_{kj} u_{il}$$

$$u_{ji} u_{lk} + u_{jk} u_{li} = u_{lk} u_{ji} + u_{li} u_{jk}$$

Now by comparing with the original formula, we obtain from this:

$$u_{li} u_{jk} = u_{jk} u_{li}$$

In order to finish, it remains to prove that the coordinates u_{ij} commute as well with their adjoints. For this purpose, we use a similar method. We have:

$$\Phi(x_i x_k^*) = \sum_{jl} x_j x_l^* \otimes u_{ji} u_{lk}^*$$

$$\Phi(x_k^* x_i) = \sum_{jl} x_l^* x_j \otimes u_{lk}^* u_{ji}$$

Since the variables on the left are equal, we deduce from this that we have:

$$\sum_{jl} x_j x_l^* \otimes u_{ji} u_{lk}^* = \sum_{jl} x_j x_l^* \otimes u_{lk}^* u_{ji}$$

Thus we have $u_{ji} u_{lk}^* = u_{lk}^* u_{ji}$, and so $G \subset U_N$, as claimed.

$\underline{S_{\mathbb{R}}^{N-1}}$. The fact that we have an action $O_N \curvearrowright S_{\mathbb{R}}^{N-1}$ is clear. In what regards the converse, this follows by combining the results that we already have, as follows:

$$\begin{aligned} G \curvearrowright S_{\mathbb{R}}^{N-1} &\implies G \curvearrowright S_{\mathbb{R},+}^{N-1}, S_{\mathbb{C}}^{N-1} \\ &\implies G \subset O_N^+, U_N \\ &\implies G \subset O_N^+ \cap U_N = O_N \end{aligned}$$

Thus, we conclude that we have $G^+(S_{\mathbb{R}}^{N-1}) = O_N$, as desired. \square

Summarizing, in relation with our plan for this chapter, we are done with the correspondences $S \rightarrow U$, modulo the fact, which is of importance, but not directly related to our axiomatization, that we still have to clarify the metric aspects of the actions $U \curvearrowright S$ that we constructed. We will discuss this at the end of this chapter.

Let us discuss now the construction $U \rightarrow S$. In the classical case the situation is very simple, because S appears by rotating the point $x = (1, 0, \dots, 0)$ by the isometries in U . Equivalently, $S = S^{N-1}$ appears from $U = U_N$ as an homogeneous space, as follows:

$$S^{N-1} = U_N / U_{N-1}$$

In functional analytic terms, all this becomes even simpler, the correspondence $U \rightarrow S$ being obtained, at the level of algebras of functions, as follows:

$$C(S^{N-1}) \subset C(U_N) \quad , \quad x_i \rightarrow u_{1i}$$

In general now, let us start with the following observation:

PROPOSITION 3.5. *For the basic spheres, we have a diagram as follows,*

$$\begin{array}{ccc} C(S) & \xrightarrow{\Phi} & C(S) \otimes C(U) \\ \downarrow \alpha & & \downarrow \alpha \otimes id \\ C(U) & \xrightarrow{\Delta} & C(U) \otimes C(U) \end{array}$$

where the map on top is the affine coaction map,

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

and the map on the left is given by $\alpha(x_i) = u_{1i}$.

PROOF. The diagram in the statement commutes indeed on the standard coordinates, the corresponding arrows being as follows:

$$\begin{array}{ccc} x_i & \longrightarrow & \sum_j x_j \otimes u_{ji} \\ \downarrow & & \downarrow \\ u_{1i} & \longrightarrow & \sum_j u_{1j} \otimes u_{ji} \end{array}$$

Thus by linearity and multiplicativity, the whole the diagram commutes. \square

We therefore have the following result:

THEOREM 3.6. *We have a quotient map and an inclusion as follows,*

$$U \rightarrow S_U \subset S$$

with S_U being the first row space of U , given by

$$C(S_U) = \langle u_{1i} \rangle \subset C(U)$$

at the level of the corresponding algebras of functions.

PROOF. At the algebra level, we have an inclusion and a quotient map as follows:

$$C(S) \rightarrow C(S_U) \subset C(U)$$

Thus, we obtain the result, by transposing. \square

We will prove in what follows that the inclusion $S_U \subset S$ is an isomorphism. This will produce the correspondence $U \rightarrow S$ that we are currently looking for.

In order to do so, we will use the uniform integration over S , which can be introduced, in analogy with what happens in the classical case, in the following way:

DEFINITION 3.7. *We endow each of the algebras $C(S)$ with its integration functional*

$$\int_S : C(S) \rightarrow C(U) \rightarrow \mathbb{C}$$

obtained by composing the morphism of algebras given by

$$x_i \rightarrow u_{1i}$$

with the Haar integration functional of the algebra $C(U)$.

In order to efficiently integrate over the sphere S , we need to know how to efficiently integrate over the corresponding quantum group U . Following [21], [33], we have:

THEOREM 3.8. *Assuming that a compact quantum group $G \subset U_N^+$ is easy, coming from a category of partitions $D \subset P$, we have the Weingarten formula*

$$\int_G u_{i_1 j_1}^{e_1} \cdots u_{i_k j_k}^{e_k} = \sum_{\pi, \sigma \in D(k)} \delta_\pi(i) \delta_\sigma(j) W_{kN}(\pi, \sigma)$$

for any indices $i_r, j_r \in \{1, \dots, N\}$ and any exponents $e_r \in \{\emptyset, *\}$, where δ are the usual Kronecker type symbols, and where

$$W_{kN} = G_{kN}^{-1}$$

is the inverse of the matrix $G_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma|}$.

PROOF. Let us arrange indeed all the integrals to be computed, at a fixed value of the exponent $k = (e_1 \dots e_k)$, into a single matrix, of size $N^k \times N^k$, as follows:

$$P_{i_1 \dots i_k, j_1 \dots j_k} = \int_G u_{i_1 j_1}^{e_1} \cdots u_{i_k j_k}^{e_k}$$

According to the construction of the Haar measure from [99], explained in chapter 2 above, this matrix P is the orthogonal projection onto the following space:

$$\text{Fix}(u^{\otimes k}) = \text{span} \left(\xi_\pi \mid \pi \in D(k) \right)$$

Consider now the following linear map:

$$E(x) = \sum_{\pi \in D(k)} \langle x, \xi_\pi \rangle \xi_\pi$$

Consider as well the inverse W of the restriction of E to:

$$\text{span} \left(T_\pi \mid \pi \in D(k) \right)$$

By a standard linear algebra computation, it follows that we have:

$$P = WE$$

But the restriction of E is the linear map corresponding to G_{kN} , so W is the linear map corresponding to W_{kN} , and this gives the result. See [21], [33]. \square

Following [3], [28], we can now integrate over the spheres S , as follows:

PROPOSITION 3.9. *The integration over the basic spheres is given by*

$$\int_S x_{i_1}^{e_1} \cdots x_{i_k}^{e_k} = \sum_{\pi} \sum_{\sigma \leq \ker i} W_{kN}(\pi, \sigma)$$

with $\pi, \sigma \in D(k)$, where $W_{kN} = G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma|}$.

PROOF. According to our conventions, the integration over S is a particular case of the integration over U , via $x_i = u_{1i}$. By using the formula in Theorem 3.8, we obtain:

$$\begin{aligned} \int_S x_{i_1}^{e_1} \dots x_{i_k}^{e_k} &= \int_U u_{1i_1}^{e_1} \dots u_{1i_k}^{e_k} \\ &= \sum_{\pi, \sigma \in D(k)} \delta_\pi(1) \delta_\sigma(i) W_{kN}(\pi, \sigma) \\ &= \sum_{\pi, \sigma \in D(k)} \delta_\sigma(i) W_{kN}(\pi, \sigma) \end{aligned}$$

Thus, we are led to the formula in the statement. \square

Following now [28], we have the following key result:

THEOREM 3.10. *The integration functional of S has the ergodicity property*

$$\left(id \otimes \int_U \right) \Phi(x) = \int_S x$$

where $\Phi : C(S) \rightarrow C(S) \otimes C(U)$ is the universal affine coaction map.

PROOF. In the real case, $x_i = x_i^*$, it is enough to check the equality in the statement on an arbitrary product of coordinates, $x_{i_1} \dots x_{i_k}$. The left term is as follows:

$$\begin{aligned} \left(id \otimes \int_U \right) \Phi(x_{i_1} \dots x_{i_k}) &= \sum_{j_1 \dots j_k} x_{j_1} \dots x_{j_k} \int_U u_{j_1 i_1} \dots u_{j_k i_k} \\ &= \sum_{j_1 \dots j_k} \sum_{\pi, \sigma \in D(k)} \delta_\pi(j) \delta_\sigma(i) W_{kN}(\pi, \sigma) x_{j_1} \dots x_{j_k} \\ &= \sum_{\pi, \sigma \in D(k)} \delta_\sigma(i) W_{kN}(\pi, \sigma) \sum_{j_1 \dots j_k} \delta_\pi(j) x_{j_1} \dots x_{j_k} \end{aligned}$$

Let us look now at the last sum on the right. The situation is as follows:

(1) In the free case we have to sum quantities of type $x_{j_1} \dots x_{j_k}$, over all choices of multi-indices $j = (j_1, \dots, j_k)$ which fit into our given noncrossing pairing π , and just by using the condition $\sum_i x_i^2 = 1$, we conclude that the sum is 1.

(2) The same happens in the classical case. Indeed, our pairing π can now be crossing, but we can use the commutation relations $x_i x_j = x_j x_i$, and the sum is again 1.

Thus the sum on the right is 1, in all cases, and we obtain:

$$\left(id \otimes \int_U \right) \Phi(x_{i_1} \dots x_{i_k}) = \sum_{\pi, \sigma \in D(k)} \delta_\sigma(i) W_{kN}(\pi, \sigma)$$

On the other hand, another application of the Weingarten formula gives:

$$\begin{aligned} \int_S x_{i_1} \cdots x_{i_k} &= \int_U u_{1i_1} \cdots u_{1i_k} \\ &= \sum_{\pi, \sigma \in D(k)} \delta_\pi(1) \delta_\sigma(i) W_{kN}(\pi, \sigma) \\ &= \sum_{\pi, \sigma \in D(k)} \delta_\sigma(i) W_{kN}(\pi, \sigma) \end{aligned}$$

Thus, we are done with the proof of the result, in the real case. In the complex case the proof is similar, by adding exponents everywhere. \square

We can now deduce an abstract characterization of the integration, as follows:

THEOREM 3.11. *There is a unique positive unital trace $tr : C(S) \rightarrow \mathbb{C}$ satisfying*

$$(tr \otimes id)\Phi(x) = tr(x)1$$

where Φ is the coaction map of the corresponding quantum isometry group,

$$\Phi : C(S) \rightarrow C(S) \otimes C(U)$$

and this is the canonical integration, as constructed in Definition 3.7.

PROOF. First of all, it follows from the Haar integral invariance condition for U that the canonical integration has indeed the invariance property in the statement, namely:

$$(tr \otimes id)\Phi(x) = tr(x)1$$

In order to prove now the uniqueness, let tr be as in the statement. We have:

$$\begin{aligned} tr \left(id \otimes \int_U \right) \Phi(x) &= \int_U (tr \otimes id)\Phi(x) \\ &= \int_U (tr(x)1) \\ &= tr(x) \end{aligned}$$

On the other hand, according to Theorem 3.10, we have as well:

$$tr \left(id \otimes \int_U \right) \Phi(x) = tr \left(\int_S x \right) = \int_S x$$

We therefore conclude that tr equals the standard integration, as claimed. \square

Getting back now to our axiomatization questions, we have:

THEOREM 3.12. *We have correspondences between the basic unitary groups*

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array}$$

and the basic noncommutative spheres,

$$\begin{array}{ccc} S_{\mathbb{R},+}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\ \uparrow & & \uparrow \\ S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{C}}^{N-1} \end{array}$$

obtained via the operation $U \rightarrow S_U$.

PROOF. We use the ergodicity formula from Theorem 3.10, namely:

$$\left(id \otimes \int_U \right) \Phi = \int_S$$

We know that \int_U is faithful on $\mathcal{C}(U)$, and that we have:

$$(id \otimes \varepsilon)\Phi = id$$

The coaction map Φ follows to be faithful as well. Thus for any $x \in \mathcal{C}(S)$ we have:

$$\int_S xx^* = 0 \implies x = 0$$

Thus \int_S is faithful on $\mathcal{C}(S)$. But this shows that we have:

$$S = S_U$$

Thus, we are led to the conclusion in the statement. \square

3c. Tori and reflections

Summarizing, in relation with our plan for this chapter, we have satisfactory correspondences $S \leftrightarrow U$. It remains to discuss the correspondence $T \rightarrow K$. Normally this can be obtained as well via affine isometries, because in the classical case, we have:

$$K = G(T)$$

In the free case, however, things are quite tricky, with the naive formula $K = G^+(T)$ being wrong. In order to discuss this, and find the fix, we must compute the quantum isometry groups of the tori that we have. We will need the following construction:

THEOREM 3.13. *The following constructions produce compact quantum groups,*

$$\begin{aligned} C(\bar{O}_N) &= C(O_N^+) / \left\langle u_{ij}u_{kl} = \pm u_{kl}u_{ij} \right\rangle \\ C(\bar{U}_N) &= C(U_N^+) / \left\langle u_{ij}\dot{u}_{kl} = \pm \dot{u}_{kl}u_{ij} \right\rangle \end{aligned}$$

with the signs corresponding to anticommutation of different entries on same rows or same columns, and commutation otherwise, and where \dot{u} stands for u or for \bar{u} .

PROOF. This is something well-known, coming from [18] and subsequent papers, where these quantum groups were first introduced, the idea being as follows:

(1) First of all, the operations $O_N \rightarrow \bar{O}_N$ and $U_N \rightarrow \bar{U}_N$ in the statement, obtained by replacing the commutation between the standard coordinates by some appropriate commutation/anticommutation, should be thought of as being $q = -1$ twistings.

(2) However, this is not exactly the $q = -1$ twisting in the sense of Drinfeld [59] and Jimbo [67], which produces non-semisimple objects, and so the result must be checked as such, independently of the $q = -1$ twisting literature related to [59], [67].

(3) But this is something elementary, which follows in the usual way, by considering the matrices $u^\Delta, u^\varepsilon, u^S$, defined by the same formulae as for O_N^+, U_N^+ , and proving that these matrices satisfy the same relations as u .

(4) Indeed, let us first discuss the construction of the quantum group \bar{O}_N . We must prove that the algebra $C(\bar{O}_N)$ obtained from $C(O_N^+)$ via the relations in the statement has a comultiplication Δ , a counit ε , and an antipode S .

(5) Regarding the construction of the comultiplication Δ , let us set:

$$U_{ij} = \sum_k u_{ik} \otimes u_{kj}$$

For $j \neq k$ we have the following computation:

$$\begin{aligned} U_{ij}U_{ik} &= \sum_{s \neq t} u_{is}u_{it} \otimes u_{sj}u_{tk} + \sum_s u_{is}u_{is} \otimes u_{sj}u_{sk} \\ &= \sum_{s \neq t} -u_{it}u_{is} \otimes u_{tk}u_{sj} + \sum_s u_{is}u_{is} \otimes (-u_{sk}u_{sj}) \\ &= -U_{ik}U_{ij} \end{aligned}$$

Also, for $i \neq k, j \neq l$ we have the following computation:

$$\begin{aligned} U_{ij}U_{kl} &= \sum_{s \neq t} u_{is}u_{kt} \otimes u_{sj}u_{tl} + \sum_s u_{is}u_{ks} \otimes u_{sj}u_{sl} \\ &= \sum_{s \neq t} u_{kt}u_{is} \otimes u_{tl}u_{sj} + \sum_s (-u_{ks}u_{is}) \otimes (-u_{sl}u_{sj}) \\ &= U_{kl}U_{ij} \end{aligned}$$

Thus, we can define a comultiplication map for $C(\bar{O}_N)$, by setting:

$$\Delta(u_{ij}) = U_{ij}$$

(6) Regarding now the counit ε and the antipode S , things are clear here, by using the same method, and with no computations needed, the formulae to be satisfied being trivially satisfied. We conclude that \bar{O}_N is a compact quantum group.

(7) The proof that the quantum space \bar{U}_N in the statement is indeed a quantum group is similar, by adding $*$ exponents everywhere in the above computations. \square

All the above might seem to be a bit ad-hoc, but there is way of doing the $q = -1$ twisting in a more conceptual way as well, by using representation theory and Tannakian duality. We will be back later to all this, in chapter 11 below, with full details.

Now back to our axiomatization questions, we have:

THEOREM 3.14. *The quantum isometry groups of the basic tori*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

are the following quantum groups,

$$\begin{array}{ccc} H_N^+ & \longrightarrow & K_N^+ \\ \vdots & & \vdots \\ \bar{O}_N & \longrightarrow & \bar{U}_N \end{array}$$

where \bar{O}_N, \bar{U}_N are the standard $q = -1$ twists of O_N, U_N .

PROOF. In all cases we must find the conditions on a closed subgroup $G \subset U_N^+$ such that the following formula defines a coaction:

$$g_i \rightarrow \sum_j g_j \otimes u_{ji}$$

Since the coassociativity of such a map is automatic, we are left with checking that the map itself exists, and this is the same as checking that the following variables satisfy the same relations as the generators $g_i \in G$:

$$G_i = \sum_j g_j \otimes u_{ji}$$

(1) For $\Gamma = \mathbb{Z}_2^N$ the relations to be checked are as follows:

$$G_i = G_i^* \quad , \quad G_i^2 = 1 \quad , \quad G_i G_j = G_j G_i$$

Regarding the first relation, namely $G_i = G_i^*$, by using $g_i = g_i^*$ this reads:

$$\sum_j g_j \otimes u_{ji} = \sum_j g_j \otimes u_{ji}^*$$

Now since the group generators g_j are linearly independent, we obtain from this relation that we must have $u_{ij} = u_{ij}^*$ for any i, j . Thus, the condition on G is:

$$G \subset O_N^+$$

We have the following formula, for the squares:

$$\begin{aligned} G_i^2 &= \sum_{kl} g_k g_l \otimes u_{ki} u_{li} \\ &= 1 + \sum_{k < l} g_k g_l \otimes (u_{ki} u_{li} + u_{li} u_{ki}) \end{aligned}$$

We have as well the following formula, for the commutants:

$$\begin{aligned} [G_i, G_j] &= \sum_{kl} g_k g_l \otimes (u_{ki} u_{lj} - u_{kj} u_{li}) \\ &= \sum_{k < l} g_k g_l \otimes (u_{ki} u_{lj} - u_{kj} u_{li} + u_{li} u_{kj} - u_{lj} u_{ki}) \end{aligned}$$

From the first relation we obtain $ab = -ba$ for $a \neq b$ on the same column of u , and by using the antipode, the same happens for rows. From the second relation we obtain:

$$[u_{ki}, u_{lj}] = [u_{kj}, u_{li}] \quad , \quad \forall k \neq l$$

Now by applying the antipode we obtain from this:

$$[u_{ik}, u_{jl}] = [u_{jk}, u_{il}] \quad , \quad \forall k \neq l$$

By relabelling, this gives the following formula:

$$[u_{ki}, u_{lj}] = [u_{li}, u_{kj}] \quad , \quad \forall i \neq j$$

Summing up, we are therefore led to the following conclusion:

$$[u_{ki}, u_{lj}] = [u_{kj}, u_{li}] = 0 \quad , \quad \forall i \neq j, k \neq l$$

Thus we must have $G \subset \bar{O}_N$, and this finishes the proof.

(2) For $\Gamma = \mathbb{Z}_2^{*N}$ the relations to be checked are as follows:

$$G_i = G_i^* \quad , \quad G_i^2 = 1$$

As before, regarding the first relation, $G_i = G_i^*$, by using $g_i = g_i^*$ this reads:

$$\sum_j g_j \otimes u_{ji} = \sum_j g_j \otimes u_{ji}^*$$

Now since the group generators g_j are linearly independent, we obtain from this relation that we must have $u_{ij} = u_{ij}^*$ for any i, j . Thus, the condition on G is:

$$G \subset O_N^+$$

Also as before, in what regards the squares, we have:

$$\begin{aligned} G_i^2 &= \sum_{kl} g_k g_l \otimes u_{ki} u_{li} \\ &= 1 + \sum_{k \neq l} g_k g_l \otimes u_{ki} u_{li} \end{aligned}$$

Thus we obtain $G \subset H_N^+$, as claimed.

(3) For $\Gamma = \mathbb{Z}^N$ the relations to be checked are as follows:

$$G_i G_i^* = G_i^* G_i = 1 \quad , \quad G_i G_j = G_j G_i$$

In what regards the first relation, we have the following formula:

$$\begin{aligned} G_i G_i^* &= \sum_{kl} g_k g_l^{-1} \otimes u_{ki} u_{li}^* \\ &= 1 + \sum_{k \neq l} g_k g_l^{-1} \otimes u_{ki} u_{li}^* \end{aligned}$$

Also, we have the following formula:

$$\begin{aligned} G_i^* G_i &= \sum_{kl} g_k^{-1} g_l \otimes u_{ki}^* u_{li} \\ &= 1 + \sum_{k \neq l} g_k^{-1} g_l \otimes u_{ki}^* u_{li} \end{aligned}$$

Finally, we have the following formula for the commutants:

$$\begin{aligned} [G_i, G_j] &= \sum_{kl} g_k g_l \otimes (u_{ki} u_{lj} - u_{kj} u_{li}) \\ &= \sum_{k < l} g_k g_l \otimes (u_{ki} u_{lj} - u_{kj} u_{li} + u_{li} u_{kj} - u_{lj} u_{ki}) \end{aligned}$$

From the first relation we obtain $ab = -ba$ for $a \neq b$ on the same column of u , and by using the antipode, the same happens for rows. From the second relation we obtain:

$$[u_{ki}, u_{lj}] = [u_{kj}, u_{li}] \quad , \quad \forall k \neq l$$

By processing these formulae as before, in the proof of (1) above, we obtain from this that we must have $G \subset \bar{U}_N$, as claimed.

(4) For $\Gamma = F_N$ the relations to be checked are as follows:

$$G_i G_i^* = G_i^* G_i = 1$$

As before, in what regards the first relation, we have the following formula:

$$\begin{aligned} G_i G_i^* &= \sum_{kl} g_k g_l^{-1} \otimes u_{ki} u_{li}^* \\ &= 1 + \sum_{k \neq l} g_k g_l^{-1} \otimes u_{ki} u_{li}^* \end{aligned}$$

Also as before, we have the following formula:

$$\begin{aligned} G_i^* G_i &= \sum_{kl} g_k^{-1} g_l \otimes u_{ki}^* u_{li} \\ &= 1 + \sum_{k \neq l} g_k^{-1} g_l \otimes u_{ki}^* u_{li} \end{aligned}$$

By processing these formulae as before, in the proof of (2) above, we obtain from this that we must have $G \subset K_N^+$, as claimed. \square

The above result is quite surprising, and does not fit with what happens in the classical case, where the classical isometry groups of the tori are the reflection groups.

In short, the above result is not exactly what we want, in view of our axiomatization purposes here. However, we can “recycle” it, as follows:

THEOREM 3.15. *The basic noncommutative tori, namely*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

produce the basic quantum reflection groups, namely

$$\begin{array}{ccc} H_N^+ & \longrightarrow & K_N^+ \\ \uparrow & & \uparrow \\ H_N & \longrightarrow & K_N \end{array}$$

via the operation $T \rightarrow G^+(T) \cap K_N^+$.

PROOF. The operation in the statement produces the following intersections:

$$\begin{array}{ccc} H_N^+ & \longrightarrow & K_N^+ \\ \uparrow & & \uparrow \\ \bar{O}_N \cap H_N^+ & \longrightarrow & \bar{U}_N \cap K_N^+ \end{array}$$

But a routine computation, coming from the fact that commutation + anticommutation means vanishing, gives the quantum groups in the statement. Indeed:

(1) In what regards $\bar{U}_N \cap K_N^+$, here as explained above we can use the fact that commutation + anticommutation means vanishing, and we obtain, as desired:

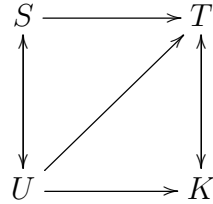
$$\begin{aligned} \bar{U}_N \cap K_N^+ &= (\bar{U}_N \cap K_N^+)_{class} \\ &= \bar{U}_N \cap K_N \\ &= K_N \end{aligned}$$

(2) In what regards $\bar{O}_N \cap H_N^+$, here we can proceed as follows:

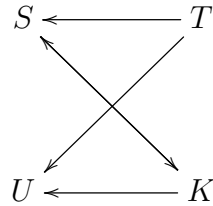
$$\begin{aligned} \bar{O}_N \cap H_N^+ &= \bar{O}_N \cap H_N^+ \cap (\bar{U}_N \cap K_N^+) \\ &= \bar{O}_N \cap H_N^+ \cap K_N \\ &= H_N \end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

As a conclusion to all this, we have now correspondences between the pairs (S, T) constructed in chapter 1 and the pairs (U, K) constructed in chapter 2, and together with the correspondences already established in chapters 1-2, our diagram looks as follows:



Thus, in order to finish our axiomatization program for the abstract noncommutative geometries, we are left with constructing correspondences as follows:



We will be back to this in the next chapter, with the construction of some of these correspondences, and with the axiomatization of the quadruplets of type (S, T, U, K) .

3d. Metric aspects

Following [28] and subsequent papers, let us comment now on the “metric” aspects of our quantum isometry group constructions.

To start with, we have the following basic definition:

DEFINITION 3.16. *Given a compact Riemannian manifold X , we denote by $\Omega^1(X)$ the space of smooth 1-forms on X , with scalar product given by*

$$\langle \omega, \eta \rangle = \int_X \langle \omega(x), \eta(x) \rangle dx$$

and we construct the Hodge Laplacian $\Delta : L^2(X) \rightarrow L^2(X)$ by setting

$$\Delta = d^*d$$

where $d : C^\infty(X) \rightarrow \Omega^1(X)$ is the usual differential map, and d^* is its adjoint.

According to a standard differential geometry result, whose proof is elementary, the classical isometry group $\mathcal{G}(X)$ of our Riemannian manifold X is then the group of diffeomorphisms $\varphi : X \rightarrow X$ whose induced action on $C^\infty(X)$ commutes with Δ .

In view of this, following now Goswami [64], we can formulate:

DEFINITION 3.17. *Associated to a compact Riemannian manifold X are:*

- (1) $\mathcal{D}^+(X)$: *the category of compact quantum groups acting on X .*
- (2) $\mathcal{G}^+(X) \in \mathcal{D}^+(X)$: *the universal object with a coaction commuting with Δ .*

Here the coactions maps $\Phi : C(X) \rightarrow C(X) \otimes C(G)$ that we consider in (1) must satisfy by definition the usual axioms for the algebraic coactions, namely:

$$\begin{aligned} (\Phi \otimes id)\Phi &= (id \otimes \Delta)\Phi \\ (id \otimes \varepsilon)\Phi &= id \end{aligned}$$

In addition, these are subject as well to the following smoothness assumption:

$$\Phi(C^\infty(X)) \subset C^\infty(X) \otimes C(G)$$

As for the commutation condition with Δ in (2) above, this regards the canonical extension of the action to the space $L^2(X)$. For details here, see [64].

Before getting further, we should mention that Definition 3.17 above does not really bring new examples of compact quantum groups, and this due to a non-trivial result of Goswami, stating that when the compact Riemannian manifold X is connected we have $\mathcal{G}^+(X) = \mathcal{G}(X)$. We refer here to [64], [65], [66] and subsequent papers.

Let us record here, however, the following well-known weak version of Goswami's main rigidity result, which is something rather elementary:

PROPOSITION 3.18. *A compact connected Riemannian manifold X cannot have genuine group dual isometries.*

PROOF. We recall that for a connected Riemannian manifold X , the eigenfunctions of the Laplacian have the domain property, namely:

$$f, g \neq 0 \implies fg \neq 0$$

This is for instance because the set of zeros of each nonzero eigenfunction of the Laplacian is known to have Hausdorff dimension $\dim X - 1$, and hence measure zero. Now assume that we have a group dual coaction, as follows:

$$\Phi : C(X) \rightarrow C(X) \otimes C^*(\Gamma)$$

Let $E = E_1 \oplus E_2$ be the direct sum of two eigenspaces of the Laplacian Δ . Pick a basis $\{x_i\}$ such that the corresponding corepresentation on E becomes diagonal, in the sense that we have, for certain group elements $g_i \in \Gamma$:

$$\Phi(x_i) = x_i \otimes g_i$$

The formula $\Phi(x_i x_j) = \Phi(x_j x_i)$ reads then:

$$x_i x_j \otimes g_i g_j = x_i x_j \otimes g_j g_i$$

Thus, by using the domain property of the eigenfunctions of Δ , we obtain:

$$g_i g_j = g_j g_i$$

Also, the formula $\Phi(x_i \bar{x}_j) = \Phi(\bar{x}_j x_i)$ reads:

$$x_i \bar{x}_j \otimes g_i g_j^{-1} = x_i \bar{x}_j \otimes g_j^{-1} g_i$$

Thus by using the domain property again, we obtain:

$$g_i g_j^{-1} = g_j^{-1} g_i$$

Thus the elements $\{g_i, g_i^{-1}\}$ pairwise commute, and with the eigenspace E varying, this shows that our group Γ must be abelian, as claimed. \square

Let us discuss now the case of the noncommutative Riemannian manifolds. We will use in what follows some very light axioms, inspired from Connes' ones from [51]:

DEFINITION 3.19. *A baby spectral triple $X = (A, H, D)$ consists of the following:*

- (1) *A is a unital C^* -algebra.*
- (2) *H is a Hilbert space, on which A acts.*
- (3) *D is an unbounded self-adjoint operator on H , with compact resolvents, such that $[D, \phi]$ has a bounded extension, for any ϕ in a dense $*$ -subalgebra $\mathcal{A} \subset A$.*

The guiding examples come from the compact Riemannian manifolds X . Indeed, associated to any such manifold X are several triples (A, H, D) , with the dense $*$ -subalgebra $\mathcal{A} \subset A$ being the algebra $C^\infty(X) \subset C(X)$, as follows:

- (1) H is the space of square-integrable spinors, and D is the Dirac operator.
- (2) H is the space of forms on X , and D is the Hodge-Dirac operator $d + d^*$.
- (3) $H = L^2(X, dv)$, dv being the Riemannian volume, and $D = d^*d$.

In this list the first example is the most interesting one, and by far, and this because under a number of supplementary axioms, a reconstruction theorem for X holds, in terms of the associated triple (A, H, D) . We refer to Connes' paper [52] for a proof of this fact, and for the definition of the "true" spectral triples as well.

In view of Definition 3.17 (2), however, the last example in the above list will be in fact the one that we will be interested in. Once again following Goswami [64], we have:

DEFINITION 3.20. *Associated to a baby spectral triple $X = (A, H, D)$ are:*

- (1) $\mathcal{D}^+(X)$: *the category of compact quantum groups acting on (A, H) .*
- (2) $\mathcal{G}^+(X) \in \mathcal{D}^+(X)$: *the universal object with a coaction commuting with D .*

In other words, $\mathcal{G}^+(X)$ must have a unitary representation U on H which commutes with D , satisfies $U1_A = 1 \otimes 1_A$, and is such that ad_U maps A'' into itself.

Now back to our spheres, we will construct a baby spectral triple, in the sense of Definition 3.19, and then compute the corresponding quantum isometry group, in the sense of Definition 3.20, with the result that this is in fact the affine isometry one.

The idea is to use the inclusion $S_{\mathbb{R}}^{N-1} \subset S_{\mathbb{R},\times}^{N-1}$, and to construct the Laplacian filtration as the pullback of the Laplacian filtration for $S_{\mathbb{R}}^{N-1}$, as follows:

PROPOSITION 3.21. *Associated to any real sphere $S_{\mathbb{R},\times}^{N-1}$ is the baby triple (A, H, D) , where $A = C(S_{\mathbb{R},\times}^{N-1})$, and where D acting on $H = L^2(A, tr)$ is defined as follows:*

(1) *Consider the following linear space:*

$$H_k = \text{span} \left(x_{i_1} \dots x_{i_r} \mid r \leq k \right)$$

(2) *Define $E_k = H_k \cap H_{k-1}^\perp$, so that we have:*

$$H = \bigoplus_{k=0}^{\infty} E_k$$

(3) *Finally, set $Dx = \lambda_k x$, for any $x \in E_k$, where λ_k are distinct numbers.*

PROOF. We have to show that the operator $[D, T_i]$ is bounded, where T_i is the left multiplication by x_i . Since $x_i \in A$ is self-adjoint, so is the corresponding operator T_i . Now since we have $T_i(H_k) \subset H_{k+1}$, by self-adjointness we get:

$$T_i(H_k^\perp) \subset H_{k-1}^\perp$$

Thus we have inclusions as follows:

$$T_i(E_k) \subset E_{k-1} \oplus E_k \oplus E_{k+1}$$

This gives a decomposition of the following type:

$$T_i = T_i^{-1} + T_i^0 + T_i^1$$

We have then $[D, T_i^\alpha] = \alpha T_i^\alpha$ for any $\alpha \in \{-1, 0, 1\}$, and this gives the result. \square

Summarizing, what we constructed above is some kind of algebraic structure on our spheres, coming from the eigenspaces of the Laplacian. This structure misses however the fine geometric aspects, coming from the eigenvalues, at least in the above formulation.

However, with our formalism, we can now prove, following [28]:

THEOREM 3.22. *We have the quantum isometry group formula*

$$\mathcal{G}^+(S_{\mathbb{R},\times}^{N-1}) = O_N^\times$$

with respect to the baby spectral triple structure constructed above.

PROOF. Consider the universal affine coaction map on our sphere:

$$\Phi : C(S_{\mathbb{R}, \times}^{N-1}) \rightarrow C(S_{\mathbb{R}, \times}^{N-1}) \otimes C(O_N^\times)$$

This coaction map extends to a unitary representation on the GNS space H , that we denote by U . We have then an inclusion, as follows:

$$\Phi(H_k) \subset H_k \otimes C(O_N^\times)$$

Now observe that this formula reads:

$$U(H_k) \subset H_k$$

By unitarity we obtain as well:

$$U(H_k^\perp) \subset H_k^\perp$$

Thus each space E_k is U -invariant, and U, D must commute. We conclude that Φ is isometric with respect to D . Finally, the universality of O_N^\times is clear. \square

There are several interesting questions in relation with the above. First, we have the question of understanding what happens for the complex spheres, and also for the tori, real or complex. Also, we have the question of understanding what the eigenvalues of the Laplacian are, and whether this resulting Laplacian can be used in order to formulate basic PDE over our spheres. We refer here to [56] and related papers, for a discussion.

3e. Exercises

As a first exercise for this chapter, in connection with the notion of affine quantum isometry, which was central here, we have:

EXERCISE 3.23. *Work out the basic theory of the classical and quantum reflection isometry groups, defined for a real algebraic manifold $X \subset S_{\mathbb{C}, +}^{N-1}$, and given by*

$$K(X) = G(X) \cap K_N$$

$$K^+(X) = G^+(X) \cap K_N^+$$

in analogy with what is known about the affine isometry groups $G(X), G^+(X)$, and their toral versions, $T(X) = G(X) \cap \mathbb{T}_N$ and $T^+(X) = G^+(X) \cap \mathbb{T}_N^+$.

To be more precise here, what is needed is an existence and uniqueness result for $K(X), K^+(X)$, obtained independently, along with some basic computations.

In relation now with the Weingarten formula, which was crucial in the above, we have:

EXERCISE 3.24. *Work out the explicit formula of the Weingarten matrix*

$$W_{kN} = G_{kN}^{-1} \quad : \quad G_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma|}$$

for the basic free quantum groups, at small values of $N \in \mathbb{N}$.

This might seem a bit anecdotal, but the exercise is very important, because every single professional user of the Weingarten formula has already spent some time on all this, and not only at very small values of $N \in \mathbb{N}$, but also at bigger values. So, why not you.

In practice now, before even starting, you will have to find a good order relation on the partitions, in order to write down the Gram and Weingarten matrices.

In relation now with the twisting operation, we have:

EXERCISE 3.25. *Find a definition for the twist of SO_3 , by using common sense and your intuition, and then prove that this twist is isomorphic to S_4^+ .*

All this might seem quite surprising, and so this exercise is definitely a must-do. As a hint here, once you found the good definition for the twist of SO_3 , by whatever ad-hoc means, try applying the Fourier transform over the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$ to the fundamental corepresentation, see what you get is magic or not.

Also, as a comment here, this is related to a difficult exercise from the previous chapter, asking to prove that the dual of S_4^+ is amenable. The above exercise does not really solve this amenability question, but makes the amenability thing quite plausible.

In relation now with differential geometry, we first have:

EXERCISE 3.26. *Prove that the eigenfunctions of the Laplacian on a connected Riemannian manifold have the domain property.*

This is something well-known, that we used in the above, when talking about the toral isometries of such manifolds. In practice now, this is quite tricky, and in case you do not know or find, search on the internet, and then write down a brief account of that.

Finally, in relation with advanced differential geometry, we have:

EXERCISE 3.27. *Try to talk about orientation in free geometry. Also, try finding eigenvalues for the eigenspaces of the Laplacian on $S_{\mathbb{R},+}^{N-1}$. Also, try constructing a spectral triple in the sense of Connes for $S_{\mathbb{R},+}^{N-1}$, and explain what works, and what fails.*

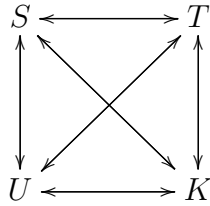
These are all difficult questions. Browsing the internet won't help much, because the answers to most of these questions is folklore. In short, this is an exercise that you will have to work out by yourself, every now and then, in order to join the community.

CHAPTER 4

Axiomatization

4a. Basic quadruplets

We finish here our axiomatization work. We recall that our goal is that of axiomatizing the quadruplets (S, T, U, K) consisting of a quantum sphere, torus, unitary group and reflection group, with a full set of correspondences between them, as follows:



In order to discuss all this, we first need precise definitions for all the objects involved. So, let us start with the following general definition:

DEFINITION 4.1. *We call quantum sphere, quantum torus, quantum unitary group and quantum reflection group the intermediate objects as follows,*

$$S_{\mathbb{R}}^{N-1} \subset S \subset S_{\mathbb{C},+}^{N-1}$$

$$T_N \subset T \subset \mathbb{T}_N^+$$

$$O_N \subset U \subset U_N^+$$

$$H_N \subset K \subset K_N^+$$

with S being an algebraic manifold, and T, U, K being compact quantum groups.

Here, as usual, all the objects are taken up to the standard equivalence relation for the quantum algebraic manifolds, discussed in chapter 1 above.

As a first observation, the above definition, with intermediate objects ranging between classical real and free complex, brings us into the “hybrid” zone, between real and complex.

One reason for doing so is that we would like to deal at the same time with the real and complex cases, in order to simplify our axiomatization work.

Also, and importantly, at a more advanced level, we will see later on that we have an isomorphism between the free real and complex projective spaces, as follows:

$$P_{\mathbb{R},+}^{N-1} = P_{\mathbb{C},+}^{N-1}$$

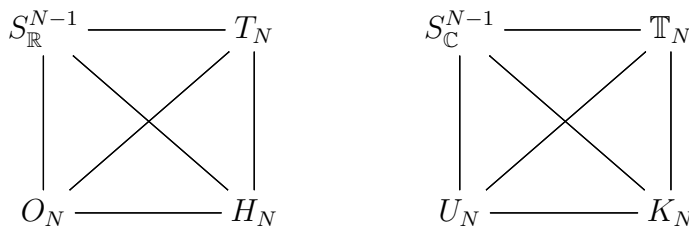
This isomorphism is something quite interesting, the conclusion being that in the free setting, the usual \mathbb{R}/\mathbb{C} dichotomy tends to become “blurred”. Thus, it is a good idea to forget about this dichotomy, and formulate things as above.

We will be back to projective geometry questions in chapter 15 below, with more explanations regarding the above isomorphism, and with other results as well.

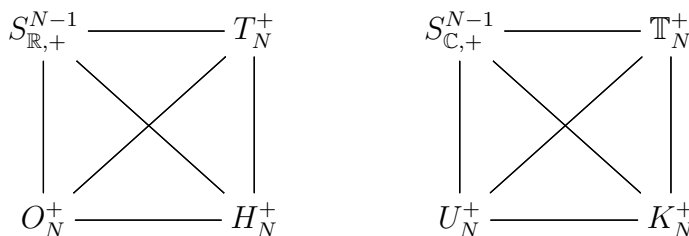
At the level of the basic examples, the situation is as follows:

PROPOSITION 4.2. *We have “basic” quadruplets (S, T, U, K) as follows:*

(1) *A classical real and a classical complex quadruplet, as follows:*



(2) *A free real and a free complex quadruplet, as follows:*



PROOF. This is more or less an empty statement, with the various objects appearing in the above diagrams being those constructed in chapters 1 and 2 above. \square

Regarding now the correspondences between our objects (S, T, U, K) , we would like to have all 12 of them axiomatized. There is still quite some work to be done here, and in order to get started, let us begin with a summary of what we have so far:

THEOREM 4.3. *For the basic quadruplets, we have correspondences as follows,*

$$\begin{array}{ccc}
 S & \longrightarrow & T \\
 \uparrow & & \nearrow \\
 U & \longrightarrow & K \\
 \downarrow & & \downarrow
 \end{array}$$

constructed via the following formulae:

- (1) $S = S_U$.
- (2) $T = S \cap \mathbb{T}_N^+ = U \cap \mathbb{T}_N^+ = K \cap \mathbb{T}_N^+$.
- (3) $U = G^+(S)$.
- (4) $K = U \cap K_N^+ = K^+(T)$.

PROOF. This is a summary of what we already have, with the fact that the 7 correspondences in the statement work well for the 4 basic quadruplets, from Proposition 4.2, coming from the various results established in chapters 1-3 above:

(1) The formula $S = S_U$ is from chapter 3, with the proof being based on an ergodicity result, ultimately coming from easiness, and the Weingarten formula.

(2) The formula $T = S \cap \mathbb{T}_N^+$ is from chapter 1, and this is something elementary, coming from definitions.

(3) The formula $T = U \cap \mathbb{T}_N^+$ is from chapter 2, and this is once again something elementary, coming from definitions.

(4) The formula $T = K \cap \mathbb{T}_N^+$ is once again from chapter 2, coming as before essentially from the definitions.

(5) The formula $U = G^+(S)$ is from chapter 3, with the proof being something quite standard, based on the tricks from [28].

(6) The formula $K = U \cap K_N^+$ is from chapter 2, and this is something elementary, coming from definitions.

(7) The formula $K = K^+(T)$ is from chapter 3, and this is definitely something quite tricky, involving $q = -1$ twists. \square

4b. Easy geometries

Our goal is that of having a full set of correspondences between our objects (S, T, U, K) . In view of the above result, a key problem is that of passing from the discrete objects (T, K) to the continuous objects (S, U) . We will solve this by doing some work at the quantum group level, in relation with the quantum groups T, K, U . To be more precise, what we would like to have are correspondences as follows:

$$T \rightarrow K \rightarrow U$$

In order to discuss this, we need some preliminaries, in relation with the intersection and generation operations for the compact quantum groups. Let us start with:

PROPOSITION 4.4. *The closed subgroups of U_N^+ are subject to operations as follows:*

- (1) *Intersection: $H \cap K$ is the biggest quantum subgroup of H, K .*
- (2) *Generation: $\langle H, K \rangle$ is the smallest quantum group containing H, K .*

PROOF. We must prove that the universal quantum groups in the statement exist indeed. For this purpose, let us pick writings as follows, with I, J being Hopf ideals:

$$C(H) = C(U_N^+)/I$$

$$C(K) = C(U_N^+)/J$$

We can then construct our two universal quantum groups, as follows:

$$C(H \cap K) = C(U_N^+)/\langle I, J \rangle$$

$$C(\langle H, K \rangle) = C(U_N^+)/(I \cap J)$$

Thus, we obtain the result. □

In practice, the operation \cap can be usually computed by using:

PROPOSITION 4.5. *Assuming $H, K \subset G$, the intersection $H \cap K$ is given by*

$$C(H \cap K) = C(G)/\{\mathcal{R}, \mathcal{P}\}$$

whenever we have formulae of type

$$C(H) = C(G)/\mathcal{R}$$

$$C(K) = C(G)/\mathcal{P}$$

with \mathcal{R}, \mathcal{P} being sets of polynomial $$ -relations between the standard coordinates.*

PROOF. This follows from Proposition 4.4 above, or rather from its proof, and from the following trivial fact, regarding relations and ideals:

$$I = \langle \mathcal{R} \rangle, J = \langle \mathcal{P} \rangle \implies \langle I, J \rangle = \langle \mathcal{R}, \mathcal{P} \rangle$$

Thus, we are led to the conclusion in the statement. □

In relation with the generation operation, let us call Hopf image of a representation $C(G) \rightarrow A$ the smallest Hopf algebra quotient $C(L)$ producing a factorization:

$$C(G) \rightarrow C(L) \rightarrow A$$

The fact that such a quotient exists indeed is routine, by dividing by a suitable ideal. This notion can be generalized to families of representations, and we have:

PROPOSITION 4.6. *Assuming $H, K \subset G$, the quantum group $\langle H, K \rangle$ is such that*

$$C(G) \rightarrow C(H \cap K) \rightarrow C(H), C(K)$$

is the joint Hopf image of the following quotient maps:

$$C(G) \rightarrow C(H), C(K)$$

PROOF. In the particular case from the statement, the joint Hopf image appears as the smallest Hopf algebra quotient $C(L)$ producing factorizations as follows:

$$C(G) \rightarrow C(L) \rightarrow C(H), C(K)$$

Thus $L = \langle H, K \rangle$, which leads to the conclusion in the statement. See [47]. \square

In the Tannakian setting now, we have the following result:

THEOREM 4.7. *The intersection and generation operations \cap and \langle, \rangle can be constructed via the Tannakian correspondence $G \rightarrow C_G$, as follows:*

- (1) *Intersection: defined via $C_{G \cap H} = \langle C_G, C_H \rangle$.*
- (2) *Generation: defined via $C_{\langle G, H \rangle} = C_G \cap C_H$.*

PROOF. This follows from Proposition 4.4, or rather from its proof, by taking I, J to be the ideals coming from Tannakian duality, in its soft form, from chapter 2 above. \square

In relation now with easiness, we first have the following result:

PROPOSITION 4.8. *Assuming that H, K are easy, then so is $H \cap K$, and we have*

$$D_{H \cap K} = \langle D_H, D_K \rangle$$

at the level of the corresponding categories of partitions.

PROOF. We have indeed the following computation:

$$\begin{aligned} C_{H \cap K} &= \langle C_H, C_K \rangle \\ &= \langle \text{span}(D_H), \text{span}(D_K) \rangle \\ &= \text{span}(\langle D_H, D_K \rangle) \end{aligned}$$

Thus, by Tannakian duality we obtain the result. \square

Regarding the generation operation, the situation is more complicated, as follows:

PROPOSITION 4.9. *Assuming that H, K are easy, we have an inclusion*

$$\langle H, K \rangle \subset \{H, K\}$$

coming from an inclusion of Tannakian categories as follows,

$$C_H \cap C_K \supset \text{span}(D_H \cap D_K)$$

where $\{H, K\}$ is the easy quantum group having as category of partitions $D_H \cap D_K$.

PROOF. This follows from the properties of the generation operation, and from:

$$\begin{aligned} C_{\langle H,K \rangle} &= C_H \cap C_K \\ &= \text{span}(D_H) \cap \text{span}(D_K) \\ &\supset \text{span}(D_H \cap D_K) \end{aligned}$$

Indeed, by Tannakian duality we obtain from this all the assertions. \square

Summarizing, we have some problems here, and we must proceed as follows:

THEOREM 4.10. *The intersection and easy generation operations \cap and $\{, \}$ can be constructed via the Tannakian correspondence $G \rightarrow D_G$, as follows:*

- (1) *Intersection: defined via $D_{G \cap H} = \langle D_G, D_H \rangle$.*
- (2) *Easy generation: defined via $D_{\{G,H\}} = D_G \cap D_H$.*

PROOF. Here (1) is an result coming from Proposition 4.8, and (2) is more of an empty statement, related to the difficulties that we met in Proposition 4.9. \square

Let us go back to the various quantum groups of type T, U, K that we are interested in. We have the following summary of the results that we have so far, along with a few new things, in relation with the intersection and generation operations:

THEOREM 4.11. *The basic quantum unitary and reflection groups,*

$$\begin{array}{ccccc} & & K_N^+ & \longrightarrow & U_N^+ \\ & \nearrow & \uparrow & & \nearrow \\ H_N^+ & \longrightarrow & O_N^+ & & \\ \uparrow & & \uparrow & & \uparrow \\ & \nearrow & K_N & \longrightarrow & U_N \\ H_N & \longrightarrow & O_N & & \end{array}$$

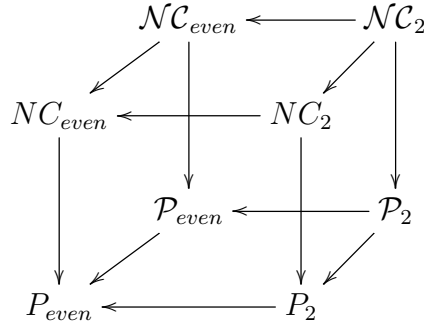
form an intersection and easy generation diagram, and their diagonal tori

$$\begin{array}{ccccc} & & \mathbb{T}_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ & \nearrow & \uparrow & & \nearrow \\ T_N^+ & \longrightarrow & T_N^+ & & \\ \uparrow & & \uparrow & & \uparrow \\ & \nearrow & \mathbb{T}_N & \longrightarrow & \mathbb{T}_N \\ T_N & \longrightarrow & T_N & & \end{array}$$

form an intersection and generation diagram.

PROOF. We have two assertions here, the idea being as follows:

(1) We know from chapter 2 above that the quantum unitary and reflection groups are all easy, the corresponding categories of partitions being as follows:



Now since these categories form an intersection and generation diagram, it follows that the quantum groups form an intersection and easy generation diagram, as claimed.

(2) Regarding now the corresponding diagonal tori, we know from chapter 2 that these are indeed the tori in the statement. As for the fact that these tori form an intersection and generation diagram, this is something well-known, and elementary. \square

It is conjectured that the above quantum group diagram should be actually a plain generation diagram. We will be back to this later.

As a first consequence of the above result, in connection with our axiomatization questions for the quadruplets (S, T, U, K) , we have the following result:

PROPOSITION 4.12. *The unitary quantum groups appear from their classical versions*

$$\begin{array}{ccc}
 O_N^+ & \longrightarrow & U_N^+ \\
 \uparrow \text{dotted} & & \uparrow \text{dotted} \\
 O_N & \longrightarrow & U_N
 \end{array}$$

via the easy liberation formula

$$G = \{G_{class}, K\}$$

where $K \subset G$ is the quantum reflection group, given by $K = G \cap K_N^+$.

PROOF. We have two formulae to be established, the idea being as follows:

(1) For the quantum group O_N^+ the classical version is O_N , and the corresponding reflection group is H_N^+ , and from the fact that the front face of the quantum group diagram in Theorem 4.11 is an easy generation diagram we obtain, as desired:

$$O_N^+ = \{O_N, H_N^+\}$$

(2) For the quantum group U_N^+ the classical version is U_N , and the corresponding reflection group is K_N^+ , and from the fact that the rear face of the quantum group diagram in Theorem 4.11 is an easy generation diagram we obtain, as desired:

$$U_N^+ = \{U_N, K_N^+\}$$

Thus, we are led to the conclusion in the statement. \square

We can further reformulate the above result, in the following way:

PROPOSITION 4.13. *The unitary quantum groups appear from their reflection subgroups*

$$\begin{array}{ccc} H_N^+ & \longrightarrow & K_N^+ \\ \uparrow & & \uparrow \\ H_N & \longrightarrow & K_N \end{array} \quad \longrightarrow \quad \begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array}$$

via the following easy generation formula

$$U = \{O_N, K\}$$

computed inside the quantum group U_N^+ .

PROOF. This is a reformulation of Proposition 4.12, as follows:

(1) In the classical orthogonal case the formula to be proved is trivial, namely:

$$O_N = \{O_N, H_N\}$$

(2) In the free orthogonal case the formula established in Proposition 4.12 is precisely the one that we need, namely:

$$O_N^+ = \{O_N, H_N^+\}$$

(3) In the classical unitary case now, the formula in the statement is as follows, coming from the fact that the bottom face of the quantum group diagram in Theorem 4.11 is an easy generation diagram:

$$U_N = \{O_N, K_N\}$$

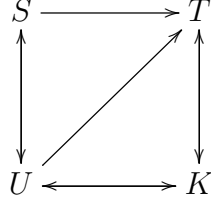
(4) In the free unitary case, we have the following computation, based on the unitary formula established in Proposition 4.12, and on the formula in (3) above:

$$\begin{aligned} U_N^+ &= \{U_N, K_N^+\} \\ &= \{\{O_N, K_N\}, K_N^+\} \\ &= \{O_N, \{K_N, K_N^+\}\} \\ &= \{O_N, K_N^+\} \end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

We can now update our main result so far, as follows:

THEOREM 4.14. *For the basic quadruplets, we have correspondences as follows,*



constructed via the following formulae:

- (1) $S = S_U$.
- (2) $T = S \cap \mathbb{T}_N^+ = U \cap \mathbb{T}_N^+ = K \cap \mathbb{T}_N^+$.
- (3) $U = G^+(S) = \{O_N, K\}$.
- (4) $K = U \cap K_N^+ = K^+(T)$.

PROOF. This is an update of Theorem 4.3, taking into account Proposition 4.13. \square

Regarding the missing correspondences, namely $T \rightarrow S, U$ and $S \leftrightarrow K$, the situation here is more complicated, and we will discuss this later.

We can however compose the correspondences that we have, and formulate, as a conclusion to what we did so far:

DEFINITION 4.15. *A quadruplet (S, T, U, K) is said to produce an easy geometry when U, K are easy, and one can pass from each object to all the other objects, as follows,*

$$\begin{aligned}
 S &= S_{\{O_N, K^+(T)\}} = S_U = S_{\{O_N, K\}} \\
 S \cap \mathbb{T}_N^+ &= T = U \cap \mathbb{T}_N^+ = K \cap \mathbb{T}_N^+ \\
 G^+(S) &= \{O_N, K^+(T)\} = U = \{O_N, K\} \\
 K^+(S) &= K^+(T) = U \cap K_N^+ = K
 \end{aligned}$$

with the usual convention that all this is up to the equivalence relation.

There are many comments that can be made here. As a first remark, if we plug the data from any axiom line into the 3 other lines, we obtain axiomatizations in terms of S, T, U, K alone, that we can try to simplify afterwards.

It is of course possible to axiomatize everything in terms of ST, SU, SK, TU, TK, UK as well, and also in terms of STU, STK, SUK, TUK , and try to simplify afterwards.

In what follows we will not bother much with this, and use Definition 4.15 as it is. We will need that 12 correspondences, as results, and whether we call such results “verifications of the axioms” or “basic properties of our geometry” is irrelevant.

Regarding now the basic examples, these are of course the classical and free, real and complex geometries. To be more precise, we have the following result:

THEOREM 4.16. *We have 4 basic easy geometries, denoted*

$$\begin{array}{ccc} \mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\ \uparrow & & \uparrow \\ \mathbb{R}^N & \longrightarrow & \mathbb{C}^N \end{array}$$

which appear from quadruplets as above, as follows:

- (1) *Classical real: produced by $(S_{\mathbb{R}}^{N-1}, T_N, O_N, H_N)$.*
- (2) *Classical complex: produced by $(S_{\mathbb{C}}^{N-1}, \mathbb{T}_N, U_N, K_N)$.*
- (3) *Free real: produced by $(S_{\mathbb{R},+}^{N-1}, T_N^+, O_N^+, H_N^+)$.*
- (4) *Free complex: produced by $(S_{\mathbb{C},+}^{N-1}, \mathbb{T}_N^+, U_N^+, K_N^+)$.*

PROOF. This is something that we already know, which follows from Theorem 4.14, as explained in the discussion preceding Definition 4.15. \square

It is possible to construct some further easy geometries in the above sense, and also to work out some structure and classification results.

To be more precise, the 4-diagram of geometries from Theorem 4.16 can be extended into a 9-diagram of geometries, as follows:

$$\begin{array}{ccccc} \mathbb{R}_+^N & \longrightarrow & \mathbb{T}\mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{R}_*^N & \longrightarrow & \mathbb{T}\mathbb{R}_*^N & \longrightarrow & \mathbb{C}_*^N \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{R}^N & \longrightarrow & \mathbb{T}\mathbb{R}^N & \longrightarrow & \mathbb{C}^N \end{array}$$

Moreover, one can prove that, under some supplementary assumptions, these 9 easy geometries are the only ones. We will be back to this.

4c. Liberation theory

Moving ahead now, if we want to improve all the above, and reach to a better axiomatization, we have two problems which are still in need to be solved, namely:

- (1) Understanding the operation $K \rightarrow U$, without reference to easiness.
- (2) Understanding the operation $T \rightarrow U$.

In short, we are back to the problem mentioned after Theorem 4.3, namely understanding the following operations, and this time without reference to easiness:

$$T \rightarrow K \rightarrow U$$

This is something quite subtle, which will take us into advanced quantum group theory. Let us start our discussion with the following definition:

DEFINITION 4.17. Consider a closed subgroup $G \subset U_N^+$, and let

$$T \subset K \subset G$$

be its diagonal torus, and its reflection subgroup. The inclusion $G_{class} \subset G$ is called:

- (1) A *soft liberation*, when $G = \langle G_{class}, K \rangle$.
- (2) A *hard liberation*, when $G = \langle G_{class}, T \rangle$.

As a first remark, given $G \subset U_N^+$, we have a diagram as follows, which is an intersection diagram, in the sense that any subsquare $P \subset Q, R \subset S$ satisfies $P = Q \cap R$:

$$\begin{array}{ccccc}
 T & \longrightarrow & K & \longrightarrow & G \\
 \uparrow & & \uparrow & & \uparrow \\
 T_{class} & \longrightarrow & K_{class} & \longrightarrow & G_{class}
 \end{array}$$

With this picture in mind, the soft liberation condition states that the square on the right $P \subset Q, R \subset S$ is a generation diagram, $\langle Q, R \rangle = S$.

As for the hard liberation condition, which is stronger, this states that the whole rectangle has this generation property.

In order to comment on all this, let us discuss now some weaker versions of the generation property, which involve spinned versions of the diagonal torus.

Let us start with the following standard result:

THEOREM 4.18. *Given a closed subgroup $G \subset U_N^+$ and a matrix $Q \in U_N$, we let $T_Q \subset G$ be the diagonal torus of G , with fundamental representation spinned by Q :*

$$C(T_Q) = C(G) / \left\langle (QuQ^*)_{ij} = 0 \mid \forall i \neq j \right\rangle$$

This torus is then a group dual, given by $T_Q = \widehat{\Lambda}_Q$, where $\Lambda_Q = \langle g_1, \dots, g_N \rangle$ is the discrete group generated by the elements

$$g_i = (QuQ^*)_{ii}$$

which are unitaries inside the quotient algebra $C(T_Q)$.

PROOF. This follows from the general results for the diagonal torus, explained in chapter 2 above, because, as said in the statement, T_Q is by definition a diagonal torus. Equivalently, since $v = QuQ^*$ is a unitary corepresentation, its diagonal entries $g_i = v_{ii}$, when regarded inside $C(T_Q)$, are unitaries, and satisfy:

$$\Delta(g_i) = g_i \otimes g_i$$

Thus $C(T_Q)$ is a group algebra, and more specifically we have $C(T_Q) = C^*(\Lambda_Q)$, where $\Lambda_Q = \langle g_1, \dots, g_N \rangle$ is the group in the statement, and this gives the result. \square

Summarizing, associated to any closed subgroup $G \subset U_N^+$ is a whole family of tori, indexed by the unitaries $U \in U_N$. We use the following terminology:

DEFINITION 4.19. *Let $G \subset U_N^+$ be a closed subgroup.*

- (1) *The tori $T_Q \subset G$ constructed above are called standard tori of G .*
- (2) *The collection of tori $T = \{T_Q \subset G \mid Q \in U_N\}$ is called skeleton of G .*

This might seem a bit awkward, but in view of various results, examples and counterexamples, to be presented below, this is perhaps the best terminology.

As a first general result now regarding these tori, coming from [98], we have:

THEOREM 4.20. *Any torus $T \subset G$ appears as follows, for a certain $Q \in U_N$:*

$$T \subset T_Q \subset G$$

In other words, any torus appears inside a standard torus.

PROOF. Given a torus $T \subset G$, we have an inclusion as follows:

$$T \subset G \subset U_N^+$$

On the other hand, we know from chapter 2 above that each torus $T \subset U_N^+$ has a fundamental corepresentation as follows, with $Q \in U_N$:

$$u = Q \text{diag}(g_1, \dots, g_N) Q^*$$

But this shows that we have $T \subset T_Q$, and this gives the result. \square

Getting back now to our generation questions, let us formulate:

DEFINITION 4.21. *A closed subgroup $G_N \subset U_N^+$, with classical version G_N^c , is called:*

(1) *Weakly generated by its tori, when:*

$$G_N = \langle G_N^c, (T_Q)_{Q \in U_N} \rangle$$

(2) *A diagonal liberation of G_N^c , when:*

$$G_N = \langle G_N^c, T_1 \rangle$$

It is elementary to check that the first property is satisfied for the groups, for the group duals, and is stable under generations, and direct products.

Regarding the second property, this is something quite interesting, which takes us away from our original generation questions. The idea here, from [46] and subsequent papers, is that such things can be proved by recurrence on $N \in \mathbb{N}$.

In order to discuss this, let us start with:

PROPOSITION 4.22. *Assume that $G = (G_N)$ is weakly uniform, let $n \in \{2, 3, \dots, \infty\}$ be minimal such that G_n is not classical, and consider the following conditions:*

- (1) *Strong generation: $G_N = \langle G_N^c, G_n \rangle$, for any $N > n$.*
- (2) *Usual generation: $G_N = \langle G_N^c, G_{N-1} \rangle$, for any $N > n$.*
- (3) *Initial step generation: $G_{n+1} = \langle G_{n+1}^c, G_n \rangle$.*

We have then (1) \iff (2) \implies (3), and (3) is in general strictly weaker.

PROOF. All the implications and non-implications are elementary, as follows:

(1) \implies (2) This follows from $G_n \subset G_{N-1}$ for $N > n$, coming from uniformity.

(2) \implies (1) By using twice the usual generation, and then the uniformity, we have:

$$\begin{aligned} G_N &= \langle G_N^c, G_{N-1} \rangle \\ &= \langle G_N^c, G_{N-1}^c, G_{N-2} \rangle \\ &= \langle G_N^c, G_{N-2} \rangle \end{aligned}$$

Thus we have a descent method, and we end up with the strong generation condition.

(2) \implies (3) This is clear, because (2) at $N = n + 1$ is precisely (3).

(3) $\not\implies$ (2) In order to construct counterexamples here, the simplest is to use group duals. Indeed, with $G_N = \widehat{\Gamma_N}$ and $\Gamma_N = \langle g_1, \dots, g_N \rangle$, the uniformity condition tells us that we must be in a projective limit situation, as follows:

$$\begin{aligned} \Gamma_1 &\leftarrow \Gamma_2 \leftarrow \Gamma_3 \leftarrow \Gamma_4 \leftarrow \dots \\ \Gamma_{N-1} &= \Gamma_N / \langle g_N = 1 \rangle \end{aligned}$$

Now by assuming for instance that Γ_2 is given and not abelian, there are many ways of completing the sequence, and so the uniqueness coming from (2) can only fail. \square

Let us introduce now a few more notions, as follows:

PROPOSITION 4.23. *Assume that $G = (G_N)$ is weakly uniform, let $n \in \{2, 3, \dots, \infty\}$ be as above, and consider the following conditions, where $I_N \subset G_N$ is the diagonal torus:*

- (1) *Strong diagonal liberation: $G_N = \langle G_N^c, I_n \rangle$, for any $N \geq n$.*
- (2) *Technical condition: $G_N = \langle G_N^c, I_{N-1} \rangle$ for any $N > n$, and $G_n = \langle G_n^c, I_n \rangle$.*
- (3) *Diagonal liberation: $G_N = \langle G_N^c, I_N \rangle$, for any N .*
- (4) *Initial step diagonal liberation: $G_n = \langle G_n^c, I_n \rangle$.*

We have then (1) \implies (2) \implies (3) \implies (4).

PROOF. Our claim is that when assuming that $G = (G_N)$ is weakly uniform, so is the family of diagonal tori $I = (I_N)$. Indeed, we have the following computation:

$$\begin{aligned} I_N \cap U_{N-1}^+ &= (G_N \cap \mathbb{T}_N^+) \cap U_{N-1}^+ \\ &= (G_N \cap U_{N-1}^+) \cap (\mathbb{T}_N^+ \cap U_{N-1}^+) \\ &= G_{N-1} \cap \mathbb{T}_{N-1}^+ \\ &= I_{N-1} \end{aligned}$$

Thus our claim is proved, and this gives the various implications in the statement. \square

We can now formulate a key theoretical observation, as follows:

THEOREM 4.24. *If $G = (G_N)$ is weakly uniform, and with $n \in \{2, 3, \dots, \infty\}$ being as above, the following conditions are equivalent, modulo their initial steps:*

- (1) *Generation: $G_N = \langle G_N^c, G_{N-1} \rangle$, for any $N > n$.*
- (2) *Strong generation: $G_N = \langle G_N^c, G_n \rangle$, for any $N > n$.*
- (3) *Diagonal liberation: $G_N = \langle G_N^c, I_N \rangle$, for any $N \geq n$.*
- (4) *Strong diagonal liberation: $G_N = \langle G_N^c, I_n \rangle$, for any $N \geq n$.*

PROOF. Our first claim is that generation plus initial step diagonal liberation imply the technical diagonal liberation condition. Indeed, the recurrence step goes as follows:

$$\begin{aligned} G_N &= \langle G_N^c, G_{N-1} \rangle \\ &= \langle G_N^c, G_{N-1}^c, I_{N-1} \rangle \\ &= \langle G_N^c, I_{N-1} \rangle \end{aligned}$$

In order to pass now from the technical diagonal liberation condition to the strong diagonal liberation condition itself, observe that we have:

$$\begin{aligned} G_N &= \langle G_N^c, G_{N-1} \rangle \\ &= \langle G_N^c, G_{N-1}^c, I_{N-1} \rangle \\ &= \langle G_N^c, I_{N-1} \rangle \end{aligned}$$

With this condition in hand, we have then as well:

$$\begin{aligned} G_N &= \langle G_N^c, G_{N-1} \rangle \\ &= \langle G_N^c, G_{N-1}^c, I_{N-2} \rangle \\ &= \langle G_N^c, I_{N-2} \rangle \end{aligned}$$

This procedure can be of course be continued. Thus we have a descent method, and we end up with the strong diagonal liberation condition.

In the other sense now, we want to prove that we have, at $N \geq n$:

$$G_N = \langle G_N^c, G_{N-1} \rangle$$

At $N = n + 1$ this is something that we already have. At $N = n + 2$ now, we have:

$$\begin{aligned} G_{n+2} &= \langle G_{n+2}^c, I_n \rangle \\ &= \langle G_{n+2}^c, G_{n+1}^c, I_n \rangle \\ &= \langle G_{n+2}^c, G_{n+1} \rangle \end{aligned}$$

This procedure can be of course be continued. Thus, we have a descent method, and we end up with the strong generation condition. \square

It is possible to prove that many interesting quantum groups have the above properties, and hence appear as diagonal liberations, but the whole subject is quite technical.

Here is however a statement, coming from [43], [46], [47], collecting most of the known results on the subject:

THEOREM 4.25. *The following happen:*

- (1) O_N^+, U_N^+ appear as soft liberations of O_N, U_N .
- (2) O_N^+, U_N^+ appear as well as hard liberations of O_N, U_N .
- (3) H_N^+, K_N^+ appear as soft liberations of H_N, K_N .
- (4) H_N^+, K_N^+ do not appear as hard liberations of H_N, K_N .

PROOF. This result, while being fundamental for us, is something quite technical. In the lack of a simple proof for all this, here is the idea:

(1) This simply follows from (2) below. Normally there should be a simpler proof for this, by using Tannakian duality, but this is something which is not known yet.

(2) A key result from [46], [47], whose proof is quite technical, not to be explained here, states that we have the following generation formula, valid at any $N \geq 3$:

$$O_N^+ = \langle O_N, O_{N-1}^+ \rangle$$

With this in hand, the hard liberation formula $O_N^+ = \langle O_N, T_N^+ \rangle$ can be proved by recurrence on N . Indeed, at $N = 1$ there is nothing to prove, at $N = 2$ this is something

well-known, and elementary, as explained for instance in [46], [47], and in general, the recurrence step $N - 1 \rightarrow N$ can be established as follows:

$$\begin{aligned}
O_N^+ &= \langle O_N, O_{N-1}^+ \rangle \\
&= \langle O_N, O_{N-1}, T_{N-1}^+ \rangle \\
&= \langle O_N, T_{N-1}^+ \rangle \\
&= \langle O_N, T_N, T_{N-1}^+ \rangle \\
&= \langle O_N, T_N^+ \rangle
\end{aligned}$$

Regarding now the hard liberation formula $U_N^+ = \langle U_N, \mathbb{T}_N^+ \rangle$, this basically follows from $O_N^+ = \langle O_N, T_N^+ \rangle$, via the following standard isomorphism:

$$PO_N^+ = PU_N^+$$

Indeed, as explained for instance in [46], [47], this latter isomorphism shows that we have an isomorphism as follows:

$$U_N^+ = \langle U_N, O_N^+ \rangle$$

By using this isomorphism, we obtain:

$$\begin{aligned}
U_N^+ &= \langle U_N, O_N^+ \rangle \\
&= \langle U_N, O_N, T_N^+ \rangle \\
&= \langle U_N, T_N^+ \rangle \\
&= \langle U_N, \mathbb{T}_N^+ \rangle
\end{aligned}$$

(3) This is something trivial, because H_N^+, K_N^+ equal their reflection subgroups.

(4) This result, which is something quite surprising, is well-known, coming from the fact that the quantum group $H_N^{[\infty]} \subset H_N^+$ constructed in [81], and its unitary counterpart $K_N^{[\infty]} \subset K_N^+$, have the same diagonal subgroups as H_N^+, K_N^+ . Thus, the hard liberation procedure stops at $H_N^{[\infty]}, K_N^{[\infty]}$, and cannot reach H_N^+, K_N^+ .

To be more precise, we can construct quantum groups $H_N^{[\infty]}, K_N^{[\infty]}$ by using the relations $\alpha\beta\gamma = 0$, for any $a \neq c$ on the same row or column of u , with the convention $\alpha = a, \alpha^* = a^*$, and so on. These quantum groups appear as intermediate liberations, as follows:

$$\begin{array}{ccccccc}
K_N & \longrightarrow & K_N^* & \longrightarrow & K_N^{[\infty]} & \longrightarrow & K_N^+ \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H_N & \longrightarrow & H_N^* & \longrightarrow & H_N^{[\infty]} & \longrightarrow & H_N^+
\end{array}$$

Moreover, these quantum groups $H_N^{[\infty]}, K_N^{[\infty]}$ are easy, the corresponding categories $\mathcal{P}_{even}^{[\infty]} \subset \mathcal{P}_{even}$ and $\mathcal{P}_{even}^{[\infty]} \subset \mathcal{P}_{even}$ being generated by the following partition:

$$\eta = \ker \begin{pmatrix} i & i & j \\ j & i & i \end{pmatrix}$$

Indeed, this is routine, by using the fact that the above relations $\alpha\beta\gamma = 0$ are equivalent to the following condition, with $|k| = 3$:

$$\eta \in \text{End}(u^{\otimes k})$$

For further details on these quantum groups, we refer to [25], [81].

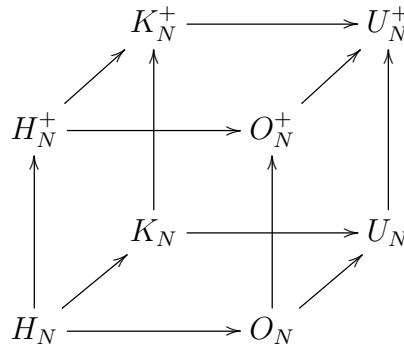
In relation with our questions, since the relations $g_i g_i g_j = g_j g_i g_i$ are trivially satisfied for real reflections, the diagonal tori of these quantum groups coincide with those for H_N^+, K_N^+ . Thus, the diagonal liberation procedure “stops” at $H_N^{[\infty]}, K_N^{[\infty]}$. \square

Before going further, let us make some comments on all this.

As a first comment, in contrast to what happens in the classical case, where we have $K = \langle H_N, T \rangle$, the correspondence $T \rightarrow K$ cannot be constructed via the hard generation formula $K = \langle H_N, T \rangle$, because this hard generation formula is wrong in the free case, due to the negative result from Theorem 4.25 (4), and more specifically to the intermediate quantum reflection groups $H_N^{[\infty]}, K_N^{[\infty]}$ used there, in the proof.

Thus, our formula $K = K^+(T)$ is the only solution to the $T \rightarrow K$ problem.

As a second comment, all the above is interesting in connection with the cube formed by the quantum unitary and reflection groups. Let us recall from Theorem 4.11 that these quantum groups form an intersection and easy generation diagram, as follows:



It is conjectured that this diagram should be a generation diagram too, and the above results prove this conjecture for 5 of the faces. For the remaining face, namely the one on the left, the corresponding formula $K_N^+ = \langle K_N, H_N^+ \rangle$ is not proved yet.

We refer to [81] and related papers for some discussion on these topics.

As yet another comment, the material in Theorem 4.25 is definitely waiting for more study. Indeed, we have the following Tannakian formulae:

$$C_{H \cap K} = \langle C_H, C_K \rangle$$

$$C_{\langle H, K \rangle} = C_H \cap C_K$$

Thus, from a Tannakian viewpoint, all the above results ultimately correspond to doing some combinatorics. To be more precise, the soft and hard generation properties in Definition 4.17 amount respectively in proving the following formulae:

$$C_G = \langle C_G, C_{U_N} \rangle \cap C_K$$

$$C_G = \langle C_G, C_{U_N} \rangle \cap C_T$$

In the easy case now, where $C_G = \text{span}(D)$, which is the case for the various quantum groups from Theorem 4.25, these two equalities reformulate as follows:

$$\text{span}(D) = \text{span}(D, \chi) \cap C_K$$

$$\text{span}(D) = \text{span}(D, \chi) \cap C_T$$

Thus, we are led into some combinatorics, which remains to be understood, in a direct way, without reference to algebra and recurrence methods.

Getting back now to our axiomatization questions, the soft and hard liberation can be thought of as being refinements of the easy liberation, and Theorem 4.25 can be regarded as being a refinement of Proposition 4.12. With this idea in mind, we have the following refinement of Proposition 4.13, dealing this time with hard liberation:

PROPOSITION 4.26. *The unitary quantum groups appear from their diagonal subgroups*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array} \quad \longrightarrow \quad \begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array}$$

via the following hard generation formula

$$U = \langle O_N, T \rangle$$

computed inside the quantum group U_N^+ .

PROOF. This comes from the results in Theorem 4.25, as follows:

- (1) In the classical real case the condition is $O_N = \langle O_N, T_N \rangle$, which is trivial.
- (2) In the free real case the condition is $O_N^+ = \langle O_N, T_N^+ \rangle$, which is exactly the hard liberation property of $O_N \subset O_N^+$, as explained in Theorem 4.25.

(3) In the classical complex case the condition is $U_N = \langle O_N, \mathbb{T}_N \rangle$. But this is something well-known, coming for instance from the fact that the inclusion of compact Lie groups $\mathbb{T}O_N \subset U_N$ is maximal. For more details here, we refer to [19].

(4) In the free complex case the condition is $U_N^+ = \langle O_N, \mathbb{T}_N^+ \rangle$. But this comes from the hard liberation formula $U_N^+ = \langle U_N, T_N^+ \rangle$ from Theorem 4.25, as follows:

$$\begin{aligned} U_N^+ &= \langle U_N, \mathbb{T}_N^+ \rangle \\ &= \langle O_N, \mathbb{T}_N, \mathbb{T}_N^+ \rangle \\ &= \langle O_N, \mathbb{T}_N^+ \rangle \end{aligned}$$

Thus, we are led to the conclusions in the statement. \square

Generally speaking, the same comments as those after Theorem 4.25 apply. In Tanakian formulation, the equalities to be proved are as follows:

$$C_U = \text{span}(P_2) \cap C_K$$

$$C_U = \text{span}(P_2) \cap C_T$$

Thus, we are led into some combinatorics, of basically the same type as the combinatorics needed for Theorem 4.25, which remains to be understood.

We can now update our main result from the general, non-easy case, as follows:

THEOREM 4.27. *For the basic quadruplets, we have correspondences as follows,*

$$\begin{array}{ccc} S & \xrightarrow{\quad} & T \\ \updownarrow & \nearrow & \updownarrow \\ U & \xleftarrow{\quad} & K \end{array}$$

constructed via the following formulae:

- (1) $S = S_U$.
- (2) $T = S \cap \mathbb{T}_N^+ = U \cap \mathbb{T}_N^+ = K \cap \mathbb{T}_N^+$.
- (3) $U = G^+(S) = \langle O_N, T \rangle = \langle O_N, K \rangle$.
- (4) $K = U \cap K_N^+ = K^+(T)$.

PROOF. This is an update of our main result so far, namely Theorem 4.3 above, taking into account the correspondences coming from Proposition 4.26. \square

Summarizing, we have now a reasonable set of correspondences between our objects, which are constructed without any reference to easiness.

4d. General axioms

As already mentioned before, in chapter 1 and afterwards, in what regards the missing correspondences, $T \rightarrow S$ and $S \leftrightarrow K$, the situation here is quite complicated.

In short, we have to give up now with our general principle of constructing all the correspondences independently of each other, and compose what we have. We are led to:

DEFINITION 4.28. *A quadruplet (S, T, U, K) is said to produce a noncommutative geometry when one can pass from each object to all the other objects, as follows,*

$$\begin{aligned} S &= S_{\langle O_N, T \rangle} = S_U = S_{\langle O_N, K \rangle} \\ S \cap \mathbb{T}_N^+ &= T = U \cap \mathbb{T}_N^+ = K \cap \mathbb{T}_N^+ \\ G^+(S) &= \langle O_N, T \rangle = U = \langle O_N, K \rangle \\ K^+(S) &= K^+(T) = U \cap K_N^+ = K \end{aligned}$$

with the usual convention that all this is up to the equivalence relation.

Observe the similarity with the axiomatics from the easy case. The same comments as those made after Definition 4.15, from the easy case, apply.

To be more precise, if we plug the data from any axiom line into the 3 other lines, we obtain axiomatizations in terms of S, T, U, K alone, that we can try to simplify afterwards.

It is of course possible to axiomatize everything in terms of ST, SU, SK, TU, TK, UK as well, and also in terms of STU, STK, SUK, TUK , and try to simplify afterwards.

In what follows we will not bother much with this, and use Definition 4.21 as it is. We will need that 12 correspondences, as results, and whether we call such results “verifications of the axioms” or “basic properties of our geometry” is irrelevant.

Observe also that the above definition is independent from Definition 4.15, in the sense that an easy geometry in the sense of Definition 4.15 does not automatically satisfy the above axioms, or vice versa. However, we do not know counterexamples here.

As another technical comment, the previous work in [17] was based on (S, T, U) triples, but as explained there, this formalism, missing a lot of restrictions coming from K , is a bit too broad. As for the subsequent work in [11], this was based on sextuplets $(S, \bar{S}, T, U, \bar{U}, K)$, with the bars standing for twists, which is perhaps something quite natural, but which leads to too many correspondences between objects, namely 30.

Regarding now the basic examples, these are of course the classical and free, real and complex geometries. To be more precise, we have the following result:

THEOREM 4.29. *We have 4 basic geometries, denoted*

$$\begin{array}{ccc}
 \mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\
 \uparrow & & \uparrow \\
 \mathbb{R}^N & \longrightarrow & \mathbb{C}^N
 \end{array}$$

which appear from quadruplets as above, as follows:

- (1) *Classical real: produced by $(S_{\mathbb{R}}^{N-1}, T_N, O_N, H_N)$.*
- (2) *Classical complex: produced by $(S_{\mathbb{C}}^{N-1}, \mathbb{T}_N, U_N, K_N)$.*
- (3) *Free real: produced by $(S_{\mathbb{R},+}^{N-1}, T_N^+, O_N^+, H_N^+)$.*
- (4) *Free complex: produced by $(S_{\mathbb{C},+}^{N-1}, \mathbb{T}_N^+, U_N^+, K_N^+)$.*

PROOF. This is something that we already know, which follows from Theorem 4.20, as explained in the discussion preceding Definition 4.21. \square

We will be back to more examples in chapters 9-12 below, and with some classification results as well, the idea being that of looking for intermediate geometries on the horizontal, and on the vertical of the above diagram, and then combining these constructions.

The conclusion there will be that the 4-diagram of geometries from Theorem 4.22 can be extended into a 9-diagram of geometries, as follows:

$$\begin{array}{ccccc}
 \mathbb{R}_+^N & \longrightarrow & \mathbb{T}\mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{R}_*^N & \longrightarrow & \mathbb{T}\mathbb{R}_*^N & \longrightarrow & \mathbb{C}_*^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{R}^N & \longrightarrow & \mathbb{T}\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
 \end{array}$$

Getting back to abstract things, and to the axioms from Definition 4.21 above, let us recall that the correspondences there were partly obtained by composing.

Here is an equivalent formulation of our axioms, which is more convenient, and that we will use in what follows, cutting some trivial redundancies:

THEOREM 4.30. *A quadruplet (S, T, U, K) produces a noncommutative geometry when*

$$\begin{aligned} S &= S_U \\ S \cap \mathbb{T}_N^+ &= T = K \cap \mathbb{T}_N^+ \\ G^+(S) &= \langle O_N, T \rangle = U \\ K^+(T) &= U \cap K_N^+ = K \end{aligned}$$

with the usual convention that all this is up to the equivalence relation.

PROOF. This follows indeed by examining the axioms in Definition 4.21 above, by cutting some trivial redundancies, and then by rescaling the whole table. \square

We will use many times the above result, in what follows, so let us comment now, a bit informally, on the 7 axioms that we have, arranged in increasing order of complexity, based on the 4 computations that we have already:

- (1) $T = S \cap \mathbb{T}_N^+$ is usually something quite trivial, and easy to check.
- (2) $T = K \cap \mathbb{T}_N^+$ is once again something quite trivial, and easy to check.
- (3) $K = U \cap K_N^+$ is of the same nature, usually some trivial algebra.
- (4) $U = G^+(S)$ is something more subtle, of algebraic geometric nature, and which usually requires some tricks, in the spirit of [38]. These tricks can actually get very complicated, and for many examples of quantum spheres S , the corresponding quantum isometry groups $G^+(S)$ are not known yet.
- (5) $K = K^+(T)$ is something in the same spirit, but more complicated, with even the simplest possible non-trivial cases, namely the free real and complex ones, requiring subtle ingredients, such as a good knowledge of the $q = -1$ twisting.
- (6) $S = S_U$ is something fairly heavy, requiring a good knowledge of the advanced representation theory and probability theory of compact quantum groups. Note that this is our only way here of getting to the sphere S .
- (7) $U = \langle O_N, T \rangle$ is something heavy too, requiring an excellent knowledge of the advanced representation theory of compact quantum groups. In fact, this is the key axiom, beating in complexity all the previous axioms, taken altogether.

Regarding now further work on these axioms, with new examples of geometries, and will classification results, we will discuss this later, in chapters 9-12 below. We will see there, among others, that under strong supplementary axioms, called “purity” and “uniformity”, the 4 main geometries, from Theorem 4.22 above, are the only ones.

In view of this, the question of developing the real and complex free geometries, which are the “main” non-classical geometries, appears. We will discuss this in chapters 5-8 below, with the construction of various “free homogeneous spaces”, and we will come back to this later as well, in chapters 13-16 below, with more advanced results.

4e. Exercises

There are many possible exercises on the material above, which was quite varied, and also on the final axioms, that we have not explored yet. As a first exercise, we have:

EXERCISE 4.31. *Work out the details of the Tannakian formulae*

$$C_{G \cap H} = \langle C_G, C_H \rangle$$

$$C_{\langle G, H \rangle} = C_G \cap C_H$$

for the intersection and generation operations for compact quantum groups.

Normally this is not something difficult, once Tannakian duality well understood.

On the same topic, but at a more advanced level, we have:

EXERCISE 4.32. *Prove that any closed subgroup $G \subset U_N^+$ has an “easy envelope”*

$$G \subset \tilde{G} \subset U_N^+$$

which is the smallest easy quantum group containing G . Then prove that

$$\langle H, K \rangle \subset \langle \widetilde{H}, \widetilde{K} \rangle \subset \{H, K\}$$

for any two easy quantum groups H, K .

Here, as for the previous exercise, the first question looks like pure algebra, and so is probably normally not something very difficult, once Tannakian duality well understood. As for the second question, this looks quite routine as well.

As a bonus exercise here, try finding some nice examples for the easy envelope operation, for instance in the classical group case. Also, as a second bonus exercise, try deciding which of the inclusions in the second question is the one to start with, meaning the “simplest”, if we want to prove that we have global isomorphism.

In relation now with the easy geometries, we have:

EXERCISE 4.33. *Try finding an easy geometry, or at least a candidate for an easy geometry, without full verification of the axioms, which is not among the 4 main ones.*

This question will be investigated later in this book, but thinking a bit at these questions, by yourself, will certainly not hurt, and will help understanding the material below. As a hint here, try finding something between $ab = ba$ and freeness.

In relation now with liberation, we have several questions, the first one being:

EXERCISE 4.34. *Establish the following isomorphism, as usual modulo equivalence*

$$PO_N^+ = PU_N^+$$

by using Tannakian duality and easiness, or any other means.

This is something quite important, and this for several reasons. First, we have used this isomorphism in the above, in checking our axioms for the free complex geometry. Also, philosophically speaking, this is important, because it tends to suggest that there is only one “free projective geometry”, which is at the same time real and complex.

In practice now, the straightforward approach is that of saying that we have an inclusion $PO_N^+ \subset PU_N^+$, and so it suffices to show that the corresponding Hom spaces, or just their dimensions, coincide. But the easiness technology is precisely there for that.

Still in relation with liberation, but at a more advanced level, we have:

EXERCISE 4.35. *Establish the following isomorphism, as usual modulo equivalence*

$$O_N^+ = \langle O_N, O_{N-1}^+ \rangle$$

by using representation theory, or any other means.

This is something quite tricky, so in case you do not find the solution, you can look for it in the literature, and write down a brief account of what you found.

In relation now with our axiomatization, you can first have some fun with:

EXERCISE 4.36. *Try axiomatizing the missing correspondences, namely*

$$T \rightarrow S$$

$$S \leftrightarrow K$$

and in case you fail, explain at least what the difficulties are.

No comment here, as the author of the present book has failed so far in axiomatizing these correspondences, and does not want either to talk about this.

As a simpler exercise now, which is instructive, we have:

EXERCISE 4.37. *Axiomatize the abstract noncommutative geometries in our sense in terms of S , or of T , or of U , or of K , only.*

This is something that we talked about in the above, that can be formally solved very quickly, simply by modifying our axioms, in the obvious way. The problem, however, is that of working out the simplifications that might appear in this way.

Part II

Free manifolds

*In the town San Domingo
As we laughed and danced all night
To the throb of flamingo guitars
Seemed a long long way from tomorrow's fight*

CHAPTER 5

Free integration

5a. Weingarten formula

We have seen so far that the two basic geometries, namely those of $\mathbb{R}^N, \mathbb{C}^N$, have free analogues, namely those of $\mathbb{R}_+^N, \mathbb{C}_+^N$. The question of “developing” these new geometries appears. To be more precise, each of our free geometries consists so far of 4 objects, namely a sphere S , a torus T , a unitary group U , and a reflection group K . We must on one hand study S, T, U, K , from a geometric perspective, and on the other hand construct other “free manifolds”, like suitable homogeneous spaces, and study them as well.

Following the operator algebra tradition, coming from von Neumann, and then Connes, Jones, Voiculescu, we will primarily regard our various manifolds X as “quantum measured spaces”, corresponding to von Neumann algebras $L^\infty(X)$. From this perspective, the main question to be investigated is the computation of the Haar functional:

$$tr : L^\infty(X) \rightarrow \mathbb{C}$$

We will investigate this question in this chapter, for S, T, U, K . Later on, in chapters 6-8 below, we will introduce other manifolds, such as quotient spaces $X = G/H$ coming from quantum groups $H \subset G \subset U$, and compute their integration functional as well.

In practice now, our first task will be that of explaining how to integrate over our objects S, T, U, K . In order to integrate over U, K , we can use the Weingarten formula [49], [96], whose quantum group formulation, from [21], [33], is as follows:

THEOREM 5.1. *Assuming that a compact quantum group $G \subset U_N^+$ is easy, coming from a category of partitions $D \subset P$, we have the Weingarten formula*

$$\int_G u_{i_1 j_1}^{e_1} \dots u_{i_k j_k}^{e_k} = \sum_{\pi, \sigma \in D(k)} \delta_\pi(i) \delta_\sigma(j) W_{kN}(\pi, \sigma)$$

for any indices $i_r, j_r \in \{1, \dots, N\}$ and exponents $e_r \in \{\emptyset, *\}$, where δ are Kronecker type symbols, and where the Weingarten matrix

$$W_{kN} = G_{kN}^{-1}$$

is the inverse of the Gram matrix $G_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma|}$.

PROOF. This is a formula that we know from chapter 3, the idea being that the matrix formed by the integrals in the statement is the projection on the following space:

$$\text{Fix}(u^{\otimes k}) = \text{span} \left(\xi_\pi \mid \pi \in D(k) \right)$$

By doing the linear algebra, this gives the result, as explained in chapter 3. \square

Regarding now the integration over the torus T , this is something very simple, because we can use here the following fact, coming again from [99]:

THEOREM 5.2. *Given a finitely generated discrete group $\Gamma = \langle g_1, \dots, g_N \rangle$, the integrals over the corresponding torus $T = \widehat{\Gamma}$ are given by*

$$\int_T g_{i_1}^{e_1} \cdots g_{i_k}^{e_k} = \delta_{g_{i_1}^{e_1} \cdots g_{i_k}^{e_k}, 1}$$

for any indices $i_r \in \{1, \dots, N\}$ and any exponents $e_r \in \{\emptyset, *\}$, with the Kronecker symbol on the right being a usual one, computed inside the group Γ .

PROOF. This is something standard, coming from the fact that the Haar integration over the torus $T = \widehat{\Gamma}$ is given by the following formula:

$$\int_T g = \delta_{g1}$$

Indeed, this formula defines a functional on the algebra $C(T) = C^*(\Gamma)$, which is left and right invariant. For details on all this, we refer to [99]. \square

Finally, regarding the associated spheres S , here the integrals appear as particular cases of the integrals over the corresponding unitary groups U , as explained in chapter 3 above, and in the easy case, we have a Weingarten formula, as follows:

THEOREM 5.3. *The integration over a noncommutative sphere S , coming from a category of pairings D , is given by the Weingarten formula*

$$\int_S x_{i_1}^{e_1} \cdots x_{i_k}^{e_k} = \sum_{\pi} \sum_{\sigma \leq \ker i} W_{kN}(\pi, \sigma)$$

with $\pi, \sigma \in D(k)$, where $W_{kN} = G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma|}$.

PROOF. This follows from the definition of the integration functional over S , as being the composition of the morphism $C(S) \rightarrow C(U)$ with the Haar integration over U :

$$\int_S : C(S) \rightarrow C(U) \rightarrow \mathbb{C}$$

Indeed, with this description of the integration functional in mind, we can compute this functional via the Weingarten formula for U , from Theorem 5.1, as follows:

$$\begin{aligned} \int_S x_{i_1}^{e_1} \cdots x_{i_k}^{e_k} &= \int_U u_{1i_1}^{e_1} \cdots u_{1i_k}^{e_k} \\ &= \sum_{\pi, \sigma \in D(k)} \delta_\pi(1) \delta_\sigma(i) W_{kN}(\pi, \sigma) \\ &= \sum_{\pi} \sum_{\sigma \leq \ker i} W_{kN}(\pi, \sigma) \end{aligned}$$

Thus, we are led to the formula in the statement. \square

5b. Free probability

Let us discuss now the explicit computation of the various integrals over our manifolds, with respect to the uniform measure. In order to formulate our results in a conceptual form, we use the modern measure theory language, namely probability theory. In the noncommutative setting, the starting definition is as follows:

DEFINITION 5.4. *Let A be a C^* -algebra, given with a trace tr .*

- (1) *The elements $a \in A$ are called random variables.*
- (2) *The moments of such a variable are the numbers $M_k(a) = tr(a^k)$.*
- (3) *The law of such a variable is the functional $\mu : P \rightarrow tr(P(a))$.*

Here $k = \circ \bullet \bullet \circ \dots$ is as usual a colored integer, and the powers a^k are defined by the usual formulae, namely $a^\emptyset = 1$, $a^\circ = a$, $a^\bullet = a^*$ and multiplicativity. As for the polynomial P , this is by definition a noncommuting $*$ -polynomial in one variable:

$$P \in \mathbb{C} \langle X, X^* \rangle$$

Observe that the law is uniquely determined by the moments, because:

$$P(X) = \sum_k \lambda_k X^k \implies \mu(P) = \sum_k \lambda_k M_k(a)$$

In the self-adjoint case, the law is a usual probability measure, supported by the spectrum of a . This follows indeed from the Gelfand theorem, and the Riesz theorem.

There are many things that can be said, at this general level, so as a more concrete objective, let us try to understand how the main result in classical probability, namely the Central Limit Theorem (CLT), can be extended in the noncommutative setting.

Let us start with the usual formulation of the CLT, which is as follows:

THEOREM 5.5 (CLT). *Given real random variables x_1, x_2, x_3, \dots , which are i.i.d., centered, and with variance $t > 0$, we have, with $n \rightarrow \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim g_t$$

where g_t is the Gaussian law of parameter t , having as density:

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

PROOF. This is something standard, the proof being in three steps, as follows:

(1) Linearization of the convolution. It well-known that the log of the Fourier transform $F_x(\xi) = \mathbb{E}(e^{i\xi x})$ does the job, in the sense that if x, y are independent, then:

$$F_{x+y} = F_x F_y$$

(2) Study of the limit. We have the following formula for a general Fourier transform $F_x(\xi) = \mathbb{E}(e^{i\xi x})$, in terms of moments:

$$F_x(\xi) = \sum_{k=0}^{\infty} \frac{i^k M_k(x)}{k!} \xi^k$$

It follows that the Fourier transform of the variable in the statement is:

$$\begin{aligned} F(\xi) &= \left[F_x \left(\frac{\xi}{\sqrt{n}} \right) \right]^n \\ &= \left[1 - \frac{t\xi^2}{2n} + O(n^{-2}) \right]^n \\ &\simeq e^{-t\xi^2/2} \end{aligned}$$

(3) Gaussian laws. The Fourier transform of the Gaussian law is given by:

$$\begin{aligned} F_{g_t}(x) &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-y^2/2t + ixy} dy \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-(y/\sqrt{2t} - \sqrt{t/2}ix)^2 - tx^2/2} dy \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-z^2 - tx^2/2} \sqrt{2t} dz \\ &= \frac{1}{\sqrt{\pi}} e^{-tx^2/2} \int_{\mathbb{R}} e^{-z^2} dz \\ &= \frac{1}{\sqrt{\pi}} e^{-tx^2/2} \cdot \sqrt{\pi} \\ &= e^{-tx^2/2} \end{aligned}$$

Thus the variables on the left and on the right in the statement have the same Fourier transform, and we obtain the result. \square

Following Voiculescu [88], in order to extend the CLT to the free setting, our starting point will be the following definition:

DEFINITION 5.6. *Given a pair (A, tr) , two subalgebras $B, C \subset A$ are called free when the following condition is satisfied, for any $b_i \in B$ and $c_i \in C$:*

$$tr(b_i) = tr(c_i) = 0 \implies tr(b_1 c_1 b_2 c_2 \dots) = 0$$

Also, two noncommutative random variables $b, c \in A$ are called free when the C^ -algebras $B = \langle b \rangle$, $C = \langle c \rangle$ that they generate inside A are free, in this sense.*

As a first observation, there is a similarity here with the classical notion of independence. Indeed, modulo some standard identifications, two subalgebras $B, C \subset L^\infty(X)$ are independent when the following condition is satisfied, for any $b \in B$ and $c \in C$:

$$tr(b) = tr(c) = 0 \implies tr(bc) = 0$$

Thus, freeness appears by definition as a kind of “free analogue” of independence. As a first result regarding this notion, we have:

PROPOSITION 5.7. *Assuming that $B, C \subset A$ are free, the restriction of tr to $\langle B, C \rangle$ can be computed in terms of the restrictions of tr to B, C . To be more precise,*

$$tr(b_1 c_1 b_2 c_2 \dots) = P\left(\{tr(b_{i_1} b_{i_2} \dots)\}_i, \{tr(c_{j_1} c_{j_2} \dots)\}_j\right)$$

where P is certain polynomial in several variables, depending on the length of the word $b_1 c_1 b_2 c_2 \dots$, and having as variables the traces of products of type

$$b_{i_1} b_{i_2} \dots$$

$$c_{j_1} c_{j_2} \dots$$

with the indices being chosen increasing, $i_1 < i_2 < \dots$ and $j_1 < j_2 < \dots$

PROOF. This is something quite theoretical, so let us begin with an example. Our claim is that if b, c are free then, exactly as in the case where we have independence:

$$tr(bc) = tr(b)tr(c)$$

Indeed, we have the following computation, with the convention $a' = a - tr(a)$:

$$\begin{aligned} tr(bc) &= tr[(b' + tr(b))(c' + tr(c))] \\ &= tr(b'c') + tr(b')tr(c) + tr(b)tr(c') + tr(b)tr(c) \\ &= tr(b'c') + tr(b)tr(c) \\ &= tr(b)tr(c) \end{aligned}$$

In general now, the situation is a bit more complicated, but the same trick applies. To be more precise, we can start our computation as follows:

$$\begin{aligned}
& tr(b_1 c_1 b_2 c_2 \dots) \\
&= tr[(b'_1 + tr(b_1))(c'_1 + tr(c_1))(b'_2 + tr(b_2))(c'_2 + tr(c_2)) \dots] \\
&= tr(b'_1 c'_1 b'_2 c'_2 \dots) + \text{other terms} \\
&= \text{other terms}
\end{aligned}$$

Observe that we have used here the freeness condition, in the following form:

$$tr(b'_i) = tr(c'_i) = 0 \implies tr(b'_1 c'_1 b'_2 c'_2 \dots) = 0$$

Now regarding the “other terms”, those which are left, each of them will consist of a product of traces of type $tr(b_i)$ and $tr(c_i)$, and then a trace of a product still remaining to be computed, which is of the following form, with $\beta_i \in B$ and $\gamma_i \in C$:

$$tr(\beta_1 \gamma_1 \beta_2 \gamma_2 \dots)$$

To be more precise, the variables $\beta_i \in B$ appear as ordered products of those $b_i \in B$ not getting into individual traces $tr(b_i)$, and the variables $\gamma_i \in C$ appear as ordered products of those $c_i \in C$ not getting into individual traces $tr(c_i)$. Now since the length of each such alternating product $\beta_1 \gamma_1 \beta_2 \gamma_2 \dots$ is smaller than the length of the original alternating product $b_1 c_1 b_2 c_2 \dots$, we are led into of recurrence, and this gives the result. \square

As a second result regarding the notion of freeness, which provides us with a useful class of examples, which can be used for various modelling purposes, we have:

PROPOSITION 5.8. *Given two algebras (B, tr) and (C, tr) , the following hold:*

- (1) *B, C are independent inside their tensor product $B \otimes C$, endowed with its canonical tensor product trace, given on basic tensors by $tr(b \otimes c) = tr(b)tr(c)$.*
- (2) *B, C are free inside their free product $B * C$, endowed with its canonical free product trace, given by the formulae in Proposition 5.7.*

PROOF. Both the assertions are clear from definitions, as follows:

- (1) This is clear, because we have by construction of the trace:

$$\begin{aligned}
tr(bc) &= tr[(b \otimes 1)(1 \otimes c)] \\
&= tr(b \otimes c) \\
&= tr(b)tr(c)
\end{aligned}$$

- (2) This is clear from definitions, the only point being that of showing that the notion of freeness, or the recurrence formulae in Proposition 5.7, can be used in order to construct a canonical free product trace, on the free product of the two algebras involved:

$$tr : B * C \rightarrow \mathbb{C}$$

But this can be checked for instance by using a GNS construction. Indeed, consider the GNS constructions for the algebras (B, tr) and (C, tr) :

$$B \rightarrow B(l^2(B)) \quad , \quad C \rightarrow B(l^2(C))$$

By taking the free product of these representations, we obtain a representation as follows, with the $*$ symbol on the right being a free product of pointed Hilbert spaces:

$$B * C \rightarrow B(l^2(B) * l^2(C))$$

Now by composing with the linear form $T \rightarrow \langle T\xi, \xi \rangle$, where $\xi = 1_B = 1_C$ is the common distinguished vector of $l^2(B)$ and $l^2(C)$, we obtain a linear form, as follows:

$$tr : B * C \rightarrow \mathbb{C}$$

It is routine then to check that tr is indeed a trace, and this is the “canonical free product trace” from the statement. Then, an elementary computation shows that B, C are indeed free inside $B * C$, with respect to this trace, and this finishes the proof. \square

Finally, we have the following more explicit modelling result:

THEOREM 5.9. *We have the following results, valid for group algebras:*

- (1) $C^*(\Gamma), C^*(\Lambda)$ are independent inside $C^*(\Gamma \times \Lambda)$.
- (2) $C^*(\Gamma), C^*(\Lambda)$ are free inside $C^*(\Gamma * \Lambda)$.

PROOF. In order to prove these results, we have two possible methods:

(1) We can either use the general results in Proposition 5.8 above, along with the following two isomorphisms, which are both standard:

$$\begin{aligned} C^*(\Gamma \times \Lambda) &= C^*(\Lambda) \otimes C^*(\Gamma) \\ C^*(\Gamma * \Lambda) &= C^*(\Lambda) * C^*(\Gamma) \end{aligned}$$

(2) Or, we can prove this directly, by using the fact that each group algebra is spanned by the corresponding group elements. Indeed, this shows that it is enough to check the independence and freeness formulae on group elements, which is in turn trivial. \square

There are many things that can be said about the analogy between independence and freeness. We have in particular the following result, due to Voiculescu [88]:

THEOREM 5.10. *Given a real probability measure μ , consider its Cauchy transform*

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t}$$

and define its R-transform as being the solution of the following equation:

$$G_\mu \left(R_\mu(\xi) + \frac{1}{\xi} \right) = \xi$$

The operation $\mu \rightarrow R_\mu$ linearizes then the free convolution operation.

PROOF. In order to prove this, we need a good model for the free convolution. The best here is to use the semigroup algebra of the free semigroup on two generators:

$$A = C^*(\mathbb{N} * \mathbb{N})$$

Indeed, we have some freeness in the semigroup setting, a bit in the same way as for the group algebras $C^*(\Gamma * \Lambda)$, from Theorem 5.7 (2), and in addition to this fact, and to what happens in the group algebra case, the following two key things happen:

(1) The variables of type $S^* + f(S)$, with $S \in C^*(\mathbb{N})$ being the shift, and with $f \in \mathbb{C}[X]$ being a polynomial, model in moments all the distributions $\mu : \mathbb{C}[X] \rightarrow \mathbb{C}$. This is indeed something elementary, which can be checked via a direct algebraic computation.

(2) Given $f, g \in \mathbb{C}[X]$, the variables $S^* + f(S)$ and $T^* + g(T)$, where $S, T \in C^*(\mathbb{N} * \mathbb{N})$ are the shifts corresponding to the generators of $\mathbb{N} * \mathbb{N}$, are free, and their sum has the same law as $S^* + (f + g)(S)$. This follows indeed by using a 45° argument.

With these results in hand, we can see that the operation $\mu \rightarrow f$ linearizes the free convolution. We are therefore left with a computation inside $C^*(\mathbb{N})$, whose conclusion is that $R_\mu = f$ can be recaptured from μ via the Cauchy transform G_μ , as stated. \square

We can now state and prove a free analogue of the CLT, from [88], as follows:

THEOREM 5.11 (FCLT). *Given self-adjoint variables x_1, x_2, x_3, \dots , which are f.i.d., centered, with variance $t > 0$, we have, with $n \rightarrow \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim \gamma_t$$

where γ_t is the Wigner semicircle law of parameter t , having density:

$$\gamma_t = \frac{1}{2\pi t} \sqrt{4t^2 - x^2} dx$$

PROOF. At $t = 1$, the R -transform of the variable in the statement can be computed by using the linearization property with respect to the free convolution, and is given by:

$$R(\xi) = nR_x \left(\frac{\xi}{\sqrt{n}} \right) \simeq \xi$$

On the other hand, some elementary computations show that the Cauchy transform of the Wigner law γ_1 satisfies the following equation:

$$G_{\gamma_1} \left(\xi + \frac{1}{\xi} \right) = \xi$$

Thus we have the following formula:

$$R_{\gamma_1}(\xi) = \xi$$

Observe that this latter formula follows as well from:

$$S^* + S \sim \gamma_1$$

But this gives the result. The passage to the general case, $t > 0$, is routine. \square

Let us discuss now the complex versions of the main limiting theorems.

In the classical case, we recall that the complex Gaussian law of parameter $t > 0$ is defined as follows, with a, b being independent, each following the law g_t :

$$G_t = \text{law} \left(\frac{1}{\sqrt{2}}(a + ib) \right)$$

With this convention, we have the following result:

THEOREM 5.12 (CCLT). *Given complex random variables x_1, x_2, x_3, \dots , which are i.i.d., centered, and with variance $t > 0$, we have, with $n \rightarrow \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim G_t$$

where G_t is the complex Gaussian law of parameter t .

PROOF. This follows indeed from the real CLT, without new computations needed, just by taking real and imaginary parts. \square

In the free case, the Voiculescu circular law of parameter $t > 0$ is defined as follows, with α, β being independent, each following the law γ_t :

$$\Gamma_t = \text{law} \left(\frac{1}{\sqrt{2}}(\alpha + i\beta) \right)$$

With this convention, we have the following result:

THEOREM 5.13 (FCCLT). *Given noncommutative random variables x_1, x_2, x_3, \dots , which are f.i.d., centered, and with variance $t > 0$, we have, with $n \rightarrow \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim \Gamma_t$$

where Γ_t is the Voiculescu circular law of parameter t .

PROOF. This follows indeed from the free real CLT, without new computations needed, just by taking real and imaginary parts. \square

With these ingredients in hand, let us go back now to our quantum groups.

We can compute the character laws for the unitary groups, as follows:

THEOREM 5.14. *With $N \rightarrow \infty$, the main characters*

$$\chi = \sum_{i=1}^N u_{ii}$$

for the basic unitary quantum groups are as follows:

- (1) O_N : real Gaussian, following g_1 .
- (2) O_N^+ : semicircular, following γ_1 .
- (3) U_N : complex Gaussian, following G_1 .
- (4) U_N^+ : circular, following Γ_1 .

PROOF. Following [1], [21], we use the moment method. For an arbitrary closed subgroup $G_N \subset U_N^+$, we have, according to the general Peter-Weyl type results from [99]:

$$\int_{G_N} \chi^k = \dim(\text{Fix}(u^{\otimes k}))$$

In the easy case now, where $G = (G_N)$ comes from a certain category of partitions D , the fixed point space on the right is spanned by the vectors T_π with $\pi \in D(k)$. Now since by [71] these vectors are linearly independent with $N \rightarrow \infty$, we have:

$$\lim_{N \rightarrow \infty} \int_{G_N} \chi^k = |D(k)|$$

Thus, we are led into some combinatorics, and the continuation is as follows:

- (1) For O_N we have $D = P_2$, so we obtain as even asymptotic moments the numbers $|P_2(2k)| = k!!$, which are well-known to be the moments of the Gaussian law.
- (2) For O_N^+ we have $D = NC_2$, so we obtain as even asymptotic moments the Catalan numbers $|NC_2(2k)| = C_k$, which are the moments of the Wigner semicircle law.
- (3) For U_N we have $D = \mathcal{P}_2$, and we can conclude as in the real case, involving O_N , by using this time moments with respect to colored integers, as in Definition 5.4.
- (4) For U_N^+ we have $D = \mathcal{NC}_2$, and once again we can conclude as in the real case, involving O_N^+ , by using moments with respect to colored integers, as in Definition 5.4. \square

5c. Truncated characters

Summarizing, we have seen so far that for O_N, O_N^+, U_N, U_N^+ , the asymptotic laws of the main characters are the laws $g_1, \gamma_1, G_1, \Gamma_1$ coming from the various CLT. This is certainly nice, but there is still one conceptual problem, coming from:

PROPOSITION 5.15. *The above convergences $\text{law}(\chi_u) \rightarrow g_1, \gamma_1, G_1, \Gamma_1$ are as follows:*

- (1) They are non-stationary in the classical case.
- (2) They are stationary in the free case, starting from $N = 2$.

PROOF. This is something quite subtle, which can be proved as follows:

(1) Here we can use an amenability argument, based on the Kesten criterion. Indeed, O_N, U_N being coamenable, the upper bound of the support of the law of $Re(\chi_u)$ is precisely N , and we obtain from this that the law of χ_u itself depends on $N \in \mathbb{N}$.

(2) Here the result follows from the fact that the linear maps T_π associated to the noncrossing pairings are linearly independent, at any $N \geq 2$. \square

Fortunately, the solution to the convergence question is quite simple. The idea will be that of improving our $g_1, \gamma_1, G_1, \Gamma_1$ results with certain $g_t, \gamma_t, G_t, \Gamma_t$ results, which will require $N \rightarrow \infty$ in both the classical and free cases, in order to hold at any t . Following [21], the definition that we will need is as follows:

DEFINITION 5.16. *Given a Woronowicz algebra (A, u) , the variable*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

is called truncation of the main character, with parameter $t \in (0, 1]$.

Our purpose in what follows will be that of proving that for O_N, O_N^+, U_N, U_N^+ , the asymptotic laws of the truncated characters χ_t with $t \in (0, 1]$ are the laws $g_t, \gamma_t, G_t, \Gamma_t$. This is something quite technical, but natural, motivated by the findings in Proposition 5.15 above, and also by a number of more advanced considerations, to become clear later on. In order to study the truncated characters, we can use:

THEOREM 5.17. *The moments of the truncated characters are given by*

$$\int_G (u_{11} + \dots + u_{ss})^k = Tr(W_{kN} G_{ks})$$

and with $N \rightarrow \infty$ this quantity equals $(s/N)^k |D(k)|$.

PROOF. The first assertion follows from the following computation:

$$\begin{aligned} \int_G (u_{11} + \dots + u_{ss})^k &= \sum_{i_1=1}^s \dots \sum_{i_k=1}^s \int u_{i_1 i_1} \dots u_{i_k i_k} \\ &= \sum_{\pi, \sigma \in D(k)} W_{kN}(\pi, \sigma) \sum_{i_1=1}^s \dots \sum_{i_k=1}^s \delta_\pi(i) \delta_\sigma(i) \\ &= \sum_{\pi, \sigma \in D(k)} W_{kN}(\pi, \sigma) G_{ks}(\sigma, \pi) \\ &= Tr(W_{kN} G_{ks}) \end{aligned}$$

We have $G_{kN}(\pi, \sigma) = N^k$ for $\pi = \sigma$, and $G_{kN}(\pi, \sigma) \leq N^{k-1}$ for $\pi \neq \sigma$. Thus with $N \rightarrow \infty$ we have $G_{kN} \sim N^k \mathbf{1}$, which gives:

$$\begin{aligned} \int_G (u_{11} + \dots + u_{ss})^k &= \text{Tr}(G_{kN}^{-1} G_{ks}) \\ &\sim \text{Tr}((N^k \mathbf{1})^{-1} G_{ks}) \\ &= N^{-k} \text{Tr}(G_{ks}) \\ &= N^{-k} s^k |D(k)| \end{aligned}$$

Thus, we have obtained the formula in the statement. See [21]. \square

In order to process the above moment formula, we will need some more probability theory. Following [84], given a random variable a , we write:

$$\begin{aligned} \log F_a(\xi) &= \sum_n k_n(a) \xi^n \\ R_a(\xi) &= \sum_n \kappa_n(a) \xi^n \end{aligned}$$

We call the above coefficients $k_n(a), \kappa_n(a)$ the cumulants, respectively free cumulants of our variable a . With this notion in hand, we can define then more general quantities $k_\pi(a), \kappa_\pi(a)$, depending on arbitrary partitions $\pi \in P(k)$, which coincide with the above ones for the 1-block partitions, and then by multiplicativity over the blocks.

With these conventions, we have the following result, from [84]:

THEOREM 5.18. *We have the classical and free moment-cumulant formulae*

$$\begin{aligned} M_k(a) &= \sum_{\pi \in P(k)} k_\pi(a) \\ M_k(a) &= \sum_{\pi \in NC(k)} \kappa_\pi(a) \end{aligned}$$

where $k_\pi(a), \kappa_\pi(a)$ are the generalized cumulants and free cumulants of a .

PROOF. This is standard, either by using the formulae of F_a, R_a , or by doing some direct combinatorics, based on the Möbius inversion formula. \square

Following [21], we can now improve our results about characters, as follows:

THEOREM 5.19. *With $N \rightarrow \infty$, the laws of truncated characters are as follows:*

- (1) For O_N we obtain the Gaussian law g_t .
- (2) For O_N^+ we obtain the Wigner semicircle law γ_t .
- (3) For U_N we obtain the complex Gaussian law G_t .
- (4) For U_N^+ we obtain the Voiculescu circular law Γ_t .

PROOF. With $s = [tN]$ and $N \rightarrow \infty$, the general moment formula in Theorem 5.17 above gives the following estimate:

$$\lim_{N \rightarrow \infty} \int_{G_N} \chi_t^k = \sum_{\pi \in D(k)} t^{|\pi|}$$

By using now the formulae in Theorem 5.18, and doing a number of standard computations, this gives the results. See [21]. \square

As an interesting consequence, related to [35], let us formulate as well:

THEOREM 5.20. *The asymptotic laws of truncated characters for the liberation operations*

$$\begin{aligned} O_N &\rightarrow O_N^+ \\ U_N &\rightarrow U_N^+ \end{aligned}$$

are in Bercovici-Pata bijection, in the sense that the classical cumulants in the classical case equal the free cumulants in the free case.

PROOF. This follows indeed from the computations in the proof of Theorem 5.19, and from the combinatorial interpretation of the Bercovici-Pata bijection [35]. \square

Let us discuss now the integration over the spheres. A basic probabilistic question regarding the spheres concerns the computation of the associated hyperspherical laws. We have here the following result, from [7], [28]:

THEOREM 5.21. *With $N \rightarrow \infty$, the rescaled coordinates of the various spheres*

$$\sqrt{N}x_i \in C(S_{\times}^{N-1})$$

are as follows, with respect to the uniform integration:

- (1) $S_{\mathbb{R}}^{N-1}$: real Gaussian.
- (2) $S_{\mathbb{R},+}^{N-1}$: semicircular.
- (3) $S_{\mathbb{C}}^{N-1}$: complex Gaussian.
- (4) $S_{\mathbb{C},+}^{N-1}$: circular.

PROOF. This follows from Theorem 5.19, but we can use as well the Weingarten formula for the spheres, from Theorem 5.3. Indeed, we have the following estimate:

$$\int_{S_{\times}^{N-1}} x_{i_1} \dots x_{i_k} dx \simeq N^{-k/2} \sum_{\sigma \in P_2^{\times}(k)} \delta_{\sigma}(i)$$

With this formula in hand, we can compute the asymptotic moments of each coordinate x_i . Indeed, by setting $i_1 = \dots = i_k = i$, all Kronecker symbols are 1, and we obtain:

$$\int_{S_{\times}^{N-1}} x_i^k dx \simeq N^{-k/2} |P_2^{\times}(k)|$$

But this gives the results, via the same combinatorics as before. See [3], [28]. \square

5d. Poisson laws

In order to discuss now the quantum reflection groups, we will need some more theory, namely Poisson limit theorems. In the classical case, we have the following result:

THEOREM 5.22 (PLT). *We have the following convergence, in moments,*

$$\left(\left(1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1 \right)^{*n} \rightarrow p_t$$

the limiting measure being

$$p_t = \frac{1}{e^t} \sum_{k=0}^{\infty} \frac{t^k \delta_k}{k!}$$

which is the Poisson law of parameter $t > 0$.

PROOF. We recall that the Fourier transform is given by:

$$F_f(x) = \mathbb{E}(e^{ixf})$$

We therefore obtain the following formula:

$$\begin{aligned} F_{p_t}(x) &= e^{-t} \sum_k \frac{t^k}{k!} F_{\delta_k}(x) \\ &= e^{-t} \sum_k \frac{t^k}{k!} e^{ikx} \\ &= e^{-t} \sum_k \frac{(e^{ix}t)^k}{k!} \\ &= \exp(-t) \exp(e^{ix}t) \\ &= \exp((e^{ix} - 1)t) \end{aligned}$$

Let us denote by μ_n the measure under the convolution sign:

$$\mu_n = \left(1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1$$

We have the following computation:

$$\begin{aligned} F_{\delta_r}(x) = e^{irx} &\implies F_{\mu_n}(x) = \left(1 - \frac{t}{n} \right) + \frac{t}{n} e^{ix} \\ &\implies F_{\mu_n^{*n}}(x) = \left(\left(1 - \frac{t}{n} \right) + \frac{t}{n} e^{ix} \right)^n \\ &\implies F_{\mu_n^{*n}}(x) = \left(1 + \frac{(e^{ix} - 1)t}{n} \right)^n \\ &\implies F(x) = \exp((e^{ix} - 1)t) \end{aligned}$$

Thus, we obtain the Fourier transform of p_t , as desired. \square

In the free case, the result is as follows:

THEOREM 5.23 (FPLT). *We have the following convergence, in moments,*

$$\left(\left(1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1 \right)^{\boxplus n} \rightarrow \pi_t$$

the limiting measure being the Marchenko-Pastur law of parameter $t > 0$,

$$\pi_t = \max(1 - t, 0) \delta_0 + \frac{\sqrt{4t - (x - 1 - t)^2}}{2\pi x} dx$$

also called free Poisson law of parameter $t > 0$.

PROOF. Consider the measure in the statement, appearing under the convolution sign:

$$\mu = \left(1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1$$

The Cauchy transform of this measure is elementary to compute, given by:

$$G_\mu(\xi) = \left(1 - \frac{t}{n} \right) \frac{1}{\xi} + \frac{t}{n} \cdot \frac{1}{\xi - 1}$$

By using the linearization results for the free convolution explained above, we want to compute the following R -transform:

$$R = R_{\mu^{\boxplus n}}(y) = nR_\mu(y)$$

The equation for this function R is as follows:

$$\left(1 - \frac{t}{n} \right) \frac{1}{y^{-1} + R/n} + \frac{t}{n} \cdot \frac{1}{y^{-1} + R/n - 1} = y$$

By multiplying by n/y , this equation can be written as:

$$\frac{t + yR}{1 + yR/n} = \frac{t}{1 + yR/n - y}$$

With $n \rightarrow \infty$ we obtain the following formula:

$$t + yR = \frac{t}{1 - y}$$

Thus $R = \frac{t}{1-y} = R_{\pi_t}$, which gives the result. \square

In order to get beyond this, let us introduce the following notions:

DEFINITION 5.24. Associated to any compactly supported positive measure ρ on \mathbb{C} are the probability measures

$$p_\rho = \lim_{n \rightarrow \infty} \left(\left(1 - \frac{c}{n}\right) \delta_0 + \frac{1}{n} \rho \right)^{*n}$$

$$\pi_\rho = \lim_{n \rightarrow \infty} \left(\left(1 - \frac{c}{n}\right) \delta_0 + \frac{1}{n} \rho \right)^{\boxplus n}$$

where $c = \text{mass}(\rho)$, called compound Poisson and compound free Poisson laws.

In what follows we will be interested in the case where ρ is discrete, as is for instance the case for $\rho = t\delta_1$ with $t > 0$, which produces the Poisson and free Poisson laws.

The following result allows one to detect compound Poisson/free Poisson laws:

THEOREM 5.25. For a discrete measure, written as

$$\rho = \sum_{i=1}^s c_i \delta_{z_i}$$

with $c_i > 0$ and $z_i \in \mathbb{R}$, we have the formulae

$$F_{p_\rho}(y) = \exp \left(\sum_{i=1}^s c_i (e^{iyz_i} - 1) \right)$$

$$R_{\pi_\rho}(y) = \sum_{i=1}^s \frac{c_i z_i}{1 - yz_i}$$

where F, R denote respectively the Fourier transform, and Voiculescu's R -transform.

PROOF. Let μ_n be the measure appearing in Definition 5.24, under the convolution signs:

$$\mu_n = \left(1 - \frac{c}{n}\right) \delta_0 + \frac{1}{n} \rho$$

In the classical case, we have the following computation:

$$F_{\mu_n}(y) = \left(1 - \frac{c}{n}\right) + \frac{1}{n} \sum_{i=1}^s c_i e^{iyz_i}$$

$$\implies F_{\mu_n^{*n}}(y) = \left(\left(1 - \frac{c}{n}\right) + \frac{1}{n} \sum_{i=1}^s c_i e^{iyz_i} \right)^n$$

$$\implies F_{p_\rho}(y) = \exp \left(\sum_{i=1}^s c_i (e^{iyz_i} - 1) \right)$$

In the free case now, we use a similar method. The Cauchy transform of μ_n is:

$$G_{\mu_n}(\xi) = \left(1 - \frac{c}{n}\right) \frac{1}{\xi} + \frac{1}{n} \sum_{i=1}^s \frac{c_i}{\xi - z_i}$$

Consider now the R -transform of the measure $\mu_n^{\boxplus n}$, which is given by:

$$R_{\mu_n^{\boxplus n}}(y) = nR_{\mu_n}(y)$$

The above formula of G_{μ_n} shows that the equation for $R = R_{\mu_n^{\boxplus n}}$ is as follows:

$$\begin{aligned} \left(1 - \frac{c}{n}\right) \frac{1}{y^{-1} + R/n} + \frac{1}{n} \sum_{i=1}^s \frac{c_i}{y^{-1} + R/n - z_i} &= y \\ \implies \left(1 - \frac{c}{n}\right) \frac{1}{1 + yR/n} + \frac{1}{n} \sum_{i=1}^s \frac{c_i}{1 + yR/n - yz_i} &= 1 \end{aligned}$$

Now multiplying by n , rearranging the terms, and letting $n \rightarrow \infty$, we get:

$$\begin{aligned} \frac{c + yR}{1 + yR/n} &= \sum_{i=1}^s \frac{c_i}{1 + yR/n - yz_i} \\ \implies c + yR_{\pi_\rho}(y) &= \sum_{i=1}^s \frac{c_i}{1 - yz_i} \\ \implies R_{\pi_\rho}(y) &= \sum_{i=1}^s \frac{c_i z_i}{1 - yz_i} \end{aligned}$$

This finishes the proof in the free case, and we are done. \square

We also have the following technical result, providing a useful alternative to Definition 5.24, in order to detect the classical and free compound Poisson laws:

THEOREM 5.26. *For a discrete measure, written as*

$$\rho = \sum_{i=1}^s c_i \delta_{z_i}$$

with $c_i > 0$ and $z_i \in \mathbb{R}$, we have the classical/free formulae

$$p_\rho/\pi_\rho = \text{law} \left(\sum_{i=1}^s z_i \alpha_i \right)$$

where the variables α_i are Poisson/free Poisson(c_i), independent/free.

PROOF. Let α be the sum of Poisson/free Poisson variables in the statement:

$$\alpha = \sum_{i=1}^s z_i \alpha_i$$

By using some well-known Fourier transform formulae, we have:

$$\begin{aligned} F_{\alpha_i}(y) = \exp(c_i(e^{iy} - 1)) &\implies F_{z_i \alpha_i}(y) = \exp(c_i(e^{iyz_i} - 1)) \\ &\implies F_{\alpha}(y) = \exp\left(\sum_{i=1}^s c_i(e^{iyz_i} - 1)\right) \end{aligned}$$

Also, by using some well-known R -transform formulae, we have:

$$\begin{aligned} R_{\alpha_i}(y) = \frac{c_i}{1-y} &\implies R_{z_i \alpha_i}(y) = \frac{c_i z_i}{1-yz_i} \\ &\implies R_{\alpha}(y) = \sum_{i=1}^s \frac{c_i z_i}{1-yz_i} \end{aligned}$$

Thus we have indeed the same formulae as those which are needed. \square

We refer to [35], [88], [90] for the general theory here, to [21], [24], [49] for representation theory aspects, and to [76], [89], [97] for random matrix aspects.

In what follows we will only need the main examples of classical and free compound Poisson laws, which are the classical and free Bessel laws.

These laws are constructed as follows:

DEFINITION 5.27. *The Bessel and free Bessel laws are the compound Poisson laws*

$$b_t^s = p_{t\varepsilon_s} \quad , \quad \beta_t^s = \pi_{t\varepsilon_s}$$

where ε_s is the uniform measure on the s -th roots unity. In particular:

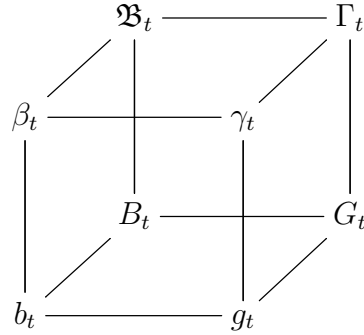
- (1) At $s = 1$ we obtain the usual Poisson and free Poisson laws, p_t, π_t .
- (2) At $s = 2$ we obtain the “real” Bessel and free Bessel laws, denoted b_t, β_t .
- (3) At $s = \infty$ we obtain the “complex” Bessel and free Bessel laws, denoted B_t, \mathfrak{B}_t .

There is a lot of theory regarding these laws, involving classical and quantum reflection groups, subfactors and planar algebras, and free probability and random matrices.

We refer here to [13], where these laws were introduced.

Let us just record here:

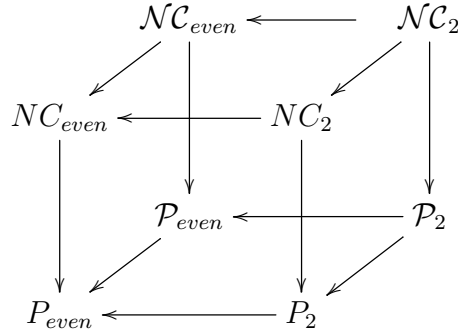
THEOREM 5.28. *The moments of the various central limiting measures, namely*



are always given by the same formula, involving partitions, namely

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

with the sets of partitions $D(k)$ in question being respectively



and with $|\cdot|$ being the number of blocks.

PROOF. This follows by putting together the various moment results that we have. We refer here to [13]. □

Getting back now to our quantum reflection groups, we first have:

THEOREM 5.29. *With $N \rightarrow \infty$, the laws of characters are as follows:*

- (1) *For H_N we obtain the Bessel law b_1 .*
- (2) *For H_N^+ we obtain the free Bessel law β_1 .*
- (3) *For K_N we obtain the complex Bessel law B_1 .*
- (4) *For K_N^+ we obtain the complex free Bessel law \mathfrak{B}_1 .*

PROOF. This is routine indeed, by counting the partitions, a bit as in the continuous case, in the proof of Theorem 5.13 above. For the full proof here, we refer to [13]. □

At the level of truncated characters, we have:

THEOREM 5.30. *With $N \rightarrow \infty$, the laws of truncated characters are as follows:*

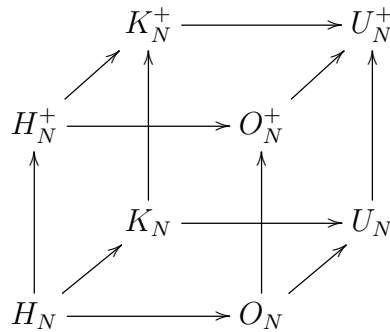
- (1) *For H_N we obtain the Bessel law b_t .*
- (2) *For H_N^+ we obtain the free Bessel law β_t .*
- (3) *For K_N we obtain the complex Bessel law B_t .*
- (4) *For K_N^+ we obtain the complex free Bessel law \mathfrak{B}_t .*

Also, we have the Bercovici-Pata bijection for truncated characters.

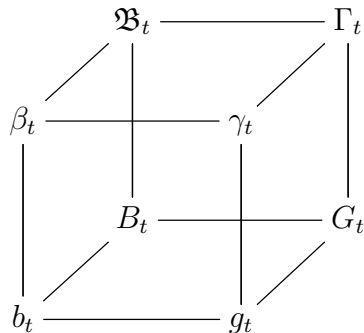
PROOF. Once again this is routine, by using the Weingarten formula, as in the continuous case, in the proof of Theorem 5.18 above. For the full proof here, we refer as before to the paper [13]. □

The results that we have so far, for the quantum unitary and reflection groups, are quite interesting, from a theoretical probability perspective, because we have:

THEOREM 5.31. *The laws of the truncated characters for the basic quantum groups,*



and the various classical and free central limiting measures, namely



in the $N \rightarrow \infty$ limit.

PROOF. This follows indeed by putting together the various results obtained above, concerning general free probability theory, and our computations here. □

Regarding now the tori, the situation here is more complicated, no longer involving the Bercovici-Pata bijection. Let us recall indeed that the basic tori are as follows:

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

These tori appear by definition as duals of the following discrete groups:

$$\begin{array}{ccc} \mathbb{Z}_2^{*N} & \longleftarrow & F_N \\ \downarrow & & \downarrow \\ \mathbb{Z}_2^N & \longleftarrow & \mathbb{Z}^N \end{array}$$

We are interested in the computation of the laws of the associated truncated characters, which are the following variables:

$$\chi_t = g_1 + g_2 + \dots + g_{[tN]}$$

By dilation we can assume $t = 1$. For the complex tori, $\mathbb{T}_N \subset \mathbb{T}_N^+$, we are led into the computation of the Kesten measures for $F_N \rightarrow \mathbb{Z}^N$, and so into the Meixner/free Meixner correspondence. As for the real tori, $T_N \subset T_N^+$, here we are led into the computation of the Kesten measures for $\mathbb{Z}_2^{*N} \rightarrow \mathbb{Z}_2^N$, and so into a real version of this correspondence.

Summarizing, we have some nice liberation results for S, T, U, K , with a technical problem, however, coming from the fact that those for S, U, K come from the Bercovici-Pata bijection, while those for T come from the Meixner/free Meixner correspondence.

5e. Exercises

As a first exercise in relation with the material in this chapter, we reproduce here a key exercise from chapter 3 above, in case you have not done it yet:

EXERCISE 5.32. *Work out the explicit formula of the Weingarten matrix*

$$W_{kN} = G_{kN}^{-1} \quad : \quad G_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma|}$$

for the basic free quantum groups, at small values of $N \in \mathbb{N}$.

In practice, before even starting, you will have to find a good order relation on the partitions, in order to write down the Gram and Weingarten matrices.

In relation now with the integration over the spheres, we have:

EXERCISE 5.33. *Work out the Weingarten formula for the classical spheres*

$$S_{\mathbb{R}}^{N-1} \quad , \quad S_{\mathbb{C}}^{N-1}$$

in general, then at small $N \in \mathbb{N}$, and at big $N \in \mathbb{N}$, and find some applications of this.

Here the application part is a bit up to you. The classical spheres are very classical objects, and they appear in connection with many questions. So, simply pick one of these questions, that you really like, and try to solve it using the Weingarten formula.

Here is now a more difficult exercise, in connection with the tori:

EXERCISE 5.34. *Compute the laws of the truncated characters for the main tori*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

and then work out the asymptotics, with $N \rightarrow \infty$.

This is something that we briefly discussed at the end of this chapter, and there is definitely some interesting work to be done here.

For the rest, in connection with all the free probability material in this chapter, we can only recommend downloading and reading a good free probability book.

CHAPTER 6

Quotient spaces

6a. Quotient spaces

In this chapter and in the next two ones we keep building on the work started in the previous chapter, by systematically developing the real and complex free geometry.

We will extend the family of objects (S, T, U, K) that we have, first by unifying S, U via a homogeneous space construction, involving row algebras for $C(U)$, and then by further building on this construction, first with more general explicit homogeneous spaces, of “quantum partial isometries”, and then with even more abstract manifolds, emerging from our study, that we will call “affine homogeneous spaces”.

We will also discuss, at the end of chapter 8, the axiomatization problem for the “free manifolds”, with a number of preliminary results on the subject.

The present chapter the next two ones are relatively independent and self-contained, based respectively on the papers [32], then [9], then [10]. However, as explained above, the constructions that we will present will generalize each other, and often in a quite substantial and abstract way, so typically for understanding the main examples of the “higher” constructions, you have to go back to the “lower” constructions.

Let us begin with some generalities regarding the quotient spaces, and more general homogeneous spaces. Regarding the quotients, we have the following construction:

PROPOSITION 6.1. *Given a quantum subgroup $H \subset G$, with associated quotient map $\rho : C(G) \rightarrow C(H)$, if we define the quotient space $X = G/H$ by setting*

$$C(X) = \left\{ f \in C(G) \mid (\rho \otimes id)\Delta f = 1 \otimes f \right\}$$

then we have a coaction map as follows,

$$\Phi : C(X) \rightarrow C(X) \otimes C(G)$$

obtained as the restriction of the comultiplication of $C(G)$. In the classical case, we obtain in this way the usual quotient space $X = G/H$.

PROOF. Observe that the linear subspace $C(X) \subset C(G)$ defined in the statement is indeed a subalgebra, because it is defined via a relation of the following type, with both the maps φ, ψ being morphisms of algebras:

$$\varphi(f) = \psi(f)$$

Observe also that in the classical case we obtain the algebra of continuous functions on the quotient space $X = G/H$, because we have:

$$\begin{aligned} & (\rho \otimes id)\Delta f = 1 \otimes f \\ \iff & (\rho \otimes id)\Delta f(h, g) = (1 \otimes f)(h, g), \forall h \in H, \forall g \in G \\ \iff & f(hg) = f(g), \forall h \in H, \forall g \in G \\ \iff & f(hg) = f(kg), \forall h, k \in H, \forall g \in G \end{aligned}$$

Regarding now the construction of Φ , observe that for $f \in C(X)$ we have:

$$\begin{aligned} (\rho \otimes id \otimes id)(\Delta \otimes id)\Delta f &= (\rho \otimes id \otimes id)(id \otimes \Delta)\Delta f \\ &= (id \otimes \Delta)(\rho \otimes id)\Delta f \\ &= (id \otimes \Delta)(1 \otimes f) \\ &= 1 \otimes \Delta f \end{aligned}$$

Thus the condition $f \in C(X)$ implies that we have:

$$\Delta f \in C(X) \otimes C(G)$$

But this gives the existence of Φ . Finally, the other assertions are clear. \square

As an illustration, in the group dual case we have:

PROPOSITION 6.2. *Assume that $G = \widehat{\Gamma}$ is a discrete group dual.*

- (1) *The quantum subgroups of G are $H = \widehat{\Lambda}$, with $\Gamma \rightarrow \Lambda$ being a quotient group.*
- (2) *For such a quantum subgroup $\widehat{\Lambda} \subset \widehat{\Gamma}$, we have $\widehat{\Gamma}/\widehat{\Lambda} = \widehat{\Theta}$, where:*

$$\Theta = \ker(\Gamma \rightarrow \Lambda)$$

PROOF. This is well-known, the idea being as follows:

(1) In one sense, this is clear. Conversely, since the algebra $C(G) = C^*(\Gamma)$ is cocommutative, so are all its quotients, and this gives the result.

(2) Consider a quotient map $r : \Gamma \rightarrow \Lambda$, and denote by $\rho : C^*(\Gamma) \rightarrow C^*(\Lambda)$ its extension. Consider a group algebra element, written as follows:

$$f = \sum_{g \in \Gamma} \lambda_g \cdot g \in C^*(\Gamma)$$

We have then the following computation:

$$\begin{aligned}
f \in C(\widehat{\Gamma}/\widehat{\Lambda}) &\iff (\rho \otimes id)\Delta(f) = 1 \otimes f \\
&\iff \sum_{g \in \Gamma} \lambda_g \cdot r(g) \otimes g = \sum_{g \in \Gamma} \lambda_g \cdot 1 \otimes g \\
&\iff \lambda_g \cdot r(g) = \lambda_g \cdot 1, \forall g \in \Gamma \\
&\iff \text{supp}(f) \subset \ker(r)
\end{aligned}$$

But this means that we have $\widehat{\Gamma}/\widehat{\Lambda} = \widehat{\Theta}$, with $\Theta = \ker(\Gamma \rightarrow \Lambda)$, as claimed. \square

Given two compact quantum spaces X, Y , we say that X is a quotient space of Y when we have an embedding of C^* -algebras, as follows:

$$\alpha : C(X) \subset C(Y)$$

With this convention, we can formulate the following definition:

DEFINITION 6.3. *We call a quotient space $G \rightarrow X$ homogeneous when*

$$\Delta(C(X)) \subset C(X) \otimes C(G)$$

where $\Delta : C(G) \rightarrow C(G) \otimes C(G)$ is the comultiplication map.

In other words, an homogeneous quotient space $G \rightarrow X$ is a quantum space coming from a subalgebra $C(X) \subset C(G)$, which is stable under the comultiplication.

The relation with the quotient spaces from Proposition 6.1 is as follows:

THEOREM 6.4. *The following results hold:*

- (1) *The quotient spaces $X = G/H$ are homogeneous.*
- (2) *In the classical case, any homogeneous space is of type G/H .*
- (3) *In general, there are homogeneous spaces which are not of type G/H .*

PROOF. Once again these results are well-known, the proof being as follows:

(1) This is clear from Proposition 6.1 above.

(2) Consider a quotient map $p : G \rightarrow X$. The invariance condition in the statement tells us that we must have an action $G \curvearrowright X$, given by:

$$g(p(g')) = p(gg')$$

Thus, we have the following implication:

$$p(g') = p(g'') \implies p(gg') = p(gg''), \forall g \in G$$

Now observe that the following subset $H \subset G$ is a subgroup:

$$H = \left\{ g \in G \mid p(g) = p(1) \right\}$$

Indeed, $g, h \in H$ implies that we have:

$$p(gh) = p(g) = p(1)$$

Thus we have $gh \in H$, and the other axioms are satisfied as well. Our claim now is that we have an identification $X = G/H$, obtained as follows:

$$p(g) \rightarrow Hg$$

Indeed, the map $p(g) \rightarrow Hg$ is well-defined and bijective, because $p(g) = p(g')$ is equivalent to $p(g^{-1}g') = p(1)$, and so to $Hg = Hg'$, as desired.

(3) Given a discrete group Γ and an arbitrary subgroup $\Theta \subset \Gamma$, the quotient space $\widehat{\Gamma} \rightarrow \widehat{\Theta}$ is homogeneous. Now by using Proposition 6.2 above, we can see that if $\Theta \subset \Gamma$ is not normal, the quotient space $\widehat{\Gamma} \rightarrow \widehat{\Theta}$ is not of the form G/H . \square

With the above formalism in hand, let us try now to understand the general properties of the homogeneous spaces $G \rightarrow X$, in the sense of Theorem 6.4.

We recall from chapter 2 that any compact quantum group G has a Haar integration functional $\int : C(G) \rightarrow \mathbb{C}$, having the following invariance properties:

$$\left(\int \otimes id \right) \Delta = \left(id \otimes \int \right) \Delta = \int (\cdot) 1$$

We have the following result, which is once again well-known:

PROPOSITION 6.5. *Assume that a quotient space $G \rightarrow X$ is homogeneous.*

(1) *We have a coaction map as follows, obtained as restriction of Δ :*

$$\Phi : C(X) \rightarrow C(X) \otimes C(G)$$

(2) *We have the following formula:*

$$\Phi(f) = f \otimes 1 \implies f \in \mathbb{C}1$$

(3) *We have as well the following formula:*

$$\left(id \otimes \int \right) \Phi f = \int f$$

(4) *The restriction of \int is the unique unital form satisfying:*

$$(\tau \otimes id)\Phi = \tau(\cdot)1$$

PROOF. These results are all elementary, the proof being as follows:

(1) This is clear from definitions, because Δ itself is a coaction.

(2) Assume indeed $f \in C(G)$ satisfies the following condition:

$$\Delta(f) = f \otimes 1$$

By applying the counit we obtain from this:

$$(\varepsilon \otimes id)\Delta f = (\varepsilon \otimes id)(f \otimes 1)$$

We conclude from this that we have, as desired:

$$f = \varepsilon(f)1$$

(3) The formula in the statement, $(id \otimes \int)\Phi f = \int f$, follows from the left invariance property of the Haar functional of $C(G)$, namely:

$$\left(id \otimes \int\right) \Delta f = \int f$$

Indeed, by restriction to $C(X)$, we obtain our formula.

(4) In order to prove the result, we can use here the right invariance property of the Haar functional of $C(G)$, namely:

$$\left(\int \otimes id\right) \Delta f = \int f$$

Indeed, we obtain from this that $tr = \int_{|C(X)}$ is G -invariant, in the sense that:

$$(tr \otimes id)\Phi f = tr(f)1$$

Conversely, assuming that $\tau : C(X) \rightarrow \mathbb{C}$ satisfies $(\tau \otimes id)\Phi f = \tau(f)1$, we have:

$$\begin{aligned} \left(\tau \otimes \int\right) \Phi(f) &= \int (\tau \otimes id)\Phi(f) \\ &= \int (\tau(f)1) \\ &= \tau(f) \end{aligned}$$

On the other hand, we can compute the same quantity as follows:

$$\begin{aligned} \left(\tau \otimes \int\right) \Phi(f) &= \tau \left(id \otimes \int\right) \Phi(f) \\ &= \tau(tr(f)1) \\ &= tr(f) \end{aligned}$$

Thus we have $\tau(f) = tr(f)$ for any $f \in C(X)$, and this finishes the proof. \square

Summarizing, we have a notion of noncommutative homogeneous space, which perfectly covers the classical case. In general, however, the group dual case shows that our formalism is more general than that of the quotient spaces G/H .

6b. Extended spaces

We have seen so far that a theory of homogeneous spaces can be developed for the compact quantum groups, by following the theory from the classical case, with a few minor twists, of algebraic nature, basically coming from the group duals.

We discuss now an extra issue, of analytic nature. The point indeed is that for one of the most basic examples of actions, $O_N^+ \curvearrowright S_{\mathbb{R},+}^{N-1}$, the associated morphism $\alpha : C(X) \rightarrow C(G)$ is not injective. The same is true for other basic actions, in the free setting.

In order to include such examples, we must relax our axioms:

DEFINITION 6.6. *An extended homogeneous space over a compact quantum group G consists of a morphism of C^* -algebras, and a coaction map, as follows*

$$\begin{aligned}\alpha &: C(X) \rightarrow C(G) \\ \Phi &: C(X) \rightarrow C(X) \otimes C(G)\end{aligned}$$

such that the following diagram commutes

$$\begin{array}{ccc} C(X) & \xrightarrow{\Phi} & C(X) \otimes C(G) \\ \alpha \downarrow & & \downarrow \alpha \otimes id \\ C(G) & \xrightarrow{\Delta} & C(G) \otimes C(G) \end{array}$$

and such that the following diagram commutes as well

$$\begin{array}{ccc} C(X) & \xrightarrow{\Phi} & C(X) \otimes C(G) \\ \alpha \downarrow & & \downarrow id \otimes f \\ C(G) & \xrightarrow{f(\cdot)^1} & C(X) \end{array}$$

where \int is the Haar integration over G . We write then $G \rightarrow X$.

As a first observation, in relation with these new axioms, when the morphism α is injective we obtain an homogeneous space in the previous sense. Thus, all the general theory developed in the above is covered by our new formalism.

The examples with α not injective, which motivate the above formalism, include the standard action $O_N^+ \curvearrowright S_{\mathbb{R},+}^{N-1}$, and the standard action $U_N^+ \curvearrowright S_{\mathbb{C},+}^{N-1}$.

Here are a few general remarks on the above axioms:

PROPOSITION 6.7. *Assume that we have morphisms of C^* -algebras*

$$\alpha : C(X) \rightarrow C(G)$$

$$\Phi : C(X) \rightarrow C(X) \otimes C(G)$$

satisfying the coassociativity condition $(\alpha \otimes id)\Phi = \Delta\alpha$.

- (1) *If α is injective on a dense $*$ -subalgebra $A \subset C(X)$, and $\Phi(A) \subset A \otimes C(G)$, then Φ is automatically a coaction map, and is unique.*
- (2) *The ergodicity type condition $(id \otimes \int)\Phi = \int \alpha(\cdot)1$ is equivalent to the existence of a linear form $\lambda : C(X) \rightarrow \mathbb{C}$ such that $(id \otimes \int)\Phi = \lambda(\cdot)1$.*

PROOF. This is something elementary, the idea being as follows:

(1) Assuming that we have a dense $*$ -subalgebra $A \subset C(X)$ as in the statement, satisfying $\Phi(A) \subset A \otimes C(G)$, the restriction $\Phi|_A$ is given by:

$$\Phi|_A = (\alpha|_A \otimes id)^{-1} \Delta\alpha|_A$$

This restriction is therefore coassociative, and unique. By continuity, Φ itself follows to be coassociative and unique, as desired.

(2) Assuming $(id \otimes \int)\Phi = \lambda(\cdot)1$, we have:

$$\left(\alpha \otimes \int \right) \Phi = \lambda(\cdot)1$$

On the other hand, we have as well the following formula:

$$\left(\alpha \otimes \int \right) \Phi = \left(id \otimes \int \right) \Delta\alpha = \int \alpha(\cdot)1$$

Thus we obtain $\lambda = \int \alpha$, as claimed. □

Given an extended homogeneous space $G \rightarrow X$, with associated map $\alpha : C(X) \rightarrow C(G)$, we can consider the image of this latter map:

$$\alpha : C(X) \rightarrow C(Y) \subset C(G)$$

Equivalently, at the level of the associated noncommutative spaces, we can factorize the corresponding quotient map $G \rightarrow Y \subset X$. With these conventions, we have:

PROPOSITION 6.8. *Consider an extended homogeneous space $G \rightarrow X$.*

- (1) $\Phi(f) = f \otimes 1 \implies f \in \mathbb{C}1$.
- (2) $tr = \int \alpha$ is the unique unital G -invariant form on $C(X)$.
- (3) The image space obtained by factorizing, $G \rightarrow Y$, is homogeneous.

PROOF. We have several assertions to be proved, the idea being as follows:

(1) This follows indeed from $(id \otimes f)\Phi(f) = \int \alpha(f)1$, which gives:

$$f = \int \alpha(f)1$$

(2) The fact that $tr = \int \alpha$ is indeed G -invariant can be checked as follows:

$$\begin{aligned} (tr \otimes id)\Phi f &= (\int \alpha \otimes id)\Phi f \\ &= (\int \otimes id)\Delta \alpha f \\ &= \int \alpha(f)1 \\ &= tr(f)1 \end{aligned}$$

As for the uniqueness assertion, this follows as before.

(3) The condition $(\alpha \otimes id)\Phi = \Delta \alpha$, together with the fact that i is injective, allows us to factorize Δ into a morphism Ψ , as follows:

$$\begin{array}{ccc} C(X) & \xrightarrow{\Phi} & C(X) \otimes C(G) \\ \alpha \downarrow & & \downarrow \alpha \otimes id \\ C(Y) & \xrightarrow{\Psi} & C(Y) \otimes C(G) \\ i \downarrow & & \downarrow i \otimes id \\ C(G) & \xrightarrow{\Delta} & C(G) \otimes C(G) \end{array}$$

Thus the image space $G \rightarrow Y$ is indeed homogeneous, and we are done. \square

Finally, we have the following result:

THEOREM 6.9. *Let $G \rightarrow X$ be an extended homogeneous space, and construct quotients $X \rightarrow X'$, $G \rightarrow G'$ by performing the GNS construction with respect to $\int \alpha, \int$. Then α factorizes into an inclusion $\alpha' : C(X') \rightarrow C(G')$, and we have an homogeneous space.*

PROOF. We factorize $G \rightarrow Y \subset X$ as above. By performing the GNS construction with respect to $\int i\alpha, \int i, \int$, we obtain a diagram as follows:

$$\begin{array}{ccc}
 C(X) & \xrightarrow{p} & C(X') \\
 \alpha \downarrow & & \downarrow \alpha' \\
 C(Y) & \xrightarrow{q} & C(Y') \\
 i \downarrow & & \downarrow i' \\
 C(G) & \xrightarrow{r} & C(G')
 \end{array}
 \begin{array}{l}
 \nearrow^{tr'} \\
 \searrow_{f'} \\
 \mathbb{C}
 \end{array}$$

Indeed, with $tr = \int \alpha$, the GNS quotient maps p, q, r are defined respectively by:

$$\begin{aligned}
 \ker p &= \left\{ f \in C(X) \mid tr(f^*f) = 0 \right\} \\
 \ker q &= \left\{ f \in C(Y) \mid \int (f^*f) = 0 \right\} \\
 \ker r &= \left\{ f \in C(G) \mid \int (f^*f) = 0 \right\}
 \end{aligned}$$

Next, we can define factorizations i', α' as above. Observe that i' is injective, and that α' is surjective. Our claim now is that α' is injective as well. Indeed:

$$\begin{aligned}
 \alpha'p(f) = 0 &\implies q\alpha(f) = 0 \\
 &\implies \int \alpha(f^*f) = 0 \\
 &\implies tr(f^*f) = 0 \\
 &\implies p(f) = 0
 \end{aligned}$$

We conclude that we have $X' = Y'$, and this gives the result. \square

Summarizing, the basic homogeneous space theory from the classical case extends to the quantum group setting, with a few twists, both of algebraic and analytic nature.

6c. Row spaces

Following [20], let us discuss now some basic examples of homogeneous spaces, which unify the spheres S with the unitary quantum groups U they come from.

We first discuss the construction in the classical case. Given a closed subgroup $G \subset U_N$ and a number $k \leq N$, we can consider the compact group $H = G \cap U_k$, computed inside

U_N , where the embedding $U_k \subset U_N$ that we use is given by:

$$g \rightarrow \begin{pmatrix} g & 0 \\ 0 & 1_{N-k} \end{pmatrix}$$

We can form the homogeneous space $X = G/H$, and we have the following result:

PROPOSITION 6.10. *Let $G \subset U_N$ be a closed subgroup, and construct as above the closed subgroup $H \subset G$ given by*

$$H = G \cap U_k$$

with the intersection being computed inside U_N . Then the subalgebra

$$C(G/H) \subset C(G)$$

that we obtain is generated by the last $N - k$ rows of coordinates on G .

PROOF. Let $u_{ij} \in C(G)$ be the standard coordinates on G , given by:

$$u_{ij}(g) = g_{ij}$$

Consider the following subalgebra of $C(G)$:

$$A = \langle u_{ij} \mid i > k, j > 0 \rangle$$

Since each coordinate function u_{ij} with $i > k$ is constant on each coset $Hg \in G/H$, we have an inclusion as follows, between subalgebras of $C(G)$:

$$A \subset C(G/H)$$

In order to prove that this inclusion is an isomorphism, as to finish, we use the Stone-Weierstrass theorem. Indeed, in view of this theorem, it is enough to show that the following family of functions separates the cosets $\{Hg \mid g \in G\}$:

$$\{u_{ij} \mid i > k, j > 0\}$$

But this is the same as saying that the condition $Hg \neq Hh$ implies, for some indices $i > k, j > 0$, that we have:

$$g_{ij} \neq h_{ij}$$

Equivalently, we must prove that the condition $g_{ij} = h_{ij}$ for any $i > k, j > 0$ must imply that we have:

$$Hg = Hh$$

Now since $Hg = Hh$ is equivalent to $gh^{-1} \in H$, the result follows from the usual matrix formula of gh^{-1} , and from the fact that g, h are unitary. \square

In the quantum case now, let $k \leq N$, and consider the embedding $U_k^+ \subset U_N^+$ given by the same formula as before, namely:

$$g \rightarrow \begin{pmatrix} g & 0 \\ 0 & 1_{N-k} \end{pmatrix}$$

That is, at the level of algebras, we use the quotient map $C(U_N^+) \rightarrow C(U_k^+)$ given by the following formula, where v is the fundamental corepresentation of U_k^+ :

$$u \rightarrow \begin{pmatrix} v & 0 \\ 0 & 1_{N-k} \end{pmatrix}$$

With this convention, we have the following definition:

DEFINITION 6.11. *Associated to any quantum subgroup $G \subset U_N^+$ and any $k \leq N$ are:*

- (1) *The compact quantum group $H = G \cap U_k^+$.*
- (2) *The algebra $C(G/H) \subset C(G)$ constructed above.*
- (3) *The algebra $C_\times(G/H) \subset C(G/H)$ generated by $\{u_{ij} | i > k, j > 0\}$.*

Regarding (3), let u, v be the fundamental corepresentations of G, H , so that the quotient map $\pi : C(G) \rightarrow C(H)$ is given by $u \rightarrow \text{diag}(v, 1_{N-k})$. We have then:

$$\begin{aligned} (\pi \otimes id)\Delta(u_{ij}) &= \sum_s \pi(u_{is}) \otimes u_{sj} \\ &= \begin{cases} \sum_{s \leq k} v_{is} \otimes u_{sj} & i \leq k \\ 1 \otimes u_{ij} & i > k \end{cases} \end{aligned}$$

In particular we see that the equality $(\pi \otimes id)\Delta f = 1 \otimes f$ defining $C(G/H)$ holds on all the coefficients $f = u_{ij}$ with $i > k$, and this justifies the inclusion appearing in (3).

Let us first try to understand what happens in the group dual case. We will do our study here in two steps, first in the “diagonal” case, and then in the general case.

We recall that given a discrete group $\Gamma = \langle g_1, \dots, g_N \rangle$, the matrix $D = \text{diag}(g_i)$ is biunitary, and produces a surjective morphism $C(U_N^+) \rightarrow C^*(\Gamma)$.

This morphism can be viewed as corresponding to a quantum embedding $\widehat{\Gamma} \subset U_N^+$, that we call “diagonal”.

We recall also that the normal closure of a subgroup $\Lambda \subset \Gamma$ is the biggest subgroup $\Lambda' \subset \Gamma$ containing Λ as a normal subgroup. Note that Λ' can be different from the normalizer $N(\Lambda)$.

With these conventions, we have the following result:

PROPOSITION 6.12. *Assume that we have a group dual $G = \widehat{\Gamma}$, with*

$$\Gamma = \langle g_1, \dots, g_N \rangle$$

diagonally embedded, and let $H = G \cap U_k^+$.

- (1) $H = \widehat{\Theta}$, where $\Theta = \Gamma / \langle g_{k+1} = 1, \dots, g_N = 1 \rangle$.
- (2) $C_\times(G/H) = C^*(\Lambda)$, where $\Lambda = \langle g_{k+1}, \dots, g_N \rangle$.
- (3) $C(G/H) = C^*(\Lambda')$, where “prime” is the normal closure.
- (4) $C_\times(G/H) = C(G/H)$ if and only if $\Lambda \triangleleft \Gamma$.

PROOF. We use the standard fact that for any group $\Gamma = \langle a_i, b_j \rangle$, the kernel of the quotient map $\Gamma \rightarrow \Gamma / \langle a_i = 1 \rangle$ is the normal closure of the subgroup $\langle a_i \rangle \subset \Gamma$.

(1) Since the map $C(U_N^+) \rightarrow C(U_k^+)$ is given on diagonal coordinates by $u_{ii} \rightarrow v_{ii}$ for $i \leq k$ and $u_{ii} \rightarrow 1$ for $i > k$, the result follows from definitions.

(2) Once again, this assertion follows from definitions.

(3) From the above and from (1) we get $G/H = \widehat{\Lambda}'$, where $\Lambda' = \ker(\Gamma \rightarrow \Theta)$. By the above observation, this kernel is exactly the normal closure of Λ .

(4) This follows from (2) and (3). □

Let us try now to understand the general group dual case.

We recall from chapter 2 above that the group dual subgroups $\widehat{\Gamma} \subset U_N^+$ appear by taking a discrete group $\Gamma = \langle g_1, \dots, g_N \rangle$ and a unitary matrix $J \in U_N$, and constructing the morphism $C(U_N^+) \rightarrow C^*(\Gamma)$ given by $u \rightarrow J D J^*$, where $D = \text{diag}(g_i)$.

With this in hand, Proposition 6.12 generalizes as follows:

THEOREM 6.13. *Assume that we have a group dual $G = \widehat{\Gamma}$, with*

$$\Gamma = \langle g_1, \dots, g_N \rangle$$

embedded via $u \rightarrow J D J^$, and let $H = G \cap U_k^+$.*

- (1) $H = \widehat{\Theta}$, where $\Theta = \Gamma / \langle g_r = 1 | \exists i > k, J_{ir} \neq 0 \rangle$, embedded $u_{ij} \rightarrow (J D J^*)_{ij}$.
- (2) $C_\times(G/H) = C^*(\Lambda)$, where $\Lambda = \langle g_r | \exists i > k, J_{ir} \neq 0 \rangle$.
- (3) $C(G/H) = C^*(\Lambda')$, where “prime” is the normal closure.
- (4) $C_\times(G/H) = C(G/H)$ if and only if $\Lambda \triangleleft \Gamma$.

PROOF. We basically follow the proof of Proposition 6.12 above:

(1) Let $\Lambda = \langle g_1, \dots, g_N \rangle$, let $J \in U_N$, and consider the embedding $\widehat{\Lambda} \subset U_N^+$ corresponding to the morphism $C(U_N^+) \rightarrow C^*(\Lambda)$ given by $u \rightarrow J D J^*$, where $D = \text{diag}(g_i)$.

Let $G = \widehat{\Lambda} \cap U_k^+$. Since we have $G \subset \widehat{\Lambda}$, the algebra $C(G)$ is cocommutative, so we have $G = \widehat{\Theta}$ for a certain discrete group Θ . Moreover, the inclusion $\widehat{\Theta} \subset \widehat{\Lambda}$ must come from a

group morphism $\varphi : \Lambda \rightarrow \Theta$. Also, since $\widehat{\Theta} \subset U_k^+$, we have a morphism $C(U_k^+) \rightarrow C^*(\Theta)$ given by $v \rightarrow V$, where V is a certain $k \times k$ biunitary over $C^*(\Theta)$.

With these observations in hand, let us look now at the intersection operation. We must have a group morphism $\varphi : \Lambda \rightarrow \Theta$ such that the following diagram commutes:

$$\begin{array}{ccc} C(U_N^+) & \longrightarrow & C(U_k^+) \\ \downarrow & & \downarrow \\ C^*(\Lambda) & \longrightarrow & C^*(\Theta) \end{array}$$

Thus we must have the following equality:

$$(id \otimes \varphi)(JDJ^*) = diag(V, 1_{N-k})$$

With $f_i = \varphi(g_i)$, we obtain from this:

$$\sum_r J_{ir} \bar{J}_{jr} f_r = \begin{cases} V_{ij} & \text{if } i, j \leq k \\ \delta_{ij} & \text{otherwise} \end{cases}$$

Now since J is unitary, the second part of the above condition is equivalent to “ $f_r = 1$ whenever there exists $i > k$ such that $J_{ir} \neq 0$ ”. Indeed, this condition is easily seen to be equivalent to the “= 1” conditions, and implies the “= 0” conditions. We claim that:

$$\Theta = \Lambda / \langle g_r = 1 \mid \exists i > k, J_{ir} \neq 0 \rangle$$

Indeed, the above discussion shows that Θ must be a quotient of the group on the right, say Θ_0 . On the other hand, since in $C^*(\Theta_0)$ we have $J_{ir} g_r = J_{ir} 1$ for any $i > k$, we obtain that $(JDJ^*)_{ij} = \delta_{ij}$ unless $i, j \leq k$, so we have, for a certain matrix V :

$$JDJ^* = diag(V, 1_{N-k})$$

But V must be a biunitary, so we have a morphism $C(U_k^+) \rightarrow C^*(\Theta_0)$ mapping $v \rightarrow V$, which completes the proof of our claim.

(2) Consider the standard generators of the algebra $C_\times(G/H)$, which are as follows, with indices $i > k, j > 0$:

$$A_{ij} = \sum_r J_{ir} \bar{J}_{jr} g_r$$

We have then the following formula:

$$\sum_j A_{ij} J_{jm} = J_{im} g_m$$

We conclude that $C_\times(G/H)$ contains any g_r such that there exists $i > k$ with $J_{ir} \neq 0$, i.e. contains any $g_r \in \Lambda$. Conversely, if $g_r \in \Gamma - \Lambda$ then $J_{ir}g_r = 0$ for any $i > k$, so g_r doesn't appear in the formula of any of the generators A_{ij} .

(3,4) The proof here is similar to the proof of Proposition 6.12 (3,4). \square

Summarizing, we have a good understanding of the row algebras, both in the classical case, and in the group dual case.

6d. Uniformity

Going now towards the easy case, and the examples of quotient spaces that are interested in, we will need the following key result, coming from [26], [32], [33]:

THEOREM 6.14. *For an easy quantum group $G_N \subset U_N^+$, the following conditions are equivalent:*

(1) $G = (G_N)$ is uniform, in the sense that we have

$$G_N \cap U_k^+ = G_k$$

for any $k \leq N$, with respect to the standard embedding $U_k^+ \subset U_N^+$.

(2) The corresponding category of partitions

$$D = (D(k, l))$$

is stable under the operation which consists in removing blocks.

PROOF. In order to establish the equivalence between the above two conditions, we will prove that $G_N \cap U_k^+ = G'_k$, where $G' = (G'_N)$ is the easy quantum group associated to the category D' generated by all subpartitions of the partitions in D .

We know that the correspondence between categories of partitions and easy quantum groups comes from Woronowicz's Tannakian duality in [100].

More precisely, the quantum group $G_N \subset O_N^+$ associated to a category of partitions $D = (D(s))$ is obtained by imposing to the fundamental representation of O_N^+ the fact that its s -th tensor power must fix ξ_π , for any $s \in \mathbb{N}$ and $\pi \in D(s)$. In other words, we have the following presentation result:

$$C(G_N) = C(O_N^+) / \left\langle \xi_\pi \in \text{Fix}(u^{\otimes s}), \forall s, \forall \pi \in D(s) \right\rangle$$

Now since $\xi_\pi \in \text{Fix}(u^{\otimes s})$ means $u^{\otimes s}(\xi_\pi \otimes 1) = \xi_\pi \otimes 1$, this condition is equivalent to the following collection of equalities, one for each multi-index $i \in \{1, \dots, N\}^s$:

$$\sum_{j_1 \dots j_s} \delta_\pi(j) u_{i_1 j_1} \dots u_{i_s j_s} = \delta_\pi(i) 1$$

Summarizing, we have the following presentation result:

$$C(G_N) = C(O_N^+) / \left\langle \sum_{j_1 \dots j_s} \delta_\pi(j) u_{i_1 j_1} \dots u_{i_s j_s} = \delta_\pi(i) 1, \forall s, \forall \pi \in D(s), \forall i \right\rangle$$

Let now $k \leq N$, assume that we have a compact quantum group $K \subset O_k^+$, with fundamental representation denoted u , and consider the following $N \times N$ matrix:

$$\tilde{u} = \begin{pmatrix} u & 0 \\ 0 & 1_{N-k} \end{pmatrix}$$

Our claim is that for any $s \in \mathbb{N}$ and any $\pi \in P(s)$, we have:

$$\xi_\pi \in \text{Fix}(\tilde{u}^{\otimes s}) \iff \xi_{\pi'} \in \text{Fix}(u^{\otimes s'}), \forall \pi' \subset \pi$$

Here $\pi' \subset \pi$ means that $\pi' \in P(s')$ is obtained from $\pi \in P(s)$ by removing some of its blocks. The proof of this claim is standard. Indeed, when making the replacement $u \rightarrow \tilde{u}$ and trying to check the condition $\xi_\pi \in \text{Fix}(\tilde{u}^{\otimes s})$, we have two cases:

– $\delta_\pi(i) = 1$. Here the $> k$ entries of i must be joined by certain blocks of π , and we can consider the partition $\pi' \in D(s')$ obtained by removing these blocks. The point now is that the collection of $\delta_\pi(i) = 1$ equalities to be checked coincides with the collection of $\delta_\pi(i) = 1$ equalities expressing the fact that we have $\xi_\pi \in \text{Fix}(u^{\otimes s'})$, for any $\pi' \subset \pi$.

– $\delta_\pi(i) = 0$. In this case the situation is quite similar. Indeed, the collection of $\delta_\pi(i) = 0$ equalities to be checked coincides, modulo some $0 = 0$ identities, which hold automatically, with the collection of $\delta_\pi(i) = 0$ equalities expressing the fact that we have $\xi_\pi \in \text{Fix}(u^{\otimes s'})$, for any $\pi' \subset \pi$.

Our second claim is that given a quantum group $K \subset O_N^+$, with fundamental representation denoted v , the algebra of functions on $H = K \cap O_k^+$ is given by:

$$C(H) = C(O_k^+) / \left\langle \xi \in \text{Fix}(\tilde{u}^{\otimes s}), \forall \xi \in \text{Fix}(v^{\otimes s}) \right\rangle$$

This follows indeed from Woronowicz's results in [99], because the algebra on the right comes from the Tannakian formulation of the intersection operation.

Now with the above two claims in hand, we can conclude that we have the following formula, where $G' = (G'_N)$ is the easy quantum group associated to the category D' generated by all the subpartitions of the partitions in D :

$$G_N \cap U_k^+ = G'_k$$

In particular we see that the condition $G_N \cap U_k^+ = G'_k$ for any $k \leq N$ is equivalent to $D = D'$, and this gives the result. \square

Let us study now the following inclusions of algebras, constructed in Definition 6.11 above, where $G = (G_n)$ is a uniform easy quantum group:

$$C_\times(G_N/G_k) \subset C(G_N/G_k)$$

We recall from [33] that the basic examples are the classical groups S, O, H, B , and their free analogues S^+, O^+, H^+, B^+ . In addition, it is known that in the free case the list of such quantum groups is precisely S^+, O^+, H^+, B^+ . See [33].

We have the following result, which is well-known:

PROPOSITION 6.15. *The defining relations for $C(G)$ are as follows, in terms of the standard generators u_{ij} :*

- (1) $G = O_N^+$: u is orthogonal, i.e. u_{ij} are self-adjoint, and $u^t = u^{-1}$.
- (2) $G = S_N^+$: u is magic, i.e. orthogonal, with u_{ij} being projections.
- (3) $G = H_N^+$: u is cubic, i.e. orthogonal, with $xy = 0$ on rows and columns.
- (4) $G = B_N^+$: u is bistochastic, i.e. orthogonal, with sum 1 on rows and columns.

PROOF. We refer to [33] for a full discussion of these relations. \square

Observe that we have “magic = cubic + bistochastic”, which follows from definitions, by using basic C^* -algebra tricks. This shows that we have inclusions as follows:

$$\begin{array}{ccc} H_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N^+ & \longrightarrow & B_N^+ \end{array}$$

Let us go back now to the inclusions $C_\times(G_N/G_k) \subset C(G_N/G_k)$. We first work out a few simple cases, where these inclusions are isomorphisms:

PROPOSITION 6.16. *For the basic easy quantum groups, the inclusion of algebras*

$$C_\times(G_N/G_k) \subset C(G_N/G_k)$$

is an isomorphism at $N = 1$, at $k = 0$, at $k = N$, as well as in the following special cases:

- (1) $G = B^+$: at $k = 1$.
- (2) $G = S^+$: at $k = 1$, and at $k = 2, N = 3$.

PROOF. First, the results at $N = 1$, at $k = 0$, and at $k = N$ are clear from definitions. Regarding now the special cases, the situation here is as follows:

- (1) Since the coordinates of B_N^+ sum up to 1 on each column, we have:

$$u_{1j} = 1 - \sum_{i>1} u_{ij}$$

Thus the following inclusion is an isomorphism:

$$C_{\times}(B_N^+/B_1^+) \subset C(B_N^+)$$

Thus the inclusion $C_{\times}(B_N^+/B_1^+) \subset C(B_N^+/B_1^+)$ must be as well an isomorphism.

(2) By using the same argument as above we obtain that the following inclusion is as well an isomorphism:

$$C_{\times}(S_N^+/S_1^+) \subset C(S_N^+/S_1^+)$$

In the remaining case $k = 2, N = 3$, or more generally at any $k \in \mathbb{N}$ and $N < 4$, it is known from Wang [94] that we have $S_N = S_N^+$, so the inclusion in the statement is:

$$C(S_N/S_k) \subset C(S_N/S_k)$$

Thus, in this case we are done again. \square

The axiomatization of the algebras $C_{\times}(G_N/G_k)$ is a quite tricky task, because these algebras have a rectangular matrix of generators, which is a transposed isometry, but not much is known about the remaining conditions to be satisfied by the generators.

However, we can axiomatize some bigger algebras, as follows:

DEFINITION 6.17. *Associated to $k \leq N$ is the universal C^* -algebra*

$$C_+(G_N/G_k)$$

generated by the entries of a rectangular matrix

$$p = (p_{ij})_{i>k, j>0}$$

subject to the following conditions:

- (1) $G = O_N^+$: p is a transposed “orthogonal isometry”, in the sense that its entries p_{ij} are self-adjoint, and $pp^t = 1$.
- (2) $G = S_N^+$: p is a transposed “magic isometry”, in the sense that p^t is an orthogonal isometry, and p_{ij} are projections, orthogonal on columns.
- (3) $G = H_N^+$: p is a transposed “cubic isometry”, in the sense that p^t is an orthogonal isometry, with $xy = 0$ for any $x \neq y$ on the same row of p
- (4) $G = B_N^+$: p is a transposed “stochastic isometry”, in the sense that p^t is an orthogonal isometry, with sum 1 on rows.

Observe that, since the entries p_{ij} of our various rectangular matrices are assumed to be self-adjoint, we have $p^* = p^t$. Thus the condition $pp^t = 1$ reads $(p^t)^*p^t = 1$, so the transposed matrix $q = p^t$ must indeed satisfy the isometry condition $q^*q = 1$.

Observe also that the cubic condition on transposed orthogonal isometry p is equivalent to the fact that the entries $x = p_{ij}$ satisfy the “cubic” condition $x^3 = x$.

Note also that we have by definition surjective maps, as follows:

$$C_+(G_N/G_k) \rightarrow C_\times(G_N/G_k)$$

Finally, observe that in the case $G = O^+$ and $k = N - 1$ we obtain the algebra of functions on the free sphere. This will be actually our guiding example.

We will need the following elementary result:

PROPOSITION 6.18. *For a transposed orthogonal isometry p , the following are equivalent:*

- (1) p is magic.
- (2) p is cubic and stochastic.

PROOF. At $k = N$ this result is well-known. In the general case the proof is similar, by using some basic C^* -algebra tricks:

(1) \implies (2). Assume indeed that p is magic. The transposed isometry condition $pp^t = 1$ tells us that we have:

$$\sum_j p_{ij}p_{kj} = \delta_{ik}$$

At $i = k$ we get $\sum_j p_{ij}^2 = 1$, and since the elements p_{ij} are projections, this condition becomes $\sum_j p_{ij} = 1$. Thus p is stochastic.

With this observation in hand, and since projections summing up to 1 must commute, we conclude that the elements p_{ij} mutually commute on rows, so p is cubic as well.

(2) \implies (1). Assume that p is cubic and stochastic. Since the elements p_{i1}, \dots, p_{iN} are self-adjoint, satisfy $xy = 0$, and sum up to 1, they are projections, and we are done. \square

We have the following result:

THEOREM 6.19. *The algebras $C_+(G_N/G_k)$ and $C_\times(G_N/G_k)$ have the following properties:*

- (1) *They have coactions of G_N , given by $\alpha(p_{ij}) = \sum_s p_{is} \otimes u_{sj}$.*
- (2) *They have unique G_N -invariant states, which are tracial.*
- (3) *Their reduced algebra versions are isomorphic.*
- (4) *Their abelianized versions are isomorphic.*

PROOF. We follow the proof in [28], where the above result was proved for $G = O^+$ and $k = N - 1$. The only problems, requiring some new ideas, will appear in (4) for $G = S^+, H^+$, and we will follow here the proof in [20]. In practice now:

(1) For $C_\times(G_N/G_k)$ this is clear, because this algebra is “embeddable”, and the coaction of G_N is simply the restriction of the comultiplication map.

For the algebra $C_+(G_N/G_k)$, consider the following elements:

$$P_{ij} = \sum_{s=1}^N p_{is} \otimes u_{sj}$$

We have to check that these elements satisfy the same relations as those in Definition 6.17, presenting the algebra $C_+(G_n/G_k)$, and the proof here goes as follows:

O^+ case. First, since p_{ij}, u_{ij} are self-adjoint, so is P_{ij} . Also, we have:

$$\begin{aligned} \sum_j P_{ij} P_{rj} &= \sum_{jst} p_{is} p_{rt} \otimes u_{sj} u_{tj} \\ &= \sum_{st} p_{is} p_{rt} \otimes \delta_{st} \\ &= \sum_s p_{is} p_{rs} \otimes 1 \\ &= \delta_{ir} \end{aligned}$$

H^+ case. The condition $xy = 0$ on rows is checked as follows ($j \neq r$):

$$\begin{aligned} P_{ij} P_{ir} &= \sum_{st} p_{is} p_{it} \otimes u_{sj} u_{tr} \\ &= \sum_s p_{is} \otimes u_{sj} u_{sr} \\ &= 0 \end{aligned}$$

B^+ case. The sum 1 condition on rows is checked as follows:

$$\begin{aligned} \sum_j P_{ij} &= \sum_{js} p_{is} \otimes u_{sj} \\ &= \sum_s p_{is} \otimes 1 \\ &= 1 \end{aligned}$$

S^+ case. Since P^t is cubic and stochastic, we just check the projection condition:

$$\begin{aligned} P_{ij}^2 &= \sum_{st} p_{is} p_{it} \otimes u_{sj} u_{tj} \\ &= \sum_s p_{is} \otimes u_{sj} \\ &= P_{ij} \end{aligned}$$

Summarizing, P satisfies the same conditions as p , so we can define a morphism of C^* -algebras, as follows:

$$\begin{aligned}\alpha : C_+(G_N/G_k) &\rightarrow C_+(G_N/G_k) \otimes C(G_N) \\ \alpha(p_{ij}) &= P_{ij}\end{aligned}$$

We have the following computation:

$$\begin{aligned}(\alpha \otimes id)\alpha(p_{ij}) &= \sum_s \alpha(p_{is}) \otimes u_{sj} \\ &= \sum_{st} p_{it} \otimes u_{ts} \otimes u_{sj}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}(id \otimes \Delta)\alpha(p_{ij}) &= \sum_t p_{it} \otimes \Delta(u_{ij}) \\ &= \sum_{st} p_{it} \otimes u_{ts} \otimes u_{sj}\end{aligned}$$

Thus our map α is coassociative. The density conditions can be checked by using dense subalgebras generated by p_{ij} and u_{st} , and we are done.

(2) For the existence part we can use the following composition, where the first two maps are the canonical ones, and the map on the right is the integration over G_N :

$$C_+(G_N/G_k) \rightarrow C_\times(G_N/G_k) \subset C(G_N) \rightarrow \mathbb{C}$$

Also, the uniqueness part is clear for the algebra $C_\times(G_N/G_k)$, as a particular case of the general properties of “embeddable” coactions, i.e. those coactions that can be realized as coactions on subalgebras of $C(G)$, via the restriction of the comultiplication.

Regarding now the uniqueness for $C_+(G_N/G_k)$, let \int be the Haar state on G_N , and φ be the G_N -invariant state constructed above. We claim that α is ergodic:

$$\left(id \otimes \int\right) \alpha = \varphi(\cdot)1$$

Indeed, let us recall that the Haar state is given by the following Weingarten formula, where $W_{sN} = G_{sN}^{-1}$, with $G_{sN}(\pi, \sigma) = N^{|\pi \vee \sigma|}$:

$$\int u_{i_1 j_1} \cdots u_{i_s j_s} = \sum_{\pi, \sigma \in D(s)} \delta_\pi(i) \delta_\sigma(j) W_{sN}(\pi, \sigma)$$

Now, let us go back now to our claim. By linearity it is enough to check the above equality on a product of basic generators $p_{i_1 j_1} \dots p_{i_s j_s}$. The left term is as follows:

$$\begin{aligned}
& \left(id \otimes \int \right) \alpha(p_{i_1 j_1} \dots p_{i_s j_s}) \\
&= \sum_{l_1 \dots l_s} p_{i_1 l_1} \dots p_{i_s l_s} \int u_{l_1 j_1} \dots u_{l_s j_s} \\
&= \sum_{l_1 \dots l_s} p_{i_1 l_1} \dots p_{i_s l_s} \sum_{\pi, \sigma \in D(s)} \delta_\pi(l) \delta_\sigma(j) W_{sN}(\pi, \sigma) \\
&= \sum_{\pi, \sigma \in D(s)} \delta_\sigma(j) W_{sN}(\pi, \sigma) \sum_{l_1 \dots l_s} \delta_\pi(l) p_{i_1 l_1} \dots p_{i_s l_s}
\end{aligned}$$

Let us look now at the sum on the right. We have to sum the elements of type $p_{i_1 l_1} \dots p_{i_s l_s}$, over all multi-indices $l = (l_1, \dots, l_s)$ which fit into our partition $\pi \in D(s)$. In the case of a one-block partition this sum is simply $\sum_l p_{i_1 l} \dots p_{i_s l}$, and we claim that:

$$\sum_l p_{i_1 l} \dots p_{i_s l} = \delta_\pi(i)$$

Indeed, by using the explicit description of the sets of diagrams $D(s)$ given above, the proof of this formula goes as follows:

O^+ case. Here our one-block partition must be a semicircle, $\pi = \cap$, and the formula to be proved, namely $\sum_l p_{il} p_{jl} = \delta_{ij}$, follows from $pp^t = 1$.

S^+ case. Here our one-block partition can be any s -block, $1_s \in P(s)$, and the formula to be proved, namely $\sum_l p_{i_1 l} \dots p_{i_s l} = \delta_{i_1, \dots, i_s}$, follows from orthogonality on columns, and from the fact that the sum is 1 on rows.

B^+ case. Here our one-block partition must be a semicircle or a singleton. We are already done with the semicircle, and for the singleton the formula to be proved, namely $\sum_l p_{il} = 1$, follows from the fact that the sum is 1 on rows.

H^+ case. Here our one-block partition must have an even number of legs, $s = 2r$, and due to the cubic condition the formula to be proved reduces to $\sum_l p_{il}^{2r} = 1$. But since $p_{il}^{2r} = p_{il}^2$, independently on r , the result follows from the orthogonality on rows.

In the general case now, since π noncrossing, the computations over the blocks will not interfere, and we will obtain the same result, namely:

$$\sum_l p_{i_1 l} \dots p_{i_s l} = \delta_\pi(i)$$

Now by plugging this formula into the computation that we have started, we get:

$$\begin{aligned}
& \left(id \otimes \int \right) \alpha(p_{i_1 j_1} \cdots p_{i_s j_s}) \\
&= \sum_{\pi, \sigma \in D(s)} \delta_\pi(i) \delta_\sigma(j) W_{sN}(\pi, \sigma) \\
&= \int u_{i_1 j_1} \cdots u_{i_s j_s} \\
&= \varphi(p_{i_1 j_1} \cdots p_{i_s j_s})
\end{aligned}$$

This finishes the proof of our claim. So, let us get back now to the original question. Let $\tau : C_+(G_N/G_k) \rightarrow \mathbb{C}$ be a linear form as in the statement. We have:

$$\begin{aligned}
\tau \left(id \otimes \int \right) \alpha(x) &= \left(\tau \otimes \int \right) \alpha(x) \\
&= \int (\tau \otimes id) \alpha(x) \\
&= \int (\tau(x) 1) \\
&= \tau(x)
\end{aligned}$$

On the other hand, according to our above claim, we have as well:

$$\begin{aligned}
\tau \left(id \otimes \int \right) \alpha(x) &= \tau(\varphi(x) 1) \\
&= \varphi(x)
\end{aligned}$$

Thus we get $\tau = \varphi$, which finishes the proof of the uniqueness assertion.

(3) This follows from the uniqueness assertions in (2), and from some standard facts regarding the reduced versions with respect to Haar states, from [99].

(4) We denote by G^- the classical version of G , given by $G^- = O, S, H, B$ in the cases $G = O^+, S^+, H^+, B^+$. We have surjective morphisms of algebras, as follows:

$$\begin{aligned}
C_+(G_N/G_k) &\rightarrow C_\times(G_k/G_k) \\
&\rightarrow C_\times(G_N^-/G_k^-) \\
&= C(G_N^-/G_k^-)
\end{aligned}$$

Thus at the level of abelianized versions, we have surjective morphisms as follows:

$$\begin{aligned}
C_+(G_N/G_k)_{comm} &\rightarrow C_\times(G_N/G_k)_{comm} \\
&\rightarrow C(G_N^-/G_k^-)
\end{aligned}$$

In order to prove our claim, namely that the first surjective morphism is an isomorphism, it is enough to prove that the above composition is an isomorphism.

Let $r = N - k$, and denote by $A_{N,r}$ the algebra on the left. This is by definition the algebra generated by the entries of a transposed $N \times r$ isometry, whose entries commute, and which is respectively orthogonal, magic, cubic, bistochastic.

We have a surjective morphism $A_{N,r} \rightarrow C(G_N^-/G_k^-)$, and we must prove that this is an isomorphism.

S^+ case. Since $\#(S_N/S_k) = N!/k!$, it is enough to prove that we have:

$$\dim(A_{N,r}) = \frac{N!}{k!}$$

Let p_{ij} be the standard generators of $A_{N,r}$. By using the Gelfand theorem, we can write $p_{ij} = \chi(X_{ij})$, where $X_{ij} \subset X$ are certain subsets of a given set X . Now at the level of sets the magic isometry condition on (p_{ij}) tells us that the matrix of sets (X_{ij}) has the property that its entries are disjoint on columns, and form partitions of X on rows.

So, let us try to understand this property for N fixed, and $r = 1, 2, 3, \dots$

– At $r = 1$ we simply have a partition $X = X_1 \sqcup \dots \sqcup X_N$. So, the universal model can be any such partition, with $X_i \neq \emptyset$ for any i .

– At $r = 2$ the universal model is best described as follows: X is the $N \times N$ square in \mathbb{R}^2 , regarded as a union of N^2 unit tiles, minus the diagonal, the sets X_{1i} are the disjoint unions on rows, and the sets X_{2i} are the disjoint unions on columns.

– At $r \geq 3$, the universal solution is similar: we can take X to be the N^r cube in \mathbb{R}^r , with all tiles having pairs of equal coordinates removed, and say that the sets X_{si} for s fixed are the various “slices” of X in the direction of the s -th coordinate of \mathbb{R}^r .

Summarizing, the above discussion tells us that $\dim(A_{N,r})$ equals the number of tiles in the above set $X \subset \mathbb{R}^r$. But these tiles correspond by definition to the various r -tuples $(i_1, \dots, i_r) \in \{1, \dots, N\}^r$ with all i_k different, and since there are exactly $N!/k!$ such r -tuples, we obtain, as desired:

$$\dim(A_{N,r}) = \frac{N!}{k!}$$

H^+ case. We can use here the same method as for S_N^+ . This time the functions p_{ij} take values in $\{-1, 0, 1\}$, and the algebra generated by their squares p_{ij}^2 coincides with the one computed above for S_N^+ , having dimension $N!/k!$. Now by taking into account the $N - k$ possible signs we obtain the following estimate, which gives the result:

$$\dim(A_{N,r}) \leq \frac{2^{N-k} N!}{k!} = \#(H_N/H_k)$$

O^+ case. We can use the same method, namely a straightforward application of the Gelfand theorem. However, instead of performing a dimension count, which is no longer

possible, we have to complete here any transposed $N \times r$ isometry whose entries commute to a $N \times N$ orthogonal matrix. But this is the same as completing a system of r orthogonal norm 1 vectors in \mathbb{R}^N into an orthonormal basis of \mathbb{R}^N , which is of course possible.

B^+ case. Since we have a surjective map $C(O_N^+) \rightarrow C(B_N^+)$, we obtain a surjective map $C_+(O_N^+/O_k^+) \rightarrow A_{N,r}$, and hence surjective maps as follows:

$$C(O_N/O_k) \rightarrow A_{N,r} \rightarrow C(B_N/B_k)$$

The point now is that this composition is the following canonical map:

$$C(O_N/O_k) \rightarrow C(B_N/B_k)$$

Now by looking at the column vector $\xi = (1, \dots, 1)^t$, which is fixed by the stochastic matrices, we conclude that the map on the right is an isomorphism, and we are done. \square

6e. Exercises

The material in this chapter has been quite abstract and technical, and so will be our exercises here. To start with, in relation with the axioms, we have:

EXERCISE 6.20. *Analyse, with some general theory, examples and counterexamples, the validity of the well-known statement “the quotient by a normal subgroup is a group” from group theory, in the compact quantum group setting.*

This is a very good exercise, the answer to it being quite folklore. Enjoy.

Along the same lines, but dealing this time with analytic questions, we have:

EXERCISE 6.21. *Verify, with a full argumentation, that the homogenous space formalism developed in the beginning of this chapter is not the correct one, in order to deal for instance with the canonical quantum isometry actions on the free spheres.*

As before with the previous exercise, the answer here is quite folklore.

In relation now with the row space construction discussed in this chapter, things here are quite technical, and as an exercise here, we have:

EXERCISE 6.22. *Extend the theory developed in the above, for the row spaces over the free quantum groups, to all the basic unitary and reflection quantum groups.*

As before, things here are quite folklore, and so this is a must-do exercise.

CHAPTER 7

Partial isometries

7a. Partial isometries

In what follows we discuss, as a continuation of the study from the previous chapter, the formalism in [9], which is quite broad, while remaining not very abstract. To be more precise, we will study the spaces of the following type:

$$X = (G_M \times G_N) / (G_L \times G_{M-L} \times G_{N-L})$$

These spaces cover indeed the quantum groups and the spheres. And also, they are quite concrete and useful objects, consisting of certain classes of “partial isometries”.

We begin with a study in the classical case. Our starting point will be:

DEFINITION 7.1. *Associated to any integers $L \leq M, N$ are the spaces*

$$O_{MN}^L = \left\{ T : E \rightarrow F \text{ isometry} \mid E \subset \mathbb{R}^N, F \subset \mathbb{R}^M, \dim_{\mathbb{R}} E = L \right\}$$

$$U_{MN}^L = \left\{ T : E \rightarrow F \text{ isometry} \mid E \subset \mathbb{C}^N, F \subset \mathbb{C}^M, \dim_{\mathbb{C}} E = L \right\}$$

where the notion of isometry is with respect to the usual real/complex scalar products.

As a first observation, it follows from definitions that at $L = M = N$ we obtain the orthogonal and unitary groups O_N, U_N :

$$O_{NN}^N = O_N$$

$$U_{NN}^N = U_N$$

Another interesting specialization is $L = M = 1$. Here the elements of O_{1N}^1 are the isometries $T : E \rightarrow \mathbb{R}$, with $E \subset \mathbb{R}^N$ one-dimensional. But such an isometry is uniquely determined by $T^{-1}(1) \in \mathbb{R}^N$, which must belong to $S_{\mathbb{R}}^{N-1}$. Thus, we have $O_{1N}^1 = S_{\mathbb{R}}^{N-1}$. Similarly, in the complex case we have $U_{1N}^1 = S_{\mathbb{C}}^{N-1}$, and so our results here are:

$$O_{1N}^1 = S_{\mathbb{R}}^{N-1}$$

$$U_{1N}^1 = S_{\mathbb{C}}^{N-1}$$

Yet another interesting specialization is $L = N = 1$. Here the elements of O_{1N}^1 are the isometries $T : \mathbb{R} \rightarrow F$, with $F \subset \mathbb{R}^M$ one-dimensional. But such an isometry is uniquely

determined by $T(1) \in \mathbb{R}^M$, which must belong to $S_{\mathbb{R}}^{M-1}$. Thus, we have $O_{M1}^1 = S_{\mathbb{R}}^{M-1}$. Similarly, in the complex case we have $U_{M1}^1 = S_{\mathbb{C}}^{M-1}$, and so our results here are:

$$\begin{aligned} O_{M1}^1 &= S_{\mathbb{R}}^{M-1} \\ U_{M1}^1 &= S_{\mathbb{C}}^{M-1} \end{aligned}$$

In general, the most convenient is to view the elements of O_{MN}^L, U_{MN}^L as rectangular matrices, and to use matrix calculus for their study. We have indeed:

PROPOSITION 7.2. *We have identifications of compact spaces*

$$\begin{aligned} O_{MN}^L &\simeq \left\{ U \in M_{M \times N}(\mathbb{R}) \mid UU^t = \text{projection of trace } L \right\} \\ U_{MN}^L &\simeq \left\{ U \in M_{M \times N}(\mathbb{C}) \mid UU^* = \text{projection of trace } L \right\} \end{aligned}$$

with each partial isometry being identified with the corresponding rectangular matrix.

PROOF. We can indeed identify the partial isometries $T : E \rightarrow F$ with their corresponding extensions $U : \mathbb{R}^N \rightarrow \mathbb{R}^M, U : \mathbb{C}^N \rightarrow \mathbb{C}^M$, obtained by setting:

$$U_{E^\perp} = 0$$

Then, we can identify these latter linear maps U with the corresponding rectangular matrices, and we are led to the conclusion in the statement. \square

As an illustration, at $L = M = N$ we recover in this way the usual matrix description of O_N, U_N . Also, at $L = M = 1$ we obtain the usual description of $S_{\mathbb{R}}^{N-1}, S_{\mathbb{C}}^{N-1}$, as row spaces over the corresponding groups O_N, U_N .

Finally, at $L = N = 1$ we obtain the usual description of $S_{\mathbb{R}}^{N-1}, S_{\mathbb{C}}^{N-1}$, as column spaces over the corresponding groups O_N, U_N .

Now back to the general case, observe that the isometries $T : E \rightarrow F$, or rather their extensions $U : \mathbb{K}^N \rightarrow \mathbb{K}^M$, with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, obtained by setting $U_{E^\perp} = 0$, can be composed with the isometries of $\mathbb{K}^M, \mathbb{K}^N$, according to the following scheme:

$$\begin{array}{ccccccc} \mathbb{K}^N & \xrightarrow{B^*} & \mathbb{K}^N & \xrightarrow{\dots U \dots} & \mathbb{K}^M & \xrightarrow{A} & \mathbb{K}^M \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ B(E) & \xrightarrow{\dots} & E & \xrightarrow{T} & F & \xrightarrow{\dots} & A(F) \end{array}$$

With the identifications in Proposition 7.2 made, the precise statement here is:

PROPOSITION 7.3. *We have an action map as follows, which is transitive,*

$$\begin{aligned} O_M \times O_N &\curvearrowright O_{MN}^L \\ (A, B)U &= AUB^t \end{aligned}$$

as well as an action map as follows, transitive as well,

$$\begin{aligned} U_M \times U_N &\curvearrowright U_{MN}^L \\ (A, B)U &= AUB^* \end{aligned}$$

whose stabilizers are respectively:

$$\begin{aligned} O_L \times O_{M-L} \times O_{N-L} \\ U_L \times U_{M-L} \times U_{N-L} \end{aligned}$$

PROOF. We have indeed action maps as in the statement, which are transitive. Let us compute now the stabilizer G of the following point:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Since $(A, B) \in G$ satisfy $AU = UB$, their components must be of the following form:

$$A = \begin{pmatrix} x & * \\ 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} x & 0 \\ * & b \end{pmatrix}$$

Now since A, B are both unitaries, these matrices follow to be block-diagonal, and so:

$$G = \left\{ (A, B) \mid A = \begin{pmatrix} x & 0 \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} x & 0 \\ 0 & b \end{pmatrix} \right\}$$

The stabilizer of U is then parametrized by triples (x, a, b) belonging respectively to:

$$\begin{aligned} O_L \times O_{M-L} \times O_{N-L} \\ U_L \times U_{M-L} \times U_{N-L} \end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

Finally, let us work out the quotient space description of O_{MN}^L, U_{MN}^L . We have here:

THEOREM 7.4. *We have isomorphisms of homogeneous spaces as follows,*

$$\begin{aligned} O_{MN}^L &= (O_M \times O_N) / (O_L \times O_{M-L} \times O_{N-L}) \\ U_{MN}^L &= (U_M \times U_N) / (U_L \times U_{M-L} \times U_{N-L}) \end{aligned}$$

with the quotient maps being given by $(A, B) \rightarrow AUB^$, where:*

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

PROOF. This is just a reformulation of Proposition 7.3 above, by taking into account the fact that the fixed point used in the proof there was $U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. \square

Once again, the basic examples here come from the cases $L = M = N$ and $L = M = 1$. At $L = M = N$ the quotient spaces at right are respectively:

$$O_N, U_N$$

At $L = M = 1$ the quotient spaces at right are respectively:

$$O_N/O_{N-1} \quad , \quad U_N/U_{N-1}$$

In fact, in the general orthogonal $L = M$ case we obtain the following spaces:

$$\begin{aligned} O_{MN}^M &= (O_M \times O_N)/(O_M \times O_{N-M}) \\ &= O_N/O_{N-M} \end{aligned}$$

Also, in the general unitary $L = M$ case we obtain the following spaces:

$$\begin{aligned} U_{MN}^M &= (U_M \times U_N)/(U_M \times U_{N-M}) \\ &= U_N/U_{N-M} \end{aligned}$$

Similarly, the examples coming from the cases $L = M = N$ and $L = N = 1$ are particular cases of the general $L = N$ case, where we obtain the following spaces:

$$\begin{aligned} O_{MN}^N &= (O_M \times O_N)/(O_M \times O_{M-N}) \\ &= O_N/O_{M-N} \end{aligned}$$

In the unitary case, we obtain the following spaces:

$$\begin{aligned} U_{MN}^N &= (U_M \times U_N)/(U_M \times U_{M-N}) \\ &= U_N/U_{M-N} \end{aligned}$$

Summarizing, we have here homogeneous spaces which unify the spheres with the unitary quantum groups.

7b. Free isometries

We can liberate the spaces O_{MN}^L, U_{MN}^L , as follows:

DEFINITION 7.5. *Associated to any integers $L \leq M, N$ are the algebras*

$$\begin{aligned} C(O_{MN}^{L+}) &= C^* \left((u_{ij})_{i=1, \dots, M, j=1, \dots, N} \mid u = \bar{u}, uu^t = \text{projection of trace } L \right) \\ C(U_{MN}^{L+}) &= C^* \left((u_{ij})_{i=1, \dots, M, j=1, \dots, N} \mid uu^*, \bar{u}u^t = \text{projections of trace } L \right) \end{aligned}$$

with the trace being by definition the sum of the diagonal entries.

Observe that the above universal algebras are indeed well-defined, as it was previously the case for the free spheres, and this due to the trace conditions, which read:

$$\sum_{ij} u_{ij} u_{ij}^* = \sum_{ij} u_{ij}^* u_{ij} = L$$

We have inclusions between the various spaces constructed so far, as follows:

$$\begin{array}{ccc} O_{MN}^{L+} & \longrightarrow & U_{MN}^{L+} \\ \uparrow & & \uparrow \\ O_{MN}^L & \longrightarrow & U_{MN}^L \end{array}$$

At the level of basic examples now, we first have the following result:

PROPOSITION 7.6. *At $L = M = 1$ we obtain the diagram*

$$\begin{array}{ccc} S_{\mathbb{R},+}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\ \uparrow & & \uparrow \\ S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{C}}^{N-1} \end{array}$$

and at $L = N = 1$ we obtain the diagram:

$$\begin{array}{ccc} S_{\mathbb{R},+}^{M-1} & \longrightarrow & S_{\mathbb{C},+}^{M-1} \\ \uparrow & & \uparrow \\ S_{\mathbb{R}}^{M-1} & \longrightarrow & S_{\mathbb{C}}^{M-1} \end{array}$$

PROOF. We recall that the various spheres involved are constructed as follows, with the symbol \times standing for “commutative” and “free”, respectively:

$$\begin{aligned} C(S_{\mathbb{R},\times}^{N-1}) &= C_{\times}^* \left(z_1, \dots, z_N \mid z_i = z_i^*, \sum_i z_i^2 = 1 \right) \\ C(S_{\mathbb{C},\times}^{N-1}) &= C_{\times}^* \left(z_1, \dots, z_N \mid \sum_i z_i z_i^* = \sum_i z_i^* z_i = 1 \right) \end{aligned}$$

Now by comparing with the definition of $O_{1N}^{1\times}, U_{1N}^{1\times}$, this proves our first claim. As for the proof of the second claim, this is similar, via standard identifications. \square

We have as well the following result:

PROPOSITION 7.7. *At $L = M = N$ we obtain the diagram*

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array}$$

consisting of the groups O_N, U_N , and their liberations.

PROOF. We recall that the various quantum groups in the statement are constructed as follows, with the symbol \times standing once again for “commutative” and “free”:

$$\begin{aligned} C(O_N^\times) &= C_\times^* \left((u_{ij})_{i,j=1,\dots,N} \mid u = \bar{u}, uu^t = u^t u = 1 \right) \\ C(U_N^\times) &= C_\times^* \left((u_{ij})_{i,j=1,\dots,N} \mid uu^* = u^* u = 1, \bar{u}u^t = u^t \bar{u} = 1 \right) \end{aligned}$$

On the other hand, according to Proposition 7.2 and to Definition 7.5 above, we have the following presentation results:

$$\begin{aligned} C(O_{NN}^{N\times}) &= C_\times^* \left((u_{ij})_{i,j=1,\dots,N} \mid u = \bar{u}, uu^t = \text{projection of trace } N \right) \\ C(U_{NN}^{N\times}) &= C_\times^* \left((u_{ij})_{i,j=1,\dots,N} \mid uu^*, \bar{u}u^t = \text{projections of trace } N \right) \end{aligned}$$

We use now the standard fact that if $p = aa^*$ is a projection then $q = a^*a$ is a projection too. We use as well the following formulae:

$$\begin{aligned} \text{Tr}(uu^*) &= \text{Tr}(u^t \bar{u}) \\ \text{Tr}(\bar{u}u^t) &= \text{Tr}(u^* u) \end{aligned}$$

We therefore obtain the following formulae:

$$\begin{aligned} C(O_{NN}^{N\times}) &= C_\times^* \left((u_{ij})_{i,j=1,\dots,N} \mid u = \bar{u}, uu^t, u^t u = \text{projections of trace } N \right) \\ C(U_{NN}^{N\times}) &= C_\times^* \left((u_{ij})_{i,j=1,\dots,N} \mid uu^*, u^* u, \bar{u}u^t, u^t \bar{u} = \text{projections of trace } N \right) \end{aligned}$$

Now observe that, in tensor product notation, and by using the normalized trace, the conditions at right are all of the form:

$$(tr \otimes id)p = 1$$

To be more precise, p is a follows, for the above conditions:

$$p = uu^*, u^* u, \bar{u}u^t, u^t \bar{u}$$

We therefore obtain, for any faithful state φ :

$$(tr \otimes \varphi)(1 - p) = 0$$

It follows from this that the following projections must be all equal to the identity:

$$p = uu^*, u^*u, \bar{u}u^t, u^t\bar{u}$$

But this leads to the conclusion in the statement. \square

Regarding now the homogeneous space structure of $O_{MN}^{L\times}, U_{MN}^{L\times}$, the situation here is more complicated in the free case than in the classical case, due to a number of reasons, of both algebraic and analytic nature.

We have the following result:

PROPOSITION 7.8. *The spaces $U_{MN}^{L\times}$ have the following properties:*

(1) *We have an action $U_M^\times \times U_N^\times \curvearrowright U_{MN}^{L\times}$, given by:*

$$u_{ij} \rightarrow \sum_{kl} u_{kl} \otimes a_{ki} \otimes b_{lj}^*$$

(2) *We have a map $U_M^\times \times U_N^\times \rightarrow U_{MN}^{L\times}$, given by:*

$$u_{ij} \rightarrow \sum_{r \leq L} a_{ri} \otimes b_{rj}^*$$

Similar results hold for the spaces $O_{MN}^{L\times}$, with all the $$ exponents removed.*

PROOF. In the classical case, consider the action and quotient maps:

$$U_M \times U_N \curvearrowright U_{MN}^L$$

$$U_M \times U_N \rightarrow U_{MN}^L$$

The transposes of these two maps are as follows, where $J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$:

$$\begin{aligned} \varphi &\rightarrow ((U, A, B) \rightarrow \varphi(AUB^*)) \\ \varphi &\rightarrow ((A, B) \rightarrow \varphi(AJB^*)) \end{aligned}$$

But with $\varphi = u_{ij}$ we obtain precisely the formulae in the statement. The proof in the orthogonal case is similar. Regarding now the free case, the proof goes as follows:

(1) Assuming $uu^*u = u$, let us set:

$$U_{ij} = \sum_{kl} u_{kl} \otimes a_{ki} \otimes b_{lj}^*$$

We have then:

$$\begin{aligned}
(UU^*U)_{ij} &= \sum_{pq} \sum_{klmnst} u_{kl}u_{mn}^*u_{st} \otimes a_{ki}a_{mq}^*a_{sq} \otimes b_{lp}^*b_{np}b_{tj}^* \\
&= \sum_{klmt} u_{kl}u_{ml}^*u_{mt} \otimes a_{ki} \otimes b_{tj}^* \\
&= \sum_{kt} u_{kt} \otimes a_{ki} \otimes b_{tj}^* \\
&= U_{ij}
\end{aligned}$$

Also, assuming that we have $\sum_{ij} u_{ij}u_{ij}^* = L$, we obtain:

$$\begin{aligned}
\sum_{ij} U_{ij}U_{ij}^* &= \sum_{ij} \sum_{klst} u_{kl}u_{st}^* \otimes a_{ki}a_{si}^* \otimes b_{lj}^*b_{tj} \\
&= \sum_{kl} u_{kl}u_{kl}^* \otimes 1 \otimes 1 \\
&= L
\end{aligned}$$

(2) Assuming $uu^*u = u$, let us set:

$$V_{ij} = \sum_{r \leq L} a_{ri} \otimes b_{rj}^*$$

We have then:

$$\begin{aligned}
(VV^*V)_{ij} &= \sum_{pq} \sum_{x,y,z \leq L} a_{xi}a_{yq}^*a_{zq} \otimes b_{xp}^*b_{yp}b_{zj}^* \\
&= \sum_{x \leq L} a_{xi} \otimes b_{xj}^* \\
&= V_{ij}
\end{aligned}$$

Also, assuming that we have $\sum_{ij} u_{ij}u_{ij}^* = L$, we obtain:

$$\begin{aligned}
\sum_{ij} V_{ij}V_{ij}^* &= \sum_{ij} \sum_{r,s \leq L} a_{ri}a_{si}^* \otimes b_{rj}^*b_{sj} \\
&= \sum_{l \leq L} 1 \\
&= L
\end{aligned}$$

By removing all the * exponents, we obtain as well the orthogonal results. \square

Let us examine now the relation between the above maps. In the classical case, given a quotient space $X = G/H$, the associated action and quotient maps are given by:

$$\begin{cases} a : X \times G \rightarrow X & : (Hg, h) \rightarrow Hgh \\ p : G \rightarrow X & : g \rightarrow Hg \end{cases}$$

Thus we have $a(p(g), h) = p(gh)$. In our context, a similar result holds:

THEOREM 7.9. *With $G = G_M \times G_N$ and $X = G_{MN}^L$, where $G_N = O_N^\times, U_N^\times$, we have*

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ p \times id \downarrow & & \downarrow p \\ X \times G & \xrightarrow{a} & X \end{array}$$

where a, p are the action map and the map constructed in Proposition 7.8.

PROOF. At the level of the associated algebras of functions, we must prove that the following diagram commutes, where Φ, α are morphisms of algebras induced by a, p :

$$\begin{array}{ccc} C(X) & \xrightarrow{\Phi} & C(X \times G) \\ \alpha \downarrow & & \downarrow \alpha \otimes id \\ C(G) & \xrightarrow{\Delta} & C(G \times G) \end{array}$$

When going right, and then down, the composition is as follows:

$$\begin{aligned} (\alpha \otimes id)\Phi(u_{ij}) &= (\alpha \otimes id) \sum_{kl} u_{kl} \otimes a_{ki} \otimes b_{lj}^* \\ &= \sum_{kl} \sum_{r \leq L} a_{rk} \otimes b_{rl}^* \otimes a_{ki} \otimes b_{lj}^* \end{aligned}$$

On the other hand, when going down, and then right, the composition is as follows, where F_{23} is the flip between the second and the third components:

$$\begin{aligned} \Delta\pi(u_{ij}) &= F_{23}(\Delta \otimes \Delta) \sum_{r \leq L} a_{ri} \otimes b_{rj}^* \\ &= F_{23} \left(\sum_{r \leq L} \sum_{kl} a_{rk} \otimes a_{ki} \otimes b_{rl}^* \otimes b_{lj}^* \right) \end{aligned}$$

Thus the above diagram commutes indeed, and this gives the result. \square

7c. Discrete extensions

Let us discuss now some extensions of the above constructions. We will be mostly interested in the quantum reflection groups, but let us first discuss, with full details, the case of the quantum groups S_N, S_N^+ . The starting point is the semigroup \tilde{S}_N of partial permutations. This is a quite familiar object in combinatorics, defined as follows:

DEFINITION 7.10. \tilde{S}_N is the semigroup of partial permutations of $\{1 \dots, N\}$,

$$\tilde{S}_N = \left\{ \sigma : X \simeq Y \mid X, Y \subset \{1, \dots, N\} \right\}$$

with the usual composition operation, $\sigma' \sigma : \sigma^{-1}(X' \cap Y) \rightarrow \sigma'(X' \cap Y)$.

Observe that \tilde{S}_N is not simplifiable, because the null permutation $\emptyset \in \tilde{S}_N$, having the empty set as domain/range, satisfies $\emptyset \sigma = \sigma \emptyset = \emptyset$, for any $\sigma \in \tilde{S}_N$. Observe also that \tilde{S}_N has a “subinverse” map, sending $\sigma : X \rightarrow Y$ to its usual inverse $\sigma^{-1} : Y \simeq X$.

A first interesting result about this semigroup \tilde{S}_N , which shows that we are dealing here with some non-trivial combinatorics, is as follows:

PROPOSITION 7.11. The number of partial permutations is given by

$$|\tilde{S}_N| = \sum_{k=0}^N k! \binom{N}{k}^2$$

that is, 1, 2, 7, 34, 209, ..., and with $N \rightarrow \infty$ we have:

$$|\tilde{S}_N| \simeq N! \sqrt{\frac{\exp(4\sqrt{N} - 1)}{4\pi\sqrt{N}}}$$

PROOF. The first assertion is clear, because in order to construct a partial permutation $\sigma : X \rightarrow Y$ we must choose an integer $k = |X| = |Y|$, then we must pick two subsets $X, Y \subset \{1, \dots, N\}$ having cardinality k , and there are $\binom{N}{k}$ choices for each, and finally we must construct a bijection $\sigma : X \rightarrow Y$, and there are $k!$ choices here. As for the estimate, which is non-trivial, this is however something standard, and well-known. \square

Another result, which is trivial, but quite fundamental, is as follows:

PROPOSITION 7.12. We have a semigroup embedding $u : \tilde{S}_N \subset M_N(0, 1)$, defined by

$$u_{ij}(\sigma) = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

whose image are the matrices having at most one nonzero entry, on each row and column.

PROOF. This is trivial from definitions, with $u : \tilde{S}_N \subset M_N(0, 1)$ extending the standard embedding $u : S_N \subset M_N(0, 1)$, that we have been heavily using, so far. \square

Let us discuss now the construction and main properties of the quantum semigroup of quantum partial permutations \widetilde{S}_N^+ , in analogy with the above. For this purpose, let us go back to the embedding $u : \widetilde{S}_N \subset M_N(0, 1)$ in Proposition 7.12.

Due to the formula $u_{ij}(\sigma) = \delta_{i\sigma(j)}$, the matrix $u = (u_{ij})$ is “submagic”, in the sense that its entries are projections, which are pairwise orthogonal on each row and column. This suggests the following definition, given in [31]:

DEFINITION 7.13. $C(\widetilde{S}_N^+)$ is the universal C^* -algebra generated by the entries of a $N \times N$ submagic matrix u , with comultiplication and counit maps given by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$$

$$\varepsilon(u_{ij}) = \delta_{ij}$$

where submagic means formed of projections, which are pairwise orthogonal on rows and columns. We call \widetilde{S}_N^+ semigroup of quantum partial permutations of $\{1, \dots, N\}$.

Here the fact that the morphisms of algebras Δ, ε as above exist indeed follows from the universality property of $C(\widetilde{S}_N^+)$, with the needed submagic checks being nearly identical to the magic checks for $C(S_N^+)$, from chapter 2 above.

Observe that the morphisms Δ, ε satisfy the usual axioms for a comultiplication and antipode, namely:

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$$

Thus, we have a bialgebra structure of $C(\widetilde{S}_N^+)$, which tells us that the underlying noncommutative space \widetilde{S}_N^+ is a compact quantum semigroup. This semigroup is of quite special type, because $C(\widetilde{S}_N^+)$ has as well a subantipode map, defined by:

$$S(u_{ij}) = u_{ji}$$

To be more precise here, this map exists because the transpose of a submagic matrix is submagic too. As for the subantipode axiom satisfied by it, this is as follows, where $m^{(3)}$ is the triple multiplication, and $\Delta^{(2)}$ is the double comultiplication:

$$m^{(3)}(S \otimes id \otimes S)\Delta^{(2)} = S$$

Observe also that Δ, ε, S restrict to $C(\widetilde{S}_N)$, and correspond there, via Gelfand duality, to the usual multiplication, unit element, and subinversion map of \widetilde{S}_N .

As a conclusion to this discussion, the basic properties of the quantum semigroup \widetilde{S}_N^+ that we constructed in Definition 7.13 can be summarized as follows:

PROPOSITION 7.14. *We have maps as follows*

$$\begin{array}{ccc} C(\tilde{S}_N^+) & \rightarrow & C(S_N^+) & & \tilde{S}_N^+ & \supset & S_N^+ \\ & & \downarrow & & & & \downarrow \\ & & C(\tilde{S}_N) & \rightarrow & C(S_N) & & \tilde{S}_N \supset S_N \end{array} \quad \begin{array}{c} : \\ \cup \\ \cup \end{array}$$

with the bialgebras at left corresponding to the quantum semigroups at right.

PROOF. This is clear from the above discussion, and from the well-known fact that projections which sum up to 1 are pairwise orthogonal. See [31]. \square

As a first example, we have $\tilde{S}_1^+ = \tilde{S}_1$. At $N = 2$ now, recall that the algebra generated by two free projections p, q is isomorphic to the group algebra of $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$. We denote by $\varepsilon : C^*(D_\infty) \rightarrow \mathbb{C}1$ the counit map, given by the following formulae:

$$\varepsilon(1) = 1$$

$$\varepsilon(\dots pqpq \dots) = 0$$

With these conventions, we have the following result, from [31]:

PROPOSITION 7.15. *We have an isomorphism*

$$C(\tilde{S}_2^+) \simeq \left\{ (x, y) \in C^*(D_\infty) \oplus C^*(D_\infty) \mid \varepsilon(x) = \varepsilon(y) \right\}$$

which is given by the formula

$$u = \begin{pmatrix} p \oplus 0 & 0 \oplus r \\ 0 \oplus s & q \oplus 0 \end{pmatrix}$$

where p, q and r, s are the standard generators of the two copies of $C^*(D_\infty)$.

PROOF. Consider an arbitrary 2×2 matrix formed by projections:

$$u = \begin{pmatrix} P & R \\ S & Q \end{pmatrix}$$

This matrix is submagic when the following conditions are satisfied:

$$PR = PS = QR = QS = 0$$

But these conditions mean that the non-unital algebras $X = \langle P, Q \rangle$ and $Y = \langle R, S \rangle$ must commute, and must satisfy $xy = 0$, for any $x \in X, y \in Y$.

Thus, if we denote by Z the universal non-unital algebra generated by two projections, we have an isomorphism as follows:

$$C(\tilde{S}_2^+) \simeq \mathbb{C}1 \oplus Z \oplus Z$$

Now since $C^*(D_\infty) = \mathbb{C}1 \oplus Z$, we obtain an isomorphism as follows:

$$C(\tilde{S}_2^+) \simeq \left\{ (\lambda + a, \lambda + b) \mid \lambda \in \mathbb{C}, a, b \in Z \right\}$$

Thus, we are led to the conclusion in the statement. See [31]. \square

Summarizing, the semigroups of partial permutations \tilde{S}_N have non-trivial liberations, a bit like the permutation groups S_N used to have non-trivial liberations, and this starting from $N = 2$ already, in the semigroup case.

In order to reach now to homogeneous spaces, in the spirit of the partial isometry spaces discussed before, we can use the following simple observation:

PROPOSITION 7.16. *Any partial permutation $\sigma : X \simeq Y$ can be factorized as*

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & Y \\ \gamma \downarrow & & \uparrow \alpha \\ \{1, \dots, k\} & \xrightarrow{\beta} & \{1, \dots, k\} \end{array}$$

with $\alpha, \beta, \gamma \in S_k$ being certain non-unique permutations, where $k = \kappa(\sigma)$.

PROOF. Since we have $|X| = |Y| = k$, we can pick two bijections, as follows:

$$\begin{aligned} X &\simeq \{1, \dots, k\} \\ \{1, \dots, k\} &\simeq Y \end{aligned}$$

We can complete then these bijections up to permutations $\gamma, \alpha \in S_N$. The remaining permutation $\beta \in S_k$ is then uniquely determined by the following formula:

$$\sigma = \alpha\beta\gamma$$

Thus, we are led to the conclusion in the statement. \square

With a bit more work, this leads to homogeneous spaces, in the spirit of the partial isometry spaces discussed before.

To be more precise, we have the following notion:

DEFINITION 7.17. *Associated to any partial permutation,*

$$\sigma : I \simeq J$$

with $I \subset \{1, \dots, N\}$ and $J \subset \{1, \dots, M\}$, is the real/complex partial isometry

$$T_\sigma : \text{span} \left(e_i \mid i \in I \right) \rightarrow \text{span} \left(e_j \mid j \in J \right)$$

given on the standard basis elements by $T_\sigma(e_i) = e_{\sigma(i)}$.

We denote by S_{MN}^L the set of partial permutations $\sigma : I \simeq J$ as above, with range $I \subset \{1, \dots, N\}$ and target $J \subset \{1, \dots, M\}$, and with:

$$L = |I| = |J|$$

In analogy with the decomposition result $H_N^s = \mathbb{Z}_s \wr S_N$, we have:

PROPOSITION 7.18. *The space of partial permutations signed by elements of \mathbb{Z}_s ,*

$$H_{MN}^{sL} = \left\{ T(e_i) = w_i e_{\sigma(i)} \mid \sigma \in S_{MN}^L, w_i \in \mathbb{Z}_s \right\}$$

is isomorphic to the quotient space

$$(H_M^s \times H_N^s) / (H_L^s \times H_{M-L}^s \times H_{N-L}^s)$$

via a standard isomorphism.

PROOF. This follows by adapting the computations in the proof of Proposition 7.3 above. Indeed, we have an action map as follows, which is transitive:

$$H_M^s \times H_N^s \rightarrow H_{MN}^{sL}$$

$$(A, B)U = AUB^*$$

Consider now the following point:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The stabilizer of this point follows to be the following group:

$$H_L^s \times H_{M-L}^s \times H_{N-L}^s$$

To be more precise, this group is embedded via:

$$(x, a, b) \rightarrow \left[\begin{pmatrix} x & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & b \end{pmatrix} \right]$$

But this gives the result. □

In the free case now, the idea is similar, by using inspiration from the construction of the quantum group $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$ in [13]. The result here is as follows:

PROPOSITION 7.19. *The compact quantum space H_{MN}^{sL+} associated to the algebra*

$$C(H_{MN}^{sL+}) = C(U_{MN}^{L+}) / \langle u_{ij} u_{ij}^* = u_{ij}^* u_{ij} = p_{ij} = \text{projections}, u_{ij}^s = p_{ij} \rangle$$

has an action map, and is the target of a quotient map, as in Theorem 7.9 above.

PROOF. We must show that if the variables u_{ij} satisfy the relations in the statement, then these relations are satisfied as well for the following variables:

$$U_{ij} = \sum_{kl} u_{kl} \otimes a_{ki} \otimes b_{lj}^*$$

$$V_{ij} = \sum_{r \leq L} a_{ri} \otimes b_{rj}^*$$

We use the fact that the standard coordinates a_{ij}, b_{ij} on the quantum groups H_M^{s+}, H_N^{s+} satisfy the following relations, for any $x \neq y$ on the same row or column of a, b :

$$xy = xy^* = 0$$

We obtain, by using these relations:

$$\begin{aligned} U_{ij}U_{ij}^* &= \sum_{klmn} u_{kl}u_{mn}^* \otimes a_{ki}a_{mi}^* \otimes b_{lj}^*b_{mj} \\ &= \sum_{kl} u_{kl}u_{kl}^* \otimes a_{ki}a_{ki}^* \otimes b_{lj}^*b_{lj} \end{aligned}$$

We have as well the following formula:

$$\begin{aligned} V_{ij}V_{ij}^* &= \sum_{r,t \leq L} a_{ri}a_{ti}^* \otimes b_{rj}^*b_{tj} \\ &= \sum_{r \leq L} a_{ri}a_{ri}^* \otimes b_{rj}^*b_{rj} \end{aligned}$$

Consider now the following projections:

$$x_{ij} = a_{ij}a_{ij}^*$$

$$y_{ij} = b_{ij}b_{ij}^*$$

$$p_{ij} = u_{ij}u_{ij}^*$$

In terms of these projections, we have:

$$U_{ij}U_{ij}^* = \sum_{kl} p_{kl} \otimes x_{ki} \otimes y_{lj}$$

$$V_{ij}V_{ij}^* = \sum_{r \leq L} x_{ri} \otimes y_{rj}$$

By repeating the computation, we conclude that these elements are projections. Also, a similar computation shows that $U_{ij}^*U_{ij}, V_{ij}^*V_{ij}$ are given by the same formulae.

Finally, once again by using the relations of type $xy = xy^* = 0$, we have:

$$\begin{aligned} U_{ij}^s &= \sum_{k_r l_r} u_{k_1 l_1} \dots u_{k_s l_s} \otimes a_{k_1 i} \dots a_{k_s i} \otimes b_{i_1 j}^* \dots b_{i_s j}^* \\ &= \sum_{kl} u_{kl}^s \otimes a_{ki}^s \otimes (b_{ij}^*)^s \end{aligned}$$

We have as well the following formula:

$$\begin{aligned} V_{ij}^s &= \sum_{r_i \leq L} a_{r_1 i} \dots a_{r_s i} \otimes b_{r_1 j}^* \dots b_{r_s j}^* \\ &= \sum_{r \leq L} a_{ri}^s \otimes (b_{rj}^*)^s \end{aligned}$$

Thus the conditions of type $u_{ij}^s = p_{ij}$ are satisfied as well, and we are done. \square

Let us discuss now the general case. We have the following result:

PROPOSITION 7.20. *The various spaces G_{MN}^L constructed so far appear by imposing to the standard coordinates of U_{MN}^{L+} the relations*

$$\sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \delta_\pi(i) \delta_\sigma(j) u_{i_1 j_1}^{e_1} \dots u_{i_s j_s}^{e_s} = L^{|\pi \vee \sigma|}$$

with $s = (e_1, \dots, e_s)$ ranging over all the colored integers, and with $\pi, \sigma \in D(0, s)$.

PROOF. According to the various constructions above, the relations defining G_{MN}^L can be written as follows, with σ ranging over a family of generators, with no upper legs, of the corresponding category of partitions D :

$$\sum_{j_1 \dots j_s} \delta_\sigma(j) u_{i_1 j_1}^{e_1} \dots u_{i_s j_s}^{e_s} = \delta_\sigma(i)$$

We therefore obtain the relations in the statement, as follows:

$$\begin{aligned} \sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \delta_\pi(i) \delta_\sigma(j) u_{i_1 j_1}^{e_1} \dots u_{i_s j_s}^{e_s} &= \sum_{i_1 \dots i_s} \delta_\pi(i) \sum_{j_1 \dots j_s} \delta_\sigma(j) u_{i_1 j_1}^{e_1} \dots u_{i_s j_s}^{e_s} \\ &= \sum_{i_1 \dots i_s} \delta_\pi(i) \delta_\sigma(i) \\ &= L^{|\pi \vee \sigma|} \end{aligned}$$

As for the converse, this follows by using the relations in the statement, by keeping π fixed, and by making σ vary over all the partitions in the category. \square

In the general case now, where $G = (G_N)$ is an arbitrary uniform easy quantum group, we can construct spaces G_{MN}^L by using the above relations, and we have:

THEOREM 7.21. *The spaces $G_{MN}^L \subset U_{MN}^{L+}$ constructed by imposing the relations*

$$\sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \delta_\pi(i) \delta_\sigma(j) u_{i_1 j_1}^{e_1} \dots u_{i_s j_s}^{e_s} = L^{|\pi \vee \sigma|}$$

with π, σ ranging over all the partitions in the associated category, having no upper legs, are subject to an action map/quotient map diagram, as in Theorem 7.9.

PROOF. We proceed as in the proof of Proposition 7.8. We must prove that, if the variables u_{ij} satisfy the relations in the statement, then so do the following variables:

$$U_{ij} = \sum_{kl} u_{kl} \otimes a_{ki} \otimes b_{lj}^*$$

$$V_{ij} = \sum_{r \leq L} a_{ri} \otimes b_{rj}^*$$

Regarding the variables U_{ij} , the computation here goes as follows:

$$\begin{aligned} & \sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \delta_\pi(i) \delta_\sigma(j) U_{i_1 j_1}^{e_1} \dots U_{i_s j_s}^{e_s} \\ = & \sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \sum_{k_1 \dots k_s} \sum_{l_1 \dots l_s} u_{k_1 l_1}^{e_1} \dots u_{k_s l_s}^{e_s} \otimes \delta_\pi(i) \delta_\sigma(j) a_{k_1 i_1}^{e_1} \dots a_{k_s i_s}^{e_s} \otimes (b_{l_s j_s}^{e_s} \dots b_{l_1 j_1}^{e_1})^* \\ = & \sum_{k_1 \dots k_s} \sum_{l_1 \dots l_s} \delta_\pi(k) \delta_\sigma(l) u_{k_1 l_1}^{e_1} \dots u_{k_s l_s}^{e_s} = L^{|\pi \vee \sigma|} \end{aligned}$$

For the variables V_{ij} the proof is similar, as follows:

$$\begin{aligned} & \sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \delta_\pi(i) \delta_\sigma(j) V_{i_1 j_1}^{e_1} \dots V_{i_s j_s}^{e_s} \\ = & \sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \sum_{l_1, \dots, l_s \leq L} \delta_\pi(i) \delta_\sigma(j) a_{l_1 i_1}^{e_1} \dots a_{l_s i_s}^{e_s} \otimes (b_{l_s j_s}^{e_s} \dots b_{l_1 j_1}^{e_1})^* \\ = & \sum_{l_1, \dots, l_s \leq L} \delta_\pi(l) \delta_\sigma(l) = L^{|\pi \vee \sigma|} \end{aligned}$$

Thus we have constructed an action map, and a quotient map, as in Proposition 7.8 above, and the commutation of the diagram in Theorem 7.9 is then trivial. \square

7d. Integration theory

Let us discuss now the integration over G_{MN}^L . We have:

DEFINITION 7.22. *The integration functional of G_{MN}^L is the composition*

$$\int_{G_{MN}^L} : C(G_{MN}^L) \rightarrow C(G_M \times G_N) \rightarrow \mathbb{C}$$

of the representation $u_{ij} \rightarrow \sum_{r \leq L} a_{ri} \otimes b_{rj}^*$ with the Haar functional of $G_M \times G_N$.

Observe that in the case $L = M = N$ we obtain the integration over G_N . Also, at $L = M = 1$, or at $L = N = 1$, we obtain the integration over the sphere. In the general case now, we first have the following result:

PROPOSITION 7.23. *The integration functional of G_{MN}^L has the invariance property*

$$\left(\int_{G_{MN}^L} \otimes id \right) \Phi(x) = \int_{G_{MN}^L} x$$

with respect to the coaction map:

$$\Phi(u_{ij}) = \sum_{kl} u_{kl} \otimes a_{ki} \otimes b_{lj}^*$$

PROOF. We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must check the following formula:

$$\left(\int_{G_{MN}^L} \otimes id \right) \Phi(u_{i_1 j_1} \cdots u_{i_s j_s}) = \int_{G_{MN}^L} u_{i_1 j_1} \cdots u_{i_s j_s}$$

Let us compute the left term. This is given by:

$$\begin{aligned} X &= \left(\int_{G_{MN}^L} \otimes id \right) \sum_{k_x l_x} u_{k_1 l_1} \cdots u_{k_s l_s} \otimes a_{k_1 i_1} \cdots a_{k_s i_s} \otimes b_{l_1 j_1}^* \cdots b_{l_s j_s}^* \\ &= \sum_{k_x l_x} \sum_{r_x \leq L} a_{k_1 i_1} \cdots a_{k_s i_s} \otimes b_{l_1 j_1}^* \cdots b_{l_s j_s}^* \int_{G_M} a_{r_1 k_1} \cdots a_{r_s k_s} \int_{G_N} b_{r_1 l_1}^* \cdots b_{r_s l_s}^* \\ &= \sum_{r_x \leq L} \sum_{k_x} a_{k_1 i_1} \cdots a_{k_s i_s} \int_{G_M} a_{r_1 k_1} \cdots a_{r_s k_s} \otimes \sum_{l_x} b_{l_1 j_1}^* \cdots b_{l_s j_s}^* \int_{G_N} b_{r_1 l_1}^* \cdots b_{r_s l_s}^* \end{aligned}$$

By using now the invariance property of the Haar functionals of G_M, G_N , we obtain:

$$\begin{aligned} X &= \sum_{r_x \leq L} \left(\int_{G_M} \otimes id \right) \Delta(a_{r_1 i_1} \cdots a_{r_s i_s}) \otimes \left(\int_{G_N} \otimes id \right) \Delta(b_{r_1 j_1}^* \cdots b_{r_s j_s}^*) \\ &= \sum_{r_x \leq L} \int_{G_M} a_{r_1 i_1} \cdots a_{r_s i_s} \int_{G_N} b_{r_1 j_1}^* \cdots b_{r_s j_s}^* \\ &= \left(\int_{G_M} \otimes \int_{G_N} \right) \sum_{r_x \leq L} a_{r_1 i_1} \cdots a_{r_s i_s} \otimes b_{r_1 j_1}^* \cdots b_{r_s j_s}^* \end{aligned}$$

But this gives the formula in the statement, and we are done. \square

We will prove now that the above functional is in fact the unique positive unital invariant trace on $C(G_{MN}^L)$. For this purpose, we will need the Weingarten formula:

THEOREM 7.24. *We have the Weingarten type formula*

$$\int_{G_{MN}^L} u_{i_1 j_1} \dots u_{i_s j_s} = \sum_{\pi \sigma \tau \nu} L^{|\pi \vee \tau|} \delta_\sigma(i) \delta_\nu(j) W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu)$$

where the matrices on the right are given by $W_{sM} = G_{sM}^{-1}$, with $G_{sM}(\pi, \sigma) = M^{|\pi \vee \sigma|}$.

PROOF. We make use of the usual quantum group Weingarten formula, for which we refer to [21], [33]. By using this formula for G_M, G_N , we obtain:

$$\begin{aligned} & \int_{G_{MN}^L} u_{i_1 j_1} \dots u_{i_s j_s} \\ &= \sum_{l_1 \dots l_s \leq L} \int_{G_M} a_{l_1 i_1} \dots a_{l_s i_s} \int_{G_N} b_{l_1 j_1}^* \dots b_{l_s j_s}^* \\ &= \sum_{l_1 \dots l_s \leq L} \sum_{\pi \sigma} \delta_\pi(l) \delta_\sigma(i) W_{sM}(\pi, \sigma) \sum_{\tau \nu} \delta_\tau(l) \delta_\nu(j) W_{sN}(\tau, \nu) \\ &= \sum_{\pi \sigma \tau \nu} \left(\sum_{l_1 \dots l_s \leq L} \delta_\pi(l) \delta_\tau(l) \right) \delta_\sigma(i) \delta_\nu(j) W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu) \end{aligned}$$

The coefficient being $L^{|\pi \vee \tau|}$, we obtain the formula in the statement. \square

We can now derive an abstract characterization of the integration, as follows:

THEOREM 7.25. *The integration of G_{MN}^L is the unique positive unital trace*

$$C(G_{MN}^L) \rightarrow \mathbb{C}$$

which is invariant under the action of the quantum group $G_M \times G_N$.

PROOF. We use a standard method, from [28], [32], the point being to show that we have the following ergodicity formula:

$$\left(id \otimes \int_{G_M} \otimes \int_{G_N} \right) \Phi(x) = \int_{G_{MN}^L} x$$

We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must verify that the following holds:

$$\left(id \otimes \int_{G_M} \otimes \int_{G_N} \right) \Phi(u_{i_1 j_1} \dots u_{i_s j_s}) = \int_{G_{MN}^L} u_{i_1 j_1} \dots u_{i_s j_s}$$

By using the Weingarten formula, the left term can be written as follows:

$$\begin{aligned}
X &= \sum_{k_1 \dots k_s} \sum_{l_1 \dots l_s} u_{k_1 l_1} \dots u_{k_s l_s} \int_{G_M} a_{k_1 i_1} \dots a_{k_s i_s} \int_{G_N} b_{l_1 j_1}^* \dots b_{l_s j_s}^* \\
&= \sum_{k_1 \dots k_s} \sum_{l_1 \dots l_s} u_{k_1 l_1} \dots u_{k_s l_s} \sum_{\pi \sigma} \delta_\pi(k) \delta_\sigma(i) W_{sM}(\pi, \sigma) \sum_{\tau \nu} \delta_\tau(l) \delta_\nu(j) W_{sN}(\tau, \nu) \\
&= \sum_{\pi \sigma \tau \nu} \delta_\sigma(i) \delta_\nu(j) W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu) \sum_{k_1 \dots k_s} \sum_{l_1 \dots l_s} \delta_\pi(k) \delta_\tau(l) u_{k_1 l_1} \dots u_{k_s l_s}
\end{aligned}$$

By using now the summation formula in Theorem 7.21, we obtain:

$$X = \sum_{\pi \sigma \tau \nu} L^{|\pi \vee \tau|} \delta_\sigma(i) \delta_\nu(j) W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu)$$

Now by comparing with the Weingarten formula for G_{MN}^L , this proves our claim.

Assume now that $\tau : C(G_{MN}^L) \rightarrow \mathbb{C}$ satisfies the invariance condition. We have:

$$\begin{aligned}
\tau \left(id \otimes \int_{G_M} \otimes \int_{G_N} \right) \Phi(x) &= \left(\tau \otimes \int_{G_M} \otimes \int_{G_N} \right) \Phi(x) \\
&= \left(\int_{G_M} \otimes \int_{G_N} \right) (\tau \otimes id) \Phi(x) \\
&= \left(\int_{G_M} \otimes \int_{G_N} \right) (\tau(x) 1) \\
&= \tau(x)
\end{aligned}$$

On the other hand, according to the formula established above, we have as well:

$$\begin{aligned}
\tau \left(id \otimes \int_{G_M} \otimes \int_{G_N} \right) \Phi(x) &= \tau(tr(x) 1) \\
&= tr(x)
\end{aligned}$$

Thus we obtain $\tau = tr$, and this finishes the proof. □

As a main application, we have:

PROPOSITION 7.26. *For a sum of coordinates*

$$\chi_E = \sum_{(ij) \in E} u_{ij}$$

which do not overlap on rows and columns we have

$$\int_{G_{MN}^L} \chi_E^s = \sum_{\pi \sigma \tau \nu} K^{|\pi \vee \tau|} L^{|\sigma \vee \nu|} W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu)$$

where $K = |E|$ is the cardinality of the indexing set.

PROOF. With $K = |E|$, we can write $E = \{(\alpha(i), \beta(i))\}$, for certain embeddings:

$$\alpha : \{1, \dots, K\} \subset \{1, \dots, M\}$$

$$\beta : \{1, \dots, K\} \subset \{1, \dots, N\}$$

In terms of these maps α, β , the moment in the statement is given by:

$$M_s = \int_{G_{MN}^L} \left(\sum_{i \leq K} u_{\alpha(i)\beta(i)} \right)^s$$

By using the Weingarten formula, we can write this quantity as follows:

$$\begin{aligned} & M_s \\ &= \int_{G_{MN}^L} \sum_{i_1 \dots i_s \leq K} u_{\alpha(i_1)\beta(i_1)} \dots u_{\alpha(i_s)\beta(i_s)} \\ &= \sum_{i_1 \dots i_s \leq K} \sum_{\pi \sigma \tau \nu} L^{|\sigma \vee \nu|} \delta_\pi(\alpha(i_1), \dots, \alpha(i_s)) \delta_\tau(\beta(i_1), \dots, \beta(i_s)) W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu) \\ &= \sum_{\pi \sigma \tau \nu} \left(\sum_{i_1 \dots i_s \leq K} \delta_\pi(i) \delta_\tau(i) \right) L^{|\sigma \vee \nu|} W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu) \end{aligned}$$

But, as explained before, the coefficient on the left in the last formula is:

$$C = K^{|\pi \vee \tau|}$$

We therefore obtain the formula in the statement. \square

We can further advance in the classical/twisted and free cases, where the Weingarten theory for the corresponding quantum groups is available from [13], [21], [21], [33]:

THEOREM 7.27. *In the context of the liberation operations*

$$O_{MN}^L \rightarrow O_{MN}^{L+}$$

$$U_{MN}^L \rightarrow U_{MN}^{L+}$$

$$H_{MN}^{sL} \rightarrow H_{MN}^{sL+}$$

the laws of the sums of non-overlapping coordinates,

$$\chi_E = \sum_{(ij) \in E} u_{ij}$$

are in Bercovici-Pata bijection, in the

$$|E| = \kappa N, L = \lambda N, M = \mu N$$

regime and $N \rightarrow \infty$ limit.

PROOF. We use the general theory in [13], [21], [21], [33]. According to Proposition 7.26, in terms of $K = |E|$, the moments of the variables in the statement are given by:

$$M_s = \sum_{\pi\sigma\tau\nu} K^{|\pi\vee\tau|} L^{|\sigma\vee\nu|} W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu)$$

We use now two standard facts, namely:

(1) The fact that in the $N \rightarrow \infty$ limit the Weingarten matrix W_{sN} is concentrated on the diagonal.

(2) The fact that we have an inequality as follows, with equality precisely when $\pi = \sigma$:

$$|\pi \vee \sigma| \leq \frac{|\pi| + |\sigma|}{2}$$

For details on all this, we refer to [21].

Let us discuss now what happens in the regime from the statement, namely:

$$K = \kappa N, L = \lambda N, M = \mu N, N \rightarrow \infty$$

In this regime, we obtain:

$$\begin{aligned} M_s &\simeq \sum_{\pi\tau} K^{|\pi\vee\tau|} L^{|\pi\vee\tau|} M^{-|\pi|} N^{-|\tau|} \\ &\simeq \sum_{\pi} K^{|\pi|} L^{|\pi|} M^{-|\pi|} N^{-|\pi|} \\ &= \sum_{\pi} \left(\frac{\kappa\lambda}{\mu} \right)^{|\pi|} \end{aligned}$$

In order to interpret this formula, we use general theory from [13], [21], [21]:

(1) For $G_N = O_N, \bar{O}_N/O_N^+$, the above variables χ_E follow to be asymptotically Gaussian/semicircular, of parameter $\frac{\kappa\lambda}{\mu}$, and hence in Bercovici-Pata bijection.

(2) For $G_N = U_N, \bar{U}_N/U_N^+$ the situation is similar, with χ_E being asymptotically complex Gaussian/circular, of parameter $\frac{\kappa\lambda}{\mu}$, and in Bercovici-Pata bijection.

(3) Finally, for $G_N = H_N^s/H_N^{s+}$, the variables χ_E are asymptotically Bessel/free Bessel of parameter $\frac{\kappa\lambda}{\mu}$, and once again in Bercovici-Pata bijection. \square

The convergence in the above result is of course in moments, and we do not know whether some stronger convergence results can be formulated. Nor do we know whether one can use linear combinations of coordinates which are more general than the sums χ_E that we consider. These are interesting questions, that we would like to raise here.

Also, there are several possible extensions of the above result, for instance by using quantum reflection groups instead of unitary quantum groups, and by using twisting

operations as well. We refer here to [8], and to [32] as well, for a number of supplementary results, which can be obtained by using the stronger formalism there.

7e. Exercises

Things have been quite advanced in this chapter, and our exercises will mostly focus on details and examples, in relation with the above. First, we have:

EXERCISE 7.28. *Work out, with full details, the particular cases of the diagram*

$$\begin{array}{ccc} O_{MN}^{L+} & \longrightarrow & U_{MN}^{L+} \\ \uparrow & & \uparrow \\ O_{MN}^L & \longrightarrow & U_{MN}^L \end{array}$$

at $L = M = 1$ and at $L = N = 1$.

This is something that we talked about in the above, but without full details. The problem now is that of doing the complete work here, with full details.

Along the same lines, but at a more advanced level, we have:

EXERCISE 7.29. *Work out, with full details, the particular cases of the diagram*

$$\begin{array}{ccc} O_{MN}^{L+} & \longrightarrow & U_{MN}^{L+} \\ \uparrow & & \uparrow \\ O_{MN}^L & \longrightarrow & U_{MN}^L \end{array}$$

at $L = M$ and at $L = N$.

Normally we should obtain here the row spaces from chapter 6 above. The problem is that of working out all this, with full details.

At an even more advanced level now, we have:

EXERCISE 7.30. *Work out, with full details, the particular cases of the spaces*

$$G_{MN}^L \subset G_{MN}^{L+}$$

at $L = M$ and at $L = N$, for all the basic easy groups.

As before, we should get row spaces here, and all this needs a complete proof.

We have seen in this chapter that the spaces S_{MN}^L and S_{MN}^{L+} appear as components of the semigroups of partial permutations \tilde{S}_N and \tilde{S}_N^+ . In view of this, the problem now is that of developing a similar theory in the continuous case, and we have here:

EXERCISE 7.31. *Develop a theory of semigroups and quantum semigroups*

$$\begin{array}{ccc} \tilde{O}_N^+ & \longrightarrow & \tilde{U}_N^+ \\ \uparrow & & \uparrow \\ \tilde{O}_N & \longrightarrow & \tilde{U}_N \end{array}$$

and explain the relation with the spaces constructed in this chapter.

Things are actually quite tricky here, in relation with the semigroup structure in the free case. Thus, the exercise asks to develop what can be indeed be developed.

In the same spirit, we have the following exercise, which is our last one on these algebraic topics, dealing with the quantum reflection group case:

EXERCISE 7.32. *Develop a theory of semigroups and quantum semigroups*

$$\begin{array}{ccc} \tilde{H}_N^+ & \longrightarrow & \tilde{K}_N^+ \\ \uparrow & & \uparrow \\ \tilde{H}_N & \longrightarrow & \tilde{K}_N \end{array}$$

and explain the relation with the spaces constructed in this chapter.

Here the solution can be found either directly, or by generalizing first to the case of H_N^s, H_N^{s+} what we know about S_N, S_N^+ , and then particularizing at $s = 2, \infty$.

In relation now with analytic aspects, we have the following exercise:

EXERCISE 7.33. *Work out the main particular cases of the probability computation for the sums of non-overlapping coordinates*

$$\chi_E = \sum_{(ij) \in E} u_{ij}$$

as to recover most of the asymptotic results obtained so far in this book.

There is quite some work to be done here, notably in re-reading the present book.

CHAPTER 8

Higher manifolds

8a. Affine spaces

We discuss in this section an abstract extension of the constructions of quantum algebraic manifolds that we have so far. The idea will be that of looking at certain classes of algebraic manifolds $X \subset S_{\mathbb{C},+}^{N-1}$, which are homogeneous spaces, of a special type.

Following [10], let us formulate the following definition:

DEFINITION 8.1. *An affine homogeneous space over a closed subgroup $G \subset U_N^+$ is a closed subset $X \subset S_{\mathbb{C},+}^{N-1}$, such that there exists an index set $I \subset \{1, \dots, N\}$ such that*

$$\alpha(x_i) = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ji}$$

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

define morphisms of C^* -algebras, satisfying:

$$\left(id \otimes \int_G \right) \Phi = \int_G \alpha(\cdot) 1$$

Observe that $U_N^+ \rightarrow S_{\mathbb{C},+}^{N-1}$ is indeed affine in this sense, with $I = \{1\}$. Also, the $1/\sqrt{|I|}$ constant appearing above is the correct one, because:

$$\begin{aligned} \sum_i \left(\sum_{j \in I} u_{ji} \right) \left(\sum_{k \in I} u_{ki} \right)^* &= \sum_i \sum_{j,k \in I} u_{ji} u_{ki}^* \\ &= \sum_{j,k \in I} (uu^*)_{jk} \\ &= |I| \end{aligned}$$

As a first general result about such spaces, we have:

THEOREM 8.2. *Consider an affine homogeneous space X , as above.*

- (1) *The coaction condition $(\Phi \otimes id)\Phi = (id \otimes \Delta)\Phi$ is satisfied.*
- (2) *We have as well the formula $(\alpha \otimes id)\Phi = \Delta\alpha$.*

PROOF. The coaction condition is clear. For the second formula, we first have:

$$\begin{aligned} (\alpha \otimes id)\Phi(x_i) &= \sum_k \alpha(x_k) \otimes u_{ki} \\ &= \frac{1}{\sqrt{|I|}} \sum_k \sum_{j \in I} u_{jk} \otimes u_{ki} \end{aligned}$$

On the other hand, we have as well:

$$\begin{aligned} \Delta\alpha(x_i) &= \frac{1}{\sqrt{|I|}} \sum_{j \in I} \Delta(u_{ji}) \\ &= \frac{1}{\sqrt{|I|}} \sum_{j \in I} \sum_k u_{jk} \otimes u_{ki} \end{aligned}$$

Thus, by linearity, multiplicativity and continuity, we obtain the result. \square

Summarizing, the terminology in Definition 8.1 is justified, in the sense that what we have there are indeed certain homogeneous spaces, of very special, “affine” type. As a second result regarding such spaces, which closes the discussion in the case where α is injective, which is something that happens in many cases, we have:

THEOREM 8.3. *When α is injective we must have $X = X_{G,I}^{min}$, where:*

$$C(X_{G,I}^{min}) = \left\langle \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ji} \mid i = 1, \dots, N \right\rangle \subset C(G)$$

Moreover, $X_{G,I}^{min}$ is affine homogeneous, for any $G \subset U_N^+$, and any $I \subset \{1, \dots, N\}$.

PROOF. The first assertion is clear from definitions. Regarding now the second assertion, consider the variables in the statement:

$$X_i = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ji} \in C(G)$$

In order to prove that we have $X_{G,I}^{min} \subset S_{C,+}^{N-1}$, observe first that we have:

$$\begin{aligned} \sum_i X_i X_i^* &= \frac{1}{|I|} \sum_i \sum_{j,k \in I} u_{ji} u_{ki}^* \\ &= \frac{1}{|I|} \sum_{j,k \in I} (uu^*)_{jk} \\ &= 1 \end{aligned}$$

We have as well the following computation:

$$\begin{aligned} \sum_i X_i^* X_i &= \frac{1}{|I|} \sum_i \sum_{j,k \in I} u_{ji}^* u_{ki} \\ &= \frac{1}{|I|} \sum_{j,k \in I} (\bar{u} u^t)_{jk} \\ &= 1 \end{aligned}$$

Thus $X_{G,I}^{min} \subset S_{\mathbb{C},+}^{N-1}$. Finally, observe that we have:

$$\begin{aligned} \Delta(X_i) &= \frac{1}{\sqrt{|I|}} \sum_{j \in I} \sum_k u_{jk} \otimes u_{ki} \\ &= \sum_k X_k \otimes u_{ki} \end{aligned}$$

Thus we have a coaction map, given by $\Phi = \Delta$. As for the ergodicity condition, namely $(id \otimes \int_G) \Delta = \int_G(\cdot)1$, this holds as well, by definition of the integration functional \int_G . \square

Our purpose now will be to show that the affine homogeneous spaces appear as follows, a bit in the same way as the discrete group algebras:

$$X_{G,I}^{min} \subset X \subset X_{G,I}^{max}$$

We make the standard convention that all the tensor exponents k are “colored integers”, that is, $k = e_1 \dots e_k$ with $e_i \in \{\circ, \bullet\}$, with \circ corresponding to the usual variables, and with \bullet corresponding to their adjoints. With this convention, we have:

PROPOSITION 8.4. *The ergodicity condition, namely*

$$\left(id \otimes \int_G \right) \Phi = \int_G \alpha(\cdot)1$$

is equivalent to the condition

$$(Px^{\otimes k})_{i_1 \dots i_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} P_{i_1 \dots i_k, j_1 \dots j_k} \quad , \quad \forall k, \forall i_1, \dots, i_k$$

where

$$P_{i_1 \dots i_k, j_1 \dots j_k} = \int_G u_{j_1 i_1}^{e_1} \dots u_{j_k i_k}^{e_k}$$

and where $(x^{\otimes k})_{i_1 \dots i_k} = x_{i_1}^{e_1} \dots x_{i_k}^{e_k}$.

PROOF. We have the following computation:

$$\begin{aligned}
\left(id \otimes \int_G \right) \Phi(x_{i_1}^{e_1} \dots x_{i_k}^{e_k}) &= \sum_{j_1 \dots j_k} x_{j_1}^{e_1} \dots x_{j_k}^{e_k} \int_G u_{j_1 i_1}^{e_1} \dots u_{j_k i_k}^{e_k} \\
&= \sum_{j_1 \dots j_k} P_{i_1 \dots i_k, j_1 \dots j_k}(x^{\otimes k})_{j_1 \dots j_k} \\
&= (Px^{\otimes k})_{i_1 \dots i_k}
\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}
\int_G \alpha(x_{i_1}^{e_1} \dots x_{i_k}^{e_k}) &= \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} \int_G u_{j_1 i_1}^{e_1} \dots u_{j_k i_k}^{e_k} \\
&= \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} P_{i_1 \dots i_k, j_1 \dots j_k}
\end{aligned}$$

But this gives the formula in the statement, and we are done. \square

As a consequence, we have the following result:

THEOREM 8.5. *We must have $X \subset X_{G,I}^{max}$, as subsets of $S_{\mathbb{C},+}^{N-1}$, where:*

$$C(X_{G,I}^{max}) = C(S_{\mathbb{C},+}^{N-1}) / \left\langle (Px^{\otimes k})_{i_1 \dots i_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} P_{i_1 \dots i_k, j_1 \dots j_k} \mid \forall k, \forall i_1, \dots, i_k \right\rangle$$

Moreover, $X_{G,I}^{max}$ is affine homogeneous, for any $G \subset U_N^+$, and any $I \subset \{1, \dots, N\}$.

PROOF. Let us first prove that we have an action $G \curvearrowright X_{G,I}^{max}$. We must show here that the variables $X_i = \sum_j x_j \otimes u_{ji}$ satisfy the defining relations for $X_{G,I}^{max}$. We have:

$$\begin{aligned}
(Px^{\otimes k})_{i_1 \dots i_k} &= \sum_{l_1 \dots l_k} P_{i_1 \dots i_k, l_1 \dots l_k}(X^{\otimes k})_{l_1 \dots l_k} \\
&= \sum_{l_1 \dots l_k} P_{i_1 \dots i_k, l_1 \dots l_k} \sum_{j_1 \dots j_k} x_{j_1}^{e_1} \dots x_{j_k}^{e_k} \otimes u_{j_1 l_1}^{e_1} \dots u_{j_k l_k}^{e_k} \\
&= \sum_{j_1 \dots j_k} x_{j_1}^{e_1} \dots x_{j_k}^{e_k} \otimes (u^{\otimes k} P^t)_{j_1 \dots j_k, i_1 \dots i_k}
\end{aligned}$$

Since by Peter-Weyl the transpose of $P_{i_1 \dots i_k, j_1 \dots j_k} = \int_G u_{j_1 i_1}^{e_1} \dots u_{j_k i_k}^{e_k}$ is the orthogonal projection onto $Fix(u^{\otimes k})$, we have $u^{\otimes k} P^t = P^t$. We therefore obtain:

$$\begin{aligned} (PX^{\otimes k})_{i_1 \dots i_k} &= \sum_{j_1 \dots j_k} P_{i_1 \dots i_k, j_1 \dots j_k} x_{j_1}^{e_1} \dots x_{j_k}^{e_k} \\ &= (Px^{\otimes k})_{i_1 \dots i_k} \\ &= \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} P_{i_1 \dots i_k, j_1 \dots j_k} \end{aligned}$$

Thus we have an action $G \curvearrowright X_{G,I}^{max}$, and since this action is ergodic by Proposition 8.4, we have an affine homogeneous space, as claimed. \square

We can now merge the results that we have, and we obtain:

THEOREM 8.6. *Given a closed quantum subgroup $G \subset U_N^+$, and a set $I \subset \{1, \dots, N\}$, if we consider the following C^* -subalgebra and the following quotient C^* -algebra,*

$$\begin{aligned} C(X_{G,I}^{min}) &= \left\langle \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ji} \mid i = 1, \dots, N \right\rangle \subset C(G) \\ C(X_{G,I}^{max}) &= C(S_{\mathbb{C},+}^{N-1}) / \left\langle (Px^{\otimes k})_{i_1 \dots i_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} P_{i_1 \dots i_k, j_1 \dots j_k} \mid \forall k, \forall i_1, \dots, i_k \right\rangle \end{aligned}$$

then we have maps as follows,

$$G \rightarrow X_{G,I}^{min} \subset X_{G,I}^{max} \subset S_{\mathbb{C},+}^{N-1}$$

the space $G \rightarrow X_{G,I}^{max}$ is affine homogeneous, and any affine homogeneous space

$$G \rightarrow X$$

appears as an intermediate space, as follows:

$$X_{G,I}^{min} \subset X \subset X_{G,I}^{max}$$

PROOF. This follows indeed from the results that we have, namely Theorem 8.3 and Theorem 8.5 above. \square

Summarizing, the situation is a bit similar to that of the full and reduced group algebras, having intermediate objects between them.

We will need one more general result from [10], namely an extension of the Weingarten integration formula [21], [49], [96], to the affine homogeneous space setting:

THEOREM 8.7. *Assuming that $G \rightarrow X$ is an affine homogeneous space, with index set $I \subset \{1, \dots, N\}$, the Haar integration functional $\int_X = \int_G \alpha$ is given by*

$$\int_X x_{i_1}^{e_1} \dots x_{i_k}^{e_k} = \sum_{\pi, \sigma \in D} K_I(\pi) \overline{(\xi_\sigma)_{i_1 \dots i_k}} W_{kN}(\pi, \sigma)$$

where $\{\xi_\pi | \pi \in D\}$ is a basis of $Fix(u^{\otimes k})$, $W_{kN} = G_{kN}^{-1}$ with

$$G_{kN}(\pi, \sigma) = \langle \xi_\pi, \xi_\sigma \rangle$$

is the associated Weingarten matrix, and:

$$K_I(\pi) = \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} (\xi_\pi)_{j_1 \dots j_k}$$

PROOF. By using the Weingarten formula for the quantum group G , we have:

$$\begin{aligned} \int_X x_{i_1}^{e_1} \dots x_{i_k}^{e_k} &= \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} \int_G u_{j_1 i_1}^{e_1} \dots u_{j_k i_k}^{e_k} \\ &= \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} \sum_{\pi, \sigma \in D} (\xi_\pi)_{j_1 \dots j_k} \overline{(\xi_\sigma)_{i_1 \dots i_k}} W_{kN}(\pi, \sigma) \end{aligned}$$

But this gives the formula in the statement, and we are done. \square

Let us go back now to the “minimal vs maximal” discussion, in analogy with the group algebras. Here is a natural example of an intermediate space $X_{G,I}^{min} \subset X \subset X_{G,I}^{max}$:

THEOREM 8.8. *Given a closed quantum subgroup $G \subset U_N^+$, and a set $I \subset \{1, \dots, N\}$, if we consider the following quotient algebra*

$$C(X_{G,I}^{med}) = C(S_{\mathbb{C},+}^{N-1}) / \left\langle \sum_{j_1 \dots j_k} \xi_{j_1 \dots j_k} x_{j_1}^{e_1} \dots x_{j_k}^{e_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} \xi_{j_1 \dots j_k} \middle| \forall k, \forall \xi \in Fix(u^{\otimes k}) \right\rangle$$

we obtain in this way an affine homogeneous space $G \rightarrow X_{G,I}$.

PROOF. We know from Theorem 8.5 above that $X_{G,I}^{max} \subset S_{\mathbb{C},+}^{N-1}$ is constructed by imposing to the standard coordinates the conditions $Px^{\otimes k} = P^I$, where:

$$\begin{aligned} P_{i_1 \dots i_k, j_1 \dots j_k} &= \int_G u_{j_1 i_1}^{e_1} \dots u_{j_k i_k}^{e_k} \\ P_{i_1 \dots i_k}^I &= \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} P_{i_1 \dots i_k, j_1 \dots j_k} \end{aligned}$$

According to the Weingarten integration formula for G , we have:

$$\begin{aligned} (Px^{\otimes k})_{i_1 \dots i_k} &= \sum_{j_1 \dots j_k} \sum_{\pi, \sigma \in D} (\xi_\pi)_{j_1 \dots j_k} \overline{(\xi_\sigma)_{i_1 \dots i_k}} W_{kN}(\pi, \sigma) x_{j_1}^{e_1} \dots x_{j_k}^{e_k} \\ P_{i_1 \dots i_k}^I &= \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} \sum_{\pi, \sigma \in D} (\xi_\pi)_{j_1 \dots j_k} \overline{(\xi_\sigma)_{i_1 \dots i_k}} W_{kN}(\pi, \sigma) \end{aligned}$$

Thus $X_{G,I}^{med} \subset X_{G,I}^{max}$, and the other assertions are standard as well. \square

We can now put everything together, as follows:

THEOREM 8.9. *Given a closed subgroup $G \subset U_N^+$, and a subset $I \subset \{1, \dots, N\}$, the affine homogeneous spaces over G , with index set I , have the following properties:*

- (1) *These are exactly the intermediate subspaces $X_{G,I}^{min} \subset X \subset X_{G,I}^{max}$ on which G acts affinely, with the action being ergodic.*
- (2) *For the minimal and maximal spaces $X_{G,I}^{min}$ and $X_{G,I}^{max}$, as well as for the intermediate space $X_{G,I}^{med}$ constructed above, these conditions are satisfied.*
- (3) *By performing the GNS construction with respect to the Haar integration functional $\int_X \alpha = \int_G \alpha$ we obtain the minimal space $X_{G,I}^{min}$.*

We agree to identify all these spaces, via the GNS construction, and denote them $X_{G,I}$.

PROOF. This follows indeed by combining the various results and observations formulated above. Once again, for full details on all these facts, we refer to [10]. \square

8b. Basic examples

Let us discuss now some basic examples of affine homogeneous spaces, namely those coming from the classical groups, and those coming from the group duals. We will need the following technical result:

PROPOSITION 8.10. *Assuming that a closed subset $X \subset S_{\mathbb{C},+}^{N-1}$ is affine homogeneous over a classical group, $G \subset U_N$, then X itself must be classical, $X \subset S_{\mathbb{C}}^{N-1}$.*

PROOF. We use the well-known fact that, since the standard coordinates $u_{ij} \in C(G)$ commute, the corepresentation $u^{\circ \circ \bullet \bullet} = u^{\otimes 2} \otimes \bar{u}^{\otimes 2}$ has the following fixed vector:

$$\xi = \sum_{ij} e_i \otimes e_j \otimes e_i \otimes e_j$$

With $k = \circ \circ \bullet \bullet$ and with this vector ξ , the ergodicity formula reads:

$$\begin{aligned} \sum_{ij} x_i x_j x_i^* x_j^* &= \frac{1}{\sqrt{|I|^4}} \sum_{i,j \in I} 1 \\ &= 1 \end{aligned}$$

By using this formula, along with $\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$, we obtain:

$$\begin{aligned}
& \sum_{ij} (x_i x_j - x_j x_i)(x_j^* x_i^* - x_i^* x_j^*) \\
&= \sum_{ij} x_i x_j x_j^* x_i^* - x_i x_j x_i^* x_j^* - x_j x_i x_j^* x_i^* + x_j x_i x_i^* x_j^* \\
&= 1 - 1 - 1 + 1 \\
&= 0
\end{aligned}$$

We conclude that for any i, j we have:

$$[x_i, x_j] = 0$$

By using now this commutation relation, plus once again the relations defining $S_{\mathbb{C},+}^{N-1}$, we have as well:

$$\begin{aligned}
& \sum_{ij} (x_i x_j^* - x_j^* x_i)(x_j x_i^* - x_i^* x_j) \\
&= \sum_{ij} x_i x_j^* x_j x_i^* - x_i x_j^* x_i^* x_j - x_j^* x_i x_j x_i^* + x_j^* x_i x_i^* x_j \\
&= \sum_{ij} x_i x_j^* x_j x_i^* - x_i x_i^* x_j^* x_j - x_j^* x_j x_i x_i^* + x_j^* x_i x_i^* x_j \\
&= 1 - 1 - 1 + 1 \\
&= 0
\end{aligned}$$

Thus we have $[x_i, x_j^*] = 0$ as well, and so $X \subset S_{\mathbb{C}}^{N-1}$, as claimed. \square

We can now formulate the result in the classical case, as follows:

THEOREM 8.11. *In the classical case, $G \subset U_N$, there is only one affine homogeneous space, for each index set $I = \{1, \dots, N\}$, namely the quotient space*

$$X = G/(G \cap C_N^I)$$

where $C_N^I \subset U_N$ is the group of unitaries fixing the following vector:

$$\xi_I = \frac{1}{\sqrt{|I|}}(\delta_{i \in I})_i$$

PROOF. Consider an affine homogeneous space $G \rightarrow X$. We already know from Proposition 8.10 above that X is classical. We will first prove that we have $X = X_{G,I}^{min}$, and then we will prove that $X_{G,I}^{min}$ equals the quotient space in the statement.

(1) We use the well-known fact that the functional $E = (id \otimes \int_G)\Phi$ is the projection onto the fixed point algebra of the action, given by:

$$C(X)^\Phi = \left\{ f \in C(X) \mid \Phi(f) = f \otimes 1 \right\}$$

Thus our ergodicity condition, namely $E = \int_G \alpha(\cdot)1$, shows that we must have:

$$C(X)^\Phi = \mathbb{C}1$$

Since in the classical case the condition $\Phi(f) = f \otimes 1$ reads $f(gx) = f(x)$ for any $g \in G$ and $x \in X$, we recover in this way the usual ergodicity condition, stating that whenever a function $f \in C(X)$ is constant on the orbits of the action, it must be constant.

Now observe that for an affine action, the orbits are closed. Thus an affine action which is ergodic must be transitive, and we deduce from this that we have $X = X_{G,I}^{min}$.

(2) We know that the inclusion $C(X) \subset C(G)$ comes via:

$$x_i = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ji}$$

Thus, the quotient map $p : G \rightarrow X \subset S_{\mathbb{C}}^{N-1}$ is given by the following formula:

$$p(g) = \left(\frac{1}{\sqrt{|I|}} \sum_{j \in I} g_{ji} \right)_i$$

In particular, the image of the unit matrix $1 \in G$ is the following vector:

$$\begin{aligned} p(1) &= \left(\frac{1}{\sqrt{|I|}} \sum_{j \in I} \delta_{ji} \right)_i \\ &= \left(\frac{1}{\sqrt{|I|}} \delta_{i \in I} \right)_i \\ &= \xi_I \end{aligned}$$

But this gives the result, and we are done. \square

Let us discuss now the group dual case. Given a discrete group $\Gamma = \langle g_1, \dots, g_N \rangle$, we can consider the embedding $\widehat{\Gamma} \subset U_N^+$ given by $u_{ij} = \delta_{ij}g_i$. We have then:

THEOREM 8.12. *In the group dual case, $G = \widehat{\Gamma}$ with $\Gamma = \langle g_1, \dots, g_N \rangle$, we have*

$$X = \widehat{\Gamma}_I$$

$$\Gamma_I = \langle g_i | i \in I \rangle \subset \Gamma$$

for any affine homogeneous space X , when identifying full and reduced group algebras.

PROOF. Assume indeed that we have an affine homogeneous space $G \rightarrow X$. In terms of the rescaled coordinates $h_i = \sqrt{|I|}x_i$, our axioms for α, Φ read:

$$\alpha(h_i) = \delta_{i \in I}g_i$$

$$\Phi(h_i) = h_i \otimes g_i$$

As for the ergodicity condition, this translates as follows:

$$\begin{aligned}
& \left(id \otimes \int_G \right) \Phi(h_{i_1}^{e_1} \dots h_{i_p}^{e_p}) = \int_G \alpha(h_{i_1}^{e_1} \dots h_{i_p}^{e_p}) \\
\iff & \left(id \otimes \int_G \right) (h_{i_1}^{e_1} \dots h_{i_p}^{e_p} \otimes g_{i_1}^{e_1} \dots g_{i_p}^{e_p}) = \int_G \delta_{i_1 \in I} \dots \delta_{i_p \in I} g_{i_1}^{e_1} \dots g_{i_p}^{e_p} \\
\iff & \delta_{g_{i_1}^{e_1} \dots g_{i_p}^{e_p}, 1} h_{i_1}^{e_1} \dots h_{i_p}^{e_p} = \delta_{g_{i_1}^{e_1} \dots g_{i_p}^{e_p}, 1} \delta_{i_1 \in I} \dots \delta_{i_p \in I} \\
\iff & \left[g_{i_1}^{e_1} \dots g_{i_p}^{e_p} = 1 \implies h_{i_1}^{e_1} \dots h_{i_p}^{e_p} = \delta_{i_1 \in I} \dots \delta_{i_p \in I} \right]
\end{aligned}$$

Now observe that from $g_i g_i^* = g_i^* g_i = 1$ we obtain in this way:

$$h_i h_i^* = h_i^* h_i = \delta_{i \in I}$$

Thus the elements h_i vanish for $i \notin I$, and are unitaries for $i \in I$. We conclude that we have $X = \widehat{\Lambda}$, where $\Lambda = \langle h_i | i \in I \rangle$ is the group generated by these unitaries.

In order to finish the proof, our claim is that for indices $i_x \in I$ we have:

$$g_{i_1}^{e_1} \dots g_{i_p}^{e_p} = 1 \iff h_{i_1}^{e_1} \dots h_{i_p}^{e_p} = 1$$

Indeed, \implies comes from the ergodicity condition, as processed above, and \impliedby comes from the existence of the morphism α , which is given by $\alpha(h_i) = g_i$, for $i \in I$. \square

Let us go back now to the general case, and discuss a number of further axiomatization issues, based on the examples that we have. We will need the following result:

PROPOSITION 8.13. *The closed subspace $C_N^{I+} \subset U_N^+$ defined via*

$$C(C_N^{I+}) = C(U_N^+) / \langle u_{\xi_I} = \xi_I \rangle$$

where $\xi_I = \frac{1}{\sqrt{|I|}} (\delta_{i \in I})_i$, is a compact quantum group.

PROOF. We must check Woronowicz's axioms, and the proof goes as follows:

(1) Let us set $U_{ij} = \sum_k u_{ik} \otimes u_{kj}$. We have then:

$$\begin{aligned}
(U_{\xi_I})_i &= \frac{1}{\sqrt{|I|}} \sum_{j \in I} U_{ij} \\
&= \frac{1}{\sqrt{|I|}} \sum_{j \in I} \sum_k u_{ik} \otimes u_{kj} \\
&= \sum_k u_{ik} \otimes (u_{\xi_I})_k
\end{aligned}$$

Since the vector ξ_I is by definition fixed by u , we obtain:

$$\begin{aligned} (U\xi_I)_i &= \sum_k u_{ik} \otimes (\xi_I)_k \\ &= \frac{1}{\sqrt{|I|}} \sum_{k \in I} u_{ik} \otimes 1 \\ &= (u\xi_I)_i \otimes 1 \\ &= (\xi_I)_i \otimes 1 \end{aligned}$$

Thus we can define indeed a comultiplication map, by $\Delta(u_{ij}) = U_{ij}$.

(2) In order to construct the counit map, $\varepsilon(u_{ij}) = \delta_{ij}$, we must prove that the identity matrix $1 = (\delta_{ij})_{ij}$ satisfies $1\xi_I = \xi_I$. But this is clear.

(3) In order to construct the antipode, $S(u_{ij}) = u_{ji}^*$, we must prove that the adjoint matrix $u^* = (u_{ji}^*)_{ij}$ satisfies $u^*\xi_I = \xi_I$. But this is clear from $u\xi_I = \xi_I$. \square

Based on the computations that we have so far, we can formulate:

THEOREM 8.14. *Given a closed quantum subgroup $G \subset U_N^+$ and a set $I \subset \{1, \dots, N\}$, we have a quotient map and an inclusion map as follows:*

$$G/(G \cap C_N^{I+}) \rightarrow X_{G,I}^{min} \subset X_{G,I}^{max}$$

These maps are both isomorphisms in the classical case. In general, they are both proper.

PROOF. Consider the quantum group $H = G \cap C_N^{I+}$, which is by definition such that at the level of the corresponding algebras, we have:

$$C(H) = C(G) / \langle u\xi_I = \xi_I \rangle$$

In order to construct a quotient map $G/H \rightarrow X_{G,I}^{min}$, we must check that the defining relations for $C(G/H)$ hold for the standard generators $x_i \in C(X_{G,I}^{min})$. But if we denote by $\rho : C(G) \rightarrow C(H)$ the quotient map, then we have, as desired:

$$\begin{aligned} (id \otimes \rho)\Delta x_i &= (id \otimes \rho) \left(\frac{1}{\sqrt{|I|}} \sum_{j \in I} \sum_k u_{ik} \otimes u_{kj} \right) \\ &= \sum_k u_{ik} \otimes (\xi_I)_k \\ &= x_i \otimes 1 \end{aligned}$$

In the classical case, Theorem 8.11 shows that both the maps in the statement are isomorphisms. For the group duals, however, these maps are not isomorphisms, in general. This follows indeed from Theorem 8.12, and from the general theory in [32]. \square

8c. Further examples

We discuss now a number of further examples. We will need:

PROPOSITION 8.15. *Given a compact matrix quantum group $G = (G, u)$, the pair*

$$G^t = (G, u^t)$$

where $(u^t)_{ij} = u_{ji}$, is a compact matrix quantum group as well.

PROOF. The construction of the comultiplication is as follows, where Σ is the flip map:

$$\begin{aligned} \Delta^t[(u^t)_{ij}] &= \sum_k (u^t)_{ik} \otimes (u^t)_{kj} \\ \iff \Delta^t(u_{ji}) &= \sum_k u_{ki} \otimes u_{jk} \\ \iff \Delta^t &= \Sigma\Delta \end{aligned}$$

As for the corresponding counit and antipode, these can be simply taken to be (ε, S) , and the axioms are satisfied. \square

We will need as well the following result, which is standard as well:

PROPOSITION 8.16. *Given two closed subgroups $G \subset U_N^+$ and $H \subset U_M^+$, with fundamental corepresentations denoted $u = (u_{ij})$ and $v = (v_{ab})$, their product is a closed subgroup*

$$G \times H \subset U_{NM}^+$$

with fundamental corepresentation $w_{ia,jb} = u_{ij} \otimes v_{ab}$.

PROOF. Our claim is that the corresponding structural maps are:

$$\begin{aligned} \Delta(\alpha \otimes \beta) &= \Delta(\alpha)_{13} \Delta(\beta)_{24} \\ \varepsilon(\alpha \otimes \beta) &= \varepsilon(\alpha) \varepsilon(\beta) \\ S(\alpha \otimes \beta) &= S(\alpha) S(\beta) \end{aligned}$$

The verification for the comultiplication is as follows:

$$\begin{aligned} \Delta(w_{ia,jb}) &= \Delta(u_{ij})_{13} \Delta(v_{ab})_{24} \\ &= \sum_{kc} u_{ik} \otimes v_{ac} \otimes u_{kj} \otimes v_{cb} \\ &= \sum_{kc} w_{ia,kc} \otimes w_{kc,jb} \end{aligned}$$

For the counit, we have:

$$\begin{aligned}\varepsilon(w_{ia,jb}) &= \varepsilon(u_{ij})\varepsilon(v_{ab}) \\ &= \delta_{ij}\delta_{ab} \\ &= \delta_{ia,jb}\end{aligned}$$

For the antipode, we have:

$$\begin{aligned}S(w_{ia,jb}) &= S(u_{ij})S(v_{ab}) \\ &= v_{ba}^*u_{ji}^* \\ &= (u_{ji}v_{ba})^* \\ &= w_{jb,ia}^*\end{aligned}$$

We refer to Wang's paper [93] for more details regarding this construction. \square

Let us call a closed quantum subgroup $G \subset U_N^+$ self-transpose when we have an automorphism $T : C(G) \rightarrow C(G)$ given by $T(u_{ij}) = u_{ji}$. Observe that in the classical case, this amounts in $G \subset U_N$ to be closed under the transposition operation $g \rightarrow g^t$.

With these notions in hand, let us go back to the affine homogeneous spaces. As a first result here, any closed subgroup $G \subset U_N^+$ appears as an affine homogeneous space over an appropriate quantum group, as follows:

THEOREM 8.17. *Given a closed subgroup $G \subset U_N^+$, we have an identification $X_{\mathcal{G},I}^{min} \simeq G$, given at the level of standard coordinates by $x_{ij} = \frac{1}{\sqrt{N}}u_{ij}$, where:*

- (1) $\mathcal{G} = G^t \times G \subset U_{N^2}^+$, with coordinates $w_{ia,jb} = u_{ji} \otimes u_{ab}$.
- (2) $I \subset \{1, \dots, N\}^2$ is the diagonal set, $I = \{(k, k) | k = 1, \dots, N\}$.

In the self-transpose case we can choose as well $\mathcal{G} = G \times G$, with $w_{ia,jb} = u_{ij} \otimes u_{ab}$.

PROOF. As a first observation, our closed subgroup $G \subset U_N^+$ appears as an algebraic submanifold of the free complex sphere on N^2 variables, as follows:

$$\begin{aligned}G &\subset S_{\mathbb{C},+}^{N^2-1} \\ x_{ij} &= \frac{1}{\sqrt{N}}u_{ij}\end{aligned}$$

Let us construct now the affine homogeneous space structure. Our claim is that, with $\mathcal{G} = G^t \times G$ and $I = \{(k, k)\}$ as in the statement, the structural maps are:

$$\begin{aligned}\alpha &= \Delta \\ \Phi &= (\Sigma \otimes id)\Delta^{(2)}\end{aligned}$$

Indeed, in what regards $\alpha = \Delta$, this is given by the following formula:

$$\begin{aligned}\alpha(u_{ij}) &= \sum_k u_{ik} \otimes u_{kj} \\ &= \sum_k w_{kk,ij}\end{aligned}$$

Thus, by dividing by \sqrt{N} , we obtain the usual affine homogeneous space formula:

$$\alpha(x_{ij}) = \frac{1}{\sqrt{|I|}} \sum_k w_{kk,ij}$$

Regarding now $\Phi = (\Sigma \otimes id)\Delta^{(2)}$, the formula here is as follows:

$$\begin{aligned}\Phi(u_{ij}) &= (\Sigma \otimes id) \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj} \\ &= \sum_{kl} u_{kl} \otimes u_{ik} \otimes u_{lj} \\ &= \sum_{kl} u_{kl} \otimes w_{kl,ij}\end{aligned}$$

Thus, by dividing by \sqrt{N} , we obtain the usual affine homogeneous space formula:

$$\Phi(x_{ij}) = \sum_{kl} x_{kl} \otimes w_{kl,ij}$$

The ergodicity condition being clear as well, this gives the first assertion.

Regarding now the second assertion, assume that we are in the self-transpose case, and so that we have an automorphism $T : C(G) \rightarrow C(G)$ given by $T(u_{ij}) = u_{ji}$.

With $w_{ia,jb} = u_{ij} \otimes u_{ab}$, the modified map $\alpha = (T \otimes id)\Delta$ is then given by:

$$\begin{aligned}\alpha(u_{ij}) &= (T \otimes id) \sum_k u_{ik} \otimes u_{kj} \\ &= \sum_k u_{ki} \otimes u_{kj} \\ &= \sum_k w_{kk,ij}\end{aligned}$$

As for the modified map $\Phi = (id \otimes T \otimes id)(\Sigma \otimes id)\Delta^{(2)}$, this is given by:

$$\begin{aligned}\Phi(u_{ij}) &= (id \otimes T \otimes id) \sum_{kl} u_{kl} \otimes u_{ik} \otimes u_{lj} \\ &= \sum_{kl} u_{kl} \otimes u_{ki} \otimes u_{lj} \\ &= \sum_{kl} u_{kl} \otimes w_{kl,ij}\end{aligned}$$

Thus we have the correct affine homogeneous space formulae, and once again the ergodicity condition being clear as well, this gives the result. \square

Let us discuss now the generalization of the above result, to the context of the spaces introduced in [32]. We recall from there that we have the following construction:

DEFINITION 8.18. *Given a closed subgroup $G \subset U_N^+$ and an integer $M \leq N$ we set*

$$C(G_{MN}) = \left\langle u_{ij} \mid i \in \{1, \dots, M\}, j \in \{1, \dots, N\} \right\rangle \subset C(G)$$

and we call row space of G the underlying quotient space $G \rightarrow G_{MN}$.

As a basic example here, at $M = N$ we obtain G itself. Also, at $M = 1$ we obtain the space whose coordinates are those on the first row of coordinates on G . See [32].

Given $G_N \subset U_N^+$ and an integer $M \leq N$, we can consider the quantum group $G_M = G_N \cap U_M^+$, with the intersection taken inside U_N^+ , and with $U_M^+ \subset U_N^+$ given by:

$$u = \text{diag}(v, 1_{N-M})$$

Observe that we have a quotient map $C(G_N) \rightarrow C(G_M)$, given by $u_{ij} \rightarrow v_{ij}$.

We have the following extension of Theorem 8.17:

THEOREM 8.19. *Given a closed subgroup $G_N \subset U_N^+$, we have an identification $X_{G,I}^{min} \simeq G_{MN}$, given at the level of standard coordinates by $x_{ij} = \frac{1}{\sqrt{M}}u_{ij}$, where:*

- (1) $\mathcal{G} = G_M^t \times G_N \subset U_{NM}^+$, where $G_M = G_N \cap U_M^+$, with coordinates $w_{ia,jb} = u_{ji} \otimes v_{ab}$.
- (2) $I \subset \{1, \dots, M\} \times \{1, \dots, N\}$ is the diagonal set, $I = \{(k, k) \mid k = 1, \dots, M\}$.

In the self-transpose case we can choose as well $\mathcal{G} = G_M \times G_N$, with $w_{ia,jb} = u_{ij} \otimes v_{ab}$.

PROOF. We will prove that the space $X = G_{MN}$, with coordinates $x_{ij} = \frac{1}{\sqrt{M}}u_{ij}$, coincides with the space $X_{G,I}^{min}$ constructed in the statement, with its standard coordinates.

For this purpose, consider the following composition of morphisms, where in the middle we have the comultiplication, and at left and right we have the canonical maps:

$$C(X) \subset C(G_N) \rightarrow C(G_N) \otimes C(G_N) \rightarrow C(G_M) \otimes C(G_N)$$

The standard coordinates are then mapped as follows:

$$\begin{aligned}
x_{ij} &= \frac{1}{\sqrt{M}} u_{ij} \\
&\rightarrow \frac{1}{\sqrt{M}} \sum_k u_{ik} \otimes u_{kj} \\
&\rightarrow \frac{1}{\sqrt{M}} \sum_{k \leq M} u_{ik} \otimes v_{kj} \\
&= \frac{1}{\sqrt{M}} \sum_{k \leq M} w_{kk,ij}
\end{aligned}$$

Thus we obtain the standard coordinates on the space $X_{G,I}^{min}$, as claimed. Finally, the last assertion is standard as well, by suitably modifying the above morphism. \square

8d. Integration results

Let us discuss now the liberation operation, in the context of the affine homogeneous spaces, and probabilistic aspects. In the easy case, we have the following result:

PROPOSITION 8.20. *When $G \subset U_N^+$ is easy, coming from a category of partitions D , the space $X_{G,I} \subset S_{\mathbb{C},+}^{N-1}$ appears by imposing the relations*

$$\sum_{i_1 \dots i_k} \delta_\pi(i_1 \dots i_k) x_{i_1}^{e_1} \dots x_{i_k}^{e_k} = |I|^{|\pi|-k/2}, \quad \forall k, \forall \pi \in D(k)$$

where $D(k) = D(0, k)$, and where $|\cdot|$ denotes the number of blocks.

PROOF. We know by easiness that $Fix(u^{\otimes k})$ is spanned by the vectors $\xi_\pi = T_\pi$, with $\pi \in D(k)$. But these latter vectors are given by:

$$\xi_\pi = \sum_{i_1 \dots i_k} \delta_\pi(i_1 \dots i_k) e_{i_1} \otimes \dots \otimes e_{i_k}$$

We deduce that $X_{G,I} \subset S_{\mathbb{C},+}^{N-1}$ appears by imposing the following relations:

$$\sum_{i_1 \dots i_k} \delta_\pi(i_1 \dots i_k) x_{i_1}^{e_1} \dots x_{i_k}^{e_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} \delta_\pi(j_1 \dots j_k), \quad \forall k, \forall \pi \in D(k)$$

Now since the sum on the right equals $|I|^{|\pi|}$, this gives the result. \square

More generally now, in view of the examples given above, making the link with [32], it is interesting to work out what happens when G is a product of easy quantum groups, and the index set I appears as $I = \{(c, \dots, c) | c \in J\}$, for a certain set J .

The result here, in its most general form, is as follows:

THEOREM 8.21. *For a product of easy quantum groups*

$$G = G_{N_1}^{(1)} \times \dots \times G_{N_s}^{(s)}$$

and with the index set used in the above being as follows

$$I = \{(c, \dots, c) | c \in J\}$$

the space $X_{G,I} \subset S_{\mathbb{C},+}^{N-1}$ appears by imposing the relations

$$\sum_{i_1 \dots i_k} \delta_\pi(i_1 \dots i_k) x_{i_1}^{e_1} \dots x_{i_k}^{e_k} = |J|^{|\pi_1 \vee \dots \vee \pi_s| - k/2}$$

for any $k \in \mathbb{N}$ and any partition of the following type

$$\pi \in D^{(1)}(k) \times \dots \times D^{(s)}(k)$$

where $D^{(r)} \subset P$ is the category of partitions associated to $G_{N_r}^{(r)} \subset U_{N_r}^+$, and where the partition

$$\pi_1 \vee \dots \vee \pi_s \in P(k)$$

is the one obtained by superposing π_1, \dots, π_s .

PROOF. Since we are in a direct product situation, $G = G_{N_1}^{(1)} \times \dots \times G_{N_s}^{(s)}$, the general theory in [93] applies, and shows that a basis for $Fix(u^{\otimes k})$ is provided by the vectors $\rho_\pi = \xi_{\pi_1} \otimes \dots \otimes \xi_{\pi_s}$ associated to the following partitions:

$$\pi = (\pi_1, \dots, \pi_s) \in D^{(1)}(k) \times \dots \times D^{(s)}(k)$$

We conclude that the space $X_{G,I} \subset S_{\mathbb{C},+}^{N-1}$ appears by imposing the following relations to the standard coordinates:

$$\sum_{i_1 \dots i_k} \delta_\pi(i_1 \dots i_k) x_{i_1}^{e_1} \dots x_{i_k}^{e_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \dots j_k \in I} \delta_\pi(j_1 \dots j_k), \quad \forall k, \forall \pi \in D^{(1)}(k) \times \dots \times D^{(s)}(k)$$

Since the conditions $j_1, \dots, j_k \in I$ read $j_1 = (l_1, \dots, l_1), \dots, j_k = (l_k, \dots, l_k)$, for certain elements $l_1, \dots, l_k \in J$, the sums on the right are given by:

$$\begin{aligned} \sum_{j_1 \dots j_k \in I} \delta_\pi(j_1 \dots j_k) &= \sum_{l_1 \dots l_k \in J} \delta_\pi(l_1, \dots, l_1, \dots, l_k, \dots, l_k) \\ &= \sum_{l_1 \dots l_k \in J} \delta_{\pi_1}(l_1 \dots l_k) \dots \delta_{\pi_s}(l_1 \dots l_k) \\ &= \sum_{l_1 \dots l_k \in J} \delta_{\pi_1 \vee \dots \vee \pi_s}(l_1 \dots l_k) \end{aligned}$$

Now since the sum on the right equals $|J|^{|\pi_1 \vee \dots \vee \pi_s|}$, this gives the result. \square

Finally, let us discuss probabilistic aspects. Following [10], we first have:

PROPOSITION 8.22. *The moments of the variable $\chi_T = \sum_{i \leq T} x_{i\dots i}$ are given by*

$$\int_X \chi_T^k \simeq \frac{1}{\sqrt{M^k}} \sum_{\pi \in D^{(1)}(k) \cap \dots \cap D^{(s)}(k)} \left(\frac{TM}{N} \right)^{|\pi|}$$

in the $N_i \rightarrow \infty$ limit, $\forall i$, where $M = |I|$, and $N = N_1 \dots N_s$.

PROOF. We have the following formula:

$$\pi(x_{i_1 \dots i_s}) = \frac{1}{\sqrt{M}} \sum_{c \in J} u_{i_1 c} \otimes \dots \otimes u_{i_s c}$$

For the variable in the statement, we therefore obtain:

$$\pi(\chi_T) = \frac{1}{\sqrt{M}} \sum_{i \leq T} \sum_{c \in J} u_{ic} \otimes \dots \otimes u_{ic}$$

Now by raising to the power k and integrating, we obtain:

$$\begin{aligned} \int_X \chi_T^k &= \frac{1}{\sqrt{M^k}} \sum_{i_1 \dots i_k \leq T} \sum_{c_1 \dots c_k \in J} \int_{G^{(1)}} u_{i_1 c_1} \dots u_{i_k c_k} \dots \int_{G^{(s)}} u_{i_1 c_1} \dots u_{i_k c_k} \\ &= \frac{1}{\sqrt{M^k}} \sum_{ic} \sum_{\pi\sigma} \delta_{\pi_1}(i) \delta_{\sigma_1}(c) W_{kN_1}^{(1)}(\pi_1, \sigma_1) \dots \delta_{\pi_s}(i) \delta_{\sigma_s}(c) W_{kN_s}^{(s)}(\pi_s, \sigma_s) \\ &= \frac{1}{\sqrt{M^k}} \sum_{\pi\sigma} T^{|\pi_1 \vee \dots \vee \pi_s|} M^{|\sigma_1 \vee \dots \vee \sigma_s|} W_{kN_1}^{(1)}(\pi_1, \sigma_1) \dots W_{kN_s}^{(s)}(\pi_s, \sigma_s) \end{aligned}$$

We use now the standard fact that the Weingarten functions are concentrated on the diagonal. Thus in the limit we must have $\pi_i = \sigma_i$ for any i , and we obtain:

$$\begin{aligned} \int_X \chi_T^k &\simeq \frac{1}{\sqrt{M^k}} \sum_{\pi} T^{|\pi_1 \vee \dots \vee \pi_s|} M^{|\pi_1 \vee \dots \vee \pi_s|} N_1^{-|\pi_1|} \dots N_s^{-|\pi_s|} \\ &\simeq \frac{1}{\sqrt{M^k}} \sum_{\pi \in D^{(1)} \cap \dots \cap D^{(s)}} T^{|\pi|} M^{|\pi|} (N_1 \dots N_s)^{-|\pi|} \\ &= \frac{1}{\sqrt{M^k}} \sum_{\pi \in D^{(1)} \cap \dots \cap D^{(s)}} \left(\frac{TM}{N} \right)^{|\pi|} \end{aligned}$$

But this gives the formula in the statement, and we are done. \square

As a consequence, we have the following result:

THEOREM 8.23. *In the context of a liberation operation for quantum groups*

$$G^{(i)} \rightarrow G^{(i)+}$$

the laws of the variables $\sqrt{M} \chi_T$ are in Bercovici-Pata bijection, in the $N_i \rightarrow \infty$ limit.

PROOF. Assume indeed that we have easy quantum groups $G^{(1)}, \dots, G^{(s)}$, with free versions $G^{(1)+}, \dots, G^{(s)+}$. At the level of the categories of partitions, we have:

$$\bigcap_i (D^{(i)} \cap NC) = \left(\bigcap_i D^{(i)} \right) \cap NC$$

Since the intersection of Hom-spaces is the Hom-space for the generated quantum group, we deduce that at the quantum group level, we have:

$$\langle G^{(1)+}, \dots, G^{(s)+} \rangle = \langle G^{(1)}, \dots, G^{(s)} \rangle^+$$

Thus the result follows from Proposition 8.22, and from the Bercovici-Pata bijection result for truncated characters for this latter liberation operation [33], [85]. \square

As a last topic, let us discuss now the axiomatization of the affine homogeneous spaces, as algebraic submanifolds of the free sphere $S_{\mathbb{C},+}^{N-1}$. We first have:

PROPOSITION 8.24. *Any affine homogeneous space $X_{G,I} \subset S_{\mathbb{C},+}^{N-1}$ is algebraic, with*

$$\sum_{i_1 \dots i_k} \xi_{i_1 \dots i_k} x_{i_1}^{e_1} \dots x_{i_k}^{e_k} = \frac{1}{\sqrt{|I|^k}} \sum_{b_1 \dots b_k \in I} \xi_{b_1 \dots b_k} \quad \forall k, \forall \xi \in \text{Fix}(u^{\otimes k})$$

as defining relations. Moreover, we can use vectors ξ belonging to a basis of $\text{Fix}(u^{\otimes k})$.

PROOF. This follows indeed from the various results above. \square

In order to reach to a more categorical description of $X_{G,I}$, the idea will be that of using Frobenius duality. We use colored indices, and we denote by $k \rightarrow \bar{k}$ the operation on the colored indices which consists in reversing the index, and switching all the colors. Also, we agree to identify the linear maps $T : (\mathbb{C}^N)^{\otimes k} \rightarrow (\mathbb{C}^N)^{\otimes l}$ with the corresponding rectangular matrices $T \in M_{N^l \times N^k}(\mathbb{C})$, written $T = (T_{i_1 \dots i_l, j_1 \dots j_k})$.

With these conventions, the precise formulation of Frobenius duality that we will need is as follows:

PROPOSITION 8.25. *We have an isomorphism of complex vector spaces*

$$T \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \longleftrightarrow \xi \in \text{Fix}(u^{\otimes l} \otimes u^{\otimes \bar{k}})$$

given by the following formulae:

$$\begin{aligned} T_{i_1 \dots i_l, j_1 \dots j_k} &= \xi_{i_1 \dots i_l, j_k \dots j_1} \\ \xi_{i_1 \dots i_l, j_1 \dots j_k} &= T_{i_1 \dots i_l, j_k \dots j_1} \end{aligned}$$

PROOF. This is a well-known result, which follows from the general theory in [99]. To be more precise, given integers $K, L \in \mathbb{N}$, consider the following standard isomorphism, which in matrix notation makes $T = (T_{IJ}) \in M_{L \times K}(\mathbb{C})$ correspond to $\xi = (\xi_{IJ})$:

$$T \in \mathcal{L}(\mathbb{C}^{\otimes K}, \mathbb{C}^{\otimes L}) \longleftrightarrow \xi \in \mathbb{C}^{\otimes L+K}$$

Given now two arbitrary corepresentations $v \in M_K(C(G))$ and $w \in M_L(C(G))$, the abstract Frobenius duality result established in [99] states that the above isomorphism restricts into an isomorphism of vector spaces, as follows:

$$T \in \text{Hom}(v, w) \longleftrightarrow \xi \in \text{Fix}(w \otimes \bar{v})$$

In our case, we can apply this result with $v = u^{\otimes k}$ and $w = u^{\otimes l}$. Since, according to our conventions, we have $\bar{v} = u^{\otimes \bar{k}}$, this gives the isomorphism in the statement. \square

With the above result in hand, we can enhance the construction of $X_{G,I}$, as follows:

THEOREM 8.26. *Any affine homogeneous space $X_{G,I}$ is algebraic, with*

$$\sum_{i_1 \dots i_l} \sum_{j_1 \dots j_k} T_{i_1 \dots i_l, j_1 \dots j_k} x_{i_1}^{e_1} \dots x_{i_l}^{e_l} (x_{j_1}^{f_1} \dots x_{j_k}^{f_k})^* = \frac{1}{\sqrt{|I|^{k+l}}} \sum_{b_1 \dots b_l \in I} \sum_{c_1 \dots c_k \in I} T_{b_1 \dots b_l, c_1 \dots c_k}$$

for any k, l , and any $T \in \text{Hom}(u^{\otimes k}, u^{\otimes l})$, as defining relations.

PROOF. We must prove that the relations in the statement are satisfied, over $X_{G,I}$. We know from Proposition 8.24 above, with $k \rightarrow lk$, that the following relation holds:

$$\sum_{i_1 \dots i_l} \sum_{j_1 \dots j_k} \xi_{i_1 \dots i_l, j_1 \dots j_k} x_{i_1}^{e_1} \dots x_{i_l}^{e_l} x_{j_1}^{\bar{f}_1} \dots x_{j_k}^{\bar{f}_k} = \frac{1}{\sqrt{|I|^{k+l}}} \sum_{b_1 \dots b_l \in I} \sum_{c_1 \dots c_k \in I} \xi_{b_1 \dots b_l, c_1 \dots c_k}$$

In terms of the matrix $T_{i_1 \dots i_l, j_1 \dots j_k} = \xi_{i_1 \dots i_l, j_1 \dots j_k}$ from Proposition 8.25, we obtain:

$$\sum_{i_1 \dots i_l} \sum_{j_1 \dots j_k} T_{i_1 \dots i_l, j_1 \dots j_k} x_{i_1}^{e_1} \dots x_{i_l}^{e_l} x_{j_1}^{\bar{f}_1} \dots x_{j_k}^{\bar{f}_k} = \frac{1}{\sqrt{|I|^{k+l}}} \sum_{b_1 \dots b_l \in I} \sum_{c_1 \dots c_k \in I} T_{b_1 \dots b_l, c_1 \dots c_k}$$

But this gives the formula in the statement, and we are done. \square

The above results suggest the following notion:

DEFINITION 8.27. *Given a submanifold $X \subset S_{\mathbb{C},+}^{N-1}$ and a subset $I \subset \{1, \dots, N\}$, we say that X is I -affine when $C(X)$ is presented by relations of type*

$$\sum_{i_1 \dots i_l} \sum_{j_1 \dots j_k} T_{i_1 \dots i_l, j_1 \dots j_k} x_{i_1}^{e_1} \dots x_{i_l}^{e_l} (x_{j_1}^{f_1} \dots x_{j_k}^{f_k})^* = \frac{1}{\sqrt{|I|^{k+l}}} \sum_{b_1 \dots b_l \in I} \sum_{c_1 \dots c_k \in I} T_{b_1 \dots b_l, c_1 \dots c_k}$$

with the operators T belonging to certain linear spaces

$$F(k, l) \subset M_{N^l \times N^k}(\mathbb{C})$$

which altogether form a tensor category $F = (F(k, l))$.

Let us study the quantum isometry groups of the manifolds $X \subset S_{\mathbb{C},+}^{N-1}$ which are I -affine. We have here the following result:

PROPOSITION 8.28. *For an I -affine manifold $X \subset S_{\mathbb{C},+}^{N-1}$ we have*

$$G \subset G^+(X)$$

where $G \subset U_N^+$ is the Tannakian dual of the associated tensor category F .

PROOF. We recall that the relations defining $G^+(X)$ are those expressing the vanishing of the following quantities:

$$P(X_1, \dots, X_N) = \sum_r \alpha_r \sum_{j_1^r \dots j_{s(r)}^r} u_{i_1^r j_1^r} \dots u_{i_{s(r)}^r j_{s(r)}^r} \otimes x_{j_1^r} \dots x_{j_{s(r)}^r}$$

In the case of an I -affine manifold, the vanishing of these relations corresponds precisely to the Tannakian relations defining $G \subset U_N^+$, and so $G \subset G^+(X)$, as claimed. \square

We have now all the needed ingredients, and we can prove:

THEOREM 8.29. *Assuming that an algebraic manifold $X \subset S_{\mathbb{C},+}^{N-1}$ is I -affine, with associated tensor category F , the following happen:*

- (1) *We have an inclusion $G \subset G^+(X)$, where G is the Tannakian dual of F .*
- (2) *X is an affine homogeneous space, $X = X_{G,I}$, over this quantum group G .*

PROOF. We have to construct the affine space morphisms α, Φ , and the proof goes as follows:

- (1) Construction of α . We want to construct a morphism, as follows:

$$\alpha : C(X) \rightarrow C(G) \quad : \quad x_i \rightarrow X_i = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ij}$$

We must therefore prove that we have:

$$\sum_{i_1 \dots i_l} \sum_{j_1 \dots j_k} T_{i_1 \dots i_l, j_1 \dots j_k} X_{i_1}^{e_1} \dots X_{i_l}^{e_l} (X_{j_1}^{f_1} \dots X_{j_k}^{f_k})^* = \frac{1}{\sqrt{|I|^{k+l}}} \sum_{b_1 \dots b_l \in I} \sum_{c_1 \dots c_k \in I} T_{b_1 \dots b_l, c_1 \dots c_k}$$

By replacing the variables X_i by their above values, we want to prove that:

$$\sum_{i_1 \dots i_l} \sum_{j_1 \dots j_k} \sum_{r_1 \dots r_l \in I} \sum_{s_1 \dots s_k \in I} T_{i_1 \dots i_l, j_1 \dots j_k} u_{i_1 r_1}^{e_1} \dots u_{i_l r_l}^{e_l} (u_{j_1 s_1}^{f_1} \dots u_{j_k s_k}^{f_k})^* = \sum_{b_1 \dots b_l \in I} \sum_{c_1 \dots c_k \in I} T_{b_1 \dots b_l, c_1 \dots c_k}$$

Now observe that from the relation $T \in \text{Hom}(u^{\otimes k}, u^{\otimes l})$ we obtain:

$$\sum_{i_1 \dots i_l} \sum_{j_1 \dots j_k} T_{i_1 \dots i_l, j_1 \dots j_k} u_{i_1 r_1}^{e_1} \dots u_{i_l r_l}^{e_l} (u_{j_1 s_1}^{f_1} \dots u_{j_k s_k}^{f_k})^* = T_{r_1 \dots r_l, s_1 \dots s_k}$$

Thus, by summing over indices $r_i \in I$ and $s_i \in I$, we obtain the desired formula.

- (2) Construction of Φ . We want to construct a morphism, as follows:

$$\Phi : C(X) \rightarrow C(G) \otimes C(X) \quad : \quad x_i \rightarrow X_i = \sum_j u_{ij} \otimes x_j$$

But this is precisely the coaction map constructed before.

(3) Proof of the ergodicity. The ergodicity condition is equivalent to a number of Tannakian conditions, which are automatic in our case. \square

The work in [29] suggests that the relevant category F should appear in a more direct way from X . Let us formulate:

DEFINITION 8.30. *Given a submanifold $X \subset S_{\mathbb{C},+}^{N-1}$ and a subset $I \subset \{1, \dots, N\}$, we let $F_{X,I}(k, l) \subset M_{N^l \times N^k}(\mathbb{C})$ be the linear space of linear maps T such that*

$$\sum_{i_1 \dots i_l} \sum_{j_1 \dots j_k} T_{i_1 \dots i_l, j_1 \dots j_k} x_{i_1}^{e_1} \dots x_{i_l}^{e_l} (x_{j_1}^{f_1} \dots x_{j_k}^{f_k})^* = \frac{1}{\sqrt{|I|^{k+l}}} \sum_{b_1 \dots b_l \in I} \sum_{c_1 \dots c_k \in I} T_{b_1 \dots b_l, c_1 \dots c_k}$$

holds over X . We say that X is I -saturated when

$$F_{X,I} = (F_{X,I}(k, l))$$

is a tensor category, and the collection of the above relations presents the algebra $C(X)$.

Observe that any I -saturated manifold is automatically I -affine. The point is that the results in [29] seem to suggest that the converse of this fact should hold, in the sense that any I -affine manifold should be automatically I -saturated. Such a result would of course substantially improve Theorem 8.29 above, and make it ready for applications.

We do not have a proof of this fact, but we would like to present now a few preliminary observations on this subject. First of all, we have the following result:

PROPOSITION 8.31. *The linear spaces $F_{X,I}(k, l) \subset M_{N^l \times N^k}(\mathbb{C})$ constructed above have the following properties:*

- (1) *They contain the units.*
- (2) *They are stable by conjugation.*
- (3) *They satisfy the Frobenius duality condition.*

PROOF. All these assertions are elementary, as follows:

(1) Consider indeed the unit map. The associated relation is:

$$\sum_{i_1 \dots i_k} x_{i_1}^{e_1} \dots x_{i_k}^{e_k} (x_{i_1}^{e_1} \dots x_{i_k}^{e_k})^* = 1$$

But this relation holds indeed, due to the defining relations for $S_{\mathbb{C},+}^{N-1}$.

(2) We have indeed the following sequence of equivalences:

$$\begin{aligned}
& T^* \in F_{X,I}(l, k) \\
\iff & \sum_{i_1 \dots i_l} \sum_{j_1 \dots j_k} T_{j_1 \dots j_k, i_1 \dots i_l}^* x_{j_1}^{f_1} \dots x_{j_k}^{f_k} (x_{i_1}^{e_1} \dots x_{i_l}^{e_l})^* = \frac{1}{\sqrt{|I|^{k+l}}} \sum_{b_1 \dots b_l \in I} \sum_{c_1 \dots c_k \in I} T_{c_1 \dots c_k, b_1 \dots b_l}^* \\
\iff & \sum_{i_1 \dots i_l} \sum_{j_1 \dots j_k} T_{i_1 \dots i_l, j_1 \dots j_k} x_{i_1}^{e_1} \dots x_{i_l}^{e_l} (x_{j_1}^{f_1} \dots x_{j_k}^{f_k})^* = \frac{1}{\sqrt{|I|^{k+l}}} \sum_{b_1 \dots b_l \in I} \sum_{c_1 \dots c_k \in I} T_{b_1 \dots b_l, c_1 \dots c_k} \\
\iff & T \in F_{X,I}(k, l)
\end{aligned}$$

(3) We have indeed a correspondence $T \in F_{X,I}(k, l) \longleftrightarrow \xi \in F_{X,I}(\emptyset, l\bar{k})$, given by the usual formulae for the Frobenius isomorphism. \square

Based on the above result, we can now formulate our observations, as follows:

THEOREM 8.32. *Given a closed subgroup $G \subset U_N^+$, and an index set $I \subset \{1, \dots, N\}$, consider the corresponding affine homogeneous space $X_{G,I} \subset S_{\mathbb{C},+}^{N-1}$.*

(1) $X_{G,I}$ is I -saturated precisely when the collection of spaces

$$F_{X,I} = (F_{X,I}(k, l))$$

is stable under compositions, and under tensor products.

(2) We have $F_{X,I} = F$ precisely when the relation

$$\sum_{j_1 \dots j_l \in I} \left(\sum_{i_1 \dots i_l} \xi_{i_1 \dots i_l} u_{i_1 j_1}^{e_1} \dots u_{i_l j_l}^{e_l} - \xi_{j_1 \dots j_l} \right) = 0$$

implies the relation

$$\sum_{i_1 \dots i_l} \xi_{i_1 \dots i_l} u_{i_1 j_1}^{e_1} \dots u_{i_l j_l}^{e_l} - \xi_{j_1 \dots j_l} = 0$$

for any j_1, \dots, j_l .

PROOF. With $F(k, l) = \text{Hom}(u^{\otimes k}, u^{\otimes l})$, we have inclusions of vector spaces, as follows:

$$F(k, l) \subset F_{X,I}(k, l)$$

Moreover, the relations coming from the elements of the category formed by the spaces $F(k, l)$ present $X_{G,I}$. Thus, the relations coming from the elements of $F_{X,I}$ present $X_{G,I}$ as well. With this in hand, our assertions follow from Proposition 8.31:

(1) According to Proposition 8.31 (1) and (2) the unit and conjugation axioms are satisfied, so the spaces $F_{X,I}(k, l)$ form a tensor category precisely when the remaining axioms, namely the composition and the tensor product one, are satisfied. Now by assuming that these two axioms are satisfied, X follows to be I -saturated, by the above observation.

(2) Since we already have inclusions in one sense, the equality $F_{X,I} = F$ from the statement means that we must have inclusions in the other sense, as follows:

$$F_{X,I}(k, l) \subset F(k, l)$$

By using now Proposition 8.31 (3), it is enough to discuss the case $k = 0$. And here, assuming that we have $\xi \in F_{X,L}(0, l)$, the following condition must be satisfied:

$$\sum_{i_1 \dots i_l} \xi_{i_1 \dots i_l} x_{i_1}^{e_1} \dots x_{i_l}^{e_l} = \sum_{j_1 \dots j_l \in I} \xi_{j_1 \dots j_l}$$

By applying now the morphism $\alpha : C(X_{G,I}) \rightarrow C(G)$, we deduce that we have:

$$\sum_{i_1 \dots i_l} \xi_{i_1 \dots i_l} \sum_{j_1 \dots j_l \in I} u_{i_1 j_1}^{e_1} \dots u_{i_l j_l}^{e_l} = \sum_{j_1 \dots j_l \in I} \xi_{j_1 \dots j_l}$$

Now recall that $F(0, l) = \text{Fix}(u^{\otimes l})$ consists of the vectors ξ satisfying:

$$\sum_{i_1 \dots i_l} \xi_{i_1 \dots i_l} u_{i_1 j_1}^{e_1} \dots u_{i_l j_l}^{e_l} = \xi_{j_1 \dots j_l}, \forall j_1, \dots, j_l$$

We are therefore led to the conclusion in the statement. □

8e. Exercises

Things got fairly complicated in this chapter, basically leading to hot research questions, and as a unique exercise on all this, in the same spirit, we have:

EXERCISE 8.33. *Axiomatize the free manifolds, as a continuation of the above.*

There is of course no need of completely solving this exercise, and some preliminary study, for some very simple classes of manifolds, of your choice, will do.

Part III

Intermediate geometries

*Give my love to Rose please, won't you mister
Take her all my money, tell her to buy some pretty clothes
Tell my boy that daddy's so proud of him
And don't forget to give my love to Rose*

CHAPTER 9

Half-liberation

9a. Spheres and tori

We have seen in chapter 4 that the quadruplets of type (S, T, U, K) can be axiomatized, and that at the level of basic examples we have 4 such quadruplets, corresponding to the usual real and complex geometries $\mathbb{R}^N, \mathbb{C}^N$, and to the free versions of these:

$$\begin{array}{ccc}
 \mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\
 \uparrow & & \uparrow \\
 \mathbb{R}^N & \longrightarrow & \mathbb{C}^N
 \end{array}$$

Our purpose in what follows will be that of extending the above diagram, with the construction of some supplementary examples. There are two methods here:

- (1) Look for intermediate geometries $\mathbb{R}^N \subset X \subset \mathbb{R}_+^N$, and their complex analogues.
- (2) Look for intermediate geometries $\mathbb{R}^N \subset X \subset \mathbb{C}^N$, and their free analogues.

We will see that, in each case, there is a “standard” solution, and that these solutions can be combined. Thus, we will end up with a total of $3 \times 3 = 9$ solutions, as follows:

$$\begin{array}{ccccc}
 \mathbb{R}_+^N & \longrightarrow & \text{TR}_+^N & \longrightarrow & \mathbb{C}_+^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{R}_*^N & \longrightarrow & \text{TR}_*^N & \longrightarrow & \mathbb{C}_*^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{R}^N & \longrightarrow & \text{TR}^N & \longrightarrow & \mathbb{C}^N
 \end{array}$$

We will see afterwards, in chapter 10 below, that under certain strong axioms, of combinatorial type, these 9 geometries are conjecturally the only ones.

Let us focus on the first question to be solved, namely finding the intermediate geometries $\mathbb{R}^N \subset X \subset \mathbb{R}_+^N$. Since such a geometry is given by a quadruplet (S, T, U, K) , we are led to 4 different intermediate object questions, as follows:

$$S_{\mathbb{R}}^{N-1} \subset S \subset S_{\mathbb{R},+}^{N-1}$$

$$T_N \subset T \subset T_N^+$$

$$O_N \subset U \subset O_N^+$$

$$H_N \subset K \subset H_N^+$$

At the sphere and torus level, there are obviously uncountably many solutions, without supplementary assumptions, and it is hard to get beyond this, with bare hands. Thus, our hopes will basically come from the unitary and reflection quantum groups, where things are more rigid than for spheres and tori.

Let us record, however, the following fact regarding the spheres, from [29], which will appear to be relevant, later on:

THEOREM 9.1. *The algebraic manifold $S^{(k)} \subset S_{\mathbb{R},+}^{N-1}$ obtained by imposing the relations $a_1 \dots a_k = a_k \dots a_1$ to the standard coordinates of $S_{\mathbb{R},+}^{N-1}$ is as follows:*

- (1) *At $k = 1$ we have $S^{(k)} = S_{\mathbb{R},+}^{N-1}$.*
- (2) *At $k = 2, 4, 6, \dots$ we have $S^{(k)} = S_{\mathbb{R}}^{N-1}$.*
- (3) *At $k = 3, 5, 7, \dots$ we have $S^{(k)} = S^{(3)}$.*

PROOF. As a first observation, the commutation relations $ab = ba$ imply the following relations, for any $k \geq 2$:

$$a_1 \dots a_k = a_k \dots a_1$$

Thus, for any $k \geq 2$, we have an inclusion $S^{(2)} \subset S^{(k)}$. It is also elementary to check that the relations $abc = cba$ imply the following relations, for any $k \geq 3$ odd:

$$a_1 \dots a_k = a_k \dots a_1$$

Thus, for any $k \geq 3$ odd, we have an inclusion $S^{(3)} \subset S^{(k)}$. Our claim now is that we have an inclusion as follows, for any $k \geq 2$:

$$S^{(k+2)} \subset S^{(k)}$$

In order to prove this, we must show that the relations $a_1 \dots a_{k+2} = a_{k+2} \dots a_1$ between x_1, \dots, x_N imply the relations $a_1 \dots a_k = a_k \dots a_1$ between x_1, \dots, x_N . But this holds

indeed, because of the following implications:

$$\begin{aligned} & x_{i_1} \dots x_{i_{k+2}} = x_{i_{k+2}} \dots x_{i_1} \\ \implies & x_{i_1} \dots x_{i_k} x_j^2 = x_j^2 x_{i_k} \dots x_{i_1} \\ \implies & \sum_j x_{i_1} \dots x_{i_k} x_j^2 = \sum_j x_j^2 x_{i_k} \dots x_{i_1} \\ \implies & x_{i_1} \dots x_{i_k} = x_{i_k} \dots x_{i_1} \end{aligned}$$

Summing up, we have proved that we have inclusions as follows:

$$\begin{aligned} S^{(2)} & \subset \dots \subset S^{(6)} \subset S^{(4)} \subset S^{(2)} \\ S^{(3)} & \subset \dots \subset S^{(7)} \subset S^{(5)} \subset S^{(3)} \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

As a conclusion, the sphere $S^{(3)}$, obtained via the relations $abc = cba$, might be the “privileged” intermediate sphere $S_{\mathbb{R}}^{N-1} \subset S \subset S_{\mathbb{R},+}^{N-1}$ that we are looking for. It is possible to go further in this direction, with a study of the spheres given by relations of type:

$$a_1 \dots a_k = a_{\sigma(1)} \dots a_{\sigma(k)}$$

But this leads to a similar conclusion, namely that the sphere $S^{(3)}$ constructed above is the only new solution. We will discuss this later, in chapter 13 below. All this remains, however, quite ad-hoc. In short, we have constructed so far a new real sphere, $S^{(3)}$, and we have some evidence for the fact that this sphere might be the only new one, under some extra combinatorial axioms, which look quite hard to find.

9b. Quantum groups

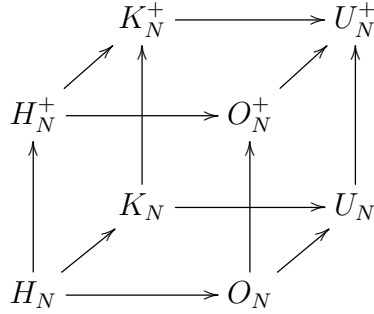
Let us focus now on the quantum groups. We will see that there is a lot more rigidity here, which makes things simpler. At the quantum group level, our goal will be that of finding the intermediate objects as follows:

$$\begin{aligned} O_N & \subset U \subset O_N^+ \\ H_N & \subset K \subset H_N^+ \end{aligned}$$

Quite surprisingly, these two questions are of quite different nature. Indeed, regarding $O_N \subset U \subset O_N^+$, there is a solution here, denoted O_N^* , coming via the relations $abc = cba$, and conjecturally nothing more. Regarding $H_N \subset K \subset H_N^+$, here it is possible to use for instance crossed products, in order to construct uncountably many solutions.

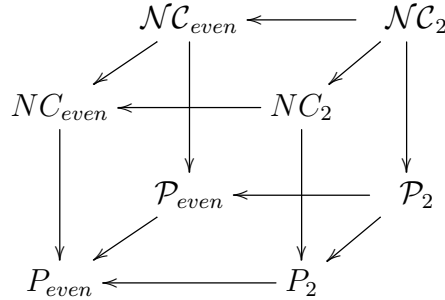
In short, in connection with our intermediate geometry question, we do have in principle our solution, coming via the relations $abc = cba$, and this is compatible with our above $S^{(3)}$ guess for the spheres. In order to get started, let us recall that we have:

THEOREM 9.2. *The basic quantum unitary and reflection groups, namely*



are all easy, coming from certain categories of partitions.

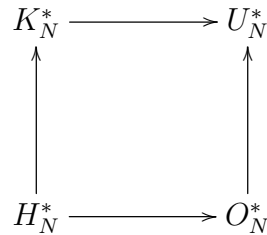
PROOF. This is something that we already discussed, in section 2 above, the corresponding categories of partitions being as follows:



Thus, we are led to the conclusion in the statement. □

Getting back now to the half-liberation question, let us start by constructing the solutions. The result here, which is well-known as well, is as follows:

THEOREM 9.3. *We have quantum groups as follows, obtained via the half-commutation relations $abc = cba$, which fit into the diagram of basic quantum groups:*



These quantum groups are all easy, and the corresponding categories of partitions fit into the diagram of categories of partitions for the basic quantum groups.

PROOF. This is standard, from [33], [34], the idea being that the half-commutation relations $abc = cba$ come from the operator T_* associated to the half-liberating partition:

$$* \in P(3, 3)$$

Thus, the quantum groups in the statement are indeed easy, obtained by adding $*$ to the corresponding categories of noncrossing partitions. We obtain the following categories, with $*$ standing for the fact that, when relabelling clockwise the legs $\circ \bullet \circ \bullet \dots$, the formula $\# \circ = \# \bullet$ must hold in each block:

$$\begin{array}{ccc} \mathcal{P}_{even}^* & \longleftarrow & \mathcal{P}_2^* \\ \downarrow & & \downarrow \\ \mathcal{P}_{even}^* & \longleftarrow & \mathcal{P}_2^* \end{array}$$

Finally, the fact that our new quantum groups and categories fit well into the previous diagrams of quantum groups and categories is clear from this. See [12]. \square

The point now is that we have the following result, from [34]:

THEOREM 9.4. *There is only one proper intermediate easy quantum group*

$$O_N \subset G \subset O_N^+$$

namely the half-classical orthogonal group O_N^ .*

PROOF. According to our definition for the easy quantum groups, we must compute here the intermediate categories of pairings, as follows:

$$NC_2 \subset D \subset P_2$$

But this can be done via some standard combinatorics, in three steps, as follows:

(1) Let $\pi \in P_2 - NC_2$, having $s \geq 4$ strings. Our claim is that:

- If $\pi \in P_2 - P_2^*$, there exists a semicircle capping $\pi' \in P_2 - P_2^*$.
- If $\pi \in P_2^* - NC_2$, there exists a semicircle capping $\pi' \in P_2^* - NC_2$.

Indeed, both these assertions can be easily proved, by drawing pictures.

(2) Consider now a partition $\pi \in P_2(k, l) - NC_2(k, l)$. Our claim is that:

- If $\pi \in P_2(k, l) - P_2^*(k, l)$ then $\langle \pi \rangle = P_2$.
- If $\pi \in P_2^*(k, l) - NC_2(k, l)$ then $\langle \pi \rangle = P_2^*$.

This can be indeed proved by recurrence on the number of strings, $s = (k + l)/2$, by using (1), which provides us with a descent procedure $s \rightarrow s - 1$, at any $s \geq 4$.

(3) Finally, assume that we are given an easy quantum group $O_N \subset G \subset O_N^+$, coming from certain sets of pairings $D(k, l) \subset P_2(k, l)$. We have three cases:

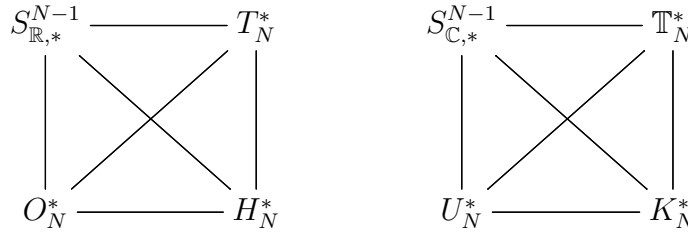
- If $D \not\subset P_2^*$, we obtain $G = O_N$.
- If $D \subset P_2, D \not\subset NC_2$, we obtain $G = O_N^*$.
- If $D \subset NC_2$, we obtain $G = O_N^+$.

Thus, we are led to the conclusion in the statement. □

It is actually believed that the above result could still hold, without the easiness assumption. We refer here to [19]. Thus, under a certain natural “easiness” assumption, and perhaps even in general, we can only have an intermediate geometry between classical real and free real, namely half-classical real.

In practice now, what we have to do is to construct this geometry, and its complex analogue as well, and check the axioms from section 4. Let us begin by constructing the corresponding quadruplets. We have the following result:

PROPOSITION 9.5. *We half-classical real and complex quadruplets, as follows,*



obtained by imposing to the standard coordinates the relations $abc = cba$.

PROOF. This is more or less an empty statement, with the quantum groups appearing in the above diagrams being those constructed above, and with the corresponding spheres and tori being constructed in a similar way, by imposing the half-commutation relations $abc = cba$ to the standard coordinates, and their adjoints. □

9c. Matrix models

In order to check now our noncommutative geometry axioms, we are in need of a better understanding of the half-liberation operation, via some supplementary results. Let us start with the following simple observation, regarding the real spheres:

PROPOSITION 9.6. *We have a morphism of C^* -algebras as follows,*

$$C(S_{\mathbb{R},*}^{N-1}) \rightarrow M_2(C(S_{\mathbb{C}}^{N-1})) \quad , \quad x_i \rightarrow \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

where z_i are the standard coordinates of $S_{\mathbb{C}}^{N-1}$.

PROOF. We have to prove that the matrices X_i on the right satisfy the defining relations for $S_{\mathbb{R},*}^{N-1}$. But these matrices are self-adjoint, and we have:

$$\begin{aligned} \sum_i X_i^2 &= \sum_i \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}^2 \\ &= \sum_i \begin{pmatrix} |z_i|^2 & 0 \\ 0 & |z_i|^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

As for the half-commutation relations, these follow from the following formula:

$$\begin{aligned} X_i X_j X_k &= \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix} \begin{pmatrix} 0 & z_j \\ \bar{z}_j & 0 \end{pmatrix} \begin{pmatrix} 0 & z_k \\ \bar{z}_k & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & z_i \bar{z}_j z_k \\ \bar{z}_i z_j \bar{z}_k & 0 \end{pmatrix} \end{aligned}$$

Indeed, the quantities on the right being symmetric in i, k , this gives the result. \square

Regarding the complex spheres, the result here is similar, as follows:

PROPOSITION 9.7. *We have a morphism of C^* -algebras as follows,*

$$C(S_{\mathbb{C},*}^{N-1}) \rightarrow M_2(C(S_{\mathbb{C}}^{N-1} \times S_{\mathbb{C}}^{N-1})) \quad , \quad x_i \rightarrow \begin{pmatrix} 0 & z_i \\ y_i & 0 \end{pmatrix}$$

where y_i, z_i are the standard coordinates of $S_{\mathbb{C}}^{N-1} \times S_{\mathbb{C}}^{N-1}$.

PROOF. We have to prove that the matrices X_i on the right satisfy the defining relations for $S_{\mathbb{C},*}^{N-1}$. We have the following computation:

$$\begin{aligned} \sum_i X_i X_i^* &= \sum_i \begin{pmatrix} 0 & z_i \\ y_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \bar{y}_i \\ \bar{z}_i & 0 \end{pmatrix} \\ &= \sum_i \begin{pmatrix} |z_i|^2 & 0 \\ 0 & |y_i|^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

We have as well the following computation:

$$\begin{aligned} \sum_i X_i^* X_i &= \sum_i \begin{pmatrix} 0 & \bar{y}_i \\ \bar{z}_i & 0 \end{pmatrix} \begin{pmatrix} 0 & z_i \\ y_i & 0 \end{pmatrix} \\ &= \sum_i \begin{pmatrix} |y_i|^2 & 0 \\ 0 & |z_i|^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

As for the half-commutation relations, these follow from the following formula:

$$\begin{aligned} X_i X_j X_k &= \begin{pmatrix} 0 & z_i \\ y_i & 0 \end{pmatrix} \begin{pmatrix} 0 & z_j \\ y_j & 0 \end{pmatrix} \begin{pmatrix} 0 & z_k \\ y_k & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & z_i y_j z_k \\ y_i z_j y_k & 0 \end{pmatrix} \end{aligned}$$

Indeed, the quantities on the right being symmetric in i, k , this gives the result. \square

Our goal now will be that of proving that the morphisms constructed above are faithful, up to the usual equivalence relation for the quantum algebraic manifolds. For this purpose, we will use some projective geometry arguments, the idea being that of proving that the above morphisms are indeed isomorphisms, at the projective version level, and then lifting these isomorphism results, to the affine setting.

We recall that $P_{\mathbb{R}}^{N-1}$ is the space of lines in \mathbb{R}^N passing through the origin. We have a quotient map $S_{\mathbb{R}}^{N-1} \rightarrow P_{\mathbb{R}}^{N-1}$, which produces an embedding $C(P_{\mathbb{R}}^{N-1}) \subset C(S_{\mathbb{R}}^{N-1})$, and the image of this embedding is the algebra generated by the variables $p_{ij} = x_i x_j$.

The complex projective space $P_{\mathbb{C}}^{N-1}$ has a similar description, and we have an embedding $C(P_{\mathbb{C}}^{N-1}) \subset C(S_{\mathbb{C}}^{N-1})$, whose image is generated by the variables $p_{ij} = x_i \bar{x}_j$.

The spaces $P_{\mathbb{R}}^{N-1}, P_{\mathbb{C}}^{N-1}$ have the following functional analytic description:

THEOREM 9.8. *We have presentation results as follows,*

$$\begin{aligned} C(P_{\mathbb{C}}^{N-1}) &= C_{comm}^* \left((p_{ij})_{i,j=1,\dots,N} \mid p = p^* = p^2, Tr(p) = 1 \right) \\ C(P_{\mathbb{R}}^{N-1}) &= C_{comm}^* \left((p_{ij})_{i,j=1,\dots,N} \mid p = \bar{p} = p^* = p^2, Tr(p) = 1 \right) \end{aligned}$$

where by C_{comm}^* we mean as usual universal commutative C^* -algebra.

PROOF. We use the elementary fact that the spaces $P_{\mathbb{C}}^{N-1}, P_{\mathbb{R}}^{N-1}$, as defined above, are respectively the spaces of rank one projections in $M_N(\mathbb{C}), M_N(\mathbb{R})$. With this picture in mind, it is clear that we have arrows \leftarrow .

In order to construct now arrows \rightarrow , consider the universal algebras on the right, A_C, A_R . These algebras being both commutative, by the Gelfand theorem we can write, with X_C, X_R being certain compact spaces:

$$A_C = C(X_C)$$

$$A_R = C(X_R)$$

Now by using the coordinate functions p_{ij} , we conclude that X_C, X_R are certain spaces of rank one projections in $M_N(\mathbb{C}), M_N(\mathbb{R})$. In other words, we have embeddings:

$$X_C \subset P_{\mathbb{C}}^{N-1}$$

$$X_R \subset P_{\mathbb{R}}^{N-1}$$

By transposing we obtain arrows \rightarrow , as desired. \square

The above result suggests constructing free projective spaces $P_{\mathbb{R},+}^{N-1}, P_{\mathbb{C},+}^{N-1}$, simply by lifting the commutativity conditions between the variables p_{ij} . However, there is something wrong with this, and more specifically with $P_{\mathbb{R},+}^{N-1}$, coming from the fact that if certain noncommutative coordinates x_1, \dots, x_N are self-adjoint, then the corresponding projective coordinates $p_{ij} = x_i x_j$ are not necessarily self-adjoint:

$$x_i = x_i^* \not\Rightarrow x_i x_j = (x_i x_j)^*$$

In short, our attempt to construct free projective spaces $P_{\mathbb{R},+}^{N-1}, P_{\mathbb{C},+}^{N-1}$ as above is not exactly correct, with the space $P_{\mathbb{R},+}^{N-1}$ being rather “irrelevant”, and with the space $P_{\mathbb{C},+}^{N-1}$ being probably the good one, but being at the same time “real and complex”. Observe that there is some similarity here with the following key result, from section 4 above:

$$PO_N^+ = PU_N^+$$

To be more precise, we have good evidence here for the fact that, in the free setting, the projective geometry is at the same time real and complex.

In view of all this, let us formulate the following definition:

DEFINITION 9.9. *Associated to any $N \in \mathbb{N}$ is the following universal algebra,*

$$C(P_+^{N-1}) = C^* \left((p_{ij})_{i,j=1,\dots,N} \mid p = p^* = p^2, \text{Tr}(p) = 1 \right)$$

whose abstract spectrum is called “free projective space”.

Observe that we have embeddings of noncommutative spaces, as follows:

$$P_{\mathbb{R}}^{N-1} \subset P_{\mathbb{C}}^{N-1} \subset P_+^{N-1}$$

Let us compute now the projective versions of the noncommutative spheres that we have, including the half-classical ones. We use the following formalism here:

DEFINITION 9.10. *The projective version of $S \subset S_{\mathbb{C},+}^{N-1}$ is the quotient space $S \rightarrow PS$ determined by the fact that*

$$C(PS) \subset C(S)$$

is the subalgebra generated by $p_{ij} = x_i x_j^$, called projective coordinates.*

In the classical case, this fits with the usual definition. We will be back with more details in section 15 below, which is dedicated to the study of projective geometry. We have the following result, coming from [4], [28], [29]:

THEOREM 9.11. *The projective versions of the basic spheres are as follows,*

$$\begin{array}{ccc}
 S_{\mathbb{R},+}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\
 \uparrow & & \uparrow \\
 S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{C},*}^{N-1} \\
 \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{C}}^{N-1}
 \end{array}
 \quad \rightarrow \quad
 \begin{array}{ccc}
 P_+^{N-1} & \longrightarrow & P_+^{N-1} \\
 \uparrow & & \uparrow \\
 P_{\mathbb{C}}^{N-1} & \longrightarrow & P_{\mathbb{C}}^{N-1} \\
 \uparrow & & \uparrow \\
 P_{\mathbb{R}}^{N-1} & \longrightarrow & P_{\mathbb{C}}^{N-1}
 \end{array}$$

modulo, in the free case, a GNS construction with respect to the uniform integration.

PROOF. The formulae on the bottom are true by definition. For the formulae on top, we have to prove first that the variables $p_{ij} = x_i x_j^*$ over the free sphere $S_{\mathbb{C},+}^{N-1}$ satisfy the defining relations for $C(P_+^{N-1})$. We first have:

$$\begin{aligned}
 (p^*)_{ij} &= p_{ji}^* \\
 &= (x_j x_i^*)^* \\
 &= x_i x_j^* \\
 &= p_{ij}
 \end{aligned}$$

We have as well the following computation:

$$\begin{aligned}
 (p^2)_{ij} &= \sum_k p_{ik} p_{kj} \\
 &= \sum_k x_i x_k^* x_k x_j^* \\
 &= x_i x_j^* \\
 &= p_{ij}
 \end{aligned}$$

Finally, we have as well the following computation:

$$\begin{aligned} \text{Tr}(p) &= \sum_k p_{kk} \\ &= \sum_k x_k x_k^* \\ &= 1 \end{aligned}$$

Thus, we have embeddings of algebraic manifolds, as follows:

$$PS_{\mathbb{R},+}^{N-1} \subset PS_{\mathbb{C},+}^{N-1} \subset P_+^{N-1}$$

Regarding now the GNS construction assertion, this follows by reasoning as in the case of the free spheres, the idea being that the uniform integration on these projective spaces comes from the uniform integration over the following quantum group:

$$PO_N^+ = PU_N^+$$

All this is quite technical, and we will not need this result, in what follows. We refer here to [29], and we will back to this in section 15 below. Finally, regarding the middle assertions, concerning the projective versions of the half-classical spheres, it is enough to prove here that we have inclusions as follows:

$$P_{\mathbb{C}}^{N-1} \subset PS_{\mathbb{R},*}^{N-1} \subset PS_{\mathbb{C},*}^{N-1} \subset P_{\mathbb{C}}^{N-1}$$

But this can be done in 3 steps, as follows:

(1) $P_{\mathbb{C}}^{N-1} \subset PS_{\mathbb{R},*}^{N-1}$. In order to prove this, we recall from Proposition 9.6 that we have a morphism as follows, where z_i are the standard coordinates of $S_{\mathbb{C}}^{N-1}$:

$$C(S_{\mathbb{R},*}^{N-1}) \rightarrow M_2(C(S_{\mathbb{C}}^{N-1})) \quad , \quad x_i \rightarrow \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

Now observe that this model maps the projective coordinates as follows:

$$p_{ij} \rightarrow P_{ij} = \begin{pmatrix} z_i \bar{z}_j & 0 \\ 0 & \bar{z}_i z_j \end{pmatrix}$$

Thus, at the level of generated algebras, our model maps:

$$\langle p_{ij} \rangle \rightarrow \langle P_{ij} \rangle = C(P_{\mathbb{C}}^{N-1})$$

We conclude from this that we have a quotient map as follows:

$$C(PS_{\mathbb{R},*}^{N-1}) \rightarrow C(P_{\mathbb{C}}^{N-1})$$

Thus at the level of corresponding spaces, we have, as desired, an inclusion:

$$P_{\mathbb{C}}^{N-1} \subset PS_{\mathbb{R},*}^{N-1}$$

(2) $PS_{\mathbb{R},*}^{N-1} \subset PS_{\mathbb{C},*}^{N-1}$. This is something trivial, coming by functoriality of the operation $S \rightarrow PS$, from the inclusion of spheres:

$$S_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{C},*}^{N-1}$$

(3) $PS_{\mathbb{C},*}^{N-1} \subset P_{\mathbb{C}}^{N-1}$. This follows from the half-commutation relations, which imply:

$$ab^*cd^* = cb^*ad^* = cd^*ab^*$$

Indeed, this shows that the projective version $PS_{\mathbb{C},*}^{N-1}$ is classical, and so:

$$PS_{\mathbb{C},*}^{N-1} \subset (P_+^{N-1})_{class} = P_{\mathbb{C}}^{N-1}$$

Thus, we are led to the conclusion in the statement. \square

We can go back now to the spheres, and we have the following result:

THEOREM 9.12. *We have a morphism of C^* -algebras as follows,*

$$C(S_{\mathbb{R},*}^{N-1}) \subset M_2(C(S_{\mathbb{C}}^{N-1})) \quad , \quad x_i \rightarrow \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

where z_i are the standard coordinates of $S_{\mathbb{C}}^{N-1}$.

PROOF. We know from Proposition 9.6 that we have a morphism as in the statement, and the injectivity follows from Theorem 9.11, by using a standard grading trick. \square

In the case of the complex spheres we have a similar result, as follows:

THEOREM 9.13. *We have a morphism of C^* -algebras as follows,*

$$C(S_{\mathbb{C},*}^{N-1}) \rightarrow M_2(C(S_{\mathbb{C}}^{N-1} \times S_{\mathbb{C}}^{N-1})) \quad , \quad x_i \rightarrow \begin{pmatrix} 0 & z_i \\ y_i & 0 \end{pmatrix}$$

where y_i, z_i are the standard coordinates of $S_{\mathbb{C}}^{N-1} \times S_{\mathbb{C}}^{N-1}$.

PROOF. We know from Proposition 9.7 that we have a morphism as in the statement, and the injectivity follows from Theorem 9.11, by using a standard grading trick. \square

We will be back later to the above results, which are quite similar to each other, with a number of unifications and generalizations.

Summarizing, we have some interesting affine and projective geometry results regarding the half-classical case, that we will use in what follows.

The point now is that the same arguments apply to the tori, and to the quantum groups. We first have the following result:

PROPOSITION 9.14. *The real half-classical quadruplet, namely*

$$\begin{array}{ccc}
 S_{\mathbb{R},*}^{N-1} & \text{---} & T_N^* \\
 | & \diagdown & | \\
 & & \diagup \\
 O_N^* & \text{---} & H_N^*
 \end{array}$$

and the complex real half-classical quadruplet, namely

$$\begin{array}{ccc}
 S_{\mathbb{C},*}^{N-1} & \text{---} & \mathbb{T}_N^* \\
 | & \diagdown & | \\
 & & \diagup \\
 U_N^* & \text{---} & K_N^*
 \end{array}$$

have 2×2 matrix models, constructed by using antidiagonal matrices, as for the spheres.

PROOF. This is something that we already know from the spheres, from the various results established above. For the other objects which form the quadruplets, this follows by suitably adapting the proof of Proposition 9.6 and Proposition 9.7. \square

Next, we have the following result:

THEOREM 9.15. *The real and complex half-classical quadruplets have the same projective version, which is as follows:*

$$\begin{array}{ccc}
 P_{\mathbb{C}}^{N-1} & \text{---} & P\mathbb{T}_N \\
 | & \diagdown & | \\
 & & \diagup \\
 PU_N & \text{---} & PK_N
 \end{array}$$

PROOF. As before, this is something that we already know from the spheres, from the various results established above. For the other objects which form the quadruplets, this follows from Proposition 9.14, by suitably adapting the proof of Theorem 9.11. \square

Finally, we have the following result:

THEOREM 9.16. *The 2×2 antidiagonal matrix models for the real and complex half-classical quadruplets, constructed above, are faithful.*

PROOF. This is something that we already know from the spheres. For the other objects, this follows by suitably adapting the proof of Theorem 9.12 and Theorem 9.13. \square

As already mentioned, all these results are part of a series of more general results, regarding the half-liberation. We will be back to this, in chapter 12 below.

9d. Axiom check

Let us check now the axioms, for these half-classical quadruplets. We first need some quantum isometry group results:

THEOREM 9.17. *The quantum isometry groups of the basic spheres are*

$$\begin{array}{ccc}
 S_{\mathbb{R},+}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\
 \uparrow & & \uparrow \\
 S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{C},*}^{N-1} \\
 \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{C}}^{N-1}
 \end{array}
 \quad \rightarrow \quad
 \begin{array}{ccc}
 O_N^+ & \longrightarrow & U_N^+ \\
 \uparrow & & \uparrow \\
 O_N^* & \longrightarrow & U_N^* \\
 \uparrow & & \uparrow \\
 O_N & \longrightarrow & U_N
 \end{array}$$

modulo identifying, as usual, the various C^ -algebraic completions.*

PROOF. We just have to prove the results in the middle.

Assume $G \curvearrowright S_{\mathbb{C},*}^{N-1}$. From $\Phi(x_a) = \sum_i x_i \otimes u_{ia}$ we obtain, with $p_{ab} = z_a \bar{z}_b$:

$$\Phi(p_{ab}) = \sum_{ij} p_{ij} \otimes u_{ia} u_{jb}^*$$

By multiplying two such arbitrary formulae, we obtain:

$$\begin{aligned}
 \Phi(p_{ab} p_{cd}) &= \sum_{ijkl} p_{ij} p_{kl} \otimes u_{ia} u_{jb}^* u_{kc} u_{ld}^* \\
 \Phi(p_{ad} p_{cb}) &= \sum_{ijkl} p_{il} p_{kj} \otimes u_{ia} u_{ld}^* u_{kc} u_{jb}^*
 \end{aligned}$$

The left terms being equal, and the first terms on the right being equal too, we deduce that, with $[a, b, c] = abc - cba$, we must have the following equality:

$$\sum_{ijkl} p_{ij} p_{kl} \otimes u_{ia} [u_{jb}^*, u_{kc}, u_{ld}^*] = 0$$

Now observe that the products of projective variables $p_{ij} p_{kl} = z_i \bar{z}_j z_k \bar{z}_l$ depend only on the following two cardinalities:

$$|\{i, k\}|, |\{j, l\}| \in \{1, 2\}$$

The point now is that this dependence produces the only relations between our variables, we are led in this way to 4 equations, as follows:

- (1) $u_{ia}[u_{jb}^*, u_{ka}, u_{lb}^*] = 0, \forall a, b.$
- (2) $u_{ia}[u_{jb}^*, u_{ka}, u_{ld}^*] + u_{ia}[u_{jd}^*, u_{ka}, u_{lb}^*] = 0, \forall a, \forall b \neq d.$
- (3) $u_{ia}[u_{jb}^*, u_{kc}, u_{lb}^*] + u_{ic}[u_{jb}^*, u_{ka}, u_{lb}^*] = 0, \forall a \neq c, \forall b.$
- (4) $u_{ia}([u_{jb}^*, u_{kc}, u_{ld}^*] + [u_{jd}^*, u_{kc}, u_{lb}^*]) + u_{ic}([u_{jb}^*, u_{ka}, u_{ld}^*] + [u_{jd}^*, u_{ka}, u_{lb}^*]) = 0, \forall a \neq c, \forall b \neq d.$

From (1,2) we conclude that (2) holds with no restriction on the indices. By multiplying now this formula to the left by u_{ia}^* , and then summing over i , we obtain:

$$[u_{jb}^*, u_{ka}, u_{ld}^*] + [u_{jd}^*, u_{ka}, u_{lb}^*] = 0$$

By applying now the antipode, then the involution, and finally by suitably relabelling all the indices, we successively obtain from this formula:

$$\begin{aligned} [u_{dl}, u_{ak}^*, u_{bj}] + [u_{bl}, u_{ak}^*, u_{dj}] = 0 &\implies [u_{dl}^*, u_{ak}, u_{bj}^*] + [u_{bl}^*, u_{ak}, u_{dj}^*] = 0 \\ &\implies [u_{ld}^*, u_{ka}, u_{jb}^*] + [u_{jd}^*, u_{ka}, u_{lb}^*] = 0 \end{aligned}$$

Now by comparing with the original relation, above, we conclude that we have:

$$[u_{jb}^*, u_{ka}, u_{ld}^*] = [u_{jd}^*, u_{ka}, u_{lb}^*] = 0$$

Thus we have reached to the formulae defining U_N^* , and we are done.

Finally, in what regards the universality of the action $O_N^* \curvearrowright S_{\mathbb{R},*}^{N-1}$, this follows from the universality of the following actions:

$$\begin{aligned} U_N^* &\curvearrowright S_{\mathbb{C},*}^{N-1} \\ O_N^+ &\curvearrowright S_{\mathbb{R},+}^{N-1} \end{aligned}$$

Indeed, we have $U_N^* \cap O_N^+ = O_N^*$, and we obtain the result. \square

Regarding now the tori, the computation here is as follows:

THEOREM 9.18. *The quantum isometry groups of the basic tori are*

$$\begin{array}{ccc} \begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N^* & \longrightarrow & \mathbb{T}_N^* \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array} & \longrightarrow & \begin{array}{ccc} H_N^+ & \longrightarrow & K_N^+ \\ \uparrow & & \uparrow \\ H_N^* & \longrightarrow & K_N^* \\ \vdots & & \vdots \\ \bar{O}_N & \longrightarrow & \bar{U}_N \end{array} \end{array}$$

with all arrows being inclusions, and with no vertical maps at bottom right.

PROOF. We just have to prove the results in the middle. In the real case, we must find the conditions on $G \subset O_N^+$ such that $g_a \rightarrow \sum_i g_a \otimes u_{ia}$ defines a coaction.

In order for this map to be a coaction, the variables $G_a = \sum_i g_a \otimes u_{ia}$ must satisfy the following relations, which define the groups in the statement:

$$G_a^2 = 1$$

$$G_a G_b G_c = G_c G_b G_a$$

In what regards the squares, we have the following formula:

$$\begin{aligned} G_a^2 &= \sum_{ij} g_i g_j \otimes u_{ia} u_{ja} \\ &= 1 + \sum_{i \neq j} g_i g_j \otimes u_{ia} u_{ja} \end{aligned}$$

As for the products, with the notation $[x, y, z] = xyz - zyx$, we have:

$$[G_a, G_b, G_c] = \sum_{ijk} g_i g_j g_k \otimes [u_{ia}, u_{jb}, u_{kc}]$$

From the first relations, $G_a^2 = 1$, we obtain $G \subset H_N^+$. In order to process now the second relations, $G_a G_b G_c = G_c G_b G_a$, we can split the sum over i, j, k , as follows:

$$\begin{aligned} [G_a, G_b, G_c] &= \sum_{i,j,k \text{ distinct}} g_i g_j g_k \otimes [u_{ia}, u_{jb}, u_{kc}] \\ &+ \sum_{i \neq j} g_i g_j g_i \otimes [u_{ia}, u_{jb}, u_{ic}] \\ &+ \sum_{i \neq j} g_i \otimes [u_{ia}, u_{jb}, u_{jc}] \\ &+ \sum_{i \neq k} g_k \otimes [u_{ia}, u_{ib}, u_{kc}] \\ &+ \sum_i g_i \otimes [u_{ia}, u_{ib}, u_{ic}] \end{aligned}$$

Our claim is that the last three sums vanish. Indeed, observe that we have:

$$[u_{ia}, u_{ib}, u_{ic}] = \delta_{abc} u_{ia} - \delta_{abc} u_{ia} = 0$$

Thus the last sum vanishes. Regarding now the fourth sum, we have:

$$\begin{aligned}
\sum_{i \neq k} [u_{ia}, u_{ib}, u_{kc}] &= \sum_{i \neq k} u_{ia} u_{ib} u_{kc} - u_{kc} u_{ib} u_{ia} \\
&= \sum_{i \neq k} \delta_{ab} u_{ia}^2 u_{kc} - \delta_{ab} u_{kc} u_{ia}^2 \\
&= \delta_{ab} \sum_{i \neq k} [u_{ia}^2, u_{kc}] \\
&= \delta_{ab} \left[\sum_{i \neq k} u_{ia}^2, u_{kc} \right] \\
&= \delta_{ab} [1 - u_{ka}^2, u_{kc}] \\
&= 0
\end{aligned}$$

The proof for the third sum is similar. Thus, we are left with the first two sums. By using $g_i g_j g_k = g_k g_j g_i$ for the first sum, the formula becomes:

$$\begin{aligned}
[G_a, G_b, G_c] &= \sum_{i < k, j \neq i, k} g_i g_j g_k \otimes ([u_{ia}, u_{jb}, u_{kc}] + [u_{ka}, u_{jb}, u_{ic}]) \\
&\quad + \sum_{i \neq j} g_i g_j g_i \otimes [u_{ia}, u_{jb}, u_{ic}]
\end{aligned}$$

In order to have a coaction, the above coefficients must vanish. Now observe that, when setting $i = k$ in the coefficients of the first sum, we obtain twice the coefficients of the second sum. Thus, our vanishing conditions can be formulated as follows:

$$[u_{ia}, u_{jb}, u_{kc}] + [u_{ka}, u_{jb}, u_{ic}] = 0, \forall j \neq i, k$$

Now observe that at $a = b$ or $b = c$ this condition reads $0 + 0 = 0$. Thus, we can formulate our vanishing conditions in a more symmetric way, as follows:

$$[u_{ia}, u_{jb}, u_{kc}] + [u_{ka}, u_{jb}, u_{ic}] = 0, \forall j \neq i, k, \forall b \neq a, c$$

We use now the trick from [38]. We apply the antipode to this formula, and then we relabel the indices $i \leftrightarrow c, j \leftrightarrow b, k \leftrightarrow a$. We succesively obtain in this way:

$$[u_{ck}, u_{bj}, u_{ai}] + [u_{ci}, u_{bj}, u_{ak}] = 0, \forall j \neq i, k, \forall b \neq a, c$$

$$[u_{ia}, u_{jb}, u_{kc}] + [u_{ic}, u_{jb}, u_{ka}] = 0, \forall b \neq a, c, \forall j \neq i, k$$

Since we have $[x, y, z] = -[z, y, x]$, by comparing the last formula with the original one, we conclude that our vanishing relations reduce to a single formula, as follows:

$$[u_{ia}, u_{jb}, u_{kc}] = 0, \forall j \neq i, k, \forall b \neq a, c$$

Our first claim is that this formula implies $G \subset H_N^{[\infty]}$, where $H_N^{[\infty]} \subset O_N^+$ is defined via the relations $xyz = 0$, for any $x \neq z$ on the same row or column of u . In order to prove this, we will just need the $c = a$ particular case of this formula, which reads:

$$u_{ia}u_{jb}u_{ka} = u_{ka}u_{jb}u_{ia}, \forall j \neq i, k, \forall a \neq b$$

It is enough to check that the assumptions $j \neq i, k$ and $a \neq b$ can be dropped. But this is what happens indeed, because at $j = i$ we have:

$$\begin{aligned} [u_{ia}, u_{ib}, u_{ka}] &= u_{ia}u_{ib}u_{ka} - u_{ka}u_{ib}u_{ia} \\ &= \delta_{ab}(u_{ia}^2 u_{ka} - u_{ka} u_{ia}^2) \\ &= 0 \end{aligned}$$

Also, at $j = k$ we have:

$$\begin{aligned} [u_{ia}, u_{kb}, u_{ka}] &= u_{ia}u_{kb}u_{ka} - u_{ka}u_{kb}u_{ia} \\ &= \delta_{ab}(u_{ia}u_{ka}^2 - u_{ka}^2 u_{ia}) \\ &= 0 \end{aligned}$$

Finally, at $a = b$ we have:

$$\begin{aligned} [u_{ia}, u_{ja}, u_{ka}] &= u_{ia}u_{ja}u_{ka} - u_{ka}u_{ja}u_{ia} \\ &= \delta_{ijk}(u_{ia}^3 - u_{ia}^3) \\ &= 0 \end{aligned}$$

Our second claim now is that, due to $G \subset H_N^{[\infty]}$, we can drop the assumptions $j \neq i, k$ and $b \neq a, c$ in the original relations $[u_{ia}, u_{jb}, u_{kc}] = 0$. Indeed, at $j = i$ we have:

$$\begin{aligned} [u_{ia}, u_{ib}, u_{kc}] &= u_{ia}u_{ib}u_{kc} - u_{kc}u_{ib}u_{ia} \\ &= \delta_{ab}(u_{ia}^2 u_{kc} - u_{kc} u_{ia}^2) \\ &= 0 \end{aligned}$$

The proof at $j = k$ and at $b = a, b = c$ being similar, this finishes the proof of our claim. We conclude that the half-commutation relations $[u_{ia}, u_{jb}, u_{kc}] = 0$ hold without any assumption on the indices, and so we obtain $G \subset H_N^*$, as claimed.

As for the proof in the complex case, this is similar. See [7]. \square

By intersecting now with K_N^+ , as required by our (S, T, U, K) axioms, we obtain:

THEOREM 9.19. *The quantum reflection groups of the basic tori are*

$$\begin{array}{ccc}
 T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\
 \uparrow & & \uparrow \\
 T_N^* & \longrightarrow & \mathbb{T}_N^* \\
 \uparrow & & \uparrow \\
 T_N & \longrightarrow & \mathbb{T}_N
 \end{array}
 \quad \rightarrow \quad
 \begin{array}{ccc}
 H_N^+ & \longrightarrow & K_N^+ \\
 \uparrow & & \uparrow \\
 H_N^* & \longrightarrow & K_N^* \\
 \uparrow & & \uparrow \\
 H_N & \longrightarrow & K_N
 \end{array}$$

with all the arrows being inclusions.

PROOF. We already know that the results on the left and on the right hold indeed. As for the results in the middle, these follow from Theorem 9.18 above. \square

We can now formulate our extension result, as follows:

THEOREM 9.20. *We have basic noncommutative geometries, as follows,*

$$\begin{array}{ccc}
 \mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\
 \uparrow & & \uparrow \\
 \mathbb{R}_*^N & \longrightarrow & \mathbb{C}_*^N \\
 \uparrow & & \uparrow \\
 \mathbb{R}^N & \longrightarrow & \mathbb{C}^N
 \end{array}$$

with each \mathbb{K}_\times^N symbol standing for the corresponding (S, T, U, K) quadruplet.

PROOF. We have to check the axioms from section 4, for the half-classical geometries.

The algebraic axioms are all clear, and the quantum isometry axioms follow from the above computations. Next in line, we have to prove the following formulae:

$$O_N^* = \langle O_N, T_N^* \rangle$$

$$U_N^* = \langle U_N, \mathbb{T}_N^* \rangle$$

By using standard generation results, it is enough to prove the first formula. Moreover, once again by standard generation results, it is enough to check that:

$$H_N^* = \langle H_N, T_N^* \rangle$$

The inclusion \supset being clear, we are left with proving the inclusion \subset . But this follows from the formula $H_N^* = T_N^* \rtimes S_N$, established in [81], as follows:

$$\begin{aligned} H_N^* &= T_N^* \rtimes S_N \\ &= \langle S_N, T_N^* \rangle \\ &\subset \langle H_N, T_N^* \rangle \end{aligned}$$

Alternatively, these formulae can be established by using the technology in [41], or by doing some combinatorial computations, using categories and easiness.

Finally, the axiom $S = S_U$ can be proved as in the classical and free cases, by using the Weingarten formula, and the following ergodicity property:

$$\left(id \otimes \int_U \right) \Phi(x) = \int_S x$$

Our claim, which will finish the proof, is that this holds as well in the half-classical case. Indeed, in the real case, where $x_i = x_i^*$, it is enough to check the above equality on an arbitrary product of coordinates, $x_{i_1} \dots x_{i_k}$. The left term is as follows:

$$\begin{aligned} &\left(id \otimes \int_{O_N^*} \right) \Phi(x_{i_1} \dots x_{i_k}) \\ &= \sum_{j_1 \dots j_k} x_{j_1} \dots x_{j_k} \int_{O_N^*} u_{j_1 i_1} \dots u_{j_k i_k} \\ &= \sum_{j_1 \dots j_k} \sum_{\pi, \sigma \in P_2^*(k)} \delta_\pi(j) \delta_\sigma(i) W_{kN}(\pi, \sigma) x_{j_1} \dots x_{j_k} \\ &= \sum_{\pi, \sigma \in P_2^*(k)} \delta_\sigma(i) W_{kN}(\pi, \sigma) \sum_{j_1 \dots j_k} \delta_\pi(j) x_{j_1} \dots x_{j_k} \end{aligned}$$

Let us look now at the last sum on the right. We have to sum there quantities of type $x_{j_1} \dots x_{j_k}$, over all choices of multi-indices $j = (j_1, \dots, j_k)$ which fit into our given pairing $\pi \in P_2^*(k)$. But by using the relations $x_i x_j x_k = x_k x_j x_i$, and then $\sum_i x_i^2 = 1$ in order to simplify, we conclude that the sum of these quantities is 1. Thus, we obtain:

$$\left(id \otimes \int_{O_N^*} \right) \Phi(x_{i_1} \dots x_{i_k}) = \sum_{\pi, \sigma \in P_2^*(k)} \delta_\sigma(i) W_{kN}(\pi, \sigma)$$

On the other hand, another application of the Weingarten formula gives:

$$\begin{aligned} \int_{S_{\mathbb{R},*}^{N-1}} x_{i_1} \dots x_{i_k} &= \int_{O_N^*} u_{1i_1} \dots u_{1i_k} \\ &= \sum_{\pi, \sigma \in P_2^*(k)} \delta_\pi(1) \delta_\sigma(i) W_{kN}(\pi, \sigma) \\ &= \sum_{\pi, \sigma \in P_2^*(k)} \delta_\sigma(i) W_{kN}(\pi, \sigma) \end{aligned}$$

Thus, we are done. In the complex case the proof is similar, by adding exponents. For further details, we refer to [28] for the real case, and to [4] for the complex case. \square

Summarizing, we have done so far half of our extension program.

9e. Exercises

There are many interesting questions regarding the half-classical geometry, which is not that far from the classical geometry, at the level of the tools that can be used, and which can be developed well beyond what was done in this chapter, with our check of the general noncommutative geometry axioms, along with a few more things.

First, in relation with what was discussed in the above, we have:

EXERCISE 9.21. *Develop a full theory of the main half-classical groups,*

$$\begin{array}{ccc} K_N^* & \longrightarrow & U_N^* \\ \uparrow & & \uparrow \\ H_N^* & \longrightarrow & O_N^* \end{array}$$

in particular by working out in detail their easiness property.

This is something that we already discussed, with the comment that the above quantum groups are indeed easy, the corresponding categories of partitions being as follows:

$$\begin{array}{ccc} \mathcal{P}_{even}^* & \longleftarrow & \mathcal{P}_2^* \\ \downarrow & & \downarrow \\ \mathcal{P}_{even}^* & \longleftarrow & \mathcal{P}_2^* \end{array}$$

Here $*$ stands for the fact that, when relabelling clockwise the legs $\circ \bullet \circ \dots$, the formula $\# \circ = \# \bullet$ must hold in each block. The problem is that of proving this.

In relation now with this, and with the tori as well, we have:

EXERCISE 9.22. *Work out the moments, and then the $N \rightarrow \infty$ asymptotics of the laws of the truncated characters*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

for the half-classical unitary and reflection groups, and for the tori as well.

In what regards the unitary groups, we can only obtain some versions of the normal laws, and in case of trouble here, a quick look at the normal law literature can only help. As for the reflection groups and the tori, the situation here is a bit similar.

In relation with the spheres, we have the following exercise:

EXERCISE 9.23. *Work out the moments, and then the $N \rightarrow \infty$ asymptotics of the laws of the individual coordinates for the half-classical spheres.*

To be more precise here, the moment problem is a calculus one, that needs to be solved. As for the $N \rightarrow \infty$ study, this requires finding first the good normalization, in order for things to converge, and then a study of the asymptotic laws, and also a formulation of the independence-type relation between the asymptotic coordinates.

More generally now, we have the following question:

EXERCISE 9.24. *Explain, both at the algebraic and the probabilistic level, how the general theory developed in chapters 5-8 above can be applied to the half-classical situation, in order to talk about more general classes of half-classical manifolds, with algebraic and probabilistic results about them, generalizing what we already have.*

There is quite some work to be done here, and as a first observation, the half-classical quantum groups are not uniform, and so the constructions from chapters 5-6, from the classical and free case, do not extend well. Thus, the idea is rather of looking at the material from chapters 7-8, to see what extends from there.

As a last comment, we have not talked in the above about more general half-classical quantum groups, and related manifolds. This is because we will come back to these questions in chapter 12 below, with a systematic study of all this.

CHAPTER 10

Hybrid geometries

10a. Spheres and tori

In order to finish the extension program outlined in the beginning of the previous chapter, we must discuss now the second question, concerning the “hybrid” case.

To be more precise, we have seen so far that have basic noncommutative geometries as follows, with each \mathbb{K}_x^N symbol standing for the corresponding (S, T, U, K) quadruplet:

$$\begin{array}{ccc}
 \mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\
 \uparrow & & \uparrow \\
 \mathbb{R}_*^N & \longrightarrow & \mathbb{C}_*^N \\
 \uparrow & & \uparrow \\
 \mathbb{R}^N & \longrightarrow & \mathbb{C}^N
 \end{array}$$

We will see in this section that there are some privileged intermediate geometries between the real and the complex ones, as follows:

$$\begin{array}{ccccc}
 \mathbb{R}_+^N & \longrightarrow & \text{TR}_+^N & \longrightarrow & \mathbb{C}_+^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{R}_*^N & \longrightarrow & \text{TR}_*^N & \longrightarrow & \mathbb{C}_*^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{R}^N & \longrightarrow & \text{TR}^N & \longrightarrow & \mathbb{C}^N
 \end{array}$$

We will see as well that, that under strong combinatorial axioms, of “easiness” and “uniformity” type, these 9 geometries are the only ones.

In order to get started, an intermediate geometry $\mathbb{R}^N \subset X \subset \mathbb{C}^N$ is given by a quadruplet (S, T, U, K) , whose components are subject to the following conditions:

$$S_{\mathbb{R}}^{N-1} \subset S \subset S_{\mathbb{C}}^{N-1}$$

$$T_N \subset T \subset \mathbb{T}_N$$

$$O_N \subset U \subset U_N$$

$$H_N \subset K \subset K_N$$

Our plan will be that of investigating first these intermediate object questions. Then, we will discuss the verification of the geometric axioms, for the solutions that we found. And then, afterwards, we will discuss the half-classical and the free cases as well.

In what regards the $S_{\mathbb{R}}^{N-1} \subset S \subset S_{\mathbb{C}}^{N-1}$ problem, there are obviously infinitely many solutions. However, we have a “privileged” solution, constructed as follows:

THEOREM 10.1. *We have an intermediate sphere as follows,*

$$S_{\mathbb{R}}^{N-1} \subset \mathbb{T}S_{\mathbb{R}}^{N-1} \subset S_{\mathbb{C}}^{N-1}$$

which appears as the affine lift of $P_{\mathbb{R}}^{N-1}$, inside the complex sphere $S_{\mathbb{C}}^{N-1}$.

PROOF. The projective version of the intermediate sphere $\mathbb{T}S_{\mathbb{R}}^{N-1}$ is given by:

$$P\mathbb{T}S_{\mathbb{R}}^{N-1} = PS_{\mathbb{R}}^{N-1} = P_{\mathbb{R}}^{N-1}$$

Conversely, assume that $S \subset S_{\mathbb{C}}^{N-1}$ satisfies:

$$PS \subset P_{\mathbb{R}}^{N-1}$$

For $x \in S$ the projective coordinates $p_{ij} = x_i \bar{x}_j$ must be real:

$$x_i \bar{x}_j = \bar{x}_i x_j$$

Thus, we must have the following equalities:

$$\frac{x_1}{\bar{x}_1} = \frac{x_2}{\bar{x}_2} = \dots = \frac{x_N}{\bar{x}_N}$$

Now if we denote by $\lambda \in \mathbb{T}$ this common number, we successively have:

$$\begin{aligned} \frac{x_i}{\bar{x}_i} = \lambda &\iff x_i = \lambda \bar{x}_i \\ &\iff x_i^2 = \lambda |x_i|^2 \\ &\iff x_i = \pm \sqrt{\lambda} |x_i| \end{aligned}$$

Thus we obtain $x \in \sqrt{\lambda} S_{\mathbb{R}}^{N-1}$, and this gives the result. □

In the case of the tori, we have a similar result, as follows:

THEOREM 10.2. *We have an intermediate torus as follows, which appears as the affine lift of the Clifford torus $PT_N = T_{N-1}$, inside the complex torus \mathbb{T}_N :*

$$T_N \subset \mathbb{T}T_N \subset \mathbb{T}_N$$

More generally, we have intermediate tori as follows, with $r \in \mathbb{N} \cup \{\infty\}$,

$$T_N \subset \mathbb{Z}_r T_N \subset \mathbb{T}_N$$

all whose projective versions equal the Clifford torus $PT_N = T_{N-1}$.

PROOF. The first assertion, regarding $\mathbb{T}T_N$, follows exactly as for the spheres, as in proof of Theorem 10.1. The second assertion is clear as well, because we have:

$$P\mathbb{Z}_r T_N = PT_N = T_{N-1}$$

Thus, we are led to the conclusion in the statement. □

In connection with the above statement, an interesting question is that of classifying the intermediate tori, which in our case are usual compact groups, as follows:

$$T_N \subset T \subset \mathbb{T}_N$$

At the group dual level, we must classify the following intermediate quotients:

$$\mathbb{Z}^N \rightarrow \Gamma \rightarrow \mathbb{Z}_2^N$$

There are many examples of such groups, and this even when imposing strong supplementary conditions, such as having an action of the symmetric group S_N on the generators. We will not go further in this direction, our main idea being anyway that of basing our study mostly on quantum group theory, and on the related notion of easiness.

10b. Quantum groups

At the group level now, the situation is of course much more rigid, and becomes quite interesting. Let us first discuss the unitary case.

We have the following result from [19], to start with:

THEOREM 10.3. *The following inclusion of compact groups is maximal,*

$$\mathbb{T}O_N \subset U_N$$

in the sense that there is no intermediate compact group in between.

PROOF. In order to prove this result, consider as well the following group:

$$\mathbb{T}SO_N = \left\{ wU \mid w \in \mathbb{T}, U \in SO_N \right\}$$

Observe that we have $\mathbb{T}SO_N = \mathbb{T}O_N$ if N is odd. If N is even the group $\mathbb{T}O_N$ has two connected components, with $\mathbb{T}SO_N$ being the component containing the identity.

Let us denote by $\mathfrak{so}_N, \mathfrak{u}_N$ the Lie algebras of SO_N, U_N . It is well-known that \mathfrak{u}_N consists of the matrices $M \in M_N(\mathbb{C})$ satisfying $M^* = -M$, and that:

$$\mathfrak{so}_N = \mathfrak{u}_N \cap M_N(\mathbb{R})$$

Also, it is easy to see that the Lie algebra of $\mathbb{T}SO_N$ is $\mathfrak{so}_N \oplus i\mathbb{R}$.

Step 1. Our first claim is that if $N \geq 2$, the adjoint representation of SO_N on the space of real symmetric matrices of trace zero is irreducible.

Let indeed $X \in M_N(\mathbb{R})$ be symmetric with trace zero. We must prove that the following space consists of all the real symmetric matrices of trace zero:

$$V = \text{span} \left\{ UXU^t \mid U \in SO_N \right\}$$

We first prove that V contains all the diagonal matrices of trace zero. Since we may diagonalize X by conjugating with an element of SO_N , our space V contains a nonzero diagonal matrix of trace zero. Consider such a matrix:

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_N \end{pmatrix}$$

We can conjugate this matrix by the following matrix:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix} \in SO_N$$

We conclude that our space V contains as well the following matrix:

$$D' = \begin{pmatrix} d_2 & & & \\ & d_1 & & \\ & & d_3 & \\ & & & \ddots \\ & & & & d_N \end{pmatrix}$$

More generally, we see that for any $1 \leq i, j \leq N$ the diagonal matrix obtained from D by interchanging d_i and d_j lies in V . Now since S_N is generated by transpositions, it follows that V contains any diagonal matrix obtained by permuting the entries of D .

But it is well-known that this representation of S_N on the diagonal matrices of trace zero is irreducible, and hence V contains all such diagonal matrices, as claimed.

In order to conclude now, assume that Y is an arbitrary real symmetric matrix of trace zero. We can find then an element $U \in SO_N$ such that UYU^t is a diagonal matrix of trace zero. But we then have $UYU^t \in V$, and hence also $Y \in V$, as desired.

Step 2. Our claim is that the inclusion $\mathbb{T}SO_N \subset U_N$ is maximal in the category of connected compact groups.

Let indeed G be a connected compact group satisfying:

$$\mathbb{T}SO_N \subset G \subset U_N$$

Then G is a Lie group. Let \mathfrak{g} denote its Lie algebra, which satisfies:

$$\mathfrak{so}_N \oplus i\mathbb{R} \subset \mathfrak{g} \subset \mathfrak{u}_N$$

Let ad_G be the action of G on \mathfrak{g} obtained by differentiating the adjoint action of G on itself. This action turns \mathfrak{g} into a G -module. Since $SO_N \subset G$, \mathfrak{g} is also a SO_N -module.

Now if $G \neq \mathbb{T}SO_N$, then since G is connected we must have:

$$\mathfrak{so}_N \oplus i\mathbb{R} \neq \mathfrak{g}$$

It follows from the real vector space structure of the Lie algebras \mathfrak{u}_N and \mathfrak{so}_N that there exists a nonzero symmetric real matrix of trace zero X such that:

$$iX \in \mathfrak{g}$$

We know that the space of symmetric real matrices of trace zero is an irreducible representation of SO_N under the adjoint action. Thus \mathfrak{g} must contain all such X , and hence $\mathfrak{g} = \mathfrak{u}_N$. But since U_N is connected, it follows that $G = U_N$.

Step 3. Let us compute now the commutant of SO_N in $M_N(\mathbb{C})$. Our first claim is that at $N = 2$, this commutant is as follows:

$$SO'_2 = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mid \alpha, \beta \in \mathbb{C} \right\}$$

As for the case $N \geq 3$, our claim here is that this commutant is as follows:

$$SO'_N = \left\{ \alpha I_N \mid \alpha \in \mathbb{C} \right\}$$

Indeed, at $N = 2$, the above formula follows from a direct computation.

At $N \geq 3$ now, an element in $X \in SO'_N$ commutes with any diagonal matrix having exactly $N - 2$ entries equal to 1 and two entries equal to -1 . Hence X is diagonal.

Now since X commutes with any even permutation matrix, and we have assumed $N \geq 3$, it commutes in particular with the permutation matrix associated with the cycle (i, j, k) for any $1 < i < j < k$, and hence all the entries of X are the same.

We conclude that X is a scalar matrix, as claimed.

Step 4. Our claim now is that the set of matrices with nonzero trace is dense in SO_N .

At $N = 2$ this is clear, since the set of elements in SO_2 having a given trace is finite.

So assume $N > 2$, and consider a matrix as follows:

$$T \in SO_N \simeq SO(\mathbb{R}^N)$$

$$\text{Tr}(T) = 0$$

Let $E \subset \mathbb{R}^N$ be a 2-dimensional subspace preserved by T , such that:

$$T|_E \in SO(E)$$

Let $\varepsilon > 0$ and let $S_\varepsilon \in SO(E)$ satisfying the following condition:

$$\|T|_E - S_\varepsilon\| < \varepsilon$$

Moreover, in the $N = 2$ case, we can assume that T satisfies as well:

$$\text{Tr}(T|_E) \neq \text{Tr}(S_\varepsilon)$$

Now define $T_\varepsilon \in SO(\mathbb{R}^N) = SO_N$ by the following formulae:

$$T_{\varepsilon|E} = S_\varepsilon$$

$$T_{\varepsilon|E^\perp} = T|_{E^\perp}$$

It is clear that we have the following estimate:

$$\|T - T_\varepsilon\| \leq \|T|_E - S_\varepsilon\| < \varepsilon$$

Also, we have the following estimate:

$$\text{Tr}(T_\varepsilon) = \text{Tr}(S_\varepsilon) + \text{Tr}(T|_{E^\perp}) \neq 0$$

Thus, we have proved our claim.

Step 5. Our claim now is that $\mathbb{T}O_N$ is the normalizer of $\mathbb{T}SO_N$ in U_N , i.e. is the subgroup of U_N consisting of the unitaries U for which, for all $X \in \mathbb{T}SO_N$:

$$U^{-1}XU \in \mathbb{T}SO_N$$

It is clear indeed that the group $\mathbb{T}O_N$ normalizes $\mathbb{T}SO_N$, so in order to prove the result, we must show that if $U \in U_N$ normalizes $\mathbb{T}SO_N$ then $U \in \mathbb{T}O_N$.

First note that U normalizes SO_N . Indeed if $X \in SO_N$ then:

$$U^{-1}XU \in \mathbb{T}SO_N$$

Thus we have a formula as follows, for some $\lambda \in \mathbb{T}$ and $Y \in SO_N$:

$$U^{-1}XU = \lambda Y$$

If $\text{Tr}(X) \neq 0$, we have $\lambda \in \mathbb{R}$ and hence:

$$\lambda Y = U^{-1}XU \in SO_N$$

The set of matrices having nonzero trace being dense in SO_N , we conclude that $U^{-1}XU \in SO_N$ for all $X \in SO_N$. Thus, we have:

$$\begin{aligned} X \in SO_N &\implies (UXU^{-1})^t(UXU^{-1}) = I_N \\ &\implies X^tU^tUX = U^tU \\ &\implies U^tU \in SO'_N \end{aligned}$$

It follows that at $N \geq 3$ we have $U^tU = \alpha I_N$, with $\alpha \in \mathbb{T}$, since U is unitary. Hence we have $U = \alpha^{1/2}(\alpha^{-1/2}U)$ with:

$$\alpha^{-1/2}U \in O_N \quad , \quad U \in \mathbb{T}O_N$$

If $N = 2$, $(U^tU)^t = U^tU$ gives again that $U^tU = \alpha I_2$, and we conclude as in the previous case.

Step 6. Our claim is that the inclusion $\mathbb{T}O_N \subset U_N$ is maximal in the category of compact groups.

Suppose indeed that $\mathbb{T}O_N \subset G \subset U_N$ is a compact group such that $G \neq U_N$. It is a well-known fact that the connected component of the identity in G is a normal subgroup, denoted G_0 . Since we have $\mathbb{T}SO_N \subset G_0 \subset U_N$, we must have:

$$G_0 = \mathbb{T}SO_N$$

But since G_0 is normal in G , the group G normalizes $\mathbb{T}SO_N$, and hence $G \subset \mathbb{T}O_N$, which finishes the proof. \square

Following [19], we have as well the following result:

THEOREM 10.4. *The following inclusion of compact groups is maximal,*

$$PO_N \subset PU_N$$

in the sense that there is no intermediate compact group in between.

PROOF. This follows from Theorem 10.3. Indeed, consider a proper intermediate subgroup, as follows:

$$PO_N \subset G \subset PU_N$$

The preimage of this subgroup under the quotient map $U_N \rightarrow PU_N$ would be then a proper intermediate subgroup of $\mathbb{T}O_N \subset U_N$, which is a contradiction. \square

Finally, still following [19], we have as well the following result:

THEOREM 10.5. *The following inclusion of compact quantum groups is maximal,*

$$O_N \subset O_N^*$$

in the sense that there is no intermediate compact quantum group in between.

PROOF. The idea is that this follows from the result regarding $PO_N \subset PU_N$, by taking affine lifts, and using algebraic techniques.

Consider indeed a sequence of surjective Hopf $*$ -algebra maps as follows, whose composition is the canonical surjection:

$$C(O_N^*) \xrightarrow{f} A \xrightarrow{g} C(O_N)$$

This produces a diagram of Hopf algebra maps with pre-exact rows, as follows:

$$\begin{array}{ccccccccc} \mathbb{C} & \longrightarrow & C(PO_N^*) & \longrightarrow & C(O_N^*) & \longrightarrow & C(\mathbb{Z}_2) & \longrightarrow & \mathbb{C} \\ & & \downarrow f_1 & & \downarrow f & & \parallel & & \\ \mathbb{C} & \longrightarrow & PA & \longrightarrow & A & \longrightarrow & C(\mathbb{Z}_2) & \longrightarrow & \mathbb{C} \\ & & \downarrow g_1 & & \downarrow g & & \parallel & & \\ \mathbb{C} & \longrightarrow & PC(O_N) & \longrightarrow & C(O_N) & \longrightarrow & C(\mathbb{Z}_2) & \longrightarrow & \mathbb{C} \end{array}$$

Consider now the following composition, with the isomorphism on the left being something well-known, coming from [41], as explained in chapter 9 above:

$$C(PU_N) \simeq C(PO_N^*) \xrightarrow{f_1} PA \xrightarrow{g_1} PC(O_N) \simeq C(PO_N)$$

This induces, at the group level, the following embedding:

$$PO_N \subset PU_N$$

Thus f_1 or g_1 is an isomorphism. If f_1 is an isomorphism we get a commutative diagram of Hopf algebra morphisms with pre-exact rows, as follows:

$$\begin{array}{ccccccccc} \mathbb{C} & \longrightarrow & C(PO_N^*) & \longrightarrow & C(O_N^*) & \longrightarrow & C(\mathbb{Z}_2) & \longrightarrow & \mathbb{C} \\ & & \parallel & & \downarrow f & & \parallel & & \\ \mathbb{C} & \longrightarrow & C(PO_N^*) & \longrightarrow & A & \longrightarrow & C(\mathbb{Z}_2) & \longrightarrow & \mathbb{C} \end{array}$$

Then f is an isomorphism. Similarly if g_1 is an isomorphism, then g is an isomorphism, and this gives the result. See [19]. \square

In connection now with our question, which is that of classifying the intermediate groups $O_N \subset G \subset U_N$, the above results lead to a dichotomy, coming from:

$$PG \in \{PO_N, PU_N\}$$

In the lack of a classification result here, which is surely well-known, but that we were unable to find in the literature, here are some basic examples of such intermediate groups, which are all well-known:

PROPOSITION 10.6. *We have compact groups $O_N \subset G \subset U_N$ as follows:*

- (1) *The following groups, depending on a parameter $r \in \mathbb{N} \cup \{\infty\}$,*

$$\mathbb{Z}_r O_N = \left\{ wU \mid w \in \mathbb{Z}_r, U \in O_N \right\}$$

whose projective versions equal PO_N , and the biggest of which is the group $\mathbb{T}O_N$, which appears as affine lift of PO_N .

- (2) *The following groups, depending on a parameter $d \in 2\mathbb{N} \cup \{\infty\}$,*

$$U_N^d = \left\{ U \in U_N \mid \det U \in \mathbb{Z}_d \right\}$$

interpolating between U_N^2 and $U_N^\infty = U_N$, whose projective versions equal PU_N .

PROOF. All the assertions are elementary, the idea being as follows:

- (1) We have indeed compact groups $\mathbb{Z}_r O_N$ with $r \in \mathbb{N} \cup \{\infty\}$ as in the statement, whose projective versions are given by:

$$P\mathbb{Z}_r O_N = PO_N$$

At $r = \infty$ we obtain the group $\mathbb{T}O_N$, and the fact that this group appears as the affine lift of PO_N follows exactly as in the sphere case, by using the computation from the proof of Theorem 10.1.

- (2) As a first observation, the following formula, with $d \in \mathbb{N} \cup \{\infty\}$, defines indeed a closed subgroup $U_N^d \subset U_N$:

$$U_N^d = \left\{ U \in U_N \mid \det U \in \mathbb{Z}_d \right\}$$

In the case where d is even, this subgroup contains the orthogonal group O_N . As for the last assertion, namely $PU_N^d = PU_N$, this follows either be suitably rescaling the unitary matrices, or by applying the result in Theorem 10.3. \square

The above results suggest that the solutions of $O_N \subset G \subset U_N$ should come from O_N, U_N , by succesively applying the following constructions:

$$G \rightarrow \mathbb{Z}_r G$$

$$G \rightarrow G \cap U_N^d$$

These operations do not exactly commute, but normally we should be led in this way to a 2-parameter series, unifying the two 1-parameter series from (1,2) above. However, some other groups like $\mathbb{Z}_N SO_N$ work too, so all this is probably a bit more complicated.

As already mentioned, all this looks like quite standard group and Lie algebra theory, but we unable to find a good reference here. So, in the lack of something better, the above results will be our final saying on the subject, along with the reference to [19].

In what follows we will be mostly interested in the group $\mathbb{T}O_N$, which fits with the spheres and tori that we already have, in view of our axiomatization purposes.

This particular group $\mathbb{T}O_N$, and the whole series $\mathbb{Z}_r O_N$ with $r \in \mathbb{N} \cup \{\infty\}$ that it is part of, is easy, the precise result being as follows:

THEOREM 10.7. *We have the following results:*

- (1) $\mathbb{T}O_N$ is easy, the corresponding category $\bar{P}_2 \subset P_2$ consisting of the pairings having the property that when flatenning, we have the following global formula:

$$\# \circ = \# \bullet$$

- (2) $\mathbb{Z}_r O_N$ is easy, the corresponding category $P_2^r \subset P_2$ consisting of the pairings having the property that when flatenning, we have the following global formula:

$$\# \circ = \# \bullet (r)$$

PROOF. These results are standard and well-known, the proof being as follows:

- (1) If we denote the standard corepresentation by $u = zv$, with $z \in \mathbb{T}$ and with $v = \bar{v}$, then in order to have $Hom(u^{\otimes k}, u^{\otimes l}) \neq \emptyset$, the z variabes must cancel, and in the case where they cancel, we obtain the same Hom-space as for O_N .

Now since the cancelling property for the z variables corresponds precisely to the fact that k, l must have the same numbers of \circ symbols minus \bullet symbols, the associated Tannakian category must come from the category of pairings $\bar{P}_2 \subset P_2$, as claimed.

- (2) This is something that we already know at $r = 1, \infty$, where the group in question is $O_N, \mathbb{T}O_N$. The proof in general is similar, by writing $u = zv$ as above. □

Quite remarkably, the above result has the following converse:

THEOREM 10.8. *The proper intermediate easy compact groups*

$$O_N \subset G \subset U_N$$

are precisely the groups $\mathbb{Z}_r O_N$, with $r \in \{2, 3, \dots, \infty\}$.

PROOF. According to our conventions for the easy quantum groups, which apply of course to the classical case, we must compute the following intermediate categories:

$$\mathcal{P}_2 \subset D \subset P_2$$

So, assume that we have such a category, $D \neq \mathcal{P}_2$, and pick an element $\pi \in D - \mathcal{P}_2$, assumed to be flat. We can modify π , by performing the following operations:

(1) First, we can compose with the basic crossing, in order to assume that π is a partition of type $\cap \dots \cap$, consisting of consecutive semicircles. Our assumption $\pi \notin \mathcal{P}_2$ means that at least one semicircle is colored black, or white.

(2) Second, we can use the basic mixed-colored semicircles, and cap with them all the mixed-colored semicircles. Thus, we can assume that π is a nonzero partition of type $\cap \dots \cap$, consisting of consecutive black or white semicircles.

(3) Third, we can rotate, as to assume that π is a partition consisting of an upper row of white semicircles, $\cup \dots \cup$, and a lower row of white semicircles, $\cap \dots \cap$. Our assumption $\pi \notin \mathcal{P}_2$ means that this latter partition is nonzero.

For $a, b \in \mathbb{N}$ consider the partition consisting of an upper row of a white semicircles, and a lower row of b white semicircles, and set:

$$\mathcal{C} = \left\{ \pi_{ab} \mid a, b \in \mathbb{N} \right\} \cap D$$

According to the above, we have $\pi \in \langle \mathcal{C} \rangle$. The point now is that we have:

(1) There exists $r \in \mathbb{N} \cup \{\infty\}$ such that \mathcal{C} equals the following set:

$$\mathcal{C}_r = \left\{ \pi_{ab} \mid a = b(r) \right\}$$

This is indeed standard, by using the categorical axioms.

(2) We have the following formula, with P_2^r being as above:

$$\langle \mathcal{C}_r \rangle = P_2^r$$

This is standard as well, by doing some diagrammatic work.

With these results in hand, the conclusion now follows. Indeed, with $r \in \mathbb{N} \cup \{\infty\}$ being as above, we know from the beginning of the proof that any $\pi \in D$ satisfies:

$$\pi \in \langle \mathcal{C} \rangle = \langle \mathcal{C}_r \rangle = P_2^r$$

We conclude from this that we have an inclusion as follows:

$$D \subset P_2^r$$

Conversely, we have as well the following inclusion:

$$P_2^r = \langle \mathcal{C}_r \rangle = \langle \mathcal{C} \rangle \subset \langle D \rangle = D$$

Thus we have $D = P_2^r$, and this finishes the proof. See [85]. □

As a conclusion, $\mathbb{T}O_N$ is definitely the “privileged” unitary group that we were looking for, with the remark that its arithmetic versions $\mathbb{Z}_r O_N$ are interesting as well.

Finally, let us discuss the reflection group case. Here the problem is that of classifying the intermediate compact groups $H_N \subset G \subset K_N$, and this looks of course well-known. In practice, however, the situation is considerably more complicated than in the continuous

group case, with the expected 2-parameter series there being replaced by an expected 3-parameter series. So, instead of getting into this quite technical subject, let us just formulate a basic result, explaining what the 3 parameters are:

PROPOSITION 10.9. *We have compact groups $H_N \subset G \subset K_N$ as follows:*

- (1) *The groups $\mathbb{Z}_r H_N$, with $r \in \mathbb{N} \cup \{\infty\}$.*
- (2) *The groups $H_N^s = \mathbb{Z}_s \wr S_N$, with $s \in 2\mathbb{N}$.*
- (3) *The groups $H_N^{sd} = H_N^s \cap U_N^d$, with $d|s$ and $s \in 2\mathbb{N}$.*

PROOF. The various constructions in the statement produce indeed closed subgroups $G \subset K_N$, for all the possible values of the parameters.

Regarding now the condition $H_N \subset G$, the situation is as follows:

- (1) Here this condition is automatic.
- (2) Here this condition follows from $s \in 2\mathbb{N}$.
- (3) Here this condition follows from $d|s$ and $s \in 2\mathbb{N}$. □

The same discussion as in the continuous case applies, the idea being that the constructions $G \rightarrow \mathbb{Z}_r G$ and $G \rightarrow G \cap H_N^{sd}$ can be combined, and that all this leads in principle to a 3-parameter series. All this is, however, quite technical, and we do not really know if it is so. We will actually not need all this, so we will just stop our study here, and recommend here [82] and subsequent papers.

As in the continuous case, a solution to these classification problems comes from the notion of easiness. We have indeed the following result, coming from [13], [85]:

THEOREM 10.10. *The following groups are easy:*

- (1) $\mathbb{Z}_r H_N$, the corresponding category $P_{even}^r \subset P_{even}$ consisting of the partitions having the property that when flattening, we have the following global formula:

$$\# \circ = \# \bullet (r)$$

- (2) $H_N^s = \mathbb{Z}_s \wr S_N$, the corresponding category $P_{even}^{(s)} \subset P_{even}$ consisting of the partitions having the property that we have the following formula, in each block:

$$\# \circ = \# \bullet (s)$$

In addition, the easy solutions of $H_N \subset G \subset K_N$ appear by combining these examples.

PROOF. All this is well-known, the idea being as follows:

(1) The computation here is similar to the one in the proof of Theorem 10.7, by writing the fundamental representation $u = zv$ as there.

(2) This is something very standard and fundamental, known since the paper [13], and which follows from a long, routine computation, performed there.

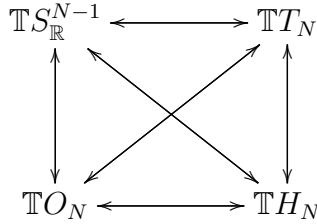
As for the last assertion, things here are quite technical, and for the precise statement and proof of the classification result, we refer here to paper [85]. \square

Summarizing, the situation here is more complicated than in the continuous group case. However, in what regards the “standard” solution, this is definitely $\mathbb{T}H_N$.

10c. Axiom check

With all this preliminary work done, let us turn now to our main question, namely constructing new geometries. We have here the following result:

THEOREM 10.11. *We have correspondences as follows,*



which produce a new geometry.

PROOF. We have indeed a quadruplet (S, T, U, K) as in the statement, produced by the various constructions above. Regarding now the verification of the axioms:

(1) We have the following computation:

$$\begin{aligned}
 P(\mathbb{T}S_{\mathbb{R}}^{N-1} \cap \mathbb{T}_N^+) &= P(\mathbb{T}S_{\mathbb{R}}^{N-1} \cap \mathbb{T}_N) \\
 &\subset P\mathbb{T}S_{\mathbb{R}}^{N-1} \cap P\mathbb{T}_N \\
 &= P_{\mathbb{R}}^{N-1} \cap \mathbb{T}_{N-1} \\
 &= T_{N-1}
 \end{aligned}$$

By lifting, we obtain from this that we have:

$$\mathbb{T}S_{\mathbb{R}}^{N-1} \cap \mathbb{T}_N^+ \subset \mathbb{T}T_N$$

The inclusion “ \supset ” being clear as well, we are done with checking the first axiom.

(2) The second axiom states that we must have:

$$\mathbb{T}H_N \cap \mathbb{T}_N^+ = \mathbb{T}T_N$$

The verification of this second axiom is similar.

(3) The third axiom states that we must have:

$$\mathbb{T}O_N \cap K_N^+ = \mathbb{T}H_N$$

But can be checked either directly, or by proceeding as above, by taking projective versions, and then lifting.

(4) The quantum isometry group axiom states that we must have:

$$G^+(\mathbb{T}S_{\mathbb{R}}^{N-1}) = \mathbb{T}O_N$$

The verification of this axiom is routine, and all this is explained for instance in [8].

(5) The quantum reflection group axiom states that we must have:

$$G^+(\mathbb{T}T_N) \cap K_N^+ = \mathbb{T}H_N$$

But this can be checked in a similar way, by adapting the computation from the classical real case.

(6) Regarding now the hard liberation axiom, this is clear, because we have:

$$\begin{aligned} \langle O_N, \mathbb{T}T_N \rangle &= \langle O_N, \mathbb{T}, T_N \rangle \\ &= \langle O_N, \mathbb{T} \rangle \\ &= \mathbb{T}O_N \end{aligned}$$

(7) Finally, we have as well the last axiom, namely:

$$S_{\mathbb{T}O_N} = \mathbb{T}S_{\mathbb{R}}^{N-1}$$

But this completes the proof. □

Let us discuss now the half-classical and free extensions of Theorem 10.11, and of some of the results preceding it. In order to have no redundant discussion and diagrams, we will talk directly about the $\times 9$ extension of the theory that we have so far.

We first need to complete our collection of spheres S , tori T , unitary groups U , and reflection groups K . In what regards the spheres, the result is as follows:

PROPOSITION 10.12. *We have noncommutative spheres as follows,*

$$\begin{array}{ccccc} S_{\mathbb{R},+}^{N-1} & \longrightarrow & \mathbb{T}S_{\mathbb{R},+}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\ \uparrow & & \uparrow & & \uparrow \\ S_{\mathbb{R},*}^{N-1} & \longrightarrow & \mathbb{T}S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{C},*}^{N-1} \\ \uparrow & & \uparrow & & \uparrow \\ S_{\mathbb{R}}^{N-1} & \longrightarrow & \mathbb{T}S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{C}}^{N-1} \end{array}$$

with the middle vertical objects coming via the relations $ab^* = a^*b$.

PROOF. We can indeed construct new spheres via the relations $ab^* = a^*b$, and these fit into previous 6-diagram of spheres as indicated. As for the fact that in the classical case we obtain the previously constructed sphere $\mathbb{T}S_{\mathbb{R}}^{N-1}$, this follows from Theorem 10.1 and its proof, because the relations used there are precisely those of type $a\bar{b} = \bar{a}b$. □

There are many things that can be done with the above spheres. As a basic result here, let us record the following fact, regarding the corresponding projective spaces:

THEOREM 10.13. *The projective spaces associated to the basic spheres are*

$$\begin{array}{ccccc}
 P_+^{N-1} & \longrightarrow & P_+^{N-1} & \longrightarrow & P_+^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 P_{\mathbb{C}}^{N-1} & \longrightarrow & P_{\mathbb{C}}^{N-1} & \longrightarrow & P_{\mathbb{C}}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 P_{\mathbb{R}}^{N-1} & \longrightarrow & P_{\mathbb{R}}^{N-1} & \longrightarrow & P_{\mathbb{C}}^{N-1}
 \end{array}$$

via the standard identifications for noncommutative algebraic manifolds.

PROOF. This is something that we already know for the 6 previous spheres. As for the 3 new spheres, this follows from the defining relations $ab^* = a^*b$, which tell us that the coordinates of the corresponding projective spaces must be self-adjoint. \square

At the torus level now, the construction is similar, as follows:

PROPOSITION 10.14. *We have noncommutative tori as follows,*

$$\begin{array}{ccccc}
 T_N^+ & \longrightarrow & \mathbb{T}T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 T_N^* & \longrightarrow & \mathbb{T}T_N^* & \longrightarrow & \mathbb{T}_N^* \\
 \uparrow & & \uparrow & & \uparrow \\
 T_N & \longrightarrow & \mathbb{T}T_N & \longrightarrow & \mathbb{T}_N
 \end{array}$$

with the middle vertical objects coming via the relations $ab^* = a^*b$.

PROOF. This is clear from Proposition 10.12, by intersecting everything with \mathbb{T}_N^+ . \square

In what regards the unitary quantum groups, the result is as follows:

THEOREM 10.15. *We have quantum groups as follows, which are all easy,*

$$\begin{array}{ccccc}
 O_N^+ & \longrightarrow & \mathbb{T}O_N^+ & \longrightarrow & U_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 O_N^* & \longrightarrow & \mathbb{T}O_N^* & \longrightarrow & U_N^* \\
 \uparrow & & \uparrow & & \uparrow \\
 O_N & \longrightarrow & \mathbb{T}O_N & \longrightarrow & U_N
 \end{array}$$

with the middle vertical objects coming via the relations $ab^ = a^*b$.*

PROOF. This is standard, indeed, the categories of partitions being as follows:

$$\begin{array}{ccccc}
 NC_2 & \longleftarrow & \bar{N}C_2 & \longleftarrow & \mathcal{N}C_2 \\
 \downarrow & & \downarrow & & \downarrow \\
 P_2^* & \longleftarrow & \bar{P}_2^* & \longleftarrow & \mathcal{P}_2^* \\
 \downarrow & & \downarrow & & \downarrow \\
 P_2 & \longleftarrow & \bar{P}_2 & \longleftarrow & \mathcal{P}_2
 \end{array}$$

Observe that our diagrams are both intersection diagrams. □

Regarding the quantum reflection groups, we have here:

THEOREM 10.16. *We have quantum groups as follows, which are all easy,*

$$\begin{array}{ccccc}
 H_N^+ & \longrightarrow & \mathbb{T}H_N^+ & \longrightarrow & K_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N^* & \longrightarrow & \mathbb{T}H_N^* & \longrightarrow & K_N^* \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N & \longrightarrow & \mathbb{T}H_N & \longrightarrow & K_N
 \end{array}$$

with the middle vertical objects coming via the relations $ab^ = a^*b$.*

PROOF. This is standard, indeed, the categories of partitions being as follows:

$$\begin{array}{ccccc}
 NC_{even} & \longleftarrow & \bar{N}C_{even} & \longleftarrow & \mathcal{N}C_{even} \\
 \downarrow & & \downarrow & & \downarrow \\
 P_{even}^* & \longleftarrow & \bar{P}_{even}^* & \longleftarrow & \mathcal{P}_{even}^* \\
 \downarrow & & \downarrow & & \downarrow \\
 P_{even} & \longleftarrow & \bar{P}_{even} & \longleftarrow & \mathcal{P}_{even}
 \end{array}$$

Observe that our diagrams are both intersection diagrams. □

Let us point out that we have some interesting questions, regarding the classification of the intermediate compact quantum groups for the following 4 inclusions:

$$\begin{array}{ccccc}
 & & K_N^+ & \longrightarrow & U_N^+ \\
 & \nearrow & \uparrow & & \nearrow \\
 H_N^+ & \longrightarrow & O_N^+ & & \\
 & \nwarrow & \downarrow & & \nwarrow \\
 & & K_N^* & \longrightarrow & U_N^* \\
 & \nearrow & \uparrow & & \nearrow \\
 H_N^* & \longrightarrow & O_N^* & &
 \end{array}$$

In what regards the half-classical questions, these can be in principle fully investigated by using the technology in [41], but we do not know what the final answer is. As for the free questions, these are more delicate, but in the easy case, they are solved by [85].

Getting back now to the verification of the axioms, we first have:

THEOREM 10.17. *The quantum isometries of the basic spheres, namely*

$$\begin{array}{ccccc}
 S_{\mathbb{R},+}^{N-1} & \longrightarrow & \mathbb{T}S_{\mathbb{R},+}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{R},*}^{N-1} & \longrightarrow & \mathbb{T}S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{C},*}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1} & \longrightarrow & \mathbb{T}S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{C}}^{N-1}
 \end{array}$$

are the basic unitary quantum groups.

PROOF. This is routine, by lifting the results that we already have. □

Regarding now the tori, we first have here:

PROPOSITION 10.18. *The quantum isometries of the basic tori are*

$$\begin{array}{ccccc}
 H_N^+ & \longrightarrow & \mathbb{T}H_N^+ & \longrightarrow & K_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N^* & \longrightarrow & \mathbb{T}H_N^* & \longrightarrow & K_N^* \\
 \uparrow & & \uparrow & & \uparrow \\
 \bar{O}_N & \longrightarrow & \mathbb{T}\bar{O}_N & \longrightarrow & \bar{U}_N
 \end{array}$$

with the bars denoting as usual Schur-Weyl twists.

PROOF. The result follows by lifting the results that we already have. □

By looking now at quantum reflections, we obtain:

THEOREM 10.19. *The quantum reflections of the tori,*

$$\begin{array}{ccccc}
 T_N^+ & \longrightarrow & \mathbb{T}T_N^+ & \longrightarrow & \mathbb{T}^+T_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 T_N^* & \longrightarrow & \mathbb{T}T_N^* & \longrightarrow & \mathbb{T}^*T_N^* \\
 \uparrow & & \uparrow & & \uparrow \\
 T_N & \longrightarrow & \mathbb{T}T_N & \longrightarrow & \mathbb{T}T_N
 \end{array}$$

are the basic quantum reflection groups.

PROOF. This is indeed routine, by intersecting. □

Finally, we have hard liberation results, as follows:

THEOREM 10.20. *We have hard liberation formulae of type*

$$U = \langle O_N, T \rangle$$

for all the basic unitary quantum groups.

PROOF. We only need to check this for the “hybrid” examples, constructed in this section. But for these hybrid examples, $U = \mathbb{T}O_N^\times$, the results follow from:

$$\begin{aligned} \mathbb{T}O_N^\times &= \langle \mathbb{T}, O_N^\times \rangle \\ &= \langle \mathbb{T}, \langle O_N, T_N^\times \rangle \rangle \\ &= \langle O_N, \langle \mathbb{T}, T_N^\times \rangle \rangle \\ &= \langle O_N, \mathbb{T}T_N^\times \rangle \end{aligned}$$

Thus, we have indeed complete hard liberation results, as claimed. □

We can now formulate our main result, as follows:

THEOREM 10.21. *We have 9 noncommutative geometries, as follows,*

$$\begin{array}{ccccc} \mathbb{R}_+^N & \longrightarrow & \mathbb{T}\mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{R}_*^N & \longrightarrow & \mathbb{T}\mathbb{R}_*^N & \longrightarrow & \mathbb{C}_*^N \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{R}^N & \longrightarrow & \mathbb{T}\mathbb{R}^N & \longrightarrow & \mathbb{C}^N \end{array}$$

with each of the \mathbb{K}^\times symbols standing for the corresponding quadruplet.

PROOF. This follows indeed by putting everything together, a bit as in the proof of Theorem 10.11, the idea being that the intersection axioms are clear, the quantum isometry axioms follow from the above computations, and the remaining axioms are elementary. Thus, we are led to the conclusion in the statement. □

Summarizing, we are done with the extension program mentioned in chapter 4, and started in the previous chapter, and this with the technical remark that, in what concerns the “hybrid” geometries, lying between real and complex, our choice of the group \mathbb{T} for “multiplying the real geometries” might be actually something a bit arbitrary, because the whole family of groups \mathbb{Z}_r with $r < \infty$ is waiting to be investigated as well.

As a second comment, it is of course possible to further develop the hybrid geometries that we found here, but the whole subject looks less interesting than the subject of further developing the half-classical geometries. Thus, we will stop our study here, and after talking next about classification results, and then in chapter 11 about twists, we will be back in chapter 12 below to the half-classical geometries.

10d. Classification results

Getting now into classification results, let us recall from chapter 4 that a geometry coming from a quadruplet (S, T, U, K) is easy when both the quantum groups U, K are easy, and when the following easy generation formula is satisfied:

$$U = \{O_N, K\}$$

Combinatorially, this gives the following statement:

PROPOSITION 10.22. *An easy geometry is uniquely determined by a pair (D, E) of categories of partitions, which must be as follows,*

$$\mathcal{NC}_2 \subset D \subset P_2$$

$$\mathcal{NC}_{\text{even}} \subset E \subset P_{\text{even}}$$

and which are subject to the following intersection and generation conditions,

$$D = E \cap P_2$$

$$E = \langle D, \mathcal{NC}_{\text{even}} \rangle$$

and to the usual axioms for the associated quadruplet (S, T, U, K) , where U, K are respectively the easy quantum groups associated to the categories D, E .

PROOF. This statement simply comes from the following conditions:

$$U = \{O_N, K\}$$

$$K = U \cap K_N^+$$

Indeed, U, K must be easy, coming from certain categories of partitions D, E . It is clear that D, E must appear as intermediate categories, as in the statement, and the fact that the intersection and generation conditions must be satisfied follows from:

$$\begin{aligned} U = \{O_N, K\} &\iff D = E \cap P_2 \\ K = U \cap K_N^+ &\iff E = \langle D, \mathcal{NC}_{\text{even}} \rangle \end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

In order to discuss now classification results, we will need some technical results regarding the intermediate easy quantum groups as follows:

$$H_N \subset G \subset K_N^+$$

$$O_N \subset G \subset U_N^+$$

Regarding the reflection groups, the complete result here known so far, from [81], concerns only the real case. This result, in a simplified form, is as follows:

THEOREM 10.23. *The easy quantum groups $H_N \subset G \subset H_N^+$ are as follows,*

$$H_N \subset H_N^\Gamma \subset H_N^{[\infty]} \subset H_N^{[r]} \subset H_N^+$$

with the family H_N^Γ covering $H_N, H_N^{[\infty]}$, and with the series $H_N^{[r]}$ covering H_N^+ .

PROOF. This is something quite technical, from [81], the idea being as follows:

(1) We have a dichotomy concerning the quantum groups $H_N \subset G \subset H_N^+$, which must fall into one of the following two classes:

$$H_N \subset G \subset H_N^{[\infty]}$$

$$H_N^{[\infty]} \subset G \subset H_N^+$$

This comes indeed from various papers, and more specifically from the final classification paper of Raum and Weber [81], where the quantum groups $S_N \subset G \subset H_N^+$ with $G \not\subset H_N^{[\infty]}$ were classified, and shown to contain $H_N^{[\infty]}$. For details here, we refer to [81].

(2) Regarding the first case, namely $H_N \subset G \subset H_N^{[\infty]}$, the result here, from [81], is quite technical. Consider a discrete group generated by real reflections, $g_i^2 = 1$:

$$\Gamma = \langle g_1, \dots, g_N \rangle$$

We call Γ uniform if each $\sigma \in S_N$ produces a group automorphism, as follows:

$$g_i \rightarrow g_{\sigma(i)}$$

In this case, we can associate to Γ a family of subsets $D(k, l) \subset P(k, l)$, which form a category of partitions, as follows:

$$D(k, l) = \left\{ \pi \in P(k, l) \mid \ker \begin{pmatrix} i \\ j \end{pmatrix} \leq \pi \implies g_{i_1} \dots g_{i_k} = g_{j_1} \dots g_{j_l} \right\}$$

Observe that we have inclusions of categories as follows, coming respectively from $\eta \in D$, and from the quotient map $\Gamma \rightarrow \mathbb{Z}_2^N$:

$$P_{even}^{[\infty]} \subset D \subset P_{even}$$

Conversely, to any category of partitions $P_{even}^{[\infty]} \subset D \subset P_{even}$ we can associate a uniform reflection group $\mathbb{Z}_2^{*N} \rightarrow \Gamma \rightarrow \mathbb{Z}_2^N$, as follows:

$$\Gamma = \left\langle g_1, \dots, g_N \mid g_{i_1} \dots g_{i_k} = g_{j_1} \dots g_{j_l}, \forall i, j, k, l, \ker \begin{pmatrix} i \\ j \end{pmatrix} \in D(k, l) \right\rangle$$

As explained in [81], the correspondences $\Gamma \rightarrow D$ and $D \rightarrow \Gamma$ constructed above are bijective, and inverse to each other, at $N = \infty$. Thus, we are done with the first case.

(3) Regarding now the second case, which is the one left, namely $H_N^{[\infty]} \subset G \subset H_N^+$, the result here, also from [81], is quite technical as well, but has a simple formulation. Let indeed $H_N^{[r]} \subset H_N^+$ be the easy quantum group coming from:

$$\pi_r = \ker \begin{pmatrix} 1 & \dots & r & r & \dots & 1 \\ 1 & \dots & r & r & \dots & 1 \end{pmatrix}$$

We have then inclusions of quantum groups as follows:

$$H_N^+ = H_N^{[1]} \supset H_N^{[2]} \supset H_N^{[3]} \supset \dots \supset H_N^{[\infty]}$$

We obtain in this way all the intermediate easy quantum groups $H_N^{[\infty]} \subset G \subset H_N^+$ satisfying the assumption $G \neq H_N^{[\infty]}$, and this finishes the proof. See [81]. \square

Let us discuss now the unitary quantum groups. Once again, there are only partial results here known so far, and we will need in what follows, for our classification purposes here, the results in [75], concerning the intermediate easy quantum groups as follows:

$$U_N \subset G \subset U_N^+$$

A first construction of such quantum groups is as follows:

PROPOSITION 10.24. *Associated to any $r \in \mathbb{N}$ is the quantum group $U_N \subset U_N^{(r)} \subset U_N^+$ coming from the category $\mathcal{P}_2^{(r)}$ of matching pairings having the property that*

$$\# \circ = \# \bullet (r)$$

holds between the legs of each string. These quantum groups have the following properties:

- (1) *At $r = 1$ we obtain the usual unitary group, $U_N^{(1)} = U_N$.*
- (2) *At $r = 2$ we obtain the half-classical unitary group, $U_N^{(2)} = U_N^*$.*
- (3) *For any $r|s$ we have an embedding $U_N^{(r)} \subset U_N^{(s)}$.*
- (4) *In general, we have an embedding $U_N^{(r)} \subset U_N^r \rtimes \mathbb{Z}_r$.*
- (5) *We have as well a cyclic matrix model $C(U_N^{(r)}) \subset M_r(C(U_N^r))$.*
- (6) *In this latter model, $\int_{U_N^{(r)}}$ appears as the restriction of $tr_r \otimes \int_{U_N^r}$.*

PROOF. This is something quite compact, summarizing various findings from [17], [74]. Here are a few brief explanations on all this:

- (1) This is clear from $\mathcal{P}_2^{(1)} = \mathcal{P}_2$, and from a well-known result of Brauer [44].
- (2) This is because $\mathcal{P}_2^{(2)}$ is generated by the partitions with implement the relations $abc = cba$ between the variables $\{u_{ij}, u_{ij}^*\}$, used in [41] for constructing U_N^* .
- (3) This simply follows from $\mathcal{P}_2^{(s)} \subset \mathcal{P}_2^{(r)}$, by functoriality.
- (4) This is the original definition of $U_N^{(r)}$, from [17]. We refer to [17] for the exact formula of the embedding, and to [74] for the compatibility with the Tannakian definition.

(5) This is also from [17], more specifically it is an alternative definition for $U_N^{(r)}$.

(6) Once again, this is something from [17], and we will be back to it. \square

Let us discuss now the second known construction of unitary quantum groups, from [75]. This construction uses an additive semigroup $D \subset \mathbb{N}$, but as pointed out there, using instead the complementary set $C = \mathbb{N} - D$ leads to several simplifications.

So, let us call “cosemigroup” any subset $C \subset \mathbb{N}$ which is complementary to an additive semigroup, $x, y \notin C \implies x + y \notin C$. The construction from [75] is then:

PROPOSITION 10.25. *Associated to any cosemigroup $C \subset \mathbb{N}$ is the easy quantum group $U_N \subset U_N^C \subset U_N^+$ coming from the category $\mathcal{P}_2^C \subset \mathcal{P}_2^{(\infty)}$ of pairings having the property*

$$\# \circ -\# \bullet \in C$$

between each two legs colored \circ, \bullet of two strings which cross. We have:

- (1) *For $C = \emptyset$ we obtain the quantum group U_N^+ .*
- (2) *For $C = \{0\}$ we obtain the quantum group U_N^\times .*
- (3) *For $C = \{0, 1\}$ we obtain the quantum group U_N^{**} .*
- (4) *For $C = \mathbb{N}$ we obtain the quantum group $U_N^{(\infty)}$.*
- (5) *For $C \subset C'$ we have an inclusion $U_N^{C'} \subset U_N^C$.*
- (6) *Each quantum group U_N^C contains each quantum group $U_N^{(r)}$.*

PROOF. Once again this is something very compact, coming from recent work in [75], with our convention that the semigroup $D \subset \mathbb{N}$ which is used there is replaced here by its complement $C = \mathbb{N} - D$. Here are a few explanations on all this:

(1) The assumption $C = \emptyset$ means that the condition $\# \circ -\# \bullet \in C$ can never be applied. Thus, the strings cannot cross, we have $\mathcal{P}_2^\emptyset = \mathcal{NC}_2$, and so $U_N^\emptyset = U_N^+$.

(2) As explained in [75], here we obtain indeed the quantum group U_N^\times , constructed by using the relations $ab^*c = cb^*a$, with $a, b, c \in \{u_{ij}\}$.

(3) This is also explained in [75], with U_N^{**} being the quantum group from [17], which is the biggest whose full projective version, in the sense there, is classical.

(4) Here the assumption $C = \mathbb{N}$ simply tells us that the condition $\# \circ -\# \bullet \in C$ in the statement is irrelevant. Thus, we have $\mathcal{P}_2^\mathbb{N} = \mathcal{P}_2^{(\infty)}$, and so $U_N^\mathbb{N} = U_N^{(\infty)}$.

(5) This is clear by functoriality, because $C \subset C'$ implies $\mathcal{P}_2^C \subset \mathcal{P}_2^{C'}$.

(6) This is clear from definitions, and from Proposition 10.24 above. \square

We have the following key result, from [75]:

THEOREM 10.26. *The easy quantum groups $U_N \subset G \subset U_N^+$ are as follows,*

$$U_N \subset \{U_N^{(r)}\} \subset \{U_N^C\} \subset U_N^+$$

with the series covering U_N , and the family covering U_N^+ .

PROOF. This is something non-trivial, and we refer here to [75]. The general idea is that $U_N^{(\infty)}$ produces a dichotomy for the quantum groups in the statement, and this leads, via combinatorial computations, to the series and the family. See [75]. \square

Now back to our abstract noncommutative geometries, as axiomatized here, in the easy case we have the following classification result, based on the above:

THEOREM 10.27. *There are exactly 4 geometries which are easy, uniform and pure, with purity meaning that the geometry must be real, classical, complex or free, namely:*

$$\begin{array}{ccc} \mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\ \uparrow & & \uparrow \\ \mathbb{R}^N & \longrightarrow & \mathbb{C}^N \end{array}$$

When lifting the uniformity and purity conditions, and replacing them with a “slicing” axiom, we have 9 such geometries, namely those in Theorem 10.21.

PROOF. All this is quite technical, the idea being as follows:

(1) Assume first that we have an easy geometry which is pure, in the sense that it lies on one of the 4 edges of the square in the statement. We know from Proposition 10.22 that its unitary group U must come from a category of pairings D satisfying:

$$D = \langle D, \mathcal{N}\mathcal{C}_{\text{even}} \rangle \cap P_2$$

But this equation can be solved by using the results in [74], [75], [81], [85], and by using the uniformity axiom, which excludes the half-liberations and the hybrids, we are led to the conclusion that the only solutions are the 4 vertices of the square.

(2) Regarding the second assertion, this can be obtained by using the same technology, by using the “slicing” axiom from [12], which amounts in saying that U , or the geometry itself, can be reconstructed from its projections on the edges of the square. See [12]. \square

10e. Exercises

Things have been very technical in this chapter, and as unique exercise, we have:

EXERCISE 10.28. *Find a better way of classifying the noncommutative geometries in our sense, say by adding some clever extra axiom, which simplifies the classification.*

Needless to say, an answer here would be very interesting. Beer offered.

CHAPTER 11

Twisted geometry

11a. Ad-hoc twists

We have seen so far that the abstract noncommutative geometries, taken in a “spherical” sense, with coordinates bounded by $\|x_i\| \leq 1$, can be axiomatized with the help of quadruplets (S, T, U, K) . There are 9 main such geometries, as follows:

$$\begin{array}{ccccc}
 \mathbb{R}_+^N & \longrightarrow & \text{TR}_+^N & \longrightarrow & \mathbb{C}_+^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{R}_*^N & \longrightarrow & \text{TR}_*^N & \longrightarrow & \mathbb{C}_*^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{R}^N & \longrightarrow & \text{TR}^N & \longrightarrow & \mathbb{C}^N
 \end{array}$$

As a first related question, we would like to investigate the $q = -1$ twists of these geometries. We will see that we have twisted geometries, as follows:

$$\begin{array}{ccccc}
 \mathbb{R}_+^N & \longrightarrow & \text{TR}_+^N & \longrightarrow & \mathbb{C}_+^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \bar{\mathbb{R}}_*^N & \longrightarrow & \text{T}\bar{\mathbb{R}}_*^N & \longrightarrow & \bar{\mathbb{C}}_*^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \bar{\mathbb{R}}^N & \longrightarrow & \text{T}\bar{\mathbb{R}}^N & \longrightarrow & \bar{\mathbb{C}}^N
 \end{array}$$

In order to get started, the best is to deform first the simplest objects that we have, namely the quantum spheres. This can be done as follows:

THEOREM 11.1. *We have quantum spheres as follows, obtained via the twisted commutation relations $ab = \pm ba$, and twisted half-commutation relations $abc = \pm cba$,*

$$\begin{array}{ccccc}
S_{\mathbb{R},+}^{N-1} & \longrightarrow & \mathbb{T}S_{\mathbb{R},+}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\
\uparrow & & \uparrow & & \uparrow \\
\bar{S}_{\mathbb{R},*}^{N-1} & \longrightarrow & \mathbb{T}\bar{S}_{\mathbb{R},*}^{N-1} & \longrightarrow & \bar{S}_{\mathbb{C},*}^{N-1} \\
\uparrow & & \uparrow & & \uparrow \\
\bar{S}_{\mathbb{R}}^{N-1} & \longrightarrow & \mathbb{T}\bar{S}_{\mathbb{R}}^{N-1} & \longrightarrow & \bar{S}_{\mathbb{C}}^{N-1}
\end{array}$$

with the precise signs being as follows:

- (1) *The signs on the bottom correspond to the anticommutation of distinct coordinates, and their adjoints. That is, with $z_i = x_i, x_i^*$ and $\varepsilon_{ij} = 1 - \delta_{ij}$, the formula is:*

$$z_i z_j = (-1)^{\varepsilon_{ij}} z_j z_i$$

- (2) *The signs in the middle come from functoriality, as for the spheres in the middle to contain those on the bottom. That is, the formula is:*

$$z_i z_j z_k = (-1)^{\varepsilon_{ij} + \varepsilon_{jk} + \varepsilon_{ik}} z_k z_j z_i$$

PROOF. As a first observation, we are using here bars in order to denote the $q = -1$ twists, and this in view of the discussion above, leading us to $q = \pm 1$, and with the $q = -1$ theory that we want to develop being different from the usual one.

- (1) Here there is nothing to prove, because we can define the spheres on the bottom by the following formulae, with $z_i = x_i, x_i^*$ and $\varepsilon_{ij} = 1 - \delta_{ij}$ being as above:

$$C(\bar{S}_{\mathbb{R}}^{N-1}) = C(S_{\mathbb{R},+}^{N-1}) / \left\langle x_i x_j = (-1)^{\varepsilon_{ij}} x_j x_i \right\rangle$$

$$C(\bar{S}_{\mathbb{C}}^{N-1}) = C(S_{\mathbb{C},+}^{N-1}) / \left\langle z_i z_j = (-1)^{\varepsilon_{ij}} z_j z_i \right\rangle$$

- (2) Here our claim is that, if we want to construct half-classical twisted spheres, via relations of type $abc = \pm cba$ between the coordinates x_i and their adjoints x_i^* , as for these spheres to contain the twisted spheres constructed in (1), the only possible choice for these relations is as follows, with $z_i = x_i, x_i^*$ and $\varepsilon_{ij} = 1 - \delta_{ij}$ being as above:

$$z_i z_j z_k = (-1)^{\varepsilon_{ij} + \varepsilon_{jk} + \varepsilon_{ik}} z_k z_j z_i$$

But this is something clear, coming from the following computation, inside of the quotient algebras corresponding to the twisted spheres constructed in (1) above:

$$\begin{aligned} z_i z_j z_k &= (-1)^{\varepsilon_{ij}} z_j z_i z_k \\ &= (-1)^{\varepsilon_{ij} + \varepsilon_{ik}} z_j z_k z_i \\ &= (-1)^{\varepsilon_{ij} + \varepsilon_{jk} + \varepsilon_{ik}} z_k z_j z_i \end{aligned}$$

Thus, we are led to the conclusion in the statement, the spheres being given by:

$$C(\bar{S}_{\mathbb{R},*}^{N-1}) = C(S_{\mathbb{R},+}^{N-1}) / \left\langle x_i x_j x_k = (-1)^{\varepsilon_{ij} + \varepsilon_{jk} + \varepsilon_{ik}} x_k x_j x_i \right\rangle$$

$$C(\bar{S}_{\mathbb{C},*}^{N-1}) = C(S_{\mathbb{C},+}^{N-1}) / \left\langle z_i z_j z_k = (-1)^{\varepsilon_{ij} + \varepsilon_{jk} + \varepsilon_{ik}} z_k z_j z_i \right\rangle$$

Thus, we have constructed our spheres, and embeddings, as desired. \square

With the above result in hand, let us go ahead now, and twist the whole quadruplets (S, T, U, K) that we have.

Things are quite tricky here, and let us start with the unitary quantum groups U .

We would like these quantum groups to act on the corresponding spheres, $U \curvearrowright S$. Thus, we would like to have morphisms of algebras, as follows:

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

Now with $z_i = x_i, x_i^*$ being as before, and with $v_{ij} = u_{ij}, u_{ij}^*$ constructed accordingly, the above formula and its adjoint tell us that we must have:

$$\Phi(z_i) = \sum_j z_j \otimes v_{ji}$$

Thus the variables $Z_i = \sum_j z_j \otimes v_{ji}$ on the right must satisfy the twisted commutation or half-commutation relations in Theorem 11.1, and this will lead us to the correct twisted commutation or half-commutation relations to be satisfied by the variables v_{ij} .

In practice now, let us first discuss the twisting of O_N, U_N .

Following [18] in the orthogonal case, and then [4] in the unitary case, the result here is as follows:

THEOREM 11.2. *We have twisted orthogonal and unitary groups, as follows,*

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ \bar{O}_N & \longrightarrow & \bar{U}_N \end{array}$$

defined via the following relations, with the convention $\alpha = a, a^*$ and $\beta = b, b^*$:

$$\alpha\beta = \begin{cases} -\beta\alpha & \text{for } a, b \in \{u_{ij}\} \text{ distinct, on the same row or column of } u \\ \beta\alpha & \text{otherwise} \end{cases}$$

These quantum groups act on the corresponding twisted real and complex spheres.

PROOF. Let us first discuss the construction of the quantum group \bar{O}_N . We must prove that the algebra $C(\bar{O}_N)$ obtained from $C(O_N^+)$ via the relations in the statement has a comultiplication Δ , a counit ε , and an antipode S . Regarding Δ , let us set:

$$U_{ij} = \sum_k u_{ik} \otimes u_{kj}$$

For $j \neq k$ we have the following computation:

$$\begin{aligned} U_{ij}U_{ik} &= \sum_{s \neq t} u_{is}u_{it} \otimes u_{sj}u_{tk} + \sum_s u_{is}u_{is} \otimes u_{sj}u_{sk} \\ &= \sum_{s \neq t} -u_{it}u_{is} \otimes u_{tk}u_{sj} + \sum_s u_{is}u_{is} \otimes (-u_{sk}u_{sj}) \\ &= -U_{ik}U_{ij} \end{aligned}$$

Also, for $i \neq k, j \neq l$ we have the following computation:

$$\begin{aligned} U_{ij}U_{kl} &= \sum_{s \neq t} u_{is}u_{kt} \otimes u_{sj}u_{tl} + \sum_s u_{is}u_{ks} \otimes u_{sj}u_{sl} \\ &= \sum_{s \neq t} u_{kt}u_{is} \otimes u_{tl}u_{sj} + \sum_s (-u_{ks}u_{is}) \otimes (-u_{sl}u_{sj}) \\ &= U_{kl}U_{ij} \end{aligned}$$

Thus, we can define a comultiplication map for $C(\bar{O}_N)$, by setting:

$$\Delta(u_{ij}) = U_{ij}$$

Regarding now the counit ε and the antipode S , things are clear here, by using the same method, and with no computations needed, the formulae to be satisfied being trivially satisfied. We conclude that \bar{O}_N is a compact quantum group, and the proof for \bar{U}_N is similar, by adding $*$ exponents everywhere in the above computations.

Finally, the last assertion is clear too, by doing some elementary computations, of the same type as above, and with the remark that the converse holds too, in the sense that if we want a quantum group $U \subset U_N^+$ to be defined by relations of type $ab = \pm ba$, and to have an action $U \curvearrowright S$ on the corresponding twisted sphere, we are led to the relations in the statement. We refer to [4] for further details on all this. \square

In order to discuss now the half-classical case, given three coordinates $a, b, c \in \{u_{ij}\}$, let us set $span(a, b, c) = (r, c)$, where $r, c \in \{1, 2, 3\}$ are the number of rows and columns spanned by a, b, c . In other words, if we write $a = u_{ij}, b = u_{kl}, c = u_{pq}$ then $r = \#\{i, k, p\}$ and $l = \#\{j, l, q\}$. With this convention, we have the following result:

THEOREM 11.3. *We have intermediate quantum groups as follows,*

$$\begin{array}{ccccc}
 O_N^+ & \longrightarrow & \mathbb{T}O_N^+ & \longrightarrow & U_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 \bar{O}_N^* & \longrightarrow & \mathbb{T}\bar{O}_N^* & \longrightarrow & \bar{U}_N^* \\
 \uparrow & & \uparrow & & \uparrow \\
 \bar{O}_N & \longrightarrow & \mathbb{T}\bar{O}_N & \longrightarrow & \bar{U}_N
 \end{array}$$

defined via the following relations, with $\alpha = a, a^*, \beta = b, b^*$ and $\gamma = c, c^*$,

$$\alpha\beta\gamma = \begin{cases} -\gamma\beta\alpha & \text{for } a, b, c \in \{u_{ij}\} \text{ with } span(a, b, c) = (\leq 2, 3) \text{ or } (3, \leq 2) \\ \gamma\beta\alpha & \text{otherwise} \end{cases}$$

which act on the corresponding twisted half-classical real and complex spheres.

PROOF. We use the same method as in the proof of Theorem 11.2, but with the combinatorics being now more complicated. Observe first that the rules for the various commutation and anticommutation signs in the statement can be summarized as follows:

$r \setminus c$	1	2	3
1	+	+	-
2	+	+	-
3	-	-	+

Let us first prove the result for \bar{O}_N^* . We must construct here morphisms Δ, ε, S , and the proof, similar to the proof of Theorem 11.2, goes as follows:

(1) We first construct Δ . For this purpose, we must prove that $U_{ij} = \sum_k u_{ik} \otimes u_{kj}$ satisfy the relations in the statement. We have the following computation:

$$\begin{aligned} U_{ia}U_{jb}U_{kc} &= \sum_{xyz} u_{ix}u_{jy}u_{kz} \otimes u_{xa}u_{yb}u_{zc} \\ &= \sum_{xyz} \pm u_{kz}u_{jy}u_{ix} \otimes \pm u_{zc}u_{yb}u_{xa} \\ &= \pm U_{kc}U_{jb}U_{ia} \end{aligned}$$

We must show that, when examining the precise two \pm signs in the middle formula, their product produces the correct \pm sign at the end. But the point is that both these signs depend only on $s = \text{span}(x, y, z)$, and for $s = 1, 2, 3$ respectively, we have:

- For a (3, 1) span we obtain $+-, +-, -+$, so a product $-$ as needed.
- For a (2, 1) span we obtain $++, ++, --$, so a product $+$ as needed.
- For a (3, 3) span we obtain $--, --, ++$, so a product $+$ as needed.
- For a (3, 2) span we obtain $+-, +-, -+$, so a product $-$ as needed.
- For a (2, 2) span we obtain $++, ++, --$, so a product $+$ as needed.

Together with the fact that our problem is invariant under $(r, c) \rightarrow (c, r)$, and with the fact that for a (1, 1) span there is nothing to prove, this finishes the proof for Δ .

(2) The construction of the counit, via the formula $\varepsilon(u_{ij}) = \delta_{ij}$, requires the Kronecker symbols δ_{ij} to commute/anticommute according to the above table. Equivalently, we must prove that the situation $\delta_{ij}\delta_{kl}\delta_{pq} = 1$ can appear only in a case where the above table indicates “+”. But this is clear, because $\delta_{ij}\delta_{kl}\delta_{pq} = 1$ implies $r = c$.

(3) Finally, the construction of the antipode, via the formula $S(u_{ij}) = u_{ji}$, is clear too, because this requires the choice of our \pm signs to be invariant under transposition, and this is true, the above table being symmetric.

We conclude that \bar{O}_N^* is indeed a compact quantum group, and the proof for \bar{U}_N^* is similar, by adding $*$ exponents everywhere in the above computations.

Finally, the last assertion is clear too, by doing some elementary computations, of the same type as above, and with the remark that the converse holds too, in the sense that if we want a quantum group $U \subset U_N^+$ to be defined by relations of type $abc = \pm cba$, and to have an action $U \curvearrowright S$ on the corresponding half-classical twisted sphere, we are led to the relations in the statement. We refer to [4] for further details on all this. \square

The above results can be summarized as follows:

THEOREM 11.4. *We have quantum groups as follows, obtained via the twisted commutation relations $ab = \pm ba$, and twisted half-commutation relations $abc = \pm cba$,*

$$\begin{array}{ccccc}
 O_N^+ & \longrightarrow & \mathbb{T}O_N^+ & \longrightarrow & U_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 \bar{O}_N^* & \longrightarrow & \mathbb{T}\bar{O}_N^* & \longrightarrow & \bar{U}_N^* \\
 \uparrow & & \uparrow & & \uparrow \\
 \bar{O}_N & \longrightarrow & \mathbb{T}\bar{O}_N & \longrightarrow & \bar{U}_N
 \end{array}$$

with the various signs coming as follows:

- (1) *The signs for \bar{O}_N correspond to anticommutation of distinct entries on rows and columns, and commutation otherwise, with this coming from $\bar{O}_N \curvearrowright \bar{S}_{\mathbb{R}}^{N-1}$.*
- (2) *The signs for $\bar{O}_N^*, \bar{U}_N, \bar{U}_N^*$ come as well from the signs for $\bar{S}_{\mathbb{R}}^{N-1}$, either via the requirement $\bar{O}_N \subset U$, or via the requirement $U \curvearrowright S$.*

PROOF. This is a summary of Theorem 11.2 and Theorem 11.3, and their proofs. \square

Moving ahead now, and back to our geometric program, we have twisted the spheres and unitary groups S, U , and we are left with twisting the tori and reflection groups T, K . But these are “discrete” objects, which can only be rigid, so let us formulate:

DEFINITION 11.5. *The twists of the basic quantum tori and reflection groups,*

$$\begin{array}{ccccc}
 T_N^+ & \longrightarrow & \mathbb{T}T_N^+ & \longrightarrow & \mathbb{T}T_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 T_N^* & \longrightarrow & \mathbb{T}T_N^* & \longrightarrow & \mathbb{T}T_N^* \\
 \uparrow & & \uparrow & & \uparrow \\
 T_N & \longrightarrow & \mathbb{T}T_N & \longrightarrow & \mathbb{T}T_N
 \end{array}
 \qquad
 \begin{array}{ccccc}
 H_N^+ & \longrightarrow & \mathbb{T}H_N^+ & \longrightarrow & K_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N^* & \longrightarrow & \mathbb{T}H_N^* & \longrightarrow & K_N^* \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N & \longrightarrow & \mathbb{T}H_N & \longrightarrow & K_N
 \end{array}$$

are by definition these tori and reflection groups themselves.

With this definition in hand, we are done with our twisting program for the triples (S, T, U, K) , and we have now candidates $\bar{\mathbb{R}}^N, \bar{\mathbb{C}}^N$ and $\bar{\mathbb{R}}_*^N, \bar{\mathbb{C}}_*^N$ for new noncommutative geometries, to be checked from our axiomatic viewpoint, and then to be developed.

11b. Schur-Weyl twists

In order to discuss these questions, we must first review the above construction of the twists of S, T, U, K , which was something quite ad-hoc, and replace all that has been said above by something more conceptual.

Let us start with something that we already know, namely:

PROPOSITION 11.6. *The intermediate easy quantum groups*

$$H_N \subset G \subset U_N^+$$

come via Tannakian duality from the intermediate categories of partitions

$$P_{\text{even}} \supset D \supset \mathcal{NC}_2$$

with $P_{\text{even}}(k, l) \subset P(k, l)$ being the category of partitions whose blocks have even size.

PROOF. This is something coming from the general easiness theory for quantum groups, discussed in section 2 above. Indeed, as explained there, the easy quantum groups appear as certain intermediate compact quantum groups, as follows:

$$S_N \subset G \subset U_N^+$$

To be more precise, such a quantum group is easy when the corresponding Tannakian category comes from an intermediate category of partitions, as follows:

$$P \supset D \supset \mathcal{NC}_2$$

Now since this correspondence makes correspond $H_N \leftrightarrow P_{\text{even}}$, once again as explained in section 2 above, we are led to the conclusion in the statement. \square

Summarizing, we must do some combinatorics, for the partitions having even blocks. Given a partition $\tau \in P(k, l)$, let us call “switch” the operation which consists in switching two neighbors, belonging to different blocks, in the upper row, or in the lower row. Also, we use the standard embedding $S_k \subset P_2(k, k)$, via the pairings having only up-to-down strings. With these conventions, we have the following result:

THEOREM 11.7. *There is a signature map $\varepsilon : P_{\text{even}} \rightarrow \{-1, 1\}$, given by*

$$\varepsilon(\tau) = (-1)^c$$

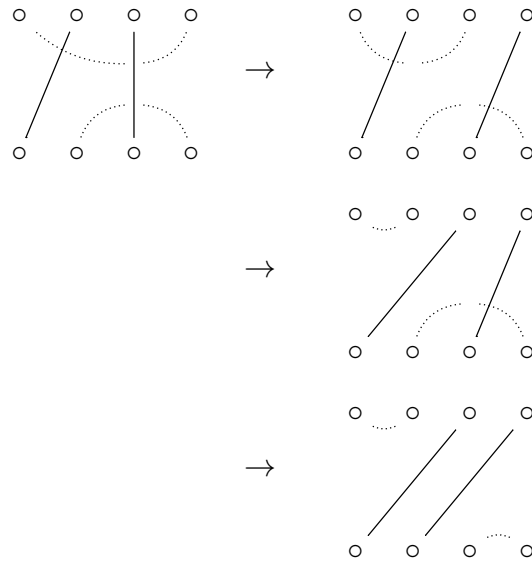
where c is the number of switches needed to make τ noncrossing. In addition:

- (1) For $\tau \in S_k$, this is the usual signature.
- (2) For $\tau \in P_2$ we have $(-1)^c$, where c is the number of crossings.
- (3) For $\tau \leq \pi \in \mathcal{NC}_{\text{even}}$, the signature is 1.

PROOF. In order to show that the signature map $\varepsilon : P_{even} \rightarrow \{-1, 1\}$ in the statement, given by $\varepsilon(\tau) = (-1)^c$, is well-defined, we must prove that the number c in the statement is well-defined modulo 2. It is enough to perform the verification for the noncrossing partitions. More precisely, given $\tau, \tau' \in NC_{even}$ having the same block structure, we must prove that the number of switches c required for the passage $\tau \rightarrow \tau'$ is even.

In order to do so, observe that any partition $\tau \in P(k, l)$ can be put in “standard form”, by ordering its blocks according to the appearance of the first leg in each block, counting clockwise from top left, and then by performing the switches as for block 1 to be at left, then for block 2 to be at left, and so on. Here the required switches are also uniquely determined, by the order coming from counting clockwise from top left.

Here is an example of such an algorithmic switching operation, with block 1 being first put at left, by using two switches, then with block 2 left unchanged, and then with block 3 being put at left as well, but at right of blocks 1 and 2, with one switch:



The point now is that, under the assumption $\tau \in NC_{even}(k, l)$, each of the moves required for putting a leg at left, and hence for putting a whole block at left, requires an even number of switches. Thus, putting τ in standard form requires an even number of switches. Now given $\tau, \tau' \in NC_{even}$ having the same block structure, the standard form coincides, so the number of switches c required for the passage $\tau \rightarrow \tau'$ is indeed even.

Regarding now the remaining assertions, these are all elementary:

- (1) For $\tau \in S_k$ the standard form is $\tau' = id$, and the passage $\tau \rightarrow id$ comes by composing with a number of transpositions, which gives the signature.

(2) For a general $\tau \in P_2$, the standard form is of type $\tau' = |\dots|_{\cap \dots \cap}^{\cup \dots \cup}$, and the passage $\tau \rightarrow \tau'$ requires $c \bmod 2$ switches, where c is the number of crossings.

(3) Assuming that $\tau \in P_{\text{even}}$ comes from $\pi \in NC_{\text{even}}$ by merging a certain number of blocks, we can prove that the signature is 1 by proceeding by recurrence. \square

We define the kernel of a multi-index $\binom{i}{j}$ to be the partition obtained by joining the equal indices.

Also, we write $\pi \leq \sigma$ if each block of π is contained in a block of σ .

With these conventions, and the above result in hand, we can now formulate:

DEFINITION 11.8. *Associated to any partition $\pi \in P_{\text{even}}(k, l)$ is the linear map*

$$\bar{T}_\pi : (\mathbb{C}^N)^{\otimes k} \rightarrow (\mathbb{C}^N)^{\otimes l}$$

given by the following formula, with e_1, \dots, e_N being the standard basis of \mathbb{C}^N ,

$$\bar{T}_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \bar{\delta}_\pi \left(\begin{matrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{matrix} \right) e_{j_1} \otimes \dots \otimes e_{j_l}$$

and where $\bar{\delta}_\pi \in \{-1, 0, 1\}$ is $\bar{\delta}_\pi = \varepsilon(\tau)$ if $\tau \geq \pi$, and $\bar{\delta}_\pi = 0$ otherwise, with:

$$\tau = \ker \left(\begin{matrix} i \\ j \end{matrix} \right)$$

In other words, what we are doing here is to add signatures to the usual formula of T_π . Indeed, observe that the usual formula for T_π can be written as follows:

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j: \ker \binom{i}{j} \geq \pi} e_{j_1} \otimes \dots \otimes e_{j_l}$$

Now by inserting signs, coming from the signature map $\varepsilon : P_{\text{even}} \rightarrow \{\pm 1\}$, we are led to the following formula, which coincides with the one given above:

$$\bar{T}_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{\tau \geq \pi} \varepsilon(\tau) \sum_{j: \ker \binom{i}{j} = \tau} e_{j_1} \otimes \dots \otimes e_{j_l}$$

We will be back later to this analogy, with more details on what can be done with it. For the moment, we must first prove a key categorical result, as follows:

PROPOSITION 11.9. *The assignment $\pi \rightarrow \bar{T}_\pi$ is categorical, in the sense that*

$$\bar{T}_\pi \otimes \bar{T}_\sigma = \bar{T}_{[\pi\sigma]} \quad , \quad \bar{T}_\pi \bar{T}_\sigma = N^{c(\pi, \sigma)} \bar{T}_{[\sigma]} \quad , \quad \bar{T}_\pi^* = \bar{T}_{\pi^*}$$

where $c(\pi, \sigma)$ are certain positive integers.

PROOF. We have to go back to the proof from the untwisted case, from section 2 above, and insert signs. We have to check three conditions, as follows:

1. Concatenation. In the untwisted case, this was based on the following formula:

$$\delta_\pi \begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_q \end{pmatrix} \delta_\sigma \begin{pmatrix} k_1 \dots k_r \\ l_1 \dots l_s \end{pmatrix} = \delta_{[\pi\sigma]} \begin{pmatrix} i_1 \dots i_p & k_1 \dots k_r \\ j_1 \dots j_q & l_1 \dots l_s \end{pmatrix}$$

In the twisted case, it is enough to check the following formula:

$$\varepsilon \left(\ker \begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_q \end{pmatrix} \right) \varepsilon \left(\ker \begin{pmatrix} k_1 \dots k_r \\ l_1 \dots l_s \end{pmatrix} \right) = \varepsilon \left(\ker \begin{pmatrix} i_1 \dots i_p & k_1 \dots k_r \\ j_1 \dots j_q & l_1 \dots l_s \end{pmatrix} \right)$$

Let us denote by τ, ν the partitions on the left, so that the partition on the right is of the form $\rho \leq [\tau\nu]$. Now by switching to the noncrossing form, $\tau \rightarrow \tau'$ and $\nu \rightarrow \nu'$, the partition on the right transforms into:

$$\rho \rightarrow \rho' \leq [\tau'\nu']$$

Now since the partition $[\tau'\nu']$ is noncrossing, we can use Theorem 11.7 (3), and we obtain the result.

2. Composition. In the untwisted case, this was based on the following formula:

$$\sum_{j_1 \dots j_q} \delta_\pi \begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_q \end{pmatrix} \delta_\sigma \begin{pmatrix} j_1 \dots j_q \\ k_1 \dots k_r \end{pmatrix} = N^{c(\pi,\sigma)} \delta_{[\pi\sigma]} \begin{pmatrix} i_1 \dots i_p \\ k_1 \dots k_r \end{pmatrix}$$

In order to prove now the result in the twisted case, it is enough to check that the signs match. More precisely, we must establish the following formula:

$$\varepsilon \left(\ker \begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_q \end{pmatrix} \right) \varepsilon \left(\ker \begin{pmatrix} j_1 \dots j_q \\ k_1 \dots k_r \end{pmatrix} \right) = \varepsilon \left(\ker \begin{pmatrix} i_1 \dots i_p \\ k_1 \dots k_r \end{pmatrix} \right)$$

Let τ, ν be the partitions on the left, so that the partition on the right is of the form $\rho \leq [\tau]$. Our claim is that we can jointly switch τ, ν to the noncrossing form. Indeed, we can first switch as for $\ker(j_1 \dots j_q)$ to become noncrossing, and then switch the upper legs of τ , and the lower legs of ν , as for both these partitions to become noncrossing. Now observe that when switching in this way to the noncrossing form, $\tau \rightarrow \tau'$ and $\nu \rightarrow \nu'$, the partition on the right transforms into:

$$\rho \rightarrow \rho' \leq [\tau']$$

Now since the partition $[\tau']$ is noncrossing, we can apply Theorem 11.7 (3), and we obtain the result.

3. Involution. Here we must prove the following formula:

$$\bar{\delta}_\pi \begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_q \end{pmatrix} = \bar{\delta}_{\pi^*} \begin{pmatrix} j_1 \dots j_q \\ i_1 \dots i_p \end{pmatrix}$$

But this is clear from the definition of $\bar{\delta}_\pi$, and we are done. □

As a conclusion, our twisted construction $\pi \rightarrow \bar{T}_\pi$ has all the needed properties for producing quantum groups, via Tannakian duality, and we can now formulate:

THEOREM 11.10. *Given a category of partitions $D \subset P_{\text{even}}$, the construction*

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{span} \left(\bar{T}_\pi \Big| \pi \in D(k, l) \right)$$

produces via Tannakian duality a quantum group $\bar{G}_N \subset U_N^+$, for any $N \in \mathbb{N}$.

PROOF. This follows indeed from the Tannakian results from section 2 above, exactly as in the easy case, by using this time Proposition 11.9 as technical ingredient.

To be more precise, Proposition 11.9 shows that the linear spaces on the right form a Tannakian category, and so the results in section 2 apply, and give the result. \square

We can unify the easy quantum groups, or at least the examples coming from categories $D \subset P_{\text{even}}$, with the quantum groups constructed above, as follows:

DEFINITION 11.11. *A closed subgroup $G \subset U_N^+$ is called q -easy, or quizzly, with deformation parameter $q = \pm 1$, when its tensor category appears as follows,*

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{span} \left(\dot{T}_\pi \Big| \pi \in D(k, l) \right)$$

for a certain category of partitions $D \subset P_{\text{even}}$, where, for $q = -1, 1$:

$$\dot{T} = \bar{T}, T$$

The Schur-Weyl twist of G is the quizzly quantum group $\bar{G} \subset U_N^+$ obtained via $q \rightarrow -q$.

We will see later on that the easy quantum group associated to P_{even} itself is the hyperochahedral group H_N , and so that our assumption $D \subset P_{\text{even}}$, replacing $D \subset P$, simply corresponds to $H_N \subset G$, replacing the usual condition $S_N \subset G$.

For the moment, our most pressing task is that of checking that, when applying the Schur-Weyl twisting to the basic unitary quantum groups, we obtain the ad-hoc twists that we previously constructed. This is indeed the case:

THEOREM 11.12. *The twisted unitary quantum groups introduced before,*

$$\begin{array}{ccccc} O_N^+ & \longrightarrow & \mathbb{T}O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ \bar{O}_N^* & \longrightarrow & \mathbb{T}\bar{O}_N^* & \longrightarrow & \bar{U}_N^* \\ \uparrow & & \uparrow & & \uparrow \\ \bar{O}_N & \longrightarrow & \mathbb{T}\bar{O}_N & \longrightarrow & \bar{U}_N \end{array}$$

appear as Schur-Weyl twists of the basic unitary quantum groups.

PROOF. This is something routine, in several steps, as follows:

(1) The basic crossing, $\ker \begin{pmatrix} ij \\ ji \end{pmatrix}$ with $i \neq j$, comes from the transposition $\tau \in S_2$, so its signature is -1 . As for its degenerated version $\ker \begin{pmatrix} ii \\ ii \end{pmatrix}$, this is noncrossing, so here the signature is 1 . We conclude that the linear map associated to the basic crossing is:

$$\bar{T}_\chi(e_i \otimes e_j) = \begin{cases} -e_j \otimes e_i & \text{for } i \neq j \\ e_j \otimes e_i & \text{otherwise} \end{cases}$$

For the half-classical crossing, namely $\ker \begin{pmatrix} ijk \\ kji \end{pmatrix}$ with i, j, k distinct, the signature is once again -1 , and by examining the signatures of the various degenerations of this half-classical crossing, we are led to the following formula:

$$\bar{T}_\chi(e_i \otimes e_j \otimes e_k) = \begin{cases} -e_k \otimes e_j \otimes e_i & \text{for } i, j, k \text{ distinct} \\ e_k \otimes e_j \otimes e_i & \text{otherwise} \end{cases}$$

(2) Our claim now is that for an orthogonal quantum group G , the following holds, with the quantum group \bar{O}_N being the one in Theorem 11.2:

$$\bar{T}_\chi \in \text{End}(u^{\otimes 2}) \iff G \subset \bar{O}_N$$

Indeed, by using the formula of \bar{T}_χ found in (1) above, we obtain:

$$\begin{aligned} (\bar{T}_\chi \otimes 1)u^{\otimes 2}(e_i \otimes e_j \otimes 1) &= \sum_k e_k \otimes e_k \otimes u_{ki}u_{kj} \\ &\quad - \sum_{k \neq l} e_l \otimes e_k \otimes u_{ki}u_{lj} \end{aligned}$$

On the other hand, we have as well the following formula:

$$u^{\otimes 2}(\bar{T}_\chi \otimes 1)(e_i \otimes e_j \otimes 1) = \begin{cases} \sum_{kl} e_l \otimes e_k \otimes u_{li}u_{ki} & \text{if } i = j \\ -\sum_{kl} e_l \otimes e_k \otimes u_{lj}u_{ki} & \text{if } i \neq j \end{cases}$$

For $i = j$ the conditions are $u_{ki}^2 = u_{ki}^2$ for any k , and $u_{ki}u_{li} = -u_{li}u_{ki}$ for any $k \neq l$. For $i \neq j$ the conditions are $u_{ki}u_{kj} = -u_{kj}u_{ki}$ for any k , and $u_{ki}u_{lj} = u_{lj}u_{ki}$ for any $k \neq l$. Thus we have exactly the relations between the coordinates of \bar{O}_N , and we are done.

(3) Our claim now is that for an orthogonal quantum group G , the following holds, with the quantum group \bar{O}_N^* being the one in Theorem 11.3:

$$\bar{T}_\chi \in \text{End}(u^{\otimes 3}) \iff G \subset \bar{O}_N^*$$

Indeed, by using the formula of \bar{T}_λ found in (1) above, we obtain:

$$\begin{aligned}
 (\bar{T}_\lambda \otimes 1)u^{\otimes 2}(e_i \otimes e_j \otimes e_k \otimes 1) &= \sum_{abc \text{ not distinct}} e_c \otimes e_b \otimes e_a \otimes u_{ai}u_{bj}u_{ck} \\
 &- \sum_{a,b,c \text{ distinct}} e_c \otimes e_b \otimes e_a \otimes u_{ai}u_{bj}u_{ck}
 \end{aligned}$$

On the other hand, we have as well the following formula:

$$\begin{aligned}
 &u^{\otimes 2}(\bar{T}_\lambda \otimes 1)(e_i \otimes e_j \otimes e_k \otimes 1) \\
 &= \begin{cases} \sum_{abc} e_c \otimes e_b \otimes e_a \otimes u_{ck}u_{bj}u_{ai} & \text{for } i, j, k \text{ not distinct} \\ -\sum_{abc} e_c \otimes e_b \otimes e_a \otimes u_{ck}u_{bj}u_{ai} & \text{for } i, j, k \text{ distinct} \end{cases}
 \end{aligned}$$

For i, j, k not distinct the conditions are $u_{ai}u_{bj}u_{ck} = u_{ck}u_{bj}u_{ai}$ for a, b, c not distinct, and $u_{ai}u_{bj}u_{ck} = -u_{ck}u_{bj}u_{ai}$ for a, b, c distinct. For i, j, k distinct the conditions are $u_{ai}u_{bj}u_{ck} = -u_{ck}u_{bj}u_{ai}$ for a, b, c not distinct, and $u_{ai}u_{bj}u_{ck} = u_{ck}u_{bj}u_{ai}$ for a, b, c distinct. Thus we have exactly the relations between the coordinates of \bar{O}_N^* , and we are done.

(4) Now with the above results in hand, we obtain that the Schur-Weyl twists of O_N, O_N^* are indeed the quantum groups \bar{O}_N, \bar{O}_N^* from Theorem 11.2 and Theorem 11.3.

(4) The proof in the unitary case is similar, by adding signs in the above computations (2,3), the conclusion being that the Schur-Weyl twists of U_N, U_N^* are indeed \bar{U}_N, \bar{U}_N^* . \square

Let us clarify now the relation between the maps T_π, \bar{T}_π . We recall that the Möbius function of any lattice, and in particular of P_{even} , is given by:

$$\mu(\sigma, \pi) = \begin{cases} 1 & \text{if } \sigma = \pi \\ -\sum_{\sigma \leq \tau < \pi} \mu(\sigma, \tau) & \text{if } \sigma < \pi \\ 0 & \text{if } \sigma \not\leq \pi \end{cases}$$

With this notation, we have the following result:

PROPOSITION 11.13. *For any partition $\pi \in P_{even}$ we have the formula*

$$\bar{T}_\pi = \sum_{\tau \leq \pi} \alpha_\tau T_\tau$$

where $\alpha_\sigma = \sum_{\sigma \leq \tau \leq \pi} \varepsilon(\tau)\mu(\sigma, \tau)$, with μ being the Möbius function of P_{even} .

PROOF. The linear combinations $T = \sum_{\tau \leq \pi} \alpha_\tau T_\tau$ acts on tensors as follows:

$$\begin{aligned} T(e_{i_1} \otimes \dots \otimes e_{i_k}) &= \sum_{\tau \leq \pi} \alpha_\tau T_\tau(e_{i_1} \otimes \dots \otimes e_{i_k}) \\ &= \sum_{\tau \leq \pi} \alpha_\tau \sum_{\sigma \leq \tau} \sum_{j: \ker(j) = \sigma} e_{j_1} \otimes \dots \otimes e_{j_i} \\ &= \sum_{\sigma \leq \pi} \left(\sum_{\sigma \leq \tau \leq \pi} \alpha_\tau \right) \sum_{j: \ker(j) = \sigma} e_{j_1} \otimes \dots \otimes e_{j_i} \end{aligned}$$

Thus, in order to have $\bar{T}_\pi = \sum_{\tau \leq \pi} \alpha_\tau T_\tau$, we must have $\varepsilon(\sigma) = \sum_{\sigma \leq \tau \leq \pi} \alpha_\tau$, for any $\sigma \leq \pi$. But this problem can be solved by using the Möbius inversion formula, and we obtain the numbers $\alpha_\sigma = \sum_{\sigma \leq \tau \leq \pi} \varepsilon(\tau) \mu(\sigma, \tau)$ in the statement. \square

With the above results in hand, let us go back now to the question of twisting the quantum reflection groups. It is convenient to include in our discussion two more quantum groups, coming from [81] and denoted $H_N^{[\infty]}, K_N^{[\infty]}$, constructed as follows:

PROPOSITION 11.14. *We have intermediate liberations $H_N^{[\infty]}, K_N^{[\infty]}$ as follows, constructed by using the relations $\alpha\beta\gamma = 0$ for any $a \neq c$ on the same row or column of u :*

$$\begin{array}{ccccccc} K_N & \longrightarrow & K_N^* & \longrightarrow & K_N^{[\infty]} & \longrightarrow & K_N^+ \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H_N & \longrightarrow & H_N^* & \longrightarrow & H_N^{[\infty]} & \longrightarrow & H_N^+ \end{array}$$

These quantum groups are both easy, with the corresponding categories of partitions, denoted $\mathcal{P}_{even}^{[\infty]} \subset \mathcal{P}_{even}$ and $\mathcal{P}_{even}^{[\infty]} \subset \mathcal{P}_{even}$, being generated by $\eta = \ker \begin{pmatrix} ij \\ jii \end{pmatrix}$.

PROOF. This is routine, by using the fact that the relations $\alpha\beta\gamma = 0$ in the statement are equivalent to the condition $\eta \in \text{End}(u^{\otimes k})$, with $|k| = 3$. For details here, and for more on these two quantum groups, which are very interesting objects, and that we have actually already met in section 4 above, we refer to the paper of Raum-Weber [81]. \square

In order to discuss now the Schur-Weyl twisting of the various quantum reflection groups that we have, we will need the following technical result:

PROPOSITION 11.15. *We have the following equalities,*

$$\begin{aligned} P_{even}^* &= \left\{ \pi \in P_{even} \mid \varepsilon(\tau) = 1, \forall \tau \leq \pi, |\tau| = 2 \right\} \\ P_{even}^{[\infty]} &= \left\{ \pi \in P_{even} \mid \sigma \in P_{even}^*, \forall \sigma \subset \pi \right\} \\ P_{even}^{[\infty]} &= \left\{ \pi \in P_{even} \mid \varepsilon(\tau) = 1, \forall \tau \leq \pi \right\} \end{aligned}$$

where $\varepsilon : P_{even} \rightarrow \{\pm 1\}$ is the signature of even permutations.

PROOF. This is routine combinatorics, from [8], [81], the idea being as follows:

(1) Given $\pi \in P_{even}$, we have $\tau \leq \pi, |\tau| = 2$ precisely when $\tau = \pi^\beta$ is the partition obtained from π by merging all the legs of a certain subpartition $\beta \subset \pi$, and by merging as well all the other blocks.

Now observe that π^β does not depend on π , but only on β , and that the number of switches required for making π^β noncrossing is $c = N_\bullet - N_\circ$ modulo 2, where N_\bullet/N_\circ is the number of black/white legs of β , when labelling the legs of π counterclockwise $\circ \bullet \circ \bullet \dots$.

Thus $\varepsilon(\pi^\beta) = 1$ holds precisely when $\beta \in \pi$ has the same number of black and white legs, and this gives the result.

(2) This simply follows from the equality $P_{even}^{[\infty]} = \langle \eta \rangle$ coming from Proposition 11.14, by computing $\langle \eta \rangle$, and for the complete proof here we refer to [81].

(3) We use the fact, also from [81], that the relations $g_i g_i g_j = g_j g_i g_i$ are trivially satisfied for real reflections. Thus, we have:

$$P_{even}^{[\infty]}(k, l) = \left\{ \ker \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} \mid g_{i_1} \dots g_{i_k} = g_{j_1} \dots g_{j_l} \text{ inside } \mathbb{Z}_2^{*N} \right\}$$

In other words, the partitions in $P_{even}^{[\infty]}$ are those describing the relations between free variables, subject to the conditions $g_i^2 = 1$.

We conclude that $P_{even}^{[\infty]}$ appears from NC_{even} by “inflating blocks”, in the sense that each $\pi \in P_{even}^{[\infty]}$ can be transformed into a partition $\pi' \in NC_{even}$ by deleting pairs of consecutive legs, belonging to the same block.

Now since this inflation operation leaves invariant modulo 2 the number $c \in \mathbb{N}$ of switches in the definition of the signature, it leaves invariant the signature $\varepsilon = (-1)^c$ itself, and we obtain in this way the inclusion “ \subset ” in the statement.

Conversely, given $\pi \in P_{even}$ satisfying $\varepsilon(\tau) = 1, \forall \tau \leq \pi$, our claim is that:

$$\rho \leq \sigma \subset \pi, |\rho| = 2 \implies \varepsilon(\rho) = 1$$

Indeed, let us denote by α, β the two blocks of ρ , and by γ the remaining blocks of π , merged altogether. We know that the partitions $\tau_1 = (\alpha \wedge \gamma, \beta), \tau_2 = (\beta \wedge \gamma, \alpha), \tau_3 = (\alpha, \beta, \gamma)$ are all even. On the other hand, putting these partitions in noncrossing

form requires respectively $s + t, s' + t, s + s' + t$ switches, where t is the number of switches needed for putting $\rho = (\alpha, \beta)$ in noncrossing form. Thus t is even, and we are done.

With the above claim in hand, we conclude, by using the second equality in the statement, that we have $\sigma \in P_{even}^*$. Thus we have $\pi \in P_{even}^{[\infty]}$, which ends the proof of “ \supset ”. \square

With the above result in hand, we can now prove:

THEOREM 11.16. *The basic quantum reflection groups, namely*

$$\begin{array}{ccccc}
 H_N^+ & \longrightarrow & \mathbb{T}H_N^+ & \longrightarrow & K_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N^* & \longrightarrow & \mathbb{T}H_N^* & \longrightarrow & K_N^* \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N & \longrightarrow & \mathbb{T}H_N & \longrightarrow & K_N
 \end{array}$$

equal their own Schur-Weyl twists.

PROOF. This result, established in [7], basically comes from the results that we have:

(1) In the real case, the verifications are as follows:

– H_N^+ . We know from Theorem 11.7 above that for $\pi \in NC_{even}$ we have $\bar{T}_\pi = T_\pi$, and since we are in the situation $D \subset NC_{even}$, the definitions of G, \bar{G} coincide.

– $H_N^{[\infty]}$. Here we can use the same argument as in (1), based this time on the description of $P_{even}^{[\infty]}$ involving the signatures found in Proposition 11.15.

– H_N^* . We have $H_N^* = H_N^{[\infty]} \cap O_N^*$, so $\bar{H}_N^* \subset H_N^{[\infty]}$ is the subgroup obtained via the defining relations for \bar{O}_N^* . But all the $abc = -cba$ relations defining \bar{H}_N^* are automatic, of type $0 = 0$, and it follows that $\bar{H}_N^* \subset H_N^{[\infty]}$ is the subgroup obtained via the relations $abc = cba$, for any $a, b, c \in \{u_{ij}\}$. Thus we have $\bar{H}_N^* = H_N^{[\infty]} \cap O_N^* = H_N^*$, as claimed.

– H_N . We have $H_N = H_N^* \cap O_N$, and by functoriality, $\bar{H}_N = \bar{H}_N^* \cap \bar{O}_N = H_N^* \cap \bar{O}_N$. But this latter intersection is easily seen to be equal to H_N , as claimed.

(2) In the complex case the proof is similar, and we refer here to [8]. \square

In relation now with the tori, we have the following result:

THEOREM 11.17. *The diagonal tori of the twisted quantum groups are*

$$\begin{array}{ccccc}
 T_N^+ & \longrightarrow & \mathbb{T}T_N^+ & \longrightarrow & \mathbb{T}^+_N \\
 \uparrow & & \uparrow & & \uparrow \\
 T_N^* & \longrightarrow & \mathbb{T}T_N^* & \longrightarrow & \mathbb{T}^*_N \\
 \uparrow & & \uparrow & & \uparrow \\
 T_N & \longrightarrow & \mathbb{T}T_N & \longrightarrow & \mathbb{T}_N
 \end{array}$$

exactly as in the untwisted case.

PROOF. This is clear for the quantum reflection groups, which are not twistable, and for the quantum unitary groups this is elementary as well, coming from definitions. \square

11c. Twisted integration

Before getting into the spheres, let us discuss integration questions. The result here, valid for any Schur-Weyl twist in our sense, is as follows:

THEOREM 11.18. *We have the Weingarten type formula*

$$\int_{\dot{G}} u_{i_1 j_1}^{e_1} \dots u_{i_k j_k}^{e_k} = \sum_{\pi, \sigma \in P_{\times}(\alpha)} \dot{\delta}_{\pi}(i_1 \dots i_k) \dot{\delta}_{\sigma}(j_1 \dots j_k) W_{kN}(\pi, \sigma)$$

where $W_{kN} = G_{kN}^{-1}$, with $G_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma|}$, for $\pi, \sigma \in D(k)$.

PROOF. This follows exactly as in the untwisted case, the idea being that the signs will cancel. Let us recall indeed from Definition 11.8 and the comments afterwards that the twisted vectors $\bar{\xi}_{\pi}$ associated to the partitions $\pi \in P_{\text{even}}(k)$ are as follows:

$$\bar{\xi}_{\pi} = \sum_{\tau \geq \pi} \varepsilon(\tau) \sum_{i: \ker(i)=\tau} e_{i_1} \otimes \dots \otimes e_{i_k}$$

Thus, the Gram matrix of these vectors is given by:

$$\begin{aligned}
 \langle \xi_{\pi}, \xi_{\sigma} \rangle &= \sum_{\tau \geq \pi \vee \sigma} \varepsilon(\tau)^2 \left| \left\{ (i_1, \dots, i_k) \mid \ker i = \tau \right\} \right| \\
 &= \sum_{\tau \geq \pi \vee \sigma} \left| \left\{ (i_1, \dots, i_k) \mid \ker i = \tau \right\} \right| \\
 &= N^{|\pi \vee \sigma|}
 \end{aligned}$$

Thus the Gram matrix is the same as in the untwisted case, and so the Weingarten matrix is the same as well as in the untwisted case, and this gives the result. \square

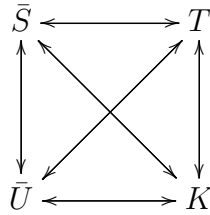
In relation now with the spheres, we have the following result:

THEOREM 11.19. *The twisted spheres have the following properties:*

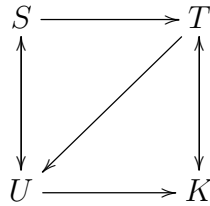
- (1) *They have affine actions of the twisted unitary quantum groups.*
- (2) *They have unique invariant Haar functionals, which are ergodic.*
- (3) *Their Haar functionals are given by Weingarten type formulae.*
- (4) *They appear, via the GNS construction, as first row spaces.*

PROOF. The proofs here are similar to those from the untwisted case, via a routine computation, by adding signs where needed, and with the main technical ingredient, namely the Weingarten formula, being available from Theorem 11.18 above. \square

As a conclusion now, we have shown that the various quadruplets (S, T, U, K) constructed in chapters 1-10 above have twisted counterparts (\bar{S}, T, \bar{U}, K) . The question that we would like to solve now is that of finding correspondences, as follows:



In order to discuss this, let us get back to the axioms from chapter 4. We have seen there that the 12 correspondences come in fact from 7 correspondences, as follows:



In the twisted case, 6 of these correspondences hold as well, but the remaining one, namely $S \rightarrow T$, definitely does not hold as stated, and must be modified. Let us begin our discussion with the quantum isometry group results. We have here:

THEOREM 11.20. *We have the quantum isometry group formula*

$$\bar{U} = G^+(\bar{S})$$

in all the 9 main twisted cases.

PROOF. The proofs here are similar to those from the untwisted case, via a routine computation, by adding signs where needed, which amounts in replacing the usual commutators $[a, b] = ab - ba$ by twisted commutators, given by:

$$[[a, b]] = ab + ba$$

There is one subtle point, however, coming from the fact that the linear independence of various products of coordinates of length 1,2,3, which was something clear in the untwisted case, is now a non-trivial question.

But this can be solved via a technical application of the Weingarten formula, from Theorem 11.18. For details here, we refer to [4]. □

Regarding now the $K = G^+(T) \cap K_N^+$ axiom, this is something that we already know. However, regarding the correspondence $S \rightarrow T$, things here fail in the twisted case. Our “fix” for this, or at least the best fix that we could find, is as follows:

THEOREM 11.21. *Given an algebraic manifold $X \subset S_{\mathbb{C},+}^{N-1}$, define its toral isometry group as being the biggest subgroup of \mathbb{T}_N^+ acting affinely on X :*

$$\mathcal{G}^+(X) = G^+(X) \cap \mathbb{T}_N^+$$

With this convention, for the 9 basic spheres S , and for their twists as well, the toral isometry group equals the torus T .

PROOF. We recall from chapter 3 that the affine quantum isometry group $G^+(X) \subset U_N^+$ of a noncommutative manifold $X \subset S_{\mathbb{C},+}^{N-1}$ coming from certain polynomial relations P is constructed according to the following procedure:

$$P(x_i) = 0 \implies P\left(\sum_j x_j \otimes u_{ji}\right) = 0$$

Similarly, the toral isometry group $\mathcal{G}^+(X) \subset \mathbb{T}_N^+$ is constructed as follows:

$$P(x_i) = 0 \implies P(x_i \otimes u_i) = 0$$

In the monomial case one can prove that the following formula holds:

$$G^+(\bar{S}) = \overline{G^+(S)}$$

By intersecting with \mathbb{T}_N^+ , we obtain from this that we have:

$$\mathcal{G}^+(\bar{S}) = \mathcal{G}^+(S)$$

The result can be of course be proved as well directly. For $\bar{S}_{\mathbb{R}}^{N-1}$ we have:

$$\Phi(x_i x_j) = x_i x_j \otimes u_i u_j$$

$$\Phi(x_j x_i) = x_j x_i \otimes u_j u_i$$

Thus we obtain $u_i u_j = -u_j u_i$ for $i \neq j$, and so the quantum group is T_N .

The proof in the complex, half-liberated and hybrid cases is similar. □

Regarding the hard liberation axiom, this seems to hold indeed in all the cases under consideration, but this is non-trivial, and not known yet. As a conclusion, we conjecturally have an extension of our (S, T, U, K) formalism, with the $S \rightarrow T$ axiom needing a modification as above, which covers the twisted objects (\bar{S}, T, \bar{U}, K) as well.

11d. Twisted geometry

There are many things that can be done in the context of the twisted geometry, going beyond what we have so far, namely some theory and computations for the spheres \bar{S} , tori T , unitary groups \bar{U} , and reflection groups K . We briefly discuss here, as a main topic, the twisted extension of the various constructions from chapter 7 above.

So, let us go back to the theory there. As a first observation, we can both liberate the spaces O_{MN}^L, U_{MN}^L , and twist them, by proceeding as follows:

DEFINITION 11.22. *Associated to any integers $L \leq M \leq N$ are the algebras*

$$\begin{aligned} C(O_{MN}^{L+}) &= C^* \left((u_{ij})_{i=1, \dots, M, j=1, \dots, N} \mid u = \bar{u}, uu^t = \text{projection of trace } L \right) \\ C(U_{MN}^{L+}) &= C^* \left((u_{ij})_{i=1, \dots, M, j=1, \dots, N} \mid uu^*, \bar{u}u^t = \text{projections of trace } L \right) \end{aligned}$$

and their quotients $C(\bar{O}_{MN}^L), C(\bar{U}_{MN}^L)$, obtained by imposing the twisting relations.

With this extended formalism, we have inclusions between the various spaces constructed so far, in chapter 7 above and then here, as follows:

$$\begin{array}{ccccc} U_{MN}^L & \longrightarrow & U_{MN}^{L+} & \longleftarrow & \bar{U}_{MN}^L \\ \uparrow & & \uparrow & & \uparrow \\ O_{MN}^L & \longrightarrow & O_{MN}^{L+} & \longleftarrow & \bar{O}_{MN}^L \end{array}$$

More generally now, and once again by following the material from chapter 7 above, we can perform the above constructions for any quizzly quantum group.

In order to discuss this, we use the Kronecker symbols $\delta_\pi(i) \in \{-1, 0, 1\}$ explained in the above, depending on a twisting parameter $q = \pm 1$, constructed as follows:

$$\delta_\sigma(i) = \begin{cases} \delta_{\ker i \leq \sigma} & \text{(untwisted case)} \\ \varepsilon(\ker i) \delta_{\ker i \leq \sigma} & \text{(twisted case)} \end{cases}$$

With this convention, we have the following result:

PROPOSITION 11.23. *The various spaces G_{MN}^L constructed so far appear by imposing to the standard coordinates of U_{MN}^{L+} the relations*

$$\sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \delta_\pi(i) \delta_\sigma(j) u_{i_1 j_1}^{e_1} \dots u_{i_s j_s}^{e_s} = L^{|\pi \vee \sigma|}$$

with $s = (e_1, \dots, e_s)$ ranging over all the colored integers, and with $\pi, \sigma \in D(0, s)$.

PROOF. According to the various constructions in chapter 7 and in the above, the relations defining G_{MN}^L can be written as follows, with σ ranging over a family of generators, with no upper legs, of the corresponding category of partitions D :

$$\sum_{j_1 \dots j_s} \delta_\sigma(j) u_{i_1 j_1}^{e_1} \dots u_{i_s j_s}^{e_s} = \delta_\sigma(i)$$

We therefore obtain the relations in the statement, as follows:

$$\begin{aligned} \sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \delta_\pi(i) \delta_\sigma(j) u_{i_1 j_1}^{e_1} \dots u_{i_s j_s}^{e_s} &= \sum_{i_1 \dots i_s} \delta_\pi(i) \sum_{j_1 \dots j_s} \delta_\sigma(j) u_{i_1 j_1}^{e_1} \dots u_{i_s j_s}^{e_s} \\ &= \sum_{i_1 \dots i_s} \delta_\pi(i) \delta_\sigma(i) \\ &= \sum_{\tau \leq \pi \vee \sigma} \sum_{\ker i = \tau} (\pm 1)^2 \\ &= \sum_{\tau \leq \pi \vee \sigma} \sum_{\ker i = \tau} 1 \\ &= L^{|\pi \vee \sigma|} \end{aligned}$$

As for the converse, this follows by using the relations in the statement, by keeping π fixed, and by making σ vary over all the partitions in the category. \square

Summarizing, we have unified the twisted and untwisted constructions, in the continuous case.

In the general case now, where $G = (G_N)$ is an arbitrary uniform quizzly quantum group, we can construct spaces G_{MN}^L by using the above relations, and we have:

THEOREM 11.24. *The spaces $G_{MN}^L \subset U_{MN}^{L+}$ constructed by imposing the relations*

$$\sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \delta_\pi(i) \delta_\sigma(j) u_{i_1 j_1}^{e_1} \dots u_{i_s j_s}^{e_s} = L^{|\pi \vee \sigma|}$$

with π, σ ranging over all the partitions in the associated category, having no upper legs, are subject to an action map/quotient map diagram, as follows

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ p \times id \downarrow & & \downarrow p \\ X \times G & \xrightarrow{a} & X \end{array}$$

exactly as in the classical case, or the free case.

PROOF. We proceed as in chapter 7 above. We must prove that, if the variables u_{ij} satisfy the relations in the statement, then so do the following variables:

$$U_{ij} = \sum_{kl} a_{ik} \otimes b_{jl}^* \otimes u_{kl}$$

$$V_{ij} = \sum_{l \leq L} a_{il} \otimes b_{jl}^*$$

Regarding the variables U_{ij} , the computation here goes as follows:

$$\begin{aligned} & \sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \delta_\pi(i) \delta_\sigma(j) U_{i_1 j_1}^{e_1} \dots U_{i_s j_s}^{e_s} \\ = & \sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \sum_{k_1 \dots k_s} \sum_{l_1 \dots l_s} \delta_\pi(i) \delta_\sigma(j) a_{i_1 k_1}^{e_1} \dots a_{i_s k_s}^{e_s} \otimes (b_{j_s l_s}^{e_s} \dots b_{j_1 l_1}^{e_1})^* \otimes u_{k_1 l_1}^{e_1} \dots u_{k_s l_s}^{e_s} \\ = & \sum_{k_1 \dots k_s} \sum_{l_1 \dots l_s} \delta_\pi(k) \delta_\sigma(l) u_{k_1 l_1}^{e_1} \dots u_{k_s l_s}^{e_s} \\ = & L^{|\pi \vee \sigma|} \end{aligned}$$

For the variables V_{ij} the proof is similar, as follows:

$$\begin{aligned} & \sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \delta_\pi(i) \delta_\sigma(j) V_{i_1 j_1}^{e_1} \dots V_{i_s j_s}^{e_s} \\ = & \sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} \sum_{l_1, \dots, l_s \leq L} \delta_\pi(i) \delta_\sigma(j) a_{i_1 l_1}^{e_1} \dots a_{i_s l_s}^{e_s} \otimes (b_{j_s l_s}^{e_s} \dots b_{j_1 l_1}^{e_1})^* \\ = & \sum_{l_1, \dots, l_s \leq L} \delta_\pi(l) \delta_\sigma(l) \\ = & L^{|\pi \vee \sigma|} \end{aligned}$$

Thus we have constructed an action map, and a quotient map, and the commutation of the diagram in the statement is then trivial. \square

Still by following the material in chapter 7 above, we can now construct a Haar integration for the above spaces, and we have the following result:

THEOREM 11.25. *We have the Weingarten type formula*

$$\int_{G_{MN}^L} u_{i_1 j_1} \dots u_{i_s j_s} = \sum_{\pi \sigma \tau \nu} L^{|\sigma \vee \nu|} \delta_\pi(i) \delta_\tau(j) W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu)$$

where $W_{sM} = G_{sM}^{-1}$, with $G_{sM}(\pi, \sigma) = M^{|\pi \vee \sigma|}$.

PROOF. We make use of the usual quantum group Weingarten formula, explained in the above in the twisted case. By using this formula for G_M, G_N , we obtain:

$$\begin{aligned}
& \int_{G_{MN}^L} u_{i_1 j_1} \dots u_{i_s j_s} \\
&= \sum_{l_1 \dots l_s \leq L} \int_{G_M} a_{i_1 l_1} \dots a_{i_s l_s} \int_{G_N} b_{j_1 l_1}^* \dots b_{j_s l_s}^* \\
&= \sum_{l_1 \dots l_s \leq L} \sum_{\pi \sigma} \delta_\pi(i) \delta_\sigma(l) W_{sM}(\pi, \sigma) \sum_{\tau \nu} \delta_\tau(j) \delta_\nu(l) W_{sN}(\tau, \nu) \\
&= \sum_{\pi \sigma \tau \nu} \left(\sum_{l_1 \dots l_s \leq L} \delta_\sigma(l) \delta_\nu(l) \right) \delta_\pi(i) \delta_\tau(j) W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu)
\end{aligned}$$

Let us compute now the coefficient appearing in the last formula. Since the signature map takes ± 1 values, for any multi-index $l = (l_1, \dots, l_s)$ we have:

$$\begin{aligned}
\delta_\sigma(l) \delta_\nu(l) &= \delta_{\ker l \leq \sigma} \varepsilon(\ker l) \cdot \delta_{\ker l \leq \nu} \varepsilon(\ker l) \\
&= \delta_{\ker l \leq \sigma \vee \nu}
\end{aligned}$$

Thus the coefficient is $L^{|\sigma \vee \nu|}$, and we obtain the formula in the statement. \square

With this formula in hand, we can derive explicit integration results for the sums of non-overlapping coordinates, exactly as in chapter 7 above. To be more precise, the laws and their asymptotics are identical in the classical and twisted cases.

11e. Exercises

As already mentioned in the above, this chapter was just an introduction to the twisted geometry, and what we did here, namely a study of the quadruplets (\bar{S}, T, \bar{U}, K) , and of some related homogeneous spaces, is just an epsilon of what can be done.

As an initiation to all the unexplored land which is left, we have:

EXERCISE 11.26. *Make a list, based on the existing $q = -1$ literature, carefully checked and doublechecked, of the compact quantum groups $G \subset U_N^+$, not necessarily easy, which can be twisted in a reasonable sense, and develop some geometry for them.*

The key words here are “careful” and “reasonable”, and this due to the fact that the standard Drinfeld-Jimbo twisting procedure, widely used in the literature, produces non-semisimple quantum groups, even at $q = -1$, and so is not very useful.

CHAPTER 12

Matrix models

12a. Matrix models

We have seen in chapter 9 above, when talking about half-liberation, that a useful technique for the study of the half-classical manifolds, $X \subset S_{\mathbb{C},*}^{N-1}$, is that of modelling the standard coordinates $x_1, \dots, x_N \in C(X)$ by certain explicit variables.

To be more precise, for certain such manifolds $X \subset S_{\mathbb{C},*}^{N-1}$, we have constructed embeddings of algebras of the following type, with Y being a certain classical manifold, and with $T_1, \dots, T_N \in M_2(C(Y))$ being certain suitable antidiagonal 2×2 matrices:

$$\pi : C(X) \subset M_2(C(Y))$$

$$x_i \rightarrow T_i$$

These models, which are quite powerful, were used afterwards in order to establish several non-trivial results on the original half-classical manifolds $X \subset S_{\mathbb{C},*}^{N-1}$.

In this chapter we discuss modelling questions for the general manifolds $X \subset S_{\mathbb{C},+}^{N-1}$, by using the same idea, suitably modified and generalized, as to cover most of the manifolds that we are interested in. Let us start with a broad definition, as follows:

DEFINITION 12.1. *A model for a real algebraic manifold $X \subset S_{\mathbb{C},+}^{N-1}$ is a morphism of C^* -algebras of the following type,*

$$\pi : C(X) \rightarrow B$$

with B being a C^ -algebra, called target algebra of the model. We say that the model is faithful if π is faithful, in the usual sense.*

Obviously, this is something too broad, because we can formally take $B = C(X)$, and we have in this way our model, which in addition is faithful:

$$id : C(X) \rightarrow C(X)$$

This model has of course no use, so the next thing to be done is that of suitably restricting the class of target algebras that we use, to algebras that we know well. However, this is something quite tricky, because if we want our model to be faithful, we cannot use simple algebras like the algebras $M_2(C(Y))$ used in the half-classical setting.

In short, we are running into some difficulties here, right from the beginning, which are of functional analytic nature, and a systematic discussion of all this is needed.

As a first objective, let us try to understand if an arbitrary manifold $X \subset S_{\mathbb{C},+}^{N-1}$ can be modelled by using familiar variables such as usual matrices, or operators. The answer here is yes, when using operators on a separable Hilbert space, with this coming from the GNS representation theorem, that we know from chapter 1, which is as follows:

THEOREM 12.2. *Any C^* -algebra A appears as closed $*$ -algebra of operators on a Hilbert space, $A \subset B(H)$, in the following way:*

- (1) *In the commutative case, where $A = C(X)$, we can set $H = L^2(X)$, with respect to some probability measure on X , and use the embedding $g \rightarrow (g \rightarrow fg)$.*
- (2) *In general, we can set $H = L^2(A)$, with respect to some faithful positive trace $tr : A \rightarrow \mathbb{C}$, and then use a similar embedding, $a \rightarrow (b \rightarrow ab)$.*

PROOF. This is something that we already know, from chapter 1 above, coming from basic measure theory and functional analysis, the idea being as follows:

(1) Let us first discuss the commutative case, $A = C(X)$. Our claim here is that if we pick a probability measure on X , we have an embedding as follows:

$$C(X) \subset B(L^2(X))$$

$$f \rightarrow (g \rightarrow fg)$$

Indeed, given a function $f \in C(X)$, consider the operator $T_f(g) = fg$, acting on $H = L^2(X)$. Observe that T_f is indeed well-defined, and bounded as well, because:

$$\|fg\|_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 dx}$$

$$\leq \|f\|_\infty \|g\|_2$$

The application $f \rightarrow T_f$ being linear, involutive, continuous, and injective as well, we obtain in this way a C^* -algebra embedding $C(X) \subset B(H)$, as claimed.

(2) In the general case, where our algebra A is arbitrary, we can use a similar idea. Indeed, assuming that a linear form $\varphi : A \rightarrow \mathbb{C}$ has suitable positivity properties, making it analogous to the integration functionals $\int_X : A \rightarrow \mathbb{C}$ from the commutative case, we can define a scalar product on A , by the following formula:

$$\langle a, b \rangle = \varphi(ab^*)$$

By completing we obtain a Hilbert space H , and we have a representation as follows, called GNS representation of our algebra with respect to the linear form φ :

$$A \rightarrow B(H)$$

$$a \rightarrow (b \rightarrow ab)$$

Moreover, when $\varphi : A \rightarrow \mathbb{C}$ has suitable faithfulness properties, making it analogous to the integration functionals $\int_X : A \rightarrow \mathbb{C}$ from the commutative case, with respect to faithful probability measures on X , this representation is faithful.

(3) The whole point now is that of proving that, given an arbitrary C^* -algebra A , there exists one faithful positive linear form, as follows:

$$\varphi : A \rightarrow \mathbb{C}$$

But in order to establish this latter result, we can take some inspiration once again from the classical case, where $A = C(X)$, and more specifically from the existence theorem for the probability measures over an arbitrary compact space X .

To be more precise, we have to convert the proof of this result in functional analysis terms, and then lift the assumption that the algebra $A = C(X)$ is commutative.

(4) But this can be done indeed, with a bit of spectral theory involved, and with the main tool being, as always in functional analysis, the Hahn-Banach theorem. The proof here takes a total of 3-4 pages, and can be found in any operator algebra book. \square

Now back to our questions, in the case of the algebras $A = C(X)$ with $X \subset S_{\mathbb{C},+}^{N-1}$ that we are interested in, the above result tells us that we can always find Hilbert space operators $T_i \in B(H)$ which model the standard coordinates $x_i \in C(X)$.

To be more precise, we have the following result:

PROPOSITION 12.3. *Given an algebraic manifold $X \subset S_{\mathbb{C},+}^{N-1}$, coming via*

$$C(X) = C(S_{\mathbb{C},+}^{N-1}) / \left\langle f_\alpha(x_1, \dots, x_N) = 0 \right\rangle$$

we have a morphism of C^ -algebras as follows,*

$$\pi : C(X) \rightarrow B(H) \quad , \quad x_i \rightarrow T_i$$

whenever the operators $T_i \in B(H)$ satisfy the following relations:

$$\sum_i T_i T_i^* = \sum_i T_i^* T_i = 1$$

$$f_\alpha(T_1, \dots, T_N) = 0$$

Moreover, we can always find a space H and operators (T_i) such that π is faithful.

PROOF. Here the first assertion is more of an empty statement, explaining the definition of the algebra $C(X)$, via generators and relations, and the second assertion is something non-trivial, coming as a consequence of the GNS theorem. \square

In practice now, all this is a bit too general, and not very useful. However, and here comes our point, by replacing the operator algebra models $C(X) \rightarrow B(H)$ by suitable models of type $C(X) \rightarrow B$, with B being a C^* -algebra, not necessarily equal to a full operator algebra over a Hilbert space, we are led to some interesting and useful theory.

In order to discuss this, we need a good family of target algebras B , that can we can say that we understand very well. And here, we can use:

DEFINITION 12.4. *A random matrix C^* -algebra is an algebra of type*

$$B = M_K(C(T))$$

with T being a compact space, and $K \in \mathbb{N}$ being an integer.

The terminology here comes from the fact that, in practice, the space T usually comes with a probability measure on it, which makes the elements of B “random matrices”. Observe that we can write our random matrix algebra as follows:

$$B = M_K(\mathbb{C}) \otimes C(T)$$

Thus, the random matrix algebras appear by definition as tensor products of the simplest types of C^* -algebras that we know, namely the full matrix algebras, $M_K(\mathbb{C})$ with $K \in \mathbb{N}$, and the commutative algebras, $C(T)$, with T being a compact space.

Getting back now to our modelling questions for manifolds, we can formulate:

DEFINITION 12.5. *A matrix model for a noncommutative algebraic manifold*

$$X \subset S_{\mathbb{C},+}^{N-1}$$

is a morphism of C^ -algebras of the following type,*

$$\pi : C(X) \rightarrow M_K(C(T))$$

with T being a compact space, and $K \in \mathbb{N}$ being an integer.

As a first observation, when X happens to be classical, we can take $K = 1$ and $T = X$, and we have a faithful model for our manifold, namely:

$$id : C(X) \rightarrow M_1(C(X))$$

In general, we will be looking of course for faithful models for our manifolds, or at least for models having some suitable, weaker faithfulness properties. For this purpose we cannot use of course $K = 1$, and the smallest value $K \in \mathbb{N}$ doing the job, if any, will correspond somehow to the “degree of noncommutativity” of our manifold.

Before getting into all this, we would like to clarify a few more abstract issues. As mentioned above, the C^* -algebras of type $B = M_K(C(T))$ are called “random matrix C^* -algebras”. The reason for this is the fact that most of the interesting compact spaces

T come by definition with a natural probability measure of them. Thus, B is a subalgebra of the algebra $B'' = M_K(L^\infty(T))$, usually known as a “random matrix algebra”.

This perspective is quite interesting for us, because most of our examples of manifolds $X \subset X_{\mathbb{C},+}^{N-1}$ appear as homogeneous spaces, and so are measured spaces too. Thus, we can further ask for our models $C(X) \rightarrow M_K(C(T))$ to extend into models of the following type, which can help in connection with integration problems:

$$L^\infty(X) \rightarrow M_K(L^\infty(T))$$

In short, time now to talk about L^∞ -functions, in the noncommutative setting.

12b. Von Neumann algebras

In order to discuss all this, we will need some basic von Neumann algebra theory, coming as a complement to the C^* -algebra theory from chapter 1 above.

Let us start with a key result in functional analysis, as follows:

PROPOSITION 12.6. *For an operator algebra $A \subset B(H)$, the following are equivalent:*

- (1) *A is closed under the weak operator topology, making each of the linear maps $T \rightarrow \langle Tx, y \rangle$ continuous.*
- (2) *A is closed under the strong operator topology, making each of the linear maps $T \rightarrow Tx$ continuous.*

In the case where these conditions are satisfied, A is closed under the norm topology.

PROOF. There are several statements here, the proof being as follows:

(1) It is clear that the norm topology is stronger than the strong operator topology, which is in turn stronger than the weak operator topology. At the level of the subsets $S \subset B(H)$ which are closed things get reversed, in the sense that weakly closed implies strongly closed, which in turn implies norm closed. Thus, we are left with proving that for any algebra $A \subset B(H)$, strongly closed implies weakly closed.

(2) But this latter fact is something standard, which can be proved via an amplification trick. Consider the Hilbert space obtained by summing n times H with itself:

$$K = H \oplus \dots \oplus H$$

The operators over K can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi : B(H) \rightarrow B(K)$, as follows:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

Assume now that we are given an operator $T \in \bar{A}$, with the bar denoting the weak closure. We have then, by using the Hahn-Banach theorem, for any $x \in K$:

$$\begin{aligned} T \in \bar{A} &\implies \pi(T) \in \overline{\pi(A)} \\ &\implies \pi(T)x \in \overline{\pi(A)x} \\ &\implies \pi(T)x \in \overline{\pi(A)x}^{\|\cdot\|} \end{aligned}$$

Now observe that the last formula tells us that for any $x = (x_1, \dots, x_n)$, and any $\varepsilon > 0$, we can find $S \in A$ such that the following holds, for any i :

$$\|Sx_i - Tx_i\| < \varepsilon$$

Thus T belongs to the strong operator closure of A , as desired. □

In the above statement the terminology, while standard, is a bit confusing, because the norm topology is stronger than the strong operator topology. As a solution to this, we agree in what follows to call the norm topology “strong”, and the weak and strong operator topologies “weak”, whenever these two topologies coincide.

With this convention, the algebras from Proposition 12.6 are those which are weakly closed, and we can formulate:

DEFINITION 12.7. *A von Neumann algebra is a *-algebra of operators*

$$A \subset B(H)$$

which is closed under the weak topology.

As basic examples, we have the algebra $B(H)$ itself, then the singly generated von Neumann algebras, $A = \langle T \rangle$, with $T \in B(H)$, and then the multiply generated von Neumann algebras, namely $A = \langle T_i \rangle$, with $T_i \in B(H)$. There are many other examples, and general methods for constructing examples, and we will discuss this later.

At the level of the general results, we first have the bicommutant theorem of von Neumann, which provides a useful alternative to Definition 12.7, as follows:

THEOREM 12.8. *For a *-algebra $A \subset B(H)$, the following are equivalent:*

- (1) *A is weakly closed, so it is a von Neumann algebra.*
- (2) *A equals its algebraic bicommutant A'' , taken inside $B(H)$.*

PROOF. Since the commutants are automatically weakly closed, it is enough to show that weakly closed implies $A = A''$. For this purpose, we will prove something a bit more general, stating that given a *-algebra of operators $A \subset B(H)$, the following holds, with A'' being the bicommutant inside $B(H)$, and with \bar{A} being the weak closure:

$$A'' = \bar{A}$$

We prove this equality by double inclusion, as follows:

“ \supset ” Since any operator commutes with the operators that it commutes with, we have a trivial inclusion $S \subset S''$, valid for any set $S \subset B(H)$. In particular, we have:

$$A \subset A''$$

Our claim now is that the algebra A'' is closed, with respect to the strong operator topology. Indeed, assuming that we have $T_i \rightarrow T$ in this topology, we have:

$$\begin{aligned} T_i \in A'' &\implies ST_i = T_iS, \forall S \in A' \\ &\implies ST = TS, \forall S \in A' \\ &\implies T \in A \end{aligned}$$

Thus our claim is proved, and together with Proposition 12.6, which allows us to pass from the strong to the weak operator topology, this gives the desired inclusion:

$$\bar{A} \subset A''$$

“ \subset ” Here we must prove that we have the following implication, valid for any $T \in B(H)$, with the bar denoting as usual the weak operator closure:

$$T \in A'' \implies T \in \bar{A}$$

For this purpose, we use the same amplification trick as in the proof of Proposition 12.5 above. Consider the Hilbert space obtained by summing n times H with itself:

$$K = H \oplus \dots \oplus H$$

The operators over K can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi : B(H) \rightarrow B(K)$, as follows:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

The idea will be that of doing the computations in this representation. First, in this representation, the image of our algebra $A \subset B(H)$ is given by:

$$\pi(A) = \left\{ \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \mid T \in A \right\}$$

We can compute the commutant of this image, exactly as in the usual scalar matrix case, and we obtain the following formula:

$$\pi(A)' = \left\{ \begin{pmatrix} S_{11} & \dots & S_{1n} \\ \vdots & & \vdots \\ S_{n1} & \dots & S_{nn} \end{pmatrix} \mid S_{ij} \in A' \right\}$$

We conclude from this that, given an operator $T \in A''$ as above, we have:

$$\begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \in \pi(A)''$$

In other words, the conclusion of all this is that we have:

$$T \in A'' \implies \pi(T) \in \pi(A)''$$

Now given a vector $x \in K$, consider the orthogonal projection $P \in B(K)$ on the norm closure of the vector space $\pi(A)x \subset K$. Since the subspace $\pi(A)x \subset K$ is invariant under the action of $\pi(A)$, so is its norm closure inside K , and we obtain from this:

$$P \in \pi(A)'$$

By combining this with what we found above, we conclude that we have:

$$T \in A'' \implies \pi(T)P = P\pi(T)$$

Now since this holds for any $x \in K$, we conclude that any $T \in A''$ belongs to the strong operator closure of A . By using now Proposition 12.5, which allows us to pass from the strong to the weak operator closure, we conclude that we have $A'' \subset \bar{A}$, as desired. \square

In order to develop now some general theory for the von Neumann algebras, let us start by investigating the finite dimensional case.

Here the ambient operator algebra is $B(H) = M_N(\mathbb{C})$, and any subspace $A \subset B(H)$ is automatically closed, for all 3 topologies from Proposition 12.6 above.

Thus, we are left with the question of investigating the $*$ -algebras of usual matrices $A \subset M_N(\mathbb{C})$. But this is a purely algebraic question, whose answer is as follows:

THEOREM 12.9. *The $*$ -algebras $A \subset M_N(\mathbb{C})$ are exactly the algebras of the form*

$$A = M_{r_1}(\mathbb{C}) \oplus \dots \oplus M_{r_k}(\mathbb{C})$$

depending on parameters $k \in \mathbb{N}$ and $r_1, \dots, r_k \in \mathbb{N}$ satisfying

$$r_1 + \dots + r_k = N$$

embedded into $M_N(\mathbb{C})$ via the obvious block embedding, twisted by a unitary $U \in U_N$.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Given numbers $r_1, \dots, r_k \in \mathbb{N}$ satisfying $r_1 + \dots + r_k = N$, we have an obvious embedding of $*$ -algebras, via matrix blocks, as follows:

$$M_{r_1}(\mathbb{C}) \oplus \dots \oplus M_{r_k}(\mathbb{C}) \subset M_N(\mathbb{C})$$

In addition, we can twist this embedding by a unitary $U \in U_N$, as follows:

$$M \rightarrow U M U^*$$

Thus, we have proved one of the implications.

(2) In the other sense now, consider an arbitrary $*$ -algebra of the $N \times N$ matrices:

$$A \subset M_N(\mathbb{C})$$

Let us first look at the center of this algebra, which given by:

$$Z(A) = A \cap A'$$

It is elementary to prove that this center, as an algebra, is of the following form:

$$Z(A) \simeq \mathbb{C}^k$$

Consider now the standard basis $e_1, \dots, e_k \in \mathbb{C}^k$, and let $p_1, \dots, p_k \in Z(A)$ be the images of these vectors via the above identification. In other words, these elements $p_1, \dots, p_k \in A$ are central minimal projections, summing up to 1:

$$p_1 + \dots + p_k = 1$$

The idea is then that this partition of the unity will eventually lead to the block decomposition of A , as in the statement. We prove this in 4 steps, as follows:

Step 1. We first construct the matrix blocks, our claim here being that each of the following linear subspaces of A are non-unital $*$ -subalgebras of A :

$$A_i = p_i A p_i$$

But this is clear, with the fact that each A_i is closed under the various non-unital $*$ -subalgebra operations coming from the projection equations $p_i^2 = p_i = p_i^*$.

Step 2. We prove now that the above algebras $A_i \subset A$ are in a direct sum position, in the sense that we have a non-unital $*$ -algebra sum decomposition, as follows:

$$A = A_1 \oplus \dots \oplus A_k$$

As with any direct sum question, we have two things to be proved here. First, by using the formula $p_1 + \dots + p_k = 1$ and the projection equations $p_i^2 = p_i = p_i^*$, we conclude that we have the needed generation property, namely:

$$A_1 + \dots + A_k = A$$

As for the fact that the sum is indeed direct, this follows as well from the formula $p_1 + \dots + p_k = 1$, and from the projection equations $p_i^2 = p_i = p_i^*$.

Step 3. Our claim now, which will finish the proof, is that each of the $*$ -subalgebras $A_i = p_i A p_i$ constructed above is a full matrix algebra. To be more precise here, with $r_i = \text{rank}(p_i)$, our claim is that we have isomorphisms, as follows:

$$A_i \simeq M_{r_i}(\mathbb{C})$$

In order to prove this claim, recall that the projections $p_i \in A$ were chosen central and minimal. Thus, the center of each of the algebras A_i reduces to the scalars:

$$Z(A_i) = \mathbb{C}$$

But this shows, either via a direct computation, or via the bicommutant theorem, that each of the algebras A_i is a full matrix algebra, as claimed.

Step 4. We can now obtain the result, by putting together what we have. Indeed, by using the results from Step 2 and Step 3, we obtain an isomorphism as follows:

$$\begin{aligned} A &= A_1 \oplus \dots \oplus A_k \\ &\simeq M_{r_1}(\mathbb{C}) \oplus \dots \oplus M_{r_k}(\mathbb{C}) \end{aligned}$$

Moreover, a careful look at the isomorphisms established in Step 3 shows that at the global level, of the algebra A itself, the above isomorphism comes by twisting the standard multimatrix embedding $M_{r_1}(\mathbb{C}) \oplus \dots \oplus M_{r_k}(\mathbb{C}) \subset M_N(\mathbb{C})$, discussed in the beginning of the proof, (1) above, by a certain unitary $U \in U_N$. Thus, we obtain the result. \square

As an application of Theorem 12.9, clarifying the relation with linear algebra, or operator theory in finite dimensions, we have the following result:

PROPOSITION 12.10. *Given an operator $T \in B(H)$ in finite dimensions, $H = \mathbb{C}^N$, the von Neumann algebra $A = \langle T \rangle$ that it generates inside $B(H) = M_N(\mathbb{C})$ is*

$$A = M_{r_1}(\mathbb{C}) \oplus \dots \oplus M_{r_k}(\mathbb{C})$$

with the sizes of the blocks $r_1, \dots, r_k \in \mathbb{N}$ coming from the spectral theory of the associated matrix $M \in M_N(\mathbb{C})$. In the normal case $TT^ = T^*T$, this decomposition comes from*

$$T = UDU^*$$

with $D \in M_N(\mathbb{C})$ diagonal, and with $U \in U_N$ unitary.

PROOF. This is standard, by using the basic linear algebra theory and spectral theory for the usual matrices $M \in M_N(\mathbb{C})$. \square

Let us get now to infinite dimensions, with Proposition 12.10 as our main source of inspiration. We have here the following result:

THEOREM 12.11. *Given an operator $T \in B(H)$ which is normal,*

$$TT^* = T^*T$$

the von Neumann algebra $A = \langle T \rangle$ that it generates inside $B(H)$ is

$$\langle T \rangle = L^\infty(\sigma(T))$$

with $\sigma(T)$ being its spectrum, formed of numbers $\lambda \in \mathbb{C}$ such that $T - \lambda$ is not invertible.

PROOF. This is standard as well, by using the spectral theory for the normal operators $T \in B(H)$, coming from chapter 1 above. \square

More generally, along the same lines, we have the following result, dealing this time with commuting families of normal operators:

THEOREM 12.12. *Given operators $T_i \in B(H)$ which are normal, and which commute, the von Neumann algebra $A = \langle T_i \rangle$ that these operators generates inside $B(H)$ is*

$$\langle T_i \rangle = L^\infty(X)$$

with X being a certain measured space, associated to the family $\{T_i\}$.

PROOF. This is once again routine, by using the spectral theory for the families of commuting normal operators $T_i \in B(H)$, coming from chapter 1 above. \square

As an interesting abstract consequence of this, we have:

THEOREM 12.13. *The commutative von Neumann algebras are the algebras of type*

$$A = L^\infty(X)$$

with X being a measured space.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) In one sense, we must prove that given a measured space X , we can realize the commutative algebra $A = L^\infty(X)$ as a von Neumann algebra, on a certain Hilbert space H . But this is something that we already know, coming from the multiplicity operators $T_f(g) = fg$ discussed in chapter 1 above, the representation being as follows:

$$L^\infty(X) \subset B(L^2(X))$$

(2) In the other sense, given a commutative von Neumann algebra $A \subset B(H)$, we must construct a certain measured space X , and an identification $A = L^\infty(X)$. But this follows from Theorem 12.12, because we can write our von Neumann algebra as follows:

$$A = \langle T_i \rangle$$

To be more precise, A being commutative, any element $T \in A$ is normal. Thus, we can pick a basis $\{T_i\} \subset A$, and then we have $A = \langle T_i \rangle$ as above, with $T_i \in B(H)$ being commuting normal operators. Thus Theorem 12.12 applies, and gives the result. \square

The above result is not the end of the story with the commutative von Neumann algebras, because we still have to understand how a given commutative algebra $A = L^\infty(X)$ can be represented as an operator algebra, $A \subset B(H)$, over the various Hilbert spaces H . The answer here is that the commutative von Neumann algebras appear as $L^\infty(X) \subset B(L^2(X))$, up to a certain multiplicity, but we will not need this here.

Moving ahead now, we can combine Proposition 12.8 with Theorem 12.13, and by building along the lines of Theorem 12.9, but this time in infinite dimensions, we are led to the following statement, due to Murray-von Neumann and Connes:

THEOREM 12.14. *Given a von Neumann algebra $A \subset B(H)$, if we write its center as*

$$Z(A) = L^\infty(X)$$

then we have a decomposition as follows, with the fibers A_x having trivial center:

$$A = \int_X A_x dx$$

Moreover, the factors, $Z(A) = \mathbb{C}$, can be basically classified in terms of the II_1 factors, which are those satisfying $\dim A = \infty$, and having a faithful trace $\text{tr} : A \rightarrow \mathbb{C}$.

PROOF. This is something that we know to hold in finite dimensions, as a consequence of Theorem 12.9 above. In general, this is something heavy, the idea being as follows:

(1) This is von Neumann's reduction theory main result, whose statement is already quite hard to understand, and whose proof uses advanced functional analysis.

(2) This is heavy, due to Murray-von Neumann and Connes, the idea being that the other factors can be basically obtained via crossed product constructions. \square

Now back to our noncommutative geometry questions, as a first application of all this, we can extend our noncommutative space setting, as follows:

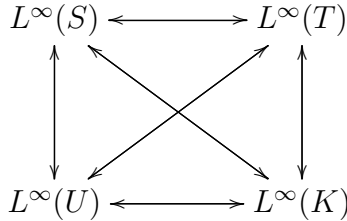
THEOREM 12.15. *Consider the category of "noncommutative measure spaces", having as objects the pairs (A, tr) consisting of a von Neumann algebra with a faithful trace, and with the arrows reversed, which amounts in writing $A = L^\infty(X)$ and $\text{tr} = \int_X$.*

- (1) *The category of usual measured spaces embeds into this category, and we obtain in this way the objects whose associated von Neumann algebra is commutative.*
- (2) *Each C^* -algebra given with a trace produces as well a noncommutative measure space, by performing the GNS construction, and taking the weak closure.*
- (3) *In what regards the finitely generated group duals, or more generally the compact matrix quantum groups, the corresponding identification is injective.*
- (4) *Even more generally, for noncommutative algebraic manifolds having an integration functional, like the spheres, the identification is injective.*

PROOF. This is clear indeed from the basic properties of the GNS construction, from Theorem 12.2, and from the general theory from Theorem 12.14. \square

Before getting into matrix modelling questions, we would like to formulate the following result, that we announced in chapter 1 above, but had not discussed yet:

THEOREM 12.16. *In the context of noncommutative geometries coming from quadruplets (S, T, U, K) , we have von Neumann algebras, with traces, as follows,*



with $L^\infty(S) \subset L^\infty(U)$ being obtained by taking the first row algebra.

PROOF. This follows indeed from the results that we already have, from chapters 1-4 above, by using the general formalism from Theorem 12.15. \square

As already suggested in chapter 1, such a statement raises the question of axiomatizing, or rather reaxiomatizing, the quadruplets (S, T, U, K) that we are interested in directly in terms of the associated von Neumann algebras, as above.

Indeed, in view of our general quantum mechanics motivations, we are after all mostly interested in integrating over our quantum manifolds, and so with this in mind, the von Neumann algebra formalism seems to be well adapted to our questions.

However, this is wrong. The above result is something theoretical, because it assumes the existence of Haar measures, which come by theorem. Thus, while all this is nice, the good way of doing things is with C^* -algebras, as we did in chapters 1-4 above.

As a side comment here, the question “does the algebra or the Hilbert space come first” is a well-known one in quantum mechanics, basically leading to 2 different schools of thought. We obviously adhere here to the “algebra comes first” school.

12c. Matrix truncations

In relation now with the modelling questions that we are interested in here, we can now go ahead with our program, and discuss von Neumann algebraic extensions.

We have the following result:

THEOREM 12.17. *Given a matrix model $\pi : C(X) \rightarrow M_K(C(T))$, with both X, T being assumed to have integration functionals, the following are equivalent:*

- (1) π is stationary, in the sense that $\int_X = (\text{tr} \otimes \int_T)\pi$.
- (2) π produces an inclusion $\pi' : C_{\text{red}}(X) \subset M_K(X(T))$.
- (3) π produces an inclusion $\pi'' : L^\infty(X) \subset M_K(L^\infty(T))$.

Moreover, in the quantum group case, these conditions imply that π is faithful.

PROOF. This is standard functional analysis, as follows:

(1) Consider the following diagram, with all the solid arrows being by definition the canonical maps between the algebras concerned:

$$\begin{array}{ccc}
 M_K(C(T)) & \xrightarrow{\hspace{10em}} & M_K(L^\infty(T)) \\
 \uparrow \pi & \swarrow \pi' & \uparrow \pi'' \\
 C(X) & \xrightarrow{\hspace{2em}} C_{red}(X) \xrightarrow{\hspace{2em}} & L^\infty(X)
 \end{array}$$

(2) With this picture in hand, the implications (1) \iff (2) \iff (3) between the conditions (1,2,3) in the statement are all clear, coming from the basic properties of the GNS construction, and of the von Neumann algebras, explained in the above.

(3) As for the last assertion, this is something more subtle, coming from the fact that if $L^\infty(G)$ is of type I, as required by (3), then G must be coamenable. See [79]. \square

The above result raises a number of interesting questions, notably in what regards the extension of the last assertion, to the case of more general homogeneous spaces.

Before going further, we would like to record as well the following key result regarding the matrix models, valid so far in the quantum group case only:

THEOREM 12.18. *Consider a matrix model $\pi : C(G) \rightarrow M_K(C(T))$ for a closed subgroup $G \subset U_N^+$, with T being assumed to be a compact probability space.*

(1) *There exists a smallest subgroup $G' \subset G$, producing a factorization of type:*

$$\pi : C(G) \rightarrow C(G') \rightarrow M_K(C(T))$$

The algebra $C(G')$ is called Hopf image of π .

(2) *When π is inner faithful, in the sense that $G = G'$, we have the formula*

$$\int_G = \lim_{k \rightarrow \infty} \sum_{r=1}^k \varphi^{*r}$$

*where $\varphi = (tr \otimes \int_T)\pi$, and $\phi * \psi = (\phi \otimes \psi)\Delta$.*

PROOF. All this is well-known, but quite specialized, the idea being as follows:

(1) This follows by dividing the algebra $C(G)$ by a suitable ideal, namely the Hopf ideal generated by the kernel of the matrix model map $\pi : C(G) \rightarrow M_K(C(T))$.

(2) This follows by suitably adapting Woronowicz’s proof for the existence and formula of the Haar integration functional from [100], to the matrix model situation. \square

The above result is quite important, for a number of reasons. Indeed, as a main application of it, while the existence of a faithful matrix model $\pi : C(G) \subset M_K(C(T))$ forces the C^* -algebra $C(G)$ to be of type I, and so G to be coamenable, as already mentioned in the proof of Theorem 12.17 above, there is no known restriction coming from the existence of an inner faithful model $\pi : C(G) \rightarrow M_K(C(T))$. See [15], [47].

In the general manifold setting, talking about such things is in general not possible, unless our manifold X has some extra special structure, as for instance being an homogeneous space, in the spirit of the spaces discussed in chapters 5-8 above.

However, in practice, such a theory has not been developed yet.

Let us go back now to our basic notion of a matrix model, from Definition 12.5 above, and develop some more general theory, in that setting. We first have:

PROPOSITION 12.19. *A 1×1 model for a manifold $X \subset S_{\mathbb{C},+}^{N-1}$ must come from a map*

$$p : T \rightarrow X_{class} \subset X$$

and π is faithful precisely when $X = X_{class}$, and when p is surjective.

PROOF. According to our conventions, a 1×1 model for a manifold $X \subset S_{\mathbb{C},+}^{N-1}$ is simply a morphism of C^* -algebras as follows:

$$\pi : C(X) \rightarrow C(T)$$

Now since the algebra $C(T)$ is commutative, this morphism must factorize through the abelianization of $C(X)$, as follows:

$$\pi : C(X) \rightarrow C(X_{class}) \rightarrow C(T)$$

Thus, our morphism π must come by transposition from a map p , as claimed. □

In order to generalize the above trivial fact, we use the following definition:

DEFINITION 12.20. *Let $X \subset S_{\mathbb{C},+}^{N-1}$. We define a closed subspace $X^{(K)} \subset X$ by*

$$C(X^{(K)}) = C(X)/J_K$$

where J_K is the common null space of matrix representations of $C(X)$, of size $L \leq K$,

$$J_K = \bigcap_{L \leq K} \bigcap_{\pi: C(X) \rightarrow M_L(\mathbb{C})} \ker(\pi)$$

and we call $X^{(K)}$ the “part of X which is realizable with $K \times K$ models”.

As a basic example here, the first such space, at $K = 1$, is the classical version:

$$X^{(1)} = X_{class}$$

Observe that we have embeddings of quantum spaces, as follows:

$$X^{(1)} \subset X^{(2)} \subset X^{(3)} \dots \subset X$$

As a first result now on these spaces, we have the following well-known fact:

THEOREM 12.21. *The increasing union of compact quantum spaces*

$$X^{(\infty)} = \bigcup_{K \geq 1} X^{(K)}$$

equals X precisely when the algebra $C(X)$ is residually finite dimensional.

PROOF. This is something well-known, coming from the general theory from [92]. We refer to [46] for a discussion on this topic, in the context of the quantum groups. \square

Getting back now to the case $K < \infty$, we first have, following [16]:

PROPOSITION 12.22. *Consider an algebraic manifold $X \subset S_{\mathbb{C},+}^{N-1}$.*

- (1) *Given a closed subspace $Y \subset X \subset S_{\mathbb{C},+}^{N-1}$, we have $Y \subset X^{(K)}$ precisely when any irreducible representation of $C(Y)$ has dimension $\leq K$.*
- (2) *In particular, we have $X^{(K)} = X$ precisely when any irreducible representation of $C(X)$ has dimension $\leq K$.*

PROOF. This follows by using the general C^* -algebra theory, as follows:

(1) If any irreducible representation of $C(Y)$ has dimension $\leq K$, then we have $Y \subset X^{(K)}$, because the irreducible representations of a C^* -algebra separate its points.

Conversely, assuming $Y \subset X^{(K)}$, it is enough to show that any irreducible representation of the algebra $C(X^{(K)})$ has dimension $\leq K$. But this is once again well-known.

(2) This follows indeed from (1). \square

The connection with the previous considerations comes from:

THEOREM 12.23. *If $X \subset S_{\mathbb{C},+}^{N-1}$ has a faithful matrix model*

$$C(X) \rightarrow M_K(C(T))$$

then we have $X = X^{(K)}$.

PROOF. This follows from the above and from standard representation theory of the C^* -algebras. For full details on all this, we refer to [16]. \square

We now discuss the universal $K \times K$ -matrix model, constructed as follows:

THEOREM 12.24. *Given $X \subset S_{\mathbb{C},+}^{N-1}$ algebraic, the category of its $K \times K$ matrix models, with $K \geq 1$ being fixed, has a universal object as follows:*

$$\pi_K : C(X) \rightarrow M_K(C(T_K))$$

That is, given a matrix model

$$\rho : C(X) \rightarrow M_K(C(T))$$

we have a diagram of type

$$\begin{array}{ccc} C(X) & \xrightarrow{\pi} & M_K(C(T_K)) \\ & \searrow \rho & \swarrow \\ & & M_K(C(T)) \end{array}$$

where the map on the right is unique and arises from a continuous map $T \rightarrow T_K$.

PROOF. Consider the universal commutative C^* -algebra generated by elements $x_{ij}(a)$, with $1 \leq i, j \leq K$, $a \in \mathcal{O}(X)$, subject to the relations ($a, b \in \mathcal{O}(X)$, $\lambda \in \mathbb{C}$, $1 \leq i, j \leq K$):

$$x_{ij}(a + \lambda b) = x_{ij}(a) + \lambda x_{ij}(b)$$

$$x_{ij}(ab) = \sum_k x_{ik}(a)x_{kj}(b)$$

$$x_{ij}(1) = \delta_{ij}$$

$$x_{ij}(a)^* = x_{ji}(a^*)$$

This is indeed well-defined because of the following relations:

$$\sum_l \sum_k x_{ik}(z_l^*)x_{ki}(z_l) = 1$$

Let T_K be the spectrum of this C^* -algebra. Since X is algebraic, we have:

$$\pi : C(X) \rightarrow M_K(C(T_K))$$

$$\pi(z_k) = (x_{ij}(z_k))$$

By construction of T_K and π , we have the universal matrix model. See [16]. □

Getting now to the case of the algebraic manifolds, we first have here:

PROPOSITION 12.25. *Let $X \subset S_{\mathbb{C},+}^{N-1}$ with X algebraic and $X_{class} \neq \emptyset$, and let*

$$\pi : C(X) \rightarrow M_K(C(T_K))$$

be the universal matrix model. Then we have

$$C(X^{(K)}) = C(X)/Ker(\pi)$$

and hence $X = X^{(K)}$ if and only if X has a faithful $K \times K$ -matrix model.

PROOF. We have to show that $\text{Ker}(\pi) = J_K$, the latter ideal being the intersection of the kernels of all matrix representations $C(X) \rightarrow M_L(\mathbb{C})$, for any $L \leq K$. For $a \notin \text{Ker}(\pi)$, we see that $a \notin J_K$ by evaluating at an appropriate element of T_K .

Conversely, assume that we are given $a \in \text{Ker}(\pi)$. Let $\rho : C(X) \rightarrow M_L(\mathbb{C})$ be a representation with $L \leq K$, and let $\varepsilon : C(X) \rightarrow \mathbb{C}$ be a representation. We can extend ρ to a representation $\rho' : C(X) \rightarrow M_K(\mathbb{C})$ by letting, for any $b \in C(X)$:

$$\rho'(b) = \begin{pmatrix} \rho(b) & 0 \\ 0 & \varepsilon(b)I_{K-L} \end{pmatrix}$$

The universal property of the universal matrix model yields that $\rho'(a) = 0$, since $\pi(a) = 0$. Thus $\rho(a) = 0$. We therefore have $a \in J_K$, and $\text{Ker}(\pi) \subset J_K$, and the first statement is proved. The last statement follows from the first one. See [16]. \square

Next, we have the following result, also from [16]:

PROPOSITION 12.26. *Let $X \subset S_{\mathbb{C},+}^{N-1}$ be algebraic, and satisfying:*

$$X_{\text{class}} \neq \emptyset$$

Then $X^{(K)}$ is algebraic as well.

PROOF. We keep the notations above, and consider the following map:

$$\begin{aligned} \pi_0 : \mathcal{O}(X) &\rightarrow M_K(C(T_K)) \\ z_l &\rightarrow (x_{ij}(z_l)) \end{aligned}$$

This induces a $*$ -algebra map, as follows:

$$\tilde{\pi}_0 : C^*(\mathcal{O}(X)/\text{Ker}(\pi_0)) \rightarrow M_K(C(T_K))$$

We need to show that $\tilde{\pi}_0$ is injective. For this purpose, observe that the universal model factorizes as follows, where p is canonical surjection:

$$\pi : C(X) \xrightarrow{p} C^*(\mathcal{O}(X)/\text{Ker}(\pi_0)) \xrightarrow{\tilde{\pi}_0} M_K(C(T_K))$$

We therefore obtain $\text{Ker}(\pi) = \text{Ker}(p)$, and we conclude that:

$$\begin{aligned} C(X^{(K)}) &= C(X)/\text{Ker}(p) \\ &= C^*(\mathcal{O}(X)/\text{Ker}(\pi_0)) \end{aligned}$$

Thus $X^{(K)}$ is indeed algebraic. Since $\mathcal{O}(X)/\text{Ker}(\pi_0)$ is isomorphic to a $*$ -subalgebra of $M_K(C(T_K))$, it satisfies the standard Amitsur-Levitski polynomial identity:

$$S_{2K}(x_1, \dots, x_{2K}) = 0$$

By density, so does $C^*(\mathcal{O}(X)/\text{Ker}(\pi_0))$.

Thus any irreducible representation of $C^*(\mathcal{O}(X)/Ker(\pi_0))$ has dimension $\leq K$. Now if $a \in C^*(\mathcal{O}(X)/Ker(\pi_0))$ is a nonzero element, we can, by the same reasoning as in the previous proof, find a representation as follows, such that $\rho(a) \neq 0$:

$$\rho : C^*(\mathcal{O}(X)/Ker(\pi_0)) \rightarrow M_K(\mathbb{C})$$

Indeed, given algebra map $\varepsilon : C(X) \rightarrow \mathbb{C}$ induces an algebra map:

$$\begin{aligned} C(T_K) &\rightarrow \mathbb{C} \\ x_{ij}(a) &\rightarrow \delta_{ij}\varepsilon(a) \end{aligned}$$

This map enables us to extend representations similarly as before. By construction the universal model space yields an algebra map as follows:

$$M_K(C(T_K)) \rightarrow M_K(\mathbb{C})$$

The composition of this map with $\tilde{\pi}_0 p = \pi$ is ρp , so $\tilde{\pi}_0(a) \neq 0$, and $\tilde{\pi}_0$ is injective. \square

Summarizing, we have proved the following result:

THEOREM 12.27. *Let $X \subset S_{\mathbb{C},+}^{N-1}$ be algebraic, satisfying:*

$$X_{class} \neq \emptyset$$

Then we have an increasing sequence of algebraic submanifolds

$$X_{class} = X^{(1)} \subset X^{(2)} \subset X^{(3)} \subset \dots \subset X$$

where $X^{(K)}$ is given by the fact that

$$C(X^{(K)}) \subset M_K(C(T_K))$$

is obtained by factorizing the universal matrix model.

PROOF. This follows indeed from the above results. See [16]. \square

12d. Half-liberation

As an illustration for the above theory, let us discuss now the half-liberation operation, which is connected to $X^{(2)}$, as a continuation of the material from chapter 9.

We first restrict the attention to the real case. Let us start with:

DEFINITION 12.28. *The half-classical version of a manifold X is given by:*

$$C(X^*) = C(X) / \left\langle abc = cba \mid \forall a, b, c \in \{x_i\} \right\rangle$$

We say that X is half-classical when $X = X^$.*

In order to understand the structure of X^* , we use an old matrix model method, which goes back to [41], and then to [40]. This is based on the following observation:

PROPOSITION 12.29. *For any $z \in \mathbb{C}^N$, the matrices*

$$X_i = \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

are self-adjoint, and half-commute.

PROOF. The matrices X_i are indeed self-adjoint, and their products are given by:

$$\begin{aligned} X_i X_j &= \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix} \begin{pmatrix} 0 & z_j \\ \bar{z}_j & 0 \end{pmatrix} \\ &= \begin{pmatrix} z_i \bar{z}_j & 0 \\ 0 & \bar{z}_i z_j \end{pmatrix} \end{aligned}$$

Also, we have the following formula:

$$\begin{aligned} X_i X_j X_k &= \begin{pmatrix} z_i \bar{z}_j & 0 \\ 0 & \bar{z}_i z_j \end{pmatrix} \begin{pmatrix} 0 & z_k \\ \bar{z}_k & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & z_i \bar{z}_j z_k \\ \bar{z}_i z_j \bar{z}_k & 0 \end{pmatrix} \end{aligned}$$

Now since this latter quantity is symmetric in i, k , we obtain from this that we have the half-commutation formula $X_i X_j X_k = X_k X_j X_i$, as desired. \square

The idea now will be that of using the matrices in Proposition 12.29 in order to model the coordinates of the arbitrary half-classical manifolds.

In order to connect the algebra of the classical coordinates z_i to that of the noncommutative coordinates X_i , we will need an abstract definition, as follows:

DEFINITION 12.30. *Given a noncommutative polynomial in N variables*

$$f \in \mathbb{R} \langle x_1, \dots, x_N \rangle$$

we define a usual polynomial in $2N$ variables

$$f^\circ \in \mathbb{R}[z_1, \dots, z_N, \bar{z}_1, \dots, \bar{z}_N]$$

according to the formula

$$f = x_{i_1} x_{i_2} x_{i_3} x_{i_4} \dots \implies f^\circ = z_{i_1} \bar{z}_{i_2} z_{i_3} \bar{z}_{i_4} \dots$$

in the monomial case, and then by extending this correspondence, by linearity.

As a basic example here, the polynomial defining the free real sphere $S_{\mathbb{R},+}^{N-1}$ produces in this way the polynomial defining the complex sphere $S_{\mathbb{C}}^{N-1}$:

$$f = x_1^2 + \dots + x_N^2 \implies f^\circ = |z_1|^2 + \dots + |z_N|^2$$

Also, given a polynomial $f \in \mathbb{R} \langle x_1, \dots, x_N \rangle$, we can decompose it into its even and odd parts, $f = g + h$, by putting into g/h the monomials of even/odd length. Observe that with $z = (z_1, \dots, z_N)$, these odd and even parts are given by:

$$g(z) = \frac{f(z) + f(-z)}{2}$$

$$h(z) = \frac{f(z) - f(-z)}{2}$$

With these conventions, we have the following result:

PROPOSITION 12.31. *Given a manifold X , coming from a family of polynomials*

$$\{f_\alpha\} \subset \mathbb{R} \langle x_1, \dots, x_N \rangle$$

we have a morphism of unital C^ -algebras as follows,*

$$\pi : C(X) \rightarrow M_2(\mathbb{C}) \quad , \quad \pi(x_i) = \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

precisely when $z = (z_1, \dots, z_N) \in \mathbb{C}^N$ belongs to the real algebraic manifold

$$Y = \left\{ z \in \mathbb{C}^N \mid g_\alpha^\circ(z_1, \dots, z_N) = h_\alpha^\circ(z_1, \dots, z_N) = 0, \forall \alpha \right\}$$

where $f_\alpha = g_\alpha + h_\alpha$ is the even/odd decomposition of f_α .

PROOF. Let X_i be the matrices in the statement. In order for $x_i \rightarrow X_i$ to define a morphism of algebras, these matrices must satisfy the equations defining X . Thus, the model space Z in the statement consists of those points $z = (z_1, \dots, z_N) \in \mathbb{C}^N$ satisfying:

$$f_\alpha(X_1, \dots, X_N) = 0 \quad , \quad \forall \alpha$$

Now observe that the matrices X_i in the statement multiply as follows:

$$X_{i_1} X_{j_1} \dots X_{i_k} X_{j_k} = \begin{pmatrix} z_{i_1} \bar{z}_{j_1} \dots z_{i_k} \bar{z}_{j_k} & 0 \\ 0 & \bar{z}_{i_1} z_{j_1} \dots \bar{z}_{i_k} z_{j_k} \end{pmatrix}$$

$$X_{i_1} X_{j_1} \dots X_{i_k} X_{j_k} X_{i_{k+1}} = \begin{pmatrix} 0 & z_{i_1} \bar{z}_{j_1} \dots z_{i_k} \bar{z}_{j_k} z_{i_{k+1}} \\ \bar{z}_{i_1} z_{j_1} \dots \bar{z}_{i_k} z_{j_k} \bar{z}_{i_{k+1}} & 0 \end{pmatrix}$$

We therefore obtain, in terms of the even/odd decomposition $f_\alpha = g_\alpha + h_\alpha$:

$$f_\alpha(X_1, \dots, X_N) = \begin{pmatrix} g_\alpha^\circ(z_1, \dots, z_N) & h_\alpha^\circ(z_1, \dots, z_N) \\ \overline{h_\alpha^\circ(z_1, \dots, z_N)} & \overline{g_\alpha^\circ(z_1, \dots, z_N)} \end{pmatrix}$$

Thus, we obtain the equations for Y from the statement. □

As a first consequence, of theoretical interest, a necessary condition for X to exist is that the manifold $Y \subset \mathbb{C}^N$ constructed above must be compact.

In order to discuss modelling questions, we will need as well:

DEFINITION 12.32. *Assuming that we are given a manifold Z , appearing via*

$$C(Z) = C^* \left(z_1, \dots, z_N \mid f_\alpha(z_1, \dots, z_N) = 0 \right)$$

we define the projective version of Z to be the quotient space $Z \rightarrow PZ$ corresponding to the subalgebra $C(PZ) \subset C(Z)$ generated by the variables $x_{ij} = z_i z_j^$.*

The relation with the half-classical manifolds comes from the fact that the projective version of a half-classical manifold is classical. Indeed, from $abc = cba$ we obtain:

$$\begin{aligned} ab \cdot cd &= (abc)d \\ &= (cba)d \\ &= c(bad) \\ &= c(dab) \\ &= cd \cdot ab \end{aligned}$$

Finally, let us call “matrix model” any morphism of unital C^* -algebras $f : A \rightarrow B$, with target algebra $B = M_K(C(Y))$, with $K \in \mathbb{N}$, and Y being a compact space.

Following [40], we have the following result:

THEOREM 12.33. *Given a half-classical manifold X which is symmetric, in the sense that all its defining polynomials f_α are even, its universal 2×2 antidiagonal model,*

$$\pi : C(X) \rightarrow M_2(C(Y))$$

where Y is the manifold constructed in Proposition 12.31, is faithful. In addition, the construction $X \rightarrow Y$ is such that X exists precisely when Y is compact.

PROOF. We can proceed as in [40]. Indeed, the universal model π in the statement induces, at the level of projective versions, a certain representation:

$$C(PX) \rightarrow M_2(C(PY))$$

By using the multiplication formulae from the proof of Proposition 12.31, the image of this representation consists of diagonal matrices, and the upper left components of these matrices are the standard coordinates of PY . Thus, we have an isomorphism:

$$PX \simeq PY$$

We can conclude as in [40], by using a grading trick. See [40]. □

As a first observation, this result shows that when X is symmetric, we have $X^* \subset X^{(2)}$. Going beyond this observation is an interesting problem.

In what follows, we will rather need a more detailed version of the above result. For this purpose, we can use the following definition:

DEFINITION 12.34. *Associated to any compact manifold $Y \subset \mathbb{C}^N$ is the real compact half-classical manifold $[Y]$, having as coordinates the following variables,*

$$X_i = \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

where z_1, \dots, z_N are the standard coordinates on Y . In other words, $[Y]$ is given by the fact that $C([Y]) \subset M_2(C(Y))$ is the algebra generated by these matrices.

Here the fact that $[Y]$ is indeed half-classical follows from the results above. As for the fact that $[Y]$ is indeed algebraic, this follows from Theorem 12.33.

We can now reformulate the result in Theorem 12.33, as follows:

THEOREM 12.35. *The symmetric half-classical manifolds X appear as follows:*

- (1) *We have $X = [Y]$, for a certain conjugation-invariant subspace $Y \subset \mathbb{C}^N$.*
- (2) *$PX = P[Y]$, and X is maximal with this property.*
- (3) *In addition, we have an embedding $C([X]) \subset C(X) \rtimes \mathbb{Z}_2$.*

PROOF. This follows from Theorem 12.33, with the embedding in (3) being constructed as in [40], by $x_i = z_i \otimes \tau$, where τ is the standard generator of \mathbb{Z}_2 . See [40]. \square

As a conclusion to all this, the half-classical geometry can be normally developed in a quite efficient way, at a technical level which is close to that of the classical one, by using 2×2 matrix models. Of course, all this still remains to be done.

As another comment, there are some higher analogues of the above developed, using $K \times K$ matrix models. We refer to [15], [40], [41] for more on these topics.

12e. Exercises

The matrix model problematics is quite exciting, making us exit the abstract algebra computations that we have been mainly doing throughout this book, and we have many exercises on the subject, for the most of research level. First, we have:

EXERCISE 12.36. *Find in the literature the complete proof of the GNS theorem, and write down a short account of that, with the main ideas explained.*

This is certainly something useful, because the GNS theorem is one of the 2 main results about the C^* -algebras, the other one being the Gelfand theorem.

Similarly, in connection with the von Neumann algebras, we have:

EXERCISE 12.37. *Find in the literature the complete statement and proof regarding the commutative case, $A = L^\infty(X)$, and write down a brief account of that.*

This is something that we talked about in the above, with a full proof of $A = L^\infty(X)$, and the problem left is that of understanding the embeddings $A \subset B(H)$.

As a third theoretical exercise, which is instructive, we have:

EXERCISE 12.38. *Try axiomatizing the quadruplets (S, T, U, K) in terms of the associated von Neumann algebras, and report on what you found.*

This is something that we already discussed in the above, with the comment that this is a “bad” exercise. However, trying to have it done is certainly instructive.

Moving ahead now towards our matrix modelling questions, we first have:

EXERCISE 12.39. *Prove that given a compact quantum group G , in order for having a faithful model $C(G) \subset M_K(C(T))$, the discrete dual $\Gamma = \widehat{G}$ must be amenable.*

As a bonus exercise here, try to fully clarify the situation in the case where $\Gamma = \widehat{G}$ is assumed to be a classical discrete group. This is something quite tricky, and in case you do not find the answer, the keyword for a search is “Thoma theorem”.

In relation with all this, here is now a difficult exercise:

EXERCISE 12.40. *Try to come up with a notion of inner faithfulness for the matrix models $C(X) \rightarrow M_K(C(T))$, in the case where $X \subset S_{\mathbb{C},+}^{N-1}$ is an homogeneous space.*

This is actually an open question, and any study on it would be very interesting.

In relation now with the matrix truncations, we have several questions, as follows:

EXERCISE 12.41. *In the context of the matrix truncations, comment on what happens when $X_{class} = \emptyset$. Also, comment on the case $X^{(\infty)} = X$. And also, comment on the case where $X = G$ is assumed to be a compact quantum group.*

Here, in what regards the first 2 questions, the answer normally requires some theory, examples, and counterexamples. As for the last question, the very first problem here is whether $G^{(2)}$ is a quantum group or not, the answer being no in general.

Here is now a very interesting, and down-to-earth question:

EXERCISE 12.42. *Develop a matrix model theory for the spaces of quantum partial isometries and partial permutations from chapter 7 above.*

Needed here would be especially interesting examples. As a hint, try first finding some interesting models for the quantum permutation groups S_N^+ , and then suitably modify your construction, as to make it work for the spaces of quantum partial permutations.

Part IV

Free coordinates

*And that seemed the end
But they caught him in vain
Cause a change came for Spain
And El Lute*

CHAPTER 13

Free coordinates

13a. Easy geometries

We discuss here and in the next 3 chapters a number of more specialized questions, of algebraic, geometric, analytic and probabilistic nature.

We will be interested in the main 9 examples of noncommutative geometries in our sense, which are as follows:

$$\begin{array}{ccccc}
 \mathbb{R}_+^N & \longrightarrow & \text{TR}_+^N & \longrightarrow & \mathbb{C}_+^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{R}_*^N & \longrightarrow & \text{TR}_*^N & \longrightarrow & \mathbb{C}_*^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{R}^N & \longrightarrow & \text{TR}^N & \longrightarrow & \mathbb{C}^N
 \end{array}$$

Our purpose will be that of going beyond the basic level, where we are now, with a number of results regarding the coordinates x_1, \dots, x_N of such spaces:

- (1) A first question, which is algebraic, is that of understanding the precise relations satisfied by these coordinates. We will see that this is related to the question of unifying the twisted and untwisted geometries, via intersection.
- (2) A second question, which is analytic, is that of understanding the fixed N behavior of these coordinates. This can be done via deformation methods. We will see as well that there is an unexpected link here with quantum permutations.

Let us begin by discussing algebraic aspects. This is something quite fundamental. Indeed, in the classical case, the algebraic manifolds X can be identified with the corresponding ideals of vanishing polynomials J , and the correspondence $X \leftrightarrow J$ is the foundation for all the known algebraic geometric theory, ancient or more modern.

In the free setting, things are in a quite primitive status, and a suitable theory of “noncommutative algebra”, useful in connection with our present considerations, is so far

missing. Computing J for the free spheres, and perhaps for some other spheres as well, is a problem which is difficult enough for us, and that we will investigate here.

As a starting point, we know that the above 9 geometries are easy, and looking in detail at this easiness property will be our first task. Let us first recall that we have:

DEFINITION 13.1. *A geometry (S, T, U, K) is called easy when U, K are easy, and*

$$U = \{O_N, K\}$$

with the operation on the right being the easy generation operation.

In other words, the easiness condition asks of course for U, K to be easy, and asks as well for the following condition to be satisfied:

$$\langle O_N, K \rangle = \{O_N, K\}$$

Here the operation on the right is the easy generation one, discussed in section 2 above, given at the level of the associated categories of partitions by:

$$D_{\{G, H\}} = D_G \cap D_H$$

The easy geometries in the above sense can be investigated by using:

PROPOSITION 13.2. *An easy geometry is uniquely determined by a pair (D, E) of categories of partitions, which must be as follows,*

$$\mathcal{NC}_2 \subset D \subset P_2$$

$$\mathcal{NC}_{\text{even}} \subset E \subset P_{\text{even}}$$

and which are subject to the following intersection and generation conditions,

$$D = E \cap P_2$$

$$E = \langle D, \mathcal{NC}_{\text{even}} \rangle$$

and to the usual axioms for the associated quadruplet (S, T, U, K) , where U, K are respectively the easy quantum groups associated to the categories D, E .

PROOF. This statement simply comes from the following conditions:

$$U = \{O_N, K\}$$

$$K = U \cap K_N^+$$

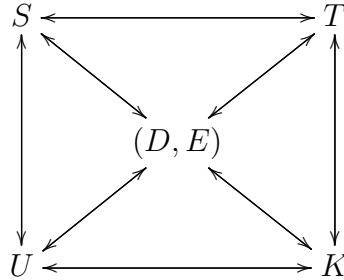
To be more precise, let us look at Definition 13.1. The main condition there tells us that U, K must be easy, coming from certain categories D, E .

It is clear that D, E must appear as intermediate categories, as in the statement, and the fact that the intersection and generation conditions must be satisfied follows from:

$$\begin{aligned} U = \{O_N, K\} &\iff D = E \cap P_2 \\ K = U \cap K_N^+ &\iff E = \langle D, \mathcal{NC}_{\text{even}} \rangle \end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

Generally speaking, the idea now is that, in the context of an easy geometry, everything can be reformulated in terms of the categories of partitions (D, E) , which must satisfy the conditions in Proposition 13.2. Thus, we have in fact a diagram as follows:



This is not surprising, because the main examples of geometries that we have seen in the above are the classical ones, which are governed by the commutation relations $ab = ba$, then the half-classical ones, coming from the half-commutation relations $abc = cba$, and then the free geometries, coming from no relations at all.

Thus, modulo technical conditions and axioms involving the quadruplets (S, T, U, K) , which are there in order for our geometry to really “work”, everything comes down to the combinatorial structure which can replace the commutation relations $ab = ba$. And the notion of category of partitions is precisely there for that.

This was for the idea. Now instead of discussing the full reformulation of our axioms in terms of categories of partitions, which technically speaking will not bring many new things, let us work out at least the construction of the quadruplet (S, T, U, K) .

In what regards the quantum groups, these come from the categories of partitions via Tannakian duality, as follows:

THEOREM 13.3. *In the context of an easy geometry (S, T, U, K) , we have:*

$$C(U) = C(U_N^+) / \left\langle T_\pi \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \mid \forall k, l, \forall \pi \in D(k, l) \right\rangle$$

Also, we have the following formula:

$$C(K) = C(K_N^+) / \left\langle T_\pi \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \mid \forall k, l, \forall \pi \in D(k, l) \right\rangle$$

In fact, these formulae simply follow from the fact that U is easy.

PROOF. This is clear indeed by applying Tannakian duality, in its “soft” form, to the unitary quantum group U , and to the quantum reflection group K , with the remark that, in what regards K , this appears indeed as a quantum subgroup of K_N^+ .

To be more precise, the Tannakian duality from [100], in its soft form from [73], which was discussed in chapter 2 above, states that for a closed subgroup $G \subset U_N$, with fundamental corepresentation denoted $v = (v_{ij})$, we have:

$$C(G) = C(U_N^+) / \left\langle T \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \mid \forall k, l, \forall T \in \text{Hom}(v^{\otimes k}, v^{\otimes l}) \right\rangle$$

But in the easy case, and in particular for the quantum groups U, K that we are interested in, this gives the formulae in the statement. \square

Regarding now the associated torus T , which is not exactly covered by the easy quantum group formalism, the result here is a bit different, as follows:

THEOREM 13.4. *In the context of an easy geometry (S, T, U, K) , we have:*

$$\Gamma = F_N / \left\langle g_{i_1} \dots g_{i_k} = g_{j_1} \dots g_{j_l} \mid \forall i, j, k, l, \exists \pi \in D(k, l), \delta_\pi \binom{i}{j} \neq 0 \right\rangle$$

In fact, this formula simply follows from the fact that U is easy.

PROOF. Let us denote by $g_i = u_{ii}$ the standard coordinates on the associated torus T , and consider the diagonal matrix formed by these coordinates:

$$g = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_N \end{pmatrix}$$

We have the following computation:

$$\begin{aligned} C(T) &= \left[C(U_N^+) / \left\langle T_\pi \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \mid \forall \pi \in D \right\rangle \right] / \left\langle u_{ij} = 0 \mid \forall i \neq j \right\rangle \\ &= \left[C(U_N^+) / \left\langle u_{ij} = 0 \mid \forall i \neq j \right\rangle \right] / \left\langle T_\pi \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \mid \forall \pi \in D \right\rangle \\ &= C^*(F_N) / \left\langle T_\pi \in \text{Hom}(g^{\otimes k}, g^{\otimes l}) \mid \forall \pi \in D \right\rangle \end{aligned}$$

Now observe that, with $g = \text{diag}(g_1, \dots, g_N)$ as before, we have:

$$T_\pi g^{\otimes k}(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_\pi \binom{i_1 \ \dots \ i_k}{j_1 \ \dots \ j_l} e_{j_1} \otimes \dots \otimes e_{j_l} \cdot g_{i_1} \dots g_{i_k}$$

On the other hand, we have as well:

$$g^{\otimes l} T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_\pi \binom{i_1 \ \dots \ i_k}{j_1 \ \dots \ j_l} e_{j_1} \otimes \dots \otimes e_{j_l} \cdot g_{j_1} \dots g_{j_l}$$

Thus, the commutation relation $T_\pi \in Hom(g^{\otimes k}, g^{\otimes l})$ reads:

$$\begin{aligned} & \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l} \cdot g_{i_1} \dots g_{i_k} \\ &= \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l} \cdot g_{j_1} \dots g_{j_l} \end{aligned}$$

Thus we obtain the formula in the statement, and the last assertion is clear. □

Finally, regarding the sphere S , which is not a quantum group, but rather an homogeneous space, here the result is a bit more complicated, as follows:

THEOREM 13.5. *In the context of an easy geometry (S, T, U, K) , we have*

$$C(S) = C(S_{\mathbb{C},+}^{N-1}) / \left\langle x_{i_1} \dots x_{i_k} = x_{j_1} \dots x_{j_k} \mid \forall i, j, k, l, \exists \pi \in D(k) \cap I_k, \delta_\pi \begin{pmatrix} i \\ j \end{pmatrix} \neq 0 \right\rangle$$

where the set on the right, $I_k \subset P_2(k, k)$, is the set of colored permutations.

PROOF. This follows indeed from Theorem 13.3 above, by applying the construction $U \rightarrow S$, which amounts in taking the first row space. □

Summarizing, in the case of an easy geometry, the central object is the pair of categories (D, E) , which are subject to the intersection and generation conditions explained above, and which produce the whole quadruplet (S, T, U, K) , in a quite simple way.

As already mentioned, it is possible to reformulate everything in terms of the pair (D, E) , by taking our axioms from chapter 4 above, and plugging in the formulae of S, T, U, K in terms of (D, E) , coming from the above results. However, this remains something quite theoretical, and we will not get into details here.

13b. Monomial spheres

Let us discuss now an alternative take on these questions, following [29], based on the notion of “monomiality”, which applies to the spheres, which are not easy.

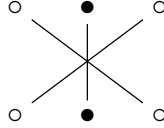
Looking back at the definition of the spheres that we have, and at the precise relations between the coordinates, we are led into the following notion:

DEFINITION 13.6. *A monomial sphere is a subset $S \subset S_{\mathbb{C},+}^{N-1}$ obtained via relations of type*

$$x_{i_1}^{e_1} \dots x_{i_k}^{e_k} = x_{i_{\sigma(1)}}^{f_1} \dots x_{i_{\sigma(k)}}^{f_k} \quad , \quad \forall (i_1, \dots, i_k) \in \{1, \dots, N\}^k$$

with $\sigma \in S_k$ being certain permutations, and with $e_r, f_r \in \{1, *\}$ being certain exponents.

This definition is quite broad, and we have for instance as example the sphere $S_{\mathbb{C},\times}^{N-1}$ coming from the relations $ab^*c = cb^*a$, corresponding to the following diagram:



This latter sphere is actually a quite interesting object, coming from the projective space considerations in [36], [37]. However, while being monomial, this sphere does not exactly fit with our noncommutative geometry considerations here.

To be more precise, according to the work in [8], [17], this sphere is part of a triple $(S_{\mathbb{C},\times}^{N-1}, \mathbb{T}_N^\times, U_N^\times)$, satisfying a simplified set of noncommutative geometry axioms. However, according to the work in [74], [75], the quantum group U_N^\times has no reflection group counterpart K_N^\times . Thus, this sphere does not exactly fit with our axiomatics here.

In view of these difficulties, we will now restrict the attention to the real case.

Let us first recall, from the various classification results established above, that we have the following result, dealing with the real case:

THEOREM 13.7. *There are exactly 3 real easy geometries, whose associated spheres, tori and quantum unitary and reflection groups are as follows,*

$$S_{\mathbb{R}}^{N-1} \subset S_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{R},+}^{N-1}$$

$$T_N \subset T_N^* \subset T_N^+$$

$$O_N \subset O_N^* \subset O_N^+$$

$$H_N \subset H_N^* \subset H_N^+$$

and whose associated categories of pairings and partitions D, E are as follows:

$$P_2 \supset P_2^* \supset NC_2$$

$$P_{even} \supset P_{even}^* \supset NC_{even}$$

PROOF. This is something that we already know, basically coming from the fact that $G = O_N^*$ is the unique intermediate easy quantum group $O_N \subset G \subset O_N^+$. \square

Let us focus now on the spheres, and try to better understand their “easiness” property, with results getting beyond what has been done above, in the general easy context. That is, our objects of interest in what follows will be the 3 real spheres, namely:

$$S_{\mathbb{R}}^{N-1} \subset S_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{R},+}^{N-1}$$

In order to talk about monomiality, it is convenient to introduce the following group:

$$S_\infty = \bigcup_{k \geq 0} S_k$$

To be more precise, this group appears by definition as an inductive limit, with the inclusions $S_k \subset S_{k+1}$ that we use being given by:

$$\sigma \in S_k \implies \sigma(k+1) = k+1$$

In terms of S_∞ , the definition of the monomial spheres reformulates as follows:

PROPOSITION 13.8. *The monomial spheres are the subsets $S \subset S_{\mathbb{R},+}^{N-1}$ obtained via relations*

$$x_{i_1} \dots x_{i_k} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}, \quad \forall (i_1, \dots, i_k) \in \{1, \dots, N\}^k$$

associated to certain elements $\sigma \in S_\infty$, where $k \in \mathbb{N}$ is such that $\sigma \in S_k$.

PROOF. We must prove that the relations $x_{i_1} \dots x_{i_k} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}$ are left unchanged when replacing $k \rightarrow k+1$. But this follows from $\sum_i x_i^2 = 1$, because:

$$\begin{aligned} & x_{i_1} \dots x_{i_k} x_{i_{k+1}} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}} x_{i_{k+1}} \\ \implies & x_{i_1} \dots x_{i_k} x_{i_{k+1}}^2 = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}} x_{i_{k+1}}^2 \\ \implies & \sum_{i_{k+1}} x_{i_1} \dots x_{i_k} x_{i_{k+1}}^2 = \sum_{i_{k+1}} x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}} x_{i_{k+1}}^2 \\ \implies & x_{i_1} \dots x_{i_k} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}} \end{aligned}$$

Thus we can indeed “simplify at right”, and this gives the result. □

Following [29], our goal in what follows will be that of proving that the 3 main spheres are the only monomial ones, in our sense.

In order to prove this result, we will use group theory methods. We call a subgroup $G \subset S_\infty$ filtered when it is stable under concatenation, in the sense that when writing $G = (G_k)$ with $G_k \subset S_k$, we have the following formula:

$$\sigma \in G_k, \pi \in G_l \implies \sigma\pi \in G_{k+l}$$

With this convention, we have the following result:

THEOREM 13.9. *The monomial spheres are the subsets $S_G \subset S_{\mathbb{R},+}^{N-1}$ given by*

$$C(S_G) = C(S_{\mathbb{R},+}^{N-1}) / \left\langle x_{i_1} \dots x_{i_k} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}, \forall (i_1, \dots, i_k) \in \{1, \dots, N\}^k, \forall \sigma \in G_k \right\rangle$$

where $G = (G_k)$ is a filtered subgroup of $S_\infty = (S_k)$.

PROOF. We know from Proposition 13.8 that the construction in the statement produces a monomial sphere. Conversely, given a monomial sphere $S \subset S_{\mathbb{R},+}^{N-1}$, let us set:

$$G_k = \left\{ \sigma \in S_k \mid x_{i_1} \dots x_{i_k} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}, \forall (i_1, \dots, i_k) \in \{1, \dots, N\}^k \right\}$$

With $G = (G_k)$ we have then the following equality:

$$S = S_G$$

Thus, it remains to prove that G is a filtered group.

Since the relations $x_{i_1} \dots x_{i_k} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}$ can be composed and reversed, each G_k follows to be stable under composition and inversion, and is therefore a group.

Also, since the relations $x_{i_1} \dots x_{i_k} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}$ can be concatenated as well, our group $G = (G_k)$ is stable under concatenation, and we are done. \square

At the level of examples, according to our definitions, the simplest filtered groups, namely $\{1\} \subset S_\infty$, produce the simplest real spheres, namely:

$$S_{\mathbb{R},+}^{N-1} \supset S_{\mathbb{R}}^{N-1}$$

In order to discuss now the half-liberated case, we need to introduce and study a certain privileged intermediate filtered group $\{1\} \subset S_\infty^* \subset S_\infty$, which will eventually produce the intermediate sphere $S_{\mathbb{R},+}^{N-1} \supset S_{\mathbb{R},*}^{N-1} \supset S_{\mathbb{R}}^{N-1}$. This can be done as follows:

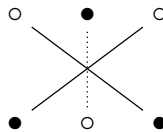
PROPOSITION 13.10. *Let $S_\infty^* \subset S_\infty$ be the set of permutations having the property that when labelling cyclically the legs as follows*



each string joins a black leg to a white leg.

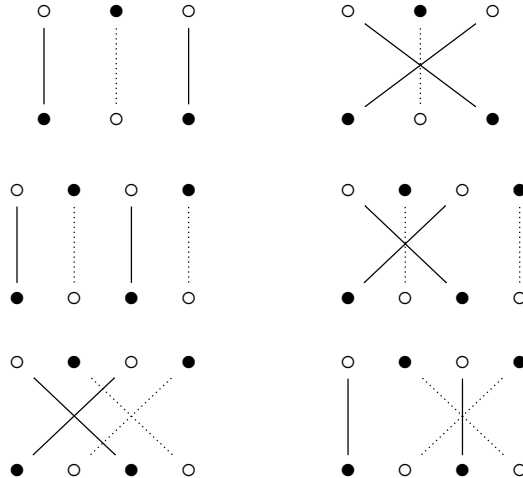
- (1) S_∞^* is a filtered subgroup of S_∞ , generated by the half-classical crossing.
- (2) We have $S_{2k}^* \simeq S_k \times S_k$, and $S_{2k+1}^* \simeq S_k \times S_{k+1}$, for any $k \in \mathbb{N}$.

PROOF. The fact that S_∞^* is indeed a subgroup of S_∞ , which is filtered, is clear. Observe now that the half-classical crossing has the “black-to-white” joining property:



Thus this crossing belongs to S_3^* , and it is routine to check, by double inclusion, that the filtered subgroup of S_∞ generated by it is the whole S_∞^* .

Regarding now the last assertion, observe first that the filtered subgroups S_3^*, S_4^* consist of the following permutations:



Thus we have $S_3^* = S_1 \times S_2$ and $S_4^* = S_2 \times S_2$, with the first component coming from dotted permutations, and with the second component coming from the solid line permutations. The same argument works in general, and gives the last assertion. \square

Now back to the main 3 real spheres, the result is as follows:

PROPOSITION 13.11. *The basic monomial real spheres, namely*

$$S_{\mathbb{R}}^{N-1} \subset S_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{R},+}^{N-1}$$

come respectively from the filtered groups

$$S_{\infty} \supset S_{\infty}^* \supset \{1\}$$

via the above correspondence.

PROOF. This is clear by definition in the classical and in the free cases. In the half-liberated case, the result follows from Proposition 13.10 (1) above. \square

Now back to the general case, with the idea in mind of proving the uniqueness of the above spheres, consider a monomial sphere $S_G \subset S_{\mathbb{R},+}^{N-1}$, with the filtered group $G \subset S_{\infty}$ taken to be maximal, as in the proof of Theorem 13.9. We have the following result:

PROPOSITION 13.12. *The filtered group $G \subset S_{\infty}$ associated to a monomial sphere $S \subset S_{\mathbb{R},+}^{N-1}$ is stable under the following operations, on the corresponding diagrams:*

- (1) *Removing outer strings.*
- (2) *Removing neighboring strings.*

PROOF. Both these results follow by using the quadratic condition:

(1) Regarding the outer strings, by summing over a , we have:

$$\begin{aligned} Xa = Ya &\implies Xa^2 = Ya^2 \\ &\implies X = Y \end{aligned}$$

We have as well the following computation:

$$\begin{aligned} aX = aY &\implies a^2X = a^2Y \\ &\implies X = Y \end{aligned}$$

(2) Regarding the neighboring strings, once again by summing over a , we have:

$$\begin{aligned} XabY = ZabT &\implies Xa^2Y = Za^2T \\ &\implies XY = ZT \end{aligned}$$

We have as well the following computation:

$$\begin{aligned} XabY = ZbaT &\implies Xa^2Y = Za^2T \\ &\implies XY = ZT \end{aligned}$$

Thus $G = (G_k)$ has both the properties in the statement. \square

We are now in position of stating and proving a main result, as follows:

THEOREM 13.13. *There is only one intermediate monomial sphere*

$$S_{\mathbb{R}}^{N-1} \subset S \subset S_{\mathbb{R},+}^{N-1}$$

namely the half-classical real sphere $S_{\mathbb{R},}^{N-1}$.*

PROOF. We will prove that the only filtered groups $G \subset S_{\infty}$ satisfying the conditions in Proposition 13.12 are those corresponding to our 3 spheres, namely:

$$\{1\} \subset S_{\infty}^* \subset S_{\infty}$$

In order to do so, consider such a filtered group:

$$G \subset S_{\infty}$$

We assume this group to be non-trivial, $G \neq \{1\}$.

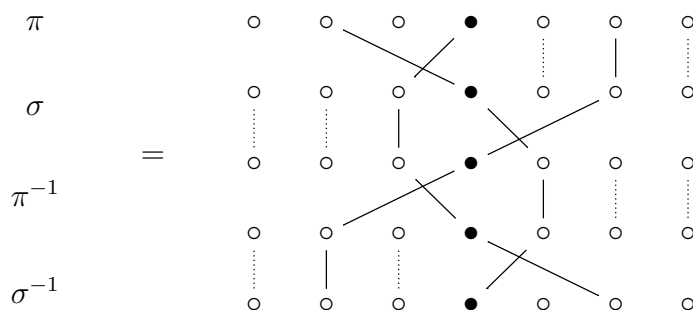
Step 1. Our first claim is that G contains a 3-cycle. Assume indeed that two permutations $\pi, \sigma \in S_{\infty}$ have support overlapping on exactly one point, say:

$$\text{supp}(\pi) \cap \text{supp}(\sigma) = \{i\}$$

The point is then that the commutator $\sigma^{-1}\pi^{-1}\sigma\pi$ is a 3-cycle, namely:

$$(i, \sigma^{-1}(i), \pi^{-1}(i))$$

Indeed the computation of the commutator goes as follows:



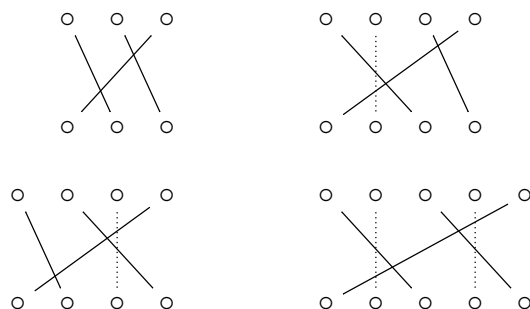
Now let us pick a non-trivial element $\tau \in G$. By removing outer strings at right and at left we obtain permutations $\tau' \in G_k, \tau'' \in G_s$ having a non-trivial action on their right/left leg, and the trick applies, with:

$$\pi = \tau' \otimes id_{s-1}$$

$$\sigma = id_{k-1} \otimes \tau''$$

Thus, G contains a 3-cycle, as claimed.

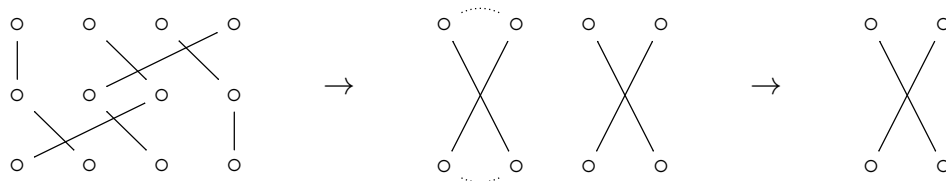
Step 2. Our second claim is G must contain one of the following permutations:



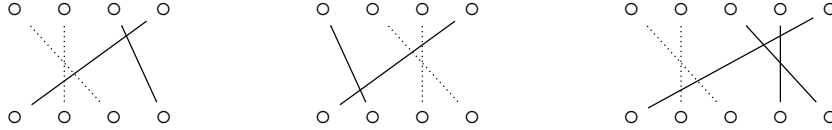
Indeed, consider the 3-cycle that we just constructed. By removing all outer strings, and then all pairs of adjacent vertical strings, we are left with these permutations.

Step 3. Our claim now is that we must have $S_\infty^* \subset G$. Indeed, let us pick one of the permutations that we just constructed, and apply to it our various diagrammatic rules.

From the first permutation we can obtain the basic crossing, as follows:



Also, by removing a suitable \times shaped configuration, which is represented by dotted lines in the diagrams below, we can obtain the basic crossing from the second and third permutation, and the half-liberated crossing from the fourth permutation:



Thus, in all cases we have a basic or half-liberated crossing, and so, as desired:

$$S_\infty^* \subset G$$

Step 4. Our last claim, which will finish the proof, is that there is no proper intermediate subgroup as follows:

$$S_\infty^* \subset G \subset S_\infty$$

In order to prove this, observe that $S_\infty^* \subset S_\infty$ is the subgroup of parity-preserving permutations, in the sense that “ i even $\implies \sigma(i)$ even”.

Now let us pick an element $\sigma \in S_k - S_k^*$, with $k \in \mathbb{N}$. We must prove that the group $G = \langle S_\infty^*, \sigma \rangle$ equals the whole S_∞ . In order to do so, we use the fact that σ is not parity preserving. Thus, we can find i even such that $\sigma(i)$ is odd. In addition, up to passing to $\sigma|$, we can assume that $\sigma(k) = k$, and then, up to passing one more time to $\sigma|$, we can further assume that k is even. Since both i, k are even we have:

$$(i, k) \in S_k^*$$

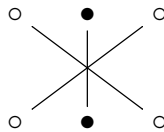
We conclude that the following element belongs to G :

$$\sigma(i, k)\sigma^{-1} = (\sigma(i), k)$$

But, since $\sigma(i)$ is odd, by deleting an appropriate number of vertical strings, $(\sigma(i), k)$ reduces to the basic crossing $(1, 2)$. Thus $G = S_\infty$, and we are done. \square

As already mentioned in the above, the story is not over with this kind of result, because the complex case still remains to be worked out.

Regarding the complex case, the problem, already discussed in the above, and that we do not know how to solve, so far, is that of fine-tuning the definition of the monomial spheres, as to avoid for instance the sphere coming from the following diagram:



Indeed, this diagram produces the relations $ab^*c = cb^*a$, and we do not want the corresponding sphere $S_{\mathbb{C}, x}^{N-1}$ to be part of our formalism, the reason behind this being the

fact that the corresponding quantum group U_N^\times has no reflection group counterpart K_N^\times , and so all this does not exactly fit with our axiomatics here.

13c. Twists, intersections

Our purpose now will be that of going beyond the above results, with a number of more specialized results regarding the coordinates x_1, \dots, x_N of our real spheres.

To be more precise, a first question that we would like to solve, which is of purely algebraic nature, is that of understanding the precise relations satisfied by these coordinates x_1, \dots, x_N over our real spheres. We will see that this is related to the question of unifying the twisted and untwisted geometries, via intersection.

Let us begin by recalling the construction of the twisted real spheres, which was discussed in chapter 11 above. This is something very simple, as follows:

DEFINITION 13.14. *The subspheres $\bar{S}_{\mathbb{R}}^{N-1}, \bar{S}_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{R},+}^{N-1}$ are constructed by imposing the following conditions on the standard coordinates x_1, \dots, x_N :*

- (1) $\bar{S}_{\mathbb{R}}^{N-1}$: $x_i x_j = -x_j x_i$, for any $i \neq j$.
- (2) $\bar{S}_{\mathbb{R},*}^{N-1}$: $x_i x_j x_k = -x_k x_j x_i$ for any i, j, k distinct, $x_i x_j x_k = x_k x_j x_i$ otherwise.

Here the fact that we have indeed $\bar{S}_{\mathbb{R}}^{N-1} \subset \bar{S}_{\mathbb{R},*}^{N-1}$ comes from the following computations, for $a, b, c \in \{x_i\}$ distinct, where x_1, \dots, x_N are the standard coordinates on $\bar{S}_{\mathbb{R}}^{N-1}$:

$$abc = -bac = bca = -cba$$

$$aab = -aba = baa$$

We refer to chapter 11 for more details regarding the above spheres.

Summarizing, we have a total of 5 real spheres, or rather a total of $3 + 3 = 6$ real spheres, with the convention that the free real sphere equals its twist:

$$S_{\mathbb{R},+}^{N-1} = \bar{S}_{\mathbb{R},+}^{N-1}$$

The point now is that we can intersect these $3 + 3 = 6$ spheres, and we end up with a total of $3 \times 3 = 9$ real spheres, in a generalized sense, as follows:

DEFINITION 13.15. *Associated to any integer $N \in \mathbb{N}$ are the generalized spheres*

$$\begin{array}{ccccc}
 S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},+}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1} \cap \bar{S}_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} \cap \bar{S}_{\mathbb{R},*}^{N-1} & \longrightarrow & \bar{S}_{\mathbb{R},*}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1} \cap \bar{S}_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} \cap \bar{S}_{\mathbb{R}}^{N-1} & \longrightarrow & \bar{S}_{\mathbb{R}}^{N-1}
 \end{array}$$

obtained by intersecting the 3 twisted real spheres and the 3 untwisted real spheres.

In order to compute the various intersections appearing above, which in general cannot be thought of as being smooth, let us introduce the following objects:

DEFINITION 13.16. *The polygonal spheres are real algebraic manifolds, defined as*

$$S_{\mathbb{R}}^{N-1,d-1} = \left\{ x \in S_{\mathbb{R}}^{N-1} \mid x_{i_0} \dots x_{i_d} = 0, \forall i_0, \dots, i_d \text{ distinct} \right\}$$

depending on integers $1 \leq d \leq N$.

These spheres are not smooth in general, but recall that we are currently doing algebraic geometry, rather than differential geometry. To be more precise, the point is that the problem that we want to solve, namely understanding the precise relations satisfied by the coordinates x_1, \dots, x_N for the real spheres, naturally leads into polygonal spheres.

More generally now, we have the following construction of “generalized polygonal spheres”, which applies to the half-classical and twisted cases too:

$$C(\dot{S}_{\mathbb{R},\times}^{N-1,d-1}) = C(\dot{S}_{\mathbb{R},\times}^{N-1}) \left/ \left\langle x_{i_0} \dots x_{i_d} = 0, \forall i_0, \dots, i_d \text{ distinct} \right\rangle \right.$$

Here the fact that in the classical case we obtain the polygonal spheres from Definition 13.16 comes from a straightforward application of the Gelfand theorem.

With these conventions, we have the following result, dealing with all the spheres that we have so far in real case, namely twisted, untwisted and intersections:

THEOREM 13.17. *The diagram obtained by intersecting the twisted and untwisted real spheres, from Definition 13.15 above, is given by*

$$\begin{array}{ccccc}
 S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},+}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1,1} & \longrightarrow & S_{\mathbb{R},*}^{N-1,1} & \longrightarrow & \bar{S}_{\mathbb{R},*}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1,0} & \longrightarrow & \bar{S}_{\mathbb{R}}^{N-1,1} & \longrightarrow & \bar{S}_{\mathbb{R}}^{N-1}
 \end{array}$$

and so all these spheres are generalized polygonal spheres.

PROOF. Consider the 4-diagram obtained by intersecting the 5 main spheres, which is as follows:

$$\begin{array}{ccc}
 S_{\mathbb{R}}^{N-1} \cap \bar{S}_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} \cap \bar{S}_{\mathbb{R},*}^{N-1} \\
 \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1} \cap \bar{S}_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} \cap \bar{S}_{\mathbb{R}}^{N-1}
 \end{array}$$

We must prove that this diagram coincides with the 4-diagram appearing at bottom left in the statement, which is as follows:

$$\begin{array}{ccc}
 S_{\mathbb{R}}^{N-1,1} & \longrightarrow & S_{\mathbb{R},*}^{N-1,1} \\
 \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1,0} & \longrightarrow & \bar{S}_{\mathbb{R}}^{N-1,1}
 \end{array}$$

But this is clear, because combining the commutation and anticommutation relations leads to the vanishing relations defining the spheres of type $\hat{S}_{\mathbb{R},\times}^{N-1,d-1}$. More precisely:

- (1) $S_{\mathbb{R}}^{N-1} \cap \bar{S}_{\mathbb{R}}^{N-1}$ consists of the points $x \in S_{\mathbb{R}}^{N-1}$ such that, for any $i \neq j$:

$$x_i x_j = -x_j x_i$$

Now since we have as well $x_i x_j = x_j x_i$, for any i, j , this relation reads $x_i x_j = 0$ for $i \neq j$, which means that we have $x \in S_{\mathbb{R}}^{N-1,0}$, as desired.

- (2) $S_{\mathbb{R}}^{N-1} \cap \bar{S}_{\mathbb{R},*}^{N-1}$ consists of the points $x \in S_{\mathbb{R}}^{N-1}$ such that, for i, j, k distinct:

$$x_i x_j x_k = -x_k x_j x_i$$

Once again by commutativity, this relation is equivalent to $x \in S_{\mathbb{R}}^{N-1,1}$, as desired.

(3) $S_{\mathbb{R},*}^{N-1} \cap \bar{S}_{\mathbb{R}}^{N-1}$ is obtained from $\bar{S}_{\mathbb{R}}^{N-1}$ by imposing to the standard coordinates the half-commutation relations $abc = cba$. On the other hand, we know from $\bar{S}_{\mathbb{R}}^{N-1} \subset \bar{S}_{\mathbb{R},*}^{N-1}$ that the standard coordinates on $\bar{S}_{\mathbb{R}}^{N-1}$ satisfy $abc = -cba$ for a, b, c distinct, and $abc = cba$ otherwise. Thus, the relations brought by intersecting with $S_{\mathbb{R},*}^{N-1}$ reduce to the relations $abc = 0$ for a, b, c distinct, and so we are led to the sphere $\bar{S}_{\mathbb{R}}^{N-1,1}$.

(4) $S_{\mathbb{R},*}^{N-1} \cap \bar{S}_{\mathbb{R},*}^{N-1}$ is obtained from $\bar{S}_{\mathbb{R},*}^{N-1}$ by imposing the relations $abc = -cba$ for a, b, c distinct, and $abc = cba$ otherwise. Since we know that $abc = cba$ for any a, b, c , the extra relations reduce to $abc = 0$ for a, b, c distinct, and so we are led to $S_{\mathbb{R},*}^{N-1,1}$. \square

Summarizing, whether we want it or not, when talking about intersections between twisted and untwisted geometries, we are led into polygonal spheres, and into non-smooth objects in general. This will be of course not an issue, in what follows.

In view of this, and also in connection with general axiomatization questions, let us find now a suitable axiomatic framework for the 9 spheres in Theorem 13.17.

We denote by $P(k, l)$ the set of partitons between an upper row of k points, and a lower row of l points, we set $P = \bigcup_{kl} P(k, l)$, and we denote by $P_{even} \subset P$ the subset of partitions having all the blocks of even size. Observe that $P_{even}(k, l) = \emptyset$ for $k + l$ odd.

We use the fact that there is a signature map $\varepsilon : P_{even} \rightarrow \{-1, 1\}$, extending the usual signature of permutations, $\varepsilon : S_{\infty} \rightarrow \{-1, 1\}$. This map is obtained by setting $\varepsilon(\pi) = (-1)^c$, where $c \in \mathbb{N}$ is the number of switches between neighbors required for making π noncrossing, and which can be shown to be well-defined modulo 2.

We have the following definition, once again from [5]:

DEFINITION 13.18. *Given variables x_1, \dots, x_N , any permutation $\sigma \in S_k$ produces two collections of relations between these variables, as follows:*

- (1) *Untwisted relations, namely, for any i_1, \dots, i_k :*

$$x_{i_1} \dots x_{i_k} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}$$

- (2) *Twisted relations, namely, for any i_1, \dots, i_k :*

$$x_{i_1} \dots x_{i_k} = \varepsilon \left(\ker \begin{pmatrix} i_1 & \dots & i_k \\ i_{\sigma(1)} & \dots & i_{\sigma(k)} \end{pmatrix} \right) x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}$$

The untwisted relations are denoted \mathcal{R}_{σ} , and the twisted ones are denoted $\bar{\mathcal{R}}_{\sigma}$.

Observe that the untwisted relations \mathcal{R}_{σ} are trivially satisfied for the standard coordinates on $S_{\mathbb{R}}^{N-1}$, for any permutation $\sigma \in S_k$. A twisted analogue of this fact holds, in the sense that the standard coordinates on $\bar{S}_{\mathbb{R}}^{N-1}$ satisfy the relations $\bar{\mathcal{R}}_{\sigma}$, for any $\sigma \in S_k$.

Indeed, by using the anticommutation relations between the distinct coordinates of these latter spheres, we must have a formula of the following type:

$$x_{i_1} \dots x_{i_k} = \pm x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}$$

But the sign \pm obtained in this way is precisely the one given above, namely:

$$\pm = \varepsilon \left(\ker \begin{pmatrix} i_1 & \dots & i_k \\ i_{\sigma(1)} & \dots & i_{\sigma(k)} \end{pmatrix} \right)$$

We have now all the needed ingredients for axiomatizing the various spheres appearing so far, namely the twisted and untwisted ones, and their intersections:

DEFINITION 13.19. *We have 3 types of quantum spheres $S \subset S_{\mathbb{R},+}^{N-1}$, as follows:*

- (1) *Monomial, namely $S_{\mathbb{R},E}^{N-1}$, with $E \subset S_\infty$, obtained via the following relations:*

$$\left\{ \dot{\mathcal{R}}_\sigma \mid \sigma \in E \right\}$$

- (2) *Mixed monomial, which appear as intersections as follows, with $E, F \subset S_\infty$:*

$$S_{\mathbb{R},E,F}^{N-1} = S_{\mathbb{R},E}^{N-1} \cap \bar{S}_{\mathbb{R},F}^{N-1}$$

- (3) *Polygonal, which are again intersections, with $E, F \subset S_\infty$, and $d \in \{1, \dots, N\}$:*

$$S_{\mathbb{R},E,F}^{N-1,d-1} = S_{\mathbb{R},E,F}^{N-1} \cap S_{\mathbb{R},+}^{N-1,d-1}$$

Here the subsphere $S_{\mathbb{R},+}^{N-1,d-1} \subset S_{\mathbb{R},+}^{N-1}$ appearing in (3) is constructed as in Definition 13.16 above, by imposing the following relations, with i_0, \dots, i_d distinct:

$$x_{i_0} \dots x_{i_d} = 0$$

With the above notions, we cover all spheres appearing so far. More precisely, the 5 basic spheres in are monomial, the 9 spheres in Theorem 13.17 are mixed monomial, and the polygonal sphere formalism covers all the examples constructed so far.

13d. Standard parametrization

Observe that the set of mixed monomial spheres is closed under intersections. The same holds for the set of polygonal spheres, because we have the following formula:

$$S_{\mathbb{R},E,F}^{N-1,d-1} \cap S_{\mathbb{R},E',F'}^{N-1,d'-1} = S_{\mathbb{R},E \cup E', F \cup F'}^{N-1, \min(d,d')-1}$$

Let us try now to understand the structure of the various types of spheres, by using the real sphere technology developed before.

We call a group of permutations $G \subset S_\infty$ filtered if, with $G_k = G \cap S_k$, we have $G_k \times G_l \subset G_{k+l}$, for any k, l .

We use the following simple fact, coming from [29]:

PROPOSITION 13.20. *The various spheres can be parametrized by groups, as follows:*

- (1) *Monomial case:* $\dot{S}_{\mathbb{R},G}^{N-1}$, with $G \subset S_\infty$ filtered group.
- (2) *Mixed monomial case:* $S_{\mathbb{R},G,H}^{N-1}$, with $G, H \subset S_\infty$ filtered groups.
- (3) *Polygonal case:* $S_{\mathbb{R},G,H}^{N-1,d-1}$, with $G, H \subset S_\infty$ filtered groups, and $d \in \{1, \dots, N\}$.

PROOF. This basically follows from the theory developed before, as follows:

(1) As explained before, in order to prove this assertion, for a monomial sphere $S = \dot{S}_{\mathbb{R},E}$, we can take $G \subset S_\infty$ to be the set of permutations $\sigma \in S_\infty$ having the property that the relations \mathcal{R}_σ hold for the standard coordinates of S . We have then $E \subset G$, we have as well $S = \dot{S}_{\mathbb{R},G}^{N-1}$, and the fact that G is a filtered group is clear as well.

(2) This follows from (1), by taking intersections.

(3) Once again this follows from (1), by taking intersections. □

The idea in what follows will be that of writing the 9 main polygonal spheres as in Proposition 13.20 (2), as to reach to a “standard parametrization” for our spheres.

We recall from the beginning of this section that the permutations $\sigma \in S_\infty$ having the property that when labelling clockwise their legs $\circ \bullet \circ \bullet \dots$, and string joins a white leg to a black leg, form a filtered group, denoted $S_\infty^* \subset S_\infty$.

This group comes from the general half-liberation considerations from chapter 9 above, and its algebraic structure is very simple, as follows:

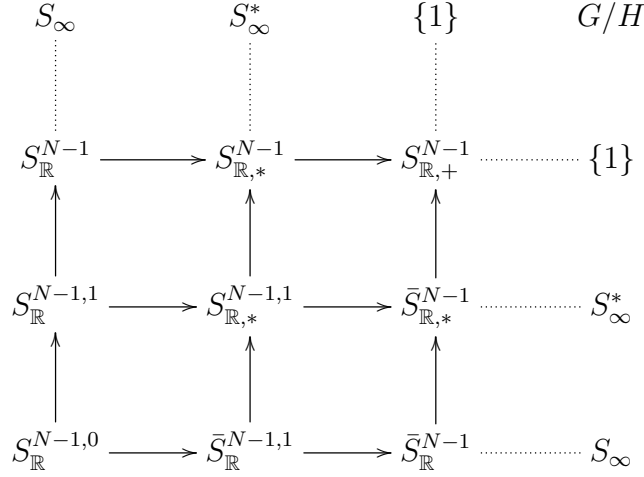
$$S_{2n}^* \simeq S_n \times S_n$$

$$S_{2n+1}^* \simeq S_n \times S_{n+1}$$

We call a mixed monomial sphere parametrization $S = S_{\mathbb{R},G,H}^{N-1}$ standard when both filtered groups $G, H \subset S_\infty$ are chosen to be maximal. In this case, Proposition 13.20 and its proof tell us that G, H encode all the monomial relations which hold in S .

With these conventions, we have the following result from [5], [7], extending some previous findings from above, regarding the untwisted spheres:

THEOREM 13.21. *The standard parametrization of the 9 main spheres is*



so these spheres come from the $3 \times 3 = 9$ pairs of groups among $\{1\} \subset S_\infty^* \subset S_\infty$.

PROOF. The fact that we have parametrizations as above is known to hold for the 5 untwisted and twisted spheres. For the remaining 4 spheres the result follows by intersecting, by using the following formula, valid for any $E, F \subset S_\infty$:

$$S_{\mathbb{R},E,F}^{N-1} \cap S_{\mathbb{R},E',F'}^{N-1} = S_{\mathbb{R},E \cup E', F \cup F'}^{N-1}$$

In order to prove now that the parametrizations are standard, we must compute the following two filtered groups, and show that we get the groups in the statement:

$$G = \left\{ \sigma \in S_\infty \mid \text{the relations } \mathcal{R}_\sigma \text{ hold over } S \right\}$$

$$H = \left\{ \sigma \in S_\infty \mid \text{the relations } \bar{\mathcal{R}}_\sigma \text{ hold over } S \right\}$$

As a first observation, by using the various inclusions between spheres, we just have to compute G for the spheres on the bottom, and H for the spheres on the left:

$$X = S_\mathbb{R}^{N-1,0}, \bar{S}_\mathbb{R}^{N-1,1}, \bar{S}_\mathbb{R}^{N-1} \implies G = S_\infty, S_\infty^*, \{1\}$$

$$X = S_\mathbb{R}^{N-1,0}, S_\mathbb{R}^{N-1,1}, S_\mathbb{R}^{N-1} \implies H = S_\infty, S_\infty^*, \{1\}$$

The results for $S_\mathbb{R}^{N-1,0}$ being clear, we are left with computing the remaining 4 groups, for the spheres $S_\mathbb{R}^{N-1}, \bar{S}_\mathbb{R}^{N-1}, S_\mathbb{R}^{N-1,1}, \bar{S}_\mathbb{R}^{N-1,1}$. The proof here goes as follows:

(1) $S_{\mathbb{R}}^{N-1}$. According to the definition of $H = (H_k)$, we have:

$$\begin{aligned} & H_k \\ &= \left\{ \sigma \in S_k \mid x_{i_1} \dots x_{i_k} = \varepsilon \left(\ker \begin{pmatrix} i_1 & \dots & i_k \\ i_{\sigma(1)} & \dots & i_{\sigma(k)} \end{pmatrix} \right) x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}, \forall i_1, \dots, i_k \right\} \\ &= \left\{ \sigma \in S_k \mid \varepsilon \left(\ker \begin{pmatrix} i_1 & \dots & i_k \\ i_{\sigma(1)} & \dots & i_{\sigma(k)} \end{pmatrix} \right) = 1, \forall i_1, \dots, i_k \right\} \\ &= \left\{ \sigma \in S_k \mid \varepsilon(\tau) = 1, \forall \tau \leq \sigma \right\} \end{aligned}$$

Now observe that for any permutation $\sigma \in S_k, \sigma \neq 1_k$, we can always find a partition $\tau \leq \sigma$ satisfying the following condition:

$$\varepsilon(\tau) = -1$$

We deduce that we have $H_k = \{1_k\}$, and so $H = \{1\}$, as desired.

(2) $\bar{S}_{\mathbb{R}}^{N-1}$. The proof of $G = \{1\}$ here is similar to the proof of $H = \{1\}$ in (1) above, by using the same combinatorial ingredient at the end.

(3) $S_{\mathbb{R}}^{N-1,1}$. By definition of $H = (H_k)$, a permutation $\sigma \in S_k$ belongs to H_k when the following condition is satisfied, for any choice of the indices i_1, \dots, i_k :

$$x_{i_1} \dots x_{i_k} = \varepsilon \left(\ker \begin{pmatrix} i_1 & \dots & i_k \\ i_{\sigma(1)} & \dots & i_{\sigma(k)} \end{pmatrix} \right) x_{i_{\sigma(1)}} \dots x_{i_{\sigma(k)}}$$

We have three cases here, as follows:

- When $|\ker i| = 1$ this formula reads $x_r^k = x_r^k$, which is true.
- When $|\ker i| \geq 3$ this formula is automatically satisfied as well, because by using the relations $ab = ba$, and $abc = 0$ for a, b, c distinct, which both hold over $S_{\mathbb{R}}^{N-1,1}$, this formula reduces to $0 = 0$.
- Thus, we are left with studying the case $|\ker i| = 2$. Here the quantities on the left $x_{i_1} \dots x_{i_k}$ will not vanish, so the sign on the right must be 1, and we therefore have:

$$H_k = \left\{ \sigma \in S_k \mid \varepsilon(\tau) = 1, \forall \tau \leq \sigma, |\tau| = 2 \right\}$$

Now by coloring the legs of σ clockwise $\circ \bullet \circ \bullet \dots$, the above condition is satisfied when each string of σ joins a white leg to a black leg. Thus $H_k = S_k^*$, as desired.

(4) $\bar{S}_{\mathbb{R}}^{N-1,1}$. The proof of $G = S_{\infty}^*$ here is similar to the proof of $H = S_{\infty}^*$ in (3) above, by using the same combinatorial ingredient at the end. \square

Summarizing, in relation with the general algebraic geometry questions formulated in the beginning of this chapter, and more precisely with the free algebra needed for developing free geometry, we have now a clear result on the subject, Theorem 13.21. This is of course just a beginning of something, dealing with the sphere case only.

We can now formulate a classification result, as follows:

THEOREM 13.22. *The following hold:*

(1) *The untwisted monomial spheres are:*

$$S_{\mathbb{R}}^{N-1} \subset S_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{R},+}^{N-1}$$

(2) *The twisted monomial spheres are:*

$$\bar{S}_{\mathbb{R}}^{N-1} \subset \bar{S}_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{R},+}^{N-1}$$

(3) *The polygonal spheres in our sense are:*

$$\begin{array}{ccccc}
 S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},+}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1,1} & \longrightarrow & S_{\mathbb{R},*}^{N-1,1} & \longrightarrow & \bar{S}_{\mathbb{R},*}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1,0} & \longrightarrow & \bar{S}_{\mathbb{R}}^{N-1,1} & \longrightarrow & \bar{S}_{\mathbb{R}}^{N-1}
 \end{array}$$

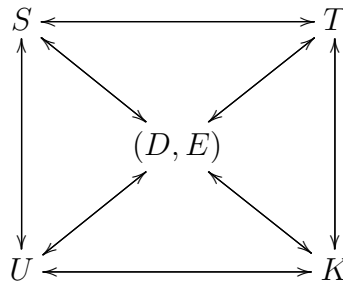
PROOF. By using standard parametrizations, the above 3 statements are equivalent. Now since (1) was proved before, all the results hold true. \square

We will be back to the polygonal spheres in the next chapter, with a better axiomatization, and with a study of the associated quantum groups as well.

13e. Exercises

There has been a lot of non-trivial algebra in this chapter, and our questions here will be on this precise topic, non-trivial algebra. First, we have:

EXERCISE 13.23. *Reformulate the noncommutative geometry axioms from chapter 4, in the easy case, in terms of (D, E) only, according to the following scheme*



for the various objects involved.

This is something that was already discussed in the above, and the problem now is that of having it done, with of course looking for simplifications.

As a second question now, which is quite difficult, we have:

EXERCISE 13.24. *Work out a theory of monomial spheres, in the complex case. Once this done, work out as well a theory of standard parametrization in the complex case.*

The difficulties in dealing with this question were already explained, in the above.

As yet another difficult question, we have:

EXERCISE 13.25. *Extend the theory of standard parametrization that we developed in the above, from the sphere case, to the case of more general manifolds.*

And this is all we have. Many things to be explored, in connection with the material developed in this chapter, which was more of an introduction to the subject.

CHAPTER 14

Polygonal spheres

14a. Polygonal spheres

In this chapter we build on the findings from the previous chapter, with the idea in mind that all this material belongs to a new and exciting area of noncommutative algebra, which can help in building an algebraic geometry theory for the free manifolds, and which therefore needs to be prioritarily developed.

As in the previous chapter, due to various technical difficulties with the complex case, at least at this stage of the things, we will basically restrict the attention to the real case. The main objects of study here are the 3 real spheres, which are as follows:

$$S_{\mathbb{R}}^{N-1} \subset S_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{R},+}^{N-1}$$

We have seen that the study of the relations between the coordinates x_1, \dots, x_N of these real spheres naturally leads to the twisted versions of these spheres, namely:

$$\bar{S}_{\mathbb{R}}^{N-1} \subset \bar{S}_{\mathbb{R},*}^{N-1} \subset \bar{S}_{\mathbb{R},+}^{N-1}$$

More precisely, the study of the algebraic relations between the coordinates x_1, \dots, x_N of the real spheres leads to the study of the various intersections between the twisted and untwisted spheres. These 3×3 intersections form a square diagram, as follows:

$$\begin{array}{ccccc}
 S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},+}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1} \cap \bar{S}_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} \cap \bar{S}_{\mathbb{R},*}^{N-1} & \longrightarrow & \bar{S}_{\mathbb{R},*}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1} \cap \bar{S}_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} \cap \bar{S}_{\mathbb{R}}^{N-1} & \longrightarrow & \bar{S}_{\mathbb{R}}^{N-1}
 \end{array}$$

We have seen as well that these intersections all appear as “polygonal spheres”, which are certain real algebraic manifolds, according to the following result:

THEOREM 14.1. *The 5 main spheres, and the intersections between them, are*

$$\begin{array}{ccccc}
 S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},+}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1,1} & \longrightarrow & S_{\mathbb{R},*}^{N-1,1} & \longrightarrow & \bar{S}_{\mathbb{R},*}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{R}}^{N-1,0} & \longrightarrow & \bar{S}_{\mathbb{R}}^{N-1,1} & \longrightarrow & \bar{S}_{\mathbb{R}}^{N-1}
 \end{array}$$

where $\dot{S}_{\mathbb{R},\times}^{N-1,d-1} \subset \dot{S}_{\mathbb{R},\times}^{N-1}$ is obtained by assuming $x_{i_0} \dots x_{i_d} = 0$, for i_0, \dots, i_d distinct.

PROOF. This is something that we know from chapter 13, the idea being that commutation and anticommutation produces vanishing relations. \square

We refer to chapter 13 for more on these spheres, their algebraic axiomatization and main properties, and the “standard parametrization” result there.

In this chapter we discuss the extension of the axiomatics that we have, in order to cover both the twisted and untwisted cases, and the intersections.

For this purpose, we are in need of some new quantum isometry group computations. In order to deal with the polygonal spheres, we will need the following standard result:

PROPOSITION 14.2. *Assume that a manifold $X \subset S_{\mathbb{R}}^{N-1}$ is invariant, for any i , under:*

$$x_i \rightarrow -x_i$$

(1) *If the coordinates x_1, \dots, x_N are linearly independent inside $C(X)$, then the group*

$$G(X) = G^+(X) \cap O_N$$

consists of the usual isometries of X .

(2) *In addition, in the case where the products of coordinates*

$$\{x_i x_j \mid i \leq j\}$$

are linearly independent inside $C(X)$, we have $G^+(X) = G(X)$.

PROOF. This follows from [38], [65], the idea being as follows:

(1) The assertion here is well-known, $G(X) = G^+(X) \cap O_N$ being by definition the biggest subgroup $G \subset O_N$ acting affinely on X . We refer to [65] for details, and for a number of noncommutative extensions of this fact, with $G(X)$ replaced by $G^+(X)$.

(2) Consider an arbitrary coaction map on the algebra $C(X)$, as follows:

$$\Phi : C(X) \rightarrow C(X) \otimes C(G)$$

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

In order to establish the result, we must prove that the variables u_{ij} commute. But this follows by using a standard trick, from [38], that we will briefly recall now. We can write the action of Φ on the commutators between the coordinates as follows:

$$\Phi([x_i, x_j]) = \sum_{k \leq l} \left(1 - \frac{\delta_{kl}}{2}\right) x_k x_l \otimes ([u_{ki}, u_{lj}] - [u_{kj}, u_{li}])$$

Now since the variables $\{x_k x_l | k \leq l\}$ were assumed to be linearly independent, we obtain from this that we have the following formula:

$$[u_{ki}, u_{lj}] = [u_{kj}, u_{li}]$$

Moreover, if we apply now the antipode we further obtain:

$$[u_{jl}, u_{ik}] = [u_{il}, u_{jk}]$$

By relabelling, this gives the following formula:

$$[u_{ki}, u_{lj}] = [u_{li}, u_{kj}]$$

Now by comparing with the original equality of commutators, from above, we conclude from this that we have a commutation relation, as follows:

$$[u_{ki}, u_{lj}] = 0$$

Thus, we are led to the conclusion in the statement. See [38]. □

With the above notion in hand, let us investigate the polygonal spheres.

We recall that, according to the various computations from the previous chapters, the quantum isometry groups of the 5 main spheres are as follows:

$$\begin{array}{ccccccccc}
 S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},+}^{N-1} & \longleftarrow & \bar{S}_{\mathbb{R},*}^{N-1} & \longleftarrow & \bar{S}_{\mathbb{R}}^{N-1} \\
 \left\{ \begin{array}{c} \text{wavy} \\ \text{line} \end{array} \right\} & & \left\{ \begin{array}{c} \text{wavy} \\ \text{line} \end{array} \right\} & & \left\{ \begin{array}{c} \text{wavy} \\ \text{line} \end{array} \right\} & & \left\{ \begin{array}{c} \text{wavy} \\ \text{line} \end{array} \right\} & & \left\{ \begin{array}{c} \text{wavy} \\ \text{line} \end{array} \right\} \\
 O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ & \longleftarrow & \bar{O}_N^* & \longleftarrow & \bar{O}_N
 \end{array}$$

In the polygonal sphere case now, we begin with the computations of the quantum isometry groups in the classical case.

We have here the following result, from [5]:

THEOREM 14.3. *The quantum isometry groups of the classical polygonal spheres*

$$S_{\mathbb{R}}^{N-1,d-1} = \left\{ x \in S_{\mathbb{R}}^{N-1} \mid x_{i_0} \dots x_{i_d} = 0, \forall i_0, \dots, i_d \text{ distinct} \right\}$$

are as follows:

- (1) At $d = 1$ we obtain the free hyperoctahedral group H_N^+ .
- (2) At $d = 2, \dots, N - 1$ we obtain the hyperoctahedral group H_N .
- (3) At $d = N$ we obtain the orthogonal group O_N .

PROOF. Observe first that the sphere $S_{\mathbb{R}}^{N-1,d-1}$ appears by definition as a union on $\binom{N}{d}$ copies of the sphere $S_{\mathbb{R}}^{d-1}$, one for each choice of d coordinate axes, among the coordinate axes of \mathbb{R}^N . We can write this decomposition as follows, with $I_N = \{1, \dots, N\}$:

$$S_{\mathbb{R}}^{N-1,d-1} = \bigcup_{I \subset I_N, |I|=d} (S_{\mathbb{R}}^{d-1})^I$$

With this observation in hand, the proof goes as follows:

- (1) At $d = 1$ our sphere is given by the following formula:

$$S_{\mathbb{R}}^{N-1,0} = \mathbb{Z}_2^{\oplus N}$$

To be more precise, what we have here is the set formed by the endpoints of the N copies of $[-1, 1]$ on the coordinate axes of \mathbb{R}^N . Thus by the free wreath product results in [18] the corresponding quantum isometry group is H_N^+ :

$$\begin{aligned} G^+(S_{\mathbb{R}}^{N-1,0}) &= G^+(\mathbb{Z}_2^{\oplus N}) \\ &= G^+(| \dots |) \\ &= \mathbb{Z}_2 \wr_* S_N^+ \\ &= H_N^+ \end{aligned}$$

- (2) In order to discuss now the case $d \geq 2$, the idea is to use Proposition 14.2 (2) above. Our claim is that the following elements are linearly independent:

$$\left\{ x_i x_j \mid i \leq j \right\}$$

Since $S_{\mathbb{R}}^{N-1,1} \subset S_{\mathbb{R}}^{N-1,d}$, we can restrict attention to the case $d = 2$. Here the above decomposition is as follows, where $\mathbb{T}^{\{i,j\}}$ denote the various copies of \mathbb{T} :

$$S_{\mathbb{R}}^{N-1,d-1} = \bigcup_{i < j} \mathbb{T}^{\{i,j\}}$$

Now observe that the following elements are linearly independent over $\mathbb{T} \subset \mathbb{R}^2$:

$$\{x^2, y^2, xy\}$$

We deduce that the following elements are linearly independent over $S_{\mathbb{R}}^{N-1,d-1}$:

$$\left\{ x_i x_j \mid i \leq j \right\}$$

Thus, our claim is proved, and so Proposition 14.2 (2) above applies, and gives:

$$G^+(X) = G(X)$$

Thus, we are left with proving the following formula, for any $d \in \{2, \dots, N - 1\}$:

$$G(X) = H_N$$

– Let us first discuss the case $d = 2$. By using the decomposition formula from the beginning of the proof, here any affine isometric action $U \curvearrowright S_{\mathbb{R}}^{N-1,1}$ must permute the $\binom{N}{2}$ circles \mathbb{T}^I , so we can write, for a certain permutation of the indices $I \rightarrow I'$:

$$U(\mathbb{T}^I) = \mathbb{T}^{I'}$$

Now since U is bijective, we deduce that for any I, J we have:

$$U(\mathbb{T}^I \cap \mathbb{T}^J) = \mathbb{T}^{I'} \cap \mathbb{T}^{J'}$$

The point now is that for $|I \cap J| = 0, 1, 2$ we have:

$$\mathbb{T}^I \cap \mathbb{T}^J \simeq \emptyset, \{-1, 1\}, \mathbb{T}$$

By taking now the union over I, J with $|I \cap J| = 1$, we deduce that:

$$U(\mathbb{Z}_2^{\oplus N}) = \mathbb{Z}_2^{\oplus N}$$

Thus we must have $U \in H_N$, and we are done with the case $d = 2$.

– In the general case now, $d \in \{2, \dots, N - 1\}$, we can proceed similarly, by recurrence. Indeed, for any subsets $I, J \subset I_N$ with $|I| = |J| = d$ we have:

$$(S_{\mathbb{R}}^{d-1})^I \cap (S_{\mathbb{R}}^{d-1})^J = (S_{\mathbb{R}}^{|I \cap J| - 1})^{I \cap J}$$

By using $d \leq N - 1$, we deduce that we have the following formula:

$$S_{\mathbb{R}}^{N-1,d-2} = \bigcup_{|I|=|J|=d, |I \cap J|=d-1} (S_{\mathbb{R}}^{|I \cap J| - 1})^{I \cap J}$$

On the other hand, by using exactly the same argument as in the $d = 2$ case, we deduce that the space on the right is invariant, under any affine isometric action on $S_{\mathbb{R}}^{N-1,d-1}$. Thus by recurrence we obtain, as desired, that we have:

$$\begin{aligned} G(S_{\mathbb{R}}^{N-1,d-1}) &= G(S_{\mathbb{R}}^{N-1,d-2}) \\ &= H_N \end{aligned}$$

(3) At $d = N$ the result is known since [28], with the proof coming from the equality $G^+(X) = G(X)$, deduced from Proposition 14.2 (2), as explained above. \square

The study in the twisted case is considerably more difficult than in the classical case, and we have complete results only at $d = 1, 2, N$, as follows:

THEOREM 14.4. *The quantum isometry groups of the twisted polygonal spheres, which are by definition given by*

$$C(\bar{S}_{\mathbb{R}}^{N-1,d-1}) = C(\bar{S}_{\mathbb{R}}^{N-1}) / \left\langle x_{i_0} \dots x_{i_d} = 0, \forall i_0, \dots, i_d \text{ distinct} \right\rangle$$

are as follows:

- (1) At $d = 1$ we obtain the free hyperoctahedral group H_N^+ .
- (2) At $d = 2$ we obtain the hyperoctahedral group H_N .
- (3) At $d = N$ we obtain the twisted orthogonal group \bar{O}_N .

PROOF. The idea is to adapt the proof of Theorem 14.3 above:

- (1) At $d = 1$ the situation is very simple, because we have:

$$\bar{S}_{\mathbb{R}}^{N-1,0} = S_{\mathbb{R}}^{N-1,0} = \mathbb{Z}_2^{\oplus N}$$

By Theorem 14.3 (1) above, coming from the free wreath product computations in [18], the corresponding quantum isometry group is indeed H_N^+ .

(2) In order to deal now with the case $d = 2$, in analogy with what was done before in the classical case, as a first ingredient, we will need the twisted analogue of the trick from [38], explained in the proof of Proposition 14.2 (2) above.

This twisted trick was already worked out in [6], for the twisted sphere $\bar{S}_{\mathbb{R}}^{N-1}$ itself, and the situation is in fact similar for any closed subset $X \subset \bar{S}_{\mathbb{R}}^{N-1}$, having the property that the following variables are linearly independent:

$$\{x_i x_j \mid i \leq j\}$$

More precisely, our claim is that under this linear independence assumption, if a quantum group $G \subset O_N^+$ acts on X , then we must have:

$$G \subset \bar{O}_N$$

Indeed, consider a coaction map, written as follows:

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

By making products, we have the following formula:

$$\Phi(x_i x_j) = \sum_k x_k^2 \otimes u_{ki} u_{kj} + \sum_{k < l} x_k x_l \otimes (u_{ki} u_{lj} - u_{li} u_{kj})$$

We deduce that with $[[a, b]] = ab + ba$ we have the following formula:

$$\Phi([[x_i, x_j]]) = \sum_k x_k^2 \otimes [[u_{ki}, u_{kj}]] + \sum_{k < l} x_k x_l \otimes ([u_{ki}, u_{lj}] - [u_{li}, u_{kj}])$$

Now assuming $i \neq j$, we have $[[x_i, x_j]] = 0$, and we therefore obtain, for any k :

$$[[u_{ki}, u_{kj}]] = 0$$

We also have, for any $k < l$, the following formula:

$$[u_{ki}, u_{lj}] = [u_{li}, u_{kj}]$$

By applying the antipode and then by relabelling, the latter relation gives:

$$[u_{ki}, u_{lj}] = 0$$

Thus we have reached to the defining relations for the quantum group \bar{O}_N , from chapter 11 above, and so we have $G \subset \bar{O}_N$, as claimed.

Our second claim is that the above trick applies to any $\bar{S}_{\mathbb{R}}^{N-1, d-1}$ with $d \geq 2$. Consider indeed the following maps, obtained by setting $x_k = 0$ for $k \neq i, j$:

$$\pi_{ij} : C(\bar{S}_{\mathbb{R}}^{N-1, d-1}) \rightarrow C(\bar{S}_{\mathbb{R}}^1)$$

By using these maps, we conclude that the following variables are indeed linearly independent over $\bar{S}_{\mathbb{R}}^{N-1, d-1}$, as desired:

$$\left\{ x_i x_j \mid i \leq j \right\}$$

Summarizing, we have proved so far that if a compact quantum group $G \subset O_N^+$ acts on a polygonal sphere $\bar{S}_{\mathbb{R}}^{N-1, d-1}$ with $d \geq 2$, then we must have:

$$G \subset \bar{O}_N$$

In order to finish, we must now adapt the second part of the proof of Proposition 14.2, and since this is quite unobvious at $d \geq 3$, due to various technical reasons, we will restrict now attention to the case $d = 2$, as in the statement.

So, consider a compact quantum group $G \subset \bar{O}_N$. The question is that of understanding when we have a coaction map, as follows:

$$\Phi : C(\bar{S}_{\mathbb{R}}^{N-1, 1}) \rightarrow C(G) \otimes C(\bar{S}_{\mathbb{R}}^{N-1, 1})$$

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

In order for this to happen, the elements $X_i = \sum_j x_j \otimes u_{ji}$ must satisfy the relations $X_i X_j X_k = 0$, for any i, j, k distinct.

So, let us compute $X_i X_j X_k$ for i, j, k distinct. We have:

$$\begin{aligned}
X_i X_j X_k &= \sum_{abc} x_a x_b x_c \otimes u_{ai} u_{bj} u_{ck} \\
&= \sum_{a,b,c \text{ not distinct}} x_a x_b x_c \otimes u_{ai} u_{bj} u_{ck} \\
&= \sum_{a \neq b} x_a^2 x_b \otimes u_{ai} u_{aj} u_{bk} + \sum_{a \neq b} x_a x_b x_a \otimes u_{ai} u_{bj} u_{ak} \\
&\quad + \sum_{a \neq b} x_b x_a^2 \otimes u_{bi} u_{aj} u_{ak} + \sum_a x_a^3 \otimes u_{ai} u_{aj} u_{ak}
\end{aligned}$$

By using $x_a x_b x_a = -x_a^2 x_b$ and $x_b x_a^2 = x_a^2 x_b$, we deduce that we have:

$$\begin{aligned}
X_i X_j X_k &= \sum_{a \neq b} x_a^2 x_b \otimes (u_{ai} u_{aj} u_{bk} - u_{ai} u_{bj} u_{ak} + u_{bi} u_{aj} u_{ak}) \\
&\quad + \sum_a x_a^3 \otimes u_{ai} u_{aj} u_{ak} \\
&= \sum_{ab} x_a^2 x_b \otimes (u_{ai} u_{aj} u_{bk} - u_{ai} u_{bj} u_{ak} + u_{bi} u_{aj} u_{ak})
\end{aligned}$$

By using now the defining relations for \bar{O}_N , which apply to the variables u_{ij} , this formula can be written in a cyclic way, as follows:

$$X_i X_j X_k = \sum_{ab} x_a^2 x_b \otimes (u_{ai} u_{aj} u_{bk} + u_{aj} u_{ak} u_{bi} + u_{ak} u_{ai} u_{bj})$$

We use now the fact that the variables on the left, namely $x_a^2 x_b$, are linearly independent. We conclude that, in order for our quantum group $G \subset \bar{O}_N$ to act on $\bar{S}_{\mathbb{R}}^{N-1,1}$, its coordinates must satisfy the following relations, for any i, j, k distinct:

$$u_{ai} u_{aj} u_{bk} + u_{aj} u_{ak} u_{bi} + u_{ak} u_{ai} u_{bj} = 0$$

By multiplying to the right by u_{kb} and then by summing over b , we deduce from this that we have, for any i, j :

$$u_{ai} u_{aj} = 0$$

Now since the quotient of $C(\bar{O}_N)$ by these latter relations is the algebra $C(H_N)$, we conclude that we have, as claimed:

$$G^+(\bar{S}_{\mathbb{R}}^{N-1,1}) = H_N$$

(3) At $d = N$ the result is already known, and its proof follows in fact from the ‘‘twisted trick’’ explained in the proof of (2) above, applied to $\bar{S}_{\mathbb{R}}^{N-1}$. \square

In general now, the idea will be that the quantum isometry groups of the intersections of the spheres will basically appear as intersections of the quantum isometry groups.

14b. Quantum groups

To start with, we must compute the intersections between the quantum orthogonal groups and their twists.

The result here, which is similar to the one for the corresponding spheres, established in chapter 13 above, is as follows:

PROPOSITION 14.5. *The 5 orthogonal groups and their twists, and the intersections between them, are as follows, at any $N \geq 3$:*

$$\begin{array}{ccccc}
 O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N & \longrightarrow & H_N^* & \longrightarrow & \bar{O}_N^* \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N & \longrightarrow & H_N & \longrightarrow & \bar{O}_N
 \end{array}$$

At $N = 2$ the same holds, with the lower left square being replaced by:

$$\begin{array}{ccc}
 O_2 & \longrightarrow & O_2^+ \\
 \uparrow & & \uparrow \\
 H_2 & \longrightarrow & \bar{O}_2
 \end{array}$$

PROOF. We have to study 4 quantum group intersections, as follows:

(1) $O_N \cap \bar{O}_N$. Here an element $U \in O_N$ belongs to the intersection when its entries satisfy $ab = 0$ for any $a \neq b$ on the same row or column of U . But this means that our matrix $U \in O_N$ must be monomial, and so we get $U \in H_N$, as claimed.

(2) $O_N \cap \bar{O}_N^*$. At $N = 2$ the defining relations for \bar{O}_N^* disappear, and so we have the following computation, which leads to the conclusion in the statement:

$$O_2 \cap \bar{O}_2^* = O_2 \cap O_2^+ = O_2$$

At $N \geq 3$ now, the following inclusion is clear:

$$H_N \subset O_N \cap \bar{O}_N^*$$

In order to prove the converse inclusion, pick $U \in O_N$ in the intersection, and assume that U is not monomial. By permuting the entries we can further assume:

$$\begin{aligned}
 U_{11} &\neq 0 \\
 U_{12} &\neq 0
 \end{aligned}$$

From $U_{11}U_{12}U_{i3} = 0$ for any i we deduce that the third column of U is filled with 0 entries, a contradiction. Thus we must have $U \in H_N$, as claimed.

(3) $O_N^* \cap \bar{O}_N$. At $N = 2$ we have the following computation, as claimed:

$$O_2^* \cap \bar{O}_2 = O_2^+ \cap \bar{O}_2 = \bar{O}_2$$

At $N \geq 3$ now, the best is to use the result in (4) below. Indeed, knowing that we have $O_N^* \cap \bar{O}_N^* = H_N^*$, our intersection is then:

$$G = H_N^* \cap \bar{O}_N$$

Now since the standard coordinates on H_N^* are known to satisfy $ab = 0$ for $a \neq b$ on the same row or column of u , the commutation/anticommutation relations defining \bar{O}_N reduce to plain commutation relations. Thus G follows to be classical, $G \subset O_N$, and by using (1) above we obtain the following formula, as claimed:

$$\begin{aligned} G &= H_N^* \cap \bar{O}_N \cap O_N \\ &= H_N^* \cap H_N \\ &= H_N \end{aligned}$$

(4) $O_N^* \cap \bar{O}_N^*$. The result here is non-trivial, and we must use the half-liberation technology from [41]. The quantum group $H_N^\times = O_N^* \cap \bar{O}_N^*$ is indeed half-classical in the sense of [41], and since we have $H_N^* \subset H_N^\times$, this quantum group is not classical. Thus the main result in [41] applies, and shows that $H_N^\times \subset O_N^*$ must come, via the crossed product construction there, from an intermediate compact group, as follows:

$$\mathbb{T} \subset G \subset U_N$$

Now observe that the standard coordinates on H_N^\times are by definition subject to the conditions $abc = 0$ when $(r, s) = (\leq 2, 3), (3, \leq 2)$, with the notations and conventions from chapter 11 above. It follows that the standard coordinates on G are subject to the conditions $\alpha\beta\gamma = 0$ when $(r, s) = (\leq 2, 3), (3, \leq 2)$, where $r, s = \text{span}(a, b, c)$, and $\alpha = a, a^*, \beta = b, b^*, \gamma = c, c^*$. Thus we have an inclusion as follows:

$$G \subset \bar{U}_N^*$$

We deduce that we have an inclusion as follows, with $K_N^\circ = U_N \cap \bar{U}_N^*$:

$$G \subset K_N^\circ$$

But this intersection can be computed exactly as in the real case, in the proof of (2) above, and we obtain $K_2^\circ = U_2$, and $K_N^\circ = \mathbb{T} \wr S_N$ at $N \geq 3$.

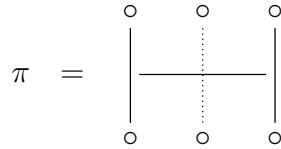
But the half-liberated quantum groups obtained from U_2 and $\mathbb{T} \wr S_N$ via the half-liberation construction in [41] are well-known, these being $O_2^* = O_2^+$ and H_N^* . Thus by functoriality we have $H_2^\times \subset O_2^+$ and $H_N^\times \subset H_N^*$ at $N \geq 3$, and since the reverse inclusions are clear, we obtain $H_2^\times = O_2^+$ and $H_N^\times = H_N^*$ at $N \geq 3$, as claimed. \square

Let us go back now to the sphere left, namely $S_{\mathbb{R},*}^{N-1,1}$. Things are quite tricky here, and we will need the following technical result:

PROPOSITION 14.6. *Let $H_N^{[\infty]} \subset O_N^+$ be the compact quantum group obtained via the relations $abc = 0$, whenever $a \neq c$ are on the same row or column of u .*

- (1) *We have inclusions $H_N^* \subset H_N^{[\infty]} \subset H_N^+$.*
- (2) *We have $ab_1 \dots b_r c = 0$, whenever $a \neq c$ are on the same row or column of u .*
- (3) *We have $ab^2 = b^2a$, for any two entries a, b of u .*

PROOF. We briefly recall the proof in [81], for future use in what follows. Our first claim is that $H_N^{[\infty]}$ comes, as an easy quantum group, from the following diagram:



Indeed, this diagram acts via the following linear map:

$$T_\pi(e_{ijk}) = \delta_{ik}e_{ijk}$$

We therefore have the following formula:

$$\begin{aligned} T_\pi u^{\otimes 3} e_{abc} &= T_\pi \sum_{ijk} e_{ijk} \otimes u_{ia} u_{jb} u_{kc} \\ &= \sum_{ijk} e_{ijk} \otimes \delta_{ik} u_{ia} u_{jb} u_{kc} \end{aligned}$$

On the other hand, we have as well the following formula:

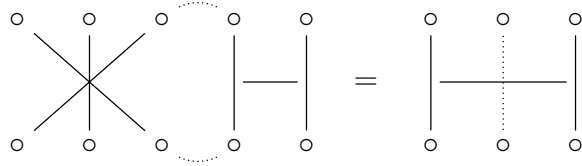
$$\begin{aligned} u^{\otimes 3} T_\pi e_{abc} &= u^{\otimes 3} \delta_{ac} e_{abc} \\ &= \sum_{ijk} e_{ijk} \otimes \delta_{ac} u_{ia} u_{jb} u_{kc} \end{aligned}$$

Thus the condition $T_\pi \in \text{End}(u^{\otimes 3})$ is equivalent to the following relations:

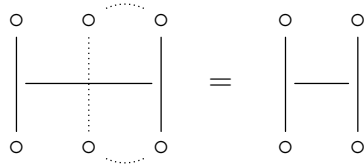
$$(\delta_{ik} - \delta_{ac})u_{ia}u_{jb}u_{kc} = 0$$

The non-trivial cases are $i = k, a \neq c$ and $i \neq k, a = c$, and these produce the relations $u_{ia}u_{jb}u_{ic} = 0$ for any $a \neq c$, and $u_{ia}u_{jb}u_{ka} = 0$, for any $i \neq k$. Thus, we have reached to the standard relations for the quantum group $H_N^{[\infty]}$.

(1) We have the following formula:



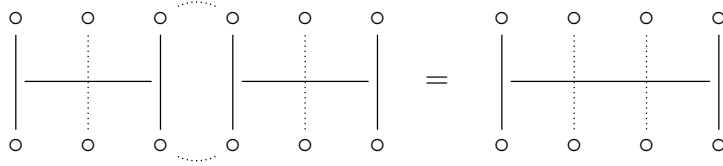
We have as well the following formula:



Thus, we obtain inclusions as desired, namely:

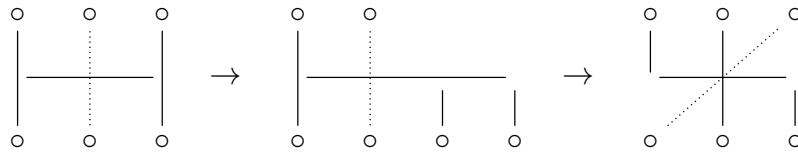
$$H_N^* \subset H_N^{[\infty]} \subset H_N^+$$

(2) At $r = 2$, the relations $ab_1b_2c = 0$ come indeed from the following diagram:



In the general case $r \geq 2$ the proof is similar, see [25] for details.

(3) We use here an idea from [81], [81]. By rotating π , we obtain:



Let us denote by σ the partition on the right. Since $T_\sigma(e_{ijk}) = \delta_{ij}e_{kji}$, we obtain:

$$\begin{aligned} T_\sigma u^{\otimes 3} e_{abc} &= T_\sigma \sum_{ijk} e_{ijk} \otimes u_{ia} u_{jb} u_{kc} \\ &= \sum_{ijk} e_{kji} \otimes \delta_{ij} u_{ia} u_{jb} u_{kc} \end{aligned}$$

On the other hand, we obtain as well the following formula:

$$\begin{aligned} u^{\otimes 3} T_\sigma e_{abc} &= u^{\otimes 3} \delta_{ab} e_{cba} \\ &= \sum_{ijk} e_{kji} \otimes \delta_{ab} u_{kc} u_{jb} u_{ia} \end{aligned}$$

Thus $T_\sigma \in \text{End}(u^{\otimes 3})$ is equivalent to the following relations:

$$\delta_{ij}u_{ia}u_{jb}u_{kc} = \delta_{ab}u_{kc}u_{jb}u_{ia}$$

Now by setting $j = i, b = a$ in this formula we obtain the commutation relations in the statement, namely $u_{ia}^2u_{kc} = u_{kc}u_{ia}^2$, and this finishes the proof. \square

The relation of $H_N^{[\infty]}$ with the polygonal spheres comes from the following fact:

PROPOSITION 14.7. *Let $X \subset S_{\mathbb{R},+}^{N-1}$ be closed, let $d \geq 2$, and set:*

$$X^{d-1} = X \cap S_{\mathbb{R},+}^{N-1,d-1}$$

Then for a quantum group $G \subset H_N^{[\infty]}$ the following are equivalent:

(1) $x_i \rightarrow \sum_j x_j \otimes u_{ji}$ defines a coaction map, as follows:

$$\Phi : C(X^{d-1}) \rightarrow C(X^{d-1}) \otimes C(G)$$

(2) $x_i \rightarrow \sum_j x_j \otimes u_{ji}$ defines a morphism, as follows:

$$\tilde{\Phi} : C(X) \rightarrow C(X^{d-1}) \otimes C(G)$$

In particular, $G^+(X) \cap H_N^{[\infty]}$ acts on X^{d-1} , for any $d \geq 2$.

PROOF. The idea here is to use the relations in Proposition 14.6 (2) above:

(1) \implies (2) This is clear, by composing Φ with the following projection map:

$$\pi : C(X) \rightarrow C(X^{d-1})$$

(2) \implies (1) In order to prove this implication, we must understand when a coaction map as follows exists:

$$C(X^{d-1}) \rightarrow C(X^{d-1}) \otimes C(G)$$

In order for this to happen, the variables $X_i = \sum_j x_j \otimes u_{ji}$ must satisfy the relations defining X , which hold indeed by (2), and must satisfy as well the following relations, with i_0, \dots, i_d distinct, which define the polygonal spheres $S_{\mathbb{R},+}^{N-1,d-1}$:

$$X_{i_0} \dots X_{i_d} = 0$$

The point now is that, under the assumption $G \subset H_N^{[\infty]}$, these latter relations are automatic. Indeed, by using Proposition 14.6 (2), for i_0, \dots, i_d distinct we obtain:

$$\begin{aligned} X_{i_0} \dots X_{i_d} &= \sum_{j_0 \dots j_d} x_{j_0} \dots x_{j_d} \otimes u_{j_0 i_0} \dots u_{j_d i_d} \\ &= \sum_{j_0 \dots j_d \text{ distinct}} 0 \otimes u_{j_0 i_0} \dots u_{j_d i_d} + \sum_{j_0 \dots j_d \text{ not distinct}} x_{j_0} \dots x_{j_d} \otimes 0 \\ &= 0 + 0 = 0 \end{aligned}$$

Thus the coaction in (1) exists precisely when (2) is satisfied, and we are done.

Finally, the last assertion is clear from (2) \implies (1), because the universal coaction of $G = G^+(X)$ gives rise to a map $\tilde{\Phi} : C(X) \rightarrow C(X^{d-1}) \otimes C(G)$ as in (2). \square

As an illustration, we have the following result:

THEOREM 14.8. *The compact quantum groups*

$$H_N, H_N, H_N^*, H_N^*, H_N^{[\infty]}$$

act respectively on the spheres

$$S_{\mathbb{R}}^{N-1,d-1}, \bar{S}_{\mathbb{R}}^{N-1,d-1}, S_{\mathbb{R},*}^{N-1,d-1}, \bar{S}_{\mathbb{R},*}^{N-1,d-1}, S_{\mathbb{R},+}^{N-1,d-1}$$

at any $d \geq 2$.

PROOF. We use Proposition 14.7. We know that the quantum isometry groups at $d = N$ are respectively equal to the following quantum groups:

$$O_N, \bar{O}_N, O_N^*, \bar{O}_N^*, O_N^+$$

Our claim is that, by intersecting these quantum groups with $H_N^{[\infty]}$, we obtain the quantum groups in the statement. Indeed:

- (1) $O_N \cap H_N^{[\infty]} = H_N$ is clear from definitions.
- (2) $\bar{O}_N \cap H_N^{[\infty]} = H_N$ follows from $\bar{O}_N \cap H_N^+ \subset O_N$, which in turn follows from the computation (3) in the proof of Proposition 14.5, with H_N^* replaced by H_N^+ .
- (3) $O_N^* \cap H_N^{[\infty]} = H_N^*$ follows from $O_N^* \cap H_N^+ = H_N^*$.
- (4) $\bar{O}_N^* \cap H_N^{[\infty]} \supset H_N^*$ is clear, and the reverse inclusion can be proved by a direct computation, similar to the computation (3) in the proof of Proposition 14.5.
- (5) $O_N^+ \cap H_N^{[\infty]} = H_N^{[\infty]}$ is clear from definitions. \square

Observe that the above result is “sharp”, in the sense that the actions there are the universal ones, in the classical case at any $d \in \{2, \dots, N-1\}$, as well as in the twisted case at $d = 2$. Indeed, this follows from the various results established above.

14c. Middle computation

Let us discuss now the computation of the quantum isometry group for the polygonal sphere which is left, namely $S_{\mathbb{R},*}^{N-1,1}$, appearing in the middle of our square diagram of polygonal spheres, given in the beginning of this chapter.

We have seen in the above that the quantum group H_N^* acts on $S_{\mathbb{R},*}^{N-1,1}$.

This action, however, is not universal, because we have:

PROPOSITION 14.9. *The discrete group dual*

$$G = \widehat{\mathbb{Z}_2^{*N}}$$

acts on the polygonal sphere $S_{\mathbb{R},*}^{N-1,1}$.

PROOF. The standard coordinates on the polygonal sphere $S_{\mathbb{R},*}^{N-1,1}$ are by definition subject to the following relations:

$$x_i x_j x_k = \begin{cases} 0 & \text{for } i, j, k \text{ distinct} \\ x_k x_j x_i & \text{otherwise} \end{cases}$$

Let us try now to understand under which assumptions on a compact quantum group G , and in particular on a group dual, we have a coaction map, as follows:

$$\Phi : C(S_{\mathbb{R},*}^{N-1,1}) \rightarrow C(S_{\mathbb{R},*}^{N-1,1}) \otimes C(G)$$

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

In order for this to happen, the following variables must satisfy the above relations:

$$X_i = \sum_j x_j \otimes u_{ji}$$

For the group dual $G = \widehat{\mathbb{Z}_2^{*N}}$ we have by definition $u_{ij} = \delta_{ij} g_i$, where g_1, \dots, g_N are the standard generators of \mathbb{Z}_2^{*N} . We therefore have:

$$X_i X_j X_k = x_i x_j x_k \otimes g_i g_j g_k$$

$$X_k X_j X_i = x_k x_j x_i \otimes g_k g_j g_i$$

Thus the formula $X_i X_k X_k = 0$ for i, j, k distinct is clear, and the formula $X_i X_j X_k = X_k X_j X_i$ for i, j, k not distinct requires $g_i g_j g_k = g_k g_j g_i$ for i, j, k not distinct, which is clear as well. Indeed, at $i = j$ this latter relation reduces to $g_k = g_k$, at $i = k$ this relation is trivial, $g_i g_j g_i = g_i g_j g_i$, and at $j = k$ this relation reduces to $g_i = g_i$. \square

More generally now, we have the following result:

PROPOSITION 14.10. *The intermediate liberation of the hyperoctahedral group*

$$G = H_N^{[\infty]}$$

acts on the polygonal sphere $S_{\mathbb{R},*}^{N-1,1}$.

PROOF. We proceed as in the proof of Theorem 14.4 above. By expanding the formula of $X_i X_j X_k$ and by using the relations for the sphere $S_{\mathbb{R},*}^{N-1,1}$, we have:

$$\begin{aligned}
X_i X_j X_k &= \sum_{abc} x_a x_b x_c \otimes u_{ai} u_{bj} u_{ck} \\
&= \sum_{a,b,c \text{ not distinct}} x_a x_b x_c \otimes u_{ai} u_{bj} u_{ck} \\
&= \sum_{a \neq b} x_a^2 x_b \otimes (u_{ai} u_{aj} u_{bk} + u_{bi} u_{aj} u_{ak}) \\
&+ \sum_{a \neq b} x_a x_b x_a \otimes u_{ai} u_{bj} u_{ak} \\
&+ \sum_a x_a^3 \otimes u_{ai} u_{aj} u_{ak}
\end{aligned}$$

Now by assuming $G = H_N^{[\infty]}$, as in the statement, and by using the various formulae in Proposition 14.6 above, we obtain, for any i, j, k distinct:

$$\begin{aligned}
X_i X_j X_k &= \sum_{a \neq b} x_a^2 x_b \otimes (0 \cdot u_{bk} + u_{bi} \cdot 0) \\
&+ \sum_{a \neq b} x_a x_b x_a \otimes 0 \\
&+ \sum_a x_a^3 \otimes (0 \cdot u_{ak}) \\
&= 0
\end{aligned}$$

It remains to prove that we have $X_i X_j X_k = X_k X_j X_i$, for i, j, k not distinct. By replacing $i \leftrightarrow k$ in the above formula of $X_i X_j X_k$, we obtain:

$$\begin{aligned}
X_k X_j X_i &= \sum_{a \neq b} x_a^2 x_b \otimes (u_{ak} u_{aj} u_{bi} + u_{bk} u_{aj} u_{ai}) \\
&+ \sum_{a \neq b} x_a x_b x_a \otimes u_{ak} u_{bj} u_{ai} \\
&+ \sum_a x_a^3 \otimes u_{ak} u_{aj} u_{ai}
\end{aligned}$$

Let us compare this formula with the above formula of $X_i X_j X_k$. The last sum being 0 in both cases, we must prove that for any i, j, k not distinct and any $a \neq b$ we have:

$$\begin{aligned}
u_{ai} u_{aj} u_{bk} + u_{bi} u_{aj} u_{ak} &= u_{ak} u_{aj} u_{bi} + u_{bk} u_{aj} u_{ai} \\
u_{ai} u_{bj} u_{ak} &= u_{ak} u_{bj} u_{ai}
\end{aligned}$$

By symmetry the three cases $i = j, i = k, j = k$ reduce to two cases, $i = j$ and $i = k$. The case $i = k$ being clear, we are left with the case $i = j$, where we must prove:

$$u_{ai}u_{ai}u_{bk} + u_{bi}u_{ai}u_{ak} = u_{ak}u_{ai}u_{bi} + u_{bk}u_{ai}u_{ai}$$

$$u_{ai}u_{bi}u_{ak} = u_{ak}u_{bi}u_{ai}$$

By using $a \neq b$, the first equality reads:

$$u_{ai}^2u_{bk} + 0 \cdot u_{ak} = u_{ak} \cdot 0 + u_{bk}u_{ai}^2$$

Since by Proposition 14.6 (3) above we have $u_{ai}^2u_{bk} = u_{bk}u_{ai}^2$, this formula is satisfied, and we are done. As for the second equality, this reads:

$$0 \cdot u_{ak} = u_{ak} \cdot 0$$

But this is true as well, and this ends the proof. \square

We will prove now that the action in Proposition 14.10 is universal. In order to do so, we need to convert the formulae of type $X_iX_jX_k = 0$ and $X_iX_jX_k = X_kX_jX_i$ into relations between the quantum group coordinates u_{ij} , and this requires a good knowledge of the linear relations between the variables $x_a^2x_b, x_ax_bx_a, x_a^3$ over the sphere $S_{\mathbb{R},*}^{N-1,1}$.

So, we must first study these variables. The answer here is given by:

PROPOSITION 14.11. *The variables*

$$\left\{ x_a^2x_b, x_ax_bx_a, x_a^3 \mid a \neq b \right\}$$

are linearly independent over the sphere $S_{\mathbb{R},*}^{N-1,1}$.

PROOF. We use a trick from [41]. Consider the 1-dimensional polygonal version of the complex sphere $S_{\mathbb{C}}^{N-1}$, which is by definition given by:

$$S_{\mathbb{C}}^{N-1,1} = \left\{ z \in S_{\mathbb{C}}^{N-1} \mid z_i z_j z_k = 0, \forall i, j, k \text{ distinct} \right\}$$

We have then a 2×2 matrix model for the coordinates of $S_{\mathbb{R},*}^{N-1,1}$, as follows:

$$x_i \rightarrow \gamma_i = \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

Indeed, the matrices γ_i on the right are all self-adjoint, their squares sum up to 1, they half-commute, and they satisfy, for i, j, k distinct:

$$\gamma_i \gamma_j \gamma_k = 0$$

Thus we have indeed a morphism as follows, as claimed:

$$C(S_{\mathbb{R},*}^{N-1,1}) \rightarrow M_2(C(S_{\mathbb{C}}^{N-1,1}))$$

$$x_i \rightarrow \gamma_i$$

We can use this model in order to prove the linear independence. Consider indeed the variables that we are interested in, namely:

$$\begin{aligned} & x_a^2 x_b \\ & x_a x_b x_a \\ & x_a^3 \end{aligned}$$

In the model, these variables are mapped to the following variables:

$$\begin{aligned} \gamma_a^2 \gamma_b &= \begin{pmatrix} 0 & |z_a|^2 z_b \\ |z_a|^2 \bar{z}_b & 0 \end{pmatrix} \\ \gamma_a \gamma_b \gamma_a &= \begin{pmatrix} 0 & z_a^2 \bar{z}_b \\ \bar{z}_a^2 z_b & 0 \end{pmatrix} \\ \gamma_a^3 &= \begin{pmatrix} 0 & |z_a|^2 z_a \\ |z_a|^2 \bar{z}_a & 0 \end{pmatrix} \end{aligned}$$

Now observe that the following variables are linearly independent over $S_{\mathbb{C}}^1$:

$$|z_1|^2 z_2, |z_2|^2 z_1, z_1^2 \bar{z}_2, z_2^2 \bar{z}_1, |z_1|^2 z_1, |z_2|^2 z_2$$

Thus the upper right entries of the above matrices are linearly independent over $S_{\mathbb{C}}^{N-1,1}$. Thus the matrices themselves are linearly independent, and this proves our result. \square

With the above result in hand, we can now reformulate the coaction problem into a purely quantum group-theoretical problem, as follows:

PROPOSITION 14.12. *A quantum group $G \subset O_N^+$ acts on $S_{\mathbb{R},*}^{N-1,1}$ precisely when its standard coordinates u_{ij} satisfy the following relations:*

- (1) $u_{ai} u_{aj} u_{bk} + u_{bi} u_{aj} u_{ak} = 0$ for any i, j, k distinct.
- (2) $u_{ai} u_{bj} u_{ak} = 0$ for any i, j, k distinct.
- (3) $u_{ai}^2 u_{bk} = u_{bk} u_{ai}^2$.
- (4) $u_{ak} u_{ai} u_{bi} = u_{bi} u_{ai} u_{ak}$.
- (5) $u_{ai} u_{bi} u_{ak} = u_{bk} u_{bi} u_{ai}$.

PROOF. We use notations from the beginning of the proof of Proposition 14.10, along with the following formula, also established there:

$$\begin{aligned} X_i X_j X_k &= \sum_{a \neq b} x_a^2 x_b \otimes (u_{ai} u_{aj} u_{bk} + u_{bi} u_{aj} u_{ak}) \\ &+ \sum_{a \neq b} x_a x_b x_a \otimes u_{ai} u_{bj} u_{ak} \\ &+ \sum_a x_a^3 \otimes u_{ai} u_{aj} u_{ak} \end{aligned}$$

In order to have an action as in the statement, these quantities must satisfy $X_i X_k X_k = 0$ for i, j, k distinct, and $X_i X_k X_k = X_k X_j X_i$ for i, j, k not distinct. Now by using Proposition 14.11, we conclude that the relations to be satisfied are as follows:

(A) For i, j, k distinct, the following must hold:

$$u_{ai}u_{aj}u_{bk} + u_{bi}u_{aj}u_{ak} = 0, \forall a \neq b$$

$$u_{ai}u_{bj}u_{ak} = 0, \forall a \neq b$$

$$u_{ai}u_{aj}u_{ak} = 0, \forall a$$

(B) For i, j, k not distinct, the following must hold:

$$u_{ai}u_{aj}u_{bk} + u_{bi}u_{aj}u_{ak} = u_{ak}u_{aj}u_{bi} + u_{bk}u_{aj}u_{ai}, \forall a \neq b$$

$$u_{ai}u_{bj}u_{ak} = u_{ak}u_{bj}u_{ai}, \forall a \neq b$$

$$u_{ai}u_{aj}u_{ak} = u_{ak}u_{aj}u_{ai}, \forall a$$

In order to simplify this set of relations, the first observation is that the last relations in both (A) and (B) can be merged with the other ones, and we are led to:

(A') For i, j, k distinct, the following must hold:

$$u_{ai}u_{aj}u_{bk} + u_{bi}u_{aj}u_{ak} = 0, \forall a, b$$

$$u_{ai}u_{bj}u_{ak} = 0, \forall a, b$$

(B') For i, j, k not distinct, the following must hold:

$$u_{ai}u_{aj}u_{bk} + u_{bi}u_{aj}u_{ak} = u_{ak}u_{aj}u_{bi} + u_{bk}u_{aj}u_{ai}, \forall a, b$$

$$u_{ai}u_{bj}u_{ak} = u_{ak}u_{bj}u_{ai}, \forall a, b$$

Observe that the relations (A') are exactly the relations (1,2) in the statement.

Let us further process the relations (B'). In the case $i = k$ the relations are automatic, and in the cases $j = i, j = k$ the relations that we obtain coincide, via $i \leftrightarrow k$. Thus (B') reduces to the set of relations obtained by setting $j = i$, which are as follows:

$$u_{ai}u_{ai}u_{bk} + u_{bi}u_{ai}u_{ak} = u_{ak}u_{ai}u_{bi} + u_{bk}u_{ai}u_{ai}$$

$$u_{ai}u_{bi}u_{ak} = u_{ak}u_{bi}u_{ai}$$

Observe that the second relation is the relation (5) in the statement. Regarding now the first relation, with the notation $[x, y, z] = xyz - zyx$, this is as follows:

$$[u_{ai}, u_{ai}, u_{bk}] = [u_{ak}, u_{ai}, u_{bi}]$$

By applying the antipode, we obtain from this:

$$[u_{kb}, u_{ia}, u_{ia}] = [u_{ib}, u_{ia}, u_{ka}]$$

By relabelling $a \leftrightarrow i$ and $b \leftrightarrow k$, this relation becomes:

$$[u_{bk}, u_{ai}, u_{ai}] = [u_{ak}, u_{ai}, u_{bi}]$$

Now since we have $[x, y, z] = -[z, y, x]$, by comparing this latter relation with the original one, a simplification occurs, and the resulting relations are as follows:

$$[u_{ai}, u_{ai}, u_{bk}] = [u_{ak}, u_{ai}, u_{bi}] = 0$$

But these are exactly the relations (3,4) in the statement, and we are done. \square

Now by solving the quantum group problem raised by Proposition 14.12, we obtain:

PROPOSITION 14.13. *We have the following formula:*

$$G^+(S_{\mathbb{R},*}^{N-1,1}) = H_N^{[\infty]}$$

PROOF. The inclusion \supset is clear from Proposition 14.10. For the converse, we already have the result at $N = 2$, so assume $N \geq 3$. We will use many times the conditions (1-5) in Proposition 14.12. By using (2), for $i \neq j$ we have:

$$\begin{aligned} & u_{ai}u_{bj}u_{ak} = 0, \forall k \neq i, j \\ \implies & u_{ai}u_{bj}u_{ak}^2 = 0, \forall k \neq i, j \\ \implies & u_{ai}u_{bj} \left(\sum_{k \neq i, j} u_{ak}^2 \right) = 0, \forall i \neq j \\ \implies & u_{ai}u_{bj}(1 - u_{ai}^2 - u_{aj}^2) = 0, \forall i \neq j \end{aligned}$$

Now by using (3), we can move the variable u_{bj} to the right. By further multiplying by u_{bj} to the right, and then summing over b , we obtain:

$$\begin{aligned} & u_{ai}u_{bj}(1 - u_{ai}^2 - u_{aj}^2) = 0, \forall i \neq j \\ \implies & u_{ai}(1 - u_{ai}^2 - u_{aj}^2)u_{bj} = 0, \forall i \neq j \\ \implies & u_{ai}(1 - u_{ai}^2 - u_{aj}^2)u_{bj}^2 = 0, \forall i \neq j \\ \implies & u_{ai}(1 - u_{ai}^2 - u_{aj}^2) = 0, \forall i \neq j \end{aligned}$$

We can proceed now as follows, by summing over $j \neq i$:

$$\begin{aligned} & u_{ia}(1 - u_{ai}^2 - u_{aj}^2) = 0, \forall i \neq j \\ \implies & u_{ai}u_{aj}^2 = u_{ai} - u_{ai}^3, \forall i \neq j \\ \implies & u_{ai}(1 - u_{ai}^2) = (N - 1)(u_{ai} - u_{ai}^3) \\ \implies & u_{ai} = u_{ai}^3 \end{aligned}$$

Thus the standard coordinates are partial isometries, and so $G \subset H_N^+$. On the other hand, we know from the proof of Proposition 14.6 (3) that the quantum subgroup $G \subset H_N^+$ obtained via the relations $[a, b^2] = 0$ is $H_N^{[\infty]}$, and this finishes the proof. \square

We have now complete results for the 9 main spheres, as follows:

THEOREM 14.14. *The quantum isometry groups of the 9 polygonal spheres are*

$$\begin{array}{ccccc}
 O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N & \longrightarrow & H_N^{[\infty]} & \longrightarrow & \bar{O}_N^* \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N^+ & \longrightarrow & H_N & \longrightarrow & \bar{O}_N
 \end{array}$$

where $H_N^+, H_N^{[\infty]}$ and $\bar{O}_N, O_N^*, \bar{O}_N^*, O_N^+$ are quantum versions of H_N, O_N .

PROOF. This follows indeed by putting together the above results. □

As a conclusion, we have a key computation available, but there are still many questions left, regarding the extension of our (S, T, U, K) formalism, as to cover the intersections between the twisted and untwisted geometries.

As explained in the present and in the previous chapter, on several occasions, all this is not exactly something theoretical, but is rather something of practical interest, in connection with the algebraic geometry of the free spheres, and other free manifolds.

14d. Complex extension

Let us discuss now a straightforward complex extension of the above results. Our starting point will be the following definition:

DEFINITION 14.15. *The complex polygonal spheres, denoted*

$$S_{\mathbb{C}}^{N-1,d-1}, \bar{S}_{\mathbb{C}}^{N-1,d-1}, \bar{S}_{\mathbb{C},*}^{N-1,d-1}, S_{\mathbb{C},+}^{N-1,d-1}$$

are constructed from $S_{\mathbb{C},+}^{N-1}$ in the same way as their real versions, namely

$$S_{\mathbb{R}}^{N-1,d-1}, \bar{S}_{\mathbb{R}}^{N-1,d-1}, \bar{S}_{\mathbb{R},*}^{N-1,d-1}, S_{\mathbb{R},+}^{N-1,d-1}$$

are constructed from $S_{\mathbb{R},+}^{N-1}$, namely by assuming that the corresponding vanishing relations hold between the variables $x_i = z_i, z_i^*$.

As in the real case, we will restrict now the attention to the 5 main spheres, coming from [8], and to their intersections. We have 9 such spheres here, as follows:

$$\begin{array}{ccccc}
 S_{\mathbb{C}}^{N-1} & \longrightarrow & S_{\mathbb{C},*}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{C}}^{N-1} \cap \bar{S}_{\mathbb{C},*}^{N-1} & \longrightarrow & S_{\mathbb{C},*}^{N-1} \cap \bar{S}_{\mathbb{C},*}^{N-1} & \longrightarrow & \bar{S}_{\mathbb{C},*}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{C}}^{N-1} \cap \bar{S}_{\mathbb{C}}^{N-1} & \longrightarrow & S_{\mathbb{C},*}^{N-1} \cap \bar{S}_{\mathbb{C}}^{N-1} & \longrightarrow & \bar{S}_{\mathbb{C}}^{N-1}
 \end{array}$$

The intersections can be computed as in the real case, and we have:

PROPOSITION 14.16. *The 5 main spheres, and the intersections between them, are*

$$\begin{array}{ccccc}
 S_{\mathbb{C}}^{N-1} & \longrightarrow & S_{\mathbb{C},*}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{C}}^{N-1,1} & \longrightarrow & S_{\mathbb{C},*}^{N-1,1} & \longrightarrow & \bar{S}_{\mathbb{C},*}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 S_{\mathbb{C}}^{N-1,0} & \longrightarrow & \bar{S}_{\mathbb{C}}^{N-1,1} & \longrightarrow & \bar{S}_{\mathbb{C}}^{N-1}
 \end{array}$$

with all the maps being inclusions.

PROOF. This is similar to the proof from the real case, by replacing in all the computations there the variables x_i by the variables $x_i = z_i, z_i^*$. □

Next, we have the following result:

THEOREM 14.17. *The quantum isometry groups of the 9 main complex spheres are*

$$\begin{array}{ccccc}
 U_N & \longrightarrow & U_N^{**} & \longrightarrow & U_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 K_N & \longrightarrow & K_N^{[\infty]} & \longrightarrow & \bar{U}_N^{**} \\
 \uparrow & & \uparrow & & \uparrow \\
 K_N^+ & \longrightarrow & K_N & \longrightarrow & \bar{U}_N
 \end{array}$$

where K_N and its versions are the complex analogues of H_N and its versions.

PROOF. The idea is that the proof here is quite similar to the proof in the real case, by replacing H_N, O_N with their complex analogues K_N, U_N . \square

As a conclusion, we have many technical results available, but there are still many questions left, regarding the extension of our (S, T, U, K) formalism, as to cover the intersections between the twisted and untwisted geometries.

14e. Exercises

As with the exercises from the previous chapter, that the present chapter continues, our exercises here will be quite technical, and algebraic. Let us start with:

EXERCISE 14.18. *Compute the quantum isometry groups of the twisted polygonal spheres, which are by definition given by*

$$C(\bar{S}_{\mathbb{R}}^{N-1, d-1}) = C(\bar{S}_{\mathbb{R}}^{N-1}) / \langle x_{i_0} \dots x_{i_d} = 0, \forall i_0, \dots, i_d \text{ distinct} \rangle$$

at the missing values of the parameter, $d = 3, 4, \dots, N - 1$.

This is something that we discussed in the above, and is a result that will be nice to have. Technically speaking, however, it is not clear how exactly to do this.

As a more elementary exercise now, we have:

EXERCISE 14.19. *Work out the full theory of the quantum group $H_N^{[\infty]}$, notably with a Brauer type result for it, and then with probability computations.*

This exercise is interesting not only because of the considerations from the present chapter, where the quantum group $H_N^{[\infty]}$ plays a central role, but also in view of the classification discussion in chapter 10, where $H_N^{[\infty]}$ was playing as well a central role.

Along the same lines, we have, in the complex case now:

EXERCISE 14.20. *Work out the full theory of the quantum group $K_N^{[\infty]}$, notably with a Brauer type result for it, and then with probability computations.*

This exercise is particularly interesting, because unlike $H_N^{[\infty]}$, which is a well-known quantum group, $K_N^{[\infty]}$ is a more recent object, not systematically studied yet.

In connection with axiomatization questions, we have:

EXERCISE 14.21. *Find suitable quantum group axioms covering the quantum isometry groups of the main polygonal spheres, namely*

$$\begin{array}{ccccc}
 O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N & \longrightarrow & H_N^{[\infty]} & \longrightarrow & \bar{O}_N^* \\
 \uparrow & & \uparrow & & \uparrow \\
 H_N^+ & \longrightarrow & H_N & \longrightarrow & \bar{O}_N
 \end{array}$$

and then do the same in the complex case.

The above quantum groups are all quizzy, and the problem is that of looking more carefully, in order to see what are the common features of these quantum groups.

In addition to these exercises, we have plenty of other interesting questions, which were discussed throughout the present chapter, and the previous one.

CHAPTER 15

Projective geometry

15a. Quantum groups

We discuss here analogues of the various structure results and axiomatization and classification questions developed in the above, in the projective geometry setting. We will present as well a few basic results regarding the structure of the quantum projective spaces, quantum projective groups, and other projective manifolds.

This chapter will be quite elementary, with full details given, even for results that we already know, to be repeated here, with the aim of making this presentation as independent as possible from the previous chapters, as a beginning of something new.

The point is that things become considerably simpler in the projective geometry setting, and this is something that we would like to explain, without much reference to the various affine considerations, which are often quite technical, developed in the above.

Consider indeed the diagram of 9 main geometries, that we found above:

$$\begin{array}{ccccc}
 \mathbb{R}_+^N & \longrightarrow & T\mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{R}_*^N & \longrightarrow & T\mathbb{R}_*^N & \longrightarrow & \mathbb{C}_*^N \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{R}^N & \longrightarrow & T\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
 \end{array}$$

As explained in chapters 9-10 above, when looking at the projective versions of these geometries, the diagram drastically simplifies.

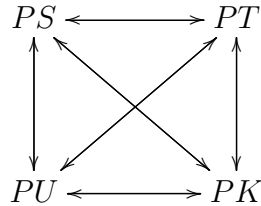
To be more precise, when looking for instance at the projective versions of the corresponding spheres, the diagram becomes as follows, consisting of 3 objects only:

$$\begin{array}{ccccc}
 P_+^{N-1} & \longrightarrow & P_+^{N-1} & \longrightarrow & P_+^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 P_{\mathbb{C}}^{N-1} & \longrightarrow & P_{\mathbb{C}}^{N-1} & \longrightarrow & P_{\mathbb{C}}^{N-1} \\
 \uparrow & & \uparrow & & \uparrow \\
 P_{\mathbb{R}}^{N-1} & \longrightarrow & P_{\mathbb{R}}^{N-1} & \longrightarrow & P_{\mathbb{C}}^{N-1}
 \end{array}$$

Thus, we are led to the conclusion that, under certain combinatorial axioms, there should be only 3 projective geometries, namely the real, complex and free one:

$$P_{\mathbb{R}}^{N-1} \subset P_{\mathbb{C}}^{N-1} \subset P_+^{N-1}$$

We will discuss this in what follows, with analogues and improvements of the affine results. Also, we would like to study the corresponding quadruplets (PS, PT, PU, PK) , and to axiomatize the projective geometries, with correspondences as follows:



Summarizing, there is a lot of work to be done, on one hand in reformulating and improving the results from the affine case, and on the other hand, in starting to develop the projective theory independently from the affine theory.

As a first topic that we would like to discuss, and which historically speaking, was at the beginning of everything, we have the following remarkable isomorphism, that we already used in the above, and that we would like to discuss now in detail:

$$PO_N^+ = PU_N^+$$

In order to get started, let us first discuss the classical case, and more specifically the precise relation between the orthogonal group O_N , and the unitary group U_N .

Contrary to the passage $\mathbb{R}^N \rightarrow \mathbb{C}^N$, or to the passage $S_{\mathbb{R}}^{N-1} \rightarrow S_{\mathbb{C}}^{N-1}$, which are both elementary, the passage $O_N \rightarrow U_N$ cannot be understood directly. In order to understand this passage we must pass through the corresponding Lie algebras, a follows:

THEOREM 15.1. *The passage $O_N \rightarrow U_N$ appears via Lie algebra complexification,*

$$O_N \rightarrow \mathfrak{o}_N \rightarrow \mathfrak{u}_n \rightarrow U_N$$

with the Lie algebra \mathfrak{u}_N being a complexification of the Lie algebra \mathfrak{o}_N .

PROOF. This is something rather philosophical, and advanced as well, that we will not really need here, the idea being as follows:

(1) The unitary and orthogonal groups U_N, O_N are both Lie groups, in the sense that they are smooth manifolds. The corresponding Lie algebras $\mathfrak{u}_N, \mathfrak{o}_N$, which are by definition the respective tangent spaces at 1, can be computed by differentiating the equations defining U_N, O_N , with the conclusion being as follows:

$$\begin{aligned}\mathfrak{u}_N &= \left\{ A \in M_N(\mathbb{C}) \mid A^* = -A \right\} \\ \mathfrak{o}_N &= \left\{ B \in M_N(\mathbb{R}) \mid B^t = -B \right\}\end{aligned}$$

(2) This was for the correspondences $U_N \rightarrow \mathfrak{u}_N$ and $O_N \rightarrow \mathfrak{o}_N$. In the other sense, the correspondences $\mathfrak{u}_N \rightarrow U_N$ and $\mathfrak{o}_N \rightarrow O_N$ appear by exponentiation, the result here stating that, around 1, the unitary matrices can be written as $U = e^A$, with $A \in \mathfrak{u}_N$, and the orthogonal matrices can be written as $U = e^B$, with $B \in \mathfrak{o}_N$.

(3) In view of all this, in order to understand the passage $O_N \rightarrow U_N$ it is enough to understand the passage $\mathfrak{o}_N \rightarrow \mathfrak{u}_N$. But, in view of the above formulae for $\mathfrak{o}_N, \mathfrak{u}_N$, this is basically an elementary linear algebra problem. Indeed, let us pick an arbitrary matrix $A \in M_N(\mathbb{C})$, and write it as follows, with $B, C \in M_N(\mathbb{R})$:

$$A = B + iC$$

In terms of B, C , the equation $A^* = -A$ defining the Lie algebra \mathfrak{u}_N reads:

$$\begin{aligned}B^t &= -B \\ C^t &= C\end{aligned}$$

(4) As a first observation, we must have $B \in \mathfrak{o}_N$. Regarding now C , let us decompose this matrix as follows, with D being its diagonal, and C' being the reminder:

$$C = D + C'$$

The matrix C' being symmetric with 0 on the diagonal, by switching all the signs below the main diagonal we obtain a certain matrix $C'_- \in \mathfrak{o}_N$. Thus, we have decomposed $A \in \mathfrak{u}_N$ as follows, with $B, C'_- \in \mathfrak{o}_N$, and with $D \in M_N(\mathbb{R})$ being diagonal:

$$A = B + iD + iC'_-$$

(5) As a conclusion now, we have shown that we have a direct sum decomposition of real linear spaces as follows, with $\Delta \subset M_N(\mathbb{R})$ being the diagonal matrices:

$$\mathfrak{u}_N \simeq \mathfrak{o}_N \oplus \Delta \oplus \mathfrak{o}_N$$

Thus, we can stop our study here, and say that we have reached the conclusion in the statement, namely that \mathfrak{u}_N appears as a “complexification” of \mathfrak{o}_N . \square

As before with many other things, that we will not really need in what follows, this was just an introduction to the subject. More can be found in any Lie group book.

In the free case now, the situation is much simpler, and we have:

THEOREM 15.2. *The passage $O_N^+ \rightarrow U_N^+$ appears via free complexification,*

$$U_N^+ = \widetilde{O}_N^+$$

where the free complexification of a pair (G, u) is by definition the pair $(\widetilde{G}, \widetilde{u})$ with

$$C(\widetilde{G}) = \langle zu_{ij} \rangle \subset C(\mathbb{T}) * C(G)$$

$$\widetilde{u} = zu$$

where $z \in C(\mathbb{T})$ is the standard generator, given by $x \rightarrow x$ for any $x \in \mathbb{T}$.

PROOF. We have embeddings as follows, with the first one coming by using the counit, and with the second one coming from the universality property of U_N^+ :

$$O_N^+ \subset \widetilde{O}_N^+ \subset U_N^+$$

We must prove that the embedding on the right is an isomorphism, and there are several ways of doing this, all instructive, as follows:

(1) The original argument, from [1], is something quick and advanced, based on character computations and free probability. Indeed, let us recall that if we denote by v, u the fundamental corepresentations of O_N^+, U_N^+ , then by easiness we have:

$$Fix(v^{\otimes k}) = span \left(\xi_\pi \mid \pi \in NC_2(k) \right)$$

$$Fix(u^{\otimes k}) = span \left(\xi_\pi \mid \pi \in \mathcal{NC}_2(k) \right)$$

Moreover, the above vectors ξ_π are known to be linearly independent at $N \geq 2$, and so the above results provide us with bases, and we obtain:

$$\dim(Fix(v^{\otimes k})) = |NC_2(k)|$$

$$\dim(Fix(u^{\otimes k})) = |\mathcal{NC}_2(k)|$$

Now since integrating the character of a corepresentation amounts in counting the fixed points, the above two formulae can be rewritten as follows:

$$\int_{O_N^+} \chi_v^k = |NC_2(k)|$$

$$\int_{U_N^+} \chi_u^k = |\mathcal{NC}_2(k)|$$

But this shows, via standard free probability theory, that χ_v must follow the Winger semicircle law γ_1 , and that χ_u must follow the Voiculescu circular law Γ_1 :

$$\begin{aligned}\chi_v &\sim \gamma_1 \\ \chi_u &\sim \Gamma_1\end{aligned}$$

On the other hand, when freely multiplying a semicircular variable by a Haar unitary we obtain a circular variable. Thus, the main character of \widetilde{O}_N^+ is circular:

$$\chi_{zv} \sim \Gamma_1$$

Now by forgetting about circular variables and free probability, the conclusion is that the inclusion $\widetilde{O}_N^+ \subset U_N^+$ preserves the law of the main character:

$$law(\chi_{zv}) = law(u)$$

Thus by Peter-Weyl we obtain that the inclusion $\widetilde{O}_N^+ \subset U_N^+$ must be an isomorphism, modulo the usual equivalence relation for quantum groups.

(2) A version of the above proof, not using any prior free probability knowledge, makes use of the easiness property of O_N^+, U_N^+ only, namely:

$$\begin{aligned}Hom(v^{\otimes k}, v^{\otimes l}) &= span \left(\xi_\pi \mid \pi \in NC_2(k, l) \right) \\ Hom(u^{\otimes k}, u^{\otimes l}) &= span \left(\xi_\pi \mid \pi \in \mathcal{NC}_2(k, l) \right)\end{aligned}$$

Indeed, let us look at the following inclusions of quantum groups:

$$O_N^+ \subset \widetilde{O}_N^+ \subset U_N^+$$

At the level of the associated Hom spaces we obtain reverse inclusions, as follows:

$$Hom(v^{\otimes k}, v^{\otimes l}) \supset Hom((zv)^{\otimes k}, (zv)^{\otimes l}) \supset Hom(u^{\otimes k}, u^{\otimes l})$$

The spaces on the left and on the right are known from easiness, the result being that these spaces are as follows:

$$span \left(T_\pi \mid \pi \in NC_2(k, l) \right) \supset span \left(T_\pi \mid \pi \in \mathcal{NC}_2(k, l) \right)$$

Regarding the spaces in the middle, these are obtained from those on the left by “coloring”, so we obtain the same spaces as those on the right. Thus, by Tannakian duality, our embedding $\widetilde{O}_N^+ \subset U_N^+$ is an isomorphism, modulo the usual equivalence relation. \square

As an interesting consequence of the above result, we have:

THEOREM 15.3. *We have an identification as follows,*

$$PO_N^+ = PU_N^+$$

modulo the usual equivalence relation for compact quantum groups.

PROOF. As before, we have several proofs for this result, as follows:

(1) This follows from Theorem 15.2, because we have:

$$PU_N^+ = P\widetilde{O}_N^+ = PO_N^+$$

(2) We can deduce this as well directly. With notations as before, we have:

$$\text{Hom}((v \otimes v)^k, (v \otimes v)^l) = \text{span} \left(T_\pi \Big| \pi \in \mathcal{NC}_2((\circ\bullet)^k, (\circ\bullet)^l) \right)$$

$$\text{Hom}((u \otimes \bar{u})^k, (u \otimes \bar{u})^l) = \text{span} \left(T_\pi \Big| \pi \in \mathcal{NC}_2((\circ\bullet)^k, (\circ\bullet)^l) \right)$$

The sets on the right being equal, we conclude that the inclusion $PO_N^+ \subset PU_N^+$ preserves the corresponding Tannakian categories, and so must be an isomorphism. \square

As a conclusion, the passage $O_N^+ \rightarrow U_N^+$ is something much simpler than the passage $O_N \rightarrow U_N$, with this ultimately coming from the fact that the combinatorics of O_N^+, U_N^+ is something much simpler than the combinatorics of O_N, U_N . In addition, all this leads as well to the interesting conclusion that the free projective geometry does not fall into real and complex, but is rather unique and “scalarless”. We will be back to this.

15b. Projective spaces

Let us discuss now the projective spaces. We begin with a short summary of the various projective geometry results that we have so far.

We will give full details here, with the aim of making the present chapter as independent as possible from the previous chapters, as a beginning of something new.

Our starting point is the following functional analytic description of the real and complex projective spaces $P_{\mathbb{R}}^{N-1}, P_{\mathbb{C}}^{N-1}$:

PROPOSITION 15.4. *We have presentation results as follows,*

$$\begin{aligned} C(P_{\mathbb{R}}^{N-1}) &= C_{comm}^* \left((p_{ij})_{i,j=1,\dots,N} \Big| p = \bar{p} = p^t = p^2, \text{Tr}(p) = 1 \right) \\ C(P_{\mathbb{C}}^{N-1}) &= C_{comm}^* \left((p_{ij})_{i,j=1,\dots,N} \Big| p = p^* = p^2, \text{Tr}(p) = 1 \right) \end{aligned}$$

for the algebras of continuous functions on the real and complex projective spaces.

PROOF. We use the elementary fact that the projective spaces $P_{\mathbb{C}}^{N-1}, P_{\mathbb{R}}^{N-1}$ can be respectively identified with the spaces of rank one projections in $M_N(\mathbb{C}), M_N(\mathbb{R})$.

With this picture in mind, it is clear that we have arrows \leftarrow .

In order to construct now arrows \rightarrow , consider the universal algebras on the right, A_C, A_R . These algebras being both commutative, by the Gelfand theorem we can write, with X_C, X_R being certain compact spaces:

$$A_C = C(X_C)$$

$$A_R = C(X_R)$$

Now by using the coordinate functions p_{ij} , we conclude that X_C, X_R are certain spaces of rank one projections in $M_N(\mathbb{C}), M_N(\mathbb{R})$. In other words, we have embeddings:

$$X_C \subset P_{\mathbb{C}}^{N-1}$$

$$X_R \subset P_{\mathbb{R}}^{N-1}$$

By transposing we obtain arrows \rightarrow , as desired. □

The above result suggests the following definition:

DEFINITION 15.5. *Associated to any $N \in \mathbb{N}$ is the following universal algebra,*

$$C(P_+^{N-1}) = C^* \left((p_{ij})_{i,j=1,\dots,N} \mid p = p^* = p^2, Tr(p) = 1 \right)$$

whose abstract spectrum is called “free projective space”.

Observe that, according to our presentation results for the real and complex projective spaces $P_{\mathbb{R}}^{N-1}$ and $P_{\mathbb{C}}^{N-1}$, we have embeddings of compact quantum spaces, as follows:

$$P_{\mathbb{R}}^{N-1} \subset P_{\mathbb{C}}^{N-1} \subset P_+^{N-1}$$

Also, the complex projective space $P_{\mathbb{C}}^{N-1}$ is the classical version of P_+^{N-1} .

Let us discuss now the relation with the spheres. Given a closed subset $X \subset S_{\mathbb{R},+}^{N-1}$, its projective version is by definition the quotient space $X \rightarrow PX$ determined by the fact that $C(PX) \subset C(X)$ is the subalgebra generated by the following variables:

$$p_{ij} = x_i x_j$$

In the classical case, we recover in this way the usual projective version.

In order to discuss now the relation with the quantum spheres, it is convenient to neglect the material from chapter 10, regarding the “hybrid” case, the projective versions of the spheres there bringing nothing new, for obvious reasons.

On the other hand, it is also convenient to neglect the material regarding the complex quantum spheres, because, as explained in chapter 9, the projective versions of these spheres bring nothing new, due to the various results worked out there.

Thus, we are left with the 3 real spheres, and we have the following result:

PROPOSITION 15.6. *The projective versions of the 3 real spheres are as follows,*

$$\begin{array}{ccccc}
 S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},+}^{N-1} \\
 \downarrow & & \downarrow & & \downarrow \\
 P_{\mathbb{R}}^{N-1} & \longrightarrow & P_{\mathbb{C}}^{N-1} & \longrightarrow & P_{+}^{N-1}
 \end{array}$$

modulo the standard equivalence relation for the quantum algebraic manifolds.

PROOF. The assertion at left is true by definition. For the assertion at right, we have to prove that the variables $p_{ij} = z_i z_j$ over the free sphere $S_{\mathbb{R},+}^{N-1}$ satisfy the defining relations for $C(P_{+}^{N-1})$, from Definition 15.5 above, namely:

$$\begin{aligned}
 p &= p^* = p^2 \\
 \text{Tr}(p) &= 1
 \end{aligned}$$

We first have the following computation:

$$\begin{aligned}
 (p^*)_{ij} &= p_{ji}^* \\
 &= (z_j z_i)^* \\
 &= z_i z_j \\
 &= p_{ij}
 \end{aligned}$$

We have as well the following computation:

$$\begin{aligned}
 (p^2)_{ij} &= \sum_k p_{ik} p_{kj} \\
 &= \sum_k z_i z_k^2 z_j \\
 &= z_i z_j \\
 &= p_{ij}
 \end{aligned}$$

Finally, we have as well the following computation:

$$\begin{aligned}
 \text{Tr}(p) &= \sum_k p_{kk} \\
 &= \sum_k z_k^2 \\
 &= 1
 \end{aligned}$$

Regarding now the middle assertion, stating that we have $PS_{\mathbb{R},*}^{N-1} = P_{\mathbb{C}}^{N-1}$, the inclusion “ \subset ” follows from the relations $abc = cba$, which imply:

$$abcd = cbad = cbda$$

In the other sense now, the point is that we have a matrix model, as follows:

$$\begin{aligned} \pi : C(S_{\mathbb{R},*}^{N-1}) &\rightarrow M_2(C(S_{\mathbb{C}}^{N-1})) \\ x_i &\rightarrow \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix} \end{aligned}$$

But this gives the missing inclusion “ \supset ”, and we are done. See [28]. □

In addition to the above result, let us mention that, as already discussed above, passing to the complex case brings nothing new. This is because the projective version of the free complex sphere is equal to the free projective space constructed above:

$$PS_{\mathbb{C},+}^{N-1} = P_+^{N-1}$$

For details on all this, we refer to chapter 9 above.

In what regards the tori, we have the following result:

PROPOSITION 15.7. *The projective versions of the 3 real tori are as follows,*

$$\begin{array}{ccccc} T_N & \longrightarrow & T_N^* & \longrightarrow & T_N^+ \\ \downarrow & & \downarrow & & \downarrow \\ PT_N & \longrightarrow & P\mathbb{T}_N & \longrightarrow & PT_N^+ \end{array}$$

modulo the standard equivalence relation for the quantum algebraic manifolds.

PROOF. This follows by using the same arguments as for the spheres. □

In what regards the unitary groups, that we will call in what follows orthogonal groups, because we are now in the real case, we have here the following result:

PROPOSITION 15.8. *The projective versions of the 3 orthogonal groups are as follows,*

$$\begin{array}{ccccc} O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \\ \downarrow & & \downarrow & & \downarrow \\ PO_N & \longrightarrow & PU_N & \longrightarrow & PO_N^+ \end{array}$$

modulo the standard equivalence relation for the compact quantum groups.

PROOF. This follows by using the same arguments as for the spheres, or for the tori. □

Finally, in what regards the reflection groups, that we will call hyperoctahedral groups, because we are now in the real case, we have here the following result:

PROPOSITION 15.9. *The projective versions of the 3 hyperoctahedral groups are as follows,*

$$\begin{array}{ccccc}
 H_N & \longrightarrow & H_N^* & \longrightarrow & H_N^+ \\
 \downarrow & & \downarrow & & \downarrow \\
 PH_N & \longrightarrow & PK_N & \longrightarrow & PH_N^+
 \end{array}$$

modulo the standard equivalence relation for the compact quantum groups.

PROOF. This follows by using the same arguments as for the spheres, or for the tori, or the quantum orthogonal groups. \square

In addition to the above results, let us mention that, as it was the case for the spheres, passing to the complex case brings nothing new.

This is indeed because we have isomorphisms as follows, which can be established by using easiness, as explained in the beginning of the present chapter for the isomorphism in the middle, and with the proof of the first and of the last isomorphism being quite similar, based respectively on elementary group theory, and on easiness:

$$\begin{aligned}
 P\mathbb{T}_N^+ &= PT_N^+ \\
 PU_N^+ &= PO_N^+ \\
 PK_N^+ &= PH_N^+
 \end{aligned}$$

Thus, passing to the complex case would bring indeed nothing new, and in what follows we will stay with the real formalism. This is of course something quite subtle, which happens only in the free case, and has no classical counterpart. For further details and comments here, we refer to the discussion in the beginning of this chapter.

Getting back now to our general program, we are done with the construction work, for the various projective geometry basic objects.

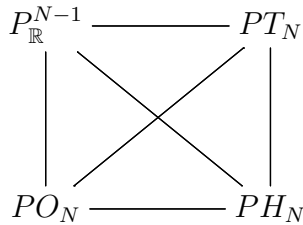
Our next task will be that of working out axiomatization and classification results, first in analogy with the affine results, and then independently of what we already have, with a number of new results, of true projective nature. Then, we will develop a bit of general theory, for more general projective manifolds.

Let us begin with a summary of the constructions discussed above, reformulated in the spirit of abstract noncommutative geometry considerations.

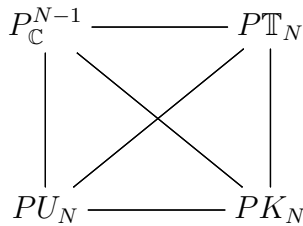
As a conclusion to what we did, with our various above constuctions and results, in the projective geometry setting, we have 3 projective quadruplets, whose construction and main properties can be summarized as follows:

THEOREM 15.10. We have “basic” projective quadruplets (P, PT, PU, PK) as follows,

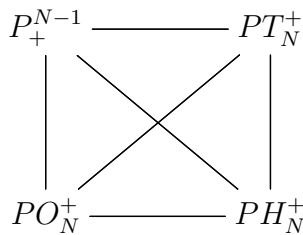
(1) A classical real quadruplet, as follows,



(2) A classical complex quadruplet, as follows,



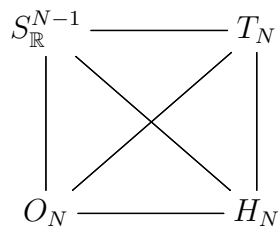
(3) A free quadruplet, as follows,



which appear as projective versions of the main 3 real quadruplets.

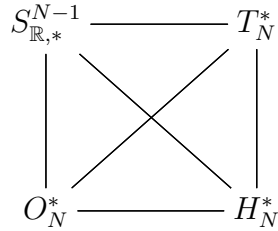
PROOF. This follows from the results that already have. To be more precise:

(1) Consider the classical affine real quadruplet, which is as follows:



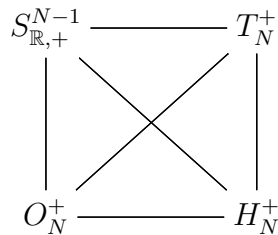
The projective version of this quadruplet is then the quadruplet in (1).

(2) Consider the half-classical affine real quadruplet, which is as follows:



The projective version of this quadruplet is then the quadruplet in (2).

(3) Consider the free affine real quadruplet, which is as follows:



The projective version of this quadruplet is then the quadruplet in (3). □

15c. Projective easiness

Getting back now to our general projective geometry program, we would like to have axiomatization and classification results for such quadruplets.

In order to do this, following [29], we can axiomatize our various projective spaces, as follows:

DEFINITION 15.11. *A monomial projective space is a closed subset $P \subset P_+^{N-1}$ obtained via relations of type*

$$p_{i_1 i_2} \cdots p_{i_{k-1} i_k} = p_{i_{\sigma(1)} i_{\sigma(2)}} \cdots p_{i_{\sigma(k-1)} i_{\sigma(k)}}, \quad \forall (i_1, \dots, i_k) \in \{1, \dots, N\}^k$$

with σ ranging over a certain subset of $\bigcup_{k \in 2\mathbb{N}} S_k$, which is stable under $\sigma \rightarrow |\sigma|$.

Observe the similarity with the corresponding notion for the spheres, from chapter 13. The only subtlety in the projective case is the stability under the operation $\sigma \rightarrow |\sigma|$, which in practice means that if the above relation associated to σ holds, then the following relation, associated to $|\sigma|$, must hold as well:

$$p_{i_0 i_1} \cdots p_{i_k i_{k+1}} = p_{i_0 i_{\sigma(1)}} p_{i_{\sigma(2)} i_{\sigma(3)}} \cdots p_{i_{\sigma(k-2)} i_{\sigma(k-1)}} p_{i_{\sigma(k)} i_{k+1}}$$

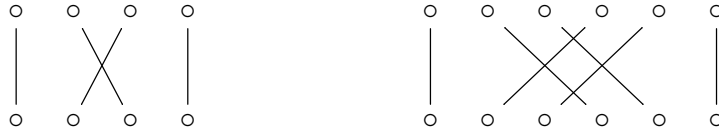
As an illustration, the basic projective spaces are all monomial:

PROPOSITION 15.12. *The 3 projective spaces are all monomial, with the permutations*



producing respectively the spaces $P_{\mathbb{R}}^{N-1}, P_{\mathbb{C}}^{N-1}$.

PROOF. We must divide the algebra $C(P_+^{N-1})$ by the relations associated to the diagrams in the statement, as well as those associated to their shifted versions, given by:



(1) The basic crossing, and its shifted version, produce the following relations:

$$p_{ab} = p_{ba}$$

$$p_{ab}p_{cd} = p_{ac}p_{bd}$$

Now by using these relations several times, we obtain:

$$\begin{aligned} p_{ab}p_{cd} &= p_{ac}p_{bd} \\ &= p_{ca}p_{db} \\ &= p_{cd}p_{ab} \end{aligned}$$

Thus, the space produced by the basic crossing is classical, $P \subset P_{\mathbb{C}}^{N-1}$. By using one more time the relations $p_{ab} = p_{ba}$ we conclude that we have $P = P_{\mathbb{R}}^{N-1}$, as claimed.

(2) The fattened crossing, and its shifted version, produce the following relations:

$$p_{ab}p_{cd} = p_{cd}p_{ab}$$

$$p_{ab}p_{cd}p_{ef} = p_{ad}p_{eb}p_{cf}$$

The first relations tell us that the projective space must be classical, $P \subset P_{\mathbb{C}}^{N-1}$. Now observe that with $p_{ij} = z_i \bar{z}_j$, the second relations read:

$$z_a \bar{z}_b z_c \bar{z}_d z_e \bar{z}_f = z_a \bar{z}_d z_e \bar{z}_b z_c \bar{z}_f$$

Since these relations are automatic, we have $P = P_{\mathbb{C}}^{N-1}$, and we are done. □

Following [29], we can now formulate our classification result, as follows:

THEOREM 15.13. *The basic projective spaces, namely*

$$P_{\mathbb{R}}^{N-1} \subset P_{\mathbb{C}}^{N-1} \subset P_+^{N-1}$$

are the only monomial ones.

PROOF. We follow the proof from the affine case, from chapter 13 above.

Let \mathcal{R}_σ be the collection of relations associated to a permutation $\sigma \in S_k$ with $k \in 2\mathbb{N}$, as in Definition 15.11. We fix a monomial projective space $P \subset P_+^{N-1}$, and we associate to it subsets $G_k \subset S_k$, as follows:

$$G_k = \begin{cases} \{\sigma \in S_k \mid \mathcal{R}_\sigma \text{ hold over } P\} & (k \text{ even}) \\ \{\sigma \in S_k \mid \mathcal{R}_{|\sigma} \text{ hold over } P\} & (k \text{ odd}) \end{cases}$$

As in the affine case, we obtain in this way a filtered group $G = (G_k)$, which is stable under removing outer strings, and under removing neighboring strings.

Thus the computations in chapter 13 apply, and show that we have only 3 possible situations, corresponding to the 3 projective spaces in Proposition 15.12 above. \square

Let us discuss now similar results for the projective quantum groups. Given a closed subgroup $G \subset O_N^+$, its projective version $G \rightarrow PG$ is by definition given by the fact that $C(PG) \subset C(G)$ is the subalgebra generated by the following variables:

$$w_{ij,ab} = u_{ia}u_{jb}$$

In the classical case we recover in this way the usual projective version:

$$PG = G/(G \cap \mathbb{Z}_2^N)$$

Let us discuss now the analogues of the classification results in chapter 13, for the quantum groups introduced above. First, we have the following key result, from [18]:

THEOREM 15.14. *We have the following results:*

- (1) *The group inclusion $\mathbb{T}O_N \subset U_N$ is maximal.*
- (2) *The group inclusion $PO_N \subset PU_N$ is maximal.*
- (3) *The quantum group inclusion $O_N \subset O_N^*$ is maximal.*

PROOF. The idea here is as follows:

- (1) This can be obtained by using standard Lie group methods.
- (2) This follows from (1), by taking projective versions.
- (3) This follows from (2), via standard algebraic lifting results.

For details on all this, we refer to [18]. \square

Our claim now is that, under suitable assumptions, O_N^* is the only intermediate object $O_N \subset G \subset O_N^+$, and PU_N is the only intermediate object $PO_N \subset G \subset PO_N^+$.

In order to formulate a precise statement here, we recall the following notion, from [33], that we already met on several occasions, in the above:

DEFINITION 15.15. *An intermediate compact quantum group*

$$O_N \subset G \subset O_N^+$$

is called easy when the corresponding Tannakian category

$$\text{span}(NC_2(k, l)) \subset \text{Hom}(u^{\otimes k}, u^{\otimes l}) \subset \text{span}(P_2(k, l))$$

comes via the following formula, using the standard $\pi \rightarrow T_\pi$ construction,

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{span}(D(k, l))$$

from a certain collection of sets of pairings $D = (D(k, l))$.

As explained in [33], by “saturating” the sets $D(k, l)$, we can assume that the collection $D = (D(k, l))$ is a category of pairings, in the sense that it is stable under vertical and horizontal concatenation, upside-down turning, and contains the semicircle. See [33].

In the projective case now, we have the following related definition:

DEFINITION 15.16. *A projective category of pairings is a collection of subsets*

$$NC_2(2k, 2l) \subset E(k, l) \subset P_2(2k, 2l)$$

stable under the usual categorical operations, and satisfying:

$$\sigma \in E \implies |\sigma| \in E$$

As basic examples here, we have the following projective categories of pairings, where P_2^* is the category of matching pairings:

$$NC_2 \subset P_2^* \subset P_2$$

This follows indeed from definitions. Now with the above notion in hand, we can formulate the following projective analogue of Definition 15.15:

DEFINITION 15.17. *An intermediate compact quantum group*

$$PO_N \subset H \subset PO_N^+$$

is called projectively easy when its Tannakian category

$$\text{span}(NC_2(2k, 2l)) \subset \text{Hom}(v^{\otimes k}, v^{\otimes l}) \subset \text{span}(P_2(2k, 2l))$$

comes via via the following formula, using the standard $\pi \rightarrow T_\pi$ construction,

$$\text{Hom}(v^{\otimes k}, v^{\otimes l}) = \text{span}(E(k, l))$$

for a certain projective category $E = (E(k, l))$.

Observe that, given any easy quantum group $O_N \subset G \subset O_N^+$, its projective version $PO_N \subset PG \subset PO_N^+$ is projectively easy in our sense.

In particular the quantum groups $PO_N \subset PU_N \subset PO_N^+$ are all projectively easy, coming from $NC_2 \subset P_2^* \subset P_2$.

We have in fact the following general result, from [29]:

THEOREM 15.18. *We have a bijective correspondence between the affine and projective categories of partitions, given by*

$$G \rightarrow PG$$

at the quantum group level.

PROOF. The construction of correspondence $D \rightarrow E$ is clear, simply by setting:

$$E(k, l) = D(2k, 2l)$$

Indeed, due to the axioms for the categories of partitions, from [33], the conditions in Definition 15.16 are satisfied.

Conversely, given $E = (E(k, l))$ as in Definition 15.16, we can set:

$$D(k, l) = \begin{cases} E(k, l) & (k, l \text{ even}) \\ \{\sigma : |\sigma \in E(k+1, l+1)\} & (k, l \text{ odd}) \end{cases}$$

Our claim is that $D = (D(k, l))$ is a category of partitions. Indeed:

(1) The composition action is clear. Indeed, when looking at the numbers of legs involved, in the even case this is clear, and in the odd case, this follows from:

$$\begin{aligned} |\sigma, |\sigma' \in E &\implies |\sigma_\tau \in E \\ &\implies \sigma_\tau \in D \end{aligned}$$

(2) For the tensor product axiom, we have 4 cases to be investigated, depending on the parity of the number of legs of σ, τ , as follows:

– The even/even case is clear.

– The odd/even case follows from the following computation:

$$\begin{aligned} |\sigma, \tau \in E &\implies |\sigma\tau \in E \\ &\implies \sigma\tau \in D \end{aligned}$$

– Regarding now the even/odd case, this can be solved as follows:

$$\begin{aligned} \sigma, |\tau \in E &\implies |\sigma|, |\tau \in E \\ &\implies |\sigma||\tau \in E \\ &\implies |\sigma\tau \in E \\ &\implies \sigma\tau \in D \end{aligned}$$

– As for the remaining odd/odd case, here the computation is as follows:

$$\begin{aligned} |\sigma, |\tau \in E &\implies ||\sigma|, |\tau \in E \\ &\implies ||\sigma||\tau \in E \\ &\implies \sigma\tau \in E \\ &\implies \sigma\tau \in D \end{aligned}$$

(3) Finally, the conjugation axiom is clear from definitions.

It is clear that both compositions $D \rightarrow E \rightarrow D$ and $E \rightarrow D \rightarrow E$ are the identities, as claimed. As for the quantum group assertion, this is clear as well. \square

We refer to [29] for further details, and comments on the above correspondence.

Now back to uniqueness issues, we have here the following result, also from [29]:

THEOREM 15.19. *We have the following results:*

(1) O_N^* is the only intermediate easy quantum group, as follows:

$$O_N \subset G \subset O_N^+$$

(2) PU_N is the only intermediate projectively easy quantum group, as follows:

$$PO_N \subset G \subset PO_N^+$$

PROOF. The idea here is as follows:

(1) The assertion regarding $O_N \subset O_N^* \subset O_N^+$ is from [34], and this is something that we already know, explained in section 11 above.

(2) The assertion regarding $PO_N \subset PU_N \subset PO_N^+$ follows from the classification result in (1), and from the duality in Theorem 15.18. \square

Summarizing, we have analogues of the various affine classification results, with the remark that everything becomes simpler in the projective setting.

In view of this, our next goal will be that of finding projective versions of the quantum isometry group results that we have in the affine setting.

15d. Quantum isometries

Let us discuss now the relation between the projective spaces and the projective orthogonal groups, with quantum isometry group computations, and then axiomatization questions for the projective quadruplets that we have.

We use the following action formalism, in the projective setting, which is quite similar to the affine action formalism introduced in chapter 3 above:

DEFINITION 15.20. Consider a closed subgroup of the free orthogonal group, $G \subset O_N^+$, and a closed subset of the free real sphere, $X \subset S_{\mathbb{R},+}^{N-1}$.

(1) We write $G \curvearrowright X$ when the formula

$$\Phi(z_i) = \sum_a z_a \otimes u_{ai}$$

defines a morphism of C^* -algebras, as follows:

$$\Phi : C(X) \rightarrow C(X) \otimes C(G)$$

(2) We write $PG \curvearrowright PX$ when the formula

$$\Phi(z_i z_j) = \sum_a z_a z_b \otimes u_{ai} u_{bj}$$

defines a morphism of C^* -algebras, as follows:

$$\Phi : C(PX) \rightarrow C(PX) \otimes C(PG)$$

Observe that the above morphisms Φ , if they exist, are automatically coaction maps. Observe also that an affine action $G \curvearrowright X$ produces a projective action $PG \curvearrowright PX$.

Finally, let us mention that given an algebraic subset $X \subset S_{\mathbb{R},+}^{N-1}$, it is routine to prove that there exist universal quantum groups $G \subset O_N^+$ acting as (1), and as in (2).

We have the following result, with respect to the above notions:

THEOREM 15.21. The quantum isometry groups of the basic real spheres and projective spaces, namely

$$\begin{array}{ccccc} S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},+}^{N-1} \\ \downarrow & & \downarrow & & \downarrow \\ P_{\mathbb{R}}^{N-1} & \longrightarrow & P_{\mathbb{C}}^{N-1} & \longrightarrow & P_+^{N-1} \end{array}$$

are the following affine and projective quantum groups,

$$\begin{array}{ccccc} O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \\ \downarrow & & \downarrow & & \downarrow \\ PO_N & \longrightarrow & PU_N & \longrightarrow & PO_N^+ \end{array}$$

with respect to the affine and projective action notions introduced above.

PROOF. The fact that the 3 quantum groups on top act affinely on the corresponding 3 spheres is known since [28], and is elementary. By restriction, the 3 quantum groups on the bottom follow to act on the corresponding 3 projective spaces.

We must prove now that all these actions are universal. At right there is nothing to prove, so we are left with studying the actions on $S_{\mathbb{R}}^{N-1}$, $S_{\mathbb{R},*}^{N-1}$ and on $P_{\mathbb{R}}^{N-1}$, $P_{\mathbb{C}}^{N-1}$.

$P_{\mathbb{R}}^{N-1}$. Consider the following projective coordinates:

$$w_{ij,ab} = u_{ai}u_{bj}$$

$$p_{ij} = z_i z_j$$

In terms of these projective coordinates, the coaction map is given by:

$$\Phi(p_{ij}) = \sum_{ab} p_{ab} \otimes w_{ij,ab}$$

Thus, we have the following formulae:

$$\Phi(p_{ij}) = \sum_{a<b} p_{ab} \otimes (w_{ij,ab} + w_{ij,ba}) + \sum_a p_{aa} \otimes w_{ij,aa}$$

$$\Phi(p_{ji}) = \sum_{a<b} p_{ab} \otimes (w_{ji,ab} + w_{ji,ba}) + \sum_a p_{aa} \otimes w_{ji,aa}$$

By comparing these two formulae, and then by using the linear independence of the variables $p_{ab} = z_a z_b$ for $a \leq b$, we conclude that we must have:

$$w_{ij,ab} + w_{ij,ba} = w_{ji,ab} + w_{ji,ba}$$

Let us apply now the antipode to this formula. For this purpose, observe that:

$$\begin{aligned} S(w_{ij,ab}) &= S(u_{ai}u_{bj}) \\ &= S(u_{bj})S(u_{ai}) \\ &= u_{jb}u_{ia} \\ &= w_{ba,ji} \end{aligned}$$

Thus by applying the antipode we obtain:

$$w_{ba,ji} + w_{ab,ji} = w_{ba,ij} + w_{ab,ij}$$

By relabelling, we obtain the following formula:

$$w_{ji,ba} + w_{ij,ba} = w_{ji,ab} + w_{ij,ab}$$

Now by comparing with the original relation, we obtain:

$$w_{ij,ab} = w_{ji,ba}$$

But, with $w_{ij,ab} = u_{ai}u_{bj}$, this formula reads:

$$u_{ai}u_{bj} = u_{bj}u_{ai}$$

Thus our quantum group $G \subset O_N^+$ must be classical:

$$G \subset O_N$$

It follows that we have $PG \subset PO_N$, as claimed.

P_C^{N-1} . Consider a coaction map, written as follows, with $p_{ab} = z_a \bar{z}_b$:

$$\Phi(p_{ij}) = \sum_{ab} p_{ab} \otimes u_{ai} u_{bj}$$

The idea here will be that of using the following formula:

$$p_{ab} p_{cd} = p_{ad} p_{cb}$$

We have the following formulae:

$$\begin{aligned} \Phi(p_{ij} p_{kl}) &= \sum_{abcd} p_{ab} p_{cd} \otimes u_{ai} u_{bj} u_{ck} u_{dl} \\ \Phi(p_{il} p_{kj}) &= \sum_{abcd} p_{ad} p_{cb} \otimes u_{ai} u_{dl} u_{ck} u_{bj} \end{aligned}$$

The terms at left being equal, and the last terms at right being equal too, we deduce that, with $[a, b, c] = abc - cba$, we must have the following formula:

$$\sum_{abcd} u_{ai} [u_{bj}, u_{ck}, u_{dl}] \otimes p_{ab} p_{cd} = 0$$

Now since the quantities $p_{ab} p_{cd} = z_a \bar{z}_b z_c \bar{z}_d$ at right depend only on the numbers $|\{a, c\}|, |\{b, d\}| \in \{1, 2\}$, and this dependence produces the only possible linear relations between the variables $p_{ab} p_{cd}$, we are led to $2 \times 2 = 4$ equations, as follows:

- (1) $u_{ai} [u_{bj}, u_{ak}, u_{bl}] = 0, \forall a, b.$
- (2) $u_{ai} [u_{bj}, u_{ak}, u_{dl}] + u_{ai} [u_{dj}, u_{ak}, u_{bl}] = 0, \forall a, \forall b \neq d.$
- (3) $u_{ai} [u_{bj}, u_{ck}, u_{bl}] + u_{ci} [u_{bj}, u_{ak}, u_{bl}] = 0, \forall a \neq c, \forall b.$
- (4) $u_{ai} [u_{bj}, u_{ck}, u_{dl}] + u_{ai} [u_{dj}, u_{ck}, u_{bl}] + u_{ci} [u_{bj}, u_{ak}, u_{dl}] + u_{ci} [u_{dj}, u_{ak}, u_{bl}] = 0, \forall a \neq c, \forall b \neq d.$

We will need in fact only the first two formulae. Since (1) corresponds to (2) at $b = d$, we conclude that (1,2) are equivalent to (2), with no restriction on the indices. By multiplying now this formula to the left by u_{ai} , and then summing over i , we obtain:

$$[u_{bj}, u_{ak}, u_{dl}] + [u_{dj}, u_{ak}, u_{bl}] = 0$$

We use now the antipode/relabel trick from [38]. By applying the antipode we obtain:

$$[u_{ld}, u_{ka}, u_{jb}] + [u_{lb}, u_{ka}, u_{jd}] = 0$$

By relabelling we obtain the following formula:

$$[u_{dl}, u_{ak}, u_{bj}] + [u_{dj}, u_{ak}, u_{bl}] = 0$$

Now by comparing with the original relation, we obtain:

$$[u_{bj}, u_{ak}, u_{dl}] = [u_{dj}, u_{ak}, u_{bl}] = 0$$

Thus our quantum group is half-classical:

$$G \subset O_N^*$$

It follows that we have $PG \subset PU_N$, and we are done. □

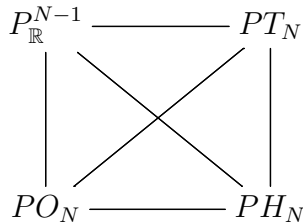
The above results can be probably improved. As an example, let us say that a closed subgroup $G \subset U_N^+$ acts projectively on PX when we have a coaction map as follows:

$$\Phi(z_i z_j) = \sum_{ab} z_a z_b \otimes u_{ai} u_{bj}^*$$

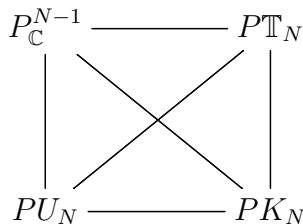
The above proof can be adapted, by putting $*$ signs where needed, and Theorem 15.21 still holds, under this general formalism. However, establishing general universality results, involving arbitrary subgroups $H \subset PO_N^+$, looks like a quite non-trivial question.

Let us discuss now the axiomatization question for the projective quadruplets of type (P, PT, PU, PK) , in the spirit of the axiomatization from chapter 4 above.

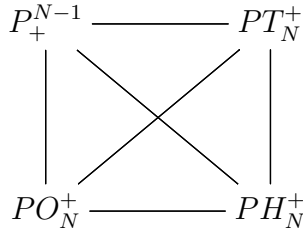
We recall that we first have a classical real quadruplet, as follows:



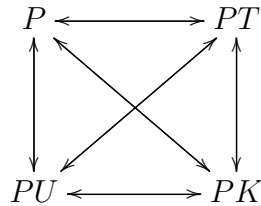
We have then a classical complex quadruplet, which can be thought of as well as being a real half-classical quadruplet, which is as follows:



Finally, we have a free quadruplet, which can be thought of as being the same time real and complex, which is as follows:



In analogy with what happens in the affine case, the problem is that of axiomatizing these geometries, with correspondences as follows:



Modulo this problem, which is for the moment open, things are potentially quite nice, because we seem to have only 3 geometries, namely real, complex and free.

Generally speaking, we are led here into several questions:

- (1) We first need functoriality results for the operations $<$, $>$ and \cap , in relation with taking the projective version, and taking affine lifts, as to deduce most of our 7 axioms, in their obvious projective formulation, from the affine ones.
- (2) Then, we need quantum isometry results in the projective setting, for the projective spaces themselves, and for the projective tori, either established ad-hoc, or by using the affine results. For the projective spaces, this was done above.
- (3) We need as well some further functoriality results, in order to axiomatize the intermediate objects that we are dealing, the problem here being whether we want to use projective objects, or projective versions, perhaps saturated, of affine objects.

In short, we need functoriality results a bit everywhere, in connection with the various questions to be solved. Modulo this, things are quite clear, with the final result being the fact that we have indeed only 3 projective geometries, in analogy with the fact that we have only 3 geometries. Technically, the proof should be using the fact that, in the easy setting, $PO_N \subset PU_N \subset PO_N^+$ are the only possible unitary groups.

Let us also mention that, in the noncommutative setting, there are several ways of defining the projective versions, with the one used here being the “simplest”. As explained in [8], [17], it is possible to construct a left projective version, a right projective version,

and a mixed projective version, with all these operations being interesting. Thus, the results and problems presented above are just the “tip of the iceberg”, with the general projective space and version problematics being much wider than this.

Yet another question concerns the study of the projective spaces associated to the twisted spheres, from chapter 13 above, and to the intersections studied in chapter 14.

Finally, at a more concrete level, the question of developing these projective geometries, and notably the free one, remains open, and extremely interesting.

In addition to the above, there is of course a lot of material which can be “imported” from the affine setting. However, at the genuine projective geometry level nothing much is known, passed a handful of quantum group results.

15e. Exercises

Projective noncommutative geometry is virtually as big as affine noncommutative geometry, and there are countless questions, in relation with the material presented above. Let us start with quantum group aspects. We first have the following exercise:

EXERCISE 15.22. *Prove that we have the free complexification formula*

$$\widetilde{O}_N^+ = U_N^+$$

directly, without character computations and free probability.

This was something that was already discussed in the above, the idea being that of comparing the corresponding Hom spaces, obtained via easiness.

Some interesting things appear at $N = 2$, where we have:

EXERCISE 15.23. *Prove that we have the free complexification formula*

$$\widetilde{SU}_2 = U_2^+$$

and then look for analogues of this formula, at arbitrary $N \in \mathbb{N}$.

Here the first formula can be established directly, say with character computations, but if you want to solve the second question as well, things become more complicated. As a hint, try first to develop of theory of “free symplectic groups”.

In relation now with the projective quantum groups, we first have:

EXERCISE 15.24. *Prove that we have the formula*

$$PO_N^+ = PU_N^+$$

directly, without using the free complexification results.

As before, this was something that was already discussed in the above, the idea being that of comparing the corresponding Hom spaces, obtained via easiness.

A fourth and final exercise on the subject, we have:

EXERCISE 15.25. *Prove that we have the formula*

$$PU_2^+ = SO_3$$

and then look for analogues of this, at arbitrary $N \in \mathbb{N}$.

As before, the first assertion is quite elementary, but if you want to solve the second question as well, things become more complicated, and you are led into free symplectic groups, and their easiness type property, which is not exactly the usual easiness.

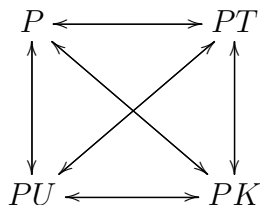
Here is now an instructive exercise, in connection with the projective spaces:

EXERCISE 15.26. *Try developing a theory of real and complex free projective spaces $P_{\mathbb{R},+}^{N-1}$ and $P_{\mathbb{C},+}^{N-1}$, and explain what fails.*

This is something quite philosophical, that we briefly discussed in the above, and in view of the importance of all this, some time spent on all this is golden.

Here is now a more difficult exercise, quite philosophical as well:

EXERCISE 15.27. *Find axioms for the projective quadruplets*



covering the real, complex and free quadruplets, constructed above.

This is something that we already discussed in the above, and good luck.

In addition to all this, there are many interesting questions regarding the development of free projective geometry, that were partly discussed in the above. The whole area is obviously very wide, and interesting, and anything here would be welcome.

CHAPTER 16

Hyperspherical laws

16a. Calculus

We discuss in this final chapter a number of more advanced results, mixing algebra, geometry, analysis and probability, twisted and untwisted objects, affine and projective manifolds, and many more. At the core of all this will be a subtle twisting result, relating the free projective orthogonal group PO_N^+ and the quantum permutation group $S_{N^2}^+$.

We believe that all this material should be relevant to certain questions in statistical mechanics and quantum physics, but all this is quite folklore, and nothing much is known here for the moment. We will comment on this at the end of this chapter.

We follow the papers [14], [20], [22], [23], [24], [28], [30], where these results were found. As a starting point, we have the very natural question, first investigated in [28], of computing the laws of individual coordinates of the main 3 real spheres, namely:

$$S_{\mathbb{R}}^{N-1} \subset S_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{R},+}^{N-1}$$

We already know from chapter 5 above the $N \rightarrow \infty$ behavior of these laws, called “hyperspherical”. To be more precise, for $S_{\mathbb{R}}^{N-1}$ we obtain the normal law, and for $S_{\mathbb{R},+}^{N-1}$ we obtain the semicircle law. As for the sphere $S_{\mathbb{R},*}^{N-1}$, this has the same projective version as $S_{\mathbb{C}}^{N-1}$, where the corresponding law becomes complex Gaussian with $N \rightarrow \infty$, as explained in chapter 5, and so we obtain a symmetrized Rayleigh variable. See [28].

The problem that we want to investigate in this chapter, and that will bring us into a lot of interesting mathematics, and in particular into advanced calculus computations, is that of computing these hyperspherical laws at fixed values of $N \in \mathbb{N}$.

Let us begin with a full discussion in the classical case.

At $N = 2$ the sphere is the unit circle \mathbb{T} , and with $z = e^{it}$ the coordinates are:

$$x = \cos t$$

$$y = \sin t$$

The integrals of products of such coordinates can be computed as follows:

THEOREM 16.1. *We have the following formula,*

$$\int_0^{\pi/2} \cos^p t \sin^q t dt = \left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)} \frac{p!!q!!}{(p+q+1)!!}$$

where $\varepsilon(p) = 1$ if p is even, and $\varepsilon(p) = 0$ if p is odd, and where

$$m!! = (m-1)(m-3)(m-5)\dots$$

with the product ending at 2 if m is odd, and ending at 1 if m is even.

PROOF. This is standard calculus, with particular cases of this formula being very familiar to everyone loving and teaching calculus, as we all should. Let us set:

$$I_p = \int_0^{\pi/2} \cos^p t dt$$

We compute I_p by partial integration. We have the following formula:

$$\begin{aligned} & (\cos^p t \sin t)' \\ = & p \cos^{p-1} t (-\sin t) \sin t + \cos^p t \cos t \\ = & p \cos^{p+1} t - p \cos^{p-1} t + \cos^{p+1} t \\ = & (p+1) \cos^{p+1} t - p \cos^{p-1} t \end{aligned}$$

By integrating between 0 and $\pi/2$, we obtain the following formula:

$$(p+1)I_{p+1} = pI_{p-1}$$

Thus we can compute I_p by recurrence, and we obtain:

$$\begin{aligned} I_p &= \frac{p-1}{p} I_{p-2} \\ &= \frac{p-1}{p} \cdot \frac{p-3}{p-2} I_{p-4} \\ &= \frac{p-1}{p} \cdot \frac{p-3}{p-2} \cdot \frac{p-5}{p-4} I_{p-6} \\ &\quad \vdots \\ &= \frac{p!!}{(p+1)!!} I_{1-\varepsilon(p)} \end{aligned}$$

Together with $I_0 = \frac{\pi}{2}$ and $I_1 = 1$, which are both clear, we obtain:

$$I_p = \left(\frac{\pi}{2}\right)^{\varepsilon(p)} \frac{p!!}{(p+1)!!}$$

Summarizing, we have proved the following formula, with one equality coming from the above computation, and with the other equality coming from this, via $t = \frac{\pi}{2} - s$:

$$\int_0^{\pi/2} \cos^p t \, dt = \int_0^{\pi/2} \sin^p t \, dt = \left(\frac{\pi}{2}\right)^{\varepsilon(p)} \frac{p!!}{(p+1)!!}$$

In relation with the formula in the statement, we are therefore done with the case $p = 0$ or $q = 0$. Let us investigate now the general case. We must compute:

$$I_{pq} = \int_0^{\pi/2} \cos^p t \sin^q t \, dt$$

In order to do the partial integration, observe that we have:

$$\begin{aligned} & (\cos^p t \sin^q t)' \\ &= p \cos^{p-1} t (-\sin t) \sin^q t \\ &+ \cos^p t \cdot q \sin^{q-1} t \cos t \\ &= -p \cos^{p-1} t \sin^{q+1} t + q \cos^{p+1} t \sin^{q-1} t \end{aligned}$$

By integrating between 0 and $\pi/2$, we obtain, for $p, q > 0$:

$$pI_{p-1, q+1} = qI_{p+1, q-1}$$

Thus, we can compute I_{pq} by recurrence. When q is even we have:

$$\begin{aligned} I_{pq} &= \frac{q-1}{p+1} I_{p+2, q-2} \\ &= \frac{q-1}{p+1} \cdot \frac{q-3}{p+3} I_{p+4, q-4} \\ &= \frac{q-1}{p+1} \cdot \frac{q-3}{p+3} \cdot \frac{q-5}{p+5} I_{p+6, q-6} \\ &= \vdots \\ &= \frac{p!!q!!}{(p+q)!!} I_{p+q} \end{aligned}$$

But the last term was already computed above, and we obtain the result:

$$\begin{aligned} I_{pq} &= \frac{p!!q!!}{(p+q)!!} I_{p+q} \\ &= \frac{p!!q!!}{(p+q)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(p+q)} \frac{(p+q)!!}{(p+q+1)!!} \\ &= \left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)} \frac{p!!q!!}{(p+q+1)!!} \end{aligned}$$

Observe that this gives the result for p even as well, by symmetry. Indeed, we have $I_{pq} = I_{qp}$, by using the following change of variables:

$$t = \frac{\pi}{2} - s$$

In the remaining case now, where both p, q are odd, we can use once again the formula $pI_{p-1, q+1} = qI_{p+1, q-1}$ established above, and the recurrence goes as follows:

$$\begin{aligned} I_{pq} &= \frac{q-1}{p+1} I_{p+2, q-2} \\ &= \frac{q-1}{p+1} \cdot \frac{q-3}{p+3} I_{p+4, q-4} \\ &= \frac{q-1}{p+1} \cdot \frac{q-3}{p+3} \cdot \frac{q-5}{p+5} I_{p+6, q-6} \\ &= \vdots \\ &= \frac{p!!q!!}{(p+q-1)!!} I_{p+q-1, 1} \end{aligned}$$

In order to compute the last term, observe that we have:

$$\begin{aligned} I_{p1} &= \int_0^{\pi/2} \cos^p t \sin t \, dt \\ &= -\frac{1}{p+1} \int_0^{\pi/2} (\cos^{p+1} t)' \, dt \\ &= \frac{1}{p+1} \end{aligned}$$

Thus, we can finish our computation in the case p, q odd, as follows:

$$\begin{aligned} I_{pq} &= \frac{p!!q!!}{(p+q-1)!!} I_{p+q-1, 1} \\ &= \frac{p!!q!!}{(p+q-1)!!} \cdot \frac{1}{p+q} \\ &= \frac{p!!q!!}{(p+q+1)!!} \end{aligned}$$

Thus, we obtain the formula in the statement, the exponent of $\pi/2$ appearing there being $\varepsilon(p)\varepsilon(q) = 0 \cdot 0 = 0$ in the present case, and this finishes the proof. \square

16b. More calculus

In order to discuss the higher spheres, we will use spherical coordinates:

THEOREM 16.2. *We have spherical coordinates in N dimensions,*

$$\begin{cases} x_1 &= r \cos t_1 \\ x_2 &= r \sin t_1 \cos t_2 \\ \vdots & \\ x_{N-1} &= r \sin t_1 \sin t_2 \dots \sin t_{N-2} \cos t_{N-1} \\ x_N &= r \sin t_1 \sin t_2 \dots \sin t_{N-2} \sin t_{N-1} \end{cases}$$

the corresponding Jacobian being given by the following formula:

$$J(r, t) = r^{N-1} \sin^{N-2} t_1 \sin^{N-3} t_2 \dots \sin^2 t_{N-3} \sin t_{N-2}$$

PROOF. The fact that we have spherical coordinates as in the statement is clear.

Regarding now the Jacobian, the proof is similar to the one from 2 or 3 dimensions, by developing the determinant over the last column, and then by proceeding by recurrence. Indeed, by developing, we have:

$$\begin{aligned} J_N &= r \sin t_1 \dots \sin t_{N-2} \sin t_{N-1} \times \sin t_{N-1} J_{N-1} \\ &+ r \sin t_1 \dots \sin t_{N-2} \cos t_{N-1} \times \cos t_{N-1} J_{N-1} \\ &= r \sin t_1 \dots \sin t_{N-2} (\sin^2 t_{N-1} + \cos^2 t_{N-1}) J_{N-1} \\ &= r \sin t_1 \dots \sin t_{N-2} J_{N-1} \end{aligned}$$

Thus, we obtain the formula in the statement, by recurrence. \square

With the above results in hand, we can now compute arbitrary polynomial integrals, over the spheres of arbitrary dimension, the result being is as follows:

THEOREM 16.3. *The spherical integral of $x_{i_1} \dots x_{i_k}$ vanishes, unless each index $a \in \{1, \dots, N\}$ appears an even number of times in the sequence i_1, \dots, i_k . We have*

$$\int_{S_{\mathbb{R}}^{N-1}} x_{i_1} \dots x_{i_k} dx = \frac{(N-1)!! l_1!! \dots l_N!!}{(N + \sum l_i - 1)!!}$$

with l_a being this number of occurrences.

PROOF. First, the result holds indeed at $N = 2$, due to the following formula proved above, where $\varepsilon(p) = 1$ when $p \in \mathbb{N}$ is even, and $\varepsilon(p) = 0$ when p is odd:

$$\int_0^{\pi/2} \cos^p t \sin^q t dt = \left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)} \frac{p!!q!!}{(p+q+1)!!}$$

In general, we can restrict attention to the case $l_a \in 2\mathbb{N}$, since the other integrals vanish. The integral in the statement can be written in spherical coordinates, as follows:

$$I = \frac{2^N}{V} \int_0^{\pi/2} \dots \int_0^{\pi/2} x_1^{l_1} \dots x_N^{l_N} J dt_1 \dots dt_{N-1}$$

In this formula, indeed:

– V is the volume of the sphere.

– J is the Jacobian.

– The 2^N factor comes from the restriction to the $1/2^N$ part of the sphere where all the coordinates are positive.

The normalization constant in front of the integral is:

$$\begin{aligned} \frac{2^N}{V} &= \frac{2^N}{N\pi^{N/2}} \cdot \Gamma\left(\frac{N}{2} + 1\right) \\ &= \left(\frac{2}{\pi}\right)^{[N/2]} (N-1)!! \end{aligned}$$

As for the unnormalized integral, this is given by:

$$\begin{aligned} I' &= \int_0^{\pi/2} \dots \int_0^{\pi/2} (\cos t_1)^{l_1} (\sin t_1 \cos t_2)^{l_2} \\ &\quad \vdots \\ &\quad (\sin t_1 \sin t_2 \dots \sin t_{N-2} \cos t_{N-1})^{l_{N-1}} \\ &\quad (\sin t_1 \sin t_2 \dots \sin t_{N-2} \sin t_{N-1})^{l_N} \\ &\quad \sin^{N-2} t_1 \sin^{N-3} t_2 \dots \sin^2 t_{N-3} \sin t_{N-2} \\ &\quad dt_1 \dots dt_{N-1} \end{aligned}$$

By rearranging the terms, we obtain:

$$\begin{aligned} I' &= \int_0^{\pi/2} \cos^{l_1} t_1 \sin^{l_2+\dots+l_N+N-2} t_1 dt_1 \\ &\quad \int_0^{\pi/2} \cos^{l_2} t_2 \sin^{l_3+\dots+l_N+N-3} t_2 dt_2 \\ &\quad \vdots \\ &\quad \int_0^{\pi/2} \cos^{l_{N-2}} t_{N-2} \sin^{l_{N-1}+l_N+1} t_{N-2} dt_{N-2} \\ &\quad \int_0^{\pi/2} \cos^{l_{N-1}} t_{N-1} \sin^{l_N} t_{N-1} dt_{N-1} \end{aligned}$$

Now by using the above-mentioned formula at $N = 2$, this gives:

$$\begin{aligned}
 I' &= \frac{l_1!!(l_2 + \dots + l_N + N - 2)!!}{(l_1 + \dots + l_N + N - 1)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(N-2)} \\
 &\quad \frac{l_2!!(l_3 + \dots + l_N + N - 3)!!}{(l_2 + \dots + l_N + N - 2)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(N-3)} \\
 &\quad \vdots \\
 &\quad \frac{l_{N-2}!!(l_{N-1} + l_N + 1)!!}{(l_{N-2} + l_{N-1} + l_N + 2)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(1)} \\
 &\quad \frac{l_{N-1}!!l_N!!}{(l_{N-1} + l_N + 1)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(0)}
 \end{aligned}$$

Now observe that the various double factorials multiply up to quantity in the statement, modulo a $(N - 1)!!$ factor, and that the $\frac{\pi}{2}$ factors multiply up to:

$$F = \left(\frac{\pi}{2}\right)^{[N/2]}$$

Thus by multiplying with the normalization constant, we obtain the result. \square

In connection now with our probabilistic questions, we have:

THEOREM 16.4. *The even moments of the hyperspherical variables are*

$$\int_{S_{\mathbb{R}}^{N-1}} x_i^k dx = \frac{(N-1)!!k!!}{(N+k-1)!!}$$

and the variables $y_i = x_i/\sqrt{N}$ become normal and independent with $N \rightarrow \infty$.

PROOF. The moment formula in the statement follows from Theorem 16.3. Now observe that with $N \rightarrow \infty$ we have the following estimate:

$$\begin{aligned}
 \int_{S_{\mathbb{R}}^{N-1}} x_i^k dx &= \frac{(N-1)!!}{(N+k-1)!!} \times k!! \\
 &\simeq N^{k/2} \times k!! \\
 &= N^{k/2} M_k(g_1)
 \end{aligned}$$

Thus we have, as claimed, the following asymptotic formula:

$$x_i/\sqrt{N} \sim g_1$$

Finally, the independence assertion follows as well from the formula in Theorem 16.3, via standard probability theory. \square

In the case of the half-classical sphere, we have the following integration result:

THEOREM 16.5. *The half-classical integral of $x_{i_1} \dots x_{i_k}$ vanishes, unless each index a appears the same number of times at odd and even positions in i_1, \dots, i_k . We have*

$$\int_{S_{\mathbb{R},*}^{N-1}} x_{i_1} \dots x_{i_k} dx = 4^{\sum l_i} \frac{(2N-1)! l_1! \dots l_n!}{(2N + \sum l_i - 1)!}$$

where l_a denotes this number of common occurrences.

PROOF. As before, we can assume that k is even, $k = 2l$. The corresponding integral can be viewed as an integral over $S_{\mathbb{C}}^{N-1}$, as follows:

$$I = \int_{S_{\mathbb{C}}^{N-1}} z_{i_1} \bar{z}_{i_2} \dots z_{i_{2l-1}} \bar{z}_{i_{2l}} dz$$

In order to get started, and prove the first assertion, let us apply to this integral transformations of the following type, with $|\lambda| = 1$:

$$p \rightarrow \lambda p$$

We conclude from this that the above integral I vanishes, unless each z_a appears as many times as \bar{z}_a does, and this gives the first assertion.

Assume now that we are in the non-vanishing case. Then the l_a copies of z_a and the l_a copies of \bar{z}_a produce by multiplication a factor $|z_a|^{2l_a}$, so we have:

$$I = \int_{S_{\mathbb{C}}^{N-1}} |z_1|^{2l_1} \dots |z_N|^{2l_N} dz$$

Now by using the standard identification $S_{\mathbb{C}}^{N-1} \simeq S_{\mathbb{R}}^{2N-1}$, we obtain:

$$\begin{aligned} I &= \int_{S_{\mathbb{R}}^{2N-1}} (x_1^2 + y_1^2)^{l_1} \dots (x_N^2 + y_N^2)^{l_N} d(x, y) \\ &= \sum_{r_1 \dots r_N} \binom{l_1}{r_1} \dots \binom{l_N}{r_N} \int_{S_{\mathbb{R}}^{2N-1}} x_1^{2l_1-2r_1} y_1^{2r_1} \dots x_N^{2l_N-2r_N} y_N^{2r_N} d(x, y) \end{aligned}$$

By using the formula in Theorem 16.3, we obtain:

$$\begin{aligned} I &= \sum_{r_1 \dots r_N} \binom{l_1}{r_1} \dots \binom{l_N}{r_N} \frac{(2N-1)!! (2r_1)!! \dots (2r_N)!! (2l_1-2r_1)!! \dots (2l_N-2r_N)!!}{(2N+2\sum l_i-1)!!} \\ &= \sum_{r_1 \dots r_N} \binom{l_1}{r_1} \dots \binom{l_N}{r_N} \frac{(2N-1)! (2r_1)! \dots (2r_N)! (2l_1-2r_1)! \dots (2l_N-2r_N)!}{(2N+\sum l_i-1)! r_1! \dots r_N! (l_1-r_1)! \dots (l_N-r_N)!} \end{aligned}$$

We can rewrite the sum on the right in the following way:

$$\begin{aligned}
& I \\
&= \sum_{r_1 \dots r_N} \frac{l_1! \dots l_N! (2N-1)! (2r_1)! \dots (2r_N)! (2l_1 - 2r_1)! \dots (2l_N - 2r_N)!}{(2N + \sum l_i - 1)! (r_1! \dots r_N! (l_1 - r_1)! \dots (l_N - r_N)!)^2} \\
&= \sum_{r_1} \binom{2r_1}{r_1} \binom{2l_1 - 2r_1}{l_1 - r_1} \dots \sum_{r_N} \binom{2r_N}{r_N} \binom{2l_N - 2r_N}{l_N - r_N} \frac{(2N-1)! l_1! \dots l_N!}{(2N + \sum l_i - 1)!}
\end{aligned}$$

The point now is that the sums on the right can be computed, by using the following well-known formula, whose proof is elementary:

$$\sum_r \binom{2r}{r} \binom{2l - 2r}{l - r} = 4^l$$

Thus the sums on the right in the last formula of I equal respectively $4^{l_1}, \dots, 4^{l_N}$, and this gives the formula in the statement. \square

As before, we can deduce from this a probabilistic result, as follows:

THEOREM 16.6. *The even moments of the half-classical hyperspherical variables are*

$$\int_{S_{\mathbb{R},*}^{N-1}} x_i^k dx = 4^k \frac{(2N-1)! k!}{(2N+k-1)!}$$

and the variables $y_i = x_i/(4N)$ become symmetrized Rayleigh with $N \rightarrow \infty$.

PROOF. The moment formula in the statement follows from Theorem 16.5. Now observe that with $N \rightarrow \infty$ we have the following estimate:

$$\begin{aligned}
\int_{S_{\mathbb{R},*}^{N-1}} x_i^k dx &= 4^k \times \frac{(N-1)!}{(N+k-1)!} \times k! \\
&\simeq 4^k \times N^k \times k! \\
&= (4N)^k M_k(|c|)
\end{aligned}$$

Here c is a standard complex Gaussian variable, and this gives the result. \square

As a comment here, it is possible to prove, based once again on the general integration formula from Theorem 16.5 above, that the rescaled variables $y_i = x_i/(4N)$ become “half-independent” with $N \rightarrow \infty$. For a discussion of the notion of half-independence, and various related topics, we refer to the series of papers [25], [26], [27].

16c. Advanced calculus

In the case of the free sphere now, the computations are substantially more complicated than those in the classical and half-classical cases. Let us start with the following result, that we basically know from chapter 5 above, and that we will recall now:

THEOREM 16.7. *For the free sphere $S_{\mathbb{R},+}^{N-1}$, the rescaled coordinates*

$$y_i = \sqrt{N}x_i$$

become semicircular and free, in the $N \rightarrow \infty$ limit.

PROOF. As explained in chapter 5 above, the Weingarten formula for the free sphere, together with the standard fact that the Gram matrix, and hence the Weingarten matrix too, is asymptotically diagonal, gives the following estimate:

$$\int_{S_{\mathbb{R},+}^{N-1}} x_{i_1} \dots x_{i_k} dx \simeq N^{-k/2} \sum_{\sigma \in NC_2(k)} \delta_{\sigma}(i_1, \dots, i_k)$$

With this formula in hand, we can compute the asymptotic moments of each coordinate x_i . Indeed, by setting $i_1 = \dots = i_k = i$, all Kronecker symbols are 1, and we obtain:

$$\int_{S_{\mathbb{R},+}^{N-1}} x_i^k dx \simeq N^{-k/2} |NC_2(k)|$$

Thus the rescaled coordinates $y_i = \sqrt{N}x_i$ become semicircular in the $N \rightarrow \infty$ limit, as claimed. As for the asymptotic freeness result, this follows as well from the above general joint moment estimate, via standard free probability theory. See [21], [28]. \square

Summarizing, we have good results for the free sphere, with $N \rightarrow \infty$.

The problem now, which is highly non-trivial, is that of computing the moments of the coordinates of the free sphere at fixed values of $N \in \mathbb{N}$.

The answer here, from [23], which is based on advanced quantum group and calculus techniques, that we will briefly explain here, is as follows:

THEOREM 16.8. *The moments of the free hyperspherical law are given by*

$$\int_{S_{\mathbb{R},+}^{N-1}} x_1^{2l} dx = \frac{1}{(N+1)^l} \cdot \frac{q+1}{q-1} \cdot \frac{1}{l+1} \sum_{r=-l-1}^{l+1} (-1)^r \binom{2l+2}{l+r+1} \frac{r}{1+q^r}$$

where $q \in [-1, 0)$ is such that $q + q^{-1} = -N$.

PROOF. This is something quite tricky. Following [23], this will follow in 4 steps, none of which is something trivial, which are as follows:

- (1) $x_1 \in C(S_{\mathbb{R},+}^{N-1})$ has the same law as $u_{11} \in C(O_N^+)$.
- (2) $u_{11} \in C(O_N^+)$ has the same law as a certain variable $w \in C(SU_2^q)$.
- (3) $w \in C(SU_2^q)$ can be in turn modelled by an explicit operator $T \in B(l^2(\mathbb{N}))$.
- (4) The law of $T \in B(l^2(\mathbb{N}))$ can be computed by using advanced calculus.

Let us first explain the relation between O_N^+ and SU_2^q . To any matrix $F \in GL_N(\mathbb{R})$ satisfying $F^2 = 1$ we associate the following universal algebra:

$$C(O_F^+) = C^* \left((u_{ij})_{i,j=1,\dots,N} \mid u = F\bar{u}F = \text{unitary} \right)$$

Observe that $O_{I_N}^+ = O_N^+$. In general, the above algebra satisfies Woronowicz's generalized axioms in [99], which do not include the strong antipode axiom $S^2 = id$.

At $N = 2$, up to a trivial equivalence relation on the matrices F , and on the quantum groups O_F^+ , we can assume that F is as follows, with $q \in [-1, 0)$:

$$F = \begin{pmatrix} 0 & \sqrt{-q} \\ 1/\sqrt{-q} & 0 \end{pmatrix}$$

Our claim is that for this matrix we have:

$$O_F^+ = SU_2^q$$

Indeed, the relations $u = F\bar{u}F$ tell us that u must be of the following special form:

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

Thus $C(O_F^+)$ is the universal algebra generated by two elements α, γ , with the relations making the above matrix u unitary. But these unitarity conditions are:

$$\begin{aligned} \alpha\gamma &= q\gamma\alpha \\ \alpha\gamma^* &= q\gamma^*\alpha \\ \gamma\gamma^* &= \gamma^*\gamma \\ \alpha^*\alpha + \gamma^*\gamma &= 1 \\ \alpha\alpha^* + q^2\gamma\gamma^* &= 1 \end{aligned}$$

We recognize here the relations in [99] defining the algebra $C(SU_2^q)$, and it follows that we have an isomorphism of Hopf C^* -algebras:

$$C(O_F^+) \simeq C(SU_2^q)$$

Now back to the general case, let us try to understand the integration over O_F^+ . Given $\pi \in NC_2(2k)$ and $i = (i_1, \dots, i_{2k})$, we set:

$$\delta_\pi^F(i) = \prod_{s \in \pi} F_{i_{s_l} i_{s_r}}$$

Here the product is over all strings of π , denoted as follows:

$$s = \{s_l \curvearrowright s_r\}$$

Our claim now is that the following family of vectors, with $\pi \in NC_2(2k)$, spans the space of fixed vectors of $u^{\otimes 2k}$:

$$\xi_\pi = \sum_i \delta_\pi^F(i) e_{i_1} \otimes \dots \otimes e_{i_{2k}}$$

Indeed, having ξ_π fixed by $u^{\otimes 2}$ is equivalent to assuming that $u = F\bar{u}F$ is unitary. By using now the above vectors, we obtain the following Weingarten formula:

$$\int_{O_F^+} u_{i_1 j_1} \dots u_{i_{2k} j_{2k}} = \sum_{\pi\sigma} \delta_\pi^F(i) \delta_\sigma^F(j) W_{kN}(\pi, \sigma)$$

With these preliminaries in hand, let us start the computation. Let $N \in \mathbb{N}$, and consider the number $q \in [-1, 0)$ satisfying:

$$q + q^{-1} = -N$$

Our claim is that we have:

$$\int_{O_N^+} \varphi(\sqrt{N+2} u_{ij}) = \int_{SU_2^q} \varphi(\alpha + \alpha^* + \gamma - q\gamma^*)$$

Indeed, the moments of the variable on the left are given by:

$$\int_{O_N^+} u_{ij}^{2k} = \sum_{\pi\sigma} W_{kN}(\pi, \sigma)$$

On the other hand, the moments of the variable on the right, which in terms of the fundamental corepresentation $v = (v_{ij})$ is given by $w = \sum_{ij} v_{ij}$, are given by:

$$\int_{SU_2^q} w^{2k} = \sum_{ij} \sum_{\pi\sigma} \delta_\pi^F(i) \delta_\sigma^F(j) W_{kN}(\pi, \sigma)$$

We deduce that $w/\sqrt{N+2}$ has the same moments as u_{ij} , which proves our claim.

In order to do now the computation over SU_2^q , we can use a matrix model due to Woronowicz [99], where the standard generators α, γ are mapped as follows:

$$\begin{aligned} \pi_u(\alpha) e_k &= \sqrt{1 - q^{2k}} e_{k-1} \\ \pi_u(\gamma) e_k &= u q^k e_k \end{aligned}$$

Here $u \in \mathbb{T}$ is a parameter, and (e_k) is the standard basis of $l^2(\mathbb{N})$. The point with this representation is that it allows the computation of the Haar functional. Indeed, if D is the diagonal operator given by $D(e_k) = q^{2k} e_k$, then the formula is as follows:

$$\int_{SU_2^q} x = (1 - q^2) \int_{\mathbb{T}} tr(D\pi_u(x)) \frac{du}{2\pi i u}$$

With the above model in hand, the law of the variable that we are interested in is of the following form:

$$\int_{SU_2^q} \varphi(\alpha + \alpha^* + \gamma - q\gamma^*) = (1 - q^2) \int_{\mathbb{T}} \text{tr}(D\varphi(M)) \frac{du}{2\pi i u}$$

To be more precise, this formula holds indeed, with:

$$M(e_k) = e_{k+1} + q^k(u - qu^{-1})e_k + (1 - q^{2k})e_{k-1}$$

The point now is that the integral on the right can be computed, by using advanced calculus methods, and this gives the result. We refer here to [23]. \square

The computation of the joint free hyperspherical laws remains an open problem. Open as well is the question of finding a more conceptual proof for the above formula.

16d. Twisting results

Following now [20], let us discuss an interesting relation of all this with the quantum permutations, and with the free hypergeometric laws.

The idea will be that of working out some abstract algebraic results, regarding twists of quantum automorphism groups, which will particularize into results relating quantum rotations and permutations, having no classical counterpart, both at the algebraic and the probabilistic level.

In order to explain this material, from [20], which is quite technical, requiring good algebraic knowledge, let us begin with some generalities. We first have:

DEFINITION 16.9. *A finite quantum space F is the abstract dual of a finite dimensional C^* -algebra B , according to the following formula:*

$$C(F) = B$$

The number of elements of such a space is $|F| = \dim B$. By decomposing the algebra B , we have a formula of the following type:

$$C(F) = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C})$$

With $n_1 = \dots = n_k = 1$ we obtain in this way the space $F = \{1, \dots, k\}$. Also, when $k = 1$ the equation is $C(F) = M_n(\mathbb{C})$, and the solution will be denoted $F = M_n$.

Following [2], we endow each finite quantum space F with its counting measure, corresponding as the algebraic level to the integration functional obtained by applying the regular representation, and then the normalized matrix trace:

$$\text{tr} : C(F) \rightarrow B(l^2(F)) \rightarrow \mathbb{C}$$

As basic examples, for both $F = \{1, \dots, k\}$ and $F = M_n$ we obtain the usual trace. In general, we can write the algebra $C(F)$ as follows:

$$C(F) = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C})$$

In terms of this writing, the weights of tr are as follows:

$$c_i = \frac{n_i^2}{\sum_i n_i^2}$$

Let us study now the quantum group actions $G \curvearrowright F$. It is convenient here to use, in order to get started, the no basis approach from [2].

If we denote by μ, η the multiplication and unit map of the algebra $C(F)$, we have the following result, from [2]:

PROPOSITION 16.10. *Consider a linear map $\Phi : C(F) \rightarrow C(F) \otimes C(G)$, written as*

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

with $\{e_i\}$ being a linear space basis of $C(F)$, orthonormal with respect to tr .

- (1) Φ is a linear space coaction $\iff u$ is a corepresentation.
- (2) Φ is multiplicative $\iff \mu \in \text{Hom}(u^{\otimes 2}, u)$.
- (3) Φ is unital $\iff \eta \in \text{Hom}(1, u)$.
- (4) Φ leaves invariant tr $\iff \eta \in \text{Hom}(1, u^*)$.
- (5) If these conditions hold, Φ is involutive $\iff u$ is unitary.

PROOF. This is a bit similar to the proof for S_N^+ from chapter 2, as follows:

(1) There are two axioms to be processed here. First, we have:

$$\begin{aligned} (id \otimes \Delta)\Phi &= (\Phi \otimes id)\Phi \iff \sum_j e_j \otimes \Delta(u_{ji}) = \sum_k \Phi(e_k) \otimes u_{ki} \\ &\iff \sum_j e_j \otimes \Delta(u_{ji}) = \sum_{jk} e_j \otimes u_{jk} \otimes u_{ki} \\ &\iff \Delta(u_{ji}) = \sum_k u_{jk} \otimes u_{ki} \end{aligned}$$

As for the axiom involving the counit, here we have as well, as desired:

$$\begin{aligned} (id \otimes \varepsilon)\Phi &= id \iff \sum_j \varepsilon(u_{ji})e_j = e_i \\ &\iff \varepsilon(u_{ji}) = \delta_{ji} \end{aligned}$$

(2) We have the following formula:

$$\Phi(e_i) = \left(\sum_{ij} e_{ji} \otimes u_{ji} \right) (e_i \otimes 1) = u(e_i \otimes 1)$$

By using this formula, we obtain the following identity:

$$\Phi(e_i e_k) = u(e_i e_k \otimes 1) = u(\mu \otimes id)(e_i \otimes e_k \otimes 1)$$

On the other hand, we have as well the following identity, as desired:

$$\begin{aligned} \Phi(e_i)\Phi(e_k) &= \sum_{jl} e_j e_l \otimes u_{ji} u_{lk} \\ &= (\mu \otimes id) \sum_{jl} e_j \otimes e_l \otimes u_{ji} u_{lk} \\ &= (\mu \otimes id) \left(\sum_{ijkl} e_{ji} \otimes e_{lk} \otimes u_{ji} u_{lk} \right) (e_i \otimes e_k \otimes 1) \\ &= (\mu \otimes id) u^{\otimes 2} (e_i \otimes e_k \otimes 1) \end{aligned}$$

(3) The formula $\Phi(e_i) = u(e_i \otimes 1)$ found above gives by linearity $\Phi(1) = u(1 \otimes 1)$. But this shows that Φ is unital precisely when $u(1 \otimes 1) = 1 \otimes 1$, as desired.

(4) This follows from the following computation, by applying the involution:

$$\begin{aligned} (tr \otimes id)\Phi(e_i) = tr(e_i)1 &\iff \sum_j tr(e_j)u_{ji} = tr(e_i)1 \\ &\iff \sum_j u_{ji}^* 1_j = 1_i \\ &\iff (u^*1)_i = 1_i \\ &\iff u^*1 = 1 \end{aligned}$$

(5) Assuming that (1-4) are satisfied, and that Φ is involutive, we have:

$$\begin{aligned}
 (u^*u)_{ik} &= \sum_l u_{li}^* u_{lk} \\
 &= \sum_{jl} \text{tr}(e_j^* e_l) u_{ji}^* u_{lk} \\
 &= (\text{tr} \otimes \text{id}) \sum_{jl} e_j^* e_l \otimes u_{ji}^* u_{lk} \\
 &= (\text{tr} \otimes \text{id})(\Phi(e_i)^* \Phi(e_k)) \\
 &= (\text{tr} \otimes \text{id}) \Phi(e_i^* e_k) \\
 &= \text{tr}(e_i^* e_k) 1 \\
 &= \delta_{ik}
 \end{aligned}$$

Thus $u^*u = 1$, and since we know from (1) that u is a corepresentation, it follows that u is unitary. The proof of the converse is standard too, by using similar tricks. \square

Following now [2], we have the following result, extending the basic theory of S_N^+ from chapter 2 to the present finite quantum space setting:

THEOREM 16.11. *Given a finite quantum space F , there is a universal compact quantum group S_F^+ acting on F , leaving the counting measure invariant. We have*

$$C(S_F^+) = C(U_N^+) / \left\langle \mu \in \text{Hom}(u^{\otimes 2}, u), \eta \in \text{Fix}(u) \right\rangle$$

where $N = |F|$ and where μ, η are the multiplication and unit maps of $C(F)$. Also:

- (1) For $F = \{1, \dots, N\}$ we have $S_F^+ = S_N^+$.
- (2) For $F = M_n$ we have $S_F^+ = PO_n^+ = PU_n^+$.

PROOF. Consider a linear map $\Phi : C(F) \rightarrow C(F) \otimes C(G)$, written as follows, with $\{e_i\}$ being a linear space basis of $C(F)$, which is orthonormal with respect to tr :

$$\Phi(e_j) = \sum_i e_i \otimes u_{ij}$$

It is routine to check, via standard algebraic computations, that Φ is a coaction precisely when u is a unitary corepresentation, satisfying the following conditions:

$$\begin{aligned}
 \mu &\in \text{Hom}(u^{\otimes 2}, u) \\
 \eta &\in \text{Fix}(u)
 \end{aligned}$$

But this gives the first assertion. Regarding now the statement about $F = \{1, \dots, N\}$ is clear. Finally, regarding $F = M_2$, here we have embeddings as follows:

$$PO_n^+ \subset PU_n^+ \subset S_F^+$$

Now since the fusion rules of all these 3 quantum groups are known to be the same as the fusion rules for SO_3 , these inclusions are isomorphisms. See [2]. \square

We have as well the following result, from [2]:

THEOREM 16.12. *The quantum groups S_F^+ have the following properties:*

- (1) *The associated Tannakian categories are $TL(N)$, with $N = |F|$.*
- (2) *The main character follows the Marchenko-Pastur law π_1 , when $N \geq 4$.*
- (3) *The fusion rules for S_F^+ with $|F| \geq 4$ are the same as for SO_3 .*

PROOF. This result is from [2], the idea being as follows:

(1) Our first claim is that the fundamental representation is equivalent to its adjoint, $u \sim \bar{u}$. Indeed, let us go back to the coaction formula from Proposition 16.10:

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

We can pick our orthogonal basis $\{e_i\}$ to be the standard multimatrix basis of $C(F)$, so that we have, for a certain involution $i \rightarrow i^*$ on the index set:

$$e_i^* = e_{i^*}$$

With this convention made, by conjugating the above formula of $\Phi(e_i)$, we obtain:

$$\Phi(e_{i^*}) = \sum_j e_{j^*} \otimes u_{ji}^*$$

Now by interchanging $i \leftrightarrow i^*$ and $j \leftrightarrow j^*$, this latter formula reads:

$$\Phi(e_i) = \sum_j e_j \otimes u_{j^*i^*}^*$$

We therefore conclude, by comparing with the original formula, that we have:

$$u_{ji}^* = u_{j^*i^*}$$

But this shows that we have an equivalence as follows, as claimed:

$$u \sim \bar{u}$$

Now with this result in hand, the proof goes as for the proof for S_N^+ , from the previous section. To be more precise, the result follows from the fact that the multiplication and unit of any complex algebra, and in particular of the algebra $C(F)$ that we are interested in here, can be modelled by the following two diagrams:

$$m = \left| \cup \right| \quad , \quad u = \cap$$

Indeed, this is certainly true algebraically, and this is something well-known. As in what regards the $*$ -structure, things here are fine too, because our choice for the trace leads to the following formula, which must be satisfied as well:

$$\mu\mu^* = N \cdot id$$

But the above diagrams m, u generate the Temperley-Lieb algebra $TL(N)$, as stated.

(2) The proof here is exactly as for S_N^+ , by using moments. To be more precise, according to (1) these moments are the Catalan numbers, which are the moments of π_1 .

(3) Once again same proof as for S_N^+ , by using the fact that the moments of χ are the Catalan numbers, which naturally leads to the Clebsch-Gordan rules. \square

Let us discuss now a number of more advanced twisting aspects, which will eventually lead us into probability, and hypergeometric laws.

Following now [20], we have the following result:

PROPOSITION 16.13. *Given a finite group F , the algebra $C(S_F^+)$ is isomorphic to the abstract algebra presented by generators x_{gh} with $g, h \in F$, with the following relations:*

$$\begin{aligned} x_{1g} &= x_{g1} = \delta_{1g} \\ x_{s,gh} &= \sum_{t \in F} x_{st^{-1},g} x_{th} \\ x_{gh,s} &= \sum_{t \in F} x_{gt^{-1}} x_{h,ts} \end{aligned}$$

The comultiplication, counit and antipode are given by the formulae

$$\begin{aligned} \Delta(x_{gh}) &= \sum_{s \in F} x_{gs} \otimes x_{sh} \\ \varepsilon(x_{gh}) &= \delta_{gh} \\ S(x_{gh}) &= x_{h^{-1}g^{-1}} \end{aligned}$$

on the standard generators x_{gh} .

PROOF. This follows indeed from a direct verification, based either on Theorem 16.11 above, or on its equivalent formulation from Wang’s paper [94]. \square

Let us discuss now the twisted version of the above result. Consider a 2-cocycle on F , which is by definition a map $\sigma : F \times F \rightarrow \mathbb{C}^*$ satisfying:

$$\begin{aligned} \sigma_{gh,s} \sigma_{gh} &= \sigma_{g,hs} \sigma_{hs} \\ \sigma_{g1} &= \sigma_{1g} = 1 \end{aligned}$$

Given such a cocycle, we can construct the associated twisted group algebra $C(\widehat{F}_\sigma)$, as being the vector space $C(\widehat{F}) = C^*(F)$, with product as follows:

$$e_g e_h = \sigma_{gh} e_{gh}$$

We have then the following generalization of Proposition 16.13:

PROPOSITION 16.14. *The algebra $C(S_{\widehat{F}_\sigma}^+)$ is isomorphic to the abstract algebra presented by generators x_{gh} with $g, h \in G$, with the relations $x_{1g} = x_{g1} = \delta_{1g}$ and:*

$$\begin{aligned} \sigma_{gh} x_{s,gh} &= \sum_{t \in F} \sigma_{st^{-1},t} x_{st^{-1},g} x_{th} \\ \sigma_{gh}^{-1} x_{gh,s} &= \sum_{t \in F} \sigma_{t^{-1},ts}^{-1} x_{gt^{-1}} x_{h,ts} \end{aligned}$$

The comultiplication, counit and antipode are given by the formulae

$$\begin{aligned} \Delta(x_{gh}) &= \sum_{s \in F} x_{gs} \otimes x_{sh} \\ \varepsilon(x_{gh}) &= \delta_{gh} \\ S(x_{gh}) &= \sigma_{h^{-1}h} \sigma_{g^{-1}g}^{-1} x_{h^{-1}g^{-1}} \end{aligned}$$

on the standard generators x_{gh} .

PROOF. Once again, this follows from a direct verification. Note that by using the cocycle identities we obtain the following formula, needed in the proof:

$$\sigma_{gg^{-1}} = \sigma_{g^{-1}g}$$

Thus we are led to the conclusion in the statement. □

In what follows, as before by following [20], we will prove that the quantum groups $S_{\widehat{F}}^+$ and $S_{\widehat{F}_\sigma}^+$ are related by a cocycle twisting operation.

Let us begin with some preliminaries. Let H be a Hopf algebra. We recall that a left 2-cocycle is a convolution invertible linear map $\sigma : H \otimes H \rightarrow \mathbb{C}$ satisfying:

$$\begin{aligned} \sigma_{x_1 y_1} \sigma_{x_2 y_2, z} &= \sigma_{y_1 z_1} \sigma_{x, y_2 z_2} \\ \sigma_{x1} &= \sigma_{1x} = \varepsilon(x) \end{aligned}$$

Note that σ is a left 2-cocycle if and only if σ^{-1} , the convolution inverse of σ , is a right 2-cocycle, in the sense that we have:

$$\begin{aligned} \sigma_{x_1 y_1, z}^{-1} \sigma_{x_1 y_2}^{-1} &= \sigma_{x, y_1 z_1}^{-1} \sigma_{y_2 z_2}^{-1} \\ \sigma_{x1}^{-1} &= \sigma_{1x}^{-1} = \varepsilon(x) \end{aligned}$$

Given a left 2-cocycle σ on H , one can form the 2-cocycle twist H^σ as follows. As a coalgebra, $H^\sigma = H$, and an element $x \in H$, when considered in H^σ , is denoted $[x]$. The product in H^σ is defined, in Sweedler notation, by:

$$[x][y] = \sum \sigma_{x_1 y_1} \sigma_{x_3 y_3}^{-1} [x_2 y_2]$$

Note that the cocycle condition ensures the fact that we have indeed a Hopf algebra. Note also that the coalgebra isomorphism $H \rightarrow H^\sigma$ given by $x \rightarrow [x]$ commutes with the respective Haar integrals, as soon as H has a Haar integral.

Following [20], we can now state a main twisting theorem, as follows:

THEOREM 16.15. *If F is a finite group and σ is a 2-cocycle on F , the Hopf algebras*

$$C(S_{\widehat{F}}^+) \quad , \quad C(S_{\widehat{F}^\sigma}^+)$$

are 2-cocycle twists of each other, in the above sense.

PROOF. In order to prove this result, we use the following Hopf algebra map:

$$\begin{aligned} \pi : C(S_{\widehat{F}}^+) &\rightarrow C(\widehat{F}) \\ x_{gh} &\rightarrow \delta_{gh} e_g \end{aligned}$$

Our 2-cocycle $\sigma : F \times F \rightarrow \mathbb{C}^*$ can be extended by linearity into a linear map as follows, which is a left and right 2-cocycle in the above sense:

$$\sigma : C(\widehat{F}) \otimes C(\widehat{F}) \rightarrow \mathbb{C}$$

Consider now the following composition:

$$\alpha = \sigma(\pi \otimes \pi) : C(S_{\widehat{F}}^+) \otimes C(S_{\widehat{F}}^+) \rightarrow C(\widehat{F}) \otimes C(\widehat{F}) \rightarrow \mathbb{C}$$

Then α is a left and right 2-cocycle, because it is induced by a cocycle on a group algebra, and so is its convolution inverse α^{-1} . Thus we can construct the twisted algebra $C(S_{\widehat{F}}^+)^{\alpha^{-1}}$, and inside this algebra we have the following computation:

$$\begin{aligned} [x_{gh}][x_{rs}] &= \alpha^{-1}(x_g, x_r) \alpha(x_h, x_s) [x_{gh} x_{rs}] \\ &= \sigma_{gr}^{-1} \sigma_{hs} [x_{gh} x_{rs}] \end{aligned}$$

By using this, we obtain the following formula:

$$\begin{aligned} \sum_{t \in F} \sigma_{st^{-1}, t} [x_{st^{-1}, g}] [x_{th}] &= \sum_{t \in F} \sigma_{st^{-1}, t} \sigma_{st^{-1}, t}^{-1} \sigma_{gh} [x_{st^{-1}, g} x_{th}] \\ &= \sigma_{gh} [x_{s, gh}] \end{aligned}$$

Similarly, we have the following formula:

$$\sum_{t \in F} \sigma_{t^{-1}, ts}^{-1} [x_{g, t^{-1}}] [x_{h, ts}] = \sigma_{gh}^{-1} [x_{gh, s}]$$

We deduce from this that there exists a Hopf algebra map, as follows:

$$\begin{aligned} \Phi : C(S_{\widehat{F}\sigma}^+) &\rightarrow C(S_{\widehat{F}}^+)^{\alpha^{-1}} \\ x_{gh} &\rightarrow [x_{g,h}] \end{aligned}$$

This map is clearly surjective, and is injective as well, by a standard fusion semiring argument, because both Hopf algebras have the same fusion semiring. \square

Summarizing, we have proved our main twisting result. Our purpose in what follows will be that of working out versions and particular cases of it. We first have:

PROPOSITION 16.16. *If F is a finite group and σ is a 2-cocycle on F , then*

$$\Phi(x_{g_1 h_1} \dots x_{g_m h_m}) = \Omega(g_1, \dots, g_m)^{-1} \Omega(h_1, \dots, h_m) x_{g_1 h_1} \dots x_{g_m h_m}$$

with the coefficients on the right being given by the formula

$$\Omega(g_1, \dots, g_m) = \prod_{k=1}^{m-1} \sigma_{g_1 \dots g_k, g_{k+1}}$$

is a coalgebra isomorphism $C(S_{\widehat{F}\sigma}^+) \rightarrow C(S_{\widehat{F}}^+)$, commuting with the Haar integrals.

PROOF. This is indeed just a technical reformulation of Theorem 16.15. \square

Here is another useful result, that we will need in what follows:

THEOREM 16.17. *Let $X \subset F$ be such that $\sigma_{gh} = 1$ for any $g, h \in X$, and consider the subalgebra*

$$B_X \subset C(S_{\widehat{F}\sigma}^+)$$

generated by the elements x_{gh} , with $g, h \in X$. Then we have an injective algebra map

$$\Phi_0 : B_X \rightarrow C(S_{\widehat{F}}^+)$$

given by $x_{g,h} \rightarrow x_{g,h}$.

PROOF. With the notations in the proof of Theorem 16.15, we have the following equality in $C(S_{\widehat{F}}^+)^{\alpha^{-1}}$, for any $g_i, h_i, r_i, s_i \in X$:

$$[x_{g_1 h_1} \dots x_{g_p h_p}] \cdot [x_{r_1 s_1} \dots x_{r_q s_q}] = [x_{g_1 h_1} \dots x_{g_p h_p} x_{r_1 s_1} \dots x_{r_q s_q}]$$

The point now is that Φ_0 can be defined to be the composition of $\Phi|_{B_X}$ with the following linear isomorphism:

$$\begin{aligned} C(S_{\widehat{F}}^+)^{\alpha^{-1}} &\rightarrow C(S_{\widehat{F}}^+) \\ [x] &\rightarrow x \end{aligned}$$

This being clearly an injective algebra map, we obtain the result. \square

Let us discuss now some concrete applications of the general results established above. Consider the group $F = \mathbb{Z}_n^2$, let $w = e^{2\pi i/n}$, and consider the following map:

$$\begin{aligned}\sigma : F \times F &\rightarrow \mathbb{C}^* \\ \sigma_{(ij)(kl)} &= w^{jk}\end{aligned}$$

It is easy to see that σ is a bicharacter, and hence a 2-cocycle on F . Thus, we can apply our general twisting result, to this situation.

In order to understand what is the formula that we obtain, we must do some computations. Let E_{ij} with $i, j \in \mathbb{Z}_n$ be the standard basis of $M_n(\mathbb{C})$. We have:

PROPOSITION 16.18. *The linear map given by*

$$\psi(e_{(i,j)}) = \sum_{k=0}^{n-1} w^{ki} E_{k,k+j}$$

defines an isomorphism of algebras $\psi : C(\widehat{F}_\sigma) \simeq M_n(\mathbb{C})$.

PROOF. Consider indeed the following linear map:

$$\psi'(E_{ij}) = \frac{1}{n} \sum_{k=0}^{n-1} w^{-ik} e_{(k,j-i)}$$

It is routine then to check that ψ, ψ' are inverse morphisms of algebras. □

As a consequence, we have the following result:

PROPOSITION 16.19. *The algebra map given by*

$$\varphi(u_{ij}u_{kl}) = \frac{1}{n} \sum_{a,b=0}^{n-1} w^{ai-bj} x_{(a,k-i),(b,l-j)}$$

defines a Hopf algebra isomorphism $\varphi : C(S_{M_n}^+) \simeq C(S_{\widehat{F}_\sigma}^+)$.

PROOF. We use the identification $C(\widehat{F}_\sigma) \simeq M_n(\mathbb{C})$ from Proposition 16.18. This identification produces a coaction map, as follows:

$$\gamma : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \otimes C(S_{\widehat{F}_\sigma}^+)$$

Now observe that this map is given by the following formula:

$$\gamma(E_{ij}) = \frac{1}{n} \sum_{ab} E_{ab} \otimes \sum_{kr} w^{ar-ik} x_{(r,b-a),(k,j-i)}$$

Thus, we obtain the isomorphism in the statement. □

We will need one more result of this type, as follows:

PROPOSITION 16.20. *The algebra map given by*

$$\rho(x_{(a,b),(i,j)}) = \frac{1}{n^2} \sum_{klrs} w^{ki+lj-ra-sb} p_{(r,s),(k,l)}$$

defines a Hopf algebra isomorphism $\rho : C(S_{\widehat{F}}^+) \simeq C(S_F^+)$.

PROOF. This follows by using the Fourier transform isomorphism over the group F , which is a map as follows:

$$C(\widehat{F}) \simeq C(F)$$

Indeed, by composing with this isomorphism, we obtain the result. □

We can now formulate a concrete twisting result, from [20], as follows:

THEOREM 16.21. *Let $n \geq 2$ and $w = e^{2\pi i/n}$. Then*

$$\Theta(u_{ij}u_{kl}) = \frac{1}{n} \sum_{ab=0}^{n-1} w^{-a(k-i)+b(l-j)} p_{ia,jb}$$

defines a coalgebra isomorphism

$$C(PO_n^+) \rightarrow C(S_{n^2}^+)$$

commuting with the Haar integrals.

PROOF. The result follows from Theorem 16.15 and Proposition 16.16, by combining them with the various isomorphisms established above. □

Here is a useful version of the above result:

THEOREM 16.22. *The following two algebras are isomorphic, via $u_{ij}^2 \rightarrow X_{ij}$:*

- (1) *The algebra generated by the variables $u_{ij}^2 \in C(O_n^+)$.*
- (2) *The algebra generated by $X_{ij} = \frac{1}{n} \sum_{a,b=1}^n p_{ia,jb} \in C(S_{n^2}^+)$*

PROOF. This follows by using Theorem 16.17, via the above identifications. □

As a probabilistic consequence now, we have:

THEOREM 16.23. *The following families of variables have the same joint law,*

- (1) $\{u_{ij}^2\} \in C(O_n^+)$,
- (2) $\{X_{ij} = \frac{1}{n} \sum_{ab} p_{ia,jb}\} \in C(S_{n^2}^+)$,

where $u = (u_{ij})$ and $p = (p_{ia,jb})$ are the corresponding fundamental corepresentations.

PROOF. This follows indeed from Theorem 16.22 above. □

In particular, we have the following result:

THEOREM 16.24. *The free hypergeometric variable*

$$X_{ij} = \frac{1}{n} \sum_{a,b=1}^n u_{ia,jb} \in C(S_{n^2}^+)$$

has the same law as the squared free hyperspherical variable, namely:

$$x_i^2 \in C(S_{\mathbb{R},+}^{N-1})$$

PROOF. This follows from Theorem 16.23. See [20]. □

As pointed out in [20], it is possible to derive as well this result directly, by using the Weingarten formula, and manipulations on the partitions, namely fattening and shrinking. We refer to [20] and subsequent papers for more details on all this.

We refer as well to [58], [24] and related papers for some further computations of this type, which are more advanced, involving this time Gram matrix determinants, and for comments, regarding the relevance of such questions.

Summarizing, there is a lot of interesting mathematics in relation with the free spheres and orthogonal groups, and with the quantum permutations and quantum reflections as well. This tends to confirm our initial thought, from the beginning of this book, that the study and axiomatization of the quadruplets (S, T, U, K) is a good question.

16e. Exercises

There has been a lot of calculus in this chapter, and some advanced mathematics as well, and as a unique exercise, we have something refreshing, namely:

EXERCISE 16.25. *Compute the integral*

$$\int_0^{\pi/2} \cos^2 t \sin^3 t \, dt$$

directly, without using any a priori knowledge.

In addition to finding the correct answer, the computation must be of course done very quickly. And with love. Once this done, there are many other interesting questions, raised throughout the present chapter, that can be investigated as well.

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