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On local path behavior of Surgailis multifractional processes

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Abstract

Multifractional processes are stochastic processes with non-stationary increments whose local regularity and self-similarity properties change from point to point. The paradigmatic example of them is the *classical* Multifractional Brownian Motions (MBM) $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$ of Benassi, Jaffard, Lévy Véhel, Peltier and Roux, which was constructed in the mid 90's just by replacing the constant Hurst parameter \mathcal{H} of the well-known Fractional Brownian Motion by a deterministic function $\mathcal{H}(t)$ having some smoothness. More than 10 years later, using a different construction method, which basically relies on nonhomogeneous fractional integration and differentiation operators, Surgailis introduced two *non-classical* Gaussian multifractional processes denoted by $\{X(t)\}_{t \in \mathbb{R}}$ and $\{Y(t)\}_{t \in \mathbb{R}}$.

In our article, under a rather weak condition on the functional parameter $\mathcal{H}(\cdot)$, we show that $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$ and $\{X(t)\}_{t \in \mathbb{R}}$ as well as $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$ and $\{Y(t)\}_{t \in \mathbb{R}}$ only differ by a part which is locally more regular than $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$ itself. Thus it turns out that the two *non-classical* multifractional processes $\{X(t)\}_{t \in \mathbb{R}}$ and $\{Y(t)\}_{t \in \mathbb{R}}$ have exactly the same local path behavior as that of the *classical* MBM $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$.

Key Words. Gaussian processes, variable Hurst parameter, local and pointwise Hölder regularity, local self-similarity

AMS Subject Classification (2010). 60G17, 60G22, 60H05

1 Introduction and statement of the main results

Let $\mathcal{H} \in (0, 1)$, the Fractional Brownian Field (FBF) of Hurst parameter \mathcal{H} , which is also called multivariate Fractional Brownian Motion, is a real-valued

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centred continuous Gaussian field on \mathbb{R}^N denoted by $\{B_{\mathcal{H}}(t)\}_{t \in \mathbb{R}^N}$ having, for all $t', t'' \in \mathbb{R}^N$, the covariance:

$$\text{Cov}(B_{\mathcal{H}}(t'), B_{\mathcal{H}}(t'')) = \mathbb{E}(B_{\mathcal{H}}(t')B_{\mathcal{H}}(t'')) = c(\mathcal{H})(|t'|^{2\mathcal{H}} + |t''|^{2\mathcal{H}} - |t' - t''|^{2\mathcal{H}}),$$

where $c(\mathcal{H})$ is a positive constant only depending on \mathcal{H} , and $|\cdot|$ is the Euclidian norm. One refers to e.g. Chapter 1 of the book [1] for a detailed presentation FBF. Notice that, up to a multiplicative constant, $\{B_{\mathcal{H}}(t)\}_{t \in \mathbb{R}^N}$ is in distribution the unique Gaussian field which satisfies the following three fundamental properties: *self-similarity* that is for all fixed positive real number a one has $\{B_{\mathcal{H}}(at)\}_{t \in \mathbb{R}^N} \stackrel{d}{=} \{a^{\mathcal{H}} B_{\mathcal{H}}(t)\}_{t \in \mathbb{R}^N}$, where the symbol $\stackrel{d}{=}$ means equality of finite-dimensional distributions; *stationarity of increments*, that is for each fixed $\mathbf{t} \in \mathbb{R}^N$ one has $\{B_{\mathcal{H}}(\mathbf{t} + t) - B_{\mathcal{H}}(\mathbf{t})\}_{t \in \mathbb{R}^N} \stackrel{d}{=} \{B_{\mathcal{H}}(t)\}_{t \in \mathbb{R}^N}$; and *isotropy*, that is for every fixed orthogonal matrix Q of size N one has $\{B_{\mathcal{H}}(Qt)\}_{t \in \mathbb{R}^N} \stackrel{d}{=} \{B_{\mathcal{H}}(t)\}_{t \in \mathbb{R}^N}$. Though FBF is a useful model, a serious limitation of it comes from the fact that local behavior of its paths does not change from point to point. More precisely, roughness of paths of a continuous nowhere differentiable real-valued stochastic field $\{Z(t)\}_{t \in \mathbb{R}^N}$ around some fixed point $\tau \in \mathbb{R}^N$ is usually measured through the *pointwise Hölder exponent at τ*

$$\rho_Z(\tau) := \sup \left\{ r \in [0, 1]; \limsup_{t \rightarrow \tau} \frac{|Z(t) - Z(\tau)|}{|t - \tau|^r} < +\infty \right\}, \quad (1.1)$$

or through the *local Hölder exponent at τ*

$$\tilde{\rho}_Z(\tau) := \sup \left\{ \tilde{r} \in [0, 1]; \limsup_{(t', t'') \rightarrow (\tau, \tau)} \frac{|Z(t') - Z(t'')|}{|t' - t''|^{\tilde{r}}} < +\infty \right\}. \quad (1.2)$$

Observe that, one always has that

$$\tilde{\rho}_Z(\tau) \leq \rho_Z(\tau), \quad \text{for all } \tau \in \mathbb{R}^N. \quad (1.3)$$

Local roughness of paths of the FBF $\{B_{\mathcal{H}}(t)\}_{t \in \mathbb{R}^N}$ does not change from point to point since it is known (see for instance [11, 4, 5, 9, 1]) that there exists an universal event Ω_* of probability 1 such that one has

$$\tilde{\rho}_{B_{\mathcal{H}}}(\tau, \omega) = \rho_{B_{\mathcal{H}}}(\tau, \omega) = \mathcal{H}, \quad \text{for all } (\tau, \omega) \in \mathbb{R}^N \times \Omega_*.$$

In order to overcome this limitation of FBF, it has been proposed in [3, 8] to replace its constant Hurst parameter \mathcal{H} by $\mathcal{H}(t)$, where $\mathcal{H}(\cdot)$ denotes a continuous function on \mathbb{R}^N with values in some compact interval included in $(0, 1)$. This idea has led to Multifractional Brownian Field $\{\mathcal{M}(t)\}_{t \in \mathbb{R}^N}$ which is more commonly called multivariate Multifractional Brownian Motion (MBM). Throughout our article we always assume that the continuous function $\mathcal{H}(\cdot)$ satisfies the usual condition:

$$\tilde{\rho}_{\mathcal{H}}(\tau) > \mathcal{H}(\tau), \quad \text{for all } \tau \in \mathbb{R}^N, \quad (1.4)$$

where $\tilde{\rho}_{\mathcal{H}}(\tau)$ denotes the local Hölder exponent of the function $\mathcal{H}(\cdot)$ at τ . Under the condition (1.4), it has shown (see [2, 1]) that there exists an universal event Ω_{**} of probability 1 such that one has

$$\tilde{\rho}_{\mathcal{M}}(\tau, \omega) = \rho_{\mathcal{M}}(\tau, \omega) = \mathcal{H}(\tau), \quad \text{for all } (\tau, \omega) \in \mathbb{R}^N \times \Omega_{**}. \quad (1.5)$$

Also, under the same condition, it has shown (see [3, 1]) that, at every fixed point $\tau \in \mathbb{R}^N$, the stochastic field $\{\mathcal{M}(t)\}_{t \in \mathbb{R}^N}$ is *strongly locally asymptotically self-similar* of exponent $\mathcal{H}(\tau)$. This means that, for some positive constant $c(\tau)$ and for any fixed positive real number T , when $\lambda \rightarrow 0_+$, the stochastic field $\{\lambda^{-\mathcal{H}(\tau)}(\mathcal{M}(\tau + \lambda u) - \mathcal{M}(\tau))\}_{u \in \mathbb{R}^N}$ converges in distribution to $\{c(\tau)B_{\mathcal{H}(\tau)}(u)\}_{u \in \mathbb{R}^N}$ in $\mathcal{C}([-T, T]^N)$ the Banach space of the real-valued continuous functions on the cube $[-T, T]^N$ equipped with the uniform norm.

From now on, we assume that $N = 1$ and that $\mathcal{H}(\cdot)$ is a continuous function on the real line with values in some compact interval included in the open interval $(1/2, 1)$. Since we are mainly concerned with the *non-classical* Gaussian multifractional processes introduced by Surgailis in his article [10], it is convenient to use from now on the same notations as in this article. Therefore, we denote by $\alpha(\cdot)$ the continuous function from \mathbb{R} into $[\alpha_{\inf}, \alpha_{\sup}] \subset (0, 1/2)$ defined as:

$$\alpha(x) = \mathcal{H}(x) - 1/2, \quad \text{for all } x \in \mathbb{R}. \quad (1.6)$$

Similarly to the article [10], we always suppose that $\alpha(\cdot)$ satisfies the uniform Dini condition:

$$\sup_{t \in \mathbb{R}} \int_{-1}^1 \frac{|\alpha(t) - \alpha(t+u)|}{|u|} du < +\infty, \quad (1.7)$$

and also the condition:

$$\tilde{\rho}_{\alpha}(\tau) > \alpha(\tau) + 1/2, \quad \text{for all } \tau \in \mathbb{R}. \quad (1.8)$$

Observe that (1.8) is nothing else than the condition (1.4) with $N = 1$ expressed in terms of the function $\alpha(\cdot)$.

We are now going to give, in terms of the function $\alpha(\cdot)$, the precise definition of the *classical* Gaussian MBM $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$ initially introduced in [8], as well as those of the two *non-classical* Surgailis Gaussian multifractional processes $\{X(t)\}_{t \in \mathbb{R}}$ and $\{Y(t)\}_{t \in \mathbb{R}}$ constructed in [10]. To this end, we make use of the usual convention:

$$\text{for all } (y, \theta) \in \mathbb{R}^2, \text{ one has } (y)_+^{\theta} := y^{\theta} \text{ if } y > 0 \text{ and } (y)_+^{\theta} = 0 \text{ else.} \quad (1.9)$$

The *classical* MBM $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$ with continuous paths is defined, for each $t \in \mathbb{R}$, through the Wiener integral:

$$\mathcal{M}(t) := \int_{\mathbb{R}} \frac{1}{\Gamma(1 + \alpha(t))} \left((t-s)_+^{\alpha(t)} - (-s)_+^{\alpha(t)} \right) dB(s), \quad (1.10)$$

where $\Gamma(\cdot)$ is the well-known "Gamma" function defined as:

$$\Gamma(x) := \int_0^{+\infty} y^{x-1} e^{-y} dy, \quad \text{for each } x \in (0, +\infty).$$

Notice that it easily follows from (1.9), (1.10), and the equality $\Gamma(x+1) = x\Gamma(x)$, for all $x \in (0, +\infty)$, that, for every $t \in \mathbb{R}$, one has

$$\mathcal{M}(t) = \int_{\mathbb{R}} \left(\int_0^t \frac{1}{\Gamma(\alpha(t))} (\tau - s)_+^{\alpha(t)-1} d\tau \right) dB(s). \quad (1.11)$$

The two *non-classical* Surgailis multifractional processes $\{X(t)\}_{t \in \mathbb{R}}$ and $\{Y(t)\}_{t \in \mathbb{R}}$ with continuous paths are, for every $t \in \mathbb{R}$, defined through the Wiener integrals:

$$X(t) := \int_{\mathbb{R}} \left(\int_0^t \frac{1}{\Gamma(\alpha(\tau))} (\tau - s)_+^{\alpha(\tau)-1} e^{H_-(s,\tau)} d\tau \right) dB(s) \quad (1.12)$$

and

$$Y(t) := \int_{\mathbb{R}} \frac{1}{\Gamma(1 + \alpha(s))} \left((t - s)_+^{\alpha(s)} e^{-H_+(s,t)} - (-s)_+^{\alpha(s)} e^{-H_+(s,0)} \right) dB(s), \quad (1.13)$$

where, for all real numbers s and t satisfying $s < t$, one has set

$$H_-(s, t) := \int_s^t \frac{\alpha(u) - \alpha(t)}{t - u} du \quad \text{and} \quad H_+(s, t) := \int_s^t \frac{\alpha(s) - \alpha(v)}{v - s} dv. \quad (1.14)$$

It is natural to seek to compare the *classical* MBM $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$ with each one of the two *non-classical* multifractional processes $\{X(t)\}_{t \in \mathbb{R}}$ and $\{Y(t)\}_{t \in \mathbb{R}}$. This leads us to introduce the two centred Gaussian processes with continuous paths $\{R(t)\}_{t \in \mathbb{R}}$ and $\{D(t)\}_{t \in \mathbb{R}}$ defined, for all $t \in \mathbb{R}$, as:

$$R(t) := X(t) - \mathcal{M}(t) \quad (1.15)$$

and

$$D(t) := Y(t) - \mathcal{M}(t). \quad (1.16)$$

The following two theorems are the two main results of our article. Roughly speaking they show that $\{R(t)\}_{t \in \mathbb{R}}$ and $\{D(t)\}_{t \in \mathbb{R}}$ are locally more regular than $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$. Thus, it turns out that $\{X(t)\}_{t \in \mathbb{R}}$ and $\{Y(t)\}_{t \in \mathbb{R}}$ have exactly the same local path behavior as $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$.

Theorem 1.1 *Assume that the conditions (1.7) and (1.8) are satisfied. Then, there exists an universal event Ω' of probability 1 such that one has*

$$\tilde{\rho}_R(\tau, \omega) > \alpha(\tau) + 1/2 = \mathcal{H}(\tau) = \rho_{\mathcal{M}}(\tau, \omega), \quad \text{for all } (\tau, \omega) \in \mathbb{R} \times \Omega'. \quad (1.17)$$

Theorem 1.2 *Assume that the conditions (1.7) and (1.8) are satisfied. Then, there exists an universal event Ω'' of probability 1 such that one has*

$$\tilde{\rho}_D(\tau, \omega) > \alpha(\tau) + 1/2 = \mathcal{H}(\tau) = \rho_{\mathcal{M}}(\tau, \omega), \quad \text{for all } (\tau, \omega) \in \mathbb{R} \times \Omega''. \quad (1.18)$$

It easily follows from Theorem 1.1, (1.15), (1.5), (1.1), (1.2) and (1.3) that:

Corollary 1.3 *Assume that the conditions (1.7) and (1.8) are satisfied, and that Ω' is the same event of probability 1 as in Theorem 1.1. Then, one has*

$$\rho_X(\tau, \omega) = \tilde{\rho}_X(\tau, \omega) = \alpha(\tau) + 1/2 = \mathcal{H}(\tau), \quad \text{for all } (\tau, \omega) \in \mathbb{R} \times \Omega'.$$

It easily follows from Theorem 1.2, (1.16), (1.5), (1.1), (1.2) and (1.3) that:

Corollary 1.4 *Assume that the conditions (1.7) and (1.8) are satisfied, and that Ω'' is the same event of probability 1 as in Theorem 1.2. Then, one has*

$$\rho_Y(\tau, \omega) = \tilde{\rho}_Y(\tau, \omega) = \alpha(\tau) + 1/2 = \mathcal{H}(\tau), \quad \text{for all } (\tau, \omega) \in \mathbb{R} \times \Omega''.$$

Corollary 1.5 *Assume that the conditions (1.7) and (1.8) are satisfied. Then, at every fixed point $\tau \in \mathbb{R}$, the stochastic process $\{X(t)\}_{t \in \mathbb{R}}$ is strongly locally asymptotically self-similar of exponent $\mathcal{H}(\tau) = \alpha(\tau) + 1/2$. More precisely, for some positive constant $c(\tau)$ and for any fixed positive real number T , when $\lambda \rightarrow 0_+$, the stochastic process $\{\lambda^{-\mathcal{H}(\tau)}(X(\tau + \lambda u) - X(\tau))\}_{u \in \mathbb{R}}$ converges in distribution to $\{c(\tau)B_{\mathcal{H}(\tau)}(u)\}_{u \in \mathbb{R}}$ in $\mathcal{C}([-T, T])$ the Banach space of the real-valued continuous functions over the compact $[-T, T]$ equipped with the uniform norm.*

Proof It follows from (1.15) that, for each fixed positive real number λ , the stochastic process $\{\lambda^{-\mathcal{H}(\tau)}(X(\tau + \lambda u) - X(\tau))\}_{u \in \mathbb{R}^N}$ can be expressed as the sum of the two processes $\{\lambda^{-\mathcal{H}(\tau)}(\mathcal{M}(\tau + \lambda u) - \mathcal{M}(\tau))\}_{u \in \mathbb{R}^N}$ and $\{\lambda^{-\mathcal{H}(\tau)}(R(\tau + \lambda u) - R(\tau))\}_{u \in \mathbb{R}^N}$. One already knows from [3, 1] that the process $\{\lambda^{-\mathcal{H}(\tau)}(\mathcal{M}(\tau + \lambda u) - \mathcal{M}(\tau))\}_{u \in \mathbb{R}^N}$ converges in distribution to $\{c(\tau)B_{\mathcal{H}(\tau)}(u)\}_{u \in \mathbb{R}}$ in $\mathcal{C}([-T, T])$ when $\lambda \rightarrow 0_+$. Thus, for proving the corollary it is enough to show that the process $\{\lambda^{-\mathcal{H}(\tau)}(R(\tau + \lambda u) - R(\tau))\}_{u \in \mathbb{R}^N}$, viewed as a random variable with values in the space $\mathcal{C}([-T, T])$, converges almost surely to 0 in this space when $\lambda \rightarrow 0_+$. The latter fact results from Theorem 1.1, (1.3) and (1.1) which entail that, for each $\omega \in \Omega'$ (the same event of probability 1 as in Theorem 1.1), there are 3 positive finite constants $C_0(\omega)$, $\varepsilon_0(\omega)$ and $\eta_0(\omega)$ such that, for all real number v satisfying $|v| \leq \eta_0(\omega)$, one has $|R(\tau + v, \omega) - R(\tau, \omega)| \leq C_0(\omega)|v|^{\mathcal{H}(\tau) + \varepsilon_0(\omega)}$. ■

Corollary 1.6 *Assume that the conditions (1.7) and (1.8) are satisfied. Then, at every fixed point $\tau \in \mathbb{R}^N$, the stochastic process $\{Y(t)\}_{t \in \mathbb{R}}$ is strongly locally asymptotically self-similar of exponent $\mathcal{H}(\tau) = \alpha(\tau) + 1/2$.*

The proof of Corollary 1.6 is skipped since it is very similar to that of Corollary 1.5 except that Theorem 1.2 and (1.16) have to be used instead of Theorem 1.1 and (1.15).

2 Proof of Theorem 1.1

Let us first point out that the proof of Theorems 1.1 mainly relies on the following proposition which is a classical result derived from the equivalence of

Gaussian moments and the well-known Kolmogorov-Chentsov Hölder continuity theorem (see e.g. [6, 7]).

Proposition 2.1 *Let $\{Z(t)\}_{t \in \mathbb{R}}$ be a real-valued Gaussian process with continuous ¹ paths. Suppose that, for some compact interval $I \subset \mathbb{R}$ and for some constants $c > 0$ and $\zeta \in (0, 1]$, the inequality*

$$\mathbb{E}|Z(t') - Z(t'')|^2 \leq c|t' - t''|^{2\zeta}$$

holds for all $t', t'' \in I$. Then, with probability 1, the paths of $\{Z(t)\}_{t \in \mathbb{R}}$ satisfy on I a uniform Hölder condition of any order $\beta \in (0, \zeta)$. More precisely, there exists Ω'_I an event of probability 1, which a priori depends on I , such that one has

$$\sup_{t', t'' \in I} \frac{|Z(t', \omega) - Z(t'', \omega)|}{|t' - t''|^\beta} < +\infty, \quad \text{for all } (\omega, \beta) \in \Omega'_I \times (0, \zeta). \quad (2.1)$$

Remark 2.2 *For proving Theorem 1.1 it is enough to show that, for all fixed $t_0 \in \mathbb{R}$, there are 3 constants $\varepsilon_{t_0} > 0$, $\eta_{t_0} > 0$ and $c_{t_0} \geq 0$, which may depend on t_0 , such that*

$$\mathbb{E}|R(t') - R(t'')|^2 \leq c_{t_0} |t' - t''|^{2(\alpha(t_0)+1/2+3\varepsilon_{t_0})}, \quad \text{for all } t', t'' \in I(t_0, \eta_{t_0}/2), \quad (2.2)$$

where $I(t_0, \eta_{t_0}/2) := [t_0 - \eta_{t_0}/2, t_0 + \eta_{t_0}/2]$. Indeed, in view of Proposition 2.1, it results from (2.2) that there exists Ω'_{t_0} an event of probability 1 such that

$$\sup_{t', t'' \in I(t_0, \eta_{t_0}/2)} \frac{|R(t', \omega) - R(t'', \omega)|}{|t' - t''|^{\alpha(t_0)+1/2+2\varepsilon_{t_0}}} < +\infty, \quad \text{for all } \omega \in \Omega'_{t_0}. \quad (2.3)$$

Observe that, using the continuity at t_0 of the function $\alpha(\cdot)$, one can choose η_{t_0} small enough so that

$$|\alpha(t) - \alpha(t_0)| \leq \varepsilon_{t_0}, \quad \text{for all } t \in I(t_0, \eta_{t_0}/2). \quad (2.4)$$

Next, let $\mathring{I}(t_0, \eta_{t_0}/2)$ be the open interval, containing t_0 and included in $I(t_0, \eta_{t_0}/2)$, defined as $\mathring{I}(t_0, \eta_{t_0}/2) := (t_0 - \eta_{t_0}/2, t_0 + \eta_{t_0}/2)$. One clearly has that $\mathbb{R} = \bigcup_{t_0 \in \mathbb{R}} \mathring{I}(t_0, \eta_{t_0}/2)$. Therefore, the local compactness of \mathbb{R} implies that

$$\mathbb{R} = \bigcup_{m \in \mathbb{N}} \mathring{I}(t_{0,m}, \eta_{t_{0,m}}/2), \quad (2.5)$$

for some sequence $(t_{0,m})_{m \in \mathbb{N}}$ of real numbers. Next, one denotes by Ω' the event of probability 1 defined as

$$\Omega' := \left(\bigcap_{m \in \mathbb{N}} \Omega_{t_{0,m}} \right) \cap \Omega_{**},$$

¹For the sake of simplicity, we can make this continuity assumption since we already know that the paths of the Gaussian processes $\{R(t)\}_{t \in \mathbb{R}}$ and $\{D(t)\}_{t \in \mathbb{R}}$, defined through (1.15) and (1.16), are continuous functions and we are interested in their local Hölder regularity. Notice that when the assumption of continuity of the paths of $\{Z(t)\}_{t \in \mathbb{R}}$ is dropped then (2.1) holds for a well-chosen modification $\{\tilde{Z}(t)\}_{t \in I}$ of $\{Z(t)\}_{t \in I}$.

where Ω_{**} is the same event of probability 1 as in (1.5) with $N = 1$. Let $(\tau, \omega) \in \mathbb{R} \times \Omega'$ be arbitrary and fixed. One knows from (2.5) that there are $\tilde{m}(\tau) \in \mathbb{N}$ and $\tilde{\eta}(\tau) \in (0, 1)$ such that

$$I(\tau, \tilde{\eta}(\tau)) \subset \mathring{I}(t_0, \tilde{m}(\tau), \eta_{t_0, \tilde{m}(\tau)}/2). \quad (2.6)$$

Thus, it follows from (2.6), (2.4) and (2.3) that

$$\begin{aligned} & \sup_{t', t'' \in I(\tau, \tilde{\eta}(\tau))} \frac{|R(t', \omega) - R(t'', \omega)|}{|t' - t''|^{\alpha(\tau) + 1/2 + \varepsilon_{t_0, \tilde{m}(\tau)}}} \\ & \leq \sup_{t', t'' \in I(t_0, \tilde{m}(\tau), \eta_{t_0, \tilde{m}(\tau)}/2)} \frac{|R(t', \omega) - R(t'', \omega)|}{|t' - t''|^{\alpha(t_0, \tilde{m}(\tau)) + 1/2 + 2\varepsilon_{t_0, \tilde{m}(\tau)}}} < +\infty. \end{aligned} \quad (2.7)$$

Finally, (2.7), (1.2), (1.5), and (1.6) imply that (1.17) is satisfied.

From now on, the goal is to prove that, for any fixed $t_0 \in \mathbb{R}$, the inequality (2.2) holds. To this end, one will make an extensive use of the following lemma borrowed from [10].

Lemma 2.3 *One assumes that the continuous function $\alpha(\cdot)$ satisfies the condition (1.7), and one denotes by $\bar{\alpha}_{\text{sup}}$ the constant in the interval $[\alpha_{\text{inf}}, \alpha_{\text{sup}}] \subset (0, 1/2)$ defined as:*

$$\bar{\alpha}_{\text{sup}} := \limsup_{t-s \rightarrow +\infty} \frac{1}{t-s} \int_s^t \alpha(u) du.$$

Then, for all fixed (strictly) positive real numbers ε and ν , there is a constant C (which depends on ε and ν) such that the inequalities

$$(t-s)^{\alpha(t)} e^{H_-(s,t)} \leq C (t-s)^{\bar{\alpha}_{\text{sup}} + \varepsilon} \quad (2.8)$$

and

$$(t-s)^{\alpha(s)} e^{-H_+(s,t)} \leq C (t-s)^{\bar{\alpha}_{\text{sup}} + \varepsilon} \quad (2.9)$$

hold for all real numbers s and t satisfying $t-s \geq \nu$.

Notice that, one knows from (1.8) that the open interval $(\alpha(t_0) + 1/2, \tilde{\rho}_\alpha(t_0))$ is non-empty. Let $\gamma \in (\alpha(t_0) + 1/2, \tilde{\rho}_\alpha(t_0))$ be arbitrary and fixed. Then, one can derive from the definition of local Hölder exponent (see (1.2)) that there are two constants $k_\alpha \geq 0$ and $\delta \in (0, 1/2]$, such that one has

$$|\alpha(x) - \alpha(y)| \leq k_\alpha |x - y|^\gamma, \quad \text{for all } x, y \in I(t_0, 2\delta) := [t_0 - 2\delta, t_0 + 2\delta]. \quad (2.10)$$

Remark 2.4 In all the sequel, one assumes that the two arbitrary and fixed positive real numbers ε and η are small enough so that they satisfy the following 4 conditions:

$$\varepsilon < 8^{-1} \times \min \{ \alpha_{\inf}, 1/2 - \alpha_{\sup}, \gamma - (\alpha(t_0) + 1/2) \}, \quad (2.11)$$

$$\eta < \delta/2 \leq 1/4, \quad (2.12)$$

$$\eta^{2\varepsilon} \leq \delta/2 - \eta/2 \quad (2.13)$$

and

$$|\alpha(x) - \alpha(y)| \leq \varepsilon, \quad \text{for all } x, y \in I(t_0, 2\eta). \quad (2.14)$$

Observe that a straightforward consequence of (2.10) and (2.12) is that

$$|\alpha(x) - \alpha(y)| \leq k_\alpha |x - y|^\gamma, \quad \text{for all } x, y \in I(t_0, 2\eta). \quad (2.15)$$

Remark 2.5 In all the sequel, $t_0 \in \mathbb{R}$ is arbitrary and fixed; one denotes by t and $t + h$, where $h \in (0, 1]$, two arbitrary real numbers belonging to the interval $I(t_0, \eta/2)$, and one sets

$$\sigma_R^2(t, h) := \mathbb{E}|R(t+h) - R(t)|^2. \quad (2.16)$$

It can easily be seen that for proving (2.2), it is enough to show that there exists a positive finite constant C_0 not depending on t and h such that one has

$$\sigma_R^2(t, h) \leq C_0 h^{2(\alpha(t_0)+1/2+\varepsilon)}. \quad (2.17)$$

Using (2.16), (1.15), (1.11), (1.12) and the isometry property of Wiener integral, one gets that

$$\begin{aligned} \sigma_R^2(t, h) &= \int_{\mathbb{R}} \left[\int_0^{t+h} \frac{1}{\Gamma(\alpha(t+h))} (\tau - s)_+^{\alpha(t+h)-1} d\tau - \int_0^t \frac{1}{\Gamma(\alpha(t))} (\tau - s)_+^{\alpha(t)-1} d\tau \right. \\ &\quad \left. - \int_t^{t+h} \frac{1}{\Gamma(\alpha(\tau))} (\tau - s)_+^{\alpha(\tau)-1} e^{H_-(s,\tau)} d\tau \right]^2 ds \\ &= \int_{\mathbb{R}} \left[\int_0^{t+h} \left(\frac{1}{\Gamma(\alpha(t+h))} (\tau - s)_+^{\alpha(t+h)-1} - \frac{1}{\Gamma(\alpha(t))} (\tau - s)_+^{\alpha(t)-1} \right) d\tau \right. \\ &\quad \left. + \int_t^{t+h} \left(\frac{1}{\Gamma(\alpha(t))} (\tau - s)_+^{\alpha(t)-1} - \frac{1}{\Gamma(\alpha(\tau))} (\tau - s)_+^{\alpha(\tau)-1} e^{H_-(s,\tau)} \right) d\tau \right]^2 ds. \end{aligned}$$

Then, it follows from the inequality

$$(u + v)^2 \leq 2u^2 + 2v^2, \quad \text{for all } (u, v) \in \mathbb{R}^2, \quad (2.18)$$

that

$$\sigma_R^2(t, h) \leq 2\lambda_1(t, h) + 2\lambda_2(t, h), \quad (2.19)$$

where

$$\lambda_1(t, h) := \int_{\mathbb{R}} \left[\int_0^{t+h} \left(\frac{1}{\Gamma(\alpha(t+h))} (\tau-s)_+^{\alpha(t+h)-1} d\tau - \frac{1}{\Gamma(\alpha(t))} (\tau-s)_+^{\alpha(t)-1} \right) d\tau \right]^2 ds \quad (2.20)$$

and

$$\lambda_2(t, h) := \int_{\mathbb{R}} \left[\int_t^{t+h} \left(\frac{1}{\Gamma(\alpha(t))} (\tau-s)_+^{\alpha(t)-1} - \frac{1}{\Gamma(\alpha(\tau))} (\tau-s)_+^{\alpha(\tau)-1} e^{H_-(s,\tau)} \right) d\tau \right]^2 ds. \quad (2.21)$$

The following lemma provides an appropriate upper bound for $\lambda_1(t, h)$.

Lemma 2.6 *There is a constant C_1 , not depending on t and h , such that*

$$\lambda_1(t, h) \leq C_1 h^{2\gamma}. \quad (2.22)$$

Proof Let us set

$$\lambda_1^1(t, h) := \int_{-\infty}^{t+h} \left(\int_0^{t+h} \left(\frac{1}{\Gamma(\alpha(t+h))} - \frac{1}{\Gamma(\alpha(t))} \right) (\tau-s)_+^{\alpha(t+h)-1} d\tau \right)^2 ds \quad (2.23)$$

and

$$\lambda_1^2(t, h) := \int_{-\infty}^{t+h} \left(\int_0^{t+h} \frac{1}{\Gamma(\alpha(t))} \left((\tau-s)_+^{\alpha(t+h)-1} - (\tau-s)_+^{\alpha(t)-1} \right) d\tau \right)^2 ds. \quad (2.24)$$

Then one can derive from (2.20), (2.23), (2.24) and (2.18) that

$$\lambda_1(t, h) \leq 2 (\lambda_1^1(t, h) + \lambda_1^2(t, h)). \quad (2.25)$$

Let us first show that one has for some constant c_0 , not depending on t and h ,

$$\lambda_1^1(t, h) \leq c_0 h^{2\gamma}. \quad (2.26)$$

Applying on the interval $[\alpha(t) \wedge \alpha(t+h), \alpha(t) \vee \alpha(t+h)] \subseteq [\alpha_{\inf}, \alpha_{\sup}] \subset (0, 1/2)$ the mean value theorem to the infinitely differentiable positive function $x \mapsto 1/\Gamma(x)$, and using the inequality (2.15), one obtains that

$$\left| \frac{1}{\Gamma(\alpha(t+h))} - \frac{1}{\Gamma(\alpha(t))} \right| \leq c_1 h^\gamma, \quad (2.27)$$

where the positive constant c_1 does not depend on t and h . Then combining (2.23) and (2.27) with inequality $\alpha(t+h) \geq \alpha_{\inf}$ one gets that

$$\begin{aligned} \lambda_1^1(t, h) &\leq c_1^2 h^{2\gamma} \int_{-\infty}^{t+h} \left(\int_0^{t+h} (\tau - s)_+^{\alpha(t+h)-1} d\tau \right)^2 ds \\ &\leq \frac{c_1^2 h^{2\gamma}}{\alpha(t+h)^2} \int_{-\infty}^{t+h} \left((t+h-s)^{\alpha(t+h)} - (-s)_+^{\alpha(t+h)} \right)^2 ds \\ &\leq \frac{c_1^2 h^{2\gamma}}{\alpha_{\inf}^2} \int_0^{+\infty} \left(s^{\alpha(t+h)} - (s-t-h)_+^{\alpha(t+h)} \right)^2 ds. \end{aligned} \quad (2.28)$$

Next, one studies two cases $t+h \geq 0$ and $t+h < 0$. In the case where $t+h \geq 0$ one has

$$\begin{aligned} &\int_0^{+\infty} \left(s^{\alpha(t+h)} - (s-t-h)_+^{\alpha(t+h)} \right)^2 ds \\ &= \int_0^{t+h} s^{2\alpha(t+h)} ds + \int_0^{+\infty} \left((s+t+h)^{\alpha(t+h)} - s^{\alpha(t+h)} \right)^2 ds \\ &= \frac{(t+h)^{2\alpha(t+h)+1}}{2\alpha(t+h)+1} + (t+h)^{2\alpha(t+h)+1} \int_0^{+\infty} \left((s+1)^{\alpha(t+h)} - s^{\alpha(t+h)} \right)^2 ds. \end{aligned} \quad (2.29)$$

Next, observe that, for each fixed real number $s \geq 1$, by applying on the interval $[s, s+1]$ the mean value theorem to the infinitely differentiable function $x \mapsto x^{\alpha(t+h)}$, one obtains that

$$\left| (s+1)^{\alpha(t+h)} - s^{\alpha(t+h)} \right| \leq s^{\alpha_{\sup}-1}, \quad \text{for all } s \geq 1.$$

Therefore, one has that

$$\int_0^{+\infty} \left((s+1)^{\alpha(t+h)} - s^{\alpha(t+h)} \right)^2 ds \leq \int_0^1 ds + \int_1^{+\infty} s^{2\alpha_{\sup}-2} ds = c_2. \quad (2.30)$$

Next, one denotes by c_0 the constant defined as:

$$c_0 := \frac{c_1^2}{\alpha_{\inf}^2} \left((|t_0| + \eta)^{2\alpha_{\inf}+1} + (|t_0| + \eta)^{2\alpha_{\sup}+1} \right) \left(\frac{1}{2\alpha_{\inf}+1} + c_2 \right).$$

Then, using (2.28), (2.29), (2.30) and the fact $t+h \in I(t_0, \eta/2) := [t_0 - \eta/2, t_0 + \eta/2]$, it follows that (2.26) is satisfied. Let us now turn to the case where

$t + h < 0$. In this case one has

$$\begin{aligned}
& \int_0^{+\infty} \left(s^{\alpha(t+h)} - (s-t-h)_+^{\alpha(t+h)} \right)^2 ds \\
&= (-(t+h))^{2\alpha(t+h)+1} \int_0^{+\infty} \left((s+1)^{\alpha(t+h)} - s^{\alpha(t+h)} \right)^2 ds \\
&\leq c_2 \left((|t_0| + \eta)^{2\alpha_{\text{inf}}+1} + (|t_0| + \eta)^{2\alpha_{\text{sup}}+1} \right) \tag{2.31}
\end{aligned}$$

Thus combining (2.28) and (2.31) it turns out that (2.26) is satisfied in this case as well.

Let us now prove that one has for some constant c_3 , not depending on t and h ,

$$\lambda_1^2(t, h) \leq c_3 h^{2\gamma}. \tag{2.32}$$

Let c_4 be a positive constant, only depending on ε , such that one has

$$|\log(x)| \leq c_4 (x \vee x^{-1})^\varepsilon, \quad \text{for all } x \in (0, +\infty). \tag{2.33}$$

Moreover, let Γ_{inf} be the positive constant defined as

$$\Gamma_{\text{inf}} := \inf_{z \in (0, +\infty)} \Gamma(z) > 0. \tag{2.34}$$

One mentions in passing that Γ_{inf} is larger than $1/2$. Using (2.24), the mean value theorem, (2.33), (2.15) and (2.11) one obtains that

$$\begin{aligned}
\lambda_1^2(t, h) &\leq \frac{c_4^2 |\alpha(t+h) - \alpha(t)|^2}{\Gamma_{\text{inf}}^2} \int_{-\infty}^{t+h} \left(\int_0^{t+h} \left((\tau-s)_+^{\alpha_{\text{inf}}-1-\varepsilon} + (\tau-s)_+^{\alpha_{\text{sup}}-1+\varepsilon} \right) d\tau \right)^2 ds \\
&\leq \left[\frac{c_4^2 k_\alpha^2}{(\alpha_{\text{inf}} - \varepsilon)^2 \Gamma_{\text{inf}}^2} \int_0^{+\infty} \left(s^{\alpha_{\text{inf}}-\varepsilon} - (s-t-h)_+^{\alpha_{\text{inf}}-\varepsilon} + s^{\alpha_{\text{sup}}+\varepsilon} - (s-t-h)_+^{\alpha_{\text{sup}}+\varepsilon} \right)^2 ds \right] h^{2\gamma}. \tag{2.35}
\end{aligned}$$

In the case where $t + h \geq 0$, one has

$$\begin{aligned}
& \int_0^{+\infty} \left(s^{\alpha_{\text{inf}}-\varepsilon} - (s-t-h)_+^{\alpha_{\text{inf}}-\varepsilon} + s^{\alpha_{\text{sup}}+\varepsilon} - (s-t-h)_+^{\alpha_{\text{sup}}+\varepsilon} \right)^2 ds \\
&= \int_0^{t+h} \left(s^{\alpha_{\text{inf}}-\varepsilon} + s^{\alpha_{\text{sup}}+\varepsilon} \right)^2 ds \\
&\quad + \int_0^{+\infty} \left((s+t+h)^{\alpha_{\text{inf}}-\varepsilon} - s^{\alpha_{\text{inf}}-\varepsilon} + (s+t+h)^{\alpha_{\text{sup}}+\varepsilon} - s^{\alpha_{\text{sup}}+\varepsilon} \right)^2 ds \\
&\leq 2 \int_0^{|t_0|+\eta} \left(s^{2(\alpha_{\text{inf}}-\varepsilon)} + s^{2(\alpha_{\text{sup}}+\varepsilon)} \right) ds + 2 \int_0^{+\infty} \left((s+t+h)^{\alpha_{\text{inf}}-\varepsilon} - s^{\alpha_{\text{inf}}-\varepsilon} \right)^2 ds \\
&\quad + 2 \int_0^{+\infty} \left((s+t+h)^{\alpha_{\text{sup}}+\varepsilon} - s^{\alpha_{\text{sup}}+\varepsilon} \right)^2 ds \\
&\leq 4(1+|t_0|+\eta)^2 + 2(t+h)^{2(\alpha_{\text{inf}}-\varepsilon)+1} \int_0^{+\infty} \left((s+1)^{\alpha_{\text{inf}}-\varepsilon} - s^{\alpha_{\text{inf}}-\varepsilon} \right)^2 ds \\
&\quad + 2(t+h)^{2(\alpha_{\text{sup}}+\varepsilon)+1} \int_0^{+\infty} \left((s+1)^{\alpha_{\text{sup}}+\varepsilon} - s^{\alpha_{\text{sup}}+\varepsilon} \right)^2 ds \\
&\leq c_5,
\end{aligned} \tag{2.36}$$

where c_5 is the finite constant, not depending on t and h , defined as:

$$\begin{aligned}
c_5 := & 4(1+|t_0|+\eta)^2 \left(1 + \int_0^{+\infty} \left((s+1)^{\alpha_{\text{inf}}-\varepsilon} - s^{\alpha_{\text{inf}}-\varepsilon} \right)^2 ds \right. \\
& \left. + \int_0^{+\infty} \left((s+1)^{\alpha_{\text{sup}}+\varepsilon} - s^{\alpha_{\text{sup}}+\varepsilon} \right)^2 ds \right).
\end{aligned}$$

In the case where $t + h < 0$, one has

$$\begin{aligned}
& \int_0^{+\infty} \left(s^{\alpha_{\text{inf}}-\varepsilon} - (s-t-h)_+^{\alpha_{\text{inf}}-\varepsilon} + s^{\alpha_{\text{sup}}+\varepsilon} - (s-t-h)_+^{\alpha_{\text{sup}}+\varepsilon} \right)^2 ds \\
&\leq 2(-(t+h))^{2(\alpha_{\text{inf}}-\varepsilon)+1} \int_0^{+\infty} \left((s+1)^{\alpha_{\text{inf}}-\varepsilon} - s^{\alpha_{\text{inf}}-\varepsilon} \right)^2 ds \\
&\quad + 2(-(t+h))^{2(\alpha_{\text{sup}}+\varepsilon)+1} \int_0^{+\infty} \left((s+1)^{\alpha_{\text{sup}}+\varepsilon} - s^{\alpha_{\text{sup}}+\varepsilon} \right)^2 ds \\
&\leq c_5.
\end{aligned} \tag{2.37}$$

Thus, (2.35), (2.36), and (2.37) entail that (2.32) holds.

Finally, combining (2.26) and (2.32) with (2.25), one gets (2.22). \blacksquare

Let us now focus on $\lambda_2(t, h)$ defined in (2.21). Using the inequality

$$(u + v + w)^2 \leq 3(u^2 + v^2 + w^2), \quad \text{for all } (u, v, w) \in \mathbb{R}^3, \quad (2.38)$$

one has that

$$\lambda_2(t, h) \leq 3\lambda_2^1(t, h) + 3\Gamma_{\inf}^{-2} \lambda_2^2(t, h) + 3\Gamma_{\inf}^{-2} \lambda_2^3(t, h), \quad (2.39)$$

where

$$\lambda_2^1(t, h) := \int_{\mathbb{R}} \left(\int_t^{t+h} \left(\frac{1}{\Gamma(\alpha(t))} - \frac{1}{\Gamma(\alpha(\tau))} \right) (\tau - s)_+^{\alpha(t)-1} d\tau \right)^2 ds, \quad (2.40)$$

$$\lambda_2^2(t, h) := \int_{\mathbb{R}} \left(\int_t^{t+h} \left| (\tau - s)_+^{\alpha(t)-1} - (\tau - s)_+^{\alpha(\tau)-1} \right| e^{H_-(s, \tau)} d\tau \right)^2 ds \quad (2.41)$$

and

$$\lambda_2^3(t, h) := \int_{\mathbb{R}} \left(\int_t^{t+h} (\tau - s)_+^{\alpha(t)-1} \left| e^{H_-(s, \tau)} - 1 \right| d\tau \right)^2 ds. \quad (2.42)$$

In view of (2.39), our next goal is to obtain three lemmas which will allow us to conveniently bound from above $\lambda_2^1(t, h)$, $\lambda_2^2(t, h)$ and $\lambda_2^3(t, h)$.

Lemma 2.7 *There is a constant C_2^1 , not depending on t and h , such that*

$$\lambda_2^1(t, h) \leq C_2^1 h^{2(\gamma + \alpha_{\inf}) + 1}.$$

Proof Similarly to (2.27), it can be shown that there is a constant c_1 , not depending on t , h and τ , such that, for all $\tau \in [t, t + h]$, one has

$$\left| \frac{1}{\Gamma(\alpha(t))} - \frac{1}{\Gamma(\alpha(\tau))} \right| \leq c_1 |t - \tau|^\gamma \leq c_1 h^\gamma. \quad (2.43)$$

Next combining (2.40) and (2.43), one gets that

$$\lambda_2^1(t, h) \leq \left[c_1^2 \int_{\mathbb{R}} \left(\int_t^{t+h} (\tau - s)_+^{\alpha(t)-1} d\tau \right)^2 ds \right] h^{2\gamma}. \quad (2.44)$$

Moreover, one has

$$\begin{aligned} \int_{\mathbb{R}} \left(\int_t^{t+h} (\tau - s)_+^{\alpha(t)-1} d\tau \right)^2 ds &= \frac{1}{\alpha(t)^2} \left(\int_0^{+\infty} \left((s+h)^{\alpha(t)} - s^{\alpha(t)} \right)^2 ds + \int_0^h s^{2\alpha(t)} ds \right) \\ &\leq \frac{h^{2\alpha_{\inf} + 1}}{\alpha_{\inf}^2} \left(\int_0^{+\infty} \left((s+1)^{\alpha(t)} - s^{\alpha(t)} \right)^2 ds + \frac{1}{2\alpha_{\inf} + 1} \right) \\ &\leq \frac{h^{2\alpha_{\inf} + 1}}{\alpha_{\inf}^2} \left(c_2 + \frac{1}{2\alpha_{\inf} + 1} \right), \end{aligned} \quad (2.45)$$

where the constant c_2 , which does not depend on t and h , is the same constant as in (2.30). Finally combining (2.44) and (2.45) one obtains the lemma. \blacksquare

Lemma 2.8 *There is a constant C_2^2 , not depending on t and h , such that*

$$\lambda_2^2(t, h) \leq C_2^2 h^{2\gamma}.$$

Proof First observe that using the mean value theorem, (2.14), (2.33) and (2.15), one has for all $\tau \in [t, t+h]$ and $s < \tau$

$$|(\tau - s)^{\alpha(t)-1} - (\tau - s)^{\alpha(\tau)-1}| \leq c_1 h^\gamma \left((\tau - s)^{\alpha(t_0)-1-2\varepsilon} + (\tau - s)^{\alpha(t_0)-1+2\varepsilon} \right), \quad (2.46)$$

where the constant c_1 does not depend on t and h . Next, one sets

$$K_-(t_0) := \sup \{ |H_-(a, b)|, (a, b) \in I(t_0, \delta) \text{ and } a < b \}. \quad (2.47)$$

Observe that $K_-(t_0)$ is a finite constant. Indeed, one can derive from (1.14) and (2.10) that, for all $(a, b) \in I(t_0, \delta)$ satisfying $a < b$, one has

$$|H_-(a, b)| \leq \int_a^b \frac{|\alpha(u) - \alpha(b)|}{b - u} du \leq k_\alpha \int_a^b \frac{du}{(b - u)^{1-\gamma}} = k_\alpha \int_0^{2\delta} \frac{dv}{v^{1-\gamma}} < +\infty.$$

It results from (2.41), (2.46), (2.47), the inequalities $(\tau - s)_+ \leq 4\eta \leq 1$, for all $(\tau, s) \in I(t_0, 2\eta)^2 \subset I(t_0, \delta)^2$ (see (2.12)), and the inequality

$$\tau - s \geq \eta/2, \quad \text{for all } (\tau, s) \in \mathbb{R}^2 \text{ s.t. } \tau \in [t, t+h] \subseteq I(t_0, \eta/2) \text{ and } s \leq t+h-3\eta/2, \quad (2.48)$$

that

$$\begin{aligned} \lambda_2^2(t, h) &\leq c_1^2 h^{2\gamma} \int_{-\infty}^{t+h} \left(\int_t^{t+h} \left((\tau - s)_+^{\alpha(t_0)-1-2\varepsilon} + (\tau - s)_+^{\alpha(t_0)-1+2\varepsilon} \right) e^{H_-(s, \tau)} d\tau \right)^2 ds \\ &\leq c_1^2 \left[4e^{2K_-(t_0)} \int_{t+h-3\eta/2}^{t+h} \left(\int_t^{t+h} (\tau - s)_+^{\alpha(t_0)-1-2\varepsilon} d\tau \right)^2 ds \right. \\ &\quad \left. + \left(1 + (2/\eta)^{4\varepsilon} \right)^2 \int_{-\infty}^{t+h-3\eta/2} \left(\int_t^{t+h} (\tau - s)^{\alpha(t_0)-1+2\varepsilon} e^{H_-(s, \tau)} d\tau \right)^2 ds \right] h^{2\gamma}. \end{aligned} \quad (2.49)$$

Let us now prove that each one of the two integrals in the right-hand side of the last inequality can be bounded from above by a finite constant not depending

on t and h . In view of (2.11) one has

$$\begin{aligned}
& \int_{t+h-3\eta/2}^{t+h} \left(\int_t^{t+h} (\tau-s)_+^{\alpha(t_0)-1-2\varepsilon} d\tau \right)^2 ds \\
& \leq \frac{1}{(\alpha_{\inf} - 2\varepsilon)^2} \int_{t+h-3\eta/2}^{t+h} \left((t+h-s)^{\alpha(t_0)-2\varepsilon} - (t-s)_+^{\alpha(t_0)-2\varepsilon} \right)^2 ds \\
& \leq \frac{1}{(\alpha_{\inf} - 2\varepsilon)^2} \int_{t+h-3\eta/2}^{t+h} (t+h-s)^{2\alpha(t_0)-4\varepsilon} ds \\
& = \frac{1}{(\alpha_{\inf} - 2\varepsilon)^2} \int_0^{3\eta/2} z^{2\alpha(t_0)-4\varepsilon} dz := c_2. \tag{2.50}
\end{aligned}$$

On the other hand, one can derive from (2.48), Lemma 2.3 (with $\nu = \eta/2$) and (2.14) that

$$\begin{aligned}
& \int_{-\infty}^{t+h-3\eta/2} \left(\int_t^{t+h} (\tau-s)^{\alpha(t_0)-1+2\varepsilon} e^{H_-(s,\tau)} d\tau \right)^2 ds \\
& = \int_{-\infty}^{t+h-3\eta/2} \left(\int_t^{t+h} (\tau-s)^{\alpha(t_0)-\alpha(\tau)} (\tau-s)^{\alpha(\tau)-1+2\varepsilon} e^{H_-(s,\tau)} d\tau \right)^2 ds \\
& \leq C^2 \int_{-\infty}^{t+h-3\eta/2} \left(\int_t^{t+h} \left((\tau-s)^{\bar{\alpha}_{\text{sup}}-1+4\varepsilon} + (\tau-s)^{\bar{\alpha}_{\text{sup}}-1+2\varepsilon} \right) d\tau \right)^2 ds \leq c_3, \tag{2.51}
\end{aligned}$$

where C is the same finite constant as in (2.8), and c_3 is the finite constant not depending on t and h defined as:

$$c_3 := \left(\frac{C}{\bar{\alpha}_{\text{sup}}} \right)^2 \int_0^{+\infty} \left((s+1)^{\bar{\alpha}_{\text{sup}}+4\varepsilon} - s^{\bar{\alpha}_{\text{sup}}+4\varepsilon} + (s+1)^{\bar{\alpha}_{\text{sup}}+2\varepsilon} - s^{\bar{\alpha}_{\text{sup}}+2\varepsilon} \right)^2 ds.$$

Finally, putting together (2.49), (2.50) and (2.51) one obtains the lemma. \blacksquare

Lemma 2.9 *There is a constant C_2^3 , not depending on t and h , such that*

$$\lambda_2^3(t, h) \leq C_2^3 h^{2(\alpha(t_0)+1/2+\varepsilon)}.$$

Proof One can derive from (2.42) that

$$\lambda_2^3(t, h) = \mu_1(t, h) + \mu_2(t, h) + \mu_3(t, h), \tag{2.52}$$

where

$$\mu_1(t, h) := \int_{-\infty}^{t-\delta/2} \left(\int_t^{t+h} (\tau-s)^{\alpha(t)-1} \left| e^{H_-(s,\tau)} - 1 \right| d\tau \right)^2 ds, \tag{2.53}$$

$$\mu_2(t, h) := \int_{t-\delta/2}^{t-h^{2\varepsilon}} \left(\int_t^{t+h} (\tau-s)^{\alpha(t)-1} \left| e^{H_-(s,\tau)} - 1 \right| d\tau \right)^2 ds, \tag{2.54}$$

and

$$\mu_3(t, h) := \int_{t-h^{2\varepsilon}}^{t+h} \left(\int_t^{t+h} (\tau-s)_+^{\alpha(t)-1} \left| e^{H_-(s,\tau)} - 1 \right| d\tau \right)^2 ds. \quad (2.55)$$

Observe that one knows from (2.13) and the inequalities $0 < h \leq \eta$, that $t - h^{2\varepsilon} > t - \delta/2$. Using (2.53), (2.18), Lemma 2.3 with $\nu = \delta/2$, (2.14), the mean value theorem and (2.11), one obtains that

$$\begin{aligned} \mu_1(t, h) &\leq 2 \int_{-\infty}^{t-\delta/2} \left(\int_t^{t+h} (\tau-s)^{\alpha(t)-1} e^{H_-(s,\tau)} d\tau \right)^2 ds + 2 \int_{-\infty}^{t-\delta/2} \left(\int_t^{t+h} (\tau-s)^{\alpha(t)-1} d\tau \right)^2 ds \\ &\leq 2C^2(2/\delta)^{4\varepsilon} \int_{-\infty}^{t-\delta/2} \left(\int_t^{t+h} (\tau-s)^{\bar{\alpha}_{\text{sup}}+2\varepsilon-1} d\tau \right)^2 ds + 2 \int_{-\infty}^{t-\delta/2} \left(\int_t^{t+h} (\tau-s)^{\alpha(t)-1} d\tau \right)^2 ds \\ &\leq \frac{2C^2(2/\delta)^{4\varepsilon}}{(\bar{\alpha}_{\text{sup}}+2\varepsilon)^2} \int_{\delta/2}^{+\infty} \left((s+h)^{\bar{\alpha}_{\text{sup}}+2\varepsilon} - s^{\bar{\alpha}_{\text{sup}}+2\varepsilon} \right)^2 ds + \frac{2}{\alpha(t)^2} \int_{\delta/2}^{+\infty} \left((s+h)^{\alpha(t)} - s^{\alpha(t)} \right)^2 ds \\ &\leq \left[2C^2(2/\delta)^{4\varepsilon} \int_{\delta/2}^{+\infty} s^{2\bar{\alpha}_{\text{sup}}+4\varepsilon-2} ds + 2 \int_{\delta/2}^{+\infty} s^{2\alpha_{\text{sup}}-2} ds \right] h^2. \end{aligned} \quad (2.56)$$

Next, observe that it follows from (2.12) and the inclusion $[t, t+h] \subseteq I(t_0, \eta/2)$ that $[t - \delta/2, t - h^{2\varepsilon}] \subset I(t_0, \delta)$. Thus, using (2.54), (2.47), the mean value theorem and (2.11), one gets that

$$\begin{aligned} \mu_2(t, h) &\leq (e^{K_-(t_0)} + 1)^2 \int_{t-\delta/2}^{t-h^{2\varepsilon}} \left(\int_t^{t+h} (\tau-s)^{\alpha(t)-1} d\tau \right)^2 ds \\ &\leq \frac{(e^{K_-(t_0)} + 1)^2}{\alpha(t)^2} \int_{h^{2\varepsilon}}^{\delta/2} \left((s+h)^{\alpha(t)} - s^{\alpha(t)} \right)^2 ds \leq (e^{K_-(t_0)} + 1)^2 h^2 \int_{h^{2\varepsilon}}^{\delta/2} s^{2\alpha(t)-2} ds \\ &\leq \frac{(e^{K_-(t_0)} + 1)^2}{1 - 2\alpha_{\text{sup}}} h^{2-2\varepsilon} \leq \frac{(e^{K_-(t_0)} + 1)^2}{1 - 2\alpha_{\text{sup}}} h^{2(\alpha(t_0)+1/2+\varepsilon)}. \end{aligned} \quad (2.57)$$

In order to bound from above $\mu_3(t, h)$, one denotes by $\mathcal{E}_-(t_0)$ the finite constant defined as:

$$\mathcal{E}_-(t_0) := \sup \left\{ \left| \frac{e^x - 1}{x} \right|, x \in \mathbb{R} \text{ and } 0 < |x| \leq K_-(t_0) \right\}. \quad (2.58)$$

Observe that the inclusion $[t - h^{2\varepsilon}, t+h] \subset I(t_0, \delta)$, (2.47), (2.58), (1.14) and (2.10) entail that, for all $(s, \tau) \in [t - h^{2\varepsilon}, t+h] \times [t, t+h]$ satisfying $s < \tau$, one has

$$\left| e^{H_-(s,\tau)} - 1 \right| \leq \mathcal{E}_-(t_0) |H_-(s,\tau)| \leq \mathcal{E}_-(t_0) k_\alpha \int_s^\tau (\tau-u)^{\gamma-1} du = \frac{\mathcal{E}_-(t_0) k_\alpha}{\gamma} (\tau-s)^\gamma. \quad (2.59)$$

Thus, one can derive from (2.55) and (2.59) that

$$\mu_3(t, h) \leq \left(\frac{\mathcal{E}_-(t_0) k_\alpha}{\gamma(\gamma + \alpha_{\inf})} \right)^2 \int_{t-h^{2\varepsilon}}^{t+h} \left((t+h-s)^{\gamma+\alpha(t)} - (t-s)_+^{\gamma+\alpha(t)} \right)^2 ds. \quad (2.60)$$

Moreover, standard computations, the mean value theorem, the inequality $\gamma > 1/2$ and the inequalities $0 < \alpha_{\inf} \leq \alpha_{\sup}$ allow to show that

$$\begin{aligned} & \int_{t-h^{2\varepsilon}}^{t+h} \left((t+h-s)^{\gamma+\alpha(t)} - (t-s)_+^{\gamma+\alpha(t)} \right)^2 ds \\ & \leq \int_0^{h^{2\varepsilon}} \left((h+s)^{\gamma+\alpha(t)} - s^{\gamma+\alpha(t)} \right)^2 ds + \int_0^h s^{2(\gamma+\alpha(t))} ds \\ & \leq h^2(\gamma + \alpha(t))^2 \int_0^{h^{2\varepsilon}} \left((s+1)^{2(\gamma+\alpha(t)-1)} + s^{-2(1-\gamma-\alpha(t))} \right) ds + h^{2(\gamma+\alpha(t))+1} \\ & \leq h^2(\gamma + \alpha_{\sup} + 1)^2 \left[\int_0^1 \left((s+1)^{2(\gamma+\alpha_{\sup}-1)} + s^{-2(1-\gamma-\alpha_{\inf})} \right) ds + 1 \right]. \end{aligned} \quad (2.61)$$

Then (2.60) and (2.61) entail that, for some constant c_1 not depending on t and h , one has

$$\mu_3(t, h) \leq c_1 h^2. \quad (2.62)$$

Finally, putting together (2.52), (2.56), (2.57), (2.62) and (2.11) one obtains the lemma. \blacksquare

We are now in a position to prove the inequality (2.17).

Remark 2.10 *Combining (2.39) and (2.11) with Lemmas 2.7, 2.8 and 2.9 one obtains that*

$$\lambda_2(t, h) \leq C_2 h^{2(\alpha(t_0)+1/2+\varepsilon)}, \quad (2.63)$$

where the constant $C_2 := 3(C_2^1 + \Gamma_{\inf}^{-2} C_2^2 + \Gamma_{\inf}^{-2} C_2^3)$. Thus, it results from (2.19), Lemma 2.6, (2.63) and (2.11) that

$$\sigma_R^2(t, h) \leq 2(C_1 + C_2) h^{2(\alpha(t_0)+1/2+\varepsilon)},$$

which shows that (2.17) is satisfied.

3 Proof of Theorem 1.2

Remark 3.1 *By arguing as in Remark 2.2 it turns out that for proving Theorem 1.2 it is enough to show that, for all fixed $t_0 \in \mathbb{R}$, there are 3 constants $\varepsilon_{t_0} > 0$, $\eta_{t_0} > 0$ and $c_{t_0} \geq 0$, which may depend on t_0 , such that*

$$\mathbb{E}|D(t') - D(t'')|^2 \leq c_{t_0} |t' - t''|^{2(\alpha(t_0)+1/2+3\varepsilon_{t_0})}, \quad \text{for all } t', t'' \in I(t_0, \eta_{t_0}/2). \quad (3.1)$$

Remark 3.2 In all the sequel, one assumes that $t_0 \in \mathbb{R}$ is arbitrary and fixed and that $\delta \in (0, 1/2]$ is the same as in (2.10). Also, one assumes that the two arbitrary and fixed positive real numbers ε and η are small enough so that they satisfy the 3 conditions (2.11), (2.12) and (2.14).

Remark 3.3 In all the sequel one denotes by t and $t+h$, where $h \in (0, 1]$, two arbitrary real numbers belonging to the interval $I(t_0, \eta/2)$, and one sets

$$\sigma_D^2(t, h) := \mathbb{E}|D(t+h) - D(t)|^2. \quad (3.2)$$

It can easily be seen that for proving (3.1), it is enough to show that there exists a positive finite constant \tilde{C}_0 not depending on t and h such that one has

$$\sigma_D^2(t, h) \leq \tilde{C}_0 h^{2(\alpha(t_0)+1/2+\varepsilon)}. \quad (3.3)$$

Using (3.2), (1.16), (1.13), (1.10), the isometry property of Wiener integral, and (2.18) one gets that

$$\sigma_D^2(t, h) \leq 2\Lambda_0(t, h) + 2\Lambda_1(t, h) + \Lambda_2(t, h) + \Lambda_3(t, h), \quad (3.4)$$

where

$$\begin{aligned} \Lambda_0(t, h) := & \int_{-\infty}^{t-\eta} \left(\frac{1}{\Gamma(1+\alpha(t+h))} \left((t+h-s)^{\alpha(t+h)} - (-s)_+^{\alpha(t+h)} \right) \right. \\ & \left. - \frac{1}{\Gamma(1+\alpha(t))} \left((t-s)^{\alpha(t)} - (-s)_+^{\alpha(t)} \right) \right)^2 ds, \\ \Lambda_1(t, h) := & \int_{-\infty}^{t-\eta} \left(\frac{1}{\Gamma(1+\alpha(s))} (t+h-s)^{\alpha(s)} e^{-H_+(s, t+h)} - \frac{1}{\Gamma(1+\alpha(s))} (t-s)^{\alpha(s)} e^{-H_+(s, t)} \right)^2 ds, \end{aligned} \quad (3.5)$$

$$\begin{aligned} \Lambda_2(t, h) := & \int_{t-\eta}^t \left(\frac{1}{\Gamma(1+\alpha(s))} (t+h-s)^{\alpha(s)} e^{-H_+(s, t+h)} - \frac{1}{\Gamma(1+\alpha(s))} (t-s)^{\alpha(s)} e^{-H_+(s, t)} \right. \\ & \left. - \frac{1}{\Gamma(1+\alpha(t+h))} (t+h-s)^{\alpha(t+h)} + \frac{1}{\Gamma(1+\alpha(t))} (t-s)^{\alpha(t)} \right)^2 ds \end{aligned} \quad (3.6)$$

and

$$\Lambda_3(t, h) := \int_t^{t+h} \left(\frac{1}{\Gamma(1+\alpha(s))} (t+h-s)^{\alpha(s)} e^{-H_+(s, t+h)} - \frac{1}{\Gamma(1+\alpha(t+h))} (t+h-s)^{\alpha(t+h)} \right)^2 ds. \quad (3.7)$$

The following lemma provides an appropriate upper bound for $\Lambda_0(t, h)$.

Lemma 3.4 There is a constant C_0 , not depending on t and h , such that

$$\Lambda_0(t, h) \leq C_0 h^{2\gamma}. \quad (3.8)$$

Proof It follows from (3.5), (2.20), (2.18), (2.34) and the equality $\Gamma(x+1) = x\Gamma(x)$, for all $x \in (0, +\infty)$, that

$$\Lambda_0(t, h) \leq 2\lambda_1(t, h) + 2\Gamma_{\inf}^{-2} \tilde{\Lambda}_0(t, h), \quad (3.10)$$

where

$$\tilde{\Lambda}_0(t, h) := \int_{-\infty}^{t-\eta} \left((t+h-s)^{\alpha(t)} - (t-s)^{\alpha(t)} \right)^2 ds.$$

Moreover, using the mean value theorem, one has that

$$\tilde{\Lambda}_0(t, h) \leq ch^2, \quad (3.11)$$

where the finite constant $c := \int_{\eta}^{+\infty} (s^{2(\alpha_{\sup}-1)} + s^{2(\alpha_{\inf}-1)}) ds$. Finally, putting together (3.10), (2.22) and (3.11), one gets (3.9). ■

The following lemma provides an appropriate upper bound for $\Lambda_1(t, h)$.

Lemma 3.5 *There is a constant C_1 , not depending on t and h , such that*

$$\Lambda_1(t, h) \leq C_1 h^2. \quad (3.12)$$

Proof One can derive from (3.6), (2.34) and (2.18) that

$$\Lambda_1(t, h) \leq 2\Gamma_{\inf}^{-2} \left(\Lambda_1^1(t, h) + \Lambda_1^2(t, h) \right), \quad (3.13)$$

where

$$\Lambda_1^1(t, h) = \int_{-t+\eta}^{+\infty} (t+h+s)^{2\alpha(-s)} \left(e^{-H_+(-s, t+h)} - e^{-H_+(-s, t)} \right)^2 ds \quad (3.14)$$

and

$$\Lambda_1^2(t, h) = \int_{-t+\eta}^{+\infty} e^{-2H_+(-s, t)} \left((t+h+s)^{\alpha(-s)} - (t+s)^{\alpha(-s)} \right)^2 ds. \quad (3.15)$$

Let us first focus on $\Lambda_1^1(t, h)$. It can easily be seen that

$$e^{-H_+(-s, t+h)} - e^{-H_+(-s, t)} = e^{-H_+(-s, t+h)} \left(1 - e^{H_+(-s, t+h) - H_+(-s, t)} \right) \quad (3.16)$$

and that

$$\sup_{|x| \leq M_0} \left| \frac{1 - e^x}{x} \right| < +\infty, \quad \text{for each fixed } M_0 \in (0, +\infty). \quad (3.17)$$

Moreover, in view of (1.14) and the fact that $\alpha(\cdot)$ is with values in $[\alpha_{\inf}, \alpha_{\sup}] \subset (0, 1/2)$ one has, for all real number $s \geq -t + \eta$, that

$$|H_+(-s, t+h) - H_+(-s, t)| = \left| \int_0^h \frac{\alpha(-s) - \alpha(v+t)}{v+t+s} dv \right| \leq \frac{h}{2(s+t)}. \quad (3.18)$$

Then, putting together (3.14), (3.16), (3.17) and (3.18), one obtains, for some finite constant c_1 not depending on t and h , that

$$\begin{aligned}\Lambda_1^1(t, h) &\leq \int_{-t+\eta}^{+\infty} (t+h+s)^{2\alpha(-s)} \left(e^{-H_+(-s, t+h)} - e^{-H_+(-s, t)} \right)^2 ds \\ &\leq c_1 h^2 \int_{-t+\eta}^{+\infty} \frac{(t+h+s)^{2\alpha(-s)} e^{-2H_+(-s, t+h)}}{(s+t)^2} ds.\end{aligned}$$

Thus, using (2.9), in which t and s are replaced by $t+h$ and $-s$, and the fact that $h \in (0, 1]$, one gets that

$$\Lambda_1^1(t, h) \leq \left[c_1 C^2 \int_{\eta}^{+\infty} \frac{(s+1)^{2\bar{\alpha}_{\text{sup}}+2\varepsilon}}{s^2} ds \right] h^2. \quad (3.19)$$

Notice that thanks to (2.11) the integral in the right-hand side of (3.19) since $\bar{\alpha}_{\text{sup}} \leq \alpha_{\text{sup}}$.

On the other hand, it follows from (3.15), the mean value theorem, and (2.9) that

$$\Lambda_1^2(t, h) \leq h^2 \int_{-t+\eta}^{+\infty} e^{-2H_+(-s, t)} (t+s)^{2\alpha(-s)-2} ds \leq \left[C^2 \int_{\eta}^{+\infty} s^{2\bar{\alpha}_{\text{sup}}+2\varepsilon-2} ds \right] h^2. \quad (3.20)$$

Finally, combining (3.19) and (3.20) with (3.13) one gets (3.12). \blacksquare

In order to derive an appropriate upper bound for $\Lambda_2(t, h)$, defined in (3.7), let us express it as:

$$\Lambda_2(t, h) = \Lambda_2^1(t, h) + \Lambda_2^2(t, h), \quad (3.21)$$

where

$$\begin{aligned}\Lambda_2^1(t, h) &:= \int_{t-\eta}^{t-\eta h^{1/2}} \left(\frac{1}{\Gamma(1+\alpha(s))} (t+h-s)^{\alpha(s)} e^{-H_+(s, t+h)} - \frac{1}{\Gamma(1+\alpha(s))} (t-s)^{\alpha(s)} e^{-H_+(s, t)} \right. \\ &\quad \left. - \frac{1}{\Gamma(1+\alpha(t+h))} (t+h-s)^{\alpha(t+h)} + \frac{1}{\Gamma(1+\alpha(t))} (t-s)^{\alpha(t)} \right)^2 ds\end{aligned} \quad (3.22)$$

and

$$\begin{aligned}\Lambda_2^2(t, h) &:= \int_{t-\eta h^{1/2}}^t \left(\frac{1}{\Gamma(1+\alpha(s))} (t+h-s)^{\alpha(s)} e^{-H_+(s, t+h)} - \frac{1}{\Gamma(1+\alpha(s))} (t-s)^{\alpha(s)} e^{-H_+(s, t)} \right. \\ &\quad \left. - \frac{1}{\Gamma(1+\alpha(t+h))} (t+h-s)^{\alpha(t+h)} + \frac{1}{\Gamma(1+\alpha(t))} (t-s)^{\alpha(t)} \right)^2 ds.\end{aligned} \quad (3.23)$$

The following lemma provides an appropriate upper bound for $\Lambda_2^1(t, h)$ defined in (3.22).

Lemma 3.6 *There is a constant C_2^1 , not depending on t and h , such that*

$$\Lambda_2^1(t, h) \leq C_2^1 (h^{2\gamma} + h^{2(\alpha(t_0)+1/2)+7\epsilon}). \quad (3.24)$$

Proof It follows from (3.22), (2.38), (2.34) and the equality $\Gamma(x+1) = x\Gamma(x)$, for all $x \in (0, +\infty)$, that

$$\Lambda_2^1(t, h) \leq 3 \left(\Gamma_{\inf}^{-2} \Lambda_2^{1,1}(t, h) + \Gamma_{\inf}^{-2} \Lambda_2^{1,2}(t, h) + \Lambda_2^{1,3}(t, h) \right), \quad (3.25)$$

where

$$\Lambda_2^{1,1}(t, h) := \int_{t-\eta}^{t-\eta h^{1/2}} (t+h-s)^{2\alpha(s)} \left(e^{-H_+(s,t+h)} - e^{-H_+(s,t)} \right)^2 ds, \quad (3.26)$$

$$\Lambda_2^{1,2}(t, h) := \int_{t-\eta}^{t-\eta h^{1/2}} \left((t+h-s)^{\alpha(s)} - (t-s)^{\alpha(s)} \right)^2 e^{-2H_+(s,t)} ds \quad (3.27)$$

and

$$\Lambda_2^{1,3}(t, h) := \int_{t-\eta}^{t-\eta h^{1/2}} \left(\frac{1}{\Gamma(1+\alpha(t+h))} (t+h-s)^{\alpha(t+h)} - \frac{1}{\Gamma(1+\alpha(t))} (t-s)^{\alpha(t)} \right)^2 ds. \quad (3.28)$$

Let us first focus on $\Lambda_2^{1,1}(t, h)$. One clearly has that

$$\left(e^{-H_+(s,t+h)} - e^{-H_+(s,t)} \right)^2 = e^{-2H_+(s,t+h)} \left(1 - e^{H_+(s,t+h)-H_+(s,t)} \right)^2. \quad (3.29)$$

Moreover, in view of the inclusions $[t, t+h] \subset I(t_0, \eta/2)$ and $[t-\eta, t-\eta h^{1/2}] \subset I(t_0, 2\eta)$, one can derive from (1.14), (2.15) and the mean value theorem that one has, for all $s \in [t-\eta, t-\eta h^{1/2}]$,

$$\begin{aligned} |H_+(s, t+h) - H_+(s, t)| &\leq \int_t^{t+h} \frac{|\alpha(s) - \alpha(v)|}{v-s} dv \\ &\leq k_\alpha \gamma^{-1} \left((t+h-s)^\gamma - (t-s)^\gamma \right) \leq k_\alpha \eta^{\gamma-1} h^{(1+\gamma)/2} \end{aligned} \quad (3.30)$$

and

$$\begin{aligned} |H_+(s, t+h)| &\leq \int_s^{t+h} \frac{|\alpha(s) - \alpha(v)|}{v-s} dv \\ &\leq k_\alpha \int_s^{t+h} (v-s)^{\gamma-1} dv = k_\alpha \gamma^{-1} (t+h-s)^\gamma \leq k_\alpha \gamma^{-1} (\eta+1)^\gamma. \end{aligned} \quad (3.31)$$

Thus, putting together (3.26), (3.29), (3.30), (3.17), (3.31) and $\alpha(s) \in [\alpha_{\inf}, \alpha_{\sup}]$, one gets, for some (finite) constant c_1 not depending on t and h , that

$$\Lambda_2^{1,1}(t, h) \leq c_1 h^{1+\gamma}. \quad (3.32)$$

As regards $\Lambda_2^{1,2}(t, h)$ defined in (3.27), one can derive from (3.31), the mean value theorem, the inclusion $[t - \eta, t - \eta h^{1/2}] \subset I(t_0, 2\eta)$, (2.14) and (2.11) that

$$\Lambda_2^{1,2}(t, h) \leq c_2 h^2 \int_{t-\eta}^{t-\eta h^{1/2}} (t-s)^{2(\alpha(t_0)-\varepsilon-1)} ds \leq c'_2 h^{2(\alpha(t_0)+1/2)+7\varepsilon}, \quad (3.33)$$

where c_2 and c'_2 are two (finite) constants not depending on t and h .

As regards $\Lambda_2^{1,3}(t, h)$ defined in (3.28), one can derive from (2.38), and (2.34) that

$$\begin{aligned} \Lambda_2^{1,3}(t, h) &\leq 3 \int_{t-\eta}^{t-\eta h^{1/2}} \left(\frac{1}{\Gamma(1+\alpha(t+h))} - \frac{1}{\Gamma(1+\alpha(t))} \right)^2 (t+h-s)^{2\alpha(t+h)} ds \\ &\quad + 3 \Gamma_{\inf}^{-2} \int_{t-\eta}^{t-\eta h^{1/2}} \left((t+h-s)^{\alpha(t+h)} - (t-s)^{\alpha(t+h)} \right)^2 ds \\ &\quad + 3 \Gamma_{\inf}^{-2} \int_{t-\eta}^{t-\eta h^{1/2}} \left((t-s)^{\alpha(t+h)} - (t-s)^{\alpha(t)} \right)^2 ds. \end{aligned} \quad (3.34)$$

Rather similarly to (2.26), it can be shown that, for some constant c_3 not depending on t and h , one has

$$\int_{t-\eta}^{t-\eta h^{1/2}} \left(\frac{1}{\Gamma(1+\alpha(t+h))} - \frac{1}{\Gamma(1+\alpha(t))} \right)^2 (t+h-s)^{2\alpha(t+h)} ds \leq c_3 h^{2\gamma}. \quad (3.35)$$

Rather similarly to (3.33), it can be shown that for some constant c_4 not depending on t and h , one has

$$\int_{t-\eta}^{t-\eta h^{1/2}} \left((t+h-s)^{\alpha(t+h)} - (t-s)^{\alpha(t+h)} \right)^2 ds \leq c_4 h^{2(\alpha(t_0)+1/2)+7\varepsilon}. \quad (3.36)$$

Rather similarly to (2.35), it can be shown that for some constant c_5 not depending on t and h , one has

$$\int_{t-\eta}^{t-\eta h^{1/2}} \left((t-s)^{\alpha(t+h)} - (t-s)^{\alpha(t)} \right)^2 ds \leq c_5 h^{2\gamma}. \quad (3.37)$$

Finally, putting together (3.25), (3.32), (3.33), (3.34), (3.35), (3.36), (3.37) and the fact that $\gamma \in (1/2, 1)$, it follows that (3.24) holds. \blacksquare

The following lemma provides an appropriate upper bound for $\Lambda_2^2(t, h)$ defined in (3.23).

Lemma 3.7 *There is a constant C_2^2 , not depending on t and h , such that*

$$\Lambda_2^2(t, h) \leq C_2^2 h^{2(\alpha(t_0)+1/2)+6\varepsilon}. \quad (3.38)$$

Proof Using (3.23) and (2.18), one has that

$$\Lambda_2^2(t, h) \leq 2\left(\Lambda_2^{2,0}(t, h) + \Lambda_2^{2,1}(t, h)\right), \quad (3.39)$$

where, for $j = 0$ or $j = 1$,

$$\begin{aligned} \Lambda_2^{2,j}(t, h) := & \int_{t-\eta h^{1/2}}^t \left(\frac{1}{\Gamma(1+\alpha(t+jh))} (t+jh-s)^{\alpha(t+jh)} \right. \\ & \left. - \frac{1}{\Gamma(1+\alpha(s))} (t+jh-s)^{\alpha(s)} e^{-H_+(s,t+jh)} \right)^2 ds. \end{aligned} \quad (3.40)$$

In view of (3.39), in order to show that (3.38) is satisfied, it is enough to prove that, for each $j \in \{0, 1\}$, the following inequality, in which c denotes a (finite) constant not depending on t and h , holds

$$\Lambda_2^{2,j}(t, h) \leq c h^{2(\alpha(t_0)+1/2)+6\varepsilon}. \quad (3.41)$$

It follows from (3.40), (2.38) and (2.34) that

$$\Lambda_2^{2,j}(t, h) \leq 3\left(\varphi_{1,j}(t, h) + \Gamma_{\inf}^{-2} \varphi_{2,j}(t, h) + \Gamma_{\inf}^{-2} \varphi_{3,j}(t, h)\right), \quad (3.42)$$

where

$$\varphi_{1,j}(t, h) := \int_{t-\eta h^{1/2}}^t \left(\frac{1}{\Gamma(1+\alpha(t+jh))} - \frac{1}{\Gamma(1+\alpha(s))} \right)^2 (t+jh-s)^{2\alpha(t+jh)} ds, \quad (3.43)$$

$$\varphi_{2,j}(t, h) := \int_{t-\eta h^{1/2}}^t (t+jh-s)^{2\alpha(t+jh)} (1 - e^{-H_+(s,t+jh)})^2 ds \quad (3.44)$$

and

$$\varphi_{3,j}(t, h) := \int_{t-\eta h^{1/2}}^t \left((t+jh-s)^{\alpha(t+h)} - (t+jh-s)^{\alpha(s)} \right)^2 e^{-2H_+(s,t+jh)} ds. \quad (3.45)$$

It results from (3.43), the mean value theorem, (2.15), (2.12), (2.14) and (2.11) that

$$\begin{aligned} \varphi_{1,j}(t, h) & \leq c_1 \int_{t-\eta h^{1/2}}^t (t+jh-s)^{2\gamma+2\alpha(t+jh)} ds \\ & \leq c'_1 h^{1/2(1+2\gamma+2\alpha(t_0)-2\varepsilon)} \leq c'_1 h^{2(\alpha(t_0)+1/2)+7\varepsilon}, \end{aligned} \quad (3.46)$$

where c_1 and c'_1 are two (finite) constants not depending on t and h . Next, observe that, similarly to (3.31), it can be shown that there exists a (finite)

constant c_2 , not depending on t , h and s , such that, for all $s \in [t - \eta h^{1/2}, t]$, one has

$$|H_+(s, t + jh)| \leq c_2 h^{\gamma/2}. \quad (3.47)$$

Thus, one can derive from (3.44), (3.47), (3.17), (2.12), (2.14) and (2.11) that

$$\varphi_{2,j}(t, h) \leq c_3 h^\gamma \int_{t-\eta h^{1/2}}^t (t + jh - s)^{2\alpha(t+jh)} ds \leq c'_3 h^{2(\alpha(t_0)+1/2)+7\varepsilon}, \quad (3.48)$$

where c_3 and c'_3 are two (finite) constants not depending on t and h . Next, using (3.45), (3.47), the mean value theorem, (2.15), (2.33), (2.14) and (2.11), one gets that

$$\begin{aligned} \varphi_{3,j}(t, h) &\leq c_4 h^\gamma \int_{t-\eta h^{1/2}}^t (t + jh - s)^{2\alpha(t_0)-4\varepsilon} ds \\ &\leq c'_4 h^{1/2(1+2\gamma+2\alpha(t_0)-4\varepsilon)} \leq c'_4 h^{2(\alpha(t_0)+1/2)+6\varepsilon}, \end{aligned} \quad (3.49)$$

where c_4 and c'_4 are two (finite) constants not depending on t and h . Finally, putting together (3.42), (3.46), (3.48) and (3.49), one obtains (3.41). \blacksquare

The following lemma provides an appropriate upper bound for $\Lambda_3(t, h)$ defined in (3.8).

Lemma 3.8 *There is a constant C_3 , not depending on t and h , such that*

$$\Lambda_3(t, h) \leq C_3 h^{2+14\varepsilon}. \quad (3.50)$$

Proof It follows from (3.8), (2.38) and (2.34) that

$$\Lambda_3(t, h) \leq 3 \left(\Lambda_3^1(t, h) + \Gamma_{\inf}^{-2} \Lambda_3^2(t, h) + \Gamma_{\inf}^{-2} \Lambda_3^3(t, h) \right), \quad (3.51)$$

where

$$\Lambda_3^1(t, h) := \int_0^h \left(\frac{1}{\Gamma(1 + \alpha(s+t))} - \frac{1}{\Gamma(1 + \alpha(t+h))} \right)^2 (h-s)^{2\alpha(s+t)} e^{-2H_+(s+t, t+h)} ds, \quad (3.52)$$

$$\Lambda_3^2(t, h) := \int_0^h (h-s)^{2\alpha(t+s)} (e^{-H_+(s+t, t+h)} - 1)^2 ds \quad (3.53)$$

and

$$\Lambda_3^3(t, h) := \int_0^h \left((h-s)^{\alpha(t+s)} - (h-s)^{\alpha(t+h)} \right)^2 ds. \quad (3.54)$$

Observe that, similarly to (3.31), it can be shown that there exists a (finite) constant c_1 , not depending on t , h and s , such that, for all $s \in [0, h]$, one has

$$|H_+(s+t, t+h)| \leq c_1 h^\gamma. \quad (3.55)$$

One can derive from (3.52), (3.55), the mean value theorem and (2.15) that

$$\Lambda_3^1(t, h) \leq c_2 h^{2\gamma} \int_0^h (h-s)^{2\alpha(s+t)} ds \leq c_2 h^{2(\gamma+\alpha_{\text{inf}})+1}, \quad (3.56)$$

where c_2 is a (finite) constant not depending on t and h . Moreover, it follows from (3.53), (3.55) and (3.17) that

$$\Lambda_3^2(t, h) \leq c_3 h^{2\gamma} \int_0^h (h-s)^{2\alpha(t+s)} ds \leq c_3 h^{2(\gamma+\alpha_{\text{inf}})+1}, \quad (3.57)$$

where c_3 is a (finite) constant not depending on t and h . Furthermore, using (3.54), the mean value theorem, (2.15), (2.33) and (2.11) one has that

$$\Lambda_3^3(t, h) \leq c_4 h^{2\gamma} \int_0^h s^{2(\alpha_{\text{inf}}-\varepsilon)} ds \leq c_4 h^{2(\gamma+\alpha_{\text{inf}}-\varepsilon)+1}, \quad (3.58)$$

where c_4 is a (finite) constant not depending on t and h . Finally, putting together (3.51), (3.56), (3.57), (3.58), the inequality $2\gamma > 1$ and the inequality $\alpha_{\text{inf}} > 8\varepsilon$ (see (2.11)), one obtains (3.50). \blacksquare

We are now in a position to prove the inequality (3.3).

Remark 3.9 *Putting together (3.4), (3.9), (3.12), (3.21), (3.24), (3.38), (3.50), the fact that $h \in (0, 1]$, the fact that $\gamma \in (1/2, 1)$ and (2.11), it follows that (3.3) is satisfied.*

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