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# Moving average Multifractional Processes with Random Exponent: lower bounds for local oscillations

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## Abstract

In the last few years Ayache, Esser and Hamonier introduced a new Multifractional Process with Random Exponent (MPRE) obtained by replacing the Hurst parameter in a moving average representation of Fractional Brownian Motion through Wiener integral by an adapted Hölder continuous stochastic process indexed by the integration variable. Thus, this MPRE can be expressed as a moving average Itô integral which is a considerable advantage with respect to another MPRE introduced a long time ago by Ayache and Taqqu. Thanks to this advantage, very recently, Loboda, Mies and Steland have derived interesting results on local Hölder regularity, self-similarity and other properties of the recently introduced moving average MPRE and generalizations of it. Yet, the problem of obtaining, on an universal event of probability 1 not depending on the location, relevant lower bounds for local oscillations of such processes has remained open. We solve it in the present article under some conditions.

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**Running Title:** On the oscillations of moving average MPRE

**Key Words.** Fractional Brownian Motion, varying Hurst parameter, pointwise Hölder regularity, Itô integral

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## 1 Introduction and statement of the main result

Roughly speaking multifractional stochastic processes (see e.g. [1]) are continuous real-valued stochastic processes with non-stationary increments which extend the well-known Fractional Brownian Motion (see e.g. [13, 20]); yet, in contrast with it, their local path roughness can be prescribed via a functional Hurst parameter and thus can change from point to point. For a generic multifractional process  $Y = \{Y(t)\}_{t \in \mathbb{R}_+}$ , path roughness in a neighborhood of any arbitrary fixed point  $\tau \in \mathbb{R}_+^* := \mathbb{R}_+ \setminus \{0\} = (0, +\infty)$  is usually measured through  $\alpha_Y(\tau)$ , the pointwise Hölder exponent at  $\tau$ , defined as:

$$\alpha_Y(\tau) := \sup \left\{ \alpha \in [0, 1] : \limsup_{r \rightarrow 0_+} r^{-\alpha} \text{Osc}_Y(\tau, r) < +\infty \right\}, \quad (1.1)$$

where, for all real number  $r > 0$  small enough,

$$\text{Osc}_Y(\tau, r) := \sup \left\{ |Y(t') - Y(t'')| : (t', t'') \in [\tau - r, \tau + r]^2 \right\} \quad (1.2)$$

is the oscillation of the process  $Y$  on the interval  $[\tau - r, \tau + r] \subset \mathbb{R}_+^*$ . When a stochastic process is the functional Hurst parameter of  $Y$ , then  $Y$  is said to be a Multifractional Process with Random Exponent (MPRE). Such kind of process turned out to be useful in stock prices modeling (see for instance [10, 11, 12]). A long time ago, [5] introduced a first type of MPRE which is given by a random wavelet series, but unfortunately cannot be represented through the usual Itô integral. In order to avoid the latter drawback, another type of MPRE was introduced in the last few years in [3], and was generalized very recently in [17]. Let  $(\Omega, (\mathcal{F}_s)_{s \in \mathbb{R}}, \mathcal{F}, \mathbb{P})$  be a filtered probability space, and let  $B = \{B(s)\}_{s \in \mathbb{R}}$  be a standard Brownian motion with respect to the filtration  $(\mathcal{F}_s)_{s \in \mathbb{R}}$ , the MPRE  $X = \{X(t)\}_{s \in \mathbb{R}_+}$  studied in [17] is defined, for each fixed  $t \in \mathbb{R}_+$ , as the Itô integral:

$$X(t) := \int_{-\infty}^t g(t, s) dB(s), \quad (1.3)$$

where, for each fixed  $t \in \mathbb{R}_+$ , the stochastic process  $\{g(t, s)\}_{s \in (-\infty, t)}$  is adapted to the filtration  $(\mathcal{F}_s)_{s \in (-\infty, t)}$  and satisfies almost surely  $\int_{-\infty}^t |g(t, s)|^2 ds < +\infty$ , which guarantees the existence of Itô integral in (1.3). In order to show that  $X$  has a modification with continuous paths and to conveniently bound from below its pointwise Hölder exponents, a well adapted extension of the Kolmogorov-Chentsov Hölder continuity Theorem has been derived in [17] (see Theorem 2.1 and Corollary 2.2 in [17]). Thus, the latter article has established that such a nice modification of  $X$  exists as soon as the associated integrand  $g$  satisfies the 3 conditions (A), (B) and (C) that we are now going to give.

**Condition (A).** For each fixed  $s \in \mathbb{R}$ , the random function  $t \mapsto g(t, s)$  is differentiable on the open interval  $(s \vee 0, +\infty)$ . Moreover, there exist  $(\mathcal{F}_s)_{s \in \mathbb{R}}$ -adapted processes  $\{H(s)\}_{s \in \mathbb{R}}$ ,  $\{L(s)\}_{s \in \mathbb{R}}$  and  $\{R(s)\}_{s \in \mathbb{R}}$ , such that  $H(s) \in (0, 1)$ ,  $L(s) > 0$ ,  $R(s) > 1/2$ , and it holds, for all  $t \in \mathbb{R}_+$ ,

$$\begin{aligned} |g(t, s)| &\leq L(s)|t - s|^{H(s)-1/2}, & \text{for every } s \in (t - 1, t), \\ |\partial_t g(t, s)| &\leq L(s)|t - s|^{H(s)-3/2}, & \text{for every } s \in (t - 1, t), \\ |\partial_t g(t, s)| &\leq L(s)|t - s|^{-R(s)}, & \text{for every } s \in (-\infty, t - 1]. \end{aligned}$$

**Condition (B).** There are deterministic real numbers  $\underline{H}$ ,  $\overline{H}$ ,  $\overline{L}$  and  $\underline{R}$ , such that, for all  $s \in \mathbb{R}$ , one has

$$0 < \underline{H} \leq H(s) \leq \overline{H} < 1, \quad 0 < L(s) \leq \overline{L} \quad \text{and} \quad R(s) \geq \underline{R} > 1/2. \quad (1.4)$$

**Condition (C).** One has almost surely, for all  $(s', s'') \in \mathbb{R}^2$ ,

$$|H(s') - H(s'')| \leq \mu(|s' - s''|), \quad (1.5)$$

where  $\mu$  is some deterministic function from  $\mathbb{R}_+$  to  $\mathbb{R}_+$  which is continuous and increasing and satisfies  $\mu(0) = 0$ .

Among many other things, the following theorem has been obtained in [17].

**Theorem 1.1** [17] *Assume that the conditions (A), (B) and (C) hold, then the process  $X$  has a modification with continuous paths which is identified with  $X$  from now on. Moreover, its pointwise Hölder exponents satisfy*

$$\mathbb{P}(\forall \tau \in \mathbb{R}_+^*, \alpha_X(\tau) \geq H(\tau)) = 1. \quad (1.6)$$

*In other words, there exists  $\Omega_1^*$  an universal event of probability 1 not depending on  $\tau$  such that one has*

$$\alpha_X(\tau, \omega) \geq H(\tau, \omega), \quad \text{for all } (\tau, \omega) \in \mathbb{R}_+^* \times \Omega_1^*. \quad (1.7)$$

In order to obtain the further result on pointwise Hölder exponents of  $X$  stated below, the article [17] has assumed that the integrand  $g$  satisfies the following additional condition (A\*) in which  $\{L(s)\}_{s \in \mathbb{R}}$  denotes the same process as in Condition (A).

**Condition (A\*).** One has, for all  $t \in \mathbb{R}_+$ ,

$$\begin{aligned} \left| g(t, s) - \sigma(s)(t - s)^{H(s)-1/2} \right| &\leq L(s)|t - s|^{H(s)-1/2+\rho}, \quad \text{for every } s \in (t - 1, t), \\ \left| \partial_t g(t, s) - \partial_t \sigma(s)(t - s)^{H(s)-1/2} \right| &\leq L(s)|t - s|^{H(s)-3/2+\rho}, \quad \text{for every } s \in (t - 1, t), \end{aligned}$$

where  $\rho$  is some positive deterministic real number which satisfies  $\overline{H} + \rho < 1$  and does not depend on  $t$  and  $s$ , and where  $\{\sigma(s)\}_{s \in \mathbb{R}}$  is a continuous  $(\mathcal{F}_s)_{s \in \mathbb{R}}$ -adapted process not depending on  $t$  and satisfying

$$0 < |\sigma(s)| \leq L(s), \quad \text{for all } s \in \mathbb{R}. \quad (1.8)$$

**Theorem 1.2** [17] *Under the conditions (A), (A\*), (B), (C) and the additional condition*

$$\lim_{\varepsilon \rightarrow 0_+} \mu(\varepsilon) \log(\varepsilon) = 0, \quad (1.9)$$

*one has*

$$\mathbb{P}(\alpha_X(\tau) = H(\tau)) = 1, \quad \text{for all } \tau \in \mathbb{R}_+^*. \quad (1.10)$$

Let us point out that the keystone of the proofs of Theorems 1.1 and 1.2 given in [17] is the important Burkholder-Davis-Gundy inequality (see for instance [18, 19]) as formulated in the following proposition:

**Proposition 1.3** *Let  $p \in [1, +\infty[$  be arbitrary and fixed. There is an universal deterministic finite constant  $a(p)$  for which the following result holds: for any  $(\mathcal{F}_s)_{s \in \mathbb{R}}$ -adapted stochastic process  $f = \{f(s)\}_{s \in \mathbb{R}}$  satisfying almost surely  $\int_{-\infty}^{+\infty} |f(s)|^2 ds < +\infty$ , one has*

$$\mathbb{E} \left( \left| \int_{-\infty}^{+\infty} f(s) dB(s) \right|^p \right) \leq a(p) \mathbb{E} \left( \left( \int_{-\infty}^{+\infty} |f(s)|^2 ds \right)^{p/2} \right), \quad (1.11)$$

where  $\int_{-\infty}^{+\infty} f(s) dB(s)$  denotes the Itô integral of  $f$  on  $\mathbb{R}$ .

**Remark 1.4** In fact the additional information brought by Theorem 1.2 with respect to Theorem 1.1 is that, for each fixed  $\tau \in \mathbb{R}_+^*$ , there exists  $\tilde{\Omega}(\tau)$  an event of probability 1 which a priori depends on  $\tau$ , such that

$$\alpha_X(\tau, \omega) \leq H(\tau, \omega), \quad \text{for all } \omega \in \tilde{\Omega}(\tau). \quad (1.12)$$

Notice that, in view of (1.1) the inequality (1.12) can equivalently be expressed as follows in terms of the local oscillations of  $X$  in the vicinity of  $\tau$ :

$$\limsup_{r \rightarrow 0_+} \left( r^{-H(\tau, \omega) - \theta} \text{Osc}_X(\tau, r, \omega) \right) = +\infty, \quad \text{for all } (\theta, \omega) \in \mathbb{R}_+^* \times \tilde{\Omega}(\tau). \quad (1.13)$$

■

The main goal of our article is to show that when the condition (C) is strengthened to the condition (C\*) given below then a significantly more strong result than (1.13) holds, namely:

**Theorem 1.5** *Suppose that the conditions (A), (A\*), (B) and (C\*) are satisfied. Then, there exists  $\Omega_2^*$  an universal event of probability 1 not depending on  $\tau$  such that one has*

$$\liminf_{r \rightarrow 0_+} \left( r^{-H(\tau, \omega) - \theta} \text{Osc}_X(\tau, r, \omega) \right) = +\infty, \quad \text{for all } (\theta, \tau, \omega) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \Omega_2^*. \quad (1.14)$$

**Condition (C\*).** There are two deterministic constants  $\kappa \in (0, +\infty)$  and  $\gamma \in (0, 1)$  such that one has almost surely, for all  $(s', s'') \in \mathbb{R}^2$ ,

$$|H(s') - H(s'')| + |\sigma(s') - \sigma(s'')| \leq \kappa |s' - s''|^\gamma. \quad (1.15)$$

It is worth mentioning that a straightforward consequence of Theorems 1.1 and 1.5 and (1.1) is that:

**Corollary 1.6** *Suppose that the conditions (A), (A\*), (B) and (C\*) hold. Let  $\Omega_1^*$  and  $\Omega_2^*$  be the two universal events of probability 1 which were introduced in Theorems 1.1 and 1.5. Then  $\Omega^* := \Omega_1^* \cap \Omega_2^*$  is an universal event of probability 1 such that*

$$\alpha_X(\tau, \omega) = H(\tau, \omega), \quad \text{for all } (\tau, \omega) \in \mathbb{R}_+^* \times \Omega^*. \quad (1.16)$$

**Remarks 1.7**

- (i) The conditions (B), (C) and (C\*) might seem restrictive, yet they can be relaxed by a localization procedure via stopping times (see for instance Section 4.4.1 in [15]) which is explained in the setting of the MPRE  $X$  in the introduction of [17]. In particular, it is important to mention that Theorem 1.5 remains true when the condition (1.15) only holds for all  $s'$  and  $s''$  belonging to every fixed compact interval  $I \subset \mathbb{R}$ , and the constant  $\kappa$  in it is no longer a deterministic real number but an almost surely finite random variable which may depend on  $I$ .
- (ii) Usually, in the literature relevant lower bounds for local oscillations of a stochastic process  $Y$  valid on an universal event of probability 1 not depending on the location (that is the point  $\tau$  in our setting) are obtained via the classical strategy (see e.g. [7, 8, 9, 21, 22, 23]) which consists in showing that the local time associated to  $Y$  is regular in the set variable uniformly in the space variable. For deriving such a regularity result for the local time explicit and exploitable formulas for the characteristic functions of the finite-dimensional distributions of  $Y$  need to be available. Unfortunately, such formulas are not available for the MPRE  $X$ . Thus, for proving Theorem 1.5 we make use of another strategy which is to some extent reminiscent of that introduced in the last few years in [2] in the framework of Hermite processes. However several significant modifications of the strategy of [2] are needed since there is a wide difference between the latter self-similar chaotic processes with stationary increments and the MPRE  $X$ . Among other things the Burkholder-Davis-Gundy inequality provided by Proposition 1.3 plays a crucial role in our proof of Theorem 1.5.
- (iii) In the framework of multifractal analysis (see e.g. [16, 6, 4]) one is very often interested in determining singularity spectra of sample paths of stochastic processes. For a generic process  $Y = \{Y(t)\}_{t \in \mathbb{R}_+}$  with nowhere differentiable paths

the singularity spectrum  $\rho_Y = \{\rho_Y(\alpha)\}_{\alpha \in [0,1]}$  is defined, for each  $\omega \in \Omega$  (the underlying probability space) and  $\alpha \in [0,1]$ , as the Hausdorff dimension (see e.g. [14]) of the level set  $\{\tau \in \mathbb{R}_+^* : \alpha_Y(\tau, \omega) = \alpha\}$ , where  $\alpha_Y(\tau)$  is the pointwise Hölder exponent of  $Y$ . Thus, in contrast with (1.10) the more strong result (1.16) draws close connections between the singularity spectrum of the MPRE  $X$  and its functional random Hurst parameter  $H$ . ■

## 2 Proof of the main result

Let  $\sigma$  and  $H$  be the same processes as in the conditions (A) and (A\*). One assumes that the condition (B) is satisfied and one denotes by  $Z = \{Z(t)\}_{t \in \mathbb{R}_+}$  and  $R = \{R(t)\}_{t \in \mathbb{R}_+}$  the two processes defined, for all  $t \in \mathbb{R}_+$ , as:

$$Z(t) := \int_{-1}^t \sigma(s)(t-s)^{H(s)-1/2} dB(s) = \int_{-\infty}^t \sigma(s)(t-s)^{H(s)-1/2} \mathbb{1}_{[-1,t)}(s) dB(s) \quad (2.1)$$

and

$$R(t) := X(t) - Z(t). \quad (2.2)$$

One can easily derive from Theorem 1.1 applied to  $Z$  that:

**Lemma 2.1** *Suppose that the conditions (B) and (C) hold, then the process  $Z$  has a modification with continuous paths which is identified with  $Z$  from now on. Moreover, its pointwise Hölder exponents satisfy*

$$\mathbb{P}(\forall \tau \in \mathbb{R}_+^*, \alpha_Z(\tau) \geq H(\tau)) = 1. \quad (2.3)$$

On the other hand, combining (1.3), (2.2) and (2.1), for all  $t \in \mathbb{R}_+$ , one has  $R(t) = \int_{-\infty}^t \tilde{g}(t,s) dB(s)$ , where  $\tilde{g}(t,s) := g(t,s) - \sigma(s)(t-s)^{H(s)-1/2} \mathbb{1}_{[-1,t)}(s)$ , for every  $s \in (-\infty, t)$ . Moreover, one can derive from the conditions (A) and (A\*), imposed to  $g$ , that  $\tilde{g}$  satisfies, for all  $t \in \mathbb{R}_+$ , a slightly modified version of the condition (A) in which  $H(s)$  is replaced by  $H(s) + \rho$ , for each  $s \in (t-1, t)$ . Thus, similarly to Theorem 1.1 it can be shown that:



**Lemma 2.2** *Assume that the conditions (A), (A\*), (B) and (C) hold, then the process  $R$  has a modification with continuous paths which is identified with  $R$  from now on. Moreover, its pointwise Hölder exponents satisfy*

$$\mathbb{P}(\forall \tau \in \mathbb{R}_+^*, \alpha_R(\tau) \geq H(\tau) + \rho) = 1. \quad (2.4)$$

**Remark 2.3** In view of Lemma 2.2, (1.1) and (1.2), in order to show that Theorem 1.5 holds, it is enough to prove it in the particular case where  $X = Z$ . On the other hand, in view of the fact that the interval  $\mathbb{R}_+^* := (0, +\infty)$  can be expressed as the countable union of the open, bounded and overlapping intervals  $(2^{-1}q, 1 + 2^{-1}q)$ ,  $q \in \mathbb{Z}_+$ , it is enough to prove the theorem for every  $\tau \in (2^{-1}q, 1 + 2^{-1}q)$ , the nonnegative integer  $q$  being arbitrary and fixed. For the sake of simplicity, we will only prove it when  $q = 0$ , that is  $\tau \in (0, 1)$ ; its proof can be done in a similar way for any other  $q$ . ■

For any integers  $j \geq 2$  and  $k \in \{0, \dots, 2^j - 1\}$ , one denotes by  $\Delta(j, k)$  the increment of the process  $Z$  such that

$$\Delta(j, k) := Z(d_{j,k+1}) - Z(d_{j,k}), \quad (2.5)$$

where  $d_{j,k+1}$  and  $d_{j,k}$  are the two dyadic numbers in the interval  $[0, 1]$  defined as:

$$d_{j,k+1} := (k + 1)/2^j \quad \text{and} \quad d_{j,k} := k/2^j. \quad (2.6)$$

Observe that, in view of (2.1), the increment  $\Delta(j, k)$  can be expressed as:

$$\Delta(j, k) = \int_{-1}^{d_{j,k+1}} \sigma(s) \left( (d_{j,k+1} - s)^{H(s)-1/2} - (d_{j,k} - s)_+^{H(s)-1/2} \right) dB(s), \quad (2.7)$$

with the convention that, for each  $(y, \alpha) \in \mathbb{R}^2$ , one has

$$y_+^\alpha := \begin{cases} y^\alpha, & \text{if } y > 0, \\ 0, & \text{else.} \end{cases} \quad (2.8)$$

In all the sequel the parameter  $b \in (0, 1/2)$  is arbitrary and fixed. For each integer  $j \geq 2$ , one sets

$$e_j := \lfloor 2^{jb} \rfloor, \quad (2.9)$$

where  $\lfloor \cdot \rfloor$  is the integer part function, and one denotes by  $\mathcal{L}^j$  the non-empty finite set of positive integers defined as:

$$\mathcal{L}^j := \mathbb{N} \cap [1, (2^j/e_j) - 1]. \quad (2.10)$$

Observe that the cardinality of  $\mathcal{L}^j$  satisfies, for some positive finite constant  $c$  not depending on  $j$ ,

$$\text{Card}(\mathcal{L}^j) \leq c 2^{j(1-b)}. \quad (2.11)$$

Also, observe that, for any  $l \in \mathcal{L}^j$ , the random variable  $\Delta(j, le_j)$ , defined through (2.7) with  $k = le_j$ , can be expressed as

$$\Delta(j, le_j) = \tilde{\Delta}(j, le_j) + \check{\Delta}(j, le_j), \quad (2.12)$$

where

$$\tilde{\Delta}(j, le_j) := \int_{d_{j, (l-1)e_j+1}}^{d_{j, le_j+1}} \sigma(s) \left( (d_{j, le_j+1} - s)^{H(s)-1/2} - (d_{j, le_j} - s)_+^{H(s)-1/2} \right) dB(s) \quad (2.13)$$

and

$$\check{\Delta}(j, le_j) := \int_{-1}^{d_{j, (l-1)e_j+1}} \sigma(s) \left( (d_{j, le_j+1} - s)^{H(s)-1/2} - (d_{j, le_j} - s)^{H(s)-1/2} \right) dB(s). \quad (2.14)$$

Let us now focus on the study of asymptotic behavior of the random variables  $|\check{\Delta}(j, le_j)|$  when  $j$  goes to  $+\infty$ .

**Lemma 2.4** *There is  $\Omega_3^*(b)$  an event of probability 1 depending on  $b$  on which one has*

$$\limsup_{j \rightarrow +\infty} \sup_{l \in \mathcal{L}^j} 2^{j(H(d_{j, le_j}) + b(1 - \bar{H})/2)} |\check{\Delta}(j, le_j)| = 0. \quad (2.15)$$

For showing that Lemma 2.4 holds one needs two preliminary results. In order to state them one first has to introduce some additional notations. Let  $\eta \in (0, 1)$  be arbitrary and fixed and such that

$$5\kappa\eta^\gamma < b(1 - \bar{H}), \quad (2.16)$$

where  $\kappa$ ,  $\gamma$ ,  $b$  and  $\bar{H}$  are as in (1.15), (2.9) and (1.4). One assumes that the integer  $\bar{J}_0 \geq 6$  is chosen so that one has  $e_j/2^j \leq 2^{-j(1-b)} < \eta/2$ , for all integer  $j \geq \bar{J}_0$ . Then, one can derive from (2.14) that

$$\check{\Delta}(j, le_j) = \check{\Delta}'(j, le_j) + \check{\Delta}''(j, le_j), \quad (2.17)$$

where

$$\check{\Delta}'(j, le_j) := \int_{-1}^{d_{j, le_j} - \eta} \sigma(s) \left( (d_{j, le_j+1} - s)^{H(s)-1/2} - (d_{j, le_j} - s)^{H(s)-1/2} \right) dB(s) \quad (2.18)$$

and

$$\check{\Delta}''(j, le_j) := \int_{d_{j,le_j}-\eta}^{d_{j,(l-1)e_j+1}} \sigma(s) \left( (d_{j,le_{j+1}} - s)^{H(s)-1/2} - (d_{j,le_j} - s)^{H(s)-1/2} \right) dB(s). \quad (2.19)$$

**Lemma 2.5** *There is a finite deterministic constant  $c > 0$  such that, for all integer  $j \geq \bar{J}_0$ , one has*

$$\sup_{l \in \mathcal{L}^j} \int_{-1}^{d_{j,le_j}-\eta} |\sigma(s)|^2 \left| (d_{j,le_{j+1}} - s)^{H(s)-1/2} - (d_{j,le_j} - s)^{H(s)-1/2} \right|^2 ds \leq c 2^{-2j}. \quad (2.20)$$

**Proof of Lemma 2.5** Using the mean value Theorem, (2.6), (1.8), and (1.4), one gets, for all  $j \geq \bar{J}_0$ ,  $l \in \mathcal{L}^j$  and  $s \in [-1, d_{j,le_j} - \eta]$ , that

$$|\sigma(s)|^2 \left| (d_{j,le_{j+1}} - s)^{H(s)-1/2} - (d_{j,le_j} - s)^{H(s)-1/2} \right|^2 \leq \bar{L}^2 \eta^{2(\underline{H}-\bar{H})} (d_{j,le_j} - s)^{2\bar{H}-3} 2^{-2j}.$$

Thus setting  $c := \bar{L}^2 \eta^{2(\underline{H}-\bar{H})} \int_{\eta}^2 x^{2\bar{H}-3} dx < +\infty$ , it follows that (2.20) holds.  $\blacksquare$

**Lemma 2.6** *There is a finite deterministic constant  $c > 0$  such that, for all integer  $j \geq \bar{J}_0 + 2b^{-1}$ , one has*

$$\begin{aligned} & \sup_{l \in \mathcal{L}^j} \int_{d_{j,le_j}-\eta}^{d_{j,(l-1)e_j+1}} 2^{2jH(d_{j,le_j}-\eta)} |\sigma(s)|^2 \left| (d_{j,le_{j+1}} - s)^{H(s)-1/2} - (d_{j,le_j} - s)^{H(s)-1/2} \right|^2 ds \\ & \leq c 2^{-2j(4b(1-\bar{H})/5)}. \end{aligned} \quad (2.21)$$

**Proof of Lemma 2.6** One can derive from the mean value Theorem, (2.6), (1.8), (1.4), (2.9) and (1.15) that, for all  $j \geq \bar{J}_0 + 2b^{-1}$ ,  $l \in \mathcal{L}^j$  and  $s \in [d_{j,le_j} - \eta, d_{j,(l-1)e_j+1}]$ ,

$$|\sigma(s)|^2 \left| (d_{j,le_{j+1}} - s)^{H(s)-1/2} - (d_{j,le_j} - s)^{H(s)-1/2} \right|^2 \leq \bar{L}^2 (d_{j,le_j} - s)^{2H(d_{j,le_j}-\eta)-2\kappa\eta^\gamma-3} 2^{-2j},$$

and consequently that

$$\begin{aligned}
& \int_{d_{j,le_j-\eta}}^{d_{j,(l-1)e_j+1}} |\sigma(s)|^2 \left| (d_{j,le_{j+1}} - s)^{H(s)-1/2} - (d_{j,le_j} - s)^{H(s)-1/2} \right|^2 ds \\
& \leq \bar{L}^2 2^{-2j} \int_{d_{j,le_j-\eta}}^{d_{j,(l-1)e_j+1}} (d_{j,le_j} - s)^{2H(d_{j,le_j}-\eta)-2\kappa\eta^\gamma-3} ds \\
& \leq \frac{\bar{L}^2 2^{-2j}}{2(1 + \kappa\eta^\gamma - H(d_{j,le_j} - \eta))} \left( 2^{-j} (\lfloor 2^j b \rfloor - 1) \right)^{2(H(d_{j,le_j}-\eta)-\kappa\eta^\gamma-1)} \\
& \leq \frac{4^{\kappa\eta^\gamma+1} \bar{L}^2 2^{-2j}}{2(1 - \bar{H})} \times 2^{-2j(1-b)(H(d_{j,le_j}-\eta)-\kappa\eta^\gamma-1)} \\
& \leq \frac{4^{\kappa\eta^\gamma+1} \bar{L}^2}{2(1 - \bar{H})} \times 2^{-2j(H(d_{j,le_j}-\eta)-(1-b)\kappa\eta^\gamma+b(1-\bar{H}))} \\
& \leq \frac{16\bar{L}^2}{2(1 - \bar{H})} \times 2^{-2j(H(d_{j,le_j}-\eta)+4b(1-\bar{H})/5)}, \tag{2.22}
\end{aligned}$$

where the last inequality follows from (2.16). Finally, (2.21) results from (2.22).  $\blacksquare$

We are now ready to prove Lemma 2.4.

**Proof of Lemma 2.4** It easily follows (2.17) and the triangle inequality that, for all integer  $j \geq \bar{J}_0 + 2b^{-1}$ , one has

$$\begin{aligned}
& \mathbb{P}\left( \sup_{l \in \mathcal{L}^j} 2^{jH(d_{j,le_j})} |\check{\Delta}(j, le_j)| > 2^{-j(8b(1-\bar{H})/15)+1} \right) \\
& \leq \mathbb{P}\left( \sup_{l \in \mathcal{L}^j} 2^{jH(d_{j,le_j})} |\check{\Delta}'(j, le_j)| > 2^{-j(8b(1-\bar{H})/15)} \right) \tag{2.23} \\
& \quad + \mathbb{P}\left( \sup_{l \in \mathcal{L}^j} 2^{jH(d_{j,le_j})} |\check{\Delta}''(j, le_j)| > 2^{-j(8b(1-\bar{H})/15)} \right).
\end{aligned}$$

In order to conveniently bound each one of the two probabilities in the right-hand side of (2.23), one denotes by  $p$  a fixed real number such that

$$p > \frac{15}{b(1 - \bar{H})} > 15. \tag{2.24}$$

One can derive from (1.4), (2.11), the Markov inequality, (1.11), (2.18) and (2.20) that

$$\begin{aligned}
& \mathbb{P}\left( \sup_{l \in \mathcal{L}^j} 2^{jH(d_{j,le_j})} |\check{\Delta}'(j, le_j)| > 2^{-j(8b(1-\bar{H})/15)} \right) \leq \mathbb{P}\left( \sup_{l \in \mathcal{L}^j} |\check{\Delta}'(j, le_j)| > 2^{-j(\bar{H}+8b(1-\bar{H})/15)} \right) \\
& \leq \sum_{l \in \mathcal{L}^j} \mathbb{P}\left( |\check{\Delta}'(j, le_j)| > 2^{-j(\bar{H}+8b(1-\bar{H})/15)} \right) \leq 2^{jp(\bar{H}+8b(1-\bar{H})/15)} \sum_{l \in \mathcal{L}^j} \mathbb{E}\left( |\check{\Delta}'(j, le_j)|^p \right) \\
& \leq c_1 2^{j((1-b)-p(1-\bar{H}-8b(1-\bar{H})/15))} \leq c_1 2^{j(1-p(7(1-\bar{H})/15))}, \tag{2.25}
\end{aligned}$$

where  $c_1$  is a constant not depending on  $j$ . Moreover, it results from (1.15), (2.16), (2.11), the Markov inequality, (1.11), (2.19) and (2.21) that

$$\begin{aligned}
& \mathbb{P}\left(\sup_{l \in \mathcal{L}^j} 2^{jH(d_{j,le_j})} |\check{\Delta}''(j, le_j)| > 2^{-j(8b(1-\bar{H})/15)}\right) \\
& \leq \mathbb{P}\left(\sup_{l \in \mathcal{L}^j} 2^{jH(d_{j,le_j}-\eta)+j\kappa\eta^\gamma} |\check{\Delta}''(j, le_j)| > 2^{-j(8b(1-\bar{H})/15)}\right) \\
& \leq \mathbb{P}\left(\sup_{l \in \mathcal{L}^j} |2^{jH(d_{j,le_j}-\eta)} \check{\Delta}''(j, le_j)| > 2^{-j(11b(1-\bar{H})/15)}\right) \\
& \leq \sum_{l \in \mathcal{L}^j} \mathbb{P}\left(|2^{jH(d_{j,le_j}-\eta)} \check{\Delta}''(j, le_j)| > 2^{-j(11b(1-\bar{H})/15)}\right) \\
& \leq 2^{jp(11b(1-\bar{H})/15)} \sum_{l \in \mathcal{L}^j} \mathbb{E}\left(|2^{jH(d_{j,le_j}-\eta)} \check{\Delta}''(j, le_j)|^p\right) \\
& \leq c_2 2^{j((1-b)-pb(1-\bar{H})/15)}, \tag{2.26}
\end{aligned}$$

where  $c_2$  is a constant not depending on  $j$ . Next, putting together (2.23), (2.24), (2.25) and (2.26), one obtains that

$$\sum_{j=2}^{+\infty} \mathbb{P}\left(\sup_{l \in \mathcal{L}^j} 2^{jH(d_{j,le_j})} |\check{\Delta}(j, le_j)| > 2^{-j(8b(1-\bar{H})/15)+1}\right) < +\infty.$$

Thus, one can derive from the Borel-Cantelli Lemma that there is  $\Omega_3^*(b)$  an event of probability 1 depending  $b$  on which one has

$$\sup_{j \geq 2} \sup_{l \in \mathcal{L}^j} 2^{j(H(d_{j,le_j})+8b(1-\bar{H})/15)} |\check{\Delta}(j, le_j)| < +\infty. \tag{2.27}$$

Finally, it is clear that (2.27) implies that (2.15) is satisfied on  $\Omega_3^*(b)$ . ■

Let us now focus on the study of asymptotic behavior of the random variables  $|\tilde{\Delta}(j, le_j)|$  (see (2.13)) when  $j$  goes to  $+\infty$ . First, one needs to introduce some additional notations. For each fixed integer  $j \geq 6$ , the integer  $M_j \geq 1$  denotes the integer part of  $j^{-1}((2^j/e_j) - 2)$ , that is

$$M_j := \lfloor j^{-1}((2^j/e_j) - 2) \rfloor. \tag{2.28}$$

Moreover,  $(U_m^j)_{m \in \{0,1,\dots,M_j\}}$  denotes the subdivision of the interval  $[1, (2^j/e_j) - 1]$  by the  $M_j + 1$  integer points such that:

$$U_{M_j}^j := (2^j/e_j) - 1 \quad \text{and} \quad U_m^j := 1 + mj, \quad \text{for all } m \in \{0, 1, \dots, M_j - 1\}; \tag{2.29}$$

notice that

$$j \leq U_{M_j}^j - U_{M_j-1}^j < 2j. \quad (2.30)$$

For all integers  $j \geq 6$  and  $m \in \{1, \dots, M_j\}$ , let  $\mathcal{L}_m^j$  be the non-empty finite set of positive integers defined as:

$$\mathcal{L}_m^j := \mathbb{N} \cap [U_{m-1}^j, U_m^j]; \quad (2.31)$$

observe that, (2.10), (2.29), (2.30) and (2.31) entail that

$$\mathcal{L}^j = \bigcup_{m=1}^{M_j} \mathcal{L}_m^j \quad (2.32)$$

and

$$j < \text{Card}(\mathcal{L}_m^j) \leq 2j, \quad \text{for all } m \in \{1, \dots, M_j\}. \quad (2.33)$$

For every integers  $j \geq 6$ ,  $m \in \{1, \dots, M_j\}$  and  $l \in \mathcal{L}_m^j$ , the 3 random variables  $\widehat{\Delta}_m(j, le_j)$ ,  $\widehat{\Delta}'_m(j, le_j)$  and  $\widehat{\Delta}''_m(j, le_j)$  are defined as:

$$\widehat{\Delta}_m(j, le_j) := \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \sigma(\zeta_{j,m}) \left( (d_{j,le_j+1} - s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j} - s)_+^{H(\zeta_{j,m})-1/2} \right) dB(s), \quad (2.34)$$

$$\widehat{\Delta}'_m(j, le_j) := \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} (\sigma(s) - \sigma(\zeta_{j,m})) \left( (d_{j,le_j+1} - s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j} - s)_+^{H(\zeta_{j,m})-1/2} \right) dB(s) \quad (2.35)$$

and

$$\begin{aligned} \widehat{\Delta}''_m(j, le_j) := & \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \sigma(s) \left( \left( (d_{j,le_j+1} - s)^{H(s)-1/2} - (d_{j,le_j} - s)_+^{H(s)-1/2} \right) \right. \\ & \left. - \left( (d_{j,le_j+1} - s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j} - s)_+^{H(\zeta_{j,m})-1/2} \right) \right) dB(s), \end{aligned} \quad (2.36)$$

where the dyadic number  $\zeta_{j,m}$  is defined as:

$$\zeta_{j,m} := d_{j,(U_{m-1}^j-1)e_j+1}. \quad (2.37)$$

Notice that, in view of (2.13), (2.34), (2.35) and (2.36), one has

$$\widetilde{\Delta}(j, le_j) = \widehat{\Delta}_m(j, le_j) + \widehat{\Delta}'_m(j, le_j) + \widehat{\Delta}''_m(j, le_j), \quad \text{for all } l \in \mathcal{L}_m^j. \quad (2.38)$$

The following lemma allows to understand the reason for which one has introduced the random variables  $\widehat{\Delta}_m(j, le_j)$ ,  $l \in \mathcal{L}_m^j$ .

**Lemma 2.7** For all integers  $j \geq 6$  and  $m \in \{1, \dots, M_j\}$ , and for each finite sequence  $(z_l)_{l \in \mathcal{L}_m^j}$  of real numbers, one has, almost surely,

$$\begin{aligned} & \mathbb{E} \left( \exp \left( i \sum_{l \in \mathcal{L}_m^j} z_l \widehat{\Delta}_m(j, le_j) \right) \middle| \mathcal{F}_{\zeta_{j,m}} \right) \\ &= \exp \left( - 2^{-1} \sum_{l \in \mathcal{L}_m^j} z_l^2 \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \left| \sigma(\zeta_{j,m}) \left( (d_{j,le_j+1} - s)^{H(\zeta_{j,m})-1/2} \right. \right. \right. \\ & \qquad \qquad \qquad \left. \left. \left. - (d_{j,le_j} - s)_+^{H(\zeta_{j,m})-1/2} \right) \right|^2 ds \right). \end{aligned} \quad (2.39)$$

Notice that (2.39) means that conditionally to the sigma-algebra  $\mathcal{F}_{\zeta_{j,m}}$ , the random variables  $\widehat{\Delta}_m(j, le_j)$ ,  $l \in \mathcal{L}_m^j$ , have independent centred Gaussian distributions with variances  $\mathbb{E}(|\widehat{\Delta}_m(j, le_j)|^2 | \mathcal{F}_{\zeta_{j,m}})$ ,  $l \in \mathcal{L}_m^j$ , satisfying

$$\begin{aligned} & \mathbb{E}(|\widehat{\Delta}_m(j, le_j)|^2 | \mathcal{F}_{\zeta_{j,m}}) \\ &= \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \left| \sigma(\zeta_{j,m}) \left( (d_{j,le_j+1} - s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j} - s)_+^{H(\zeta_{j,m})-1/2} \right) \right|^2 ds \\ &\geq 2^{-1} |\sigma(\zeta_{j,m})|^2 2^{-2jH(\zeta_{j,m})}. \end{aligned} \quad (2.40)$$

**Proof of Lemma 2.7** The main idea of the proof consists in the observation that, for all integers  $j \geq 6$ ,  $m \in \{1, \dots, M_j\}$  and  $l \in \mathcal{L}_m^j$ , the Brownian motion  $B$  in (2.34) can be replaced by the Brownian motion  $W_{j,m} = \{W_{j,m}(x)\}_{x \in \mathbb{R}_+} := \{B(x + \zeta_{j,m}) - B(\zeta_{j,m})\}_{x \in \mathbb{R}_+}$  which is independent on the sigma-algebra  $\mathcal{F}_{\zeta_{j,m}}$ . Therefore  $W_{j,m}$  is independent on the integrand in (2.34), denoted by  $K_{j,m,l}$ , which is  $\mathcal{F}_{\zeta_{j,m}}$ -measurable. Having made this observation the proof becomes classical: it can be done in a standard way by approximating, for each fixed  $l \in \mathcal{L}_m^j$ , the integrand  $K_{j,m,l} = \{K_{j,m,l}(s)\}_{s \in [d_{j,(l-1)e_j+1}, d_{j,le_j+1}]}$  by a sequence  $(K_{j,m,l}^n)_{n \in \mathbb{N}} = (\{K_{j,m,l}^n(s)\}_{s \in [d_{j,(l-1)e_j+1}, d_{j,le_j+1}]})_{n \in \mathbb{N}}$  of elementary processes of the form:

$$K_{j,m,l}^n(s) = \sum_{p=0}^{q-1} A_p \mathbb{1}_{[t_p, t_{p+1})}(s),$$

where the random variables  $A_p$ ,  $0 \leq p < q$ , are  $\mathcal{F}_{\zeta_{j,m}}$ -measurable, and the finite sequence  $(t_p)_{0 \leq p \leq q}$  is a subdivision of the interval  $[d_{j,(l-1)e_j+1}, d_{j,le_j+1}]$ .  $\blacksquare$

For every fixed integer  $j \geq 6$ , let  $(\Lambda_{j,m})_{m \in \{1, \dots, M_j\}}$ ,  $(\check{\Lambda}_{j,m})_{m \in \{1, \dots, M_j\}}$ ,  $(\tilde{\Lambda}_{j,m})_{m \in \{1, \dots, M_j\}}$ ,  $(\hat{\Lambda}_{j,m})_{m \in \{1, \dots, M_j\}}$ ,  $(\hat{\Lambda}'_{j,m})_{m \in \{1, \dots, M_j\}}$  and  $(\hat{\Lambda}''_{j,m})_{m \in \{1, \dots, M_j\}}$  be the 6 finite sequences of non-negative finite random variables defined, for all  $m \in \{1, \dots, M_j\}$ , as:

$$\Lambda_{j,m} := \sup \left\{ |\Delta(j, le_j)| : l \in \mathcal{L}_m^j \right\}, \quad (2.41)$$

$$\check{\Lambda}_{j,m} := \sup \left\{ |\check{\Delta}(j, le_j)| : l \in \mathcal{L}_m^j \right\}, \quad (2.42)$$

$$\tilde{\Lambda}_{j,m} := \sup \left\{ |\tilde{\Delta}(j, le_j)| : l \in \mathcal{L}_m^j \right\}, \quad (2.43)$$

$$\hat{\Lambda}_{j,m} := \sup \left\{ |\hat{\Delta}_m(j, le_j)| : l \in \mathcal{L}_m^j \right\}, \quad (2.44)$$

$$\hat{\Lambda}'_{j,m} = \sup \left\{ |\hat{\Delta}'_m(j, le_j)| : l \in \mathcal{L}_m^j \right\}, \quad (2.45)$$

and

$$\hat{\Lambda}''_{j,m} := \sup \left\{ |\hat{\Delta}''_m(j, le_j)| : l \in \mathcal{L}_m^j \right\}. \quad (2.46)$$

The main ingredient of the proof of Theorem 1.5 is the following proposition.

**Proposition 2.8** *There exists  $\Omega_2^*(b)$  an event of probability 1 depending on  $b$  on which one has*

$$\liminf_{j \rightarrow +\infty} \inf_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|^{-1} \Lambda_{j,m} \geq 1/4. \quad (2.47)$$

For proving Proposition 2.8 one needs several preliminary results.

**Lemma 2.9** *Let  $\Omega_3^*(b)$  be the same event of probability 1 as in Lemma 2.4. Then, one has on  $\Omega_3^*(b)$*

$$\limsup_{j \rightarrow +\infty} \sup_{1 \leq m \leq M_j} 2^{j(H(\zeta_{j,m}) + b(1-\bar{H})/2)} \check{\Lambda}_{j,m} = 0. \quad (2.48)$$

**Proof of Lemma 2.9** It follows from (2.32), (2.42), (1.15), (2.37), (2.9), (2.6), (2.31), (2.29) and (2.30) that, for all integer  $j \geq 6$ , one has

$$\begin{aligned} & \sup_{l \in \mathcal{L}^j} 2^{j(H(d_{j,le_j}) + b(1-\bar{H})/2)} |\check{\Delta}(j, le_j)| = \sup_{1 \leq m \leq M_j} \sup_{l \in \mathcal{L}_m^j} 2^{j(H(d_{j,le_j}) + b(1-\bar{H})/2)} |\check{\Delta}(j, le_j)| \\ & \geq 2^{-2\kappa} j^{\gamma+1} 2^{-j(1-b)\gamma} \sup_{1 \leq m \leq M_j} 2^{j(H(\zeta_{j,m}) + b(1-\bar{H})/2)} \check{\Lambda}_{j,m} \\ & \geq c_0 \sup_{1 \leq m \leq M_j} 2^{j(H(\zeta_{j,m}) + b(1-\bar{H})/2)} \check{\Lambda}_{j,m}, \end{aligned} \quad (2.49)$$

where the strictly positive constant  $c_0 := \inf_{j \in \mathbb{N}} 2^{-2\kappa} j^{\gamma+1} 2^{-j(1-b)\gamma} > 0$ . Then combining (2.15) and (2.49) one obtains (2.48).  $\blacksquare$



**Lemma 2.10** *There exists  $\Omega_4^*(b)$  an event of probability 1 depending on  $b$  on which one has*

$$\liminf_{j \rightarrow +\infty} \inf_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|^{-1} \widehat{\Lambda}_{j,m} \geq 1/4. \quad (2.50)$$

**Proof of Lemma 2.10** Let  $\Omega_4^*(b)$  be the event defined as:

$$\Omega_4^*(b) := \bigcup_{J=6}^{+\infty} \bigcap_{j=J}^{+\infty} \bigcap_{m=1}^{M_j} \left\{ \omega \in \Omega : 2^{jH(\zeta_{j,m,\omega})} |\sigma(\zeta_{j,m,\omega})|^{-1} \widehat{\Lambda}_m^j(\omega) \geq 1/4 \right\}. \quad (2.51)$$

In order to show that the lemma holds, it is enough to prove that  $\mathbb{P}(\Omega_4^*(b)) = 1$  which is equivalent to prove that

$$\mathbb{P}(\Omega \setminus \Omega_4^*(b)) = 0. \quad (2.52)$$

Notice that, in view of (2.51) the event  $\Omega \setminus \Omega_4^*(b)$  can be expressed as:

$$\Omega \setminus \Omega_4^*(b) := \bigcap_{J=6}^{+\infty} \bigcup_{j=J}^{+\infty} \bigcup_{m=1}^{M_j} \left\{ \omega \in \Omega : \widehat{\Lambda}_m^j(\omega) < 4^{-1} 2^{-jH(\zeta_{j,m,\omega})} |\sigma(\zeta_{j,m,\omega})| \right\}.$$

Thus, one knows from the Borel-Cantelli Lemma that in order to derive (2.52) it is enough to prove that

$$\sum_{j=6}^{+\infty} \mathbb{P} \left( \bigcup_{m=1}^{M_j} \left\{ \omega \in \Omega : \widehat{\Lambda}_m^j(\omega) < 4^{-1} 2^{-jH(\zeta_{j,m,\omega})} |\sigma(\zeta_{j,m,\omega})| \right\} \right) < +\infty. \quad (2.53)$$

One clearly has, for every  $j \geq 6$ ,

$$\begin{aligned} & \mathbb{P} \left( \bigcup_{m=1}^{M_j} \left\{ \omega \in \Omega : \widehat{\Lambda}_m^j(\omega) < 4^{-1} 2^{-jH(\zeta_{j,m,\omega})} |\sigma(\zeta_{j,m,\omega})| \right\} \right) \\ & \leq \sum_{m=1}^{M_j} \mathbb{P} \left( \widehat{\Lambda}_m^j < 4^{-1} 2^{-jH(\zeta_{j,m,\omega})} |\sigma(\zeta_{j,m,\omega})| \right) \\ & = \sum_{m=1}^{M_j} \mathbb{P} \left( \bigcap_{l \in \mathcal{L}_m^j} \left\{ \omega \in \Omega : |\widehat{\Delta}(j, l e_j, \omega)| < 4^{-1} 2^{-jH(\zeta_{j,m,\omega})} |\sigma(\zeta_{j,m,\omega})| \right\} \right), \end{aligned} \quad (2.54)$$

where the last equality follows from (2.44). Moreover, using the fact that, for each  $m \in \{1, \dots, M_j\}$ ,

$$\begin{aligned} & \mathbb{P}\left(\bigcap_{l \in \mathcal{L}_m^j} \left\{ \omega \in \Omega : |\widehat{\Delta}(j, le_j, \omega)| < 4^{-1} 2^{-jH(\zeta_{j,m}, \omega)} |\sigma(\zeta_{j,m}, \omega)| \right\}\right) \\ &= \mathbb{E}\left(\prod_{l \in \mathcal{L}_m^j} \mathbb{1}_{\{|\widehat{\Delta}(j, le_j)| < 4^{-1} 2^{-jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|\}\right)} \\ &= \mathbb{E}\left(\mathbb{E}\left(\prod_{l \in \mathcal{L}_m^j} \mathbb{1}_{\{|\widehat{\Delta}(j, le_j)| < 4^{-1} 2^{-jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|\}\right) \middle| \mathcal{F}_{\zeta_{j,m}}\right)\end{aligned}$$

and Lemma 2.7, one gets that

$$\begin{aligned} & \mathbb{P}\left(\bigcap_{l \in \mathcal{L}_m^j} \left\{ \omega \in \Omega : |\widehat{\Delta}(j, le_j, \omega)| < 4^{-1} 2^{-jH(\zeta_{j,m}, \omega)} |\sigma(\zeta_{j,m}, \omega)| \right\}\right) \\ &= \mathbb{E}\left(\prod_{l \in \mathcal{L}_m^j} \left(2\pi \mathbb{E}(|\widehat{\Delta}_m(j, le_j)|^2 | \mathcal{F}_{\zeta_{j,m}})\right)^{-1/2}\right. \\ & \quad \left. \times \int_{\mathbb{R}} \exp\left(-2^{-1} \left(\mathbb{E}(|\widehat{\Delta}_m(j, le_j)|^2 | \mathcal{F}_{\zeta_{j,m}})\right)^{-1} x^2\right) \mathbb{1}_{\{|x| < 4^{-1} 2^{-jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|\}} dx\right) \\ &\leq \mathbb{E}\left(\prod_{l \in \mathcal{L}_m^j} \pi^{-1/2} |\sigma(\zeta_{j,m})|^{-1} 2^{jH(\zeta_{j,m})} \int_{\mathbb{R}} \mathbb{1}_{\{|x| < 4^{-1} 2^{-jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|\}} dx\right) \\ &\leq (2^{-1} \pi^{-1/2})^{\text{Card}(\mathcal{L}_m^j)} < (2^{-1} \pi^{-1/2})^j, \end{aligned} \tag{2.55}$$

where the first inequality and the last inequality respectively follow from (2.40) and (2.33). Finally, putting together (2.54), (2.55) and (2.28), one obtains for every  $j \geq 6$ , that

$$\mathbb{P}\left(\bigcup_{m=1}^{M_j} \left\{ \omega \in \Omega : \widehat{\Lambda}_m^j(\omega) < 4^{-1} 2^{-jH(\zeta_{j,m}, \omega)} |\sigma(\zeta_{j,m}, \omega)| \right\}\right) \leq M_j (2^{-1} \pi^{-1/2})^j \leq \pi^{-j/2},$$

which shows that (2.53) holds. ■

In view of Lemma 2.10 and of the fact that one has, almost surely

$$|\sigma|_{\inf} := \inf_{s \in [0,1]} |\sigma(s)| > 0, \tag{2.56}$$

the following two lemmas basically show that  $\sup_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} \widehat{\Lambda}'_{j,m}$  and  $\sup_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} \widehat{\Lambda}''_{j,m}$  are almost surely asymptotically negligible with respect to  $\inf_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|^{-1} \widehat{\Lambda}_{j,m}$  when  $j$  goes to  $+\infty$ .

**Lemma 2.11** *There is  $\Omega_5^*(b)$  an event of probability 1 depending on  $b$  on which one has*

$$\limsup_{j \rightarrow +\infty} \sup_{1 \leq m \leq M_j} 2^{j(H(\zeta_{j,m}) + (1-b)\gamma/2)} \widehat{\Lambda}'_{j,m} = 0. \quad (2.57)$$

**Lemma 2.12** *There is  $\Omega_6^*(b)$  an event of probability 1 depending on  $b$  on which one has*

$$\limsup_{j \rightarrow +\infty} \sup_{1 \leq m \leq M_j} 2^{j(H(\zeta_{j,m}) + (1-b)\gamma/2)} \widehat{\Lambda}''_{j,m} = 0. \quad (2.58)$$

Before proving Lemmas 2.11 and 2.12 let us give the proof of Proposition 2.8.

**Proof of Proposition 2.8** First notice that the event of probability 1  $\Omega_2^*(b)$  is defined as:  $\Omega_2^*(b) := \bigcap_{q=3}^6 \Omega_q^*(b)$ , where the  $\Omega_q^*(b)$ ,  $3 \leq q \leq 6$ , are the same events of probability 1 as in Lemmas 2.9, 2.10, 2.11 and 2.12. Using (2.12), (2.41), (2.42), (2.43) and the triangle inequality, one obtains, for all integers  $j \geq 6$  and  $m \in \{1, \dots, M_j\}$ , that  $\Lambda_{j,m} \geq \widetilde{\Lambda}_{j,m} - \check{\Lambda}_{j,m}$ . Thus, combining the latter inequality with (2.56) one gets that

$$\begin{aligned} & \inf_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|^{-1} \Lambda_{j,m} \\ & \geq \inf_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|^{-1} \widetilde{\Lambda}_{j,m} - |\sigma|_{\inf}^{-1} \sup_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} \check{\Lambda}_{j,m}. \end{aligned}$$

Then, one can derive from Lemma 2.9 that the following inequality holds on  $\Omega_2^*(b)$ :

$$\liminf_{j \rightarrow +\infty} \inf_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|^{-1} \Lambda_{j,m} \geq \liminf_{j \rightarrow +\infty} \inf_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|^{-1} \widetilde{\Lambda}_{j,m}. \quad (2.59)$$

Moreover, (2.38), (2.43), (2.44), (2.45), (2.46) and the triangle inequality entail, for all integer  $j \geq 6$  and  $m \in \{1, \dots, M_j\}$ , that  $\widetilde{\Lambda}_{j,m} \geq \widehat{\Lambda}_{j,m} - \widehat{\Lambda}'_{j,m} - \widehat{\Lambda}''_{j,m}$ . Thus, combining the latter inequality with (2.56) one obtains that

$$\begin{aligned} & \inf_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|^{-1} \widetilde{\Lambda}_{j,m} \\ & \geq \inf_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|^{-1} \widehat{\Lambda}_{j,m} \\ & \quad - |\sigma|_{\inf}^{-1} \left( \sup_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} \widehat{\Lambda}'_{j,m} + \sup_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} \widehat{\Lambda}''_{j,m} \right). \end{aligned}$$

Then, one can derive from Lemmas 2.10, 2.11 and 2.12 that the following inequality holds on  $\Omega_2^*(b)$ :

$$\liminf_{j \rightarrow +\infty} \inf_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} |\sigma(\zeta_{j,m})|^{-1} \widetilde{\Lambda}_{j,m} \geq 1/4. \quad (2.60)$$

Finally, (2.47) results from (2.59) and (2.60). ■

For proving Lemmas 2.11 and 2.12 one needs some preliminary results.

**Lemma 2.13** *Let  $\bar{J}_1 \geq 6$  be a fixed integer such that*

$$\kappa j^\gamma 2^{-j(1-b)\gamma} \leq 4^{-1} \underline{H} \quad \text{for all } j \geq \bar{J}_1. \quad (2.61)$$

*There is a finite deterministic constant  $c > 0$  such that, for all integers  $j \geq \bar{J}_1$  and  $m \in \{1, \dots, M_j\}$ , one has*

$$\begin{aligned} & 2^{2jH(\zeta_{j,m})} \sup_{l \in \mathcal{L}_m^j} \left\{ \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} |\sigma(s)|^2 \left| \left( (d_{j,le_{j+1}} - s)^{H(s)-1/2} - (d_{j,le_j} - s)_+^{H(s)-1/2} \right) \right. \right. \\ & \quad \left. \left. - \left( (d_{j,le_{j+1}} - s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j} - s)_+^{H(\zeta_{j,m})-1/2} \right) \right|^2 ds \right\} \\ & \leq c j^{2(1+\gamma)} 2^{-2j(1-b)\gamma}. \end{aligned} \quad (2.62)$$

**Proof of Lemma 2.13** Throughout the proof the integers  $j \geq \bar{J}_1$ ,  $m \in \{1, \dots, M_j\}$  and  $l \in \mathcal{L}_m^j$  are arbitrary and fixed. Let  $s \in [d_{j,(l-1)e_j+1}, d_{j,le_j})$  be arbitrary and fixed; using the mean value Theorem, (1.15), the inequalities

$$0 < (d_{j,le_j} - s) < (d_{j,le_{j+1}} - s) < 1,$$

(2.37), (2.9), (2.6), (2.31), (2.29), (2.30) and the triangle inequality one obtains, for some real number  $\beta$  belonging to interval  $[H(s) \wedge H(\zeta_{j,m}), H(s) \vee H(\zeta_{j,m})]$ , that

$$\begin{aligned}
& \left| \left( (d_{j,le_{j+1}} - s)^{H(s)-1/2} - (d_{j,le_j} - s)^{H(s)-1/2} \right) \right. \\
& \quad \left. - \left( (d_{j,le_{j+1}} - s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j} - s)^{H(\zeta_{j,m})-1/2} \right) \right| \\
& \leq \left| (d_{j,le_{j+1}} - s)^{\beta-1/2} \log(d_{j,le_{j+1}} - s) - (d_{j,le_j} - s)^{\beta-1/2} \log(d_{j,le_j} - s) \right| |H(s) - H(\zeta_{j,m})| \\
& \leq 2\kappa j^\gamma 2^{-j(1-b)\gamma} \left| (d_{j,le_{j+1}} - s)^{\beta-1/2} \log(d_{j,le_{j+1}} - s) - (d_{j,le_j} - s)^{\beta-1/2} \log(d_{j,le_j} - s) \right| \\
& \leq 2\kappa j^\gamma 2^{-j(1-b)\gamma} \left( (d_{j,le_{j+1}} - s)^{H(\zeta_{j,m})-2\kappa j^\gamma 2^{-j(1-b)\gamma}-1/2} |\log(d_{j,le_{j+1}} - s)| \right. \\
& \quad \left. + (d_{j,le_j} - s)^{H(\zeta_{j,m})-2\kappa j^\gamma 2^{-j(1-b)\gamma}-1/2} |\log(d_{j,le_j} - s)| \right) \mathbb{1}_{[d_{j,le_{j-1}}, d_{j,le_j}]}(s) \\
& \quad + 2\kappa j^\gamma 2^{-j(1+(1-b)\gamma)} (d_{j,le_j} - s)^{H(\zeta_{j,m})-2\kappa j^\gamma 2^{-j(1-b)\gamma}-3/2} (1 + |\log(d_{j,le_j} - s)|) \mathbb{1}_{[d_{j,(l-1)e_{j+1}}, d_{j,le_{j-1}}]}(s).
\end{aligned} \tag{2.63}$$

Moreover, notice that

$$c_1 := \sup_{j \in \mathbb{N}} 2^{2\kappa j^\gamma + 1} 2^{-j(1-b)\gamma} < +\infty. \tag{2.64}$$

One can derive from (2.63), (1.8), (1.4), the change of variables  $u = 2^j(s - d_{j,le_j})$  and  $v = -u$ , (2.64), and (2.61) that

$$\begin{aligned}
& \int_{d_{j,(1-b)e_j+1}}^{d_{j,le_j}} |\sigma(s)|^2 \left| \left( (d_{j,le_{j+1}} - s)^{H(s)-1/2} - (d_{j,le_j} - s)^{H(s)-1/2} \right) \right. \\
& \quad \left. - \left( (d_{j,le_{j+1}} - s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j} - s)^{H(\zeta_{j,m})-1/2} \right) \right|^2 ds \\
& \leq 4\kappa^2 \bar{L}^2 j^{2\gamma} 2^{-j(1+2(1-b)\gamma)} \int_{-1}^0 \left( (2^{-j} - 2^{-j}u)^{H(\zeta_{j,m})-2\kappa j^\gamma 2^{-j(1-b)\gamma}-1/2} |\log(2^{-j} - 2^{-j}u)| \right. \\
& \quad \left. + (-2^{-j}u)^{H(\zeta_{j,m})-2\kappa j^\gamma 2^{-j(1-b)\gamma}-1/2} |\log(-2^{-j}u)| \right)^2 du \\
& \quad + 4\kappa^2 \bar{L}^2 j^{2\gamma} 2^{-j(3+2(1-b)\gamma)} \int_{-\infty}^{-1} (-2^{-j}u)^{2H(\zeta_{j,m})-4\kappa j^\gamma 2^{-j(1-b)\gamma}-3} (1 + |\log(-2^{-j}u)|)^2 du \\
& \leq c_2 j^{2(1+\gamma)} 2^{-2j(H(\zeta_{j,m})+(1-b)\gamma)} \int_0^1 \left( (1+v)^{H(\zeta_{j,m})-2\kappa j^\gamma 2^{-j(1-b)\gamma}-1/2} (1 + |\log(1+v)|) \right. \\
& \quad \left. + v^{H(\zeta_{j,m})-2\kappa j^\gamma 2^{-j(1-b)\gamma}-1/2} (1 + |\log(v)|) \right)^2 dv \\
& \quad + c_2 j^{2(1+\gamma)} 2^{-2j(H(\zeta_{j,m})+(1-b)\gamma)} \int_1^{+\infty} v^{2H(\zeta_{j,m})-4\kappa j^\gamma 2^{-j(1-b)\gamma}-3} (2 + |\log(v)|)^2 dv \\
& \leq c_3 j^{2(1+\gamma)} 2^{-2j(H(\zeta_{j,m})+(1-b)\gamma)}, \tag{2.65}
\end{aligned}$$

where the constant finite constant  $c_2 := 4\kappa^2 \bar{L}^2 c_1^2$  and the finite constant

$$\begin{aligned}
c_3 := c_2 \int_0^1 & \left( (1+v)^{\bar{H}-1/2} (1 + |\log(1+v)|) + v^{\underline{H}/2-1/2} (1 + |\log(v)|) \right)^2 dv \\
& + c_2 \int_1^{+\infty} v^{2\bar{H}-3} (2 + |\log(v)|)^2 dv.
\end{aligned}$$

Similarly to (2.65) it can be shown that

$$\begin{aligned}
& \int_{d_{j,le_j}}^{d_{j,le_{j+1}}} |\sigma(s)|^2 \left| (d_{j,le_{j+1}} - s)^{H(s)-1/2} - (d_{j,le_{j+1}} - s)^{H(\zeta_{j,m})-1/2} \right|^2 ds \\
& \leq c_4 j^{2(1+\gamma)} 2^{-2j(H(\zeta_{j,m})+(1-b)\gamma)}, \tag{2.66}
\end{aligned}$$

where the finite constant

$$c_4 := c_2 \int_0^1 (1-u)^{\bar{H}-1} (2 + |\log(1-u)|)^2 du.$$

Finally, (2.62) results from (2.8), (2.65) and (2.66). ■

**Lemma 2.14** *There are two finite deterministic constants  $0 < c' < c''$  such that, for all integers  $j \geq 6$ ,  $m \in \{1, \dots, M_j\}$  and  $l \in \mathcal{L}_m^j$ , one has*

$$c' 2^{-2jH(\zeta_{j,m})} \leq \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \left| (d_{j,le_j+1} - s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j} - s)_+^{H(\zeta_{j,m})-1/2} \right|^2 ds \leq c'' 2^{-2jH(\zeta_{j,m})}. \quad (2.67)$$

**Proof of Lemma 2.14** In view of (2.6), it is clear that

$$\int_{d_{j,le_j}}^{d_{j,le_j+1}} (d_{j,le_j+1} - s)^{2H(\zeta_{j,m})-1} ds = \frac{2^{-2jH(\zeta_{j,m})}}{2H(\zeta_{j,m})}. \quad (2.68)$$

Thus, setting  $c' := (2\bar{H})^{-1} > 2^{-1}$  and noticing that (see (2.8))

$$\begin{aligned} & \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} \left| (d_{j,le_j+1} - s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j} - s)_+^{H(\zeta_{j,m})-1/2} \right|^2 ds \\ &= \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j}} \left| (d_{j,le_j+1} - s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j} - s)^{H(\zeta_{j,m})-1/2} \right|^2 ds \\ & \quad + \int_{d_{j,le_j}}^{d_{j,le_j+1}} (d_{j,le_j+1} - s)^{2H(\zeta_{j,m})-1} ds, \end{aligned} \quad (2.69)$$

it clearly follows from (2.68) and (1.4) that the first inequality in (2.67) is satisfied. Thus, from now on, our goal is to show that the second inequality in (2.67) holds. Using (2.6), the change of variable  $x = 2^j(d_{j,le_j} - s)$ , and standard computations, one gets that

$$\begin{aligned} & \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j}} \left| (d_{j,le_j+1} - s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j} - s)^{H(\zeta_{j,m})-1/2} \right|^2 ds \\ & \leq 2^{-2jH(\zeta_{j,m})} \int_0^{+\infty} \left| (x+1)^{H(\zeta_{j,m})-1/2} - x^{H(\zeta_{j,m})-1/2} \right|^2 dx. \end{aligned} \quad (2.70)$$

One can easily derive from (1.4) that

$$\int_0^1 \left| (x+1)^{H(\zeta_{j,m})-1/2} - x^{H(\zeta_{j,m})-1/2} \right|^2 dx \leq 2 \int_0^1 (x+1)^{2\bar{H}-1} dx + 2 \int_0^1 x^{2\bar{H}-1} dx. \quad (2.71)$$

Moreover, it follows from the mean value Theorem and (1.4) that

$$\int_1^{+\infty} \left| (x+1)^{H(\zeta_{j,m})-1/2} - x^{H(\zeta_{j,m})-1/2} \right|^2 dx \leq \int_1^{+\infty} x^{2\bar{H}-3} dx. \quad (2.72)$$

Finally, letting  $c''$  be the finite constant defined as

$$c'' := (2\underline{H})^{-1} + 2 \int_0^1 (x+1)^{2\overline{H}-1} dx + 2 \int_0^1 x^{2\underline{H}-1} dx + \int_1^{+\infty} x^{2\overline{H}-3} dx,$$

one can derive from (1.4) and (2.68) to (2.72) that the second inequality in (2.67) holds. ■

**Lemma 2.15** *There is a finite deterministic constant  $c > 0$  such that, for all integers  $j \geq 6$  and  $m \in \{1, \dots, M_j\}$ , one has*

$$\begin{aligned} & 2^{2jH(\zeta_{j,m})} \sup_{l \in \mathcal{L}_m^j} \left\{ \int_{d_{j,(l-1)e_j+1}}^{d_{j,le_j+1}} |\sigma(s) - \sigma(\zeta_{j,m})|^2 \right. \\ & \quad \left. \times \left| (d_{j,le_j+1} - s)^{H(\zeta_{j,m})-1/2} - (d_{j,le_j} - s)_+^{H(\zeta_{j,m})-1/2} \right|^2 ds \right\} \\ & \leq c j^{2\gamma} 2^{-2j(1-b)\gamma}. \end{aligned} \quad (2.73)$$

**Proof of Lemma 2.15** Let  $j \geq 6$ ,  $m \in \{1, \dots, M_j\}$ ,  $l \in \mathcal{L}_m^j$  and  $s \in [d_{j,(l-1)e_j+1}, d_{j,le_j+1})$  be arbitrary and fixed. It follows from (1.15), (2.37), (2.9), (2.6), (2.31), (2.29) and (2.30) that

$$|\sigma(s) - \sigma(\zeta_{j,m})| \leq 2\kappa j^\gamma 2^{-j(1-b)\gamma}. \quad (2.74)$$

Thus combining (2.74) and the second inequality in (2.67) one obtains (2.73). ■

We are now ready to prove Lemmas 2.11 and 2.12.

**Proof of Lemma 2.11** Let  $p$  be a fixed real number such that

$$p > 3/\gamma > 1. \quad (2.75)$$



It follows from (2.45), the Markov inequality, (1.11), (2.35), (2.73), (2.28), (2.9) and (2.33), that one has, for all integer  $j \geq 6$ ,

$$\begin{aligned}
& \mathbb{P}\left(\sup_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} \widehat{\Lambda}'_{j,m} > 2^{-2j(1-b)\gamma/3}\right) \leq \sum_{m=1}^{M_j} \mathbb{P}\left(2^{jH(\zeta_{j,m})} \widehat{\Lambda}'_{j,m} > 2^{-2j(1-b)\gamma/3}\right) \\
& \leq \sum_{m=1}^{M_j} \sum_{l \in \mathcal{L}_m^j} \mathbb{P}\left(2^{jH(\zeta_{j,m})} |\widehat{\Delta}'_m(j, le_j)| > 2^{-2j(1-b)\gamma/3}\right) \\
& \leq 2^{2jp(1-b)\gamma/3} \sum_{m=1}^{M_j} \sum_{l \in \mathcal{L}_m^j} \mathbb{E}\left(|2^{jH(\zeta_{j,m})} \widehat{\Delta}'_m(j, le_j)|^p\right) \leq c_1 M_j \text{Card}(\mathcal{L}_m^j) j^{p\gamma} 2^{-jp(1-b)\gamma/3} \\
& \leq c_2 j^{p\gamma} 2^{-j(1-b)(p\gamma/3-1)}, \tag{2.76}
\end{aligned}$$

where  $c_1$  and  $c_2$  are two finite constants not depending on  $j$ . Next, combining (2.76) and (2.75), one obtains that

$$\sum_{j=6}^{+\infty} \mathbb{P}\left(\sup_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} \widehat{\Lambda}'_{j,m} > 2^{-2j(1-b)\gamma/3}\right) < +\infty.$$

Thus, one can derive from the Borel-Cantelli Lemma that there is  $\Omega_5^*(b)$  an event of probability 1 depending  $b$  on which one has

$$\sup_{j \geq 6} \sup_{1 \leq m \leq M_j} 2^{j(H(\zeta_{j,m})+2(1-b)\gamma/3)} \widehat{\Lambda}'_{j,m} < +\infty. \tag{2.77}$$

Finally, it is clear that (2.77) implies that (2.57) is satisfied on  $\Omega_5^*(b)$ .  $\blacksquare$

**Proof of Lemma 2.12** Let  $p$  be a fixed real number satisfying (2.75). It follows from (2.46), the Markov inequality, (1.11), (2.36), (2.62), (2.28), (2.9) and (2.33), that one has, for all integer  $j \geq \bar{J}_1$  (see (2.61)),

$$\begin{aligned}
& \mathbb{P}\left(\sup_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} \widehat{\Lambda}''_{j,m} > 2^{-2j(1-b)\gamma/3}\right) \leq \sum_{m=1}^{M_j} \mathbb{P}\left(2^{jH(\zeta_{j,m})} \widehat{\Lambda}''_{j,m} > 2^{-2j(1-b)\gamma/3}\right) \\
& \leq \sum_{m=1}^{M_j} \sum_{l \in \mathcal{L}_m^j} \mathbb{P}\left(2^{jH(\zeta_{j,m})} |\widehat{\Delta}''_m(j, le_j)| > 2^{-2j(1-b)\gamma/3}\right) \\
& \leq 2^{2jp(1-b)\gamma/3} \sum_{m=1}^{M_j} \sum_{l \in \mathcal{L}_m^j} \mathbb{E}\left(|2^{jH(\zeta_{j,m})} \widehat{\Delta}''_m(j, le_j)|^p\right) \leq c_1 M_j \text{Card}(\mathcal{L}_m^j) j^{p(1+\gamma)} 2^{-jp(1-b)\gamma/3} \\
& \leq c_2 j^{p(1+\gamma)} 2^{-j(1-b)(p\gamma/3-1)}, \tag{2.78}
\end{aligned}$$

where  $c_1$  and  $c_2$  are two finite constants not depending on  $j$ . Next, combining (2.78) and (2.75), one obtains that

$$\sum_{j=\bar{J}_1}^{+\infty} \mathbb{P} \left( \sup_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} \widehat{\Lambda}_{j,m}'' > 2^{-2j(1-b)\gamma/3} \right) < +\infty.$$

Thus, one can derive from the Borel-Cantelli Lemma that there is  $\Omega_6^*(b)$  an event of probability 1 depending  $b$  on which one has

$$\sup_{j \geq \bar{J}_1} \sup_{1 \leq m \leq M_j} 2^{j(H(\zeta_{j,m})+2(1-b)\gamma/3)} \widehat{\Lambda}_{j,m}'' < +\infty. \quad (2.79)$$

Finally, it is clear that (2.79) implies that (2.58) is satisfied on  $\Omega_6^*(b)$ . ■

We are now in position to complete the proof of Theorem 1.5.

**End of the proof of Theorem 1.5** In view of Remark 2.3 we will only prove that (1.14) holds in the case where  $X$  is the process  $Z$  defined through (2.1) and  $\tau \in (0, 1)$ . For any integer  $j \geq 6$ , one sets

$$l_j(\tau) := \lfloor 2^j \tau / e_j \rfloor, \quad (2.80)$$

where  $\lfloor \cdot \rfloor$  denotes as usual the integer part function and  $e_j$  is as in (2.9). It is clear that  $l_j(\tau)$  satisfies

$$0 \leq \tau - \frac{l_j(\tau)e_j}{2^j} < \frac{e_j}{2^j} \leq 2^{-j(1-b)}. \quad (2.81)$$

Moreover, it easily follows from (2.80), the inequalities  $0 < b < 1/2$ ,  $0 < \tau < 1$ , (2.10) and (2.9) that there exists a positive integer  $\bar{J}_2(\tau) \geq 6$ , such that, for all integer  $j \geq \bar{J}_2(\tau)$ , one has

$$l_j(\tau) \in \mathcal{L}^j \quad (2.82)$$

and  $[\tau - 12j2^{-j(1-b)}, \tau + 12j2^{-j(1-b)}] \subset [\tau - 2^{-j(1-2b)+1}, \tau + 2^{-j(1-2b)+1}] \subset (0, 1)$ . Next observe that in view of (2.82) and (2.32), there is  $m_j(\tau) \in \{1, \dots, M_j\}$  such that

$$l_j(\tau) \in \mathcal{L}_{m_j(\tau)}^j. \quad (2.83)$$

Thus, one can derive from (2.83), (2.31), (2.29) and (2.30) that

$$|l_j(\tau) - l| < 2j, \quad \text{for all } l \in \mathcal{L}_{m_j(\tau)}^j. \quad (2.84)$$

Next, combining (2.84), (2.81), (2.9) and the triangle inequality, one gets that

$$\left| \tau - \frac{le_j}{2^j} \right| < \frac{(2j+1)e_j}{2^j} \leq \frac{3je_j}{2^j} \leq 3j2^{-j(1-b)}, \quad \text{for all } l \in \mathcal{L}_{m_j(\tau)}^j, \quad (2.85)$$

and

$$\left| \tau - \frac{le_j + 1}{2^j} \right| < \frac{(2j+1)e_j + 1}{2^j} \leq \frac{3je_j}{2^j} \leq 3j2^{-j(1-b)}, \quad \text{for all } l \in \mathcal{L}_{m_j(\tau)}^j. \quad (2.86)$$

Next it follows from (2.41), (2.5), (2.6), (1.2), (2.85), (2.86), (2.81), (2.83), (2.31), (2.37), (2.9), (2.29), (2.30) and (1.15) that

$$\begin{aligned} 2^{jH(\tau)} \text{Osc}_Z(\tau, 3j2^{-j(1-b)}) &\geq 2^{jH(\tau)} \Lambda_{m_j(\tau)}^j \geq 2^{-j|H(\tau) - H(\zeta_{j,m_j(\tau)})|} 2^{jH(\zeta_{j,m_j(\tau)})} \Lambda_{m_j(\tau)}^j \\ &\geq 2^{-j\kappa(2j+1)^\gamma 2^{-j\gamma(1-b)}} \inf_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} \Lambda_m^j \geq c_1 \inf_{1 \leq m \leq M_j} 2^{jH(\zeta_{j,m})} \Lambda_m^j, \end{aligned} \quad (2.87)$$

where the deterministic constant  $c_1 := \inf_{j \in \mathbb{N}} 2^{-j\kappa(2j+1)^\gamma 2^{-j\gamma/2}} > 0$  (recall that  $b \in (0, 1/2)$ ). Thus, one can derive from (2.87), Proposition 2.8, (2.56) and (1.2) that the following inequalities hold on  $\Omega_2^*(b)$  (the event of probability 1 introduced in Proposition 2.8):

$$\liminf_{j \rightarrow +\infty} 2^{jH(\tau)} \text{Osc}_Z(\tau, 2^{-j(1-2b)}) \geq \liminf_{j \rightarrow +\infty} 2^{jH(\tau)} \text{Osc}_Z(\tau, 3j2^{-j(1-b)}) \geq 4^{-1} c_1 |\sigma|_{\text{inf}}.$$

Hence, for all  $\omega \in \Omega_2^*(b)$ , there exists an integer  $j_3 = j_3(\tau, \omega) \geq \bar{J}_2(\tau)$  such that, for all integer  $j \geq j_3$ , one has

$$2^{jH(\tau, \omega)} \text{Osc}_Z(\tau, 2^{-j(1-2b)}, \omega) \geq 5^{-1} c_1 |\sigma|_{\text{inf}}(\omega). \quad (2.88)$$

Next, let  $\rho$  be an arbitrary positive real number such  $\rho \leq 2^{-j_3}$ , one sets

$$j^*(\rho) := \lfloor -\log_2 \rho \rfloor. \quad (2.89)$$

One clearly has that  $j^*(\rho) \geq j_3$  and

$$2^{j^*(\rho)} \leq \rho^{-1} < 2^{j^*(\rho)+1}. \quad (2.90)$$

Then, (2.88), (2.89), (2.90) and (1.2) imply that, for all  $\rho \in [j_3, +\infty)$ , one has

$$\rho^{-H(\tau, \omega)} \text{Osc}_Z(\tau, (2\rho)^{1-2b}, \omega) \geq 2^{j^*(\rho)H(\tau, \omega)} \text{Osc}_Z(\tau, 2^{-j^*(\rho)(1-2b)}, \omega) \geq 5^{-1} c_1 |\sigma|_{\text{inf}}(\omega).$$

Thus, for all  $r \in [2j_3, +\infty)$ , one has

$$r^{-H(\tau, \omega)} \text{Osc}_Z(\tau, r^{1-2b}, \omega) \geq 10^{-1} c_1 |\sigma|_{\text{inf}}(\omega). \quad (2.91)$$

It clearly follows from (2.91) that

$$\liminf_{r \rightarrow 0_+} r^{-H(\tau)} \text{Osc}_Z(\tau, r^{1-2b}) \geq 10^{-1} c_1 |\sigma|_{\text{inf}}, \quad \text{on } \Omega_2^*(b),$$

which amounts to saying that

$$\liminf_{r \rightarrow 0_+} r^{-H(\tau)/(1-2b)} \text{Osc}_Z(\tau, r) \geq 10^{-1} c_1 |\sigma|_{\text{inf}}, \quad \text{on } \Omega_2^*(b). \quad (2.92)$$

Finally, denoting by  $\Omega_2^*$  the event of probability 1 defined as  $\Omega_2^* := \bigcap_{b \in (0, 1/2) \cap \mathbb{Q}} \Omega_2^*(b)$ , where  $\mathbb{Q}$  is the countable set of the rational numbers, it results from (2.92), the inequality  $c_1 |\sigma|_{\text{inf}} > 0$  and the equality  $\lim_{b \rightarrow 0_+} H(\tau)/(1-2b) = H(\tau)$  that (1.14) holds when  $X = Z$  and  $\tau \in (0, 1)$ . ■

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