

THE MEAN-FIELD LIMIT FOR HYBRID MODELS OF COLLECTIVE MOTIONS WITH CHEMOTAXIS,

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ABSTRACT. In this paper we study a general class of hybrid mathematical models of collective motions of cells under the influence of chemical stimuli. The models are hybrid in the sense that cells are discrete entities given by ODE, while the chemoattractant is considered as a continuous signal which solves a diffusive equation. For this model we prove the mean-field limit in the Wasserstein distance to a system given by the coupling of a Vlasov-type equation with the chemoattractant equation. Our approach is not based on empirical measures and we show the limit with explicit bounds, by proving also existence and uniqueness for the limit system. In the monokinetic case we derive pressureless nonlocal Euler-type model with chemotaxis.

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1. INTRODUCTION

A collective motion occurs when the behaviour of a group of individuals is dominated by the mutual interaction between them. This behaviour arises in many different contexts both for non-living and living systems, as for instance nematic fluids, simple

robots, bacteria colonies, flocks of birds, schools of fishes, human crowds, see for instance [40]. In a nutshell, all microscopic mathematical models of collective motion are based on one or more of the following elementary mechanisms: *alignment*, see [39], [7], and references therein, *separation* and *cohesion* [11, 36]. Concerning alignment models a popular one is represented by the Cucker-Smale model [7], which was originally proposed to describe the dynamics in a flock of birds, but then it was extended to cover more general phenomena, as for instance animal herding [10]. The hypothesis of the Cucker-Smale model is that the force acting on every individual is a weighted average of the differences of its velocity with those of the others, and this force decays when the distance between the individuals increases. Some preliminary analytical results about the time asymptotic behaviour of the model has been proven in [7, 22], and in the following a lot of papers investigated the behaviour of this dynamical model in many directions, see for instance [6] and [32] for a comprehensive list of references.

In recent years, there was a lot of interest about collective motion of cells driven by chemical stimuli, see [37, 3, 2, 35, 28, 9, 34], and the reviews [24, 30]. Focusing on the family of Cucker-Smale models, in [12] a model for the morphogenesis in the zebrafish lateral line primordium was proposed, where a Cucker-Smale model was coupled with other cell mechanisms (chemotaxis, attraction-repulsion, damping effects) to describe the formation of neuromasts, see [16, 27] for the experimental basis of this model. The description of the cell behaviour is hybrid: while particles are considered discrete entities, endowed of a radius R describing their circular shape, the chemical signal φ is supposed to be continuous and its time derivative is equal to a diffusion term, a source term depending on the position of each particle, and a degradation term. A simplified version of the model in [12] was proposed in [13] to allow a full analytical investigation. This simplified model reads as follows:

$$(1) \quad \begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \frac{\beta}{N} \sum_{j=1}^N \frac{1}{\left(1 + \frac{\|x_i - x_j\|^2}{R^2}\right)^\sigma} (v_j - v_i) + \eta \nabla_x \varphi(x_i), \\ \partial_t \varphi = D \Delta \varphi - \kappa \varphi + f(x, X(t)), \end{cases}$$

Initial data are given by initial position and velocity for each particle:

$$X(0) = X_0, \quad V(0) = V_0,$$

with $X = (x_1, \dots, x_N)$, $V = (v_1, \dots, v_N)$, and by the initial concentration of signal, that it is assumed

$$(2) \quad \varphi(x, 0) := \varphi_0 = 0.$$

Here x_i, v_i are the position and velocity of the i -th cell and φ stands for a generic chemical signal produced by the cells themselves and such that the cells are attracted towards the direction where $\nabla_x \varphi$ is growing. For this simple model in [13] a full analytical theory was developed in the two-dimensional case with a fixed but arbitrary

number N of particles, and results of globally in time existence and uniqueness of solutions were proved, as well as the time-asymptotic linear stability. Other analytical results, for more general hybrid models, can be found in [31].

In this paper we aim to prove the mean-field limit of a general class of models including (1) towards Vlasov type kinetic equations together with the hydrodynamic mean-field limit of such models towards Euler type equations coupled with chemotaxis. To our knowledge, both limits, and the related kinetic and Euler equations are new in the literature.

Let us describe the class of particle systems we will handle in the present article.

Consider on $\mathbf{R}^{2dN} \ni ((x_i(t))_{i=1,\dots,N}, (v_i(t))_{i=1,\dots,N}) := (X(t), V(t))$ the following vector field

$$(3) \quad \begin{cases} \dot{x}_i(t) = v_i \\ \dot{v}_i(t) = F_i(t, X(t), V(t)) \end{cases} \quad i = 1, \dots, N, \quad (X(0), V(0)) = (X^{in}, V^{in}) :$$

where

$$(4) \quad F_i(t, X, V) = \frac{1}{N} \sum_{j=1}^N \gamma(v_i - v_j, x_i - x_j) + \eta \nabla_x \varphi^t(x_i) + F_{ext}(x_i),$$

γ is the collective interaction function, F_{ext} is an external force and φ satisfies the equation

$$(5) \quad \partial_s \varphi^s(x) = D \Delta_x \varphi - \kappa \varphi + f(x, X(s)), \quad s \in [0, t], \quad \varphi^{s=0} = \varphi^{in}$$

for some $\kappa, D, \eta \geq 0$ and function f of the form

$$(6) \quad f(x, X) = \frac{1}{N} \sum_{j=1}^N \chi(x - x_j), \quad \chi \in \mathcal{C}_c^1(B_R).$$

The function $\gamma : \mathbf{R}^d \times \mathbf{R}^b \rightarrow \mathbf{R} \times \mathbf{R}^d$ is supposed to be Lipschitz continuous.

The case $\gamma(y, w) = \Phi(y)w$. $F = \varphi = 0$, covers the standard case of Cucker-Smale models.

For any t, N we define the mapping $\Phi_N^t = \Phi^t$ by

$$(7) \quad \begin{cases} \Phi_N^t : & \mathbf{R}^{2dN} \longrightarrow \mathbf{R}^{2dN} \\ Z^{in} = (X^{in}, V^{in}) & \longrightarrow Z(t) = (X(t), V(t)) \text{ solution of (3)}. \end{cases}$$

Note that Φ_N^t is not a flow.

We would like to derive a kinetic model corresponding to the system (3), that is the one particle (non-linear) PDE satisfied by the first marginal of the push-forward¹ $\Phi_N^t \# \rho^{in}$ where $\rho^{in} \in \mathcal{P}(\mathbf{R}^{2dN})$ and Φ_N^t is the mapping defined by (7).

¹We recall that the pushforward of a measure μ by a measurable function Φ is $\Phi \# \mu$ defined by $\int \varphi d(\Phi \# \mu) := \int (\varphi \circ \Phi) d\mu$ for every measurable function φ .

The first difficulty is the fact that $\rho_N^t := \Phi_N^t \# \rho_N^{in}$ does not satisfy a closed PDE, except if

$$(8) \quad \rho_N^{in} = \rho_{\bar{Z}} = \frac{1}{N!} \sum_{\sigma \in \sigma_N} \delta_{\sigma(\bar{Z})}, \quad \bar{Z} := (\bar{X}, \bar{V}) \in \mathbf{R}^{2dN},$$

where σ_N is the group of permutations in the N variables $(\bar{x}_i, \bar{v}_i)_{i=1, \dots, N}$.

In this case $\rho_N^t := \rho_{\Phi^t(\bar{Z})}$ satisfies

$$(9) \quad \partial_t \rho_N^t + V \cdot \nabla_X \rho_N^t = \sum_{i=1}^N \nabla_{v_i} \cdot G_i \rho_N^t$$

where

$$(10) \quad G_i(t, X, V) = \frac{1}{N} \sum_{j=1}^N \gamma(v_i - v_j, x_i - x_j) + \eta \nabla_x \Psi^t(x_i) + F_{ext}(x_i),$$

Ψ (and therefore G_i too) depends on the solution ρ_N^t and satisfies the equation

$$(11) \quad \partial_t \Psi^t(x) = D \Delta_x \Psi - \kappa \Psi + f(x, \rho_{N;1}^s), \quad s \in [0, t],$$

with g given by

$$(12) \quad f(x, \rho_{N;1}^s) = \int_{\mathbf{R}^{2d}} \chi(x - y) \rho_{N;1}^s(y, \xi) dy d\xi$$

with, denoting $\Phi_N^t(\bar{Z}) = (\bar{x}_1(t), \dots, \bar{x}_N(t), \bar{v}_1(t), \dots, \bar{v}_N(t))$,

$$\begin{aligned} \rho_{N;1}^s(y, \xi) &:= \int_{\mathbf{R}^{2d(N-1)}} \rho_N^t(y, x_2, \dots, x_N, \xi, v_2, \dots, v_N) dx_2 \dots dx_N dv_2 \dots dv_N \\ &= \int_{\mathbf{R}^{2d(N-1)}} \rho_{\Phi_N^t(\bar{Z})}(y, x_2, \dots, x_N, \xi, v_2, \dots, v_N) dx_2 \dots dx_N dv_2 \dots dv_N \quad (\text{see Lemma 5.1}) \\ &= \frac{1}{N} \sum_{i=1}^N \delta(y - \bar{x}_i(t)) \delta(\xi - \bar{v}_i(t)) \quad (\text{see Lemma 5.1 below}) \\ &:= \mu_{\Phi_N^t(\bar{Z})} \end{aligned}$$

In turn, this suggests that the (non local in time) Vlasov equation associated to the particle system (3, 4, 5, 6) is

$$(13) \quad \partial_t \rho^t + v \cdot \nabla_x \rho^t = \nabla_v (\nu(t, x, v) \rho^t), \quad \rho^0 = \rho^{in}$$

where

$$(14) \quad \nu(t, x, v) = \gamma(x, v) * \rho^t + \eta \nabla_x \psi^t(x) + F_{ext}(x)$$

and ψ satisfies

$$(15) \quad \partial_s \psi^s(z) = D \Delta_z \psi - \kappa \psi^s + g(z, \rho^s), \quad \psi^0 = \varphi^{in}.$$

x-special/nautilus-clipboard copy file:///home/paulth/TThe kinetic equation associated to Cucker-Smale systems, introduced in [23], has been derived in [22, 20] and, for generalizations of type (3) with $\varphi = 0$ in [32], **without chemotaxis interaction**. We refer to [20, 32] for a large bibliography on the subject.

2. THE MAIN RESULT

Theorem 2.1. *Let ρ^{in} be a compactly supported probability on \mathbf{R}^{2dN} , let Φ_N^t be the mapping generated by the particles system (3, 4, 5, 6) as defined by (7), and let τ be the function defined in (31).*

Then, for any $t \geq 0$,

$$W_2 \left((\Phi_N^t \# (\rho^{in})^{\otimes N})_{N;1}, \rho^t \right)^2 \leq \tau(t) \begin{cases} N^{-\frac{1}{2}} & d = 1 \\ N^{-\frac{1}{2}} \log N & d = 2 \\ N^{-\frac{1}{d}} & d > 2 \end{cases}$$

where ρ^t is the solution of the Vlasov equation (13, 14, 15) with initial condition ρ^{in} provided by Theorem 9.1 below and W_2 is the quadratic Wasserstein distance whose definition is recalled in Definition 4.1.

Moreover, let us denote by $\varphi_{Z^{in}}^t$ the chemical density solution of (3, 4, 5, 6) with initial data (Z^{in}, φ^{in}) and by $\psi_{\rho^{in}}^t$ the one solution of (13, 14, 15) with initial data $(\rho^{in}, \varphi^{in})$.

Then

$$\int_{\mathbf{R}^{2dN}} \|\nabla \varphi_{Z^{in}}^t - \nabla \psi_{\rho^{in}}^t\|_{\infty}^2 (\rho^{in})^{\otimes N} (dZ^{in}) \leq \tau_c(t) \begin{cases} N^{-\frac{1}{2}} & d = 1 \\ N^{-\frac{1}{2}} \log N & d = 2 \\ N^{-\frac{1}{d}} & d > 2 \end{cases}$$

where $\tau_c = 5t^2 \text{Lip}(\nabla \chi)^2 (\tau(t) + Ce^{\Gamma(t)})^2$ with $\Gamma(t)$ given by (40) and C defined in Theorem 5.2.

Finally, the function $\tau(t)$ depends only on t , $\text{Lip}(\gamma)$, $\text{Lip}(\chi)$, $\text{Lip}(\nabla \chi)$, and the supports of $\Phi_N^t \# (\rho^{in})$ and ρ^t , and satisfies the following estimate for all $t \in \mathbf{R}$,

$$\tau(t) \leq e^{Ct}$$

for some constant C , depending on $\text{Lip}(\gamma)$, $\text{Lip}(\chi)$, $\text{Lip}(\nabla \chi)$ and $|\text{supp}(\rho^{in})|$.

Corollary 2.2 (Hydrodynamic Euler limit).

x-special/nautilus-clipboard copy file:///home/paulth/T Let $\mu^{in}, u^{in}, \varphi^{in}$ be such that the Euler system

$$\begin{cases} \partial_t \mu^t + \nabla_x (u^t \mu^t) = 0 \\ \partial_t (\mu^t u^t) + \nabla (\mu^t (u^t)^{\otimes 2}) = \mu^t \int \gamma(\cdot - y, u^t(\cdot) - u^t(y)) \mu^t(y) dy + \nabla \psi^t + F \\ \partial_s \psi^s(z) = D \Delta_z \psi - \kappa \psi + \chi * \mu^s, \quad s \in [0, t], \\ (\mu^0, u^0, \psi^0) = (\mu^{in}, u^{in}, \varphi^{in}) \in H^s, \quad s > \frac{d}{2} + 1. \end{cases}$$

²x-special/nautilus-clipboard copy file:///home/paulth/T through this paper we define $\text{Lip}(f)$ for $f : \mathbf{R}^n \rightarrow \mathbf{R}^m, m, n \in \mathbf{N}$, as $\text{Lip}(f) := \max_{1 \leq i \leq m} \text{Lip}(f_i)$

has a unique solution $\mu^t, u^t \in C([0, t]; H^s) \cap C^1([0, T]; H^{s-1})$, $\psi^t \in C([0, t]; H^s) \cap C^1([0, T]; H^{s-2}) \cap L^2(0, T; H^{s+1})$ and let

$$\rho^{in} = \mu^{in}(x)\delta(v - u^{in}(x)).$$

Then, for any $t \in [0, T]$,

$$W_2((\Phi_N^t \# (\rho^{in})^{\otimes N})_{N;1}, \mu^t(x)\delta(v - u^t(x)))^2 \leq \tau(t) \begin{cases} N^{-\frac{1}{2}} & d = 1 \\ N^{-\frac{1}{2}} \log N & d = 2 \\ N^{-\frac{1}{d}} & d > 2 \end{cases}$$

Moreover,

$$\int_{\mathbf{R}^{dN}} \|\nabla \varphi_{(X^{in}, u^{\otimes N}(X^{in}))}^t - \nabla \psi_{\rho^{in}}^t\|_{\infty}^2 (\mu^{in})^{\otimes N}(dX^{in}) \leq \tau_c(t) \begin{cases} N^{-\frac{1}{2}} & d = 1 \\ N^{-\frac{1}{2}} \log N & d = 2 \\ N^{-\frac{1}{d}} & d > 2 \end{cases}.$$

. Let us describe the strategy of the proof of Theorem 2.1.

Clearly Theorem 2.1 links the dynamics of the particle system (3, 4, 5, 6) and the one driven by the Vlasov system (13, 14, 15). As an intermediate step we will consider the N -body Liouville type one defined by (9, 10, 11, 12).

We will proceed in several steps.

Step 1: we will show that the marginal $(\rho_{N;1}^t)$ of the solution ρ_N^t of (9, 10, 11, 12), is close to the solution of a Vlasov type closed equation (13, 14, 15) derived below in Wasserstein metric: this is the content of Proposition 6.1 in Section 6 below.

Step 2: we will show that the marginal $(\Phi_{\#}^t \rho_N^{in})_{N;1}$ and $(\rho_N^t)_{N;1}$ are close as $N \rightarrow \infty$ in the same Wasserstein topology: this result is precisely Proposition 5.3 in Section 5 below.

Step 3 [proof of Theorem 2.1]: we will use the triangular inequality for W_2 :

$$W_2((\Phi_N^t \# (\rho^{in})^{\otimes N})_{N;1}, \rho^t) \leq W_2((\Phi_N^t \# (\rho^{in})^{\otimes N})_{N;1}, (\rho_N^t)_{N;1}) + W_2((\rho_N^t)_{N;1}, \rho^t).$$

Theorem 2.1 will then be given by the estimate on $W_2((\Phi_N^t \# (\rho^{in})^{\otimes N})_{N;1}, (\rho_N^t)_{N;1})$ given by Proposition 5.3 and the one on $W_2((\rho_N^t)_{N;1}, \rho^t)$ given by Proposition 6.1.

The chemical density estimate is obtained by the triangle inequality

$$\|\nabla \varphi_{Z^{in}}^t - \nabla \psi_{\rho^{in}}^t\|_{\infty}^2 \leq 5(\|\nabla \varphi_{Z^{in}}^t - \nabla \psi_{\mu_{Z^{in}}}^t\|_{\infty}^2 + \|\nabla \psi_{\mu_{Z^{in}}}^t - \nabla \psi_{\rho^{in}}^t\|_{\infty}^2)$$

where $\mu_{Z^{in}} := \frac{1}{N} \sum_{l=1}^N \delta_{z_l^{in}}$ (empirical measure). Both squares are estimated by Corollary 4.6, and $W_2((\Phi_N^t \# \mu_{Z^{in}}, \rho_{\mu_{Z^{in}}}^t)^2$ is estimated by the first estimate of the Theorem, while $W_2(\rho_{\mu_{Z^{in}}}^t, (\rho^{in})^t)^2$ by the Dobrushin estimate in Theorem 9.1.

The estimate for $\tau(t)$ is proven at the end of Section 6 (see formula (32)). .

x-special/nautilus-clipboard copy file:///home/paulth/TThe proof of the Corollary is straightforward. \square

Remark 2.3. *As it is clear from the step 3 above, an alternative to the second statement in Theorem 2.1 is the following.*

$$\|\nabla\varphi_{Z^{in}}^t - \nabla\psi_{\rho^{in}}^t\|_\infty^2 \leq \|\nabla\psi_{\mu_{Z^{in}}}^t - \nabla\psi_{\rho^{in}}^t\|_\infty^2 + 5t^2 \text{Lip}(\nabla\chi)^2\tau(t) \begin{cases} N^{-\frac{1}{2}} & d = 1 \\ N^{-\frac{1}{2}}\log N & d = 2 \\ N^{-\frac{1}{d}} & d > 2 \end{cases}$$

3. THE THREE DYNAMICS

Let us be more precise and fix the notations. We define the three following dynamics, denoted by (P) for (*Particles*), (LV) for (*Liouville-Vlasov*) and (V) for (*Vlasov*).

$$(P) \left\{ \begin{array}{l} \dot{x}_i = v_i, \quad \dot{v}_i = F_i(t, X(t), V(t)), \quad (X(0), V(0)) = Z(0) = Z^{in} \in \mathbf{R}^{2dN} \\ F_i(t, Y, W) = \frac{1}{N} \sum_{j=1}^N \gamma(w_i - w_j, y_i - y_j) + \eta \nabla_z \varphi^t(z)|_{z=y_i} + F_{ext}(y_i), \\ \partial_s \varphi^s(z) = D\Delta_z \varphi - \kappa \varphi + f(z, X(s)), \quad s \in [0, t], \quad \varphi^0 = \varphi^{in} \\ f(z, X) = \frac{1}{N} \sum_{j=1}^N \chi(z - x_j). \end{array} \right.$$

$$(LV) \left\{ \begin{array}{l} \partial_t \rho_N^t + V \cdot \nabla_X \rho_N^t = \sum_{i=1}^N \nabla_{v_i} \cdot G_i \rho_N^t, \quad \rho_N^0 = \rho_N^{in} = (\rho^{in})^{\otimes N} \in \mathcal{P}(\mathbf{R}^{2dN}) \\ G_i(t, Y, W) = \frac{1}{N} \sum_{j=1}^N \gamma(w_i - w_j, y_i - y_j) + \eta \nabla_z \Psi^t(z)|_{z=y_i} + F_{ext}(y_i), \\ \partial_s \Psi^s(z) = D\Delta_z \Psi - \kappa \Psi + g(z, \rho_{N;1}^s), \quad s \in [0, t], \quad \Psi^0 = \varphi^{in}, \\ g(z, \rho_{N;1}^s) = \int_{\mathbf{R}^{2d}} \chi(z - x) \rho_{N;1}^s(x, v) dx dv \end{array} \right.$$

$$(V) \left\{ \begin{array}{l} \partial_t \rho^t + v \cdot \nabla_x \rho^t = \nabla_v (\nu(t, x, v) \rho^t), \quad \rho^0 = \rho^{in} \in \mathcal{P}(\mathbf{R}^{2d}) \\ \nu(t, x, v) = \gamma(x, v) * \rho^t + \eta \nabla_x \psi^t(x) + F_{ext}(x), \\ \partial_s \psi^s(z) = D\Delta_z \psi - \kappa \psi + g(z, \rho^s), \quad \psi^0 = \varphi^{in}. \end{array} \right.$$

Note that $(\chi * \rho_{N;1}(t))(x) = (\tilde{\chi} * \rho)(x, \dots, x)$, $\tilde{\chi}(X) = \frac{1}{N} \sum_{j=1}^N \chi(x_j)$.

Let us recall the strategy of our approach: we believe that, in some Wasserstein topology, with $\rho_N^{in} = (\rho^{in})^{\otimes N}$,

$$\begin{cases} (\Phi_N^t \# \rho_N^{in})_{N;1} \sim (\rho_N^t)_{N;1}(t), & \Phi_N^t \text{ solution of } (L), \rho_N^t \text{ of } (LV) \text{ with } \rho_N^0 = \rho_N^{in} \\ \rho_N^{in} = (\rho^{in})^{\otimes N} \\ (\rho_N^t)_{N;1} \sim \rho^t, & \rho^t \text{ solution of } (V) \text{ with } \rho^0 = \rho^{in}, \end{cases}$$

so that, by triangle inequality,

$$(\Phi_N^t \# \rho_N^{in})_{N;1} \sim \rho^t$$

with Φ_N^t solution of (L) and ρ^t solution of (V) with $\rho^0 = (\rho_N^{in})_{N;1}$.

4. TECHNICAL PRELIMINARIES

In this section we establish or recall several results which will be intensively used in the core of the proof of Theorem 2.1.

4.1. Wasserstein distances. Let us start this section by recalling the definition of the second order Wasserstein distance W_2 (see [41, 42]).

Definition 4.1 (quadratic Wasserstein distance). *The Wasserstein distance of order two between two probability measures μ, ν on \mathbf{R}^m with finite second moments is defined as*

$$W_2(\mu, \nu)^2 = \inf_{\gamma \in \Gamma(\mu, \nu)} \int_{\mathbf{R}^m \times \mathbf{R}^m} |x - y|^2 \gamma(dx, dy)$$

where $\Gamma(\mu, \nu)$ is the set of probability measures on $\mathbf{R}^m \times \mathbf{R}^m$ whose marginals on the two factors are μ and ν .

Likewise is the Wasserstein distance of order 1 W_1 two between two probability measures μ, ν on \mathbf{R}^m with finite moments is defined by the following.

Definition 4.2.

$$W_1(\mu, \nu) := \sup \left\{ \int_{\mathbf{R}^{2d}} f(\mu - \nu) \mid f \in C^\infty(\mathbf{R}^{2d}), \text{Lip}(f) \leq 1 \right\}.$$

Lemma 4.3.

- $W_1(\mu, \nu) \leq W_2(\mu, \nu)$,
- $\sup_{\text{Lip } f \leq 1} \int f(\mu - \nu) = W_1(\mu, \nu)$,
- *The convergence in the weak topology (i.e., in the duality with $C_b(\mathbf{R}^{2d})$) of sequences of probability measures with supports equibounded is equivalent to the convergence with respect to the distance W_p , $p = 1, 2$ (in fact with respect to Wasserstein of all orders),*

Proof. The first and second items are exactly formulas (7.1) and (7.3) in [41], The third item is a straightforward consequence of [41, Theorem 7.12 (iii)], since the weak convergence of equisupported sequences of measures implies the convergence of all of their moments \square

Finally, we will need below the following result.

Lemma 4.4. *For any measures $\mu, \nu \in \mathcal{P}(\mathbf{R}^{2dN})$ with finite second moments and any area preserving mapping $\Phi : \mathbf{R}^{2dN} \rightarrow \mathbf{R}^{2dN}$,*

$$W_2(\Phi\#\mu, \Phi\#\nu) \leq \text{Lip}(\Phi)W_2(\mu, \nu),$$

where $\text{Lip}(\Phi) := \max_{1 \leq i \leq N} \text{Lip}(\Phi_i)$.

Moreover, for any $N \in \mathbf{N}$,

$$W_2(\mu^{\otimes N}, \nu^{\otimes N})^2 \leq NW_2(\mu, \nu)^2.$$

Proof. Let π an optimal coupling for μ and ν . Obviously $(\Phi \otimes \Phi)\#\pi$ is a coupling for $\Phi\#\mu$ and $\Phi\#\nu$.

Therefore,

$$\begin{aligned} W_2(\Phi\#\mu, \Phi\#\nu)^2 &\leq \int_{\mathbf{R}^{4dN}} |z - z'|^2 (\Phi \otimes \Phi)\#\pi(dz, dz') \\ &\leq \int_{\mathbf{R}^{4dN}} |\Phi(z) - \Phi(z')|^2 \pi(dz, dz') \\ &\leq \text{Lip}(\Phi)^2 \int_{\mathbf{R}^{4dN}} |z - z'|^2 \pi(dz, dz') = \text{Lip}(\Phi)^2 W_2(\mu, \nu)^2 \end{aligned}$$

by optimality of π .

Moreover, $\pi^{\otimes N}$ is obviously a coupling for $\mu^{\otimes N}$ and $\nu^{\otimes N}$. Therefore

$$\begin{aligned} W_2(\mu^{\otimes N}, \nu^{\otimes N}) &\leq \int_{\mathbf{R}^{4dN}} |Z - Z'|^2 \pi^{\otimes N}(dZ, dZ') \\ &= \int_{\mathbf{R}^{4dN}} \sum_{i=1}^N |z_i - z'_i|^2 \pi^{\otimes N}(dZ, dZ') \\ &= N \int_{\mathbf{R}^{4d}} |z - z'|^2 \pi(dz, dz') = NW_2(\mu, \nu) \end{aligned}$$

by optimality of π again. \square

4.2. The diffusion term. The three equations

$$(16) \quad \begin{cases} \partial_s \varphi^s(z) = D\Delta_z \varphi - \kappa \varphi + f(z, Y(s)), & \varphi^0 = \varphi^{in} \\ \partial_s \Psi^s(z) = D\Delta_z \Psi - \kappa \Psi + g(z, \rho_{N;1}^s), & \Psi^0 = \varphi^{in} \\ \partial_s \psi^t(z) = D\Delta_z \psi - \kappa \psi + g(z, \rho^s), & \psi^0 = \varphi^{in} \end{cases}$$

can be “solved” by (we denote $\mathbf{I} = (1, 1, 1)^T$)

$$(17) \quad \begin{pmatrix} \varphi^t(z) \\ \Psi^t(z) \\ \psi^t(z) \end{pmatrix} = e^{-\kappa t} \int_0^t e^{(t-s)D\Delta_z} \begin{pmatrix} f(z, X(s)) \\ g(z, \rho_{N;1}^s) \\ g(z, \rho^s) \end{pmatrix} ds + e^{-\kappa t} e^{tD\Delta} \varphi^{in} \mathbf{I}$$

$$= e^{-\kappa t} \int_0^t \int_{\mathbf{R}^d} \frac{e^{-\frac{(z-z')^2}{4D(t-s)}}}{(4\pi D(t-s))^{\frac{d}{2}}} \begin{pmatrix} f(z', X(s)) \\ g(z', \rho_{N;1}^s) \\ g(z', \rho^s) \end{pmatrix} ds dz' + e^{-\kappa t} e^{tD\Delta} \varphi^{in} \mathbf{I}$$

Note that $\nabla_z \begin{pmatrix} \varphi^t(z) \\ \Psi^t(z) \\ \psi^t(z) \end{pmatrix}$ is given by the same formula with χ replaced by $\nabla\chi$. Therefore, if

$$\chi = \prod_{l=1}^d \chi_{[-a_l, a_l]}$$

($\chi_{[-a_l, a_l]}$ the characteristic function on $[-a_l, a_l]$), one has

$$\nabla_z \begin{pmatrix} \varphi^t(z) \\ \Psi^t(z) \\ \psi^t(z) \end{pmatrix} = e^{-\kappa t} \int_0^t \begin{pmatrix} (4\pi D(t-s))^{-\frac{d}{2}} \sum_{j=1}^N \left(e^{-\frac{(z-x_j+a)^2}{4D(t-s)}} - e^{-\frac{(z-x_j-a)^2}{4D(t-s)}} \right) \\ e^{(t-s)D\Delta_z} \rho_{N;1}^s(z+a) - e^{(t-s)D\Delta_z} \rho_{N;1}^s(z-a) \\ e^{(t-s)D\Delta_z} \rho^s(z+a) - e^{(t-s)D\Delta_z} \rho^s(z-a) \end{pmatrix} ds + e^{-\kappa t} e^{tD\Delta} \varphi^{in} \mathbf{I}.$$

The following lemma will be systematically used in the forthcoming sections.

Lemma 4.5. *Let $\rho, \rho' \in \mathcal{P}(\mathbf{R}^d)$ and $\mu \in \text{Lip}(\mathbf{R}^d)$. Then, for all $t \geq 0$,*

$$\|(e^{t\Delta}\mu) * (\rho - \rho')\|_{L^\infty(\mathbf{R}^d)} \leq \text{Lip}(\mu) W_p(\rho, \rho'), \quad p = 1, 2.$$

Proof. One has

$$\begin{aligned} |(e^{t\Delta}\nabla\chi) * (\rho - \rho')(x_i)| &= \left| \int (z)(e^{t\Delta}\nabla\chi)(x_i - z)(\rho - \rho') dz \right| \\ &\leq \text{Lip}((e^{t\Delta}\nabla\chi)(x_i - \cdot)) W_2(\rho, \rho') \\ &\leq \text{Lip}(e^{t\Delta}\nabla\chi) W_2(\rho, \rho') \\ &\leq \text{Lip} \nabla\chi W_2(\rho, \rho') \end{aligned}$$

since, by Lemma 4.3,

$$\sup_{\text{Lip } f \leq 1} \left| \int f(d\mu - d\nu) \right| = W_1(\mu, \nu) \leq W_2(\mu, \nu),$$

and

$$\begin{aligned} |(e^{t\Delta}\nabla\chi)(x) - (e^{t\Delta}\nabla\chi)(y)| &= |(e^{t\Delta}(\nabla\chi(x - \cdot) - \nabla\chi(y - \cdot)))(0)| \\ &\leq |\nabla\chi(x) - \nabla\chi(y)|. \end{aligned}$$

□

Corollary 4.6.

$$\|\nabla\varphi^t - \nabla\psi^t\|_{L^\infty(\mathbf{R}^d)} \leq t \text{Lip}(\nabla\chi) W_2((\Phi_N^t \# (\rho^{in})^{\otimes N})_{N:1}, \rho^t).$$

4.3. Propagation of Wasserstein type estimates. In this paragraph, we establish a result used later as a black box, concerning the propagation of estimates in Wasserstein topology under general transport equation including the several types used in this paper.

Theorem 4.7. *x-special/nautilus-clipboard copy file:///home/paulth/T Let us suppose that the two equations*

$$(18) \quad \partial_t \rho_i^t + V \cdot \nabla_X = \nabla_V \cdot (\mathbf{v}_i(t, X, V) \rho_i^t), \quad \rho_i^0 = (\rho_i^{in})^{\otimes N}, \quad \rho_i^{in} \in \mathcal{P}(\mathbf{R}^{2d}), \quad i = 1, 2.$$

have existence and uniqueness of solution in $C^0(\mathbf{R}^+, \mathcal{P}_c(\mathbf{R}^{2dN}))$.

Here $\mathbf{v}_i(t, X, V)$ might or might not depend on the solution ρ_i^s for $0 \leq s \leq t$, is supposed to be invariant by permutations of the variables (x_j, v_j) , $j = 1, \dots, N$, and is Lipschitz continuous with respect to (X, V) and satisfies the estimate

$$\mathbf{v}_i(t, X, V) \leq \gamma_0 \|V\|, \quad X, V \in \mathbf{R}^{2dN}, \quad i = 1, 2.$$

Let moreover ρ_i^t , $i = 1, 2$, be two solutions of the equations (18) and let π_N^t be the unique (measure) solution to the following linear transport equation

$$(19) \quad \partial_t \pi_N + V \cdot \nabla_X \pi_N + \Xi \cdot \nabla_Y \pi_N = \nabla_V \cdot (\mathbf{v}_1(t, X, V) \pi_N^t + \nabla_\Xi \cdot (\mathbf{v}_2(t, Y, \Xi) \pi_N^t)$$

with $\pi_N^{t=0} = \pi_N^{op}$ optimal coupling between $(\rho_1^{in})^{\otimes N}$ and $(\rho_2^{in})^{\otimes N}$.

Let us define, for $i = 1, 2$,

$$(\rho_i^t)_{N:1}(x, v) := \begin{cases} \int_{\mathbf{R}^{2d(N-1)}} \rho^t(x, x_2, \dots, x_n; v, v_2, \dots, v_N) dx_2 \dots dx_N dv_2 \dots dv_N & N > 1 \\ \rho_i^t(x, v), & N = 1. \end{cases}$$

Then, for all $t \in \mathbf{R}^+$,

$$\begin{aligned} W_2((\rho_1^t)_{N:1}, (\rho_2^t)_{N:1})^2 &\leq e^{\int_0^t L(s) ds} W_2(\rho_1^{in}, \rho_2^{in})^2 \\ &+ 2 \int_0^t \int_{\mathbf{R}^{2dN}} |\mathbf{v}_1(s, X, V) - \mathbf{v}_2(s, X, V)|^2 (\rho_2^{in})^{\otimes N}(dX, dV) e^{\int_s^t l(u) du} ds \end{aligned}$$

with

$$L(u) = 2 + 2 \min_{i=1,2} \sup_{(X,V) \in \text{supp}(\rho_i^u)} \text{Lip}(\mathbf{v}_1(u, X, V)) + \text{Lip}(\mathbf{v}_2(u, X, V))$$

The proof on Theorem 4.7 is given in Appendix A.

5. FROM PARTICLES TO LIOUVILLE-VLASOV

In this section we estimate $W_2((\Phi_N^t \# \rho^{in})_{N:1}, (\rho_N^t)_{N:1})$, where Φ_N^t is generated by the particle system (3, 4, 5, 6) and ρ_N^t is the solution of the N -body Liouville type one defined by (9, 10, 11, 12) with initial data ρ^{in} .

By Theorem 4.7 we get easily that

$$\begin{aligned} & W_2((\Phi_N^t \# \rho^{in})_{N;1}, (\rho_N^t)_{N;1}) \\ & \leq 4\eta \sum_{i=1}^N \int_0^t \int_{\mathbf{R}^{2dN}} |(\nabla \varphi^s(x_i) - \nabla \Psi^s(x_i))|^2 (\Phi_N^t \# (\rho^{in})^{\otimes N})(dX, dV) e^{\int_s^t \bar{l}(u) du} ds \end{aligned}$$

with

$$(20) \quad \bar{L}(u) = 2 + 2 \min_{\rho \in \{\Phi_N^t \# (\rho^{in})^{\otimes N}, \rho_N^t\}} \left(\sup_{\substack{i,l=1,\dots,N \\ X,V,Y,\Xi \in \text{supp}(\rho)}} \text{Lip}(\gamma)_{(x_i - x_l, v_i - v_l)}^2 \right) + 2\eta \text{Lip}(\nabla \chi)^2.$$

Therefore, we have to estimate

$$(21) \quad \int_{\mathbf{R}^{2dN}} |(\nabla \varphi^t(x_i) - \nabla \Psi^t(x_i))|^2 (\Phi_N^t \# (\rho^{in})^{\otimes N})(X, V) dX dV.$$

We first remark that, in (16),

$$f(\cdot, X) = \chi * \mu_Z,$$

where, for any $Z = (z_1, \dots, z_N) \in \mathbf{R}^{2dN}$, the empirical measure μ_Z is defined by

$$(22) \quad \mu_Z := \frac{1}{N} \sum_{k=1}^N \delta_{z_k}$$

Therefore, by (17),

$$\nabla \varphi^t(\cdot) = e^{-\eta t} \int_0^t e^{(t-s)\Delta} \nabla \chi * \mu_{\Phi^s(Z_0(X,V))} ds$$

where $Z_0(X, V)$ is defined by $\Phi^t(Z_0(X, V)) = (X, V)$.

Note that such $Z_0(X, V)$ exists for any (X, V) appearing in the integral in (21) since one integrates with respect to the measure $(\Phi_N^t \# (\rho^{in})^{\otimes N})(dX, dV)$ i.e. $Z_0(X, V) \in \text{supp}((\rho^{in})^{\otimes N})$.

Now,

$$\begin{aligned} & \int |(\nabla \varphi^t(x_i) - \nabla \Psi^t(x_i))|^2 (\Phi_N^t \# (\rho^{in})^{\otimes N})(X, V) dX dV \\ & = e^{-2\kappa t} \int \left| \int_0^t (e^{(t-s)\Delta} \nabla \chi) * (\mu_{\Phi^s(Z_0(X,V))} - (\rho_N^s)_{N;1})(x_i) \right|^2 (\Phi_N^t \# (\rho^{in})^{\otimes N})(X, V) dX dV \end{aligned}$$

So that, denoting

$$(23) \quad \Phi_N^t(X, V) =: (x_1^t(V, V), \dots, x_N^t(X, V), v_1^t(X, V), \dots, v_N^t(X, V))$$

i.e. $x_i^t(X, V)$ being the x_i -component of $\Phi^t(X, V)$, one has, since $\Phi^t(Z_0(X, V)) = (X, V)$,

$$\begin{aligned}
& \int |(\nabla\varphi^t(x_i) - \nabla\Psi^t(x_i))|^2 (\Phi_N^t \# (\rho^{in})^{\otimes N})(dX, dV) \\
&= e^{-2\kappa t} \int \left| \int_0^t \left(e^{(t-s)\Delta} \nabla\chi * (\mu_{\Phi^s(X, V)} - (\rho_N^s)_{N;1}) \right) (x_i^t(X, V)) \right|^2 (\rho^{in})^{\otimes N}(dX, dV) \\
&= e^{-2\kappa t} \int \left| \int_0^t \left(e^{(t-s)\Delta} \nabla\chi * (\mu_{\Phi^s(X, V)} - (\Phi^s \# (\rho^{in})^{\otimes N})_{N;1} \right. \right. \\
&\quad \left. \left. + (\Phi^s \# (\rho^{in})^{\otimes N})_{N;1} - (\rho_N^s)_{N;1} \right) (x_i^t(X, V)) \right|^2 (\rho^{in})^{\otimes N}(dX, dV) \\
(24) \quad &\leq 2e^{-2\kappa t} \int \left(\left| \int_0^t \left(e^{(t-s)\Delta} \nabla\chi \right) * (\mu_{\Phi^s(X, V)} - (\Phi^s \# (\rho^{in})^{\otimes N})_{N;1}) (x_i^t(X, V)) \right|^2 \right. \\
(25) \quad &\left. + \left| \int_0^t \left(e^{(t-s)\Delta} \nabla\chi \right) * ((\Phi^s \# (\rho^{in})^{\otimes N})_{N;1} - (\rho_N^s)_{N;1}) (x_i^t(X, V)) \right|^2 \right) (\rho^{in})^{\otimes N}(dX, dV).
\end{aligned}$$

As before,

$$(e^{(t-s)\Delta} \nabla\chi) * ((\Phi^s \# (\rho^{in})^{\otimes N})_{N;1} - (\rho_N^s)_{N;1})(x_i) \leq \text{Lip } \nabla\chi W_2((\Phi^s \# (\rho^{in})^{\otimes N})_{N;1}, (\rho_N^s)_{N;1}),$$

which gives a second contribution to (21) out of the right hand-side of (25) given by:

$$2 \text{Lip } \nabla\chi^2 e^{-2\kappa t} \int_0^t \int_0^t W_2((\Phi^s \# (\rho^{in})^{\otimes N})_{N;1}, (\rho_N^s)_{N;1}) W_2((\Phi^{s'} \# (\rho^{in})^{\otimes N})_{N;1}, (\rho_N^{s'})_{N;1}) ds ds'.$$

Lemma 5.1.

$$\mu_Z = \left(\frac{1}{N!} \sum_{\sigma \in \Sigma_N} \delta_{\sigma(Z)} \right)_{N;1}$$

where Σ_N is the group of permutations of N elements and

$$\sigma(Z) := (z_{\sigma(1)}, \dots, z_{\sigma(N)}).$$

Proof. Let us recall that $\Sigma_N = \{\sigma : \{1, \dots, N\} \rightarrow \{1, \dots, N\}, \sigma \text{ one-to-one}\}$ so that $\#\Sigma_N = N!$. We have

$$\int \cdots \int \delta_{\sigma(Z)} dz_2 \cdots dz_N = \int \cdots \int \delta_{\sigma(Z)} \prod_{l \neq \sigma(1)} dz_l$$

Therefore

$$\begin{aligned}
\left(\frac{1}{N!} \sum_{\sigma \in \Sigma_N} \delta_{\sigma(Z)} \right)_{N:1} &= \frac{1}{N!} \sum_{l=1}^N \sum_{\substack{\sigma \in \Sigma_N \text{ int} \\ \sigma(l)=1}} \int \cdots \int \delta_{\sigma(Z)} \prod_{l \neq \sigma(1)} dz_l \\
&= \sum_{l=1}^N \frac{\#\Sigma_{N-1}}{N!} \delta_{z_l} \\
&= \sum_{l=1}^N \frac{(N-1)!}{N!} \delta_{z_l} = \mu_Z.
\end{aligned}$$

□

x-special/nautilus-clipboard copy file:///home/paulth/TLooking now at the contribution out of (24), we have, using successively (23), Lemma 5.1, the definition of

marginals and the fact that $\sum_{\sigma \in \Sigma_N} \delta_{\sigma(\Phi_N^t(Z))} = \sum_{\sigma \in \Sigma_N} \delta_{\Phi_N^t(\sigma(Z))}$ since the system (3) preserves the indiscernability of the particles,

$$\begin{aligned}
& \sum_{i=1}^N \int \left| \int_0^t (e^{(t-s)\Delta} \nabla \chi) * (\mu_{\Phi^s(X,V)} - (\Phi^s \# (\rho^{in})^{\otimes N})_{N;1})(x_i^t(X,V)) ds \right|^2 (\rho^{in})^{\otimes N}(dX, dV) \\
&= \sum_{i=1}^N \int \left| \int_0^t e^{(t-s)\Delta} \nabla \chi(x_i^t(X,V) - x') (\mu_{\Phi^s(X,V)} - (\Phi^s \# (\rho^{in})^{\otimes N})_{N;1})(dx', dv') ds \right|^2 \\
&\quad \times (\rho^{in})^{\otimes N}(dZ) \\
&= \sum_{i=1}^N \int \left| \int_0^t e^{(t-s)\Delta} \nabla \chi(x_i^t(Z) - x') \left(\left(\frac{1}{N!} \sum_{\sigma} \delta_{\sigma(\Phi_N^t(Z))} \right)_{N;1} - (\Phi^s \# (\rho^{in})^{\otimes N})_{N;1} \right) (dx', dv') ds \right|^2 \\
&\quad \times (\rho^{in})^{\otimes N}(dZ) \\
&= \sum_{i=1}^N \int \left| \int_0^t e^{(t-s)\Delta} \nabla \chi(x_i^t(Z) - x') \left(\left(\frac{1}{N!} \sum_{\sigma} \delta_{\Phi_N^t(\sigma(Z))} \right)_{N;1} - (\Phi^s \# (\rho^{in})^{\otimes N})_{N;1} \right) (dx', dv') ds \right|^2 \\
&\quad \times (\rho^{in})^{\otimes N}(dZ) \\
&= \sum_{i=1}^N \int \left| \int_0^t e^{(t-s)\Delta} \nabla \chi(x_i^t(Z) - x') \left(\Phi_N^t \# \left(\frac{1}{N!} \sum_{\sigma} \delta_{\sigma(Z)} - (\rho^{in})^{\otimes N} \right) \right)_{N;1} (dx', dv') ds \right|^2 \\
&\quad \times (\rho^{in})^{\otimes N}(dZ) \\
&\leq \sum_{i=1}^N \int \left| \int_0^t ds \int \frac{1}{N} H(t-s, \Phi^t(Z) - Z') \left(\Phi_N^t \# \left(\frac{1}{N!} \sum_{\sigma} \delta_{\sigma(Z)} - (\rho^{in})^{\otimes N} \right) \right) (dZ') \right|^2 \\
&\quad \times (\rho^{in})^{\otimes N}(dZ) \\
&\leq \int \left| \int_0^t ds \int H(t-s, \Phi^t(Z) - Z') \left(\Phi_N^t \# \left(\frac{1}{N!} \sum_{\sigma} \delta_{\sigma(Z)} - (\rho^{in})^{\otimes N} \right) \right) (dZ') \right|^2 (\rho^{in})^{\otimes N}(dZ).
\end{aligned}$$

where (let us recall the notation $Z = (x_1, \dots, x_N : v_1, \dots, v_N)$,

$$H(t-s, (X, V)) = \sum_{i=1}^N (e^{(t-s)\Delta} \nabla \chi)(x_i)$$

so that

$$\text{Lip } H(t-s, \cdot) := \max_{i=1, \dots, N} \sup_{X \in \mathbf{R}^{dN}} \partial_{x_i} H(t-s, X) \leq \text{Lip } \nabla \chi.$$

One has

$$\begin{aligned}
& \sum_{i=1}^N \int \left| \int_0^t (e^{(t-s)\Delta} \nabla \chi) * (\mu_{\Phi_N^s(X,V)} - (\Phi_N^s \# (\rho^{in})^{\otimes N})_{N;1})(x_i^t(X,V)) ds \right|^2 (\rho^{in})^{\otimes N}(dX, dV) \\
(26) \quad & \leq \text{Lip } \nabla \chi^2 t^2 \int |W_2(\Phi_N^s \# \frac{1}{N!} \sum_{\sigma} \delta_{\sigma Z}, \Phi_N^s \# (\rho^{in})^{\otimes N})|^2 (\rho^{in})^{\otimes N}(dZ).
\end{aligned}$$

Applying Lemma 4.4 to (26) we get

$$\begin{aligned}
& \sum_{i=1}^N \int \left| \int_0^t (e^{(t-s)\Delta} \nabla \chi) * (\mu_{\Phi_N^s(X,V)} - (\Phi_N^s \# (\rho^{in})^{\otimes N})_{N;1})(x_i^t(X,V)) ds \right|^2 (\rho^{in})^{\otimes N}(dX, dV) \\
& \leq \text{Lip } \nabla \chi^2 \sup_{s \leq t} \text{Lip}(\Phi_N^s)^2 t^2 \int |W_2((\mu_Z)^{\otimes N}, (\rho^{in})^{\otimes N})|^2 (\rho^{in})^{\otimes N}(dZ) \\
& \leq \text{Lip } \nabla \chi^2 \sup_{s \leq t} \text{Lip}(\Phi_N^s)^2 t^2 N \int |W_2(\mu_Z, \rho^{in})|^2 (\rho^{in})^{\otimes N}(dZ) \\
& \leq \text{Lip } \nabla \chi^2 \sup_{s \leq t} \text{Lip}(\Phi_N^s)^2 t^2 N \int_{\mathbf{R}^{2d}} (x^2 + v^2) \rho^{in}(dx, dv) C \begin{cases} N^{-\frac{1}{2}} & d = 1 \\ N^{-\frac{1}{2}} \log N & d = 2 \\ N^{-\frac{1}{d}} & d > 2 \end{cases}.
\end{aligned}$$

thanks to the following result by Fournier and Guillin:

Theorem 5.2 (Theorem 1 in [15]). *Let μ satisfy*

$$\int_{\mathbf{R}^{2d}} (x^2 + v^2) \mu(dx, dv) := M_2(\mu) < \infty.$$

and let $\mu_{(X,V)}$, $(X, V) \in \mathbf{R}^{2dN}$, be the empirical measure defined by (22). Then

$$\int_{\mathbf{R}^{2dN}} W_2(\mu_{(X,V)}, \mu)^2 \mu^{\otimes N}(dX dV) \leq C_d(N) M_2(\mu),$$

where

$$C_d(N) := C \begin{cases} N^{-\frac{1}{2}} & d = 1 \\ N^{-\frac{1}{2}} \log N & d = 2 \\ N^{-\frac{1}{d}} & d > 2 \end{cases}$$

where C depends only on d .

Collecting the estimates on the right hand-sides of (24) and (25) we get

$$\begin{aligned}
& W_2((\Phi_N^t \# (\rho^{in})^{\otimes N})_{N;1}, (\rho_N^t)_{N;1})^2 \\
& \leq D(t) C_d(N) + \eta \text{Lip } \nabla \chi^2 \\
& \times \int_0^t e^{\int_s^t \bar{L}(u) du - 2\kappa s} ds \int_0^s ds' \int_0^s ds'' \\
& W_2((\Phi_N^{s'} \# (\rho^{in})^{\otimes N})_{N;1}, (\rho_N^{s'})_{N;1}) W_2((\Phi_N^{s''} \# (\rho^{in})^{\otimes N})_{N;1}, (\rho_N^{s''})_{N;1}).
\end{aligned}$$

with

$$D(t) = 2\eta \text{Lip}(\nabla \chi)^2 M_2(\rho^{in}) \int_0^t s^2 \sup_{u \leq s} \text{Lip}(\Phi_N^u)^2 e^{\int_s^t \bar{L}(u) du} ds.$$

Hence, for any $T \geq 0$, we have

$$\begin{aligned}
& \sup_{t \leq T} W_2((\Phi_N^t \# (\rho^{in})^{\otimes N})_{N;1}, (\rho_N^t)_{N;1})^2 \\
& \leq \sup_{t \leq T} D(t) C_d(N) + \eta \text{Lip} \nabla \chi^2 \\
& \quad \times \int_0^T e^{\int_s^T \bar{L}(u) du - 2\kappa s \frac{s^2}{2}} \sup_{u \leq s} W_2((\Phi_N^s \# (\rho^{in})^{\otimes N})_{N;1}, (\rho_N^s)_{N;1})^2 ds \\
& \leq \sup_{t \leq T} D(t) C_d(N) + \eta \text{Lip} \nabla \chi^2 \\
& \quad \times \int_0^T e^{T \sup_{u \leq T} \bar{L}(u)} \frac{T^2}{2} \sup_{u \leq s} W_2((\Phi_N^s \# (\rho^{in})^{\otimes N})_{N;1}, (\rho_N^s)_{N;1})^2 ds.
\end{aligned}$$

By the Grönwall Lemma, we get immediately that

$$\begin{aligned}
(27) \quad W_2((\Phi_N^t \# (\rho^{in})^{\otimes N})_{N;1}, (\rho_N^t)_{N;1})^2 & \leq \sup_{s \leq t} D(t) C_d(N) e^{\eta \text{Lip} \nabla \chi^2 \int_0^t e^{\int_0^s \sup_{u \leq s} \bar{L}(u)} \frac{s^2}{2} ds} \\
& := \beta(t)^2 C_d(N)
\end{aligned}$$

and the final result of this section.

Proposition 5.3.

$$W_2((\Phi_N^t \# \rho^{in})_{N;1}, \rho^t)^2 \leq \beta(t)^2 C_d(N)$$

6. FROM LIOUVILLE-VLASOV TO VLASOV

In this section we estimate $W_2((\rho_N^t)_{N;1}, \rho^t)$, where ρ_N^t is the solution of the N -body Liouville type one defined by (9, 10, 11, 12) and ρ^t is the solution of the Vlasov system (13, 14, 15), with initial data $(\rho^{in})^{\otimes N}$ and ρ^{in} .

By Theorem 4.7 we get easily that

$$\begin{aligned}
W_2((\rho_1^t)_{N;1}, (\rho_2^t)_{N;1})^2 & \leq 4 \sum_{i=1}^N \int_0^t \int_{\mathbf{R}^{2dN}} \left(\left| \frac{1}{N} \sum_{j=1}^N \gamma(x_i - x_j, v_i - v_j) - \gamma * \rho^s(x_i, v_i) \right|^2 \right. \\
& \quad \left. + \eta^2 |(\nabla \psi^s(x_i) - \nabla \Psi^s(x_i))|^2 \right) (\rho^s)^{\otimes N} (dX, dV) e^{\int_s^t \bar{l}(u) du} ds
\end{aligned}$$

with

$$(28) \quad \bar{L}(u) = 2 + 2 \min_{\rho \in \{\rho_N^t, (\rho^t)^{\otimes N}\}} \left(\sup_{\substack{i,l=1,\dots,N \\ (X,V) \Xi \in \text{supp}(\rho)}} \text{Lip}(\gamma)_{(x_i - x_l, v_i - v_l)}^2 \right) + 2\eta \text{Lip}(\nabla \chi)^2.$$

The first term in the integral has been estimated in [32, Lemma 3.5, Section 3] and we get

$$\begin{aligned} & \int_{\mathbf{R}^{2dN}} \left| \frac{1}{N} \sum_{j=1}^N \gamma(x_i - x_j, v_i - v_j) - \gamma * \rho^s(x_i, v_i) \right|^2 (\rho^s)^{\otimes N}(dX, dV) \\ & \leq \frac{4}{N} \sup_{(x,v),(x',v') \in \text{supp}(\rho^t)} |\gamma(x - x', v - v')|^2. \end{aligned}$$

Therefore, we have to estimate

$$(29) \quad \int_{\mathbf{R}^{2dN}} |(\nabla \psi^t(x_i) - \nabla \Psi^t(x_i))|^2 (\rho^t)^{\otimes N}(dX, dV).$$

We have

$$\begin{aligned} & |\nabla \Psi^t(x_i) - \nabla \psi^t(x_i)|^2 \\ & = e^{-2\kappa t} \left| \int_0^t (e^{(t-s)\Delta} \nabla \chi) * ((\rho_N^{s'})_{N;1} - \rho^s)(x_i) \right|^2 \\ & = e^{-2\kappa t} \int_0^t ds \int_0^t ds' ((e^{(t-s)\Delta} \nabla \chi) * ((\rho_N^{s'})_{N;1} - \rho^s)(x_i)) ((e^{(t-s')\Delta} \nabla \chi) * ((\rho_N^{s'})_{N;1} - \rho^{s'})(x_i)). \end{aligned}$$

But, by Lemma 4.5,

$$\left| (e^{(t-s)\Delta} \nabla \chi) * ((\rho_N^{s'})_{N;1} - \rho^{s'})(x_i) \right| \leq \text{Lip } \nabla \chi W_2((\rho_N^{s'})_{N;1}, \rho^{s'}).$$

Therefore

$$|\nabla \Psi^t(x_i) - \nabla \psi^t(x_i)|^2 \leq \text{Lip } \nabla \chi^2 e^{-2\kappa t} \int_0^1 \int_0^1 W_2((\rho_N^s)_{N;1}, \rho^s) W_2((\rho_N^{s'})_{N;1}, \rho^{s'}) ds ds',$$

and we get that the contribution given by (29) has the form

$$\text{Lip } \nabla \chi^2 e^{-2\kappa t} \int_0^t \int_0^t W_2((\rho_N^s)_{N;1}, \rho^s) W_2((\rho_N^{s'})_{N;1}, \rho^{s'}) ds ds',$$

Therefore, we get

$$\begin{aligned} W_2((\rho_N^t)_{N;1}, \rho^t)^2 & \leq \frac{4}{N} \int_0^t \sup_{(x,v),(x',v') \in \text{supp}(\rho^t)} |\gamma(x - x', v - v')|^2 e^{\int_s^t L(u) du} ds \\ & + \int_0^t e^{\int_s^t L(u) du} \int_0^s ds' \int_0^s ds'' W_2((\rho_N^{s'})_{N;1}, \rho^{s'}) W_2((\rho_N^{s''})_{N;1}, \rho^{s''}) ds \\ & := \frac{C(t)}{N} + \eta \text{Lip } \nabla \chi^2 \int_0^t e^{-2\kappa s} e^{\int_s^t \bar{L}(u) du} ds \times \\ & \int_0^s ds' \int_0^s ds'' W_2((\rho_N^{s'})_{N;1}, \rho^{s'}) W_2((\rho_N^{s''})_{N;1}, \rho^{s''}) \\ (30) \quad & \leq \frac{\sup_{s \leq t} C(s)}{N} e^{\eta \text{Lip } \nabla \chi^2 \int_0^t e^{\sup_{u \leq s} \bar{L}(u)} \frac{s^2}{2} ds} := \frac{\alpha(t)^2}{N} \end{aligned}$$

by the same Grönwall type argument than in the proof of Proposition 5.3 in Section 5.

We get the final result of this section.

Proposition 6.1.

$$W_2((\rho_N^t)_{N;1}, \rho^t)^2 \leq \frac{\alpha(t)^2}{N}.$$

Out of α, β , defined in (27)-(30) we define

$$(31) \quad \tau(t) = \alpha(t)^2 + \beta(t)^2.$$

Estimating $\tau(t)$.

By the same type of arguments than in the proof of [32, Corollary 2.6] one can easily estimate $\tau(t)$ by using the estimates of Φ_N^t established in Proposition 8.2 and Theorem 8.3, on the support of ρ_N^t given by Theorem 9.1 and the support of ρ^t in Theorem 9.1. We omit the details here.

...

We get. for some time independent constant C , depending explicitly on and only on $\text{Lip}(\gamma, \text{Lip}(\chi), \text{Lip}(\nabla\chi)$ and $|\text{supp}(\rho^{in})|$,

$$(32) \quad \tau(t) \leq e^{e^{Ct}}.$$

7. HYDRODYNAMIC LIMIT

x-special/nautilus-clipboard copy file:///home/paulth/T The hydrodynamic limit of Cucker-Smale models has provided up to now a large litterature, whose exhaustive quotation is beyond the scope of the present paper. We refer to [6] and the large bibliography therein. In [6], the corresponding Euler equation is derived for Cucker-Smale systems with friction, using the empirical measures formalism and in a modulated energy topology.

Our approach and results are different: we consider generalizations of frictionless Cucker-Smale models, coupled to chemotaxis through a diffusive interaction, for large numbers N of particles and we provide explicit rates of convergences in the quadratic Wasserstein metric towards Euler type equations.

Our result happens to be a simple corollary of our main result Theorem 2.1 in the case where ρ^{in} is monokinetic, i.e.

$$\rho^{in}(x, v) = \mu^{in}(x)\delta(v - u^{in}(x))$$

thanks to the following result: the monokinetic form is preserved by the Valsov equation (13) and the solution is furnished by the solution of a Euler type equation..

Theorem 7.1. *Let μ^t, u^t, ψ^t solves the following system*

$$\begin{cases} \partial_t \mu^t + \nabla_x(u^t \mu^t) = 0 \\ \partial_t(\mu^t u^t) + \nabla(\mu^t (u^t)^{\otimes 2}) = \mu^t \int \gamma(\cdot - y, u^t(\cdot) - u^t(y)) \mu^t(y) dy + \nabla \psi^t + F \\ \partial_s \psi^s(z) = D\Delta_z \psi - \kappa \psi + \chi * \mu^s, \quad s \in [0, t], \\ (\mu^0, u^0, \psi^0) = (\mu^{in}, u^{in}, \psi^{in}) \in H^s, \quad s > \frac{d}{2} + 1. \end{cases}$$

where $\mu^t, u^t \in C([0, t]; H^s) \cap C^1([0, T]; H^{s-1})$, $\psi^t \in C([0, t]; H^s) \cap C^1([0, T]; H^{s-2}) \cap L^2(0, T; H^{s+1})$ ³.

Then $\rho^t(x, v) := \mu^t(x) \delta(v - u^t(x))$ solves the following system

$$\begin{cases} \partial_t \rho^t + v \cdot \nabla_x \rho^t = \nabla_v(\nu(t, x, v) \rho^t), \\ \nu(t, x, v) = \gamma(x, v) * \rho^t + \eta \nabla_x \psi^t(x) + F_{ext}(x), \\ \partial_s \psi^s(z) = D\Delta_z \psi - \kappa \psi + g(z, \rho^s), \quad \psi^0 = \psi^{in} \\ \rho^0(x, v) = \mu^{in}(x) \delta(v - u^{in}(x)). \end{cases}$$

Proof. x-special/nautilus-clipboard copy file:///home/paulth/TWithout the chemical density ψ^t , the derivation of (33) out of (4.5) is standard, see e.g. [6, Section 1.2]. The addition of the term $\eta \Psi$ is a straightforward generalization. \square

8. ESTIMATES ON THE SOLUTION OF THE PARTICLE SYSTEM (3,4,5,6)

Global existence and uniqueness for the system (P) has been proved when γ is exactly the Cucker-Smale field in [12]. It is straightforward to adapt the proofs to the case of a general γ satisfying the hypothesis of the present paper. This situation is anyway fully included in [31, Theorem 6] and we have the following result.

Theorem 8.1. *Let $\text{Lip}(\gamma), \text{Lip}(\nabla \chi) < \infty$ and let $Z^{in} \in \mathbf{R}^{2dN}$. Then, for any $N \in \mathbf{N}$, the Cauchy problem*

$$(P) \quad \left\{ \begin{array}{l} \dot{x}_i = v_i, \quad \dot{v}_i = F_i(t, X(t), V(t)), \quad (X(0), V(0)) = Z(0) = Z^{in} \in \mathbf{R}^{2dN} \\ F_i(t, Y, W) = \frac{1}{N} \sum_{j=1}^N \gamma(w_i - w_j, y_i - y_j) + \eta \nabla_z \varphi^t(z)|_{z=y_i} + F_{ext}(y_i), \\ \partial_s \varphi^s(z) = D\Delta_z \varphi - \kappa \varphi + f(z, X(s)), \quad s \in [0, t], \\ f(z, X) = \frac{1}{N} \sum_{j=1}^N \chi(z - x_j). \end{array} \right.$$

has a unique global solution in $C^0(\mathbf{R}, \mathbf{R}^{2dN})$.

Estimates on the solution of (P) can be easily obtained by the same kind of proof that in [32, Appendix A]. We get the following result.

³x-special/nautilus-clipboard copy file:///home/paulth/TWe suppose this regularity because it is somehow standard for mixed hyperbolic-parabolic systems (see [29, Theorem 2.9 p. 34], one certainly could low it down.

Proposition 8.2. *Let $\gamma_0 = \text{Lip}(\gamma) + \text{Lip}(\nabla\chi)$. Then, for all $t \in \mathbf{R}$, the solution of (P) satisfies*

$$|v_i(t)| \leq \max_{j=1,\dots,N} |v_j(0)| e^{2\gamma_0 t}, \quad i = 1, \dots, N,$$

$$\left| |x_1(t)| - |x_i(0)| \right| \leq \max_{j=1,\dots,N} |v_j(0)| \frac{e^{2\gamma_0 t} - 1}{2\gamma_0}, \quad i = 1, \dots, N.$$

Finally, we will need the following estimate on the derivative of the flow generated by the system (P).

Theorem 8.3. *Let $z_i(t) = x_i(t), v_i(t)$, $i = 1, \dots, N$ be the solution of (P) with initial date $z_i(0) = z_i^{in}$. Then, for all $T \in \mathbf{R}$,*

$$\sup_{t \leq T} \left| \frac{\partial z_i(t)}{\partial z_j^{in}} \right| \leq e^{(\gamma_1 + \gamma_2 T)t}, \quad i, j = 1, \dots, N$$

with $\gamma_1 = \text{Lip}(\gamma), \gamma_2 = \text{Lip}(\nabla\chi)$.

In other words,

$$\sup_{t \leq T} \|d\Phi_N^t\|_\infty \leq e^{(\gamma_1 + \gamma_2 T)t}.$$

Proof. One easily get that, for each $i, j = 1, \dots, N$,

$$\left| \partial_t \frac{\partial z_i(t)}{\partial z_j^{in}} \right| \leq \text{Lip}(\gamma) \frac{1}{N} \sum_{l=1}^N \left| \frac{\partial z_l(t)}{\partial z_j^{in}} \right| + \eta \text{Lip}(\nabla\chi) \frac{1}{N} \sum_{l=1}^N \int_0^t \left| \frac{\partial z_l(s)}{\partial z_j^{in}} \right| ds$$

Therefore, since the right hand-side of the preceding equality doesn't depend on i ,

$$\left| \partial_t \sum_{i=1}^N \frac{\partial z_i(t)}{\partial z_j^{in}} \right| \leq \text{Lip}(\gamma) \sum_{l=1}^N \left| \frac{\partial z_l(t)}{\partial z_j^{in}} \right| + \eta \text{Lip}(\nabla\chi) \int_0^t \sum_{l=1}^N \left| \frac{\partial z_l(s)}{\partial z_j^{in}} \right| ds$$

so that, since

$$\sum_{i=1}^N \frac{\partial z_i(0)}{\partial z_j^{in}} = \sum_{i=1}^N \frac{\partial z_i^{in}}{\partial z_j^{in}} = \sum_{i=1}^N \delta_{i,j} = 1,$$

$$\left| \sum_{i=1}^N \frac{\partial z_i(t)}{\partial z_j^{in}} - 1 \right| \leq \text{Lip}(\gamma) \int_0^t \sum_{l=1}^N \left| \frac{\partial z_l(u)}{\partial z_j^{in}} \right| du + \eta \text{Lip}(\nabla\chi) \int_0^t \int_0^u \sum_{l=1}^N \left| \frac{\partial z_l(s)}{\partial z_j^{in}} \right| ds du$$

$$\leq (\text{Lip}(\gamma) + T\eta \text{Lip}(\nabla\chi)) \int_0^t \sum_{l=1}^N \left| \frac{\partial z_l(u)}{\partial z_j^{in}} \right| du$$

and, by Grönwall Lemma,

$$\left| \frac{\partial z_i(t)}{\partial z_j^{in}} \right| \leq \left| \sum_{i=1}^N \frac{\partial z_i(t)}{\partial z_j^{in}} \right| \leq e^{\gamma_1 t + \gamma_2 T t}.$$

□

9. EXISTENCE, UNIQUENESS AND DOBRUSHIN STABILITY FOR THE VLASOV SYSTEM (13,14,15)

Theorem 9.1. *Let $\text{Lip}(\gamma), \text{Lip}(\nabla\chi) < \infty$ and let $\nu^{in} \in \mathcal{P}_c(\mathbf{R}^{2d})$, the set of compactly supported probability measures. Then the Cauchy problem*

$$(V) \begin{cases} \partial_t \rho^t + v \cdot \nabla_x \rho^t = \nabla_v(\nu(t, x, v) \rho^t), & \rho^0 = \rho^{in} \\ \nu(t, x, v) = \gamma(x, v) * \rho^t + \eta \nabla_x \psi^t(x) + F_{ext}(x), \\ \partial_s \psi^s(z) = D\Delta_z \psi - \kappa \psi + g(z, \rho^s), & \psi^0 = \varphi^{in}. \end{cases}$$

has a unique solution $t \rightarrow \begin{pmatrix} \rho^t \\ \psi^t \end{pmatrix}$ in $C^0(\mathbf{R}, \mathcal{P}_c(\mathbf{R}^{2d}) \times W^{1,\infty}(\mathbf{R}^d))$.

Moreover, if ρ^{in} is supported in the ball $B(0, R^0)$ of \mathbf{R}^{2d} centered at the origin of radius R^0 , ρ^t is supported in $B(0, R^t)$ with

$$R^t = e^{(\text{Lip}(\gamma) + \|F_{ext}\|_{L^\infty(\mathbf{R}^d)} + \eta \text{Lip}(\chi))t} (R^0 + \text{Lip}(\gamma) + \|F_{ext}\|_{L^\infty(\mathbf{R}^d)} + \eta \text{Lip}(\chi)).$$

Finally, if ρ_1^t, ρ_2^t are the solutions of (V) with initial conditions ρ_1^{in}, ρ_2^{in} , then the following Dobrushin type estimate holds true

$$W_2(\rho_1^t, \rho_2^t)^2 \leq 2e^{\Gamma(t)} W_2(\rho_1^{in}, \rho_2^{in})^2$$

where $\Gamma(t)$ is given by (40).

Proof. The proof will follow closely the proof of Theorem 2.3 in [33, Appendix A]. The main difference is that ν is not only non-local in space as in [33], it is also non-local in time as $\nu(t, x, v)$ involves the whole history of the solution $\{\rho^s, 0 \leq s \leq t\}$. In fact

$$(33) \quad \nu(t, z) = \mathbf{v}(t, (x, v), [\rho^{in}]^{\leq t})$$

where $[\rho^{in}]^{\leq t} : s \in [0, t] \rightarrow \rho^s$ solution of (V) with initial data ρ^{in} .

We will first need the following Lemma

Lemma 9.2. *For any $T \geq 0$, there exist $L', M', K' < \infty$ such that, for any $t, t_1, t_2 \leq T$, $z, z' \in \mathbf{R}^{2d}$ and any $\rho^{in}, \rho_1^{in}, \rho_2^{in} \in \mathcal{P}(\mathbf{R}^{2d})$,*

$$\begin{aligned} \|\mathbf{v}(t, z, [\rho^{in}]^{\leq t}) - \mathbf{v}(t, z', [\rho^{in}]^{\leq t})\| &\leq L' \|z - z'\|, \\ \|\nu(t, z)\| &\leq M'(1 + \|z\|) \\ \|\mathbf{v}(t_1, z, [\rho_1^{in}]^{\leq t_1}) - \mathbf{v}(t_2, z, [\rho_2^{in}]^{\leq t_2})\| &\leq K' \sup_{s \leq \min(t_1, t_2)} W_1(\rho_1^s, \rho_2^s) + \eta \|\nabla\chi\|_{L^\infty} |t_1 - t_2|. \end{aligned}$$

where ρ_1^t, ρ_2^t are the solutions of (V) with initial conditions ρ_1^{in}, ρ_2^{in} .

Here W_1 is the Wasserstein distance of order 1 defined whose definition is recalled in Definition 4.2:

The proof is immediate with $L' = \text{Lip}(\gamma) + \text{Lip}(F_{ext}) + T\eta \text{Lip}(\nabla\chi)$, $M' = \text{Lip}(\gamma) + \|F_{ext}\|_{L^\infty(\mathbf{R}^d)} + \eta \text{Lip}(\chi) + \text{Lip}(\nabla\varphi^{in})$ and $K' = \text{Lip}(\gamma) + \eta \text{Lip}(\chi)$.

Let us fix $T > 0$. For $k \in \mathbf{N}$ we define $\tau_k = T2^{-k}$.

Let ρ_k^t be defined by $\rho_k^{t=0} = \rho^{in}$ and, for $l = 0, \dots, 2^k - 1$, $u \in [0, \tau_k)$,

$$(V_k) \begin{cases} \partial_u \rho_k^{l\tau_k+u}(x, v) + v \cdot \nabla_x \rho_k^{l\tau_k+u}(x, v) = \nabla_v \cdot \nu_k(l\tau_k, x, v) \rho_k^{l\tau_k+u}(x, v) \\ \nu_k(l\tau_k, x, v) = \gamma(x, v) * \rho_k^{l\tau_k} + \eta \nabla_x \psi_k(l\tau_k, x) + F_{ext}(x), \\ \partial_s \psi_k(s, z) = D \Delta_z \psi_k - \kappa \psi_k + g(z, \rho_k^s), \quad 0 \leq s \leq l\tau_k, \quad \psi_k^0 = \varphi^{in}. \end{cases}$$

with

$$(34) \quad \nu_k(t, z) = \mathbf{v}(t, z, [\rho_k^{in}]_k^{\leq t})$$

where $[\rho_k^{in}]_k^{\leq t} : s \in [0, t] \rightarrow \rho_k^s$ solution of (V_k) with initial data ρ^{in} (note that $\mathbf{v}(\cdot, \cdot, \cdot)$, defined by (41), is independent of k).

Note that we have obviously the following corollary of Lemma 9.2.

Corollary 9.3. *For any $T \geq 0$, $z, z' \in \mathbf{R}^{2d}$ and $k \in \mathbf{N}$, one has, with the same constant L', M', K' than in Lemma 9.2,*

$$\begin{aligned} \|\nu_k(t, z) - \nu_k(t, z')\| &\leq L' \|z - z'\|, \\ \|\nu_k(t, z)\| &\leq M'(1 + \|z\|), \end{aligned}$$

Moreover, if $\nu_k(t, z), \rho_k^t$ satisfies (V_k) with $\rho_k^{t=0} = \rho^{in}$ and $\nu'_k(t, z), \rho_k^t$ satisfies (V_k) with $\rho_k^{t=0} = \rho'^{in}$, then

$$\|\nu_k(t, z) - \nu'_k(t', z)\|_{L^\infty(\mathbf{R}; C^0(\mathbf{R}^{2d}))} \leq K' \sup_{0 \leq s \leq \min(t, t')} W_1(\rho_k(s), \rho'_k(s)) + K'' |t - t'|.$$

We first show that the support of the sequence ρ_k^t is equibounded.

One easily checks that, since ρ^{in} is compactly supported, so is ρ_k^t for all k, t by construction. So $\text{supp}(\rho_k^t) \subset B(0, R_k^t)$ for some $R_k(t)$. One can estimate R_k^t as follows.

$$\begin{aligned} \text{supp}(\rho_k^{l\tau_k}) \subset B(0, R_k^{l\tau_k}) &\Rightarrow \|\nu_k(t, z)\|_\infty = \|\mathbf{v}(t, z, [\rho_k^{in}]_k^{\leq t})\|_\infty \leq M'(1 + R_k^{l\tau_k}) \\ &\Rightarrow \text{supp}(\rho_k^{l\tau_k+u}) \subset B(0, R_k^{l\tau_k} + uM'(1 + R_k^{l\tau_k})), \quad u \in [0, \tau_k] \\ &\Rightarrow \text{supp}(\rho_k^{(l+1)\tau_k}) \subset B(0, (1 + \tau_k)R_k^{l\tau_k} + \tau_k M'). \end{aligned}$$

Therefore, one can choose $R_k^{l\tau_k}$ satisfying

$$\begin{aligned} R_k^{l\tau_k} &\leq (1 + M'\tau_k)R_k^{(l-1)\tau_k} + \tau_k M' \\ &\leq (1 + M'\tau_k)^2 R_k^{(l-2)\tau_k} + \tau_k M'(1 + (1 + M'\tau_k)) \\ &\leq (1 + M'\tau_k)^l R^0 + M'(((1 + M'\tau_k)^l - 1)) \\ &\leq (1 + M'T2^{-k})^{2k} R^0 + M'((1 + M'T2^{-k})^{2k} - 1) \leq e^{M'T}(M' + R^0) := R^T. \end{aligned}$$

Here R^0 is such that $\text{supp}(\rho_k^0 := \rho^{in}) \subset B(0, R^0)$

Hence the sequences $(\rho_k^t)_{k \in \mathbf{N}}$ are compactly supported in $B(0, R^T)$ uniformly in k for all $t \in [0, T]$. Therefore there are tight for all $t \in [0, T]$. By Prokhorov's Theorem, this is equivalent to the compactness of $(\rho_k^t)_{k \in \mathbf{N}}$ with respect to the weak topology of probability measures (i.e., in the duality with $C_b(\mathbf{R}^{2d})$, the space of bounded continuous functions). Hence, up to extracting a subsequence that we will omit to mention, $\rho_k^t \rightarrow$

ρ_*^t weakly. By Lemma 4.3, this convergence is equivalent to the convergence with respect to the distance W_1 , so that we just proved that

$$W_1(\rho_k^t, \rho_*^t) \rightarrow 0 \text{ as } k \rightarrow \infty \text{ for all } t \in [0, T].$$

By [33, Proposition A.1 2] and the first inequality in Lemma 9.2, we get that, for $l = 0, \dots, 2^k - 1$, $s \in [0, \tau_k)$,

$$W_1(\rho_k^{l\tau_k}, \rho_k^{l\tau_k+s}) \leq sL'.$$

Hence, by the triangle inequality,

$$W_1(\rho_k^t, \rho_k^{t'}) \leq L'|t - t'|, \forall t, t' \in \mathbf{R}.$$

Therefore, since L' and $\rho_k^{t=0}$ don't depend on k , the sequence ρ_k^t is equi-Lipschitz continuous with respect to W_1 . This implies, by the triangular inequality again, that, for all $t, t' \in [0, T]$, $k \in \mathbf{N}$,

$$\begin{aligned} W_1(\rho_*^t, \rho_*^{t'}) &\leq W_1(\rho_*^t, \rho_k^t) + W_1(\rho_k^t, \rho_k^{t'}) + W_1(\rho_k^{t'}, \rho_*^{t'}) \\ &\leq L'|t - t'| + W_1(\rho_*^t, \rho_k^t) + W_1(\rho_k^{t'}, \rho_*^{t'}) \rightarrow L'|t - t'| \text{ as } k \rightarrow \infty. \end{aligned}$$

Therefore ρ_*^t is L' -Lipschitz and, in particular,

$$\rho_*^t \in C_0([0, T], \mathcal{P}_c(\mathbf{R}^{2d})).$$

What is left is to prove that ρ_*^t solves (V) and that the solution of (V) is unique.

To prove that ρ_*^t is a solution of (V), it suffices to prove that

$$(V) \begin{cases} \int_0^T \int_{\mathbf{R}^{2d}} (\partial_t f + v \cdot \nabla_x f - \nabla_v f \cdot \mathbf{v}(t, z, \rho_*^{\leq t})) \rho_*^t(dZ) dt = 0 \\ \mathbf{v}(t, (x, v), \rho_*^{\leq t}) = \gamma(x, v) * \rho_*^t + \eta \nabla_x \psi^t(x) + F_{ext}(x), \\ \partial_s \psi^s(z) = D\Delta_z \psi - \kappa \psi + g(z, \rho_*^s). \end{cases}$$

for each $f \in C_c^\infty([0, T] \times \mathbf{R}^{2d})$.

By construction, we have

$$\sum_{l=0}^{2^k-1} \int_{l\tau_k}^{(l+1)\tau_k} \int_{\mathbf{R}^{2d}} (\partial_u f(u, z) + v \cdot \nabla_x f - \nabla_v f \cdot \mathbf{v}(l\tau_k, z, \rho_k^{\leq l\tau_k})) \rho_k^u(dz) du = 0$$

for every $k \in \mathbf{N}$.

The equation

$$(35) \quad \int_0^T \int_{\mathbf{R}^{2d}} (\partial_t f + v \cdot \nabla_x f - \nabla_v f \cdot \mathbf{v}(t, z, \rho_*^{\leq t})) \rho_*^t(dZ) dt = 0$$

will be proven through the three following limits:

$$(36) \quad \lim_{k \rightarrow \infty} \int_0^T \int_{\mathbf{R}^{2d}} (\partial_t f + v \cdot \nabla_x f)(\rho_*^t - \rho_k^t)(dz) dt = 0$$

$$(37) \quad \lim_{k \rightarrow \infty} \sum_{l=0}^{2^k-1} \int_{l\tau_k}^{(l+1)\tau_k} \int_{\mathbf{R}^{2d}} \nabla_v f \cdot \left(\mathbf{v}(l\tau_k, z, \rho_k^{\leq l\tau_k}) - \mathbf{v}(u, z, \rho_*^{\leq l\tau_k}) \right) \rho_*^u(dz) du = 0$$

$$(38) \quad \lim_{k \rightarrow \infty} \sum_{l=0}^{2^k-1} \int_{l\tau_k}^{(l+1)\tau_k} \int_{\mathbf{R}^{2d}} \nabla_v f \cdot \mathbf{v}(l\tau_k, z, \rho_*^{\leq l\tau_k}) (\rho_k^u - \rho_*^u)(dz) du = 0$$

To prove (36) and (38) we remark that, since $f \in C_c^\infty([0, T] \times \mathbf{R}^{2d})$ and by the Lipschitz property of $\mathbf{v}(l\tau_k, z, \rho_*^{\leq l\tau_k})$ we have, by the Kantorovich-Rubinstein- Theorem, that the absolute value of the right hand side of (36) satisfies

$$\begin{aligned} & \left| \lim_{k \rightarrow \infty} \int_0^T \int_{\mathbf{R}^{2d}} (\partial_t f + v \cdot \nabla_x f)(\rho_*^t - \rho_k^t)(dz) dt \right| \\ & \leq \lim_{k \rightarrow \infty} T (\text{Lip}(\partial_t f) + \text{Lip}(v \cdot \nabla_x f)) \sup_{t \leq T} W_1(\rho_k^t, \rho_*^t) = 0, \end{aligned}$$

and the absolute value of the right hand side of (38) satisfies

$$\begin{aligned} & \left| \lim_{k \rightarrow \infty} \sum_{l=0}^{2^k-1} \int_{l\tau_k}^{(l+1)\tau_k} \int_{\mathbf{R}^{2d}} \nabla_v f \cdot \mathbf{v}(l\tau_k, z, \rho_*^{\leq l\tau_k}) (\rho_k^u - \rho_*^u)(dz) du \right| \\ & \leq \lim_{k \rightarrow \infty} T \text{Lip}(\nabla f) |\text{supp}(f)| M' \sup_{t \leq T} W_1(\rho_k^t, \rho_*^t) = 0, \end{aligned}$$

Let us finally prove (37). By the third inequality in Lemma 9.2 and the L' -Lipschitz continuity of ρ_k we get that, for each $u \in [l\tau_k, (l+1)\tau_k]$,

$$\left| \mathbf{v}(l\tau_k, z, \rho_k^{\leq l\tau_k}) - \mathbf{v}(u, z, \rho_*^{\leq l\tau_k}) \right| \leq K' \sup_{0 \leq s \leq l\tau_k} W_1(\rho_k^{l\tau_k}, \rho_*^{l\tau_k}) + K'' \frac{T}{2^k}$$

so that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \left| \sum_{l=0}^{2^k-1} \int_{l\tau_k}^{(l+1)\tau_k} \int_{\mathbf{R}^{2d}} \nabla_v f \cdot \left(\mathbf{v}(l\tau_k, z, \rho_k^{\leq l\tau_k}) - \mathbf{v}(u, z, \rho_*^{\leq l\tau_k}) \right) \rho_*^u(dz) du \right| \\ & \lim_{k \rightarrow \infty} T \|\nabla f\|_{L^\infty(\mathbf{R}^{2d})} \left(\sup_{0 \leq t \leq T} W_1(\rho_k^t, \rho_*^t) + K'' \frac{T}{2^k} \right) = 0 \quad . \end{aligned}$$

The proof of the uniqueness of the solution of (V) is the same as in [33, Section A.2]. We omit it here.

In order to prove the Dobrushin stability result, we take two initial conditions $\rho_l^{in}, l = 1, 2$.

By Theorem 4.7, we have that

$$\begin{aligned}
& W_2(\rho_1^t, \rho_2^t)^2 \\
& \leq e^{\int_0^t L(s)ds} W_2(\rho_1^{in}, \rho_2^{in})^2 \\
& + \int_0^t \int_{\mathbf{R}^{2dN}} 2 \|\mathbf{v}(s, Z, [\rho_1^{in}]^{\leq s}) - \mathbf{v}(s, Z, [\rho_2^{in}]^{\leq s})\|^2 (\rho_2^{in})^{\otimes N} (dX, dV) e^{\int_s^t l(u)du} ds \\
& \leq e^{\int_0^t L(s)ds} W_2(\rho_1^{in}, \rho_1^{in})^2 + \int_0^t 2 \|\mathbf{v}(s, Z, [\rho_1^{in}]^{\leq s}) - \mathbf{v}(s, Z, [\rho_2^{in}]^{\leq s})\|_\infty^2 e^{\int_s^t l(u)du} ds \\
& \leq e^{\int_0^t L(s)ds} W_2(\rho_1^{in}, \rho_2^{in})^2 + 2 \int_0^t e^{\int_s^t l(u)du} (K')^2 \sup_{u \leq s} W_2(\rho_1^u, \rho_2^u)^2 ds
\end{aligned}$$

by the third inequality in Lemma 9.2.

Here, by Theorem 4.7 and the first inequality in Lemma 9.2,

$$\begin{aligned}
L(u) &= 2 + 2 \min_{i=1,2} \sup_{(x,v) \in \text{supp}(\rho_i^u)} \text{Lip}(\mathbf{v}_1(u, x, v)) + \text{Lip}(\mathbf{v}_2(u, x, v)) \\
(39) \quad &= 2 + 2 \min_{l=1,2} \left(\sup_{\substack{i,l=1,\dots,N \\ (x,v) \in \text{supp}(\rho_l^u)}} \text{Lip}(\gamma)_{(x,v)}^2 \right) + 2(L')^2
\end{aligned}$$

where we denote by $\text{Lip}(\gamma)_{(x,v)}$ the Lipschitz constant of γ at the point (x, v) .

Therefore, for any $T \geq 0$,

$$\begin{aligned}
& \sup_{t \leq T} W_2(\rho_1^t, \rho_2^t)^2 \\
& \leq \sup_{t \leq T} e^{\int_0^t L(s)ds} W_2(\rho_1^{in}, \rho_2^{in})^2 + 2 \sup_{t \leq T} \int_0^t e^{\int_s^t l(u)du} (K')^2 \sup_{u \leq s} W_2(\rho_1^u, \rho_2^u)^2 ds \\
& \leq e^{\int_0^T L(s)ds} W_2(\rho_1^{in}, \rho_2^{in})^2 + 2 \int_0^T e^{\int_s^T l(u)du} (K')^2 \sup_{u \leq s} W_2(\rho_1^u, \rho_2^u)^2 ds.
\end{aligned}$$

By the Grönwall Lemma, we get immediatly

$$W_2(\rho_1^t, \rho_2^t)^2 \leq e^{\Gamma(t)} W_2(\rho_1^{in}, \rho_2^{in})^2$$

with

$$(40) \quad \Gamma(t) := t \left(\sup_{s \leq t} L(s) + (K')^2 e^{t \sup_{s \leq t} L(s)} \right)$$

Theorem 9.1 is proved. □

10. EXISTENCE AND UNIQUENESS FOR THE LIOUVILLE-VLASOV SYSTEM (9,10,11,12)

Theorem 10.1. *Let $\text{Lip}(\gamma), \text{Lip}(\nabla \chi) < \infty$ and let $\nu^{in} \in \mathcal{P}_c(\mathbf{R}^{2d})$, the set of compactly supported probability measures. Then, for every $N \in \mathbf{N}$, the Cauchy problem*

$$(LV) \begin{cases} \partial_t \rho_N^t + V \cdot \nabla_X \rho_N^t = \sum_{i=1}^N \nabla_{v_i} \cdot G_i \rho_N^t, \quad \rho_N^0 = (\rho^{in})^{\otimes N} \\ G_i(t, Y, W) = \frac{1}{N} \sum_{j=1}^N \gamma(w_i - w_j, y_i - y_j) + \eta \nabla_z \Psi^t(z)|_{z=y_i} + F_{ext}(y_i), \\ \partial_s \Psi^s(z) = D \Delta_z \Psi - \kappa \Psi + g(z, \rho_{N;1}^s), \quad s \in [0, t], \quad \Psi^0 = \varphi^{in} \\ g(z, \rho_{N;1}^s) = \int_{\mathbf{R}^{2d}} \chi(z-x) \rho_{N;1}^s(x, v) dx dv \end{cases}$$

has a unique solution $t \rightarrow \begin{pmatrix} \rho_N^t \\ \Psi^t \end{pmatrix}$ in $C^0(\mathbf{R}, \mathcal{P}_c(\mathbf{R}^{2dN}) \times W^{1,\infty}(\mathbf{R}^d))$.

Moreover, if $(\rho^{in})^{\otimes N}$ is supported in the ball $B(0, R^0)$ of \mathbf{R}^{2dN} centered at the origin of radius R^0 , ρ_N^t is supported in $B(0, R^t)$ with

$$R^t = e^{(\text{Lip}(\gamma) + \|F_{ext}\|_{L^\infty(\mathbf{R}^d)} + \eta \text{Lip}(\chi))t} (R^0 + \text{Lip}(\gamma) + \|F_{ext}\|_{L^\infty(\mathbf{R}^d)} + \eta \text{Lip}(\chi)).$$

Proof. The proof of Theorem 10.1 is easily attainable by a straightforward modification of the one of Theorem 9.1. Let us define the vector G with components $G_i, i = 1, \dots, N$. This time, G has the form

$$(41) \quad G(t, Z) = \mathbf{\Gamma}(t, Z, ([(\rho^{in})^{\otimes N}]^{\leq t})_{N;1}) \in \mathbf{R}^N$$

where $[(\rho^{in})^{\otimes N}]^{\leq t} : s \in [0, t] \rightarrow \rho^s$ solution of (LV) with initial data $(\rho^{in})^{\otimes N}$.

One easily check that Lemme 9.2 still holds true for $\mathbf{\Gamma}$ with the same constants L', M', K' .

Lemma 10.2. *For any $T \geq 0$, there exist $L', M', K' < \infty$ such that, for any $t, t_1, t_2 \leq T$, $z, z' \in \mathbf{R}^{2d}$ and any $\rho^{in}, \rho_1^{in}, \rho_2^{in} \in \mathcal{P}(\mathbf{R}^{2d})$,*

$$\begin{aligned} \|\mathbf{\Gamma}(t, Z, ([(\rho^{in})^{\otimes N}]^{\leq t})_{N;1}) - \mathbf{\Gamma}(t, Z', ([(\rho^{in})^{\otimes N}]^{\leq t})_{N;1})\| &\leq L' \|Z - Z'\|, \quad 2 \\ \|\mathbf{\Gamma}(t, Z, ([(\rho^{in})^{\otimes N}]^{\leq t})_{N;1})\| &\leq M'(1 + \|Z\|) \\ \|\mathbf{\Gamma}(t, Z, ([(\rho_1^{in})^{\otimes N}]^{\leq t})_{N;1}) - \mathbf{\Gamma}(t, Z, ([(\rho_2^{in})^{\otimes N}]^{\leq t})_{N;1})\| &\leq K' \sup_{s \leq \min(t_1, t_2)} W_1(\rho_{1N}^s, \rho_{2N}^s) \\ &\quad + \eta \|\nabla \chi\|_{L^\infty} |t_1 - t_2|. \end{aligned}$$

where ρ_{1N}^t, ρ_{2N}^t are the solutions of (LV) with initial conditions $(\rho_1^{in})^{\otimes N}, (\rho_2^{in})^{\otimes N}$.

Here W_1 is the Wasserstein distance of order 1 defined in Definition 4.2.

For $T > 0, k \in \mathbf{N}$ we define $\tau_k = T2^{-k}$ and, with a slight abuse of notation, ρ_k^t by $\rho_k^{t=0} = (\rho^{in})^{\otimes N}$ and, for $l = 0, \dots, 2^k - 1$, $u \in [0, \tau_k)$ (remember $Z := (X, V) \in \mathbf{R}^{2dN}$),

$$(LV_k) \begin{cases} \partial_u \rho_k^{l\tau_k+u}(Z) + V \cdot \nabla_X \rho_k^{l\tau_k+u}(Z) = \nabla_V \cdot (\mathbf{\Gamma}^k(t, Z, ([(\rho^{in})^{\times N}]_k^{\leq t})^{N:1})) \rho_k^{l\tau_k+u}(Z) \\ \mathbf{\Gamma}_i^k(t, (Y, W), ([(\rho^{in})^{\times N}]_k^{\leq t})^{N:1}) = \frac{1}{N} \sum_{j=1}^N \gamma(w_i - w_j, y_i - y_j) + \eta \nabla_z \Psi^t(z)|_{z=y_i} + F_{ext}(y_i), \\ \partial_s \Psi^s(z) = D\Delta_z \Psi - \kappa \Psi + g(z, (\rho_k^s)_{N:1}), \quad s \in [0, t], \end{cases}$$

where $[\rho^{in}]_k^{\leq t} : s \in [0, t] \rightarrow \rho_k^s$ solution of (V_k) with initial data ρ^{in} .

As before, $\mathbf{\Gamma}^k$ satisfies the same estimates than $\mathbf{\Gamma}$ and we have the following result.

Corollary 10.3. *For any $T \geq 0$, $k = 1, \dots, N$, $t, t_1, t_2 \leq T$, $Z, Z' \in \mathbf{R}^{2dN}$ and $\rho^{in}, \rho_1^{in}, \rho_2^{in} \in \mathcal{P}(\mathbf{R}^{2d})$,*

$$\begin{aligned} \|\mathbf{\Gamma}^k(t, Z, ([(\rho^{in})^{\times N}]_k^{\leq t})^{N:1}) - \mathbf{\Gamma}^k(t, Z', ([(\rho^{in})^{\times N}]_k^{\leq t})^{N:1})\| &\leq L' \|Z - Z'\|, \quad 2 \\ \|\mathbf{\Gamma}^k(t, Z, ([(\rho^{in})^{\times N}]_k^{\leq t})^{N:1})\| &\leq M'(1 + \|Z\|) \\ \|\mathbf{\Gamma}^k(t, Z, ([(\rho_1^{in})^{\times N}]_k^{\leq t})^{N:1}) - \mathbf{\Gamma}^k(t, Z, ([(\rho_2^{in})^{\times N}]_k^{\leq t})^{N:1})\| &\leq K' \sup_{s \leq \min(t_1, t_2)} W_1(\rho_{1k}^s, \rho_{2k}^s) \\ &\quad + \eta \|\nabla \chi\|_{L^\infty} |t_1 - t_2|. \end{aligned}$$

where ρ_{1k}^t, ρ_{2k}^t are the solutions of (LV_k) with initial conditions $(\rho_1^{in})^{\otimes N}, (\rho_2^{in})^{\otimes N}$.

At this point, we remark that the part of the proof of Theorem 9.1 uses only the content of Corollary 9.3. Since Corollary 10.3 holds true with the same constants L', M', K' , we conclude that the end of the proof of Theorem 10.1 is the same, modulo a straightforward adaptation, as the one of Theorem 9.1. \square

APPENDIX A. PROOF OF THEOREM 4.7

We will denote $X = (x_1, \dots, x_N), V = (v_1, \dots, v_N), Y = (y_1, \dots, y_N), \Xi = (\xi_1, \dots, \xi_N)$, all of them belonging to \mathbf{R}^{2dN} .

Let π^{in} be an optimal coupling for ρ_1^{in}, ρ_2^{in} .

Obviously $\pi_N^{in} := (\pi^{in})^{\otimes N}$ is a coupling for $(\rho_1^{in})^{\otimes N}, (\rho_2^{in})^{\otimes N}$.

The following first Lemma is equivalent to [32, Lemma 3.1]. It consists in evolving a π_N^{in} by the two dynamics of ρ_i^t , $i = 1, 2$. The proof is very similar to the one of [32, Lemma 3.1], we omit it here.

Lemma A.1. *Let π_N^t be the unique (measure) solution to (19).*

Then, for all $t \in \mathbf{R}$, π_N^t is a coupling between ρ_1^t and ρ_2^t .

Proof. One easily check that the two marginal of π_N^t satisfy the two equations (18) when π_N^t solves (19). Therefore, the Lemma holds true by unicity of these solutions. \square

By a slight modification of the proof of [32, Lemma 3.2] we arrive easily to the following.

Lemma A.2. *Let*

$$D_N(t) := \frac{1}{N} \int ((X - Y)^2 + (V - \Xi)^2) d\pi_N^t.$$

Then

$$\frac{dD_N}{dt} \leq L(t)D_N + \frac{1}{N} \int_{]bR^{2dN}} |\mathbf{v}_1(t, X, V) - \mathbf{v}_2(t, X, V)|^2 \rho_1^t(dX, dV).$$

where

$$L(t) = 2(1 + \text{Lip}(\mathbf{v}_1) + \text{Lip}(\mathbf{v}_2))$$

Proof. As already mentioned, the proof is very similar to the one of [32, Lemma 3.2]. Plugging into the definition of $D_N(t)$ the equation (19) satisfied by π_N^t , integrating by part and $2U \cdot V \leq U^2 + V^2, \forall U, V \in \mathbf{R}^{2d}$ (see the proof of [32, Lemma 3.2] for details), we get

$$\begin{aligned} \frac{dD_N}{dt} &\leq \frac{1}{N} \int_{\mathbf{R}^{4dN}} \left(2(|X - Y|^2 + |V - \Xi|^2) \right. \\ &\quad \left. + |\mathbf{v}_1(t, X, V) - \mathbf{v}_2(t, Y, \Xi)|^2 \right) \pi_N^t(dX, dV, dY, d\Xi) \\ &\leq 2D_N(t) + \frac{2}{N} \int_{\mathbf{R}^{4dN}} |\mathbf{v}_1(t, X, V) - \mathbf{v}_1(t, Y, \Xi)|^2 \pi_N^t(dX, dV, dY, d\Xi) \\ &\quad + \frac{2}{N} \int_{\mathbf{R}^{4dN}} |\mathbf{v}_1(t, X, V) - \mathbf{v}_2(t, X, V)|^2 \pi_N^t(dX, dV, dY, d\Xi) \\ &\leq L(t)D_N + \frac{2}{N} \int_{]bR^{2dN}} |\mathbf{v}_1(t, X, V) - \mathbf{v}_2(t, X, V)|^2 \rho_1^t(dX, dV). \end{aligned}$$

□

Therefore, by Grönwall Lemma,

$$(42) \quad \begin{aligned} D_N(t) &\leq e^{\int_0^t L(s)ds} D_N(0) \\ &\quad + \int_0^t \int_{\mathbf{R}^{2dN}} |\mathbf{v}_1(s, X, V) - \mathbf{v}_2(s, X, V)| (\rho_2^{in})^{\otimes N}(dX, dV) e^{\int_s^t l(u)du} ds \end{aligned}$$

We now remark that, since both π_N^{in} and $\mathbf{v}_1, \mathbf{v}_2$ are invariant by permutations of the variables (x_j, v_j) , $j = 1, \dots, N$, so is π_N^t for all $t \in \mathbf{R}$. This implies that, in fact,

$$D_N(t) = \int_{\mathbf{R}^{2d}} (|x - v|^2 + |y - \xi|^2) d\pi_{N:1}^t,$$

where $\pi_{N:1}^t$ is the measure on $\mathbf{R}^{2d} \times \mathbf{R}^{2d}$ defined, for every test function $\varphi(x, v; y, \xi)$, by

$$\int_{\mathbf{R}^{2dN} \times \mathbf{R}^{2dN}} \varphi \pi_N(dX, dV; dY, d\Xi) = \int_{\mathbf{R}^{2d} \times \mathbf{R}^{2d}} \varphi(x, v; y, \xi) \pi_{N:1}^t(dx, dv, ddy, d\xi).$$

Moreover, straightforward computation (see [32, Lemma 3.4] shows that $(\pi_{N:i}^t)$ is a coupling between $(\rho_1^t)_{N:1}$ and $(\rho_2^t)_{N:1}$. Consequently, one has

$$W_2((\rho_1^t)_{N:1}, (\rho_2^t)_{N:1}) \leq \int_{\mathbf{R}^{2d}} (|x - v|^2 + |y - \xi|^2) d\pi_{N:1}^t = D_N(t)$$

and the conclusion follows by (42).

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