TOPOLOGY OF 1-PARAMETER DEFORMATIONS OF NON-ISOLATED REAL SINGULARITIES

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Abstract. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function germ with non isolated singularities and let $F : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}, 0)$ be a 1-parameter deformation of f. Let $f_t^{-1}(0) \cap B_\epsilon^n$, $0 < |t| \ll \epsilon$, be the "generalized" Milnor fibre of the deformation F. Under some conditions on F, we give a topological degree formula for the Euler characteristic of this fibre. This generalizes a result of Fukui.

1. Introduction

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function germ with an isolated critical point at 0. The Khimshiashvili formula (see [8]) states that

$$\chi(f^{-1}(\delta) \cap B_\epsilon^n) = 1 - \operatorname{sign}(-\delta)^n \deg_0 \nabla f,$$

where $0 < |\delta| \ll \epsilon \ll 1$, B_ϵ^n is the closed ball of radius ϵ centered at 0, ∇f is the gradient of f and $\deg_0 \nabla f$ is the topological degree of the mapping $\frac{\nabla f}{|\nabla f|} : S_\epsilon^{n-1} \to S^{n-1}$. If $F : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ is a 1-parameter deformation of f and the map $H = (F, \frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n})$ has an isolated zero at the origin, Fukui [7] (see also [4]) proves that

$$\chi(f_t^{-1}(0) \cap B_\epsilon^n) = 1 - \deg_0 \nabla f,$$

if n is even and

$$\chi(f_t^{-1}(0) \cap B_\epsilon^n) = 1 - \deg_0 \nabla F - \operatorname{sign}(t) \deg_0 H,$$

if n is odd.

As a corollary of the Khimshiashvili formula, by a result of Arnol'd [1] and Wall [13] we have that

$$\chi(\{f \leq 0\} \cap S_\epsilon^{n-1}) = 1 - \deg_0 \nabla f,$$

$$\chi(\{f \geq 0\} \cap S_\epsilon^{n-1}) = 1 + (-1)^{n-1} \deg_0 \nabla f,$$

and

$$\chi(\{f = 0\} \cap S_\epsilon^{n-1}) = 2 - 2 \deg_0 \nabla f,$$

if n is even.

Szafraniec [11] generalized the results of Arnol'd and Wall to the case of a function germ f with non-isolated singularities and in [12] he improved this result for a

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weighted homogeneous polynomial $f : \mathbb{R}^n \to \mathbb{R}$, constructing polynomials g_1 and g_2 with algebraically isolated critical point at the origin, which makes the topological degree of their gradient vector fields computable by the Eisenbud-Levine formula. We note that similar results were also obtained in the homogeneous case by Bruce in [2].

In [5] Lemma 2.5, the first author proves a new relation between the topology of the positive (resp. negative) real Milnor fibre of an analytic function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and the topology of the link of the set $\{f \leq 0\}$ (resp. $\{f \geq 0\}$). Using Szafraniec's results, he deduces a generalization of the Khimshiashvili formula for non-isolated singularities. Namely he proves that if $0 < \delta \ll \epsilon$, then

$$\chi(f^{-1}(-\delta) \cap B_\epsilon^n) = 1 - (-1)^n \deg_0 \nabla g_-,$$

and

$$\chi(f^{-1}(\delta) \cap B_\epsilon^n) = 1 - (-1)^n \deg_0 \nabla g_+,$$

with $g_- = -f - \omega^d$, $g_+ = f - \omega^d$, $\omega(x) = x_1^2 + \cdots + x_n^2$ and d is an integer big enough. Moreover, if $f : \mathbb{R}^n \to \mathbb{R}$ is weighted homogeneous with $\nabla f(0) = 0$, then

$$\chi(f^{-1}(-1)) = 1 - (-1)^n \deg_0 \nabla g_2,$$

and

$$\chi(f^{-1}(1)) = 1 - (-1)^n \deg_0 \nabla g_1,$$

where g_1 and g_2 are the above polynomials constructed by Szafraniec.

In this paper we adapt the methods developed by Szafraniec in [11, 12] and by the first author in [5] to establish a generalization of the Fukui formula for non-isolated singularities. We consider an analytic function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with non isolated singularities and a 1-parameter deformation $F : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}, 0)$ of f, $F(t, x) = f_t(x)$. We set $X = F^{-1}(0)$, $\Sigma_F = \nabla F^{-1}((0, 0))$ and we assume that $\Sigma_F \subset \{t = 0\}$.

Let $\omega : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function germ such that $\omega(0) = 0$ and $\omega(x) > 0$ if $x \neq 0$. We assume $|t| > \omega$ on $\Gamma_{\omega,t}^X$ where

$$\Gamma_{\omega,t}^X = \big\{(t, x) \in X \mid t \neq 0,\ \operatorname{rank}(e_0, \nabla F(t, x), \nabla \omega(x)) = 2\big\},$$

and $e_0 = (1, 0, \ldots, 0)$ in $\mathbb{R}^{1+n}$. In Corollary 2.11 we show that if n is even then

$$\chi\big(f_t^{-1}(0) \cap B_\epsilon^n\big) = 1 - \deg_0 \nabla g,$$

for $0 < t \ll \epsilon$, where

$$g(x) = G(0, x) \text{ and } G(t, x) = F(t + \omega(x), x).$$

A similar formula holds for $f_{-t}^{-1}(0) \cap B_\epsilon^n$ (see Corollary 2.12).

In order to obtain a formula in the case n odd, we assume that F has an isolated critical point at $(0, 0)$. Under this assumption, we show in Corollary 2.18 that

$$\chi\big(f_t^{-1}(0) \cap B_\epsilon^n\big) = 1 - \deg_0 \nabla F - \deg_0 H,$$

where

$$H(t, x) = \big(F(t, x), W_1(t, x), \ldots, W_n(t, x)\big),$$

and for $i \in \{1, \ldots, n\}$

$$W_i(t, x) = \frac{\partial F}{\partial t}(t, x) \frac{\partial \omega}{\partial x_i}(x) + \frac{\partial F}{\partial x_i}(t, x).$$

A similar formula holds for $f_{-t}^{-1}(0) \cap B_\epsilon^n$ (see Corollary 2.19). All these formulas generalize the Fukui formula [7].

Then we explain how to construct a function germ ω that satisfies the condition of collinearity in the general case and in the special case of weighted homogeneous polynomials, having generalizations for the results of Szafraniec [11], [12], Bruce [2] and the first author [5].

The paper is organized as follows. In the second section we prove the main results (Corollaries 2.11, 2.12, 2.18 and 2.19) and, as applications, in Section 3 we present the construction for ω in the general and in the weighted homogeneous cases.

2. The general situation

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function germ with arbitrary critical set Σ_f. Let $F : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}, 0)$ be an analytic function germ such that $F(0, x) = f(x)$. It is a 1-parameter deformation of f.

We use the notation $f_t(x) = F(t, x)$ and we call $f_t^{-1}(0) \cap B_\epsilon^n$, $0 < |t| \ll \epsilon$, the "generalized" Milnor fibre of the deformation F.

We set $X = F^{-1}(0)$, $\Sigma_F = \{(t, x) \mid \nabla F(t, x) = 0\}$ and make the hypothesis that $\Sigma_F \subset \{t = 0\}$.

Lemma 2.1. *For $t \neq 0$, $f_t^{-1}(0)$ is smooth near the origin.*

Proof. We have the following equivalences:

$$x \in f_\delta^{-1}(0) \Leftrightarrow F(\delta, x) = 0 \Leftrightarrow (\delta, x) \in X \cap \{t = \delta\}.$$

So $f_\delta^{-1}(0)$ is diffeomorphic to $X \cap \{t = \delta\}$. But $X \cap \{t = \delta\} = (X \setminus \Sigma_F) \cap \{t = \delta\}$ and by the Curve Selection Lemma, $\{t = \delta\}$ intersects $X \setminus \Sigma_F$ transversally. □

Hence the deformation f_t of f is actually a smoothing of f.

Let $\omega : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be analytic such that $\omega(0) = 0$ and $\omega(x) > 0$ for $x \neq 0$.

Definition 2.2. We set

$$\Gamma_{\omega,t}^X = \big\{(t, x) \in X \mid t \neq 0,\ \operatorname{rank}(e_0, \nabla F(t, x), \nabla\omega(x)) = 2\big\},$$

where $e_0 = (1, 0, \ldots, 0)$ in $\mathbb{R}^{1+n}$.

Remark 2.3. If $(t, x) \in X$ and $t \neq 0$ then $\operatorname{rank}(e_0, \nabla F(t, x)) = 2$ and so

$$\Gamma_{\omega,t}^X = \big\{(t, x) \in X \mid t \neq 0,\ \operatorname{rank}(e_0, \nabla F(t, x), \nabla\omega(x)) = \operatorname{rank}(e_0, \nabla F(t, x))\big\}.$$

We make the following assumption:

$$(\mathbf{A}): \quad |t| > \omega \text{ on } \Gamma_{\omega,t}^X.$$

Let $\phi : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ be the diffeomorphism defined by $\phi(t, x) = (t + \omega(x), x)$. Let $G : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}, 0)$ be defined by $G(t, x) = F(\phi(t, x))$, let $Y = G^{-1}(0)$ and let $\Sigma_G = \{(t, x) \mid \nabla G(t, x) = 0\}$. Then $\phi(Y) = X$ and $\phi^{-1}(X) = Y$. We note that $\frac{\partial G}{\partial t}(t, x) = \frac{\partial F}{\partial t}(t + \omega(x), x)$ and that for $i \in \{1, \ldots, n\}$,

$$\frac{\partial G}{\partial x_i}(t, x) = \frac{\partial F}{\partial t}(t + \omega(x), x)\frac{\partial \omega}{\partial x_i}(x) + \frac{\partial F}{\partial x_i}(t + \omega(x), x).$$

Let us study first the restriction of t to Y.

Lemma 2.4. *We have $\{t = 0\} \cap \Sigma_G \subset \{(0, 0)\}$.*

Proof. Let us study the critical points of G. A point (t_0, x) belongs to Σ_G if and only if $\frac{\partial F}{\partial t}(t_0 + \omega(x), x) = 0$ and for $i = 1, \ldots, n$,

$$\frac{\partial F}{\partial t}(t_0 + \omega(x), x)\frac{\partial \omega}{\partial x_i}(x) + \frac{\partial F}{\partial x_i}(t_0 + \omega(x), x) = 0.$$

This implies that $(t_0 + \omega(x), x) \in \Sigma_F$ and so that $t_0 + \omega(x) = 0$. Therefore $\Sigma_G \subset \{t + \omega = 0\}$. This implies that $\{t = 0\} \cap \Sigma_G \subset \{(0, 0)\}$. □

Hence the intersection $\{t = 0\} \cap \big(Y \setminus \{(0, 0)\}\big)$ is included in $Y \setminus \Sigma_G$. Moreover this intersection is tranverse as explained in the following lemma.

Lemma 2.5. *The hyperplane $\{t = 0\}$ intersects $Y \setminus \{(0, 0)\}$ transversally.*

Proof. Let $(0, x)$, with $x \neq 0$, be a critical point of $t_{|Y\setminus\{0\}}$. We have $\nabla G(0, x) \neq 0$ and $\operatorname{rank}(e_0, \nabla G(0, x)) = 1$. The point $\phi(0, x) = (\omega(x), x)$ belongs to $X \setminus \Sigma_F$ since $(0, x) \in Y \setminus \Sigma_G$. We have

$$\frac{\partial G}{\partial t}(0, x) = \frac{\partial F}{\partial t}(\omega(x), x),$$

and

$$\frac{\partial G}{\partial x_i}(0, x) = \frac{\partial F}{\partial t}(\omega(x), x)\frac{\partial \omega}{\partial x_i}(x) + \frac{\partial F}{\partial x_i}(\omega(x), x).$$

Since $(0, x)$ is a critical point of $t_{|Y\setminus\{0\}}$, $\frac{\partial G}{\partial x_i}(0, x) = 0$ for $i \in \{1, \ldots, n\}$ and $\frac{\partial G}{\partial t}(0, x) \neq 0$. Hence $\frac{\partial F}{\partial t}(\omega(x), x) \neq 0$ and we have

$$\nabla\omega(x) = -\frac{\nabla_x F(\omega(x), x)}{\frac{\partial F}{\partial t}(\omega(x), x)}.$$

Since $x \neq 0$, $\omega(x) \neq 0$ and so $(\omega(x), x)$ belongs to $\Gamma^X_{\omega,t}$. This is not possible because $|t| > \omega(x)$ on $\Gamma^X_{\omega,t}$ by Assumption (**A**). □

For any subanalytic set $Z \subset \mathbb{R}^{1+n}$, we denote by $\operatorname{Lk}(Z)$ the link of Z at $(0, 0)$.

Lemma 2.6. *We have*

$$\chi\big(\operatorname{Lk}(X \cap \{t \leq 0\})\big) = \chi\big(\operatorname{Lk}(Y \cap \{t \leq 0\})\big).$$

Proof. We note that the function $(t, x) \mapsto t^2 + \omega$ is a non-negative analytic function which vanishes only at $(0, 0)$. Therefore, for any closed subanalytic set $Z \subset \mathbb{R}^{1+n}$ such that $(0, 0) \in Z$, $\operatorname{Lk}(Z)$ is homeomorphic to $Z \cap \{t^2 + \omega = \epsilon\}$ for $0 < \epsilon \ll 1$. This is explained for example in [3] in the semi-algebraic case, but the arguments also work in the subanalytic case. Hence the set $\operatorname{Lk}(Y \cap \{t \leq 0\})$ is homeomorphic to $Y \cap \{t^2 + \omega = \epsilon^2\} \cap \{t \leq 0\}$ and $\operatorname{Lk}(X \cap \{t \leq 0\})$ is homeomorphic to $X \cap \{t^2 + \omega = \epsilon^2\} \cap \{t \leq 0\}$ where $0 < \epsilon \ll 1$. Now

$$(t_0, x) \in \operatorname{Lk}(Y \cap \{t \leq 0\}) \Leftrightarrow \left\{ \begin{array}{c} (t_0, x) \in Y \\ t_0^2 + \omega(x) = \epsilon^2 \\ t_0 \leq 0 \end{array} \right. .$$

Recall that $\phi : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ is the diffeomorphism $(t_0, x) \mapsto (u, x) = (t_0 + \omega(x), x)$. Then

$$(u, x) = \phi(t_0, x) \in \phi\left(\operatorname{Lk}(Y \cap \{t \leq 0\})\right) \Leftrightarrow \left\{ \begin{array}{c} (t_0, x) \in Y \\ (u - \omega(x))^2 + \omega(x) = \epsilon^2 \\ u - \omega(x) \leq 0 \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{c} (u,x) \in X \\ (u-\omega(x))^2 + \omega(x) = \epsilon^2 \\ u \leq \omega(x) \end{array} \right. .$$

Since the function $(u,x) \mapsto (u-\omega(x))^2 + \omega$ is analytic, non-negative and vanishes only at $(0,0)$, we can conclude that $\mathrm{Lk}(Y \cap \{t \leq 0\})$ is homeomorphic to $\mathrm{Lk}(X \cap \{t \leq \omega(x)\})$. Now

$$(t_0,x) \in \mathrm{Lk}(X \cap \{t \leq \omega(x)\}) \Leftrightarrow \left\{ \begin{array}{c} (t_0,x) \in X \\ t_0^2 + \omega(x) = \epsilon^2 \\ t_0 \leq \epsilon^2 - t_0^2 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} (t_0,x) \in X \\ t_0^2 + \omega(x) = \epsilon^2 \\ t_0 + t_0^2 \leq \epsilon^2 \end{array} \right. .$$

Similarly

$$(t_0,x) \in \mathrm{Lk}(X \cap \{t \leq 0\}) \Leftrightarrow \left\{ \begin{array}{c} (t_0,x) \in X \\ t_0^2 + \omega(x) = \epsilon^2 \\ t_0 \leq 0 \end{array} \right. .$$

But, since $|t_0| < \epsilon$ and ϵ is small, we have that $1 + t_0 \geq 0$ and so

$$(t_0,x) \in \mathrm{Lk}(X \cap \{t \leq 0\}) \Leftrightarrow \left\{ \begin{array}{c} (t_0,x) \in X \\ t_0^2 + \omega(x) = \epsilon^2 \\ t_0^2 + t_0 \leq 0 \end{array} \right. .$$

Let p be a critical point of $t^2 + t$ restricted to $X \cap \{t^2 + \omega = \epsilon^2\}$, and such that $(t^2+t)(p) > 0$. Then p belongs to $\Gamma^X_{\omega,t}$ and so $t(p) > \omega(p)$ and $(t+t^2)(p) > \omega(p) + t^2(p) = \epsilon^2$. Therefore the function $t^2 + t$ restricted to $X \cap \{t^2 + \omega = \epsilon^2\}$ has no critical point on the set $\{0 < t^2 + t \leq \epsilon^2\}$. So the set $X \cap \{t^2 + \omega = \epsilon^2\} \cap \{t \leq 0\}$ is a deformation retract of $X \cap \{t^2 + \omega = \epsilon^2\} \cap \{t \leq \omega\}$. This implies that $\chi(\mathrm{Lk}(Y \cap \{t \leq 0\})) = \chi(\mathrm{Lk}(X \cap \{t \leq 0\}))$. □

Lemma 2.7. *Let $0 < \delta \ll \epsilon \ll 1$. We have:*

$$\chi\big(X \cap B^{n+1}_\epsilon \cap \{t = -\delta\}\big) - \chi\big(X \cap S^n_\epsilon \cap \{t = -\delta\}\big) = \chi\big(\mathrm{Lk}(X)\big) - \chi\big(\mathrm{Lk}(X \cap \{t \geq 0\})\big),$$

$$\chi\big(X \cap B^{n+1}_\epsilon \cap \{t = \delta\}\big) - \chi\big(X \cap S^n_\epsilon \cap \{t = \delta\}\big) = \chi\big(\mathrm{Lk}(X)\big) - \chi\big(\mathrm{Lk}(X \cap \{t \leq 0\})\big).$$

Proof. If $X \cap \{t = -\delta\} \cap B^{n+1}_\epsilon$ is empty for $0 < \delta \ll \epsilon \ll 1$, which means that $t \geq 0$ on $X \cap B^{n+1}_\epsilon$, then the result is trivial for $\mathrm{Lk}(X) = \mathrm{Lk}(X \cap \{t \geq 0\})$.

Let us treat the case when $X \cap \{t = -\delta\} \subset \overset{\circ}{B}{}^{n+1}_\epsilon$, i.e. $X \cap \{t = -\delta\} \cap S^n_\epsilon$ is empty for $0 < \delta \ll \epsilon \ll 1$. Let C be a connected component of $X \setminus \{0\}$. Then $C \cap \{t = -\delta\} \cap S^n_\epsilon$ is empty for $\delta \in]0, \delta_\epsilon[$ for some $\delta_\epsilon > 0$. By conic structure, $C \cap S^n_\epsilon$ is connected and so by continuity, either $t \geq 0$ on $C \cap S^n_\epsilon$ or $t < 0$ on $C \cap S^n_\epsilon$.

Let $\gamma : [0, \nu[\to \overline{C}$ be an analytic arc such that $\gamma(0) = 0$. Then there exists $\epsilon_1 > 0$ such that either $t < 0$ on $\gamma(]0,\nu[) \cap B^{n+1}_{\epsilon_1}$, either $t = 0$ on $\gamma(]0,\nu[) \cap B^{n+1}_{\epsilon_1}$ or $t > 0$ on $\gamma(]0,\nu[) \cap B^{n+1}_{\epsilon_1}$. By the above observation, and taking a smaller ϵ_1 if necessary, we conclude that either $t < 0$ on $C \cap B^{n+1}_{\epsilon_1}$ or $t \geq 0$ on $C \cap B^{n+1}_{\epsilon_1}$. Let $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) be the set of connected components of $X \setminus \{0\}$ on which $t \geq 0$ (resp. $t < 0$). Then $\chi(\{t \geq 0\} \cap X \cap S^n_\epsilon) = \sum_{C \in \mathcal{C}^+} \chi(C \cap S^n_\epsilon)$ and

$$\chi(X \cap \{t = -\delta\} \cap B^{n+1}_\epsilon) = \sum_{C \in \mathcal{C}^-} \chi(C \cap S^n_\epsilon),$$

for if $C \in \mathcal{C}^-$ then the function $-t$ is strictly positive on C and $C \cap S^n_\epsilon$ and $C \cap \{t = -\delta\}$ are homeomorphic, as explained in Lemma 2.6. We conclude with

the remark that

$$\sum_{C\in\mathcal{C}^-}\chi(C\cap S^n_\epsilon)=\chi(\mathrm{Lk}(X))-\sum_{C\in\mathcal{C}^+}\chi(C\cap S^n_\epsilon).$$

Now let us assume that $X\cap\{t=-\delta\}\cap S^n_\epsilon$ is not empty for $0<\delta\ll\epsilon\ll 1$. Since $\Sigma_F\subset\{t=0\}$, we can use the deformation argument due to Milnor [10] and get that $X\cap\{t=-\delta\}\cap B^{n+1}_\epsilon$ is homeomorphic to $X\cap\{t\leq-\delta\}\cap S^n_\epsilon$. Then we use the fact that

$$\chi\big(\mathrm{Lk}(X)\big)=\chi\big(X\cap S^n_\epsilon\big)=\chi\big(\{t\leq-\delta\}\cap X\cap S^n_\epsilon\big)$$
$$+\chi\big(\{t\geq-\delta\}\cap X\cap S^n_\epsilon\big)-\chi\big(\{t=-\delta\}\cap X\cap S^n_\epsilon\big).$$

But the inclusion $\{t\geq 0\}\cap X\cap S^n_\epsilon\subset\{t\geq-\delta\}\cap X\cap S^n_\epsilon$ is a homotopy equivalence, and so $\chi\big(\{t\geq-\delta\}\cap X\cap S^n_\epsilon\big)=\chi\big(\mathrm{Lk}(X\cap\{t\geq 0\}\big)$. □

A similar result is true for Y, but only for positive values of t.

Lemma 2.8. *Let $0<\delta\ll\epsilon\ll 1$. We have:*

$$\chi\big(Y\cap B^{n+1}_\epsilon\cap\{t=\delta\}\big)-\chi\big(Y\cap S^n_\epsilon\cap\{t=\delta\}\big)=\chi\big(\mathrm{Lk}(Y)\big)-\chi\big(\mathrm{Lk}(Y\cap\{t\leq 0\})\big).$$

Proof. We remark that $\Sigma_G\subset\{t\leq 0\}$ since, as explained in Lemma 2.4, Σ_G is included in $\{t+\omega=0\}$. By a Curve Selection Lemma argument, this implies that the fibre $\{t=\delta\}\cap Y$ is smooth in a neighborhood of the origin for $0<\delta\ll\epsilon$. Then the proof is the same as in the previous lemma. □

Corollary 2.9. *Let $0<\delta\ll\epsilon\ll 1$. We have*

$$\chi\big(\{t=\delta\}\cap X\cap B^{n+1}_\epsilon\big)-\chi\big(\{t=\delta\}\cap X\cap S^n_\epsilon\big)$$
$$=\chi\big(\{t=\delta\}\cap Y\cap B^{n+1}_\epsilon\big)-\chi\big(\{t=\delta\}\cap Y\cap S^n_\epsilon\big).$$

Proof. We know that $X=\phi(Y)$ so X and Y are homeomorphic. By the conic structure, $\mathrm{Lk}(X)$ and $\mathrm{Lk}(Y)$ are homeomorphic. The result is just a consequence of Lemmas 2.6, 2.7 and 2.8. □

Let $g:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ be the analytic function germ defined by $g(x)=G(0,x)$. We note that G is a 1-parameter deformation of g.

Lemma 2.10. *The function g has an isolated critical point at the origin and $g_t^{-1}(0)$ is smooth near the origin for $0<t\ll\epsilon$.*

Proof. If $x\neq 0$ is a critical point of g then for $i\in\{1,\dots,n\}$, $\frac{\partial G}{\partial x_i}(0,x)=0$. Moreover since $\{t=0\}\cap\Sigma_G\subset\{(0,0)\}$ by Lemma 2.4, $\frac{\partial G}{\partial t}(0,x)\neq 0$. Therefore $(0,x)$ is a critical point of $t_{|Y\setminus\{(0,0)\}}$. This is not possible for $\{t=0\}$ intersects $Y\setminus\{(0,0)\}$ transversally by Lemma 2.5. This means that g has an isolated critical point at 0.

We have explained in Lemma 2.8 that the fibre $\{t=\delta\}\cap Y$ is smooth in a neighborhhod of $(0,0)$ for $0<\delta\ll 1$. But this fibre is diffeomorphic to $g_\delta^{-1}(0)$. □

Applying Corollary 2.9, we obtain

Corollary 2.11. *Let $0<t\ll\epsilon\ll 1$. We have*

$$\chi\big(f_t^{-1}(0)\cap B^n_\epsilon\big)=\chi\big(g_t^{-1}(0)\cap B^n_\epsilon\big).$$

Moreover if n is even, $\chi\big(f_t^{-1}(0)\cap B^n_\epsilon\big)=1-\deg_0\nabla g$.

Proof. We apply Corollary 2.9 and we use the fact that if $(M, \partial M)$ is a compact manifold with boundary then $\chi(\partial M) = 0$ if the manifold is even-dimensional and $\chi(\partial M) = 2\chi(M)$ if it is odd-dimensional.

If n is even, we use the Arnold-Wall formula [1, 13] to get that

$$1 - \deg_0 \nabla g = \frac{1}{2}\chi\big(\mathrm{Lk}(\{g = 0\})\big) = \frac{1}{2}\chi\big(g_t^{-1}(0) \cap S_\epsilon^{n-1}\big) = \chi\big(g_t^{-1}(0) \cap B_\epsilon^n\big).$$

□

In order to get the corresponding result for $f_{-t}^{-1}(0) \cap B_\epsilon^n$, we apply the same procedure to the functions $G'(t, x) = F(t - \omega(x), x)$, $g'(x) = G'(0, x)$ and to the set $Y' = G'^{-1}((0, 0))$. We find that $\nabla g'$ has an isolated zero at the origin. Furthermore applying the method of Lemma 2.6, we get that

$$\chi\big(\mathrm{Lk}(\{t \geq 0\} \cap X)\big) = \chi\big(\mathrm{Lk}(\{t \geq 0\} \cap Y')\big).$$

Corollary 2.12. *Let $0 < t \ll \epsilon \ll 1$. We have*

$$\chi\big(f_{-t}^{-1}(0) \cap B_\epsilon^n\big) = \chi\big(g'^{-1}_{-t}(0) \cap B_\epsilon^n\big).$$

Moreover if n is even, $\chi\big(f_{-t}^{-1}(0) \cap B_\epsilon^n\big) = 1 - \deg_0 \nabla g'$.

This method only gives a topological degree formula when n is even. In order to get a topological degree formula when n is odd, we need to impose a stronger condition on Σ_F.

From now on we make the hypothesis that $\Sigma_F \subset \{0\}$, i.e. F has an isolated critical point at the origin.

Let

$$\Sigma_t^X = \left\{(t, x) \in X \mid \frac{\partial F}{\partial x_i}(t, x) = 0,\ i = 1, \ldots, n\right\}.$$

The set Σ_t^X is the set of critical points of $t_{|X}$, and so $\Sigma_t^X \subset X \cap \{t = 0\}$ by the Curve Selection Lemma.

Lemma 2.13. *The set Σ_f is diffeomorphic to Σ_t^X.*

Proof. We see that $(t, x) \in \Sigma_t^X$ if and only if $t = 0$ and $\frac{\partial F}{\partial x_i}(0, x) = 0$ for $i \in \{1, \ldots, n\}$, that is if and only if $t = 0$ and $x \in \Sigma_f$. Therefore $\Sigma_t^X = \psi(\Sigma_f)$ where ψ is the diffeomorphism

$$\begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^{1+n} \cap \{t = 0\} \\ x & \mapsto & (0, x). \end{array}$$

□

We recall that $G : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}, 0)$ is defined by $G(t, x) = F(t + \omega(x), x)$ and that $Y = G^{-1}(0)$.

Lemma 2.14. *The function G has an isolated critical point at $(0, 0)$.*

Proof. Since $(\nabla F)^{-1}((0, 0)) \subset \{(0, 0)\}$ and $G(t, x) = F(t + \omega(x), x)$, we find that $(\nabla G)^{-1}((0, 0)) \subset \{(0, 0)\}$ using the relations between the partial derivatives of G and F. □

Lemma 2.15. *We have $\deg_0 \nabla F = \deg_0 \nabla G$.*

Proof. We have $\nabla G(t,x) = A(t,x)\nabla F(\phi(t,x))$ where

$$A(t,x) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{\partial \omega}{\partial x_1}(x) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \omega}{\partial x_n}(x) & 0 & \cdots & 1 \end{pmatrix}.$$

□

Let $\tilde{H} : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}^{1+n}, 0)$ be defined by $\tilde{H} = (G, \frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n})$.

Corollary 2.16. *We have* $\tilde{H}^{-1}((0,0)) = \{(0,0)\}$.

Proof. Since by Lemma 2.14 G has an isolated critical point at the origin, the mapping $\tilde{H}$ has an isolated zero if and only if the function $t_{|Y}$ has an isolated critical point at the origin. But $(0,x)$ is a critical point of $t_{|Y}$ if and only if x is a critical point of g. □

We note that $t_{|Y}$ has an isolated critical point at the origin, whereas $t_{|X}$ can have an arbitrary set of critical points.

Let us consider the map H :

$$H : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}^{1+n}, 0),\ (t,x) \mapsto (F(t,x), W_1(t,x), \ldots, W_n(t,x)),$$

where for each $i \in \{1, \ldots, n\}$,

$$W_i(t,x) = \frac{\partial F}{\partial t}(t,x)\frac{\partial \omega}{\partial x_i}(x) + \frac{\partial F}{\partial x_i}(t,x).$$

Lemma 2.17. *We have* $H^{-1}((0,0)) = \{(0,0)\}$ *and* $\deg_0 \tilde{H} = \deg_0 H$.

Proof. We have $\tilde{H}(t,x) = H(\phi(t,x))$ where we recall that $\phi(t,x) = (t + \omega(x), x)$. So we get the result, for ϕ preserves orientation. □

Now we can give the topological degree formulas for $\chi\big(f_t^{-1}(0) \cap B_\epsilon^n\big)$.

Corollary 2.18. *Let* $0 < t \ll \epsilon$. *If* n *is even then*

$$\chi\big(f_t^{-1}(0) \cap B_\epsilon^n\big) = 1 - \deg_0 \nabla g,$$

and if n *is odd then*

$$\chi\big(f_t^{-1}(0) \cap B_\epsilon^n\big) = 1 - \deg_0 \nabla F - \deg_0 H.$$

Proof. By Corollary 2.11, we know that

$$\chi\big(f_t^{-1}(0) \cap B_\epsilon^n\big) = \chi\big(g_t^{-1}(0) \cap B_\epsilon^n\big).$$

If n is even, we have already explained that $\chi\big(g_t^{-1}(0) \cap B_\epsilon^n\big) = 1 - \deg_0 \nabla g$ in the same corollary. In the case n odd, the Fukui formula [7, 4] states that :

$$\chi\big(g_t^{-1}(0) \cap B_\epsilon^n\big) = 1 - \deg_0 \nabla G - \deg_0 \tilde{H}.$$

We use Lemmas 2.15 and 2.17 to conclude. □

To get the corresponding formulas for $f_{-t}^{-1}(0) \cap B_\epsilon^n$, we apply the same procedure to the function $G'(t,x) = F(t - \omega(x), x)$. We define H' by $H' = (F, W_1', \ldots, W_n')$ where for each $i \in \{1, \ldots, n\}$,

$$W_i'(t,x) = -\frac{\partial F}{\partial t}(t,x)\frac{\partial \omega}{\partial x_i}(x) + \frac{\partial F}{\partial x_i}(t,x).$$

We find that H', $\nabla G'$ have an isolated zero at the origin and that $\deg_0 \nabla G' = \deg_0 \nabla F$.

Corollary 2.19. *Let $0 < t \ll \epsilon$. If n is even then*

$$\chi\big(f_{-t}^{-1}(0) \cap B_\epsilon^n\big) = 1 - \deg_0 \nabla g',$$

and if n is odd then

$$\chi\big(f_{-t}^{-1}(0) \cap B_\epsilon^n\big) = 1 - \deg_0 \nabla F - \deg_0 H'.$$

3. Applications

In this section, we give some results on the existence of the function ω and we give some explicit examples.

3.1. **General case.** We keep the notations of the previous section.

Lemma 3.1. *There exists $d \in \mathbb{N}$ such that $|t| > \omega^d$ on $\Gamma^X_{\omega,t} \setminus \{\omega = 0\}$ close to the origin.*

Proof. We adapt to our situation the method developped in [11]. Recall that

$$\Gamma^X_{\omega,t} = \big\{(t,x) \in X \mid t \neq 0,\ \operatorname{rank}(e_0, \nabla F(t,x), \nabla\omega(x)) = 2\big\}.$$

Let Σ be the following closed analytic set:

$$\Sigma = \{(t,x;r,y) \in X \times \mathbb{R} \times \mathbb{R} \mid \omega(x) = r, \text{all } \Delta_{i,j}(t,x) = 0, t = y\},$$

where for $(i,j) \in \{1,\dots,n\}^2$,

$$\Delta_{i,j}(t,x) = \begin{vmatrix} \frac{\partial F}{\partial x_i}(t,x) & \frac{\partial F}{\partial x_j}(t,x) \\ \frac{\partial \omega}{\partial x_i}(x) & \frac{\partial \omega}{\partial x_j}(x) \end{vmatrix}.$$

Let $\pi : X \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the natural projection. The mapping $\pi_{|\Sigma} : \Sigma \to \pi(\Sigma)$ is proper, and so $\pi(\Sigma)$ is closed and semi-analytic (see [9], p.127). Let us set $Y_1 = \mathbb{R} \times \{0\}$ and $Y_2 = \overline{\pi(\Sigma) \setminus Y_1}$. The set Y_2 is semi-analytic. If $r > 0$ then

$$\pi(\Sigma) \cap \{r\} \times \mathbb{R} = \{r\} \times \{\text{critical values of } t_{|X \cap \omega^{-1}(r)}\},$$

and $\pi(\Sigma) \cap \{r\} \times \mathbb{R}$ is a discrete set of points, since the set of critical values of an analytic function is discrete. Hence Y_2 has dimension less than or equal to 1 in a neighborhood of 0. So 0 is isolated in $Y_1 \cap Y_2$. By the Łojasiewicz inequality [9], there are positive constants C, α such that $|y| \geq Cr^\alpha$ for $(r,y) \in Y_2$ sufficiently close to the origin. If $d \in \mathbb{N}$ is such that $d > \alpha$, then $Cr^\alpha > r^d$ if $r > 0$ is sufficiently small. Therefore $|y| > r^d$ for $(r,y) \in Y_2$ sufficiently close to the origin.

If (t,x) belongs to $\Gamma^X_{\omega,t}$ then $(t,x,\omega(x),t)$ belongs to Σ and (ω,t) lies in $\pi(\Sigma)$. Hence, if $t \neq 0$ then $|t| > \omega(x)^d$ for $(t,x) \in \Gamma^X_{\omega,t}$ sufficiently close to the origin. □

Corollary 3.2. *We have $|t| > \omega(x)^d$ on $\Gamma^X_{\omega^d,t}$.*

Proof. We have $\nabla\omega^d = d\omega^{d-1}\nabla\omega$ so

$$\Gamma^X_{\omega^d,t} \cap \{\omega \neq 0\} = \Gamma^X_{\omega,t} \cap \{\omega \neq 0\}.$$

We conclude remarking that the inequality is trivial if $\omega^d = 0$. □

Therefore we can apply our method to the functions $G(t,x) = F(t + \omega^d(x), x)$ and $G'(t,x) = F(t - \omega^d(x), x)$. This generalizes the results of Szafraniec [11] and the first author [5].

3.2. **Weighted homogeneous case.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a weighted homogeneous polynomial and let $F : \mathbb{R}^{1+n} \to \mathbb{R}$ be a 1-parameter weighted homogeneous deformation of f such that $\Sigma_F \subset \{t = 0\}$.

We assume that F is of type $(d; d_0, d_1, \ldots, d_n)$ where $d \geq 2$. Since $F(0, x) = f(x)$ then f is of type $(d; d_1, \ldots, d_n)$. Let p be the smallest positive integer such that $2p > d$ and each d_i, with $1 \leq i \leq n$, divides p. Let $a_i = p/d_i$ and let $\omega(x) = \frac{x_1^{2a_1}}{2a_1} + \ldots + \frac{x_n^{2a_n}}{2a_n}$. We denote $\lambda \cdot (t, x) = (\lambda^{d_0} t, \lambda^{d_1} x_1, \ldots, \lambda^{d_n} x_n)$.

Lemma 3.3. *We have $|t| > \omega$ on $\Gamma^X_{\omega,t}$.*

Proof. We adapt to our situation the method developped in [12]. Recall that

$$\Gamma^X_{\omega,t} = \{(t, x) \in X \mid t \neq 0,\ \operatorname{rank}(e_0, \nabla F(t, x), \nabla\omega(x)) = 2\}.$$

Let

$$\Sigma = \big\{(t, x; r, y) \in X \times \mathbb{R} \times \mathbb{R} \mid \omega(x) = r^{2p}, \text{all } \Delta_{i,j}(t, x) = 0, t = y\big\},$$

where for $(i, j) \in \{1, \ldots, n\}^2$,

$$\Delta_{i,j}(t, x) = \begin{vmatrix} \frac{\partial F}{\partial x_i}(t, x) & \frac{\partial F}{\partial x_j}(t, x) \\ \frac{\partial \omega}{\partial x_i}(x) & \frac{\partial \omega}{\partial x_j}(x) \end{vmatrix}.$$

Let $\pi : X \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the natural projection. If $r > 0$ then

$$\pi(\Sigma) \cap \{r\} \times \mathbb{R} = \{r\} \times \{\text{critical values of } t_{|X \cap \omega^{-1}(r^{2p})}\},$$

and $\pi(\Sigma) \cap \{r\} \times \mathbb{R}$ is a finite set of points, since a polynomial function has a finite number of critical values.

Let $((t, x), r, y) \in \Sigma$ and let $\lambda \in \mathbb{R}$. We have $\omega(\lambda \cdot x) = \lambda^{2p}\omega(x) = (\lambda r)^{2p}$,

$$\Delta_{i,j}(\lambda \cdot (t, x)) = \lambda^{2p+d-d_i-d_j} \Delta_{i,j}((t, x)),$$

and $F(\lambda \cdot (t, x)) = \lambda^d F(t, x)$. So if $((t, x), r, y) \in \Sigma$ then $(\lambda \cdot (t, x), \lambda r, \lambda^{d_0} y) \in \Sigma$ too. Hence $\pi(\Sigma)$ is a finite union of curves and if $(r, y) \in \pi(\Sigma)$ and $\lambda \in \mathbb{R}$ then $(\lambda r, \lambda^{d_0} y) \in \pi(\Sigma)$.

Because $2p > d \geq d_0$ then $|y| > r^{2p}$ for every point $(r, y) \in \pi(\Sigma)$ with $y \neq 0$, sufficiently close to 0. Then if $(t, x) \in \Gamma^X_{\omega,t}$ is close to $(0, 0)$ then $((t, x), \omega(x)^{\frac{1}{2p}}, t) \in \Sigma$ and $|t| = |y| > r^{2p} = \omega(x)$. □

Therefore we can apply our method to the functions $G(t, x) = F(t + \omega(x), x)$ and $G'(t, x) = F(t - \omega(x), x)$. In [2, 12], the topological degrees appearing in the formulas can be expressed as signatures, because the mappings have algebraically isolated zeros. In the sequel, we will see that this is also the case here under the assumption that the complexification of F has an isolated critical point at $(0, 0)$.

We use the following notation: for any real analytic map $h : \mathbb{R}^m \to \mathbb{R}^p$, we denote by $h_{\mathbb{C}} : \mathbb{C}^m \to \mathbb{C}^p$ its complexification.

Let $\mathcal{L}$ be the following constructible set:

$$\mathcal{L} = \big\{x \in \mathbb{C}^n \setminus \{0\} \mid \nabla f_{\mathbb{C}}(x) \neq 0 \text{ and all } m_{i,j}(x) = 0\big\},$$

where for $(i, j) \in \{1, \ldots, n\}^2$,

$$m_{i,j}(x) = \begin{vmatrix} \frac{\partial f_{\mathbb{C}}}{\partial x_i}(x) & \frac{\partial f_{\mathbb{C}}}{\partial x_j}(x) \\ \frac{\partial \omega_{\mathbb{C}}}{\partial x_i}(x) & \frac{\partial \omega_{\mathbb{C}}}{\partial x_j}(x) \end{vmatrix}.$$

Let $C = \{x \in \mathbb{C}^n \mid \nabla\omega_{\mathbb{C}}(x)+\nabla f_{\mathbb{C}}(x) = 0\}$. It is proved in [12] that C is an algebraic compact subset of $\mathbb{C}^n$, hence it is a finite set of points.

Lemma 3.4. *A point x belongs to $\mathcal{L}$ if and only if there exist $z \in C \setminus \{0\}$ and $\lambda \in \mathbb{C}^*$ such that $x = \lambda \cdot z$.*

Proof. If $z \in C \setminus \{0\}$ then $\nabla\omega_{\mathbb{C}}(z) \neq 0$ and so $\nabla f_{\mathbb{C}}(z) \neq 0$. Since $\frac{\partial\omega_{\mathbb{C}}}{\partial x_i}(\lambda \cdot z) = \lambda^{2p-d_i}\frac{\partial\omega_{\mathbb{C}}}{\partial x_i}(z)$ and $\frac{\partial f_{\mathbb{C}}}{\partial x_i}(\lambda \cdot z) = \lambda^{d-d_i}\frac{\partial f_{\mathbb{C}}}{\partial x_i}(z)$, it is clear that $\lambda \cdot z$ belongs to $\mathcal{L}$ for $\lambda \neq 0$.

Reciprocally if $x \in \mathcal{L}$ then there is $\alpha \neq 0$ such that $\alpha\nabla\omega_{\mathbb{C}}(x)+\nabla f_{\mathbb{C}}(x) = 0$. This implies that $\lambda \cdot x$ belongs to $C \setminus \{0\}$ where $\lambda^{2p-d} = \alpha$. □

Lemma 3.5. *If $F_{\mathbb{C}}$ has an isolated critical point at the origin, then $g_{\mathbb{C}}$ and $G_{\mathbb{C}}$ have an isolated critical point at the origin and $H_{\mathbb{C}}$ has an isolated zero at the origin.*

Proof. Let x be a critical point of $g_{\mathbb{C}}$. For $i \in \{1, \ldots, n\}$, we have

$$\frac{\partial F_{\mathbb{C}}}{\partial t}(0, x)\frac{\partial\omega_{\mathbb{C}}}{\partial x_i}(x) + \frac{\partial F_{\mathbb{C}}}{\partial x_i}(0, x) = 0.$$

If $\frac{\partial F_{\mathbb{C}}}{\partial t}(0, x) = 0$ then $(0, x)$ is a critical point of $F_{\mathbb{C}}$ and $x = 0$. If $\frac{\partial F_{\mathbb{C}}}{\partial t}(0, x) \neq 0$ and $\nabla\omega_{\mathbb{C}}(x) = 0$ then $x = 0$. If $\frac{\partial F_{\mathbb{C}}}{\partial t}(0, x) \neq 0$ and $\nabla\omega_{\mathbb{C}}(x) \neq 0$ then $\nabla f_{\mathbb{C}}(x) \neq 0$ and x lies in $\mathcal{L}$. So if 0 is not an isolated critical point of $g_{\mathbb{C}}$, then there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{L}$ such that $\nabla g_{\mathbb{C}}(x_n) = 0$ and $\|x_n\| \to 0$. By Lemma 3.4 and the fact that C is finite, there exist a sequence $(\lambda_n)_{n\in\mathbb{N}}$, $\lambda_n \in \mathbb{C}^*$, and a point $z \in C \setminus \{0\}$ such that $\nabla g_{\mathbb{C}}(\lambda_n \cdot z) = 0$ and $|\lambda_n| \to 0$. By weighted homogeneity, for $i \in \{1, \ldots, n\}$ we have

$$\frac{\partial F_{\mathbb{C}}}{\partial t}(0, \lambda_n \cdot z)\frac{\partial\omega_{\mathbb{C}}}{\partial x_i}(\lambda_n \cdot z) + \frac{\partial f_{\mathbb{C}}}{\partial x_i}(\lambda_n \cdot z)$$

$$= \lambda_n^{d-d_0}\lambda_n^{2p-d_i}\frac{\partial F_{\mathbb{C}}}{\partial t}(0, z)\frac{\partial\omega_{\mathbb{C}}}{\partial x_i}(z) + \lambda_n^{d-d_i}\frac{\partial f_{\mathbb{C}}}{\partial x_i}(z) = 0.$$

Therefore $\lambda_n^{2p-d_0}\frac{\partial F_{\mathbb{C}}}{\partial t}(0, z)\frac{\partial\omega_{\mathbb{C}}}{\partial x_i}(z) + \frac{\partial f_{\mathbb{C}}}{\partial x_i}(z) = 0$. Since z is in $C \setminus \{0\}$, we get that $\lambda_n^{2p-d_0}\frac{\partial F_{\mathbb{C}}}{\partial t}(0, z) - 1 = 0$. But $2p - d_0 \geq 1$ and so $\lim_{n\to+\infty}\lambda_n^{2p-d_0}\frac{\partial F_{\mathbb{C}}}{\partial t}(0, z) = 0$, which gives a contradiction. We conclude that $g_{\mathbb{C}}$ has an isolated critical point at the origin.

The fact that $G_{\mathbb{C}}$ has an isolated critical point at $(0, 0)$ is proved as in the real case. Let (t_0, x) be a zero of $\tilde{H}_{\mathbb{C}}$. This means that $G_{\mathbb{C}}(t_0, x) = 0$ and $\frac{\partial G_{\mathbb{C}}}{\partial x_i}(t_0, x) = 0$ for $i \in \{1, \ldots, n\}$. If $\frac{\partial G_{\mathbb{C}}}{\partial t}(t_0, x) = 0$ then $(t_0, x) = (0, 0)$. If $\frac{\partial G_{\mathbb{C}}}{\partial t}(t_0, x) \neq 0$ then (t_0, x) is a critical point of $t_{\mathbb{C}|Y_{\mathbb{C}}}$, because $Y_{\mathbb{C}} = G_{\mathbb{C}}^{-1}(0)$ has at worst an isolated singularity at $(0, 0)$. Therefore $t_0 = 0$ and so x is a critical point of $g_{\mathbb{C}}$, which implies that $x = 0$. We conclude that $\tilde{H}_{\mathbb{C}}$ has an isolated zero at $(0, 0)$ and, as in the real case, that $H_{\mathbb{C}}$ has an isolated zero at $(0, 0)$.

□

Therefore when $F_{\mathbb{C}}$ has an isolated critical point at the origin, the topological degrees can be expressed as signatures. This generalizes the results of Bruce [2], Szafraniec [12] and the first author [5]. We end with some explicit examples in the weighted homogeneous case. We have computed the signatures using *Singular*.

Example 3.6. Let f be the weighted homogeneous polynomial defined by $f(x,y) = x^2y^2 + y^3$ and let F be the weighted homogenous 1-parameter deformation of f defined by $F(t,x,y) = x^2y^2 + y^3 + t^2y$. We have $\nabla F_{\mathbb{C}}^{-1}(0,0,0) \subset \{t=0\}$. The function F is of type $(6;2,1,2)$ and according to Section 3, we can take $\omega(x,y) = \frac{x^8}{8} + \frac{y^4}{4}$. Then, $G(t,x) = x^2y^2 + y^3 + (t + \frac{x^8}{8} + \frac{y^4}{4})^2y$ and $G'(t,x) = x^2y^2 + y^3 + (t - \frac{x^8}{8} - \frac{y^4}{4})^2y$

Applying Corollaries 2.11 and 2.12 for n even we get that

$$\chi(f_t^{-1}(0) \cap B_\epsilon^2) = 1 - \deg_0 \nabla g = 3$$

and

$$\chi(f_{-t}^{-1}(0) \cap B_\epsilon^2) = 1 - \deg_0 \nabla g' = 3.$$

Example 3.7. Let f be the weighted homogeneous polynomial defined by $f(x,y) = x^4 + x^2y$ and let F be the weighted homogenous 1-parameter deformation of f defined by $F(t,x,y) = x^4 + x^2y + t^2y$. We have with $\nabla F_{\mathbb{C}}^{-1}(0,0,0) \subset \{t=0\}$. The function F is of type $(4;1,1,2)$ and according to Section 3, we can take $\omega(x,y) = \frac{x^8}{8} + \frac{y^4}{4}$. Then, $G(t,x) = x^4 + x^2y + (t + \frac{x^8}{8} + \frac{y^4}{4})^2y$ and $G'(t,x) = x^4 + x^2y + (t - \frac{x^8}{8} - \frac{y^4}{4})^2y$.

Applying Corollaries 2.11 and 2.12 for n even we get that

$$\chi(f_t^{-1}(0) \cap B_\epsilon^2) = 1 - \deg_0 \nabla g = -1$$

and

$$\chi(f_{-t}^{-1}(0) \cap B_\epsilon^2) = 1 - \deg_0 \nabla g' = -1.$$

We note that in the above examples, the functions g and g' have an algebraically isolated critical point at the origin, which makes the degrees computable thanks to the Eisenbud-Levine formula. However under the assumption $\nabla F_{\mathbb{C}} \subset \{t=0\}$, we do not know in general if g and g' have an algebraically isolated critical point at the origin.

Example 3.8. Let f be the weighted homogeneous polynomial defined by $f(x,y) = x^2y^2 + y^3$ and let F be the weighted homogenous 1-parameter deformation of f defined by $F(t,x,y) = x^2y^2 + y^3 + tx$. We have $\nabla F_{\mathbb{C}}^{-1}(0,0,0) = (0,0,0)$. The function F is of type $(6;5,1,2)$ and according to Section 3, we can take $\omega(x,y) = \frac{x^8}{8} + \frac{y^4}{4}$. Then, $G(t,x,y) = x^2y^2 + y^3 + tx + \frac{x^9}{8} + \frac{y^4x}{4}$ and $G'(t,x,y) = x^2y^2 + y^3 + tx - \frac{x^9}{8} - \frac{y^4x}{4}$.

Applying Corollaries 2.11 and 2.12 for n even, we get that

$$\chi(f_t^{-1}(0) \cap B_\epsilon^2) = 1 - \deg_0 \nabla g = 1$$

and

$$\chi((f_{-t}^{-1}(0) \cap B_\epsilon^2)) = 1 - \deg_0 \nabla g' = 1.$$

Example 3.9. Let f be the weighted homogeneous polynomial defined by $f(x,y,z) = x^2y^2 - y^2 - yz^2$ weighted homogeneous and let F be the weighted homogenous 1-parameter deformation of f defined by $F(t,x,y,z) = x^2y^2 - y^2 - yz^2 - tx$. We have $\nabla F_{\mathbb{C}}^{-1}(0,0,0,0) = (0,0,0,0)$. The function F is of type $(4;3,1,2,1)$ and $\omega(x,y,z) = \frac{x^8}{8} + \frac{y^4}{4} + \frac{z^8}{8}$. Then, $G(t,x,y,z) = x^2y^2 - y^2 - yz^2 - tx + \frac{x^9}{8} + \frac{y^4x}{4} + \frac{z^8x}{8}$ and $G'(t,x,y,z) = x^2y^2 - y^2 - yz^2 - tx - \frac{x^9}{8} - \frac{y^4x}{4} - \frac{z^8x}{8}$.

Applying Corollaries 2.18 and 2.19 for n odd, we get that

$$\chi(f_t^{-1}(0) \cap B_\epsilon^3) = 1 - \deg_0 \nabla F - \deg_0 H = 1 - 1 - 2 = -2$$

and

$$\chi(f_{-t}^{-1}(0) \cap B_\epsilon^3) = 1 - \deg_0 \nabla F - \deg_0 H' = 1 - 1 - (-2) = 2.$$

Example 3.10. Let f be the weighted homogeneous polynomial defined by $f(x, y, z) = x^2 + xy^2 - xz$ and let F be the weighted homogenous 1-parameter deformation of f defined by $F(t, x, y, z) = x^2 + xy^2 - xz + ty$. We have $\nabla F_{\mathbb{C}}^{-1}(0, 0, 0, 0) = (0, 0, 0, 0)$. The function F is of type $(4; 3, 2, 1, 2)$ and $\omega(x, y) = \frac{x^4}{4} + \frac{y^8}{8} + \frac{z^4}{4}$.

Applying Corollaries 2.18 and 2.19 for n odd, we get that

$$\chi(f_t^{-1}(0) \cap B_\epsilon^3) = 1 - \deg_0 \nabla F - \deg_0 H = 1 - 1 - 1 = -1$$

and

$$\chi(f_{-t}^{-1}(0) \cap B_\epsilon^3) = 1 - \deg_0 \nabla F - \deg_0 H' = 1 - 1 - (-1) = 1.$$

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