Generalized SU(1,1) Equivariant Convolution on Fock-Bargmann Spaces for Robust Radar Doppler Signal Classification

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Abstract. Classifying radar Doppler signals with Deep Learning algorithms is a challenging task, in particular because of the noisy nature of the data (clutter, thermal noise, etc.). Equivariant Neural Networks (ENN) have already been shown very promising in this context by coupling hyperbolic embedding techniques with dedicated SU(1, 1) convolution operators in order to achieve local robustness by-design. In this paper, we introduce a generalized SU(1, 1) equivariant convolution operator on the Fock-Bargmann spaces by leveraging on the representations of SU(1, 1) over these functional Hilbert spaces. We further give a new way of sampling over SU(1, 1) for Monte-Carlo computations by using a generalization of the Bloch-Messiah decomposition of elements of the symplectic group $\mathrm{SL}(2, \mathbb{R})$ to those of SU(1, 1). We finally illustrate our approach on the problem of radar clutter classification and demonstrate in this context that SU(1, 1) ENN achive better performance results than conventional approaches from both accuracy and robustness standpoints.

Keywords: Equivariant convolution · Group representation · Monte-Carlo sampling · Radar clutter classification

1 Introduction

Geometric Deep Learning [1] is an emerging field getting more and more traction because of its successful application to a wide range of domains [8, 12, 5, 6]. In this context, Equivariant Neural Networks (ENN) [9] have been shown to be superior to conventional Deep Learning approaches from both accuracy and robustness standpoints and appear as a natural alternative to data augmentation techniques to achieve geometrical robustness with respect to semantically preserving transforms such as isometries.

Achieving equivariance with respect to the SU(1, 1) group is of particular interest in the context of radar Doppler signal classification when representing the signals as complex covariance matrices [2, 3] and leveraging on hyperbolic embedding techniques to represent the input data as graphs of functionals defined

on the Poincaré disk $\mathbb{D}$ [14]. In particular, the authors have proposed in [13] using the equivariant convolution operator defined in [11] for functions $f : \mathbb{D} \to \mathbb{C}$ and to rely on a regular action of $\mathrm{SU}(1,1)$ on those functions. However, other group actions may need to be envisioned to better account for plausible real-world deformations of the original input data and to improve robustness accordingly, as shown in [14] with respect to thermal noise effects.

In this paper, we introduce a new $\mathrm{SU}(1,1)$ equivariant convolution operator by leveraging on Unitary Irreducible Representations (UIR) of $\mathrm{SU}(1,1)$ on the Fock-Bargmann Hilbert spaces, as described in [7]. Leveraging on recent results of [10] with respect to the extension of Bloch-Messiah decomposition of symplectic matrices to $\mathrm{SU}(1,1)$, we also propose an alternative sampling method to that used in [13] for computing Monte-Carlo estimations of $\mathrm{SU}(1,1)$-based convolution operators. Finally, we illustrate the approach in the context of radar clutter classification by working on data simulated according to a realistic model.

2 Mathematical Background

We denote $\mathbb{D}$ the Poincaré unit disk $\mathbb{D} = \{z = x + iy \in \mathbb{C} / |z| < 1\}$ and then consider the following Lie Group:

$$\mathrm{SU}(1,1) = \left\{ g_{\alpha,\beta} = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}, |\alpha|^2 - |\beta|^2 = 1, \alpha, \beta \in \mathbb{C} \right\} \tag{1}$$

We can endow $\mathbb{D}$ with a transitive action $\circ$ of $\mathrm{SU}(1,1)$ defined as it follows

$$\forall g_{\alpha,\beta} \in \mathrm{SU}(1,1),\ \forall z \in \mathbb{D},\ g_{\alpha,\beta} \circ z = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \tag{2}$$

As highlighted in [7], this action can be extended to functions of the Fock-Bargmann Hilbert space $\mathcal{FB}_\eta$, for $\eta = 1, \frac{3}{2}, 2, \frac{5}{2}, ...$, through the UIR representation ρ^η of $\mathrm{SU}(1,1)$ on $\mathcal{FB}_\eta$ which is defined as it follows, for $f \in \mathcal{FB}_\eta$ and $z \in \mathbb{D}$:

$$\left[\rho^\eta \left(g_{\alpha,\beta}\right)(f)\right](z) = \frac{1}{\left(\alpha - \bar{\beta} z\right)^{2\eta}} f\left(\frac{\bar{\alpha} z - \beta}{\alpha - \bar{\beta} z}\right) = \frac{1}{\left(\alpha - \bar{\beta} z\right)^{2\eta}} f\left(g_{\alpha,\beta}^{-1} \circ z\right) \tag{3}$$

Figure 1 illustrates this action of $\mathrm{SU}(1,1)$ and provides a comparison with the regular action considered in [13] and defined by $\rho^{reg}\left(g_{\alpha,\beta}\right)(f) = f\left(g_{\alpha,\beta}^{-1} \circ z\right)$, showing in particular that several perturbations can be captured through the representations ρ^η as η varies.

3 Generalized Convolution

We generalize here the convolution operator considered in [13] and show that it allows achieving equivariance with respect to the action of $G = \mathrm{SU}(1,1)$ represented by ρ^η.

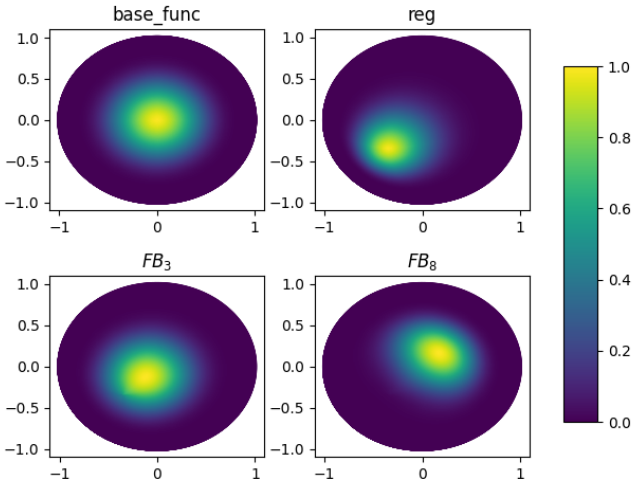

Fig. 1. Example of the action of $g_{\alpha,\beta} \in \mathrm{SU}(1,1)$ on the Fock-Bargmann Hilbert spaces $\mathcal{FB}_\eta$. From top to bottom and left to right: the original function f being a Gaussian kernel on $\mathbb{D}$, the transformed function $\rho^{reg}\left(g_{\alpha,\beta}\right)(f)$ by the regular action and the transformed functions $\rho^3\left(g_{\alpha,\beta}\right)(f)$ and $\rho^8\left(g_{\alpha,\beta}\right)(f)$.

To do so, we define for $z \in \mathbb{D}$,

$$\psi^\eta_{f,k}(z) = \int_G \left[\rho^\eta(g)(k)\right](z) \left[\rho^\eta(g)^{-1}(f)\right](0_\mathbb{D})\, d\mu^G(g) \tag{4}$$

with $0_\mathbb{D}$ the center of $\mathbb{D}$ and where μ^G refers to the Haar measure of G that is normalized according to $\int_G F(g \circ 0_\mathbb{D})\, d\mu^G(g) = \int_\mathbb{D} F(z)\, dm(z)$ for all $F \in L^1(\mathbb{D}, dm)$, and where the measure dm is given for $z = z_1 + iz_2$ by $dm(z) = \frac{dz_1 dz_2}{\left(1-|z|^2\right)^2}$.

Proposition 1. *The operator $f \to \psi^\eta_{f,k}$ is equivariant with respect to the action of* $\mathrm{SU}(1,1)$ *represented by ρ^η, in the following sense,*

$$\forall g_{\alpha_0,\beta_0} \in \mathrm{SU}(1,1),\ \rho^\eta\left(g_{\alpha_0,\beta_0}\right)\left(\psi^\eta_{f,k}\right) = \psi^\eta_{\rho^\eta\left(g_{\alpha_0,\beta_0}\right)(f),k} \tag{5}$$

Proof. $\forall g_0 = g_{\alpha_0,\beta_0} \in G$, $\forall z \in \mathbb{D}$, we have:

$$\rho^\eta(g_0)\left(\psi^\eta_{f,k}\right)(z) = \frac{1}{\left(\alpha_0 - \bar{\beta}_0 z\right)^{2\eta}} \int_G \left[\rho^\eta(g)(k)\right]\left(g_0^{-1} \circ z\right)\left[\rho^\eta(g)^{-1}(f)\right](0_\mathbb{D})\, d\mu^G(g)$$

$\forall g = g_{\alpha,\beta} \in G$, it is also possible to write

$$\rho^\eta(g)(k)\left(g_0^{-1} \circ z\right) = \frac{\left(\alpha_0 - \bar{\beta}_0 z\right)^{2\eta}}{\left(A - \bar{B} z\right)^{2\eta}} k\left((g_0 g)^{-1} \circ z\right)$$

with $A = \alpha\alpha_0 + \bar{\beta}\beta_0$ and $\bar{B} = \bar{\beta}\bar{\alpha_0} + \alpha\bar{\beta_0}$, so that $\alpha = A\bar{\alpha_0} - \beta_0\bar{B}$. We then have

$$\begin{aligned}
\rho^\eta(g_0)\left(\psi^\eta_{f,k}\right)(z) &= \int_G \frac{1}{\left(A - \bar{B}z\right)^{2\eta}} k\left((g_0 g)^{-1} \circ z\right) \frac{1}{\bar{\alpha}^{2\eta}} f(g \circ 0_{\mathbb{D}})\, d\mu^G(g) \\
&= \int_G \frac{1}{\left(A - \bar{B}z\right)^{2\eta}} k\left((g_0 g)^{-1} \circ z\right) \frac{1}{\left(\alpha_0\bar{A} - B\bar{\beta}_0\right)^{2\eta}} f(g \circ 0_{\mathbb{D}})\, d\mu^G(g) \\
&= \int_G \frac{1}{\left(A - \bar{B}z\right)^{2\eta}} k\left(\tilde{g}^{-1} \circ z\right) \frac{1}{\left(\alpha_0\bar{A} - B\bar{\beta}_0\right)^{2\eta}} f\left(\left(\tilde{g}^{-1} g_0\right)^{-1} \circ 0_{\mathbb{D}}\right) d\mu^G(\tilde{g}) \\
&= \int_G \left[\rho^\eta(\tilde{g})(k)\right](z) \left[\rho^\eta(\tilde{g})^{-1}\left(\rho^\eta(g_0)(f)\right)\right](0_{\mathbb{D}})\, d\mu^G(\tilde{g}) \\
&= \psi^\eta_{\rho^\eta(g_0)(f),k}(z)
\end{aligned}$$

where we have used the change of variable $\tilde{g}_{A,B} = \tilde{g} = g_0 g$ and the invariance property of the Haar measure.

4 Numerical Computation

In order to numerically compute the convolution (4), we can use a Monte-Carlo technique following the approach introduced in [8] and then consider the following estimator

$$\psi^{\eta,N}_{f,k}(z) = \frac{1}{N}\sum_{i=1}^{N} \left[\rho^\eta(g_i)(k)\right](z) \left[\rho^\eta(g_i)^{-1}(f)\right](0_{\mathbb{D}}) \tag{6}$$

where the samples g_i are drawn according to the Haar measure μ^G of G.

Motivated by the Cartan decomposition of G, [13] proposes sampling in $\mathrm{SU}(1,1)$ by first drawing elements in $\mathbb{D}$ seen as the cosets space $\mathrm{SU}(1,1)/\mathrm{U}(1)$ and then lifting to $\mathrm{SU}(1,1)$ by drawing random elements in the rotation group $\mathrm{U}(1)$. We propose here an alternative approach by leveraging on the result of [10] with respect to the extension of the Bloch-Messiah decomposition of symplectic matrices to $\mathrm{SU}(1,1)$ elements.

More precisely, the group elements can actually be parameterized by two angles γ and γ' and one real parameter d, so that

$$\begin{aligned}
g_{\rho,\gamma,\gamma'} &= \begin{bmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{bmatrix} \begin{bmatrix} \cosh\rho & \sinh\rho \\ \sinh\rho & \cosh\rho \end{bmatrix} \begin{bmatrix} e^{i\gamma'} & 0 \\ 0 & e^{-i\gamma'} \end{bmatrix} \\
&= \begin{bmatrix} e^{i\left(\gamma+\gamma'\right)}\cosh\rho & e^{i\left(\gamma-\gamma'\right)}\sinh\rho \\ e^{-i\left(\gamma-\gamma'\right)}\sinh\rho & e^{-i\left(\gamma+\gamma'\right)}\cosh\rho \end{bmatrix} = g_{\alpha,\beta}
\end{aligned}$$

with $\alpha = e^{i\left(\gamma+\gamma'\right)}\cosh\rho$ and $\beta = e^{i\left(\gamma-\gamma'\right)}\sinh\rho$. The following proposition gives the corresponding Haar measure that could then be used to sample elements $g_{\rho,\gamma,\gamma'} \in \mathrm{SU}(1,1)$ to compute the Monte-Carlo estimator (6).

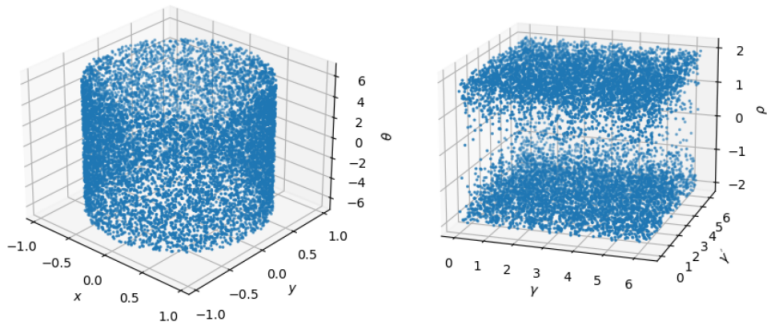

Fig. 2. Left: sampling according to the Cartan parameterization for which $\mathrm{SU}(1,1) \ni g_{\alpha,\beta} = g_{x,y,\theta}$, with $z = x + iy = \frac{\beta}{\alpha}$ and $\theta = 2\arg\alpha$. Right: sampling according to the Bloch-Messiah parameterization for which $\mathrm{SU}(1,1) \ni g_{\alpha,\beta} = g_{d,\gamma,\gamma'}$, with $\alpha = e^{i\left(\gamma+\gamma'\right)}\cosh\rho$ and $\beta = e^{i\left(\gamma-\gamma'\right)}\sinh\rho$.

Proposition 2. *The normalized Haar measure corresponding to the Bloch-Messiah parameterization of* $\mathrm{SU}(1,1)$ *is given by*

$$d\mu^G\left(g_{\rho,\gamma,\gamma'}\right) = \frac{1}{4\pi^2}\left|\sinh 2\rho\right| d\rho d\gamma d\gamma' \tag{7}$$

Proof. Based on [4], we remind ourselves that an element $g_{\alpha,\beta} \in \mathrm{SU}(1,1)$ can be written as $g_{\alpha,\beta} = t\mathbf{1} + iz\sigma_z + x\sigma_x + y\sigma_y$, with $\mathbf{1}$ the identity matrix and $\sigma_x, \sigma_y, \sigma_z$ the three Pauli matrices, where we have used the notations $\alpha = t + iz$ and $\beta = x - iy$ with $t^2 + z^2 - x^2 - y^2 = 1$. With such a parameterization, we can define the invariant Haar measure of the group by

$$d\mu^G\left(g_{\alpha,\beta}\right) = \frac{1}{\sqrt{1 + x^2 + y^2 - z^2}} dxdydz \tag{8}$$

If we consider the Bloch-Messiah parameterization for which $\alpha = e^{i\left(\gamma+\gamma'\right)}\cosh\rho$ and $\beta = e^{i\left(\gamma-\gamma'\right)}\sinh\rho$, we then have to consider the following change of variables, $x = \sinh\rho\cos\left(\gamma - \gamma'\right)$, $y = -\sinh\rho\sin\left(\gamma - \gamma'\right)$ and $z = \cosh\rho\sin\left(\gamma + \gamma'\right)$, for which the absolute determinant of the Jacobian matrix is $2\cosh^2\rho\left|\sinh\rho\cos\left(\gamma + \gamma'\right)\right|$ We then have

$$d\mu^G\left(g_{\rho,\gamma,\gamma'}\right) = \frac{2\cosh^2\rho\left|\sinh\rho\cos\left(\gamma + \gamma'\right)\right|}{\sqrt{\cosh^2\rho\cos\left(\gamma + \gamma'\right)}} d\rho d\gamma d\gamma' = \left|\sinh 2\rho\right| d\rho d\gamma d\gamma' \tag{9}$$

The measure stated in (7) is then obtained after re-normalizing the above equality for the angular part.

Figure 2 illustrates the sampling of $\mathrm{SU}(1,1)$ according to the Cartan (left) and Bloch-Messiah (right) parameterizations, the two pictures representing the same group elements but with different parameterizations. It is also interesting to notice that as the left-handside of Figure 2 can be folded along its θ axis, the Cartan parameterization actually corresponds to a torus with $\mathbb{D}$ as orthogonal sections.

5 Application to Radar Clutter Classification

In the following, we focus on radar clutter classification and consider the set-up introduce in [3], in which the signals are represented as Toeplitz Hermitian Positive Definite (THPD) covariance matrices of dimension n. A $\mathrm{SU}(1,1)$ equivariant neural network can operate on the corresponding data by leveraging on the Trench-Verblunsky theorem allowing to identify n-dimensional THPD matrices with $n-1$ reflection coefficients $\mu_i \in \mathbb{D}$ after adequate rescaling. A lifting step as introduced in [13] is then used to represent a THPD matrix Γ as a complex signal f_Γ on the Poincaré disk $\mathbb{D}$.

More precisely, we represent each spatial cell by its THPD auto-correlation matrix Γ, our goal being to predict the corresponding clutter $c \in \{1, ..., n_c\}$ from the observation of Γ. Within our formalism, the training samples are of the form (f_{Γ_i}, c_i) and the input data have been obtained by simulating a given cell according to

$$Z = \sqrt{\tau} R^{1/2} x + b_{radar} \tag{10}$$

where τ is a positive random variable corresponding to the clutter texture, R a THPD matrix associated with a given clutter, $x \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^x)$ and $b_{radar} \sim \mathcal{N}_{\mathbb{C}}(0, \sigma)$, with $\mathcal{N}_{\mathbb{C}}(0, t)$ referring to the complex gaussian distribution with mean 0 and standard deviation t. In the following, b_{radar} will be considered as a source of thermal noise inherent to the sensor.

We have instanciated a simple neural network constituted of one $\mathrm{SU}(1,1)$ regular convolutional layer with two filters and ReLu activation functions, followed by one fully connected layer and one softmax layer operating on the complex numbers represented as 2-dimensional tensors. The kernel functions are modeled as a neural networks with one layer of 16 neurons with swish activation functions, combined with the Riemannian logarithm of $\mathbb{D}$. The two convolution maps have been evaluated on the same grid constituted of 100 elements of $\mathbb{D}$ sampled according to the corresponding volume measure.

To appreciate the improvement provided by our approach, we will compare the obtained results with those corresponding to the use of a conventional neural network with roughly the same number of trainable parameters and operating on the complex reflection coefficients. In the following, we will denote $\mathcal{N}_\sigma^G$ (resp. $\mathcal{N}_\sigma^{FC}$) the neural network with $\mathrm{SU}(1,1)$ equivariant convolutional (resp. fully connected) layers and trained on 400 THPD matrices of dimension 10 corresponding to 4 different classes (100 samples in each class) which have been simulated according to (10) with a thermal noise standard deviation σ.

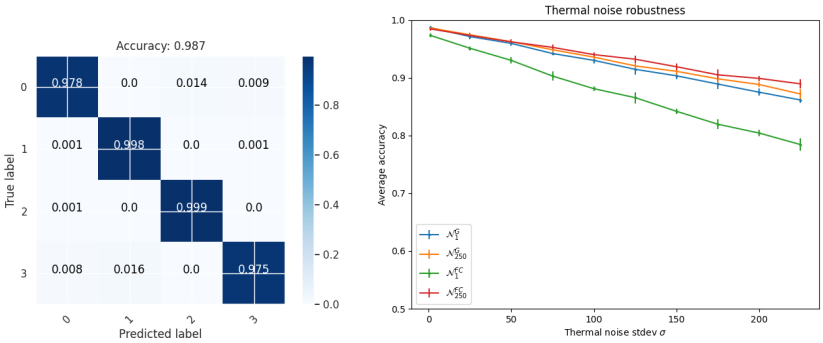

Fig. 3. Left handside: confusion matrix corresponding to the evaluation of $\mathcal{N}_1^G$ on the testing set T_1, averaged over 10 realizations. Right handside: average accuracy results of the algorithms $\mathcal{N}_1^G$, $\mathcal{N}_1^{FC}$, $\mathcal{N}_{250}^G$ and $\mathcal{N}_{250}^{FC}$ on the testing sets T_σ shown as a function of σ, together with the corresponding standard deviation as error bars

In order to evaluate the algorithms $\mathcal{N}_\sigma^G$ and $\mathcal{N}_\sigma^{FC}$, we have considered several testing sets T_σ consisting in 2000 THPD matrices of dimension 10 (500 samples in each of the 4 classes) simulated according to (10) with a thermal noise standard deviation σ. The obtained results are shown on Figure 3 where it can in particular be seen that $\mathcal{N}_1^G$ reaches similar performances as $\mathcal{N}_{250}^G$ and $\mathcal{N}_{250}^{FC}$ while significantly outperforming $\mathcal{N}_1^{FC}$ as σ increases, meaning that our approach allows to achieve some degree of robustness with respect to the variation of σ through the use of equivariant layers.

6 Conclusion and Further Work

Motivated by the sucessuful application of SU(1, 1) ENN to Doppler signal classification [14], we have generalized the convolution operator considered in [13] in order to handle more general group actions through the representation of SU(1, 1) on the Fock-Bargmann spaces. We have shown that our generalized operator is equivariant with respect to the considered action of SU(1, 1), so that it could be used to build equivariant layers of ENN. We have then proposed a sampling method for computing convolution operators with Monte-Carlo estimators by leveraging on the Bloch-Messiah parameterization of SU(1, 1), an approach complementary to that relying on the Cartan decomposition. We have finally illustrated our approach on simulated clutter data and shown its superiority with respect conventional Deep Learning algorithms from both accuracy and robustness standpoints.

Further work will include the study of numerical methods other than Monte-Carlo approaches which suffer from scalability issues when the convolution operators are used within deep ENN architectures and establishing some links with the coadjoint representation theory [15] may be useful in this context. Also, by

leveraging on the isomorphism between $\mathrm{SU}(1,1)$ and $\mathrm{SL}(2,\mathbb{R})$, we will investigate extending the approach presented in this paper to cover the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}_2$ and to build corresponding ENN in order to achieve robustness to a wider range of real-world perturbations.

References

1. Bronstein, M.M., Bruna, J., Cohen, T., Veličković, P.: Geometric deep learning: Grids, groups, graphs, geodesics, and gauges (2021)
2. Brooks, D., Schwander, O., Barbaresco, F., Schneider, J., Cord, M.: A hermitian positive definite neural network for micro-doppler complex covariance processing. In: 2019 International Radar Conference (RADAR). pp. 1–6 (2019). https://doi.org/10.1109/RADAR41533.2019.171277
3. Cabanes, Y., Barbaresco, F., Arnaudon, M., Bigot, J.: Toeplitz hermitian positive definite matrix machine learning based on fisher metric. In: Nielsen, F., Barbaresco, F. (eds.) Geometric Science of Information. pp. 261–270. Springer International Publishing, Cham (2019)
4. Chiribella, G., D'Ariano, G.M., Perinotti, P.: Applications of the group su(1, 1) for quantum computation and tomography. Laser Physics **16**(11), 1572–1581 (Nov 2006). https://doi.org/10.1134/s1054660x06110119, http://dx.doi.org/10.1134/S1054660X06110119
5. Cohen, T.S., Geiger, M., Köhler, J., Welling, M.: Spherical cnns. CoRR **abs/1801.10130** (2018), http://arxiv.org/abs/1801.10130
6. Cohen, T.S., Weiler, M., Kicanaoglu, B., Welling, M.: Gauge equivariant convolutional networks and the icosahedral cnn (2019)
7. del Olmo, M.A., Gazeau, J.P.: Covariant integral quantization of the unit disk. Journal of Mathematical Physics **61**(2), 022101 (Feb 2020). https://doi.org/10.1063/1.5128066
8. Finzi, M., Stanton, S., Izmailov, P., Wilson, A.G.: Generalizing convolutional neural networks for equivariance to lie groups on arbitrary continuous data (2020)
9. Gerken, J.E., Aronsson, J., Carlsson, O., Linander, H., Ohlsson, F., Petersson, C., Persson, D.: Geometric deep learning and equivariant neural networks (2021)
10. Grain, J., Vennin, V.: Canonical transformations and squeezing formalism in cosmology. Journal of Cosmology and Astroparticle Physics **2020**(02), 022–022 (Feb 2020). https://doi.org/10.1088/1475-7516/2020/02/022, http://dx.doi.org/10.1088/1475-7516/2020/02/022
11. Helgason, S.: Groups and geometric analysis (1984)
12. Lafarge, M.W., Bekkers, E.J., Pluim, J.P.W., Duits, R., Veta, M.: Roto-translation equivariant convolutional networks: Application to histopathology image analysis (2020)
13. Lagrave, P.Y., Cabanes, Y., Barbaresco, F.: "su(1,1) equivariant neural networks and application to robust toeplitz hermitian positive definite matrix classification". In: Nielsen, F., Barbaresco, F. (eds.) Geometric Science of Information. pp. 577–584. Springer International Publishing, Cham (2021)
14. Lagrave, P.Y., Cabanes, Y., Barbaresco, F.: An equivariant neural network with hyperbolic embedding for robust doppler signal classification. In: 2021 21st International Radar Symposium (IRS). pp. 1–9 (2021). https://doi.org/10.23919/IRS51887.2021.9466226
15. Rieffel, M.A.: Lie group convolution algebras as deformation quantizations of linear poisson structures. American Journal of Mathematics **112**(4), 657–685 (1990)