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# Propagation of velocity moments and uniqueness for the magnetized Vlasov–Poisson system

Alexandre Rege\*

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## Abstract

In this paper we present two results regarding the three-dimensional Vlasov–Poisson system in the full space with a general bounded magnetic field. First, we study the propagation of velocity moments for solutions to the system. We rely on Pallard’s optimal result regarding the unmagnetized Vlasov–Poisson system and we combine it with an induction procedure depending on the cyclotron frequency  $T_c = \frac{1}{\|B\|_\infty}$ . This induction procedure, similar to the one used by the author in the case of a constant magnetic field, is necessary because we can only get satisfactory estimates on a small time scale compared to the cyclotron frequency. Second, we manage to extend a result by Miot regarding uniqueness for Vlasov–Poisson to the magnetized case. This result relied heavily on the second-order structure of the Cauchy problem for the characteristics. The main difficulty in the magnetized case is that we lose this second-order structure.

## 1 Introduction

We study the Cauchy problem for the three-dimensional Vlasov–Poisson system with a general bounded magnetic field (which we will call magnetized Vlasov–Poisson system for the rest of the paper), given by the following set of equations:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + (E + v \wedge B) \cdot \nabla_v f = 0, \\ f(0, x, v) = f^{in}(x, v) \geq 0. \end{cases} \quad (1.1)$$

where  $f^{in}$  is a positive measurable function and  $f := f(t, x, v)$  is the distribution function of particles at time  $t \in \mathbb{R}_+$ , position  $x \in \mathbb{R}^3$  and velocity  $v \in \mathbb{R}^3$ . The self-consistent electric field  $E := E(t, x)$  is given by:

$$E = -\nabla_x \mathcal{G}_3 * \rho, \quad (1.2)$$

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\*Laboratoire Jacques-Louis Lions (LJLL), Sorbonne Université, CNRS, Université de Paris, 4 place Jussieu 75005 Paris, France

with  $\mathcal{G}_3 = \frac{1}{4\pi|x|}$  the Green function for the Laplacian and  $\rho(t, x) := \int_{\mathbb{R}^3} f(t, x, v) dv$  the macroscopic particle density.

$B := B(t, x)$  is an magnetic field which will be considered bounded and lipschitz throughout this work:

$$B \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^3). \quad (1.3)$$

This system models the evolution of a set of charged particles that interact through the Coulomb force, and thus it is relevant for the study of various physical systems, most notably plasmas.

The mathematical theory for the unmagnetized ( $B = 0$ ) Vlasov–Poisson system has been studied and developed in a great number of different works. In the three-dimensional framework, Arsenev [1] was the first to prove the existence of global weak solutions through a regularization procedure that preserves the main a priori estimates. The existence of global classical solutions for general initial data was established at the beginning of nineties in two separate works by Pfaffelmoser [18] and Lions, Perthame [12]. The first approach, extended and developed in [11, 21], relies on a very sharp study of the characteristics of the Vlasov–Poisson system while the latter approach, extended and developed in [4, 8, 16, 17, 20], is based upon the propagation of velocity moments. Even if these two approaches differ greatly, in both cases a key condition is to limit the influence of high velocities on the dynamics, by either considering initial data with compact support [18] or having a finite velocity moment of sufficiently high order ( $k > 3$ ) [12]. More recently, Pallard combined the two approaches in [15], where he showed how to exploit the first approach by Pfaffelmoser to prove propagation of velocity moments, extending the main result of [12] by showing that this propagation property is true for moments of order  $2 < k \leq 3$ .

Going back to the magnetized Vlasov–Poisson system (1.1), the propagation of velocity moments for (1.1) with constant magnetic field  $B = (0, 0, \omega)$  (with  $\omega > 0$  the cyclotron frequency) was proven by the author in [19] by extending the method of propagation of velocity moments from [12] and adapting the uniqueness condition explicated in [12]. In order to extend the moment method, an important point in [19] was to establish a representation formula for the macroscopic density  $\rho$ . This was carried out by explicitly computing the characteristics of the transport equation

$$\partial_t f + v \cdot \nabla_x f + v \wedge B \cdot \nabla_v f = 0 \quad (1.4)$$

and then using the Duhamel formula, which just meant considering the Vlasov equation as a transport equation with the non-linear term  $E \cdot \nabla_v f$  seen as a source term. Furthermore, in this analysis singularities at times  $t = 0, \frac{2\pi}{\omega}, \frac{4\pi}{\omega}, \dots$ , which are just multiples of the cyclotron period  $T_\omega = \frac{2\pi}{\omega}$ , appeared in the velocity moment estimates because of the added magnetic field. This was remedied by the fact that all the estimates depended only on quantities conserved for all time and on the initial velocity moment, allowing for an induction argument to prove propagation of moments for all time. Unfortunately, in our configuration with a general magnetic field, this analysis breaks down at the first hurdle, simply because we can't explicitly compute the characteristics of (1.4), even with a smooth  $B$ .

The first main result of this paper, which is the continuation of [19], is to prove the propagation of velocity moments for (1.1) with a general magnetic field  $B$ . We manage to obtain this result by combining Pallard’s method [15] with an induction argument using the cyclotron period similar to the one in [19]. However, in this paper we don’t obtain explicit singularities like in [19] because we use the ”Lagrangian” point of view (study of the characteristics) from [18] instead of using the ”Eulerian” point of view (study of the distribution function) from [12, 19]. This means we need to work on a small time scale compared to the cyclotron period  $T_c = \frac{1}{\|B\|_\infty}$  to obtain estimates analogous to those in [15]. This is due to the fact that on time scales comparable to  $T_c$  or greater than  $T_c$ , the variations of the characteristics are large and so the method in [15] fails. The second difficulty is to show that the estimates from [15] are also valid in our framework up to a certain time depending on  $T_c$ , and that these estimates depend only on quantities that don’t prevent us from using the induction argument. Finally, propagation of velocity moments also implies propagation of the regularity of the initial data which means we have existence of classical solutions to (1.1). This result is detailed in [19] (theorem 2.5) for a constant magnetic field under additional conditions on  $f^{in}$  ( $f^{in}$  decays faster in velocity) but can be easily extended to the case of a general  $B$ .

Now we turn to results regarding uniqueness. First let’s mention the major contribution by Loeper [13] who proved, for the unmagnetized Vlasov–Poisson system, that the set of solutions with bounded microscopic density was a uniqueness class. This result was also extended to (1.1) for a constant  $B$  in [19] and we discuss how to prove a similar result for a general  $B$  below. Then in [14], Miot improved the result by showing uniqueness under the condition that the  $L^p$  norms of the macroscopic density grow at most linearly with respect to  $p$ . This allows for solutions with unbounded macroscopic density, more precisely with logarithmic blow-up.

This paper’s second main result is to prove that this uniqueness condition is also valid for (1.1), but only with added assumptions on the velocity moments of the initial data. In [14], a key point was exploiting the second-order structure of the characteristics of the Vlasov–Poisson system. This explains why the uniqueness condition from [14] doesn’t apply to the two-dimensional Euler model for incompressible fluids, which presents many similarities with Vlasov–Poisson, whereas the condition from [13] works for both models. It is simply due to the fact that the characteristics of the Euler model only verify a first-order ODE. In our case, the main difficulty is that the added magnetic field breaks the second-order structure of the Cauchy problem for the characteristics. We manage to get around this by proving that the characteristics in the magnetized case can be controlled by assuming more regularity on  $B$  and with the additional assumptions on the moments of the initial data mentioned above. With these additional assumptions, we deduce a new uniqueness condition which is actually the same as the sufficient condition imposed on the initial data to verify the uniqueness criterion in [14, theorem 1.2]

**Outline of the paper:** This paper will be organized as follows. We will finish this section by giving some notations and the classical a priori estimates satisfied by (1.1). In section 2

the main results of the paper will be presented. Then section 3 will be devoted to the proof of propagation of velocity moments (1.1). More precisely, we will explain how we find estimates that are equivalent to those in [15] up to a time  $T_B$  depending on the cyclotron frequency and show how we can then use the same induction argument as in [19] to conclude. We finish with section 4 where we detail our proof of uniqueness for solutions to (1.1), highlighting how additional assumptions on the moments of the initial data allow us to control the added terms due to the added magnetic field in the analysis.

## 1.1 Preliminaries

Let's first detail the two main a priori bounds that we can deduce from system (1.1). The first bound is a direct consequence of the Vlasov equation where the coefficients are divergence-free.

$$\|f(t)\|_p = \|f^{in}\|_p \quad (1.5)$$

for all time  $t$  and exponents  $p \in [0, +\infty]$ .

The second bound is the conservation of the energy  $\mathcal{E}(t)$  of the system, with

$$\mathcal{E}(t) := \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f(t, x, v) dx dv + \frac{1}{2} \int_{\mathbb{R}^3} |E(t, x)|^2 dx = \mathcal{E}(0) < +\infty. \quad (1.6)$$

Furthermore, thanks to the conservation of the energy  $\mathcal{E}(t)$ , we have the following bounds.

**LEMMA 1.1.** *For all  $t \geq 0$ , we have  $M_2(t) \leq C_1$  and  $\|\rho(t)\|_{\frac{5}{3}} \leq C_2$  with the constants  $C_1, C_2$  depending only on  $\mathcal{E}(0), \|f^{in}\|_1, \|f^{in}\|_\infty$ .*

We also present the standard notation for velocity moments: for any  $k > 2$  and  $t \geq 0$  we define:

$$M_k(t) = \sup_{0 \leq s \leq t} \iint |v|^k f(s, x, v) dv dx. \quad (1.7)$$

As said before we will use the Lagrangian formulation detailed in [18], so we define the characteristics  $(X, V)$  of (1.1) which are solutions to the following Cauchy problem:

$$\begin{cases} \frac{d}{ds} X(s; t, x, v) = V(s; t, x, v), \\ \frac{d}{ds} V(s; t, x, v) = E(s, X(s; t, x, v)) + V(s; t, x, v) \wedge B(s, X(s; t, x, v)), \end{cases} \quad (1.8)$$

with

$$(X(t; t, x, v), V(t; t, x, v)) = (x, v). \quad (1.9)$$

Then like in [15], we define for any  $t > 0$  and  $\delta \in ]0, t[$ .

$$Q(t, \delta) := \sup \left\{ \int_{t-\delta}^t |E(s, X(s; 0, x, v))| ds, (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \right\}. \quad (1.10)$$

For the unmagnetized Vlasov–Poisson system,  $Q(t, \delta)$  quantifies the evolution of the characteristics on the interval  $[t - \delta, t]$ . However, in our context with the added magnetic field,  $Q(t, \delta)$  will only quantify a part of the characteristic evolution.

## 2 Results

We now give the first main result of this section, which is the propagation of velocity moments of order  $k > 2$ , extending theorem 1 in [15] to the magnetized Vlasov–Poisson system.

**THEOREM 2.1** (Propagation of moments). *Let  $k_0 > 2, T > 0, f^{in} = f^{in}(x, v) \geq 0$  a.e. with  $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$  and assume that*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{k_0} f^{in} dx dv < \infty. \quad (2.11)$$

*Then there exists a weak solution*

$$f \in C(\mathbb{R}_+; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \quad (2.12)$$

*( $1 \leq p < +\infty$ ) to the Cauchy problem for the Vlasov–Poisson system with magnetic field (1.1) in  $\mathbb{R}^3 \times \mathbb{R}^3$  such that*

$$\sup_{0 \leq t \leq T} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{k_0} f(t, x, v) dv dx \leq C \quad (2.13)$$

*with  $C$  that depends only on*

$$T, k_0, \|B\|_\infty, \mathcal{E}(0), \|f^{in}\|_1, \|f^{in}\|_\infty, \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{k_0} f^{in} dx dv. \quad (2.14)$$

**REMARK 2.2.** *If  $f^{in}$  satisfies the assumptions of the previous theorem, then all the moments of order  $k$  such that  $0 \leq k < k_0$  are also propagated for the solution  $f$ , simply because of the following Hölder inequality*

$$\iint |v|^k f(t, x, v) dv dx \leq \|f\|_1^{\frac{k_0-k}{k_0}} \left( \iint |v|^{k_0} f(t, x, v) dv dx \right)^{\frac{k}{k_0}} \quad (2.15)$$

*where we use the decomposition  $|v|^k f = f^{\frac{k_0-k}{k_0}} |v|^k f^{\frac{k}{k_0}}$  and the exponents  $p = \frac{k_0-k}{k_0}, q = \frac{k_0}{k}$ .*

Like in [12, 19], we assume we have smooth solutions to conduct the proof in section 3, and since the a priori estimates depend only on (2.14) we can pass to the limit in the approximate Vlasov–Poisson system first introduced in [1]. In fact, theorem 2.1 will be a consequence of the main estimate in this paper which will only hold for these smooth solutions because it is an estimate on  $Q$  (given in (1.10)), which isn't necessarily well-defined for functions in Lebesgue spaces. We now give this estimate on  $Q$ .

**Main estimate on  $Q$ :**

For all  $T > 0$  we have

$$N(T) := \sup_{0 \leq t \leq T} Q(t, t) \leq C, \quad (2.16)$$

with  $C$  that depends on the constants in (2.14). In the following remark, we explain how theorem 2.1 is a consequence of (3.31).

**REMARK 2.3.** *The estimate on propagation of velocity moments (2.13) in theorem 2.1 follows from the estimate on  $N(T)$  (2.16) because we have:*

$$\begin{aligned} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f(t, x, v) dv dx &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |V(t; 0, x, v)|^k f^{in}(x, v) dv dx \\ &\leq \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|v| + N(T))^k \exp(kt \|B\|_\infty) f^{in}(x, v) dv dx \\ &\leq 2^{k-1} \exp(kt \|B\|_\infty) \left( \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f^{in}(x, v) dv dx + N(T)^k \|f^{in}\|_1 \right) \end{aligned}$$

The first inequality above is obtained through a Grönwall inequality on  $|V(t; 0, x, v)|$ , indeed thanks to (1.8) we can write

$$V(t; 0, x, v) = v + \int_0^t E(s, X(s; 0, x, v)) ds + \int_0^t V(s; 0, x, v) \wedge B(s, X(s; 0, x, v)) ds \quad (2.17)$$

which implies

$$\begin{aligned} |V(t; 0, x, v)| &\leq |v| + Q(t, t) + \|B\|_\infty \int_0^t |V(s; 0, x, v)| ds \\ &\leq |v| + N(T) + \|B\|_\infty \int_0^t |V(s; 0, x, v)| ds \end{aligned}$$

This is the classical Grönwall inequality which allows us to conclude that

$$|V(t; 0, x, v)| \leq (|v| + N(T)) \exp(t \|B\|_\infty). \quad (2.18)$$

The second inequality is just due to the fact that  $2^{k-1}(1+x^k) \geq (1+x)^k$  for  $x \geq 0$ .

We finish this section by mentioning that, contrary to what is done in [15], we don't state any results on to the periodic case. This is simply explained by the fact that there are no results related to the existence of weak solutions to (1.1) in the periodic case. One possibility would be to adapt [2] to the magnetized case, and then combine the results from [15] for the periodic case and the proof of theorem 2.1 to show propagation of moments in the periodic case.

## 2.1 Uniqueness

For all the uniqueness results and so also in section 4, we assume the same regularity on  $B$  (1.3).

We first mention a very important result by Loeper [13] where it was shown that the boundedness of the macroscopic density  $\rho$  was a sufficient condition for uniqueness in the Vlasov–Poisson system. This result was extended to the Vlasov–Poisson system with constant magnetic field in [19] and the proof can be adapted to the case of a general magnetic field verifying (1.3). However, in the magnetized case, we require extra conditions on the velocity moments and space moments of the initial data  $f^{in}$ , as shown in the following theorem:

**THEOREM 2.4.** *Let  $B$  verify (1.3), let  $f^{in} = f^{in}(x, v) \geq 0$  a.e. with  $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$  and assume that*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^6 f^{in} dx dv < \infty \text{ and } \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |x|^4 f^{in} dx dv < \infty. \quad (2.19)$$

Assume further that the weak solution  $f$  to (1.1) provided by theorem 2.1 satisfies

$$\rho \in L^\infty([0, T] \times \mathbb{R}_x^3) \quad (2.20)$$

for all  $T > 0$ .

Then the weak solution  $f$  is unique.

However, to exploit this result, one needs to build solutions to (1.1) with bounded macroscopic density. Hence, in the next proposition, we give an explicit condition that guarantees the existence of solutions to (1.1) with a bounded  $\rho$ .

**PROPOSITION 2.5.** *Let  $B$  verify (1.3) and let  $f^{in}$  satisfy the assumptions of theorem 2.1 with  $k_0 > 6$ . We also assume that  $f^{in}$  is such that for all  $R > 0$  and  $T > 0$*

$$g_R(t, x, v) \in L^\infty([0, T] \times \mathbb{R}_x^3, L^1(\mathbb{R}_v^3)), \quad (2.21)$$

where

$$g_R(t, x, v) = \sup_{(y, w) \in S_{t, x, v, R}} f^{in}(y + vt, w) \quad (2.22)$$

with

$$S_{t, x, v, R} = \left\{ (y, w) : |y - x| \leq (R + \|B\|_\infty |v|) t^2 e^{\|B\|_\infty t}, |w - v| \leq (R + \|B\|_\infty |v|) t e^{\|B\|_\infty t} \right\}. \quad (2.23)$$

Then the weak solution  $f$  of (1.1) provided by theorem 2.1 verifies

$$\rho \in L^\infty([0, T] \times \mathbb{R}_x^3)$$

for all  $T > 0$ .

This proposition was shown in [19, proposition 2.7] in the case of a constant magnetic field and remains unchanged when we take a general  $B$ .

Now we present a theorem which is the second main result of this paper, where we show that the uniqueness criterion proved in [14] (theorems 1.1 and 1.2) also applies to (1.1) with  $B$  verifying (1.3), improving theorem 2.4 because it allows for solutions with unbounded macroscopic density.

**THEOREM 2.6.** *Let  $T > 0$  and  $B$  verify (1.3). Furthermore, let  $f^{in} \geq 0$  a.e. with  $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$  and such that*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^m f^{in}(x, v) dx dv < +\infty, \quad (2.24)$$

for some  $m > 6$ .

Now let  $f \in L^\infty([0, T], L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$  be a weak solution provided by theorem 2.1 with initial data  $f^{in}$ . If  $f^{in}$  satisfies

$$\forall k \geq 1, \quad \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f^{in}(x, v) dx dv \leq (C_0 k)^{\frac{k}{3}}, \quad (2.25)$$

for some constant  $C_0$  independent of  $k$ , then  $f$  the solution to the Cauchy problem for the magnetized Vlasov–Poisson system is unique and verifies

$$\sup_{[0, T]} \sup_{p \geq 1} \frac{\|\rho(t)\|_p}{p} < +\infty. \quad (2.26)$$

**REMARK 2.7.** *In our framework, an important difference with [14] is that the uniqueness criterion isn't given by the inequality on the macroscopic density (2.26) but rather the stronger assumption on the moments of the solution (2.25).*

As mentioned above, the assumptions of theorem 2.6 are less restrictive than the condition (2.20) and thus allow us to consider initial data with unbounded macroscopic density. This result is illustrated by the following theorem ([14, theorem 1.3]):

**THEOREM 2.8** (Miot, [14]). *There exists  $f^{in} \geq 0$  a.e. such that  $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$  satisfying the assumptions of theorem 2.6 and such that*

$$\rho_0(x) = \frac{4\pi}{3} \ln_-(|x|), \quad \forall x \in \mathbb{R}^3, \quad (2.27)$$

where  $\ln_- = \max(-\ln(x), 0)$  is the negative part of the function  $\ln$ .

In section 4, we will detail the proof of theorem 2.6 first because it is the main result of this section. Then we will present the proof of theorem 2.4 where ingredients from the proof of theorem 2.6 are used, notably the boundedness of the velocity characteristic. However in theorem 2.4 we require a condition on the space moment of the initial data (2.19) which isn't the case in theorem 2.6.

Finally, with regards to uniqueness for Vlasov–Poisson, a major open problem is finding a uniqueness condition in the periodic case. Indeed, it would be very interesting to see if the conditions found in [13, 14] could be adapted to the periodic case.

### 3 Proofs regarding propagation of velocity moments

In this section, we shall denote by  $C$  a constant that can change from one line to another but that only depends on

$$\mathcal{E}(0), \|f^{in}\|_1, \|f^{in}\|_\infty. \quad (3.28)$$

As mentioned above, the whole proof is conducted using smooth solutions.

We consider  $k > 2$  and  $\varepsilon > 0$  small enough, say  $\varepsilon \in ]0, \varepsilon_0[$  with  $\varepsilon_0 \leq \frac{(k-2)}{2k}$ . As said in the introduction, the main difference with the analysis in [15] is that we're going to show propagation of moments for all time by using an induction argument using the cyclotron period  $T_c = \frac{1}{\|B\|_\infty}$ . We begin with the initialization, so we're first going to consider  $T > 0$  with  $T \leq T_B$ , where  $T_B$  is the unique real number such that

$$T_B \in \mathbb{R}_+^* \text{ and } \|B\|_\infty T_B \exp(T_B \|B\|_\infty) = a, \quad (3.29)$$

with  $a > 0$ . In our method, since we can only obtain estimates on  $Q$  for  $T_B \ll T_c$ , we just need  $a$  small enough so we set  $a = 2^{-10}$ .

Thus, we show propagation of velocity moments on  $[0, T_B]$  using the following result.

**PROPOSITION 3.1.** *For all  $T > 0$  such that  $T \leq T_B$ , (2.16) is verified. More precisely we have the following estimate on  $Q(t, t)$*

$$Q(t, t) \leq C \exp(T \|B\|_\infty)^{\frac{2}{5}} (T^{\frac{1}{2}} + T^{\frac{7}{5}}) \quad (3.30)$$

with  $C$  that only depends on

$$k, \mathcal{E}(0), \|f^{in}\|_1, \|f^{in}\|_\infty, M_k(0).$$

**REMARK 3.2.** *This estimate is the analogous of the estimate (13) in [15]. In our magnetized framework, we only manage to generalize this result up to the time  $T_B$ .*

The following section will be devoted to the proof of this proposition, and just like in [15] the proof is done in three steps which correspond to proposition 3.3, proposition 3.6 and proposition 3.7.

#### 3.1 The case $T \leq T_B$

**PROPOSITION 3.3.** *For any  $0 \leq \delta \leq t \leq T \leq T_B$  we have:*

$$Q(t, \delta) \leq C(\delta Q(t, \delta))^{\frac{4}{3}} + \delta^{\frac{1}{2}}(1 + M_{2+\varepsilon}(T))^{\frac{1}{2}} \quad (3.31)$$

*Proof.* Let  $(t, x_*, v_*) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3$  and set  $(X_*, V_*)(s) = (X, V)(s; t, x_*, v_*)$ . For any  $\delta \in [0, t]$  we have by definition of  $E$

$$\int_{t-\delta}^t |E(s, X(s; t, x_*, v_*))| ds \leq \int_{t-\delta}^t \int \frac{\rho(s, x) dx}{4\pi |x - X_*(s)|^2} ds$$

Our objective in the rest of this section will be to estimate the integral:

$$I_*(t, \delta) := \int_{t-\delta}^t \int \frac{\rho(s, x) dx}{|x - X_*(s)|^2} ds = \int_{t-\delta}^t \iint \frac{f(s, x, v) dv dx}{|x - X_*(s)|^2} ds \quad (3.32)$$

Now we will use a procedure that is inspired from [21] which consists in splitting  $[t - \delta, t] \times \mathbb{R}^3 \times \mathbb{R}^3$  into three parts. Here the partition is slightly different because, following [15], we introduce  $\varepsilon > 0$ .

$$\begin{aligned} G &= \{(s, x, v) : \min(|v|, |v - V_*(s)|) < P\}, \\ B &= \{(s, x, v) : |x - X_*(s)| \leq \Lambda_\varepsilon(s, v)\} \setminus G, \\ U &= [t - \delta, t] \times \mathbb{R}^3 \times \mathbb{R}^3 \setminus (G \cup B), \end{aligned}$$

with

$$P = 2^{10} Q(t, \delta) \exp(\delta \|B\|_\infty) \text{ and } \Lambda_\varepsilon(s, v) = L(1 + |v|^{2+\varepsilon})^{-1} |v - V_*(s)|^{-1} \quad (3.33)$$

and  $L > 0$  to be fixed later. The main difference here with [15] is the definition of  $P$ , because the added magnetic modifies the evolution of the characteristic in velocity  $V(s)$ . Furthermore, we take the same numerical constant  $2^{10}$  in the definition of  $P$  as in [15] is (in truth this constant just needs to be large enough). Using obvious notations, we write  $I_* = I_*^G + I_*^B + I_*^U$ . The first two integrals will be more straightforward to estimate than  $I_*^U$ , which involves the set  $U$  considered as the "ugly set" according to [9].

The first two contributions  $I_*^G, I_*^B$  are treated the same in both magnetized and unmagnetized cases, simply because the modifications made to the sets  $G, B$  to take into account the added magnetic field don't change the computations required to estimate  $I_*^G$  and  $I_*^B$ . We succinctly present how to control both integrals following the calculations from [15]. The first bound is obtained by using a standard functional inequality.

For  $\kappa \in L^\infty(\mathbb{R}^3) \cap L^{\frac{5}{3}}(\mathbb{R}^3)$  we have

$$\left\| \kappa * |\cdot|^{-2} \right\|_\infty \leq c \|\kappa\|_{\frac{5}{3}}^{\frac{5}{9}} \|\kappa\|_\infty^{\frac{4}{9}}, \quad (3.34)$$

with  $c$  a numerical constant.

We apply (3.34) to the quantity:

$$\rho_G(s, x) = \int_{B(0, P) \cup B(V_*(s), P)} f(s, x, v) dv \leq \rho(s, x), \quad (3.35)$$

which implies the following control on  $I_*^G(t, \delta)$ :

$$I_*^G(t, \delta) \leq C(\delta P^{\frac{4}{3}}). \quad (3.36)$$

To estimate the contribution on  $B$ , we first integrate in the space variable using a spherical change of variable.

$$\begin{aligned}
I_*^B(t, \delta) &\leq \int_{t-\delta}^t \int_v \int_{|x-X_*(s)| \leq \Lambda_\varepsilon(s,v)} \frac{f(s, x, v) dx}{|x - X_*(s)|^2} dv ds, \\
&\leq \int_{t-\delta}^t \int_v \left( 4\pi \int_0^{\Lambda_\varepsilon(s,v)} \frac{\|f^{in}\|_\infty r^2 dr}{r^2} \right) dv ds, \\
&\leq C \int_{t-\delta}^t \int_v \frac{L}{(1 + |v|^{2+\varepsilon}) |v - V_*(s)|} dv ds, \\
&\leq C \int_{t-\delta}^t \left( \int_{|v| \leq |v - V_*(s)|} \frac{L}{(1 + |v|^{2+\varepsilon}) |v|} dv + \int_{|v| > |v - V_*(s)|} \frac{L}{(1 + |v - V_*(s)|^{2+\varepsilon}) |v - V_*(s)|} dv \right) ds, \\
&\leq C \int_{t-\delta}^t \int_v \frac{L}{(1 + |v|^{2+\varepsilon}) |v|} dv ds, \\
&\leq C \int_{t-\delta}^t \int_{\mathbb{R}^3} \frac{Lr}{(1 + r^{2+\varepsilon})} dr ds, \\
&\leq C\delta L.
\end{aligned}$$

The last contribution  $I_*^U(t, \delta)$  can be written

$$I_*^U(t, \delta) = \int_{t-\delta}^t \iint \frac{f(s, x, v) \mathbf{1}_U(s, x, v)}{|x - X_*(s)|^2} dv dx ds = \iint \int_{t-\delta}^t \frac{\mathbf{1}_U(s, X(s), V(s))}{|X(s) - X_*(s)|^2} ds f(t, x, v) dv dx \quad (3.37)$$

where we have the obvious notation  $(X, V)(s) = (X, V)(s; t, x, v)$ . Estimating this quantity is difficult and will occupy us for the rest of the proof of proposition 3.3.

The following lemma is very important in our proof because it highlights why we need to use the induction procedure mentioned above. In the unmagnetized case, we estimate  $I_*^U$  by noticing that because of the definition of  $U$ , the characteristic  $V(s)$  stays close to  $v$  on  $[t - \delta, t]$  because  $v$  is large compared to  $P$  and  $P$  is much larger  $Q(t, \delta)$  which quantifies the total variation of  $V(s)$  on  $[t - \delta, t]$ . However in the magnetized case, this stays true only under the condition (3.29) because if the magnetic field is large than the variations of  $V(s)$  on  $[t - \delta, t]$  can also be very large compared to  $P$ .

**LEMMA 3.4.** *Let  $s_1 \in [t - \delta, t]$  such that  $(s_1, X(s_1), V(s_1)) \in U$ , then for all  $s \in [t - \delta, t]$  we have*

$$2^{-1} |v| \leq |V(s)| \leq 2 |v|, \quad (3.38)$$

and

$$2^{-1} |v - v_*| \leq |V(s) - V_*(s)| \leq 2 |v - v_*|. \quad (3.39)$$

*Proof.* First, because of the definition of  $U$ , we can write

$$\min(|V(s_1)|, |V(s_1) - V_*(s_1)|) \geq P. \quad (3.40)$$

Let's start by proving the first bound (3.38), thanks to (1.8) we have for all  $s \in [t - \delta, t]$

$$V(s) - V(s_1) = \int_{s_1}^s E(\tau, X(\tau)) d\tau + \int_{s_1}^s V(\tau) \wedge B(\tau, X(\tau)) d\tau$$

Furthermore, one of the properties of the characteristics is that we have for all  $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$  and  $\tau, t \in \mathbb{R}_+$

$$X(\tau; t, x, v) = X(\tau; 0, X(0; t, x, v), V(0; t, x, v))$$

and also that the function  $(x, v) \mapsto X(\tau; t, x, v)$  is a  $C^1$ -diffeomorphism which means that

$$\sup \left\{ \int_{t-\delta}^t |E(s, X(s; t, x, v))| ds, (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \right\} = \sup \left\{ \int_{t-\delta}^t |E(s, X(s; 0, x, v))| ds, (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \right\} \\ = Q(t, \delta).$$

Thus we can write

$$|V(s) - V(s_1)| \leq Q(t, \delta) + \|B\|_\infty \left( \delta \|V(s_1)\| + \left| \int_{s_1}^s |V(\tau) - V(s_1)| d\tau \right| \right)$$

which is a Grönwall inequality, so we finally have for all  $s \in [t - \delta, t]$

$$\| |V(s)| - |V(s_1)| \| \leq |V(s) - V(s_1)| \leq (Q(t, \delta) + \|B\|_\infty \delta |V(s_1)|) \exp(\delta \|B\|_\infty). \quad (3.41)$$

This last inequality highlights the main difference with the unmagnetized case, indeed when we use  $V(s_1)$  as a reference point to quantify the variation of  $V(s)$ , we see that the added term  $\|B\|_\infty \delta |V(s_1)| \exp(\delta \|B\|_\infty)$ , which is just the added variation of the velocity characteristic resulting from the magnetic field, is potentially unbounded. This is due to the fact that even if  $0 \leq \delta \leq T$ ,  $\|B\|_\infty$  is potentially large. This is the reason we introduce the time  $T_B$  which depends on the cyclotron frequency  $\|B\|_\infty$ .

Now using this last inequality, thanks to the relation between  $P$  and  $Q(t, \delta)$  given in (3.33), to (3.40), and to (3.29) (because  $t \leq T_B$ ) we have

$$|V(s)| \leq |V(s_1)| (1 + \|B\|_\infty \delta \exp(\delta \|B\|_\infty)) + 2^{-10} P \\ \leq |V(s_1)| (1 + 2^{-10} + 2^{-10}) = |V(s_1)| (1 + 2^{-9})$$

and using the same relations but this time for  $-|V(s_1)|$  we can write

$$|V(s_1)| (1 - 2^{-10} - 2^{-10}) \leq |V(s_1)| (1 - \|B\|_\infty \delta \exp(\delta \|B\|_\infty)) - 2^{-10} P \\ \leq |V(s_1)| (1 - \|B\|_\infty \delta \exp(\delta \|B\|_\infty)) - Q(t, \delta) \exp(\delta \|B\|_\infty) \\ \leq |V(s)|$$

These inequalities are valid for all  $s \in [t - \delta, t]$  and so in particular for  $s = t$ . And so we can write

$$2^{-1} |V(s)| \leq |V(s)| \frac{1 - 2^{-9}}{1 + 2^{-9}} \leq |v| \leq |V(s)| \frac{1 + 2^{-9}}{1 - 2^{-9}} \leq 2 |V(s)| \quad (3.42)$$

which is equivalent to (3.38).

Now let's look at the inequality (3.39). Like for (3.38), we try to write a Grönwall inequality but this time on  $Z(s) = |(V(s) - V_*(s)) - (V(s_1) - V_*(s_1))|$ .

$$\begin{aligned} (V(s) - V_*(s)) - (V(s_1) - V_*(s_1)) &= \int_{s_1}^s E(\tau, X(\tau))d\tau + \int_{s_1}^s V(\tau) \wedge B(\tau, X(\tau))d\tau \\ &\quad - \int_{s_1}^s E(\tau, X_*(\tau))d\tau - \int_{s_1}^s V_*(\tau) \wedge B(\tau, X_*(\tau))d\tau. \end{aligned}$$

This allows us to write

$$\begin{aligned} Z(s) &\leq 2Q(t, \delta) + \left| \int_{s_1}^s (V(\tau) \wedge B(\tau, X(\tau)) - V_*(\tau) \wedge B(\tau, X_*(\tau))) d\tau \right| \\ &\leq 2Q(t, \delta) + \int_{s_1}^s |V(\tau) \wedge (B(\tau, X(\tau)) - B(\tau, X_*(\tau)))| + |(V_*(\tau) - V(\tau)) \wedge B(\tau, X_*(\tau))| d\tau \\ &\leq 2Q(t, \delta) + 2\delta \|B\|_\infty 2|V(s_1)| \\ &\quad + \|B\|_\infty \left( \delta |(V_*(s_1) - V(s_1))| + \int_{s_1}^s |(V(\tau) - V_*(\tau)) - (V(s_1) - V_*(s_1))| d\tau \right) \end{aligned}$$

where in the second term in the last inequality we used the bound  $|V(s)| \leq |V(s_1)|(1+2^{-9}) \leq 2|V(s_1)|$  that we established just before.

Thus we have our Grönwall inequality on  $Z(s)$  which gives us

$$Z(s) \leq (2Q(t, \delta) + 4\delta \|B\|_\infty |V(s_1)| + \|B\|_\infty \delta |(V_*(s_1) - V(s_1))|) \exp(\delta \|B\|_\infty). \quad (3.43)$$

We notice the term  $4\delta \|B\|_\infty |V(s_1)|$  in the inequality that will make the analysis a bit more complicated compared to what was done to obtain (3.38), because we will have to compare  $|V(s) - V_*(s)|$  to both  $|V(s_1) - V_*(s_1)|$  and  $|V(s_1)|$ . To do this we distinguish between the two cases  $|V(s_1) - V_*(s_1)| \geq |V(s_1)|$  and  $|V(s_1) - V_*(s_1)| < |V(s_1)|$ .

- *First case*  $|V(s_1) - V_*(s_1)| \geq |V(s_1)|$ :

From (3.43), we deduce

$$Z(s) \leq (2Q(t, \delta) + 5 \|B\|_\infty \delta |(V_*(s_1) - V(s_1))|) \exp(\delta \|B\|_\infty), \quad (3.44)$$

and so by following exactly the same method as above to obtain (3.38), namely using (3.33), (3.40), and (3.29), from the last inequality we can write

$$\begin{aligned} |V(s) - V_*(s)| &\leq |V(s_1) - V_*(s_1)| (1 + 5 \|B\|_\infty \delta \exp(\delta \|B\|_\infty)) + 2^{-9}P \\ &\leq |V(s_1) - V_*(s_1)| (1 + 5 \cdot 2^{-10} + 2^{-9}) \leq |V(s_1) - V_*(s_1)| (1 + 2^{-6}) \end{aligned}$$

and

$$\begin{aligned}
& |V(s_1) - V_*(s_1)| (1 - 2^{-7} - 2^{-7}) \\
& \leq |V(s_1) - V_*(s_1)| (1 - 5 \cdot 2^{-10} - 2^{-9}) \\
& \leq |V(s_1) - V_*(s_1)| (1 - 5 \|B\|_\infty \delta \exp(\delta \|B\|_\infty)) - 2^{-9}P \\
& \leq |V(s_1) - V_*(s_1)| (1 - 5 \|B\|_\infty \delta \exp(\delta \|B\|_\infty)) - 2Q(t, \delta) \exp(\delta \|B\|_\infty) \\
& \leq |V(s) - V_*(s)|
\end{aligned}$$

Now we look at the other case.

- *Second case*  $|V(s_1) - V_*(s_1)| < |V(s_1)|$ :

From (3.43), we deduce

$$Z(s) \leq (2Q(t, \delta) + 5 \|B\|_\infty \delta |V(s_1)|) \exp(\delta \|B\|_\infty). \quad (3.45)$$

We once again use (3.33), (3.40), and (3.29) to bound  $|V(s) - V_*(s)|$  but now we also use the hypothesis  $|V(s_1) - V_*(s_1)| < |V(s_1)|$ . Thanks to the last inequality this allows us to write

$$\begin{aligned}
|V(s) - V_*(s)| & \leq |V(s_1) - V_*(s_1)| + |V(s_1)| 5 \|B\|_\infty \delta \exp(\delta \|B\|_\infty) + 2^{-9}P \\
& \leq |V(s_1)| (1 + 5 \cdot 2^{-10} + 2^{-9}) \leq |V(s_1)| (1 + 2^{-6}).
\end{aligned}$$

We use exactly the same inequalities to get a lower bound on  $|V(s) - V_*(s)|$ , but this time the hypothesis is used in the form  $-|V(s_1) - V_*(s_1)| > -|V(s_1)|$

$$\begin{aligned}
& |V(s_1) - V_*(s_1)| (1 - 2^{-7} - 2^{-7}) \\
& \leq |V(s_1) - V_*(s_1)| (1 - 5 \cdot 2^{-10} - 2^{-9}) \\
& \leq |V(s_1) - V_*(s_1)| - |V(s_1)| (5 \cdot 2^{-10} + 2^{-9}) \\
& \leq |V(s_1) - V_*(s_1)| (1 - 5 \|B\|_\infty \delta \exp(\delta \|B\|_\infty)) - 2^{-9}P \\
& \leq |V(s_1) - V_*(s_1)| (1 - 5 \|B\|_\infty \delta \exp(\delta \|B\|_\infty)) - 2Q(t, \delta) \exp(\delta \|B\|_\infty) \\
& \leq |V(s) - V_*(s)|
\end{aligned}$$

□

**LEMMA 3.5.** *For any  $(x, v) \in \mathbb{R}^6$  we have*

$$\int_{t-\delta}^t \frac{\mathbf{1}_U(s, X(s), V(s))}{|X(s) - X_*(s)|^2} ds \leq C \left( \frac{1 + |v|^{2+\varepsilon}}{L} \right). \quad (3.46)$$

*Proof.* If  $(s, X(s), V(s)) \notin U$  for all  $s \in [t - \delta, t]$  then the estimate (3.46) is verified. Now we assume that there exists  $s_1 \in [t - \delta, t]$  such that  $(s_1, X(s_1), V(s_1)) \in U$ , then thanks to lemma 3.4 we can write

$$\Lambda_\varepsilon(s, V(s)) \geq L(1 + (2|v|)^{2+\varepsilon})^{-1}(2|v - v_*|)^{-1} \geq 2^{-3-\varepsilon}\Lambda_\varepsilon(t, v) \quad (3.47)$$

and hence

$$\frac{\mathbf{1}_U(s, X(s), V(s))}{|X(s) - X_*(s)|^2} \leq \frac{\mathbf{1}_{\mathbb{R}^3 \setminus B(X_*(s), 2^{-3-\varepsilon}\Lambda_\varepsilon(t, v))}(X(s))}{|X(s) - X_*(s)|^2} \leq h(|Y(s)|), \quad (3.48)$$

where  $Y(s) = X(s) - X_*(s)$  and  $h(u) = \min(|u|^{-2}, 4^{3+\varepsilon}\Lambda_\varepsilon(t, v)^{-2})$ . Since  $h$  is a non-increasing function, we look for a lower bound on  $|Y(s)|$ .

For any  $s_0 \in [t - \delta, t]$  we have, thanks to (3.39)

$$\begin{aligned} |Y(s)| &\geq |Y(s_0) + (s - s_0)Y'(s_0)| - \left| \int_{s_0}^s (s - u)Y''(u)du \right| \\ &\geq |Y(s_0) + (s - s_0)Y'(s_0)| - 2|s - s_0|(Q(t, \delta) + \delta \|B\|_\infty |v - v_*|). \end{aligned}$$

Now we consider  $s = s_0$  that minimizes  $|Y(s)|^2$  when  $s \in [t - \delta, t]$ , then this implies  $(s - s_0)Y(s_0) \cdot Y'(s_0) \geq 0$  and so

$$|Y(s_0) + (s - s_0)Y'(s_0)|^2 \geq |Y'(s_0)|^2 |s - s_0|^2 \quad (3.49)$$

and thanks to (3.39) we get  $|Y'(s_0)| \geq 2^{-1}|v - v_*|$  and when we evaluate (3.39) in  $s_1$  this also yields  $Q(t, \delta) \leq 2^{-9}|v - v_*|\exp(-\delta \|B\|_\infty)$  so we have

$$\begin{aligned} |Y'(s_0)| - 2(Q(t, \delta) + \delta \|B\|_\infty |v - v_*|) &\geq |v - v_*|(2^{-1} - 2^{-8}|v - v_*|\exp(-\delta \|B\|_\infty) - 2\delta \|B\|_\infty) \\ &\geq |v - v_*|(2^{-1} - 2^{-8} - 2\delta \|B\|_\infty). \end{aligned}$$

We need the quantity  $(2^{-1} - 2^{-8} - 2\delta \|B\|_\infty)$  to be strictly positive and so once again we need the condition (3.29) for  $\delta \|B\|_\infty$  to be small and this inequality to be verified. Now we have  $|Y(s)| \geq \alpha |v - v_*| |s - s_0|$  with  $\alpha > 0$ . Just as in [15], we bring this inequality into (3.48), integrate with respect to the time variable and estimate the integral as follows to obtain

(3.46):

$$\begin{aligned}
\int_{t-\delta}^t h(|Y(s)|) ds &\leq \int_{t-\delta}^t h(\alpha |v - v_*| |s - s_0|) ds, \\
&\leq \int_0^{+\infty} h(\alpha |v - v_*| r) dr, \\
&= (\alpha |v - v_*|)^{-1} \int_0^{+\infty} h(r) dr, \\
&= (\alpha |v - v_*|)^{-1} \left( \int_0^{2^{-3-\varepsilon} \Lambda_\varepsilon(t, v)} 4^{3+\varepsilon} \Lambda_\varepsilon(t, v)^{-2} dr + \int_{2^{-3-\varepsilon} \Lambda_\varepsilon(t, v)}^{+\infty} \frac{1}{r^2} dr \right), \\
&= (\alpha |v - v_*|)^{-1} (2^{3+\varepsilon} \Lambda_\varepsilon(t, v)^{-1} + 2^{3+\varepsilon} \Lambda_\varepsilon(t, v)^{-1}), \\
&\leq C \left( \frac{1 + |v|^{2+\varepsilon}}{L} \right).
\end{aligned}$$

□

Now integrating in  $x, v$  and using the mass conservation, we finally obtain

$$I_*^U(t, \delta) \leq CL^{-1}(1 + M_{2+\varepsilon}(T)). \quad (3.50)$$

We gather all the above estimates to conclude

$$\begin{aligned}
I_*(t, \delta) &\leq C(\delta(Q(t, \delta) \exp(\delta \|B\|_\infty))^{\frac{4}{3}} + \delta L + L^{-1}(1 + M_{2+\varepsilon}(T))) \\
&\leq C(\delta Q(t, \delta)^{\frac{4}{3}} + \delta L + L^{-1}(1 + M_{2+\varepsilon}(T)))
\end{aligned}$$

where the last inequality is justified by the fact that thanks to (3.29) we have  $\exp(\delta \|B\|_\infty) \leq \exp(g(2^{-10}))$  where  $g$  is the inverse of the function  $x \mapsto x \exp(x)$  on  $\mathbb{R}_+$ . We conclude in the same way as in [15], firstly by optimizing the parameter  $L$  and then by noticing that the pair  $(x_*, v_*)$  is arbitrary so that we have

$$\sup \{I_*(t, \delta), (x_*, v_*) \in \mathbb{R}^3 \times \mathbb{R}^3\} \geq Q(t, \delta). \quad (3.51)$$

Finally, we obtain (3.31). □

The next two propositions allow us to conclude. The proof of proposition 3.6 is identical to the one in [15] because it doesn't rely on the characteristics of the system, but rather on real analysis arguments. In an effort of clarity, and also because some arguments of the proof are more detailed in this paper than in [15], we place the proof of proposition 3.6 in the appendix.

**PROPOSITION 3.6.** *For any  $t \in [0, T]$  with  $T \leq T_B$ , we have*

$$Q(t, t) \leq C(t^{\frac{1}{2}} + t)(1 + M_{2+\varepsilon}(T))^{\frac{4}{7}}. \quad (3.52)$$

Now we state the last result necessary in the proof of proposition 3.1.

**PROPOSITION 3.7.** *There exists  $\tau(\varepsilon, k) > 0$  such that for any  $t \in [0, T]$  we have*

$$Q(t, t) \leq C(1 + M_k(0))^{\tau(\varepsilon, k)} \exp(T \|B\|_\infty)^{\frac{2}{5}} (T^{\frac{1}{2}} + T^{\frac{7}{5}}). \quad (3.53)$$

**REMARK 3.8.** *We notice that we obtain the same estimate (3.53) as in [15] when the magnetic field  $B$  is zero.*

*Proof.* Using the same argument as in (2.3), we can write

$$M_k(t) \leq 2^{k-1} \exp(kt \|B\|_\infty) \left( M_k(0) + N(T)^k \|f^{in}\|_1 \right) \quad (3.54)$$

with  $N(T)$  defined in theorem 2.1.

Now to obtain the desired estimate we have to bound  $M_{2+\varepsilon}(t)$ , which we manage with the Hölder inequality. Thus for any  $t \in [0, T]$  we have

$$\iint |v|^{2+\varepsilon} f(t, x, v) dx dv \leq \left( \iint |v|^2 f(t, x, v) dx dv \right)^{\frac{2+\varepsilon-k}{2-k}} \left( \iint |v|^k f(t, x, v) dx dv \right)^{\frac{\varepsilon}{k-2}}. \quad (3.55)$$

With the conservation of the energy  $E(t)$ , this Hölder inequality implies that

$$M_{2+\varepsilon}(T) \leq C M_k(T)^{\frac{\varepsilon}{k-2}} \quad (3.56)$$

and bringing this inequality into (3.54) it yields

$$M_{2+\varepsilon}(T) \leq C \exp\left(\frac{k\varepsilon}{k-2} T \|B\|_\infty\right) \left( M_k(0) + N(T)^k \|f^{in}\|_1 \right)^{\frac{\varepsilon}{k-2}}. \quad (3.57)$$

Now thanks to (3.52) we can deduce

$$\begin{aligned} M_{2+\varepsilon}(T) &\leq C \exp\left(\frac{k\varepsilon}{k-2} T \|B\|_\infty\right) \left( M_k(0) + N(T)^k \|f^{in}\|_1 \right)^{\frac{\varepsilon}{k-2}} \\ &\leq C \exp\left(\frac{k\varepsilon}{k-2} T \|B\|_\infty\right) \left( M_k(0) + (T^{\frac{1}{2}} + T)^k (1 + M_{2+\varepsilon}(T))^{\frac{4k}{7}} \|f^{in}\|_1 \right)^{\frac{\varepsilon}{k-2}} \\ &\leq C \exp\left(\frac{k\varepsilon}{k-2} T \|B\|_\infty\right) (1 + M_k(0))^{\frac{\varepsilon}{k-2}} (T^{\frac{1}{2}} + T)^{\frac{k\varepsilon}{k-2}} (1 + M_{2+\varepsilon}(T))^{\frac{4k\varepsilon}{7(k-2)}} \end{aligned}$$

Like in [15], we write  $\sigma(\varepsilon, k) = \frac{4k\varepsilon}{7(k-2)}$  and notice that if we take  $\varepsilon$  small enough we have  $\sigma(\varepsilon, k) \in ]0, 1[$ . More precisely, if  $\varepsilon < \varepsilon_0$  then  $\sigma(\varepsilon, k) \leq \frac{2}{7}$ ,  $\frac{k\varepsilon}{k-2} \leq \frac{1}{2}$  and  $\frac{\varepsilon}{k-2} \leq \frac{1}{2k}$  and we find

$$M_{2+\varepsilon}(T) \leq C(1 + M_k(0))^{\frac{1}{2k}} \exp(T \|B\|_\infty)^{\frac{1}{2}} (1 + T)^{\frac{1}{2}} (1 + M_{2+\varepsilon}(T))^{\frac{2}{7}}, \quad (3.58)$$

where we used  $T^{\frac{1}{2}} + T \leq 2(1 + T)$ . Now since the right term in the last inequality is larger than 1 up to a constant  $C$ , we deduce that

$$(1 + M_{2+\varepsilon}(T))^{\frac{5}{7}} \leq C(1 + M_k(0))^{\frac{1}{2k}} \exp(T \|B\|_\infty)^{\frac{1}{2}} (1 + T)^{\frac{1}{2}} \quad (3.59)$$

which finally yields

$$1 + M_{2+\varepsilon}(T) \leq C(1 + M_k(0))^{\frac{7}{10k}} \exp(T \|B\|_\infty)^{\frac{7}{10}} (1 + T)^{\frac{7}{10}}. \quad (3.60)$$

Then, using (3.52) again we deduce

$$\begin{aligned} Q(t, t) &\leq C(T^{\frac{1}{2}} + T)(1 + M_k(0))^{\frac{2}{5k}} \exp(T \|B\|_\infty)^{\frac{2}{5}} (1 + T)^{\frac{2}{5}} \\ &\leq C(1 + M_k(0))^{\frac{2}{5k}} \exp(T \|B\|_\infty)^{\frac{2}{5}} (T^{\frac{1}{2}} + T^{\frac{7}{5}}). \end{aligned}$$

This concludes the proof of proposition 3.7 and proposition 3.1.  $\square$

### 3.2 The case $T \geq T_B$

We conclude the proof of (2.16) by showing that  $Q(t, t)$  is bounded for all time.

**PROPOSITION 3.9.** *The inequality (2.16) is valid for all  $T \geq T_B$ .*

*Proof.* For all  $t \in [0, T]$ , we write  $t = nT_B + t_r$  with  $n \in \mathbb{N}$  and  $t_r \in [0, T_B[$ . Since the constant  $C$  in proposition 3.1 depends only on  $T, k, \|B\|_\infty, \|f^{in}\|_1, \|f^{in}\|_\infty, \mathcal{E}(0)$  and  $M_k(0)$ , we can reiterate the procedure on any time interval  $I_p = [pT_B, (p+1)T_B]$ . Indeed,  $T, k$  and  $\|B\|_\infty$  are constants  $\|f(t)\|_1$  and  $\|f(t)\|_\infty$  are conserved in time and the energy  $E(t)$  is bounded. This means we can write

$$\begin{aligned} Q(t, t) &\leq \sum_{p=0}^{n-1} Q((p+1)T_B, T_B) + Q(t, t_r) \\ &\leq C \sum_{p=0}^n (1 + M_k(pT_B))^{\frac{2}{5k}} \exp(T_B \|B\|_\infty)^{\frac{2}{5}} (T_B^{\frac{1}{2}} + T_B^{\frac{7}{5}}). \end{aligned}$$

Furthermore, we can show by an immediate induction that for all  $p \in \mathbb{N}$  with  $p \leq n$ ,  $M_k(pT_B)$  is bounded such that

$$M_k(pT_B) \leq C_p(k, \|B\|_\infty, \mathcal{E}(0), \|f^{in}\|_1, \|f^{in}\|_\infty, M_k(0)). \quad (3.61)$$

This is just because  $M_k(pT_B) \leq C_1 \Rightarrow Q((p+1)T_B, T_B) \leq C_2 \Rightarrow M_k((p+1)T_B) \leq C_3$  with  $C_1, C_2, C_3$  depending on (2.14). This concludes the proof of proposition 3.9 and theorem 2.1.  $\square$

## 4 Proofs regarding uniqueness

The subsections 4.1, 4.2 and 4.3 will be devoted to the proof of theorem 2.6 and subsection 4.4 will be devoted to the proof of theorem 2.4. In this section, we shall denote by  $C$  a constant that can change from one line to another but that only depends on

$$\mathcal{E}(0), \|f^{in}\|_1, \|f^{in}\|_\infty, T, \iint |v|^m f^{in}. \quad (4.62)$$

#### 4.1 Proof of the estimate on the $L^p$ norms of $\rho$ (2.26)

We consider  $f^{in}$  that satisfies the assumptions of theorem 2.6 and let  $f$  be the solution given by theorem 2.1 with initial data  $f^{in}$ . By construction, we have propagation of moments:

$$\sup_{t \in [0, T]} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^m f(t, x, v) dx dv < +\infty. \quad (4.63)$$

Now thanks to a classical velocity moment inequality, we show how to control the  $L^p$  norms of the macroscopic density with velocity moments, this inequality is given by

$$\|\rho(t)\|_{\frac{k+3}{3}} \leq C \|f(t)\|_{\infty}^{\frac{k}{k+3}} M_k(t)^{\frac{3}{k+3}}. \quad (4.64)$$

with  $C$  independent of  $k$ . Since we want  $\rho$  to verify (2.26), this means that we need to prove

$$\forall k \geq 1, \sup_{t \in [0, T]} (\|f(t)\|_{\infty}^{\frac{k}{k+3}} M_k(t)^{\frac{3}{k+3}}) \leq Ck. \quad (4.65)$$

Since the solution  $f \in L^\infty([0, T], L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ , we finally need to show

$$\forall k \geq 1, \sup_{t \in [0, T]} M_k(t)^{\frac{3}{k+3}} \leq Ck. \quad (4.66)$$

First, we recall that thanks to (4.63) where  $m > 6$  we can infer that  $\rho \in L^\infty([0, T], L^p(\mathbb{R}^3))$  with  $p = \frac{m+3}{3} > 3$  and following (4.86) we have  $E \in L^\infty([0, T], L^\infty(\mathbb{R}^3))$ . Then we write

$$\frac{d}{dt} |V(t, x, v)|^k = k |V(t, x, v)|^{k-1} \frac{\dot{V}(t, x, v) \cdot V(t, x, v)}{|V(t, x, v)|},$$

and thanks to the bound on  $E$  and the definition of the characteristics (1.8) we can infer that for all  $k > m$

$$\begin{aligned} |V(t, x, v)|^k &\leq |v|^k + k \int_0^t |V(s, x, v)|^{k-1} \frac{(E(s, X(s, x, v)) + V(s, x, v) \wedge B(s, X(s, x, v))) \cdot V(s, x, v)}{|V(s, x, v)|} ds \\ &\leq |v|^k + k \|E\|_\infty \int_0^t |V(s, x, v)|^{k-1} ds. \end{aligned}$$

Since the contribution of magnetic field  $B$  vanishes, the following computations are the same as in the unmagnetized case [14]. In an effort to be clear, we explicit these computations nonetheless.

Integrating this last inequality with respect to  $f^{in}(x, v) dx dv$  we get

$$M_k(t) \leq M_k(0) + k \|E\|_\infty \int_0^t M_{k-1}(s) ds. \quad (4.67)$$

Thus by induction we deduce that  $\sup_{t \in [0, T]} M_k(t)$  is finite for all  $k > m$ . furthermore, by another classical velocity moment inequality we obtain that

$$M_{k-1}(s) \leq \|f(s)\|_1^{\frac{1}{k}} M_k(s)^{\frac{k-1}{k}}. \quad (4.68)$$

Since  $\|f(t)\|_1$  is conserved, we get

$$M_k(t) \leq M_k(0) + Ck \int_0^t M_k(s)^{\frac{k-1}{k}} ds, . \quad (4.69)$$

Differentiating this inequality allows us to write

$$M'_k(t) \leq Ck M_k(t)^{\frac{k-1}{k}} \Leftrightarrow \frac{d}{dt} (M_k(t)^{\frac{1}{k}}) \leq C \Rightarrow \sup_{t \in [0, T]} M_k(t)^{\frac{1}{k}} \leq M_k(0)^{\frac{1}{k}} + C.$$

By assumption on  $M_k(0)$  we find for all  $t \in [0, T]$

$$M_k(t)^{\frac{1}{k}} \leq (C_0 k)^{\frac{1}{3}} + C \leq (Ck)^{\frac{1}{3}} \leq (Ck)^{\frac{1}{3} + \frac{1}{k}}, \quad (4.70)$$

which finally implies that

$$\sup_{t \in [0, T]} M_k(t)^{\frac{3}{k+3}} \leq Ck. \quad (4.71)$$

## 4.2 Estimate on the characteristics

We consider two solutions  $f_1, f_2 \in L^\infty([0, T], L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$  such that  $\rho_1, \rho_2$  verify

$$\rho_1, \rho_2 \in L^\infty([0, T], L^p(\mathbb{R}^3)) \quad (4.72)$$

for some  $p > 3$ . This regularity on  $\rho_{1,2}$  is guaranteed by the condition (2.24) thanks to the estimate (4.64). Then we write  $Y_1 = (X_1, V_1)$  and  $Y_2 = (X_2, V_2)$  for the corresponding characteristics, which are both solutions to (1.8) with  $t = 0$ . This means we can simplify the notation and will write  $Y_i(t; 0, x, v) = Y_i(t, x, v)$ ,  $i = 1, 2$ . Regarding the existence of such characteristics, the condition (4.72) yields sufficient regularity on the electric field  $E_i$ ,  $i = 1, 2$ , so that with the added regularity assumption on the magnetic field (1.3) we can define weak characteristics thanks to theorem III.2 (section III.2) in [6].

Now we introduce the distance

$$D(t) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |X_1(t, x, v) - X_2(t, x, v)| f^{in}(x, v) dx dv. \quad (4.73)$$

From (1.8) we can infer that

$$\begin{aligned} X_1(t, x, v) - X_2(t, x, v) &= \int_0^t \int_0^s E_1(\tau, X_1(\tau, x, v)) - E_2(\tau, X_2(\tau, x, v)) \\ &+ V_1(\tau, x, v) \wedge B(\tau, X_1(\tau, x, v)) - V_2(\tau, x, v) \wedge B(\tau, X_2(\tau, x, v)) d\tau ds \end{aligned} \quad (4.74)$$

which yields that

$$\begin{aligned}
D(t) &\leq \int_0^t \int_0^s \int_{\mathbb{R}^6} |E_1(\tau, X_1(\tau, x, v)) - E_2(\tau, X_2(\tau, x, v))| \\
&\quad + |V_1(\tau, x, v) \wedge B(\tau, X_1(\tau, x, v)) - V_2(\tau, x, v) \wedge B(\tau, X_2(\tau, x, v))| f^{in}(x, v) dx dv d\tau ds \\
&\leq \int_0^t \int_0^s \int_{\mathbb{R}^6} |E_1(\tau, X_1(\tau, x, v)) - E_2(\tau, X_2(\tau, x, v))| f^{in}(x, v) dx dv d\tau ds \\
&\quad + \|B\|_\infty \int_0^t \int_0^s \int_{\mathbb{R}^6} |V_1(\tau, x, v) - V_2(\tau, x, v)| f^{in}(x, v) dx dv d\tau ds \\
&\quad + \int_0^t \int_0^s \int_{\mathbb{R}^6} |V_2(\tau, x, v)| |B(\tau, X_1(\tau, x, v)) - B(\tau, X_2(\tau, x, v))| f^{in}(x, v) dx dv d\tau ds \\
&= I(t) + J(t) + K(t)
\end{aligned}$$

The term  $I(t)$  is the quantity estimated thanks to the method in [14]. As for the other two terms  $J(t)$  and  $K(t)$ , since we want to use the same method as in [14] which is to exploit the fact that the characteristics of the Vlasov equation verify an ODE of order 2, we need them to be controlled by  $\int_0^t \int_0^s D(\tau)^{1-\frac{3}{p}} d\tau ds$ . This is true and the estimates are given in the following proposition.

**PROPOSITION 4.1.** *For all  $t \in [0, T]$  and for all  $p > 3$ , we have the following estimates:*

$$I(t) \leq CpC_{\rho_1, \rho_2} \int_0^t \int_0^s D(\tau)^{1-\frac{3}{p}} d\tau ds \quad (4.75)$$

$$K(t) \leq (CpK_B + K_{B,p}) \int_0^t \int_0^s D(\tau)^{1-\frac{3}{p}} d\tau ds \quad (4.76)$$

$$\begin{aligned}
J(t) &\leq \|B\|_\infty \int_0^t \int_0^s \int_0^\tau (CpC_{\rho_1, \rho_2} + (CpK_B + K_{B,p})) D(u)^{1-\frac{3}{p}} du d\tau ds \\
&\quad + \|B\|_\infty^2 \exp(T \|B\|_\infty) \int_0^t \int_0^s \int_0^\tau \int_0^u (CpC_{\rho_1, \rho_2} + (CpK_B + K_{B,p})) D(w)^{1-\frac{3}{p}} dw du d\tau ds.
\end{aligned} \quad (4.77)$$

with

$$\begin{aligned}
C_{\rho_1, \rho_2} &= \max \left( 1 + \|\rho_1\|_{L^\infty([0, T], L^p)}, 1 + \|\rho_2\|_{L^\infty([0, T], L^p)} \right), \\
K_{B,p} &= 2 \|B\|_{W^{1, \infty}} \|E_2\|_\infty \exp(T \|B\|_\infty), \\
K_B &= 2 \|B\|_{W^{1, \infty}} \exp(T \|B\|_\infty),
\end{aligned}$$

and where  $C$  denotes a constant that depends only on  $T, \|f^{in}\|_\infty, \|f^{in}\|_1$ .

*Proof of proposition 4.1.* As said above, the term  $I(t)$  is the quantity estimated thanks to the method in [14], so we treat it identically to find the estimate (4.75).

Let's first look at the term  $K(t)$ . Since  $B \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^3)$  then for all  $t \in [0, T]$  and  $\alpha \in ]0, 1]$

$$B(t) \in C^{0,\alpha}(\mathbb{R}^3) \quad (4.78)$$

with Hölder coefficient  $C_{B(t)}$  verifying  $C_{B(t)} \leq \max(2 \|B\|_\infty, \|\nabla B\|_\infty) \leq 2 \|B\|_{W^{1,\infty}}$ .

Then we simply have for all  $p > 3$

$$K(t) \leq 2 \|B\|_{W^{1,\infty}} \int_0^t \int_0^s \int_{\mathbb{R}^6} |V_2(\tau, x, v)| |X_1(\tau, x, v) - X_2(\tau, x, v)|^{1-\frac{3}{p}} f^{in}(x, v) dx dv d\tau ds \quad (4.79)$$

Now we need to estimate the velocity characteristic  $V_2$ , and using (1.8) we can write once again

$$\begin{aligned} |V(t, x, v)| &\leq |v| + \int_0^t |E(s, X(s, x, v))| ds + \|B\|_\infty \int_0^t |V(s, x, v)| ds \\ &\leq |v| + T \|E\|_\infty + \|B\|_\infty \int_0^t |V(s, x, v)| ds. \end{aligned}$$

This classical Grönwall inequality yields for all  $t \in [0, T]$

$$|V(t, x, v)| \leq (|v| + T \|E\|_\infty) \exp(t \|B\|_\infty). \quad (4.80)$$

So that we can write

$$\begin{aligned} K(t) &\leq 2 \|B\|_{W^{1,\infty}} \exp(T \|B\|_\infty), \\ &\int_0^t \int_0^s \int_{\mathbb{R}^6} (|v| + T \|E_2\|_\infty) |X_1(\tau, x, v) - X_2(\tau, x, v)|^{1-\frac{3}{p}} f^{in}(x, v) dx dv d\tau ds, \\ &= K_1(t) + K_2(t). \end{aligned} \quad (4.81)$$

By applying Jensen's inequality for concave functions to  $x \mapsto x^{1-\frac{3}{p}}$  we obtain

$$K_2(t) \leq K_{B,p} \int_0^t \int_0^s D(\tau)^{1-\frac{3}{p}} d\tau ds. \quad (4.82)$$

Then we estimate  $K_1(t)$  by writing  $f^{in} = (f^{in})^{\frac{1}{p}} (f^{in})^{\frac{1}{p'}}$  where  $\frac{1}{p} + \frac{1}{p'} = 1$ , so that with the Hölder inequality applied to  $|v| (f^{in})^{\frac{1}{p}}$  and  $|X_1(\tau, x, v) - X_2(\tau, x, v)|^{1-\frac{3}{p}} (f^{in})^{\frac{1}{p'}}$  with the exponents  $p$  and  $p'$  we have

$$K_1(t) \leq K_B \left( \int_{\mathbb{R}^6} |v|^p f^{in}(x, v) dx dv \right)^{\frac{1}{p}} \int_0^t \int_0^s \left( \int_{\mathbb{R}^6} |X_1(\tau, x, v) - X_2(\tau, x, v)|^{(1-\frac{3}{p})p'} f^{in}(x, v) dx dv \right)^{\frac{1}{p'}} d\tau ds.$$

Using (2.25) we have  $(\int_{\mathbb{R}^6} |v|^p f^{in}(x, v) dx dv)^{\frac{1}{p}} \leq (C_0 p)^{\frac{1}{3}} \leq Cp$ . Furthermore, we can once again use the Jensen inequality because  $(1 - \frac{3}{p})p' = \frac{p-3}{p} \frac{p}{p-1} = \frac{p-3}{p-1} < 1$ , which gives us

$$K_1(t) \leq Cp K_B \int_0^t \int_0^s D(\tau)^{1-\frac{3}{p}} d\tau ds. \quad (4.83)$$

This concludes the proof of (4.77).

To estimate the last term  $J(t)$ , we also use (1.8) to obtain a Grönwall inequality on  $|V_1(t) - V_2(t)|$ , and since the computations are complicated we write  $V_{1,2}(s), X_{1,2}(s)$  for the characteristics. First we write

$$\begin{aligned} |V_1(t) - V_2(t)| &\leq \int_0^t |E_1(s, X_1(s)) - E_2(s, X_2(s))| ds + \|B\|_\infty \int_0^t |V_1(s) - V_2(s)| ds \\ &\quad + \int_0^t |V_2(s)| |B(s, X_1(s)) - B(s, X_2(s))| ds. \end{aligned}$$

Now using (4.78) and (4.80) we deduce

$$\begin{aligned} |V_1(t) - V_2(t)| &\leq \int_0^t |E_1(s, X_1(s)) - E_2(s, X_2(s))| ds + \|B\|_\infty \int_0^t |V_1(s) - V_2(s)| ds \\ &\quad + (K_B |v| + K_{B,p}) \int_0^t |X_1(s) - X_2(s)|^{1-\frac{3}{p}} ds \end{aligned}$$

which is just the Grönwall inequality on  $|V_1(t) - V_2(t)|$  we were looking for and which yields

$$\begin{aligned} |V_1(t) - V_2(t)| &\leq \int_0^t |E_1(s, X_1(s)) - E_2(s, X_2(s))| + (K_B |v| + K_{B,p}) |X_1(s) - X_2(s)|^{1-\frac{3}{p}} ds \\ &\quad + \int_0^t \left( \int_0^s |E_1(\tau, X_1(\tau)) - E_2(\tau, X_2(\tau))| + (K_B |v| + K_{B,p}) |X_1(\tau) - X_2(\tau)|^{1-\frac{3}{p}} d\tau \right) \\ &\quad \times \|B\|_\infty \exp((t-s)\|B\|_\infty) ds. \end{aligned}$$

Now we insert this inequality in the definition of  $J(t)$  to obtain

$$\begin{aligned} J(t) &\leq \|B\|_\infty \int_0^t \int_0^s \int_0^\tau \int_{\mathbb{R}^6} \left( |E_1(u, X_1(u)) - E_2(u, X_2(u))| + (K_B |v| + K_{B,p}) |X_1(u) - X_2(u)|^{1-\frac{3}{p}} \right) \\ &\quad \times f^{in}(x, v) dx dv du d\tau ds \\ &\quad + \|B\|_\infty^2 \exp(T \|B\|_\infty) \times \\ &\quad \int_0^t \int_0^s \int_0^\tau \int_0^u \int_{\mathbb{R}^6} \left( |E_1(w, X_1(w)) - E_2(w, X_2(w))| + (K_B |v| + K_{B,p}) |X_1(w) - X_2(w)|^{1-\frac{3}{p}} \right) \\ &\quad \times f^{in}(x, v) dx dv dw du d\tau ds. \end{aligned} \tag{4.84}$$

Like previously, we can use the Jensen inequality to bound the terms  $K_{B,p} |X_1 - X_2|^{1-\frac{3}{p}}$ , the relation (4.75) to bound the terms  $|E_1 - E_2|$  and the Hölder inequality used to estimate  $K_1(t)$  to bound  $K_B |v| |X_1 - X_2|^{1-\frac{3}{p}}$ .

This gives the desired estimate (4.76) on  $J(t)$ :

$$\begin{aligned} J(t) &\leq \|B\|_\infty \int_0^t \int_0^s \int_0^\tau (CpC_{\rho_1, \rho_2} + (CpK_B + K_{B,p})) D(u)^{1-\frac{3}{p}} du d\tau ds \\ &\quad + \|B\|_\infty^2 \exp(T \|B\|_\infty) \int_0^t \int_0^s \int_0^\tau \int_0^u (CpC_{\rho_1, \rho_2} + (CpK_B + K_{B,p})) D(w)^{1-\frac{3}{p}} dw du d\tau ds. \end{aligned}$$

□

### 4.3 A second order inequality on $D(t)$

We begin by looking at the dependence of  $K_{B,p}$  with respect to  $p$ . The only term in  $K_{B,p}$  which depends on  $p$  is  $\|E_2\|_\infty$ , and since  $\rho_2 \in L^\infty([0, T], L^p(\mathbb{R}^3))$  with  $p > 3$ , then we can deduce the desired  $L^\infty$  bound on  $E_2$  because for all  $t \in [0, T]$

$$\|E_2(t)\|_\infty \leq \|\mathbf{1}_{|x| \geq 1} \nabla \mathcal{G}_3\|_\infty \|\rho_2(t)\|_1 + \|\mathbf{1}_{|x| < 1} \nabla \mathcal{G}_3\|_q \|\rho_2(t)\|_p \quad (4.85)$$

with  $\frac{1}{p} + \frac{1}{q} = 1$ .

From this last inequality we can finally deduce

$$\|E_2\|_\infty \leq C(1 + \|\rho_2\|_{L^\infty([0, T], L^p)}), \quad (4.86)$$

where  $C$  depends only on  $\|f^{in}\|_1$ .

Now we consider that the solutions  $f_1, f_2$  verify the assumptions of theorem 2.6. This means that  $\max(\|\rho_1\|_{L^\infty([0, T], L^p)}, \|\rho_2\|_{L^\infty([0, T], L^p)}) \leq Cp$  for all  $p \geq 1$ , and so thanks to (4.86) and proposition 4.1 we have for all  $p > 3$

$$\begin{aligned} D(t) &\leq C_1 p^2 \int_0^t \int_0^s D(\tau)^{1-\frac{3}{p}} d\tau ds \\ &+ C_2(p^2 + p) \int_0^t \int_0^s \int_0^\tau D(u)^{1-\frac{3}{p}} du d\tau ds \\ &+ C_3(p^2 + p) \int_0^t \int_0^s \int_0^\tau \int_0^u D(w)^{1-\frac{3}{p}} dw du d\tau ds, \end{aligned} \quad (4.87)$$

where  $C_1, C_2, C_3$  are constants that depend on  $T, \|f^{in}\|_\infty, \|f^{in}\|_1, \|B\|_\infty, \|\nabla B\|_\infty$ .

Let  $\mathcal{F}(t) = \int_0^t \int_0^s D(\tau)^{1-\frac{3}{p}} d\tau ds$ . Since  $\mathcal{F}$  is increasing by construction and since  $p > 3$  we can finally conclude that

$$D(t) \leq Cp^2 \int_0^t \int_0^s D(\tau)^{1-\frac{3}{p}} d\tau ds, \quad (4.88)$$

with  $C$  that depends on  $T, \|f^{in}\|_\infty, \|f^{in}\|_1, \|B\|_\infty, \|\nabla B\|_\infty$ .

Finally, we obtain the same second order differential inequality as in [14], for all  $t \in [0, T]$  we have:

$$\mathcal{F}''(t) \leq Cp^2 \mathcal{F}(t). \quad (4.89)$$

From this inequality, we use the same method as in [14] to conclude that for all  $t \in [0, T]$  we have  $f_1(t) = f_2(t)$  a.e. on  $\mathbb{R}^3 \times \mathbb{R}^3$ . This concludes the proof of theorem 2.6.

#### 4.4 Proof of theorem 2.4

We finish this section with the proof of theorem 2.4, which is the extension of Loeper's result [13] to the magnetized Vlasov–Poisson system. As said above, this proof was already done for  $B$  constant in [19]. In fact in [19], it was already proved that under the assumptions (2.21), (2.22), and (2.23), the macroscopic density  $\rho$  is bounded (2.20), and this proof doesn't change in the case of a general magnetic field.

Like in the theorem 2.6, we require additional assumptions on the moments of  $f^{in}$  to obtain uniqueness (compared to the unmagnetized case). However, these assumptions on the moments aren't as strong as in theorem 2.6 because the boundedness of  $\rho$  is already a strong assumption.

To prove our theorem, we only need to adapt subsection 3.2 from [13]. Thus we consider two solutions of (1.1)  $f_1, f_2$  with initial datum  $f^{in}$  that verifies the assumptions of theorem 2.4. Like in the previous proof, we write the corresponding densities, electric fields, and characteristics  $\rho_1, \rho_2, E_1, E_2$ , and  $Y_1(t, x, v), Y_2(t, x, v) = (X_1(t, x, v), V_1(t, x, v)), (X_2(t, x, v), V_2(t, x, v))$ . To simplify the presentation, we will write  $Y_i(t)$  for the characteristics. We define the following quantity  $Q$ :

$$Q(t) = \frac{1}{2} \int_{\mathbb{R}^6} f^{in}(x, v) |Y_1(t, x, v) - Y_2(t, x, v)|^2 dx dv. \quad (4.90)$$

Now we differentiate  $Q$  (which we couldn't do with the distance  $D$  (4.73)) splitting the magnetic part of the Lorentz force  $V \wedge B$  like in the previous section:

$$\begin{aligned} \dot{Q}(t) &= \int_{\mathbb{R}^6} f^{in}(x, v) (Y_1(t) - Y_2(t)) \cdot \partial_t (Y_1(t) - Y_2(t)) dx dv, \\ &= \int_{\mathbb{R}^6} f^{in}(x, v) (X_1(t) - X_2(t)) \cdot (V_1(t) - V_2(t)) dx dv \\ &\quad + \int_{\mathbb{R}^6} f^{in}(x, v) (V_1(t) - V_2(t)) \cdot (E_1(t, X_1(t)) - E_2(t, X_2(t))) dx dv \\ &\quad + \int_{\mathbb{R}^6} f^{in}(x, v) (V_1(t) - V_2(t)) \cdot [V_2(t) \wedge (B_1(t, X_1(t)) - B_2(t, X_2(t)))] dx dv \\ &\quad + \int_{\mathbb{R}^6} f^{in}(x, v) (V_1(t) - V_2(t)) \cdot [(V_1(t) - V_2(t)) \wedge B(t, X_1(t))] dx dv. \end{aligned}$$

First, we notice that the last term is null, which means we only need to control the second to last term (due to the added magnetic field) which we denote  $P(t)$ . The first term is bounded by  $Q(t)$  and the second term can be estimated using the analysis from [13] and is bounded by  $Q(t) \ln(\frac{1}{Q(t)})$ . To control  $P(t)$  we first use the bound on the velocity characteristic (4.80).

$$\begin{aligned} P(t) &\leq \|B\|_{W^{1,\infty}} \int_{\mathbb{R}^6} f^{in}(x, v) |V_1(t) - V_2(t)| |V_2(t)| |X_1(t) - X_2(t)| dx dv, \\ &\leq \|B\|_{W^{1,\infty}} \int_{\mathbb{R}^6} f^{in}(x, v) |V_1(t) - V_2(t)| (|v| + T \|E_2\|_\infty) e^{T\|B\|_\infty} |X_1(t) - X_2(t)| dx dv, \\ &= R(t) + S(t). \end{aligned}$$

We recall that since  $\|\rho_{1,2}\|_\infty \leq +\infty$  we can bound  $\|E_{1,2}\|_\infty$  thanks to (4.86) and the interpolation inequality:

$$\|E_i\|_\infty \leq C(\|\rho\|_1, \|\rho\|_\infty) := C_\rho, \quad (4.91)$$

with  $i = 1, 2$ .

This means we can simply estimate  $S(t)$  with the Cauchy–Schwarz inequality applied on the functions  $(f^{in})^{\frac{1}{2}} |V_1(t) - V_2(t)|$  and  $(f^{in})^{\frac{1}{2}} |X_1(t) - X_2(t)|$ .

$$\begin{aligned} S(t) &\leq TC_\rho C_{B,T} \left( \int_{\mathbb{R}^6} f^{in}(x, v) |V_1(t) - V_2(t)|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^6} f^{in}(x, v) |X_1(t) - X_2(t)|^2 \right)^{\frac{1}{2}} \\ &\leq TC_\rho C_{B,T} Q(t), \end{aligned}$$

with  $C_{B,T} = \|B\|_{W^{1,\infty}} e^{T\|B\|_\infty}$ .

To control  $R(t)$  we first use the Cauchy–Schwarz inequality and then the bound on the velocity characteristic (4.80), which also gives us a bound on the position characteristic.

$$\begin{aligned} R(t) &\leq C_{B,T} \int_{\mathbb{R}^6} f^{in}(x, v) |v| |Y_1(t) - Y_2(t)|^2 dx dv \\ &\leq C_{B,T} \int_{\mathbb{R}^6} f^{in}(x, v) |v| |Y_1(t) - Y_2(t)| \left( |V_1|^2 + |V_2|^2 + |X_1|^2 + |X_2|^2 \right)^{\frac{1}{2}} dx dv \\ &\leq C_{B,T} Q(t) \int_{\mathbb{R}^6} f^{in}(x, v) |v|^2 \left( |V_1|^2 + |V_2|^2 + |X_1|^2 + |X_2|^2 \right) dx dv \\ &\leq C_{B,T} Q(t) \underbrace{\int_{\mathbb{R}^6} f^{in}(x, v) |v|^2 2 \left( (|v| + TC_\rho)^2 e^{2T\|B\|_\infty} + (|x| + T(|v| + TC_\rho) e^{T\|B\|_\infty})^2 \right) dx dv}_{=I}. \end{aligned}$$

Thanks to the assumption (2.19) of theorem 2.4,  $I$  is bounded because we have

$$I \leq C \left( \int f^{in} |v|^6, \int f^{in} |x|^4 \right). \quad (4.92)$$

From these estimates, we conclude that

$$\frac{d}{dt} Q(t) \leq C Q(t) \left( 1 + \ln \frac{1}{Q(t)} \right) \quad (4.93)$$

with  $C := C \left( T, \|B\|_{W^{1,\infty}}, \|\rho\|_1, \|\rho\|_\infty, \int f^{in} |v|^6, \int f^{in} |x|^4 \right)$ .

With this inequality we can show, using standard Grönwall type arguments, that  $Q(0) = 0 \Rightarrow Q(t) = 0$  for all  $t \geq 0$ , which concludes the proof of theorem 2.4.

## Appendix

As said above, we present a slightly more detailed version of the proof of proposition 3.6 compared to the one found in [15].

*Proof of proposition 3.6.* Let  $t \in [0, T]$ . We note here  $H = 1 + M_{2+\varepsilon}(T)$  and for any  $\delta \in [0, t]$  we define  $N_1(t, \delta) = \delta Q(t, \delta)^{\frac{4}{3}}$  and  $N_2(t, \delta) = (\delta H)^{\frac{1}{2}}$  as in the left hand side of inequality (3.31). We set:

$$I = \{\delta \in [0, t] : N_1(t, \delta) \geq N_2(t, \delta)\}. \quad (4.94)$$

First let's suppose that  $I$  is empty. Then  $Q(t, \delta) \lesssim N_2(t, \delta)$  thanks to (3.31) for any  $\delta \in [0, t]$ , which means that

$$Q(t, \delta) \lesssim (\delta H)^{\frac{1}{2}} \leq t^{\frac{1}{2}}(1 + M_{2+\varepsilon}(T))^{\frac{4}{7}}, \quad (4.95)$$

so that (3.52) is automatically verified. Now we suppose that there exists  $\delta_*(t) \in ]0, t]$  such that  $N_1(t, \delta_*(t)) = N_2(t, \delta_*(t))$ . It comes:

$$Q(t, \delta_*(t)) = (\delta_*(t)^{-1} H)^{\frac{3}{8}}. \quad (4.96)$$

Then we use the inequality (3.31) again so  $Q(t, \delta_*(t)) \lesssim N_1(t, \delta_*(t)) + N_2(t, \delta_*(t)) = 2N_2(t, \delta_*(t)) \lesssim (\delta H)^{\frac{1}{2}}$ , which implies that

$$H^{-\frac{1}{7}} \lesssim \delta_*(t) \quad (4.97)$$

and again using (4.96) we obtain

$$Q(t, \delta_*(t)) \lesssim H^{\frac{3}{7}}.$$

Now let  $c_*^{-1}$  be the implicit constant in (4.97), which depends only on the constants in (3.28), thanks to (4.97) we can write for any  $t \in [c_* H^{-\frac{1}{7}}, T]$

$$Q(t, c_* H^{-\frac{1}{7}}) \lesssim H^{\frac{3}{7}} \quad (4.98)$$

Then for any such  $t$ , we can write  $t = n c_* H^{-\frac{1}{7}} + r$  with  $n \in \mathbb{N}^*$  and  $r < c_* H^{-\frac{1}{7}}$  and thanks to the last inequality we obtain

$$\begin{aligned} Q(t, t) &\leq Q(r, r) + \sum_{p=1}^n Q(p c_* H^{-\frac{1}{7}} + r, c_* H^{-\frac{1}{7}}) \\ &\lesssim (rH)^{\frac{1}{2}} + n H^{\frac{3}{7}} \\ &\lesssim c_* (rH)^{\frac{1}{2}} + n c_* H^{-\frac{1}{7}} H^{\frac{4}{7}} \\ &\lesssim c_* t^{\frac{1}{2}} H^{\frac{1}{2}} + t H^{\frac{4}{7}} \end{aligned}$$

So that finally for all  $t \in [c_* H^{-\frac{1}{7}}, T]$  we have

$$Q(t, t) \lesssim (t^{\frac{1}{2}} + t) H^{\frac{4}{7}}. \quad (4.99)$$

Lastly, if  $t \leq c_* H^{-\frac{1}{7}}$  then thanks to (4.95) and (4.97) we can write

$$Q(t, t) \lesssim (tH)^{\frac{1}{2}}. \quad (4.100)$$

This concludes the proof of proposition 3.6 because  $H > 1$ .  $\square$

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