Remarks on a numerical method for unilateral contact including friction

Christian Licht, Elaine Pratt, Michel Raous

REMARKS ON A NUMERICAL METHOD FOR UNILATERAL CONTACT INCLUDING FRICTION

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1. Introduction

We consider a Signorini problem (unilateral problem) with Coulomb friction in elasticity, under a small strain hypothesis. For proportional loading, the problem is equivalent to the static friction one introduced by Duvaut and Lions ,[72].The general problem has been numerically treated through an incremental formulation in Raous et al ,[88].

For the static problem, Duvaut,[80],and Cocu,[84], give an existence and uniqueness result for small values of the friction coefficient μ and for a regularized problem obtained using a non local definition for the contact forces. Using the discretization of the regularized problem, Jeannin ,[85],shows the convergence of the approximate solution when the discretization step h tends to zero. Using a compliance regularization, allowing a controlled penetration, Oden - Martins,[85],and Klarbring et al,[88],give an existence result and obtain uniqueness under the hypothesis that the friction coefficient is small.

For the initial problem without regularization, Necas et al,[80], give an existence result for small values of μ for a strip. This is done through the introduction of a special Sobolev space for the displacement which gives a compactness property. Jarusek ,[83], extends the result to bounded domains.

A mixed finite element approximation has been studied by Haslinger,[83], a relationship between the continuous problem and its discretization is established.

In this paper, we propose an algorithm concerning a discretized problem associated with the initial one without the regularization of Duvaut and we give a theorem of existence. For small values of μ a fixed point theorem may be used which implies the uniqueness of the solution as well as the convergence of the algorithm. However, the relationship between the continuous problem and the discretization proposed here is still an open problem.

Numerical tests on the convergence illustrate these theoretical results. Special attention is paid to the influence of the friction coefficient and to the dependence on the mesh size. The numerical convergence is better than the theoretical previsions : the conditions required by the fixed point theorem seem to be very strong.

2. The continuous problem

2.1 The mechanical problem for proportional loading

We consider an elastic solid occupying a bounded regular domain Ω of $\mathbb{R}^n$ with a regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and submitted to a volume force density $\lambda(t)\ \varphi_1(x)$ in Ω and to a surface force density $\lambda(t)\ \varphi_2(x)$ on Γ_2 (t and x denote respectively the time and the space variables and λ is an increasing positive mapping such that $\lambda(0) = 0$). The solid is clamped on one part of its boundary Γ_1. On Γ_3 the solid is in unilateral contact with friction with a rigid obstacle.

For the quasi-static problem with initial conditions equal to zero, because of the specific nature of the loading, the displacement $U(x,t)$ and the constraint $\Sigma(x,t)$ may be sought as $(U(x,t), \Sigma(x,t)) = \lambda(t)(u(x), \sigma(x))$ where the equations relating u to σ are as follows :

(1) $\sigma_{ij} = K_{ijkh}\ e_{kh}$ in Ω ,

(2) $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ in Ω ,

(3) $\sigma_{ij,j} = -\varphi_{1i}$ in Ω ,

(4) $\sigma_{ij} n_j = \varphi_{2i}$ on Γ_2 ,

(5) $u_i = 0$ on Γ_1 ,

(6) $\sigma_{ij} n_j = F_i$

(7) $u_N \leqslant 0$

(8) $F_N \leqslant 0$

(9) $u_N . F_N = 0$

(10) $|F_T| \leqslant \mu\ |F_N|$

} on Γ_3 ,

with

(11) $|F_T| < \mu\ |F_N| \Rightarrow u_T = 0$,

and

(12) $|F_T| = \mu\ |F_N| \Rightarrow -u_T$ is collinear with F_T and with the same direction ,

where :

- K_{ijkh} are the elasticity coefficients ,
- n_j are the components of the external normal to Γ ,
- $u_N = u_i n_i$ and $u_{Ti} = u_i - u_N n_i$,
- $F_N = \sigma_{ij} n_i n_j$ and $F_{Ti} = \sigma_{ij} n_j - F_N n_i$,
- μ is the friction coefficient.

2.2 A mathematical formulation

The following assumptions are set on the data :

$$(13)\qquad \begin{cases} K_{ijkh} = K_{jikh} = K_{ijhk} \in L^{\infty}(\Omega) , \\ \exists\, \alpha_0 : \quad K_{ijkh}\, \tau_{kh}\, \tau_{ij} \geqslant \alpha_0\, \tau_{ij}\, \tau_{ij} \qquad \forall\, \tau_{ij} = \tau_{ji} ; \end{cases}$$

$$(14)\qquad \mu \in C^1\left(\overline{\Gamma_3}\right) ,$$

$$\exists\, \mu_0 > 0 \quad \text{such that } \mu(x) \geqslant \mu_0 > 0 \qquad \forall\, x \in \overline{\Gamma_3} ,$$

$$(15)\qquad \varphi_1 \in L^2(\Omega)^n \quad , \quad \varphi_2 \in L^2(\Gamma_2)^n \quad .$$

We then set :

$$(16)\qquad V = \{\, v \in H^1(\Omega)^n \;/\; v = 0 \text{ on } \Gamma_1 \,\} ,$$

$$(17)\qquad K = \{\, v \in V \;/\; v_N \leqslant 0 \text{ on } \Gamma_3 \,\} ,$$

$$(18)\qquad a(u,v) = \int_{\Omega} K_{ijkh}\, e_{kh} e_{ij}\, dx ,$$

$$(19)\qquad L(v) = \int_{\Omega} \varphi_1 v\, dx + \int_{\Gamma_2} \varphi_2 v\, ds .$$

Let ω be an element of V . If $s = K e(\omega)$ is such that $\operatorname{div} s = -\varphi_1$, we can generalize (on the basis of Green's theorem) the notion of stress vector $s.n$ on Γ_3 by introducing the element $F(\omega) \in H^{-\frac{1}{2}}(\Gamma_3)^n$ defined by :

$$(20)\qquad \langle F(\omega),\, v\rangle = a(\omega,v) - L(v) \qquad \forall\, v \in V \;;\; v = 0 \text{ on } \Gamma_1 \cup \Gamma_2 ,$$

where $\langle .,. \rangle$ is the duality product between $H^{-\frac{1}{2}}(\Gamma_3)^n$ and $H^{\frac{1}{2}}(\Gamma_3)^n$. We shall maintain the same notation, i.e. $\langle .,. \rangle$, for the duality product between $H^{-\frac{1}{2}}(\Gamma_3)$ and $H^{\frac{1}{2}}(\Gamma_3)$.

Because of the regularity of the boundary Γ, we can generalize the notion of normal component $F_N(\omega)$ of the stress vector by

$$(21)\qquad \langle F_N(\omega) \,,\, \varphi \rangle = \langle F(\omega),\, \varphi n \rangle \qquad \forall\, \varphi \in H^{\frac{1}{2}}(\Gamma_3) .$$

A weak formulation of the system of local relations (1) to (12) is then the following problem (cf Necas ,[80], Jarusek ,[83]) :

$$(P)\qquad \left[\begin{array}{l} \text{find } u \in K \text{ such that} \\ \\ a(u,v-u) - \langle F_N(u),\, \mu\,(|v_T| - |u_T|)\rangle \geqslant L(v-u) \qquad \forall\, v \in K . \end{array}\right.$$

By introducing an extra assumption on μ ,Necas, Haslinger and Jarusek have shown (cf Necas et al ,[80], Jarusek,[83]), that this implicit variational inequation (P) has a solution.

2.3 Remark on the contact stresses written in terms of measures

Through classical arguments, equation (3) is shown to be verified in the distribution sense together with the limit conditions (4) (resp.(5)) on Γ_2 (resp. Γ_1) in the trace sense.

By choosing $v=u+\psi$, ψ being regular and with a normal (resp. tangential) component negative (resp. null) on Γ_3, one obtains that F_N belongs to $H^{-\frac{1}{2}}(\Gamma_3)$; this negative distribution is therefore a negative measure, evidently equal to the opposite of its absolute value. Then by choosing once again $v=u+\psi$, ψ being regular but this time with a normal component null on Γ_3 one obtains :

$$\langle F_T, \psi_T \rangle \leq \langle \mu |F_N|, |\psi_T| \rangle ,$$

which establishes that F_T (and therefore F) is a measure and that condition (10), which expresses the fact that the stress vector along Γ_3 belongs to the Coulomb cone, is satisfed in a measure sense.

3. A finite dimensional problem

To problem (P) is associated a discretised problem by the use of a finite element method .The discrete problem is shown to have a solution which, under a condition on the friction coefficient μ, is unique.

3.1 Formulation of the problem in a finite dimensionnal space U^h

We shall adopt the following notations :

- $\mathcal{C}^h$ is a regular triangulation of the domain Ω : $\bar{\Omega} = \cup_{K\in\mathcal{C}^h} K$ (cf Ciarlet [78]) ,

- Let us set :

$$X^h = \left\{ w \in C^0(\bar{\Omega}) / \; w\Big|_{K\in\mathcal{C}^h} \in P_1 \right\} ,$$

where P_1 is the set of polynomials of degree 1 ,

- $V^h = \left\{ v \in (X^h)^n \;\; / \; v = 0 \text{ on } \Gamma_1 \right\}$,
- $K^h = \left\{ v \in V^h \;\; / \; v_n \leq 0 \text{ on } \Gamma_3 \right\}$,
- $\tilde{X}^h$ is the space of traces of X^h on Γ_3 ,
- $\tilde{V}^h$ is the space of traces of V^h on Γ_3 .

We then set (w_i) (resp.(v_i)), $i = 1 \ldots M$ (resp. P) a basis of X^h (resp. V^h) and $\{\tilde{w}_i\}$ (resp. $\{\tilde{v}_i\}$), $i = 1 \ldots \tilde{M}$ (resp. $\tilde{P}$), a basis of $\tilde{X}^h$ (resp. $\tilde{V}^h$) constructed from the non-zero traces of the w_i (resp. v_i) on Γ_3 . The fact that one uses P_1 finite elements enables us to assume the w_i to be positive. To each $\tilde{v}_i$ one can associate a unique element $R\tilde{v}_i = v_i$ on Γ_3 and equal to zero on each interior vertex of $\mathcal{C}^h$. This defines a linear mapping R from $\tilde{V}^h$ to V^h.

We now define a discrete version F^h and F_N^h of the mappings F and F_N by :

$$(22) \qquad < F^h(v), \tilde{v} > = a(v, R\tilde{v}) - L(R\tilde{v}) \qquad \forall \tilde{v} \in \tilde{V}^h, \ \forall v \in V^h ,$$

$$(23) \qquad < F_N^h(v), \tilde{w} > = < F^h(v), \tilde{w}_n > \qquad \forall \tilde{w} \in \tilde{X}^h , \ \forall v \in V^h .$$

Finally, let Π^h denote the restriction to Γ_3 of the interpolation operator associated with X^h. Note that Π^h is such that :

$$(24) \qquad \exists\, c(h): \quad \left| \Pi^h(|v^h|) \right|_{L^2(\Gamma_3)} \leq c(h) \, |v^h|_{L^2(\Gamma_3)} \qquad \forall v^h \in \tilde{V}^h .$$

And once again because of the use of P_1 finite elements :

(25) $\Pi^h(v)$ is positive for all positive continuous v on Γ_3 .

A discrete formulation of problem (P) in V^h is then :

$$(26) \qquad (P^h) \quad \left[\begin{array}{c} \text{Find } u^h \in K^h \text{ such that : } \forall v \in K^h \\ a(u^h, v - u^h) - < \mu F_N^h(u^h), \Pi^h(|v_T| - |u_T^h|) > \geq L(v - u^h) . \end{array} \right.$$

Remark : In order to simplify the notation we suppose here that the friction coefficient μ is constant. The general case can be obtained by setting $\Pi^h \mu$ instead of μ.

3.2 <u>An equivalent fixed point problem</u> :

Let G be the set of positive linear mappings on $\tilde{X}^h$.
For all $g \in G$ the following problem :

$$(27) \qquad (P_g^h) \quad \left[\begin{array}{c} \text{Find } u_g^h \in K^h \text{ such that : } \forall v \in K^h \\ a(u_g^h, v - u_g^h) + < g, \Pi^h(|v_T| - |u_{gT}^h|) > \geq L(v - u_g^h) . \end{array} \right.$$

has a unique solution. This is because the use of P_1 finite elements implies that the mapping $v \mapsto < g, \left|\Pi^h(|v_T|)\right| >$ is convex and therefore problem P_g^h is a classical minimization problem of a convex functional with quadratic growth.

Let us now define the mapping T by :

$$T(g) = - \mu F_N^h(u_g^h) \qquad \text{for } g \in G .$$

We shall now make sure that T takes its values in G, that is to say that T(g) is a positive linear mapping on $\tilde{X}^h$.

$$< T(g), \tilde{w}_i > = - < \mu F_N^h(u_g^h), \tilde{w}_i >$$

$$= \mu < F^h(u_g^h), - \tilde{w}_i n>$$

$$= \mu \left[a (u_g^h, R(-\tilde{w}_i n)) - L(R(-\tilde{w}_i n)) \right] .$$

The $\tilde{w}_i$ are positive because of the use of P_1 finite elements, so that $R(-\tilde{w}_i n)$ is negative. By setting $v = u_g^h + R(-\tilde{w}_i n)$ (which is an element of K^h) in inequality (27) one obtains :

$$a(u_g^h, R(-\tilde{w}_i n)) - L(R(-\tilde{w}_i n)) \geq - <g, \Pi^h(\left| (u_g^h + R(-\tilde{w}_i n))_T \right| - \left| u_{gT}^h \right|)> .$$

And finally as $(R(\tilde{w}_i n))_T = 0$,

$$< T(g), \tilde{w}_i > \geq 0 .$$

Therefore the existence of a fixed point of T is equivalent to the existence of a solution u^h of problem P^h.

3.3 A constructive existence result in U^h

We shall begin by establishing two lemmas concerning mapping T:

Lemma 1 :There exists C(h), positive constant depending on h, such that ,

$\forall \; g_1, g_2 \in G$,

$$|T(g_2) - T(g_1)|_* \leq \mu \, C(h) \, |g_2 - g_1|_* ,$$

where $$|g|_* = \sup_{v^h \in \tilde{V}^h} \frac{|< g, v^h >|}{|v^h|_{H^{1/2}(\Gamma_3)}}$$

Proof :Taking g_1 , g_2 $\in G$ and u_1, u_2 the corresponding solutions of $P_{g_1}^h$ (resp. $P_{g_2}^h$), by adding the two inequalities (27) one obtains :

$$a (u_2 - u_1, u_2 - u_1) \leq < g_2 - g_1, \Pi^h(|u_{2T}| - |u_{1T}|) \geq .$$

Because of the ellipticity of the elasticity matrix (13) one has :

$$|u_2 - u_1|_{L^2(\Gamma_3)} \leq \frac{1}{\alpha_0} |g_2 - g_1|_* \left| \Pi^h (|u_{2T}| - |u_{1T}|) \right|_{L^2(\Gamma_3)} ,$$

and by using (25) followed by (24) :

$$(28) \qquad |u_2 - u_1|^2_{L^2(\Gamma_3)} \leq \frac{c(h)}{\alpha_0} |g_2 - g_1|_* .$$

By the definition of F_N^h :

$$< F_N^h(u_1) - F_N^h(u_2), \tilde{w}_i > = a(u_1 - u_2, R\tilde{w}_i) ,$$

so that :

$$(29) \qquad \left|F_N^h(u_1) - F_N^h(u_2)\right|_* \leq C_1 \; |u_1 - u_2|_{H^{1/2}(\Gamma_3)} .$$

The lemma may now be easily deduced from (28) and (29) because the norms $H^{1/2}$ and L^2 are equivalent on the finite dimensional space $\tilde{V}^h$.

Lemma 2 : $\exists\, M > 0$ such that $|T(g)|_* \leq M \quad \forall\, g \in G$.

Proof : By setting v=0 in inequality (27) one obtains :

$$(30) \qquad \left|u_g^h\right|_{H^1} \leq C_1 ,$$

and because :

$$|T(g)|_* = \mu \sup_{v^h \in \tilde{V}^h} \frac{|< F_N^h(u_g^h), v^h >|}{|v^h|_{H^{1/2}(\Gamma_3)}}$$

$$= \mu \sup_{v^h \in \tilde{V}^h} \frac{\left|a(u_g^h, Rv^h n) - L(Rv^h n)\right|}{|v^h|_{H^{1/2}(\Gamma_3)}} ,$$

using (30) one concludes.

Theorem 1 : Problem P^h has a solution .

Proof : Let H be the intersection of G with the ball of centre 0 and radius M. H is a compact convex of the dual of $\tilde{V}^h$. Because of lemma 2, $T(H) \subset H$ and T is continuous by Lemma 1.

The existence of a solution u_g^h of P^h is deduced by applying Brower's theorem to the mapping T. Lemma 1 induces the following proposition.

Proposition : If $\mu < \frac{1}{C(h)}$, the mapping T is a contraction and there exists a unique fixed point g^* of T obtained as the limit of the following sequence :

$$\begin{cases} - \ g^0 \ \text{given} \ , \\ - \ g^{n+1} = T(g^n) \ . \end{cases}$$

Theorem 2 : If $\mu < \frac{1}{C(h)}$ problem P^h has a unique solution u_g^h .

4. Algorithm

We determine the fixed point of the application T introduced in section 3 .
For each given value of g, we solve problem P_g^h to get the solution u_g^h. Because of the symmetry of the bilinear form a(u,v), the variational inequality (27) is equivalent to the minimization of the functionnal $J^h(v)$ on the convex K^h with ,

$$J^h(v) = \frac{1}{2} a(v,v) - L(v) + < g, \Pi^h(|v_T|) > .$$

As before we introduce a mesh with P_1 triangles with linear interpolation for the displacements.

Denoting by I_N, I_T the set of the indices concerning the normal and tangential components of the contact displacements respectively, the convex K^h is written as :

$$K^h = \prod_{i=1}^{2N} K_i \ , \qquad \text{with} \quad K_i = \begin{cases} \mathbb{R}^- \ \text{if} \ i \in I_N \ , \\ \mathbb{R} \ \text{if} \ i \notin I_N \ . \end{cases}$$

N being the total number of nodes.

Note : this is due to the positiveness of the shape functions for P_1 elements. For higher order finite elements, K^h would be much more complicated.

We must find the vector $\bar{u}_g \in K^h$ minimizing in K^h the functional:

$$J(\bar{v}) = \frac{1}{2} \bar{v}^T A \bar{v} - \bar{F}^T . \bar{v} + \bar{G} . \left| \bar{v}_T \right| ,$$

where
- $\bar{v}$ is the vector of nodal displacements ,
- A is the finite element matrix of general term $a_{ij} = a(w_i, w_j)$ and where w_i are the shape functions ,
- $\bar{F}$ is the loading vector: $F_i = L(v_i)$,

- $\bar{G}$ is the sliding limit vector. We underline the essential role of the restriction mapping Π^h :

$$< g, \Pi^h \left(\left| \sum_i v_{Ti}\ w_i \right| \right) > - < g, \sum_i w_i\ |v_{Ti}| > \\ - \sum_i < g, w_i > |v_{Ti}| ,$$

$$G_i = < g, w_i > .$$

On the basis of the successive overrelaxation method(S.S.O.R.) we introduce the following specific algorithm which takes into account both the unilateral condition and the undifferentiable term. The minimization on one component v_i after the other leads to considering 3 cases :

If $i \notin I_N$ and $i \notin I_T$, we have the classical minimization of a quadratic functional on $\mathbb{R}$. The overrelaxation iteration is written :

$$u_i^{n+\frac{1}{2}} = \frac{1}{a_{ii}} \left(F_i - \sum_{j=1}^{i-1} a_{ij}\ u_j^{n+1} - \sum_{j=i+1}^{N} a_{ij}\ u_j^n \right) ,$$

$$u_i^{n+1} = (1-\omega)\ u_i^n + \omega\ u_i^{n+\frac{1}{2}} \text{ (ω is the overrelaxation coefficient)} .$$

We underline that the term $\bar{G} \left| \bar{v}_T \right|$ contains only components relative to indices belonging to I_T.

If $i \in I_N$, we have the minimization of the same functional on the convex $K_i = \mathbb{R}^-$, u_i^{n+1} is projected onto $\mathbb{R}^-$.

$$u_i^{n+1} = \text{Proj}_{\mathbb{R}^-} \left((1-\omega)\ u_i^n + \omega\ u^{n+\frac{1}{2}}_i \right) .$$

If $i \in I_T$, we have the minimization on $\mathbb{R}$ of the functional including the absolute value which is simply treated by considering the two possibilities for u_i^{n+1} to be either positive or negative.

In Raous et al,[88], several acceleration procedures are given such as a diagonal process on the determination of g, a condensation of the problem to the contact variables alone, and a special matrix storage. So as not to alterate the study of the convergence of the fixed point method presented in the next paragraph, we will not use the diagonal process wich consists in partial resolution for the first determinations of g.

We use an optimum relaxation coefficient determined through a dichotomic research.

5. Where results of section 3 are tested

To begin with we note that, applied to models connected to physical problems, a good behaviour of the algorithm and of its convergence has been observed.

We will focus here on how the convergence of the fixed point iteration is influenced by the value of the friction coefficient (even for unrealistic values), by mesh discretization, and by the choice of the initial condition g^0 .

The discussion is not exhaustive because it is limited to two examples and to relatively few meshes; however it suggests a few comments on the behaviour of the algorithm with regard to the results of section 3.

5.1 Description of the examples

Example A : contact of a long bar with a plane surface (plane strain hypothesis).

A section of the bar is represented on Fig. 1 . A force density $f = -5$ daN/mm^2 is applied on the side GE, and a force density $F = 10$ daN/mm^2 is applied on AG. On the side AD, unilateral conditions with Coulomb friction are assumed. On the side ED, boundary conditions are imposed to assume the symmetry ($u_x = 0$) and to fix the point D ($u_x = u_y = 0$). Plane strain elasticity is assumed with a Young modulus E equal to 13 000 daN/mm^2 and a Poisson coefficient ν equal to 0.2.

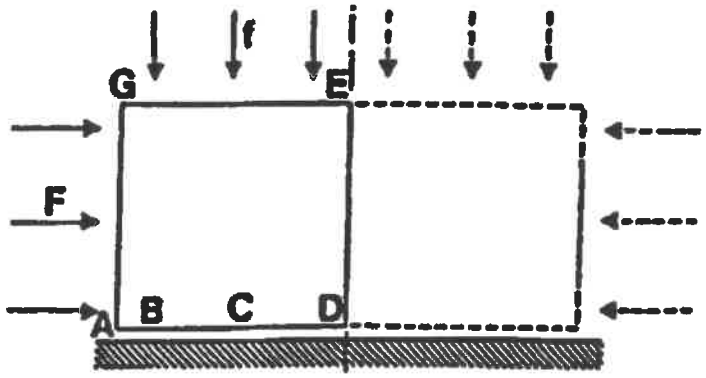

Fig. 1 : geometry of example A .

Details of the solution for different values of μ are given in Raous et al [88].

The test will be done on five different meshes . On Fig. 2, the mesh A1 has a local refinement on the contact boundary :N= 230 and NC= 33. N denotes the total number of nodes and NC the number of contact nodes.On Figs. 3 to 6, four uniform meshes are given denoted by A2, A3, A4, A5.

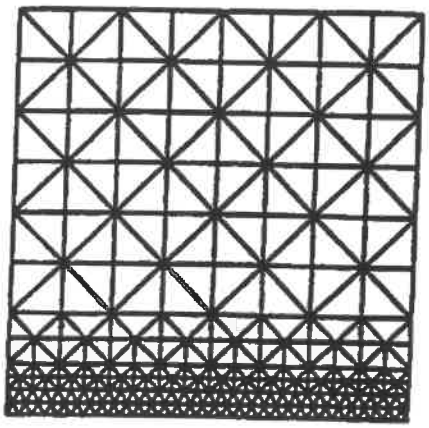

Fig. 2 : mesh A1 of example A ,
N = 230 , NC = 33 .

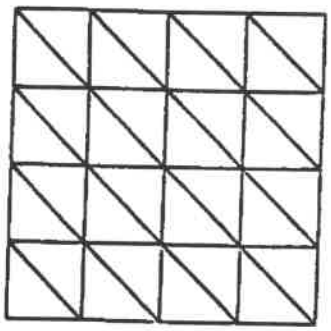

Fig. 3 : mesh A2 of example A ,
N = 25 , NC = 5 .

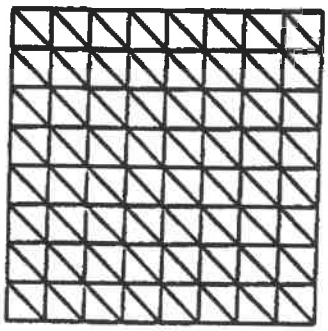

Fig. 4 : mesh A3 of example A,
N = 81 , NC = 9 .

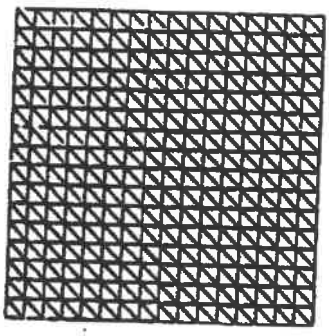

Fig. 5 : mesh A4 of example A ,
N = 289 , NC = 17 .

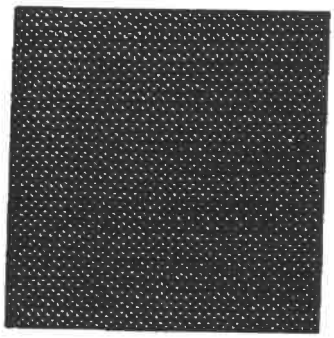

Fig. 6 : mesh A5 of example A,
N = 1089 , NC = 33 .

Example B : a dovetail assembling .

The geometry is given in Fig. 7. This is a two body contact problem with an oblique contact zone. We show in Raous et al ,[88], how to generalize the formulation and the algorithm for such a problem. In the same paper, the complete solution is given. Plane stress hypothesis is assumed. The meshes given in Fig. 8 are used (they have respectively 19 and 9 couples of contact nodes).

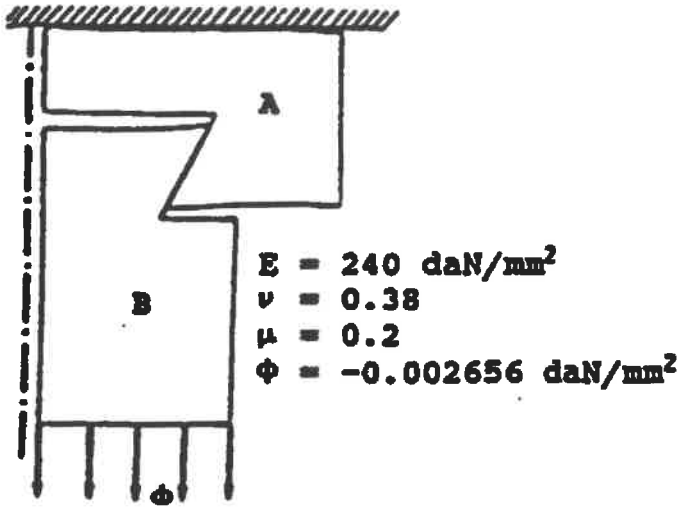

Fig. 7 : example B .
Dovetail assembling (displacements amplified twenty times).

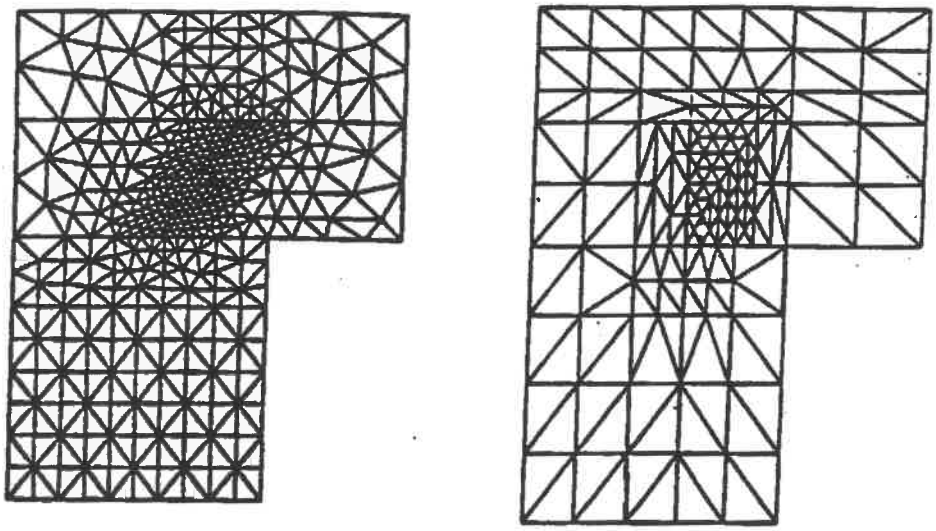

Fig. 8 :meshes B1 (N=384 NC=38) and B2 (N=148 NC=18) of example B .

5.2 The Influence of the magnitude of μ

In the next two tables we give the number of iterations on g needed for the fixed point process to converge for different values of μ and for different meshes. Some values of μ are very large and have no mechanical sense : the point is to check the apparent restriction of theorem 2 on the magnitude of μ.

Table I concerns example A and table II concerns example B.

The indication "no convergence" means that the tests of convergence were not satisfied after a given number of iterations depending on the size of the problem.

The test of convergence on g is :

$$\frac{|g^{k+1} - g^k|_\infty}{|g^k|_\infty} < 10^{-3} \quad , \qquad \text{with } |g|_\infty = \sup_{i \in I_T} |g_i| \ .$$

The test of convergence for the relaxation procedure is :

$$\frac{\|u^{k+1} - u^{k}\|_1}{\|u^{k}\|_1} < .5\ 10^{-5} \quad , \quad \text{with} \quad \|u^k\|_1 = \sum_{i=1}^{2N} \left|u_i^k\right| .$$

mesh / value of μ	A1	A2	A3	A4	A5
0.01	4	4	4	4	4
0.1	6	5	5	6	6
0.2	7	6	6	6	7
0.4	7	7	7	8	8
0.6	8	8	8	8	9
1	10	9	9	9	10
2	14	11	11	12	15
5	no conv	5	no conv	no conv	no conv
50	8	5	5	no conv	no conv
100	8	5	5	no conv	no conv

Table I: number of fixed point iterations for different meshes of example A.

mesh / value of μ	B1	B2
0.01	4	4
0.1	6	7
0.2	9	9
0.4	21	22
0.6	no conv	no conv
1	no conv	no conv
2	no conv	no conv
5	14	no conv
10	22	no conv
50	6	no conv
100	6	no conv

Table II: number of fixed point iterations for different meshes of example B.

Comments :

* In the range of physical admissible values of the friction coefficient, the solution is obtained in less than 10 iterations on g and the influence of μ is weak for this range of values. When the problem is close to a frictionless one the number of iterations to get convergence is small (4 iterations), in other cases it is about twice as much (between 6 and 10 iterations).

* For large values of μ (between 1 and 100), convergence is not obtained for special ranges which are not especially the largest values of μ. By checking the values of g in these cases, we observe that g oscillates between two neighbouring values: the difference resides in one or two nodes located at the boundary between the separated zone and the sliding one. Theses nodes are oscillating between contact with small contact force (u_N=0, F_N small) and a small separation (u_N small, F_N=0). The first state introduces a small tangential force whereas the second one gives a zero tangential force. The difference ($g^{k+1} - g^k$) does not succeed in passing the convergence test(10^{-3}), and the algorithm does not stop. This behaviour seems to be associated to the difficulty to compute the limit case of contact without transmission of normal forces ($u_N = 0$, $F_N = 0$). This has been observed also on an example presented in Raous et al [88] concerning a metal forming process with a small friction coefficient ($\mu = 0.2$).

* The mesh seems to have some influence because cases with large fricion coefficients (μ = 100 for example) converge for some grids and not for others. The limit between the separated zone and the clamped zone (for μ = 100 there is no sliding zone) introduces a singularity of the same kind as that of a crack in an elastic medium and that we are using values of g computed in this part.

We have to underline that the presence of a sliding zone introduces a tangential force in the neighbourhood of the clamped part and gives a regularization of the previous singularity. The solution is smoother for small friction coefficient and the convergence is the same for the different meshes.

5.3 Influence of the choice of the initial condition g^0

As presented in table III, the choice of the initial condition g^0 seems to have no influence on the number of iterations on g. TableIII concerns example A with μ = 1 treated on the mesh A1, for which a good convergence has been observed.

initial condition g^0	number of iterations on g	Total number of relaxation iterations
(0,0,... ,0)	10	207
(1,1,... ,1)	10	205
(10,10,... ,10)	9	172
(100,100,... ,100)	9	172
(solution computed for μ = 0.6)	9	172
(small perturbation of the right solution)	9	171

Table III : influence of the initial condition g^0 on the convergence for example A with μ = 1 treated on the mesh A1 (N = 230, NC = 33).

5-4 Influence of the mesh on the convergence and on the solution

Fig. 9 gives a representation of the contact boundary of the solution of example A for the different meshes. A good coherence of the solutions is observed. One observes numerically the convergence of the solution as h decreases, for the contact aera.

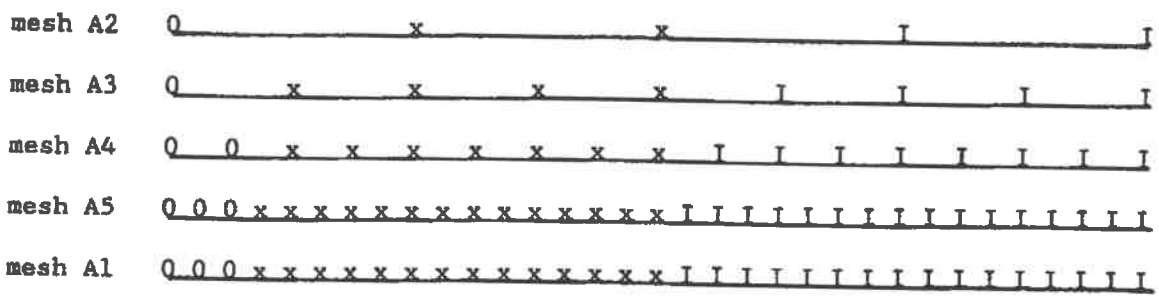

Fig. 9 : representation of the contact boundary for example A ($\mu = 1$) for the different meshes :
0 separate node
x sliding node
I stick node .

6. Conclusions

We submit in this article a dicretization by classical finite elements P1 of the unilateral contact problem with Coulomb friction. Existence, as well as uniquenesss when the value of the friction coefficent μ is small, has been obtained. A fixed point theorem is used to establish unicity for μ small, therefore convergence of the successive approximation algorithm follows. The use of a classical finite element method implies that the implementation is simple (at each iteration a problem of minimization of a convex functional is solved) and therefore allows for many numerical tests. One observes in particular that for realistic values of the friction coefficent convergence is obtained in less than ten iterations. For these same values of μ the discrete solution behaves very well with regard to the mesh refinement, which seems to indicate that the solution of the discrete problem has a limit when the mesh step tends to zero .

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