



On two variational inequalities arising from a periodic viscoelastic unilateral problem

M Raous

► To cite this version:

M Raous. On two variational inequalities arising from a periodic viscoelastic unilateral problem. R.W. Cottle; F. Gianessi; J.-L. Lions. Variational inequalities and complementarity problems - Theory and Applications, Wiley, 1980, 0-471-27610-3. hal-03308110

HAL Id: hal-03308110

<https://hal.science/hal-03308110>

Submitted on 29 Jul 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Chapter 19

On Two Variational Inequalities Arising from a Periodic Viscoelastic Unilateral Problem

M. Raous

1. INTRODUCTION

We study the periodic problem connected with the behaviour of a viscoelastic cracked solid submitted to periodic loads. The unilateral conditions on the edges of the crack lead to a complementarity problem coupled with a differential equation for time. We solve the problem in the case of Maxwell behaviour equations. In the linear case, the Maxwell law is among the most difficult because of the presence of a secular term of displacements and strains.

We choose a model where coefficients are functions of time (often, via the temperature, but sometimes also directly (aging concrete [1], for example), and of space (non-homogeneous material). We show how this problem leads to two coupled variational inequalities: the first one is a stationary variational inequality for the secular term; the second one is a variational inequality coupled with a differential equation for the periodic parts of the solutions.

Here, we solve these two problems with suitable numerical methods: a finite element method for space discretization, a Runge-Kutta method on time, an overrelaxation method with projection to solve the inequalities. A convenient combination of these discretization methods is realized and a particular solving method proposed. The associated algorithms are given and applied to an example: we give numerical results and discuss them from a mechanical point of view.

2. THE MECHANICAL PROBLEM

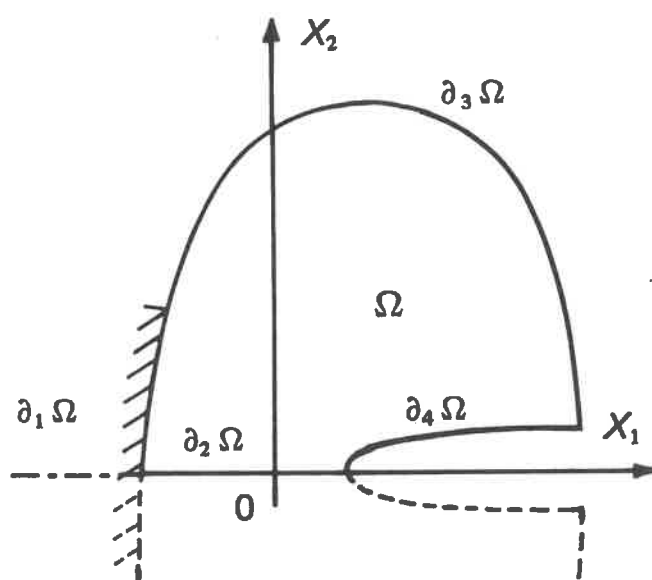
This work takes place in the study of the adaptation of a cracked solid under cyclic thermal fatigue: we try to show from a theoretical point of

view that a thermic crack is not brittle. It will be possible with a suitable choice of damage law (generally empirical) to calculate, from the stress solution of the problem with a given length of crack, the progression velocity of the crack.

Here, we calculate the stress and displacement fields for the periodic problem with a given crack configuration. The study we present here is related to the oligocyclic fatigue in cracked turbine blades. The period T will be, for example, the period of the plane flight.

Since the hypothesis of contact without friction is unrealistic in physics, the crack is supposed to lie in a plane of symmetry, which allows us to formulate conditions of contact mathematically identical to those of a contact without friction. The periodic problem leads to a weakly coupled system of variational inequalities equivalent to two complementary systems.

Let us consider a plane medium for which the classical hypotheses of *infinitesimal displacement* and *plane stresses* are assumed. For these reasons,



we suppose the problem to be symmetrical with respect to the axis Ox_1 , so that we need only consider the part located in the positive x_2 half-plane. We denote by Ω the open set limiting the medium, $\partial\Omega$ being its boundary.

All the fields considered, given or unknown, will be functions of time t and of position $x = (x_1, x_2)$: for example, the condition

$$\forall (t, x) \in [0, T] \times \partial_1 \Omega \quad u(t, x) = u^0(t, x)$$

will be written $u = u^0$ on $\partial_1 \Omega$.

The unknown fields of displacements, stresses and strains will be respectively denoted u , s , and e .

A prescribed strain field e^0 is given on Ω . Here, it is a dilation field:

$$(1) \quad e^0 = -\chi\theta$$

where θ is a temperature field and χ a field of thermal dilation symmetrical tensor. We set

$$(2) \quad e = Du + e^0$$

that means

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \chi_{ij}\theta$$

with tensor notations.

As for now, we shall write the constitutive law under the symbolic form

$$(3) \quad s = \mathcal{K}(e)$$

which represents a functional relation between the stress and strain fields will be specified in section 3.

Equilibrium equations are written

$$(4) \quad s_{ij,j} = -\phi_{1i}$$

where ϕ_1 denotes a given field of forces per unit area on $[0, T] \times \Omega$.

The boundary is divided into four disjoint open arcs. We define

$$(5) \quad u = 0 \quad \text{on } \partial_1 \Omega$$

The symmetry hypothesis is written in the form

$$(6) \quad u_2 = 0 \quad s_{12} = 0 \quad \text{on } \partial_2 \Omega$$

On $\partial_3 \Omega$, we give a field ϕ_2 of forces per unit length:

$$(7) \quad s_{ij} n_j = \phi_{2i}$$

The arc $\partial_4 \Omega$ is the upper edge of the crack: it is submitted to contact forces from the lower edge (they are vertical due to the symmetry). These

forces are represented by a field of forces $(0, g)$ per unit length on $[0, T] \times \partial_4 \Omega$. Then, the equilibrium equations on $\partial_4 \Omega$ are written:

$$(8) \quad s_{12} = 0 \quad -s_{22} = g \quad \text{on } \partial_4 \Omega$$

Let us note that g is unknown.

Now, we write the following unilateral constraints that will lead to the complementarity problem:

$$(9) \quad u_N \geq 0$$

$$(10) \quad g \geq 0 \quad \text{on } \partial_4 \Omega$$

$$(11) \quad u_N \cdot g = 0$$

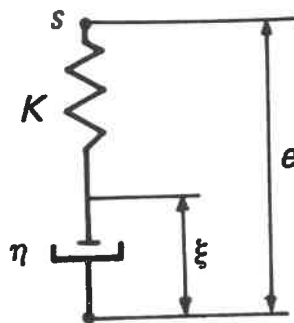
where u_N is the second component of the trace of u under $\partial_4 \Omega$.

The condition (9) means that the edges of the crack can be parted but cannot interpenetrate each other; g is a compression force.

Finally, the condition (11) means that the contact force g vanishes when the edges are parted.

3. THE CONSTITUTIVE LAW

We choose for (3) a Maxwell behaviour law that allows one to take into account the flow phenomenon. We write it with the strain parameter ξ (see [4]):



$$(12) \quad \begin{aligned} s(t, x) &= \eta(t, x) \dot{\xi}(t, x) \\ s(t, x) &= K(t, x) [e(t, x) - \xi(t, x)] \end{aligned}$$

η and K are mappings defined on $[0, T] \times \Omega$ with values in the space of 3×3 symmetrical matrices: K is the stiffness matrix, and η is the viscosity matrix.

We shall look for a periodic stress solution. We see immediately from equation (12) that ξ will be periodic, but in general with a mean value different from zero. Therefore, the strains ξ and e take the form:

$$(13) \quad \xi(t, x) = \alpha(x)t + r(t, x)$$

$$(14) \quad e(t, x) = \alpha(x)t + p(t, x)$$

where $\alpha(x)$ is a strain secular term.

Consequently, the displacement will have also the form:

$$(15) \quad u(t, x) = \beta(x)t + q(t, x)$$

where $\beta(x)$ is a displacement secular term. These secular terms $\alpha(x)$ and $\beta(x)$ mean that, under constant loads, the strains and the displacements are linear with time (flow phenomenon).

The constitutive law (12) shows that the strains will be defined with an arbitrary additive constant on time. Then, the solutions will not be unique. Consequently, we may choose $r(t, x)$, $p(t, x)$ and $q(t, x)$ as periodic functions of time with mean value zero.

4. FUNCTIONAL SCHEME

The mathematical approach of that Signorini problem (see unilateral conditions (9), (10), (11)) are treated in [2, 3] by duality and virtual works techniques. We recall here the functional scheme.

The situation is summed up in the following:

$$\begin{array}{ccccccc} u_N \in L(C) & L(C) \subset \Gamma & (\cdot, \cdot) & G & g \in G \\ & L \uparrow & & \downarrow {}^t L & \\ u \in C & C \subset U & \langle \cdot, \cdot \rangle & \Phi \subset U' & \phi \in \Phi \\ & D \downarrow & & \uparrow {}^t D & \\ e, \xi \in E & E & \langle \cdot, \cdot \rangle & S & s \in S \end{array}$$

with $U = H_0^1(\Omega)^2$ space of displacement defined by:

$$U = \{u = (u_1, u_2); u \in H^1(\Omega); u = 0 \text{ on } \partial_1 \Omega; u_2 = 0 \text{ on } \partial_2 \Omega\}$$

C is the cone defined as the set of displacements which have the second component of their trace on $\partial_4 \Omega$ positive or zero:

Variational Inequalities and Complementarity Problems

$$C = \{u \in U : Lu \geq 0\}$$

The mapping L associates to a displacement u the second component of its trace on $\partial_4 \Omega$. We set $\Gamma = L(U)$. U' is the topological dual of U . A classical subset of U' is a set isomorphic to $\Phi = L^2(\Omega)^2 \times L^2(\partial_3 \Omega)^2$. Φ is the load space.

$E = L^2(\Omega)^3$ is the space of symmetrical strain tensors. $S = L^2(\Omega)^3$ is the space of symmetrical stress tensors. D is the strain operator defined by relation (2): $e = Du + e^0$. tD transposed of D for the duality, is the equilibrium mapping. It is defined by the relations (4), (7), and (8):

$${}^tDs = \phi + g$$

means

$$\begin{aligned} -\operatorname{div} s &= \phi_1 && \text{on } \Omega \\ s \cdot n &= \phi_2 && \text{on } \partial_3 \Omega \\ s \cdot n &= g && \text{on } \partial_4 \Omega \end{aligned}$$

(n is the normal vector to the considered arc).

The duality bilinear forms are:

$$\begin{aligned} \langle\langle u, \phi \rangle\rangle &= \int_{\Omega} u(x) \phi_1(x) dx + \int_{\partial_2 \Omega} \gamma u(x) \phi_2(x) dx' \\ \langle e, s \rangle &= \int_{\Omega} e(x) s(x) dx \end{aligned}$$

(γu denotes the trace of u on the part $\partial_2 \Omega$).

From a mechanical point of view, $\langle\langle u, \phi \rangle\rangle$ is the work of the loads ϕ in the displacement u , and $-\langle e, s \rangle$ is the work of the stress s in the strain e .

5. THE EQUATIONS

Now, we set the following asymptotic problem because of the presence of the secular terms.

5.1. Problem P1

Let the given data satisfy: $\phi(t)$ is T periodic and $e^0(t)$ is T periodic. Find the time t_0 , the stress $s(t)$ (T periodic), the displacement $u(t) = \beta \cdot t + q(t)$

where

$$\bar{g} = \frac{1}{T} \int_0^T g(t) dt$$

From now on, we shall always denote the mean value of a T -periodic function f :

$$\bar{f} = \frac{1}{T} \int_0^T f(t) dt$$

5.3. Parting the problem P2 into two coupled problems

We introduce the restrictive hypothesis that the viscosity does not depend on the time t .

Note In fact, we can solve the problems where the viscosity may be written (see [3]):

$$\eta(t, x) = \eta_0(t) \eta(x)$$

where $\eta_0(t)$ is a smooth, scalar, positive, T -periodic function.

Then, using the first equations of the problem P1, we can write:

$$s = \eta \dot{\xi} \quad \text{with} \quad \dot{\xi} = D\beta + \dot{r}$$

and then

$$\bar{s} = \eta D\beta$$

It follows that the problem P2 leads to two coupled complementarity problems; we give here, immediately, the variational form of these problems.

Problem P3

Let $\bar{\phi}$ be given. Find $\beta \in C$ such that

$$(i) \quad \forall v \in C \quad \langle\langle v - \beta, {}^t D\eta \ D\beta - \bar{\phi} \rangle\rangle \geq 0$$

This is a classical elliptic problem of Signorini which is stationary. The operator ${}^t D\eta D$ is similar to the elasticity operator and has also all the good properties (coercivity, symmetry, . . .). There will be no difficulties to solve it. The solution gives us the secular term of displacement β .

It can be shown that, in the case where only the stiffness matrix K is a function of time, the secular terms are zero if the mean values of the loads are zero.

with $q(t)$ T periodic, the strain $e(t) = \alpha t + p(t)$ with $p(t)$ T periodic, the strain $\xi(t) = \alpha t + r(t)$ with $r(t)$ T periodic, and the force $g(t)$, such that

$$\begin{aligned} e &= Du + e^0 \\ {}^tDs &= \phi + {}^tLg \\ s &= \eta \dot{\xi} & \forall t \in [n+1), n \in N \\ s &= K(e - \xi) \\ u_N &= Lu \\ u_N &\geq 0 \\ g &\geq 0 & \forall t > t_0 \\ u_N \cdot g &= 0 \end{aligned}$$

Comment Indeed, at the points $x \in \partial_4 \Omega$ of the crack where β is not zero, the unilateral conditions are satisfied only after a time t_0 .

This is a periodic problem of Signorini type coupled with a differential equation. We shall transform this problem into two coupled complementarity problems: the first one on the secular terms, and the other one on the periodic parts of the solution.

5.2. Parting the unilateral conditions

Theorem 1

The problem P1 is equivalent to problem P2.

Problem P2

Let $\phi(t)$ and $e^0(t)$ be given as in P1. Find $s(t)$, $u(t)$, $e(t)$, $\xi(t)$, $g(t)$ as in P1 such that:

$$\begin{aligned} e &= Du + e^0 \\ {}^tDs &= \phi + {}^tLg \\ s &= \eta \dot{\xi} \\ s &= K(e - \xi) & \forall t \in [0, T] \\ \beta_N &= L\beta \\ q_N &= Lq \\ \beta_N &\geq 0 & q_N &\geq 0 \\ g &\geq 0 & \text{and} & g &\geq 0 \\ \beta_N \cdot g &= 0 & q_N \cdot g &= 0 \end{aligned}$$

Problem P4

Let $\phi(t)$ and $e^0(t)$ be given as in P1. Find $q(t) \in C$, and $r(t)$ as in P1, such that

$$(j) \quad \forall v \in C \quad \langle\langle v - q(t), {}^tDK(t) Dq(t) + {}^tDK(t) [e^0(t) - r(t)] - \phi(t) \rangle\rangle \geq 0$$

$$(jj) \quad \eta \dot{r}(t) = -K(t)r(t) + K(t) [Dq(t) + e^0(t)] - \eta D\beta$$

with $s(t) = \eta D\beta + \eta \dot{r}$ T -periodic.

The connection between the variational inequation (j) and the differential equation (jj) is realized by the variables $r(t)$ and $q(t)$.

6. DISCRETIZATION AND NUMERICAL TREATMENT

6.1. Problem P5

This problem is a classical elliptic problem connected with a stationary variational inequality. We use a finite element method to discretize this problem. We then replace the initial problem with the following approximate problem:

Problem P5

Let $\bar{\phi}$ be given. Find

$$\beta^h = \sum_i \beta_i^h \omega_i^h$$

such that:

$$\forall v^h \in C^h \quad \langle\langle v^h - \beta^h, {}^tD\eta D\beta^h - \bar{\phi} \rangle\rangle \geq 0$$

C^h is a convex subset of U^h which is conveniently assumed to realize an approximation of the given cone C . U^h is the finite-dimensional space depending on a parameter h connected with the mesh.

Let ω_i^h be a basis in U^h ($i = 1, \dots, n$). The *approximate problem* P5 leads to a *discrete problem* written in the n -dimensional vector space \mathbb{R}^n (n being the dimension of U^h). We thus exhibit an $n \times n$ matrix N^h given by:

$$N_{ij}^h = \langle D\omega_i, \eta D\omega_j \rangle$$

It is completely similar to the classical elasticity matrix because the viscosity matrix η has the same properties as the stiffness matrix K .

We solve that first classical problem by an overrelaxation method with projection on the cone C^h at each step [6, 7]. We are only confronted with

the usual numerical difficulties inherent to this problem.

6.2. Problem P6

This problem is more difficult because of the coupling between the variational inequality and the differential equation. Moreover, it is a periodic problem.

A result about the asymptotic stability of the Cauchy problem solution, i.e. of the associated problem with initial condition (instead of the periodic condition) can be found in [2, 3]. Consequently, we have an *explicit result* about the asymptotic convergence of the Cauchy solution towards the periodic solution. That explicit result (see [5]) enables us to estimate *a priori* the length of time of the Cauchy solution to be calculated to have a 'sufficiently' periodic solution. For the examples treated, this time was always less than $10 \times T$ (where T is the period).

Remark 1 For the cases where that convergence would be very slow, we shall consider a direct treatment with a Fourier decomposition on time. The formulation leads to a very large system which is sparse and banded by block. We have treated a one-dimensional example with this method for a case without constraints. The numerical difficulty comes from the very large dimension of that matrix: we shall prefer to use the first method each time it will be reasonable for the computation time.

Remark 2 On the other hand, we have tried to use the Poincaré mapping (or mapping of translation) which is classical for the ordinary periodic differential equation in a Hilbert space. For this, we introduce an integral operator of which the discretization is alas not numerically 'reasonable'. So, that trial fails. However, that formulation enables us to exhibit the asymptotic stability of the Cauchy solution and also the stability condition (because of the use of an explicit integration method) and to give the error estimation (in the case without constraint, for instance).

So, we shall solve the Cauchy problem associated with problem P4, where the condition (find $s = \eta D\beta + \eta \dot{r}$ periodic) is replaced by an initial condition ($r(0) = r_0$ given). And then, we calculate the Cauchy solution until $\|s(nT) - s((n+1)T)\|_s < \varepsilon$.

The approximate problem. On space, we use a *finite element* method: we write the problem P4 in the finite-dimensional space U^h defined above. To integrate the coupled differential equation, we use a *Runge-Kutta* method.

6.2.1 Semi-discretization on time

After several comparative trials with an equivalent number of operations, on several Runge-Kutta methods (order 1: Euler; order 2: Heunn; order 4) and also on a semi-implicit method (prediction-correction), we choose the

Heunn method with a fitting discretization on time. This method is preferable to the order-4 Runge-Kutta method because of the variable discretization on time. Indeed, with an equivalent number of operations, the discretization is more adjusted in this case.

We set

$$f(r, q, t) = -\eta^{-1} K(t) r(t) + \eta^{-1} K(t) q(t) + \eta^{-1} K(t) e^0(t) - D\beta$$

The Heunn relation written on the equation (jj) leads to:

$$(11) \quad r_{k+1} = r_k + (1 - \rho) \Delta t_k f_k + \rho \Delta t_k f_{k'}$$

with $\rho = 3/4$, $f_k = f(r_k, q_k, t_k)$

$$f_{k'} = f(r_k + \frac{\Delta t_k}{2\rho} f_k, q_{k'}, t_{k'}) \quad \text{and} \quad t_{k'} = t_k + \frac{\Delta t_k}{2\rho}$$

$q_{k'}$, is a solution of the equation (j) with $t = t_{k'}$, and

$$r_{k'} = r_k + \frac{\Delta t_k}{2\rho} f_k$$

Let M be the number of samples of the discretization on time ($k = 1, \dots, M$)

6.2.2. Spatial discretization

A finite element scheme applied to the equation (j) leads to relation (2) similar to the problem P5 (with regard to the notations)

$$\forall t_k \quad k = 1, \dots, M$$

$$q_k^h = \sum_{i=1}^n q_{ki}^h \omega_i^h$$

$$(1) \quad \forall v^h \in C^h \quad \langle \langle v^h - q_k^h, A_k q_k^h - B_k r_k^h + F_k^h \rangle \rangle \geq 0$$

where A_k is the elasticity operator ${}^tDK_k D$, B_k is the operator tDK_k , F_k^h is the term connected with the loads ϕ_k and the prescribed strain e_k^0 .

We associate to the problem P4 the following approximate problem P6.

6.2.3. Problem P6

Let $\phi(t, x)$ and $e^0(t, x)$ be given T periodic. Find $q_k^h(x) \in C^h$ with

$$q_k^h(x) = \sum_{i=1}^n q_{ki}^h \omega_i^h(x)$$

with $k = 1, \dots, M$ such that:

$$(1) \quad \forall v^h \in C^h \quad \langle v^h - q_k^h, A_k q_k^h - B_k r_k^h + F_k^h \rangle \geq 0$$

$$(11) \quad r_{k+1}^h = r_k^h + (1 - \rho) \Delta t_k f_k + \rho \Delta t_k f_{k'}$$

where r_1^h is given, remembering that h is the symbol for space discretization and k is the symbol for discretization on time.

The problem P6 is the *approximate problem* associated with problem P4: we do not give here the explicit form of the *discrete problem* in the n -dimensional vector space \mathbb{R}^n that we solve in fact, so as not to complicate this paper with heavy notations. The exhibited $n \times n$ matrix associated with the operator A is the classical elasticity matrix:

$$\tilde{A}_{ij} = \langle D\omega_i, KD\omega_j \rangle$$

Therefore, the *algorithm* is:

- we choose an initial condition r_1^h ;
- we solve the variational inequality problem (1) by an overrelaxation method with projection on the cone C^h at each step;
- as r_k^h and q_k^h are now known, we use the explicit relation (11) to calculate r_{k+1}^h (in fact, we must solve the variational inequality (1) m times, if m is the order of the used Runge-Kutta method: here $m = 2$, we must solve the inequality (1) on time again to calculate $q_{k'}$, and then $f_{k'}$);
- and so on, until the periodicity of the stress is 'sufficient' i.e.

$$\|s_{(a+1)M}^h - s_{aM}^h\|_\infty < \varepsilon$$

Technical remark We have to solve several times the variational inequality (1); the matrix A remains constant if K does not depend of time, or changes little between two successive times t_k and t_{k+1} . Thus, at each step we have a good initial condition for the overrelaxation method with the result of the preceding calculation. We use iterative methods for this reason.

7. EXAMPLE

After several tests, the application of the method to a first simplified example gives the following results.

It is an example where the secular term is zero. Indeed, we have solved several problems on secular terms (solution $\beta > 0$, or $\beta = 0, \dots$): there is no specific difficulty. So, we present here an example where $\beta = 0$.

We study a square domain with a 75 nodes mesh (see Figure 1) with the plane stress hypothesis (bidimensional problem). This calculation is a prelimi-

nary approach to an example that should allow a comparison with an experimental study on a rectangular symmetric piece of IN 100 (turbine blade) at 1100°C ; this experience will be performed by ONERA.

Here, the temperature is then constant and in consequence all the parameters are constants too. Period $T = 600$ s. Relaxation time $\tau = 60$ s. Young's modulus $E = 13000$ kg/mm². Poisson's coefficient $\nu = 0.2$. On AB, boundary conditions $u_2 = 0$ and, for the node B, $u_1 = 0$ $u_2 = 0$. On BC, the unilateral conditions. On ED, we give the following loads

$$\phi_{2x_1}(t, x) = 0$$

$$\phi_{2x_2}(t, x) = F_0 \sin \frac{2\pi}{T} t \quad \text{with} \quad F_0 = 10 \text{ kg/mm}^2$$

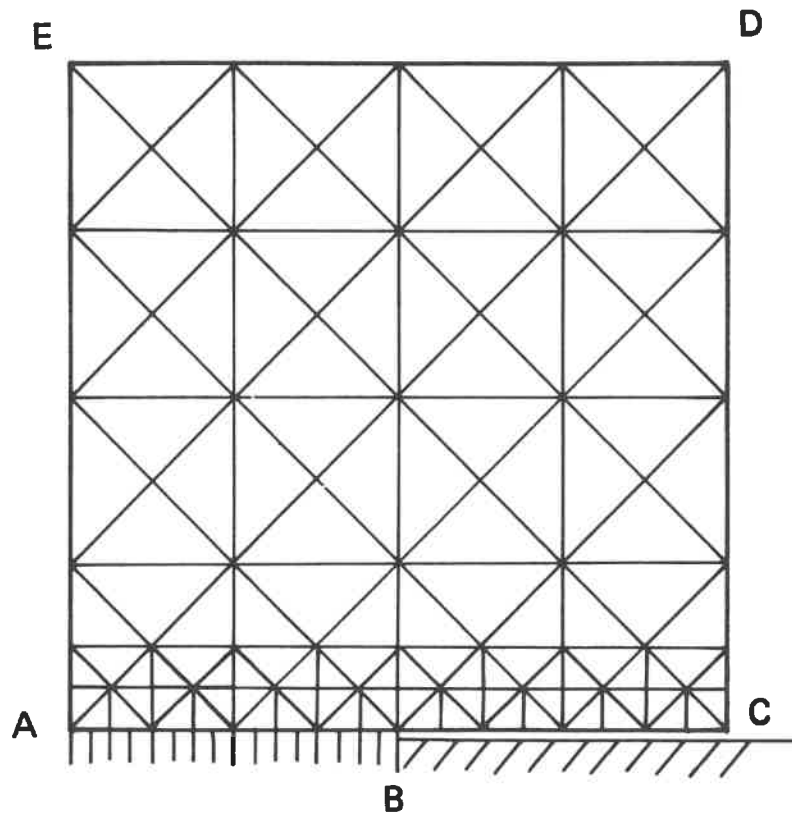


Figure 1 The mesh.

on Ω , $\phi_1(t, x) = 0$ (no volumic loads), $e^0(t, x) = 0$ (constant temperature).

Short computational aspect The dimension of the matrix \tilde{A} is 150: the band-width is 34. Determination of the *relaxation coefficient*: we find after a dichotomic analysis of the interval $]0, 2[$, $\tilde{\omega} = 1.919$. We give in table 1, the number of necessary iterations to obtain, in a case without constraints, the same residue as that with the use of a direct pivoting method. We note the great sensitivity of the method with regard to the relaxation coefficient.

Table 1 Optimal relaxation coefficients.

$\tilde{\omega}$	1.7	1.8	1.9	1.91	1.919	1.93	1.95
Number of iterations	496	341	167	144	115	123	155

We calculate the solution of the Cauchy problem on 3 periods: the stress s is then periodic with a precision of 3%. On time, we take $M=41$. The overall computation time (including the exploitation of the results) is of 8' on the UNIVAC 1110. We use 15k-words of the central memory and store the solutions in a file of 30K-words.

The solution First, we look at the stress and displacement variations along a period, at one fixed point (here, the node C (Figure 1)), to show the unilateral aspect of the solution *before reaching the asymptotic state*. (In this case, for the asymptotic solution, we have contact with no stresses along the crack because the coefficients are constant and the mean value of the load is zero).

With plane stress hypothesis, the stress tensor takes the form:

$$s = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

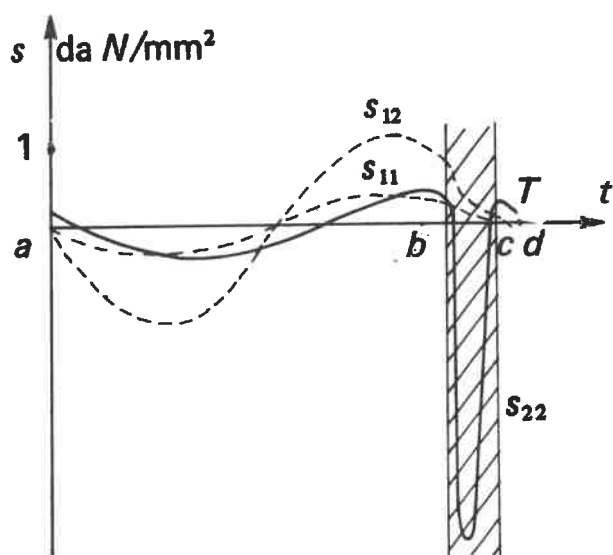


Figure 2 Stress $s = (s_{11}, s_{22}, s_{12})$ at the node C along a period.

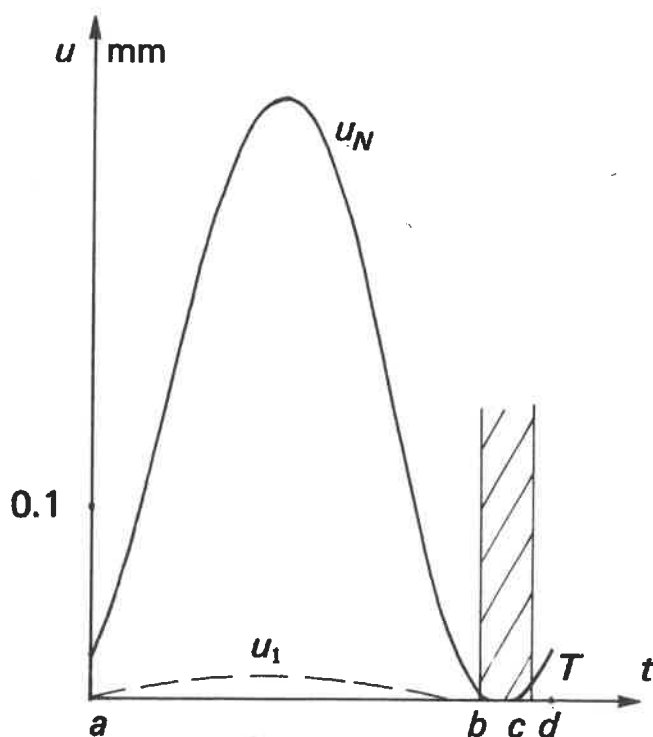


Figure 3 Displacement U_N of the node C along a period.

Figures 2 and 3 show the unilateral aspect of the solution on the edges of the crack: when $t \in [a, b]$ or $t \in [c, d]$, the crack is *open* and we have $u_N > 0$ and $g = 0$; when $t \in [b, c]$, the crack is *closed* and we have $u_N = 0$ and $g > 0$: the sudden variation of s_{22} shows the compression effect on the edge of the crack.

On the other hand, we can look at the displacement and the stress field in the whole domain at one fixed time.

On Figure 4, we show the stress level curves (3 components) for $t = T/4$ (maximum stress).

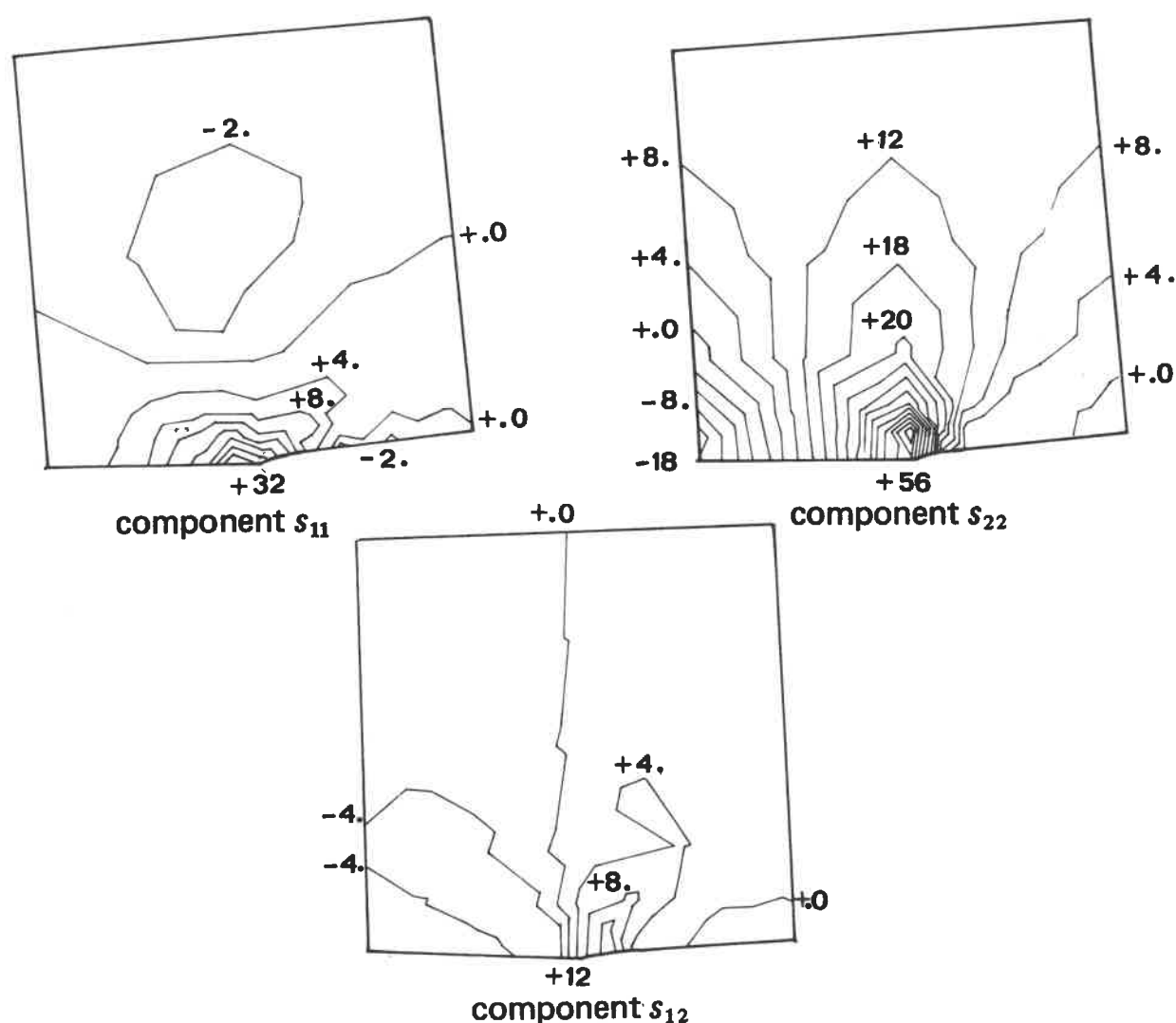


Figure 4 Stress tensor components for $t = T/4$

On figures 4,5 and 6, the displacements are multiplied by a coefficient 10 to see them clearly.

Figure 5 gives a more synthetic aspect of the stress state of the plate with the principal stresses (again for $t = T/4$).

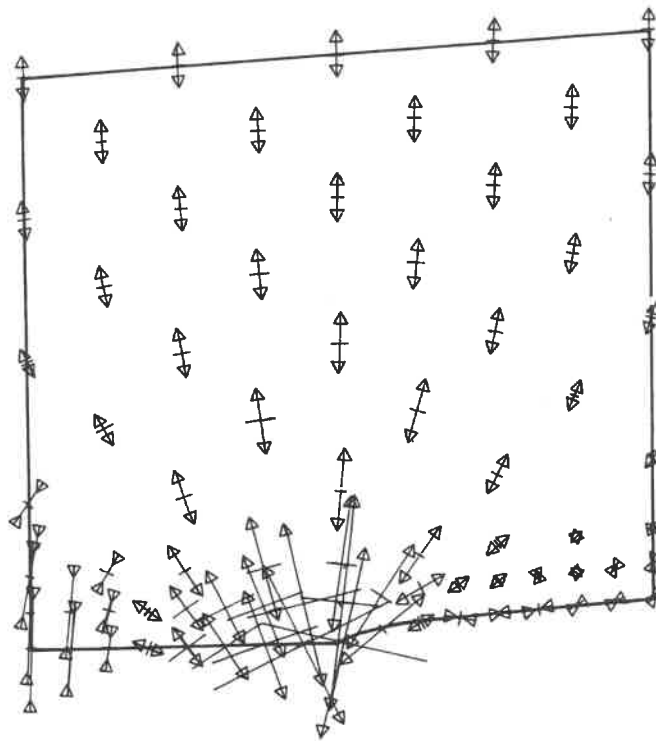


Figure 5 Principal stresses for $t = T/4$

In the Figure 6, we give the stress level curves of the second invariant of the stress deviator tensor in all the cracked plate. The quantity plotted in Figure 6 is usually used as a classical plasticity test; here, it shows the solicitation levels in the plate.

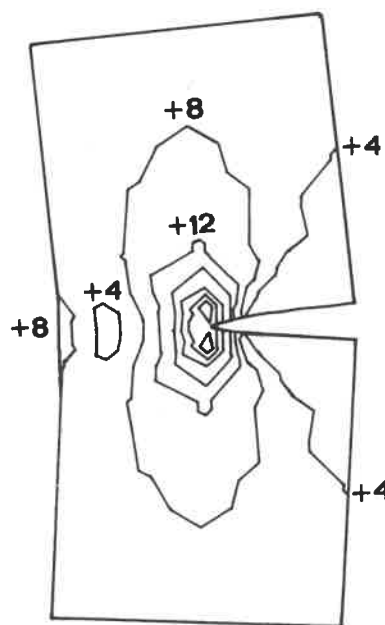


Figure 6 Second invariant of the stress deviator tensor in all the cracked plate for $t = T/4$

These presentations exhibit the high stress gradient at the top of the crack. For a better approach to the stress on that part, we shall use special elements (special functions defined on that zone; see [8]).

8. OTHER ASPECTS AND REFERENCES

The functional analysis aspect of the problem can be found in [2, 3]. In this paper, existence and 'uniqueness' theorems for the Cauchy Signorini problem and for the periodic Signorini problem are given; a result on asymptotic stability of the Cauchy problem is also established.

Moreover, we have established several numerical analysis results with Geymonat [5] in the case where $C=U$ (i.e. the variational inequalities become only *variational equalities*): that is the case of the problem without crack. In this paper, we first give results of existence and asymptotic stability for the problems by another means than in [3]. But the main result given in [5] is to obtain the numerical stability condition under an explicit form.

Furthermore, the condition on the choice of Δt is independent of the space discretization: $\Delta t_k < 2\tau$ (where τ is the smallest value of the relaxation time). This interesting result occurs because we have a bounded operator (order zero). We also give for this case, the error estimates for space and time discretization.

REFERENCES

- [1] E.P. Bazant, and S.T. Wu. 'Rate type creep law for aging concrete based on Maxwell chain'. *Rilem Matériaux et Constructions*, 7 (1974) 45-60.
- [2] R. Bouc, G. Geymonat, M. Jean, and B. Nayroles. Hilbertian unilateral problems in viscoelasticity'. *Lecture notes in mathematics*, n. 503, Springer Verlag, Berlin (1976).
- [3] R. Bouc, G. Geymonat, M. Jean, and B. Nayroles. 'Cauchy and periodic unilateral problems for aging linear viscoelastic materials'. *J. Math. Anal. Appl.*, 61, 1977.
- [4] P. Germain. 'Course de mécanique des milieux continus', Masson, Paris (1973).
- [5] G. Geymonat, and M. Raous. 'Eléments finis en viscoélasticité périodique' *Lecture notes in Mathematics*, n. 606, Springer Verlag, Berlin (1977). (*International meeting on mathematical aspect of finite element method*, Dec. 10-12, 1975.)
- [6] R. Glowinski, J.L. Lions, and R. Trémolieres. 'Analyse numérique des inéquations variationnelles'. Dunod, Paris (1976).
- [7] U. Mosco. 'An introduction to the approximate solutions of variational inequalities'. Course CIME 'Constructive aspect of functional analysis', 1974 Cremonese, Roma (1973).

- [8] G. Strang, and J. Fix. '*Analysis of the finite element method*'. Prentice Hall, New York (1973).
- [9] G. Duvaut, and J.L. Lions. '*Les inéquations en mécanique et en physique*'. Dunod, Paris (1972).
- [10] I.C. Corneau, and O.C. Zienkiewicz. 'Visco-plasticity — plasticity and creep in elastic solids — a unified numerical solution approach'. *Int. J. Numer. Meth. Eng.*, 8, (1974) 821-45.