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# A REVERSE ISOPERIMETRIC INEQUALITY FOR PLANAR $(\alpha, \beta)$ -CONVEX BODIES

GISELLA CROCE, ZAKARIA FATTAH, AND GIOVANNI PISANTE

ABSTRACT. In this paper, we study a reverse isoperimetric inequality for planar convex bodies whose radius of curvature is between two positive numbers  $0 < \alpha < \beta$ , called  $(\alpha, \beta)$ -convex bodies. We show that among planar  $(\alpha, \beta)$ -convex bodies of fixed perimeter, the extremal shape is a domain whose boundary is composed by two arcs of circles of radius  $\alpha$  joined by two arcs of circles of radius  $\beta$ .

## 1. INTRODUCTION

The isoperimetric inequality is one of the oldest problems in mathematics. In the plane it states that the circle encloses the maximal area among all curves of the same length. One can refer to [18] for a history of this inequality.

The reverse problem, that is, find, within the class of curves of given length, the one enclosing the minimal area, is not well-posed. Indeed the area of any minimizing sequence converges to 0 (one can think of a sequence of rectangles converging to a segment). Therefore one has to find a more restricted framework to set the reverse isoperimetric problem in a proper way. In the literature one can find different ways to define a well-posed problem.

In [3], K. Ball proposed to consider classes of affinely equivalent convex bodies. Modulo affine transformations he showed that among all convex bodies in  $\mathbb{R}^n$ , the  $n$ -dimensional tetrahedron has largest surface area for a given volume, while among symmetric convex bodies, the cube is extremal.

Other restrictions can involve the curvature. For example, [25] Zhang (see also the references therein) considered the locus of curvature centers of the boundary, showing the inequality

$$L^2 \leq 4\pi A + (1 + \varepsilon)\pi|\tilde{A}_1|$$

where  $\tilde{A}_1$  is the oriented area of the domain enclosed by the locus of curvature centers of the boundary of the planar set. One could also think of considering bounds on the values of the curvature, with the aim of minimizing the area enclosed by plane curves of given length. R. Howard and A. Treibergs in [16] studied planar domains enclosed by curves whose curvature  $k$  satisfies the inequality  $|k| \leq 1$ . A. Gard in [13], extended this result to surfaces of revolution in  $\mathbb{R}^3$  whose principal curvatures are bounded in absolute value by 1. See also [7] for a reverse isoperimetric inequality in  $\mathbb{R}^3$  involving the Willmore energy. Other recent directions of research related to reverse isoperimetric type inequalities can be found for example in [12] and [19].

Within the family of planar convex sets, several papers can be found in the literature, with assumptions on the upper or lower bounds on the values of the curvature  $k$ . A.

Borisenko and K. Drach [8] considered convex sets whose curvature  $k$  is greater or equal  $\frac{1}{\lambda}$ , where  $\lambda$  is a strictly positive constant. The extremal shape is the intersection of two disks of radius  $\lambda$  (see also [?] for surfaces of revolution in  $\mathbb{R}^3$ ). R. Chernov, K. Drach and K. Tatarko [9] studied the complement interval in terms of curvature. Indeed, they considered  $0 \leq k \leq \lambda$ . The extremal shape is the convex hull of two balls of radius  $\frac{1}{\lambda}$ . Their work is developed not only for planar bodies, but in  $\mathbb{R}^n$ ,  $n \geq 2$ .

In this paper we analyse the same reverse isoperimetric inequality within the family of convex bodies of fixed perimeter and whose boundary has a radius of curvature between two positive numbers  $\alpha < \beta$ . We will call such domains  $(\alpha, \beta)$ -convex bodies. We prove that the following inequality for the area  $A(K)$  holds true

$$A(K) \geq \frac{1}{2}(\beta + \alpha)(P(K) - 2\pi\alpha) + \pi\alpha^2 - (\beta - \alpha)^2 \sin\left(\frac{P(K) - 2\pi\alpha}{2(\beta - \alpha)}\right)$$

for any planar  $(\alpha, \beta)$ -convex body such the the perimeter  $P(K)$  is between  $2\pi\alpha$  and  $2\pi\beta$ . Moreover the equality holds if and only if  $K$  is the  $(\alpha, \beta)$ -egg, that is, the  $(\alpha, \beta)$ -convex body whose boundary is composed by two arcs of circles of radius  $\alpha$  joined by two arcs of circles of radius  $\beta$ . Our result together with [8] and [9] gives a complete picture of the reverse isoperimetric inequalities for convex bodies in the plane when one adds bounds on the values of the curvature.

Our strategy uses the support function of the boundary of a convex set. Applying the Pontryagin Maximum principle, roughly speaking, we can show that the radius of curvature of an optimal set takes only the values  $\alpha$  and  $\beta$ . More precisely, the boundary of an optimal set consists of a finite number of congruent pairs of arcs of circles of radius  $\alpha$  and  $\beta$  joined each other at corner points, where the support function takes always the same value, say  $\lambda$ . We will call such sets  $\alpha - \beta$  regular  $N$ -gons. Solving the ordinary differential equation  $x + x''$  equal to  $\alpha$  or  $\beta$  in the various intervals where  $x$  is less or greater than  $\lambda$ , we can express the area and the perimeter as a function of the number of the pairs of arcs and then minimize the area (keeping constant the value of the perimeter).

The optimality condition described above is very similar to that one found in [8], in terms of support function, as well. However, A. Borisenko and K. Drach use it in a geometrical argument to write the area and the perimeter of an optimal body. Our study is analytical, instead.

The paper is organized as follows. In Section 2 we will prove some regularity properties of the  $(\alpha, \beta)$ -convex sets and some geometrical features of  $(\alpha, \beta)$ -regular  $N$ -gons. We also derive some consequences of the Pontryagin Maximum Principle that we will use to write down the optimality conditions. In Section 3 we present the main result of this paper. We first prove the existence of an optimal body, we exploit the optimality conditions showing that an optimal body is an  $\alpha - \beta$  N-gone and then we establish the reverse isoperimetric inequality, finding the  $\alpha - \beta$ -egg as the optimal shape.

## 2. PRELIMINARIES

With the notation  $\mathfrak{S}(A, B)$ , where  $A$  and  $B$  are two sets and  $\mathfrak{S}$  is one of the standard symbol for a functional space (such as  $C^{0,1}$  for Lipschitz functions,  $W^{1,1}$  for absolutely

continuous, etc.), we mean the space of maps defined on  $A$  with values in  $B$ . Moreover for  $I = [a, b] \subset \mathbb{R}$  with  $\mathfrak{S}_{per}(I, B)$  we denote the  $(b-a)$ -periodic functions in  $\mathfrak{S}_{per}(\mathbb{R}, B)$ .

**2.1. Convex bodies and  $(\alpha, \beta)$ -convexity.** In this section we recall some basic properties of convex sets in euclidean spaces, we introduce the class of competitors for our optimisation problem and prove some useful properties such as regularity and compactness.

Throughout the paper we will denote with  $B_r$  the closed ball with radius  $r > 0$  centred in the origin. By a convex body we shall mean a compact convex set  $K \subset \mathbb{R}^n$  with non-empty interior. With  $\mathcal{K}^n$  we will denote the class of convex bodies in  $\mathbb{R}^n$ . The *support function* of  $K$  is the real-valued function defined on the unit sphere  $\mathbb{S}^{n-1}$  by

$$h_K(v) := \max_{k \in K} \langle k, v \rangle, \quad v \in \mathbb{S}^{n-1}.$$

We recall that the support function  $h_K$  characterises the set  $K$  and any function  $h : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ , such that its 1-homogeneous extension is convex, is the support function of a convex body (cf. [21, §1.7]). Moreover  $K \in \mathcal{K}^n$  is strictly convex if and only if the 1-homogeneous extension of its support function belongs to  $C^1(\mathbb{R}^n \setminus \{0\})$  (see [21, Cor. 1.7.3 & §2.5]).

A convenient way to endow  $\mathcal{K}^n$  with a topology is to use the Hausdorff distance between two non-empty compact sets, denoted by  $d^H(\cdot, \cdot)$  (cf. [21, §1.8]). Indeed, we recall that the perimeter and the area functionals are continuous with respect to the Hausdorff topology on  $\mathcal{K}^n$  (see also [17, Thm. 23 and Thm. 26]). By [21, Lem. 1.8.14], given  $K, M \in \mathcal{K}^n$ , we can characterize the Hausdorff distance of  $K$  from  $M$  in terms of their support functions:

$$(2.1) \quad d^H(K, M) = \|h_K - h_M\|_{L^\infty(\mathbb{S}^{n-1})}.$$

Moreover, by the *Blaschke selection theorem* (cf. [21, Theorem 1.8.7]), every bounded sequence of convex bodies has a subsequence that converges to a convex body in the Hausdorff topology.

Following [21, pag. 157] we say that the convex body  $L$  is *locally embeddable* in the convex body  $K$  if for each point  $x \in \partial K$  there are a point  $y \in L$  and a neighbourhood  $U$  of  $y$  such that

$$(L \cap U) + x - y \subset K.$$

The concept of local embeddability when  $K$  or  $L$  is a ball has been studied and used in several contexts (cf. Remark 2.3 below). The following lemmata provide two classical regularity properties related to this concept.

**Lemma 2.1.** *If a convex body  $K$  is locally embeddable in a ball, then its support function is of class  $C^{1,1}$ .*

*Proof.* From the local embeddability of  $K$  in a ball, say  $B$ , it follows that  $K$  is a strictly convex body. Therefore by [21, Theorem 3.2.3], there exists a convex body  $M \in \mathcal{K}^n$  such that  $B = K + M$  (i.e.  $K$  is a *summand* of  $B$ ) which is equivalent to say that  $K$  *slides freely inside*  $B$  (cf. [21, Theorem 3.2.2]). The result follows by the characterization of convex bodies with support function of class  $C^{1,1}$  (cf. [15, Proposition 2.3]).  $\square$

**Lemma 2.2.** *Let  $K$  be a convex body. If a ball is locally embeddable in  $K$ , then its boundary  $\partial K$  is of class  $C^{1,1}$ .*

*Proof.* Let  $B$  a ball locally embeddable in  $K$ . Since  $B$  is strictly convex, by [21, Theorem 3.2.3] there exists a convex body  $M \in \mathcal{K}^n$  such that  $K = B + M$  (i.e.  $B$  is a *summand* of  $K$ ), that is equivalent to  $\partial K$  being of class  $C^{1,1}$  (cf. for example [14, Proposition 2.4.3]).  $\square$

Let  $\alpha$  and  $\beta$  be two real numbers with  $0 < \alpha < \beta$ . We say that a convex body  $K$  is  $(\alpha, \beta)$ -convex if  $K$  is locally embeddable in  $B_\beta$  and  $B_\alpha$  is locally embeddable in  $K$ . From the previous lemmata it follows that an  $(\alpha, \beta)$ -convex body is a  $C^{1,1}$  strictly convex set with support function of class  $C^{1,1}$ .

**Remark 2.3.** *An equivalent definition of  $(\alpha, \beta)$ -convexity can be given in terms of the following notions of  $\beta$ -convexity and  $\alpha$ -concavity. A convex body  $K$  is said to be  $\alpha$ -concave if the ball of radius  $1/\alpha$  is locally embeddable in  $K$  (cf. [9, Definition 1.2]). While  $K$  is said to be  $\beta$ -convex if for each point  $y \in \partial K$ , there exist a point  $x \in \partial B_\beta$  and a neighbourhood  $U$  of  $x$  such that  $(K \cap U) + y - x \subset B_\beta$ . As a matter of fact, as a consequence of [10, Theorem 1.9], a convex body  $K$  is  $\beta$ -convex if and only if  $K$  is locally embeddable in  $B_\beta$ . Therefore  $K$  is  $(\alpha, \beta)$ -convex if it is at the same time  $\beta$ -convex and  $1/\alpha$ -concave. These geometrical assumptions on local behaviour of the boundary of convex bodies have been recently used as main assumptions to prove different forms of reverse isoperimetric inequalities (cf. [9]).*

We will mostly work in the two dimensional setting, dealing with planar convex sets, therefore we recall some useful preliminary results on planar convex geometry. First we note that it is often convenient to work with the so called *parametric support function*, i.e.  $p_K(t) := h_K \circ \sigma(t)$  where  $\sigma(t) = (\cos(t), \sin(t))$  and  $t \in [0, 2\pi]$  (note that this is how the support function of a planar convex body is defined in classical literature, cf. for example [20] and [23]). In the next propositions we recall some well known and useful properties related to the parametric support function of a planar convex body (see also [6] for a recent survey on the subject). We remark that similar results holds under weaker assumption regularity assumptions (cf. [24], [21]) but we restrict our attention to what will be sufficient for our purposes.

**Proposition 2.4.** *Let  $K \in \mathcal{K}^2$  be a strictly convex planar body and  $p_K$  its parametrized support function. Assume that  $p_K \in C_{\text{per}}^{1,1}(0, 2\pi)$ . Then the radius of curvature of the boundary  $\partial K$ ,  $\rho_K(t)$ , satisfies for a.e.  $t \in (0, 2\pi)$  the equation*

$$(2.2) \quad \rho_K(t) = p_K(t) + p_K''(t) \geq 0.$$

*Viceversa, if  $h \in C_{\text{per}}^{1,1}(0, 2\pi)$  is a function satisfying (2.2), then there exists a convex body  $K$  such that  $h$  is its parametric support function.*

**Proposition 2.5.** *Under the same assumptions of Proposition 2.4 the boundary  $\partial K$  can be parametrized by*

$$\begin{cases} x(t) = h_K(t) \cos(t) - h_K'(t) \sin(t) \\ y(t) = h_K(t) \sin(t) + h_K'(t) \cos(t) \end{cases}$$

*Moreover the perimeter and the area of  $K$  can be computed by the following formulae*

$$P(K) = \int_0^{2\pi} (p_K(t) + p_K''(t)) dt = \int_0^{2\pi} \rho_K(t) dt,$$

$$A(K) = \frac{1}{2} \int_0^{2\pi} (p_K(t) + p_K''(t)) p_K(t) dt = \frac{1}{2} \int_0^{2\pi} \rho_K(t) p_K(t) dt.$$

The  $(\alpha, \beta)$ -convexity for planar domains can be expressed in terms of parametrized support function. Indeed a convex body  $K \in \mathcal{K}^2$  is  $(\alpha, \beta)$ -convex if and only if its parametrized support function  $p_K$  satisfies the inequalities

$$\alpha \leq p_K(t) + p_K''(t) \leq \beta \quad \text{a.e. in } (0, 2\pi).$$

It easily follows, by Proposition 2.5, that the perimeter of any  $(\alpha, \beta)$ -convex body in the plane satisfies  $2\pi\alpha \leq P(K) \leq 2\pi\beta$ . Moreover this characterization allows us to prove that for planar domains the  $(\alpha, \beta)$ -convexity is preserved by Hausdorff convergence.

**Lemma 2.6.** *Let  $0 < \alpha < \beta < \infty$  and  $\{K_n\}_{n \in \mathbb{N}} \subset \mathcal{K}^2$  be a sequence of  $(\alpha, \beta)$ -convex bodies. Assume that the sequence  $K_n$  converges to  $K$  in the Hausdorff topology. Then  $K$  is  $(\alpha, \beta)$ -convex.*

*Proof.* Let  $p_{K_n}$  be the parametric support function of  $K_n$  and  $p_K$  the parametric support function of  $K$ . As  $K_n$  is  $(\alpha, \beta)$ -convex body then  $p_{K_n}$  is of class  $C^{1,1}$  and the radius of curvature of  $\partial K_n$  exists almost everywhere. Therefore, the parametric support function satisfies

$$(2.3) \quad \alpha \leq p_{K_n} + p_{K_n}'' \leq \beta \quad \text{a.e. in } [0, 2\pi].$$

By formula 2.1, as  $K_n$  converges to  $K$  in the Hausdorff distance,  $p_{K_n}$  converges to  $p_K$  in  $L^\infty([0, 2\pi])$ . Since  $p_{K_n}$  is bounded in  $L^\infty([0, 2\pi])$ , we deduce from inequality (2.3) that  $p_{K_n}''$  is bounded in  $L^\infty([0, 2\pi])$ .

By Proposition 2.5

$$p_{K_n}'(t) = -x_n(t) \sin(t) + y_n(t) \cos(t),$$

where  $(x_n(t), y_n(t))$  is the parametrization of the boundary of  $K_n$ . As  $(x_n(t), y_n(t)) \in \partial K_n$  and all the  $K_n$  are contained in a ball, then  $p_{K_n}'$  is bounded in  $L^\infty([0, 2\pi])$ . Therefore  $p_{K_n}'$  is bounded in the Sobolev space  $W^{1,\infty}([0, 2\pi])$ . By the Rellich-Kondrachov theorem, there exists  $w \in W^{1,\infty}([0, 2\pi])$  such that, up to a subsequence,  $p_{K_n}' \overset{*}{\rightharpoonup} w$  in  $W^{1,\infty}([0, 2\pi])$  and  $p_{K_n}' \rightarrow w$  in  $L^\infty([0, 2\pi])$ .

Since  $p_{K_n}$  is bounded in  $W^{1,\infty}([0, 2\pi])$ , by using again the Rellich-Kondrachov theorem, there exists  $g \in W^{1,\infty}([0, 2\pi])$  such that, up to a subsequence,  $p_{K_n} \overset{*}{\rightharpoonup} g$  in  $W^{1,\infty}([0, 2\pi])$  and  $p_{K_n} \rightarrow g$  in  $L^\infty([0, 2\pi])$ .

Since  $K_n \rightarrow K$  in the Hausdorff distance, the limit of  $p_{K_n}$  to  $p_K$  in  $L^\infty([0, 2\pi])$  implies that  $g = p_K$ ,  $w = p_K'$ . Thus, we can extract a subsequence still denoted  $p_{K_n}'$  such that  $p_{K_n}' \overset{*}{\rightharpoonup} p_K'$  in  $W^{1,\infty}([0, 2\pi])$ . This implies that  $p_K \in W^{2,\infty}([0, 2\pi])$  which means  $p_K$  is  $C^{1,1}$ . As  $p_K$  is the parametric support function of  $K$ , then  $p_K$  and  $p_K''$  are  $2\pi$ -periodic. Moreover, (2.3) implies

$$\int \alpha \phi \leq \int (p_{K_n} + p_{K_n}'') \phi \leq \int \beta \phi \quad \text{for all } \phi \text{ smooth and non negative.}$$

Using the weak- $*$  convergence in  $W^{2,\infty}([0, 2\pi])$ , we have

$$\int \alpha \phi \leq \int (p_K + p_K'') \phi \leq \int \beta \phi \quad \text{for all } \phi \text{ smooth and non negative.}$$

Thus, by the Fundamental lemma in the calculus of variations, we deduce that

$$\alpha \leq p_K + p_K'' \leq \beta \quad \text{a.e. in } [0, 2\pi],$$

that is,  $K$  is  $(\alpha, \beta)$ -convex.  $\square$

We conclude this section with an example of a family of planar  $(\alpha, \beta)$ -convex bodies that will play an important role in the sequel.

**Example 2.7** ( $(\alpha, \beta)$ -eggs). *An example of  $(\alpha, \beta)$ -convex bodies in  $\mathcal{K}^2$  is a family of sets that we will call  $(\alpha, \beta)$ -eggs. They are symmetric with respect to the Cartesian axes, their boundary is composed by 4 arcs of circles with radii  $\alpha$  and  $\beta$  alternatively and their centers are chosen in such a way to ensure the regularity of  $\partial K$  (see Lemma 2.2). Given  $0 < \alpha < \beta < +\infty$  and  $l \in (\alpha\pi, \beta\pi)$ , the  $(\alpha, \beta)$ -egg, with perimeter  $P = 2l$ , can be parametrised as follows. We set  $\tau = \frac{1}{2} \frac{l - \pi\alpha}{\beta - \alpha}$ ,  $\kappa_1 = (\beta - \alpha) \cos(\tau)$  and  $\kappa_2 = (\beta - \alpha) \sin(\tau)$ . We note that  $\tau \in (0, \frac{\pi}{2})$  and  $\kappa_1 \cdot \kappa_2 > 0$ . We define the points  $\mathbf{c}_1 = (-\kappa_1, 0)$ ,  $\mathbf{c}_2 = (0, \kappa_2)$ ,  $\mathbf{c}_3 = (\kappa_1, 0)$ ,  $\mathbf{c}_4 = (0, -\kappa_2)$ . Then the boundary of the  $(\alpha, \beta)$ -egg is parametrised by*

$$\gamma(t) = \begin{cases} \mathbf{c}_1 + \beta\sigma(t), & t \in (-\tau, \tau) \\ \mathbf{c}_2 + \alpha\sigma(t), & t \in (\tau, \pi - \tau) \\ \mathbf{c}_3 + \beta\sigma(t), & t \in (\pi - \tau, \pi + \tau) \\ \mathbf{c}_4 + \alpha\sigma(t), & t \in (\pi + \tau, \pi - \tau) \end{cases}.$$

**Remark 2.8.** *An  $(\alpha, \beta)$ -egg is an example of convex set whose radius of curvature  $\rho_K$  is piecewise constant and assumes alternatively the two values  $\alpha$  and  $\beta$ . One could consider in general a wider class of planar  $(\alpha, \beta)$ -convex sets that satisfy this property. i.e. considering the class of sets whose boundary is a finite union of arcs of circles with radii  $\alpha$  and  $\beta$ . Due to the regularity of the boundary given by Lemma 2.2, one can easily infer that two consecutive arcs cannot have the same radius of curvature and at least four arcs are needed. It follows therefore that the arcs forming the boundary of  $K$  have to be even in number.*

**Example 2.9** ( $(\alpha, \beta)$ -regular  $N$ -gone). *Given  $N \in \mathbb{N}$  with  $N \geq 3$ , we call  $(\alpha, \beta)$ -regular  $N$ -gone the  $(\alpha, \beta)$ -convex planar set  $K$  whose boundary  $\partial K$  is made up of  $2N$  arcs of circles alternating the radii between  $\alpha$  and  $\beta$  and such that the length of all the arcs with the same radius is constant. In order to write the parametrized radius of curvature of a general  $(\alpha, \beta)$ -regular  $N$ -gone,  $K$ , fix  $\sigma, \tau > 0$  such that  $N(\sigma + \tau) = 2\pi$  and define, for  $i \in \{1, 2, \dots, 2N\}$ ,*

$$t_i := \begin{cases} \frac{i-1}{2}(\sigma + \tau) + \sigma & \text{if } i \text{ is odd} \\ \frac{i}{2}(\sigma + \tau) & \text{if } i \text{ is even} \end{cases}.$$

The parametrized radius of curvature of  $K$  can be written as

$$\rho_K(t) = \begin{cases} \beta, & t \in [t_{2i+1}, t_{2i+2}] \\ \alpha, & t \in [t_{2i+2}, t_{2i+3}] \end{cases}.$$

Let  $P(K) = L$  be the perimeter of  $K$ . By Proposition 2.5 we easily get

$$(2.4) \quad P(K) = (\beta\sigma + \alpha\tau)N = L$$

and therefore

$$(2.5) \quad \sigma = \frac{L - \alpha 2\pi}{N(\beta - \alpha)}, \quad \tau = \frac{2\pi\beta - L}{N(\beta - \alpha)}.$$

The parametric support function of  $K$  can consequently be written, by (2.2), as

$$p_K(t) = \begin{cases} C_1^{2i+1} \cos t + C_2^{2i+1} \sin t + \beta, & t \in [t_{2i+1}, t_{2i+2}] \\ C_1^{2i+2} \cos t + C_2^{2i+2} \sin t + \alpha, & t \in [t_{2i+2}, t_{2i+3}] \end{cases}, \quad i \in \{0, 1, \dots, N-1\}.$$

Define  $\lambda_j := p_K(t_j)$  for  $j \in \{0, 1, \dots, N-1\}$ . The continuity of  $p_K$  in any  $t_j$  ensures us that

$$\begin{aligned} C_1^{2i+1} &= \frac{(\lambda_{2i+2} - \beta) \sin(t_{2i+1}) - (\lambda_{2i+1} - \beta) \sin(t_{2i+2})}{\sin(t_{2i+2} - t_{2i+1})}, \\ C_2^{2i+1} &= \frac{(\lambda_{2i+2} - \beta) \cos(t_{2i+1}) - (\lambda_{2i+1} - \beta) \cos(t_{2i+2})}{\sin(t_{2i+1} - t_{2i+2})}, \\ C_1^{2i} &= \frac{(\lambda_{2i} - \alpha) \sin(t_{2i+1}) - (\lambda_{2i+1} - \alpha) \sin(t_{2i})}{\sin(t_{2i+1} - t_{2i+2})}, \\ C_2^{2i} &= \frac{(\lambda_{2i} - \alpha) \cos(t_{2i+1}) - (\lambda_{2i+1} - \alpha) \cos(t_{2i})}{\sin(t_{2i} - t_{2i+1})}. \end{aligned}$$

**2.2. Some easy consequences of the Pontryagin principle.** We will reformulate our constrained shape optimisation problem as an optimal control problem and we will exploit the optimality conditions given by the Pontryagin principle. The optimal control approach for shape optimisation problems is classical (see for example the monograph [2] for a wide introduction on the subject and [1] for a more contemporary approach) and recently has been fruitfully applied to deal with constrained optimisation problems for convex domains (see [5], [4]). Here we summarise the elementary notions on control theory and we state the version of Pontryagin optimality conditions suited for our purposes, considering indeed only autonomous problems with periodic phase variables valued in  $\mathbb{R}^2$ .

Let  $I = [a, b] \subset \mathbb{R}$  be a given interval,  $J_u \subset \mathbb{R}$  be a compact set. For given maps  $f, g \in C^1(\mathbb{R}^3)$  and  $\mathbf{h} \in C^1(\mathbb{R}^3, \mathbb{R}^2)$  consider the problem of minimizing the functional

$$F(\mathbf{x}, u) := \int_a^b f(\mathbf{x}(t), u(t)) dt$$

among all pairs  $(\mathbf{x}(t), u(t)) \in W_{per}^{1,1}(I, \mathbb{R}^2) \times L^\infty(I, J)$  that satisfy for almost every  $t \in I$  the differential constraint

$$(2.6) \quad \mathbf{x}'(t) = \mathbf{h}(\mathbf{x}(t), u(t))$$

as well as the integral constraint

$$(2.7) \quad G(\mathbf{x}, u) := \int_a^b g(\mathbf{x}(t), u(t)) dt = C_0$$

for a given constant  $C_0$ . The previous constrained extremal problem is a typical example of *optimal control problem*,  $u$  in the so called *control variable*,  $\mathbf{x}$  takes the name of *phase variable* and any pair  $(\mathbf{x}, u)$  that satisfy (2.6) will be called a *controlled process*. A controlled process that minimises (locally in a  $C(I)$ -neighbourhood of  $\mathbf{x}$ ) the functional



$F(\mathbf{x}, u)$  among the controlled processes satisfying (2.7) will be called an *optimal process* for  $F(\mathbf{x}, u)$  under (2.6) and (2.7). Following the Euler's terminology, integral constraints of the type (2.7) are often named *isoperimetric constraints* and we will follow this convention, motivated by the fact that in the next section we will rephrase a geometrical isoperimetric problem as an optimal control problem and (2.7) will play exactly the role of the constraint on the perimeter. As it is customary, we will use the self-explanatory notations  $\nabla_{\mathbf{x}}f$ ,  $\nabla_{\mathbf{x}}g$ ,  $\nabla_{\mathbf{x}}\mathbf{h}$ ,  $\partial_u f$ ,  $\partial_u g$ ,  $\partial_u \mathbf{h}$  and so on, to denote the partial derivatives of  $f$ ,  $g$ ,  $\mathbf{h}$ .

**Theorem 2.10** (Pontryagin Principle). *Let  $(\mathbf{x}, u) \in W_{per}^{1,1}(I, \mathbb{R}^2) \times L^\infty(I, J)$  be an optimal process  $F(\mathbf{x}, u)$  under (2.6) and (2.7). Then there exist  $\lambda \geq 0$ ,  $\mu \in \mathbb{R}$  and  $\mathbf{p} \in W^{1,1}(I, \mathbb{R}^2)$  not all of them trivial such that, for almost all  $t \in I$ ,*

$$(2.8) \quad \dot{\mathbf{p}}(t) = \mathbf{p} \cdot \nabla_{\mathbf{x}}\mathbf{h}(\mathbf{x}(t), u(t)) + \mu \nabla_{\mathbf{x}}g(\mathbf{x}(t), u(t)) - \lambda \nabla_{\mathbf{x}}f(\mathbf{x}(t), u(t))$$

and the optimal control  $u$  satisfies, for all  $t \in I$ , the optimality condition

$$(2.9) \quad \begin{aligned} & \mathbf{p}(t) \cdot \mathbf{h}(\mathbf{x}(t), u(t)) + \mu g(\mathbf{x}(t), u(t)) - \lambda f(\mathbf{x}(t), u(t)) \\ & = \max_{v \in J} \{ \mathbf{p}(t) \cdot \mathbf{h}(\mathbf{x}(t), v) + \mu g(\mathbf{x}(t), v) - \lambda f(\mathbf{x}(t), v) \}. \end{aligned}$$

The differential system (2.8) takes the name of *adjoint system* and it is nothing but the Euler-Lagrange equation derived as a stationarity condition on the Lagrangian of the optimal problem (cf. [2, §4.2.2]).

We will use a couple of consequences of Theorem 2.10 when applied to constrained problems arising in plane convex geometry. To this aim in the following corollary we specify the Pontryagin's conditions for optimality in one dimensional control problems with a second order differential constraint.

**Corollary 2.11.** *Given  $f \in C^1(\mathbb{R}^2)$  and  $g \in C^1(\mathbb{R}^2)$  and a constraint  $C_0$ . Let the pair  $(x, u) \in W_{per}^{2,1}(I, \mathbb{R}) \times L^\infty(I, J)$  be a minimizer of the functional*

$$F(x, u) := \int_a^b f(x(t), u(t)) dt$$

among all the admissible pairs satisfying the differential constraint

$$(2.10) \quad x(t) + \ddot{x}(t) = u(t)$$

and the integral constraint

$$(2.11) \quad \int_a^b g(x(t), u(t)) dt = C_0.$$

Then there exist  $\lambda \geq 0$ ,  $\mu \in \mathbb{R}$  and  $p \in W^{1,1}(I, \mathbb{R})$  not all of them trivial such that, for almost all  $t \in I$ ,  $p$  is a solution of the equation

$$(2.12) \quad \ddot{p}(t) + p(t) = +\mu \partial_x g(x(t), u(t)) - \lambda \partial_x f(x(t), u(t))$$

and the optimal control  $u$  satisfies, for all  $t \in I$ , the optimality condition

$$(2.13) \quad \begin{aligned} & p(t)u(t) + \mu g(x(t), u(t)) - \lambda f(x(t), u(t)) \\ & = \max_{v \in J} \{ p(t)v + \mu g(x(t), v) - \lambda f(x(t), v) \}. \end{aligned}$$

*Proof.* The proof easily follows by rewriting the differential constraint as a system of first order equation and applying Theorem 2.10 with the phase variable  $\mathbf{x}(t) = (x_1(t), x_2(t))$  that satisfies the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u - x_1 \end{cases}.$$

The functionals involved are independent of the auxiliary variable  $x_2$  and Theorem 2.10 provides the existence of the multipliers exist  $\lambda, \mu$  and  $\mathbf{p} := (p_1, p_2)$  satisfying the adjoint system

$$\begin{cases} \dot{p}_1 = p_2 + \mu \partial_x g(x(t), u(t)) - \lambda \partial_x f(x(t), u(t)) \\ \dot{p}_2 = -p_1 \end{cases}$$

and the maximality condition

$$\begin{aligned} & p_1(t)x_2(t) + p_2(t)(u(t) - x_1(t)) + \mu g(x_1(t), u(t)) - \lambda f(x_1(t), u(t)) \\ &= \max_{v \in J} \{p_1(t)x_2(t) + p_2(t)(v - x_1(t)) + \mu g(x_1(t), v) - \lambda f(x_1(t), v)\}. \\ &= p_1(t)x_2(t) - p_2(t)x_1(t) + \max_{v \in J} \{p_2(t)v + \mu g(x_1(t), v) - \lambda f(x_1(t), v)\}. \end{aligned}$$

These equations are easily seen to be equivalent to (2.12) and (2.13) setting  $p = p_2$  and  $x = x_1$ . □

In the special case when the functional  $F$  and the isoperimetric constraint are linear in the control variable  $u$ , we can further deduce a *bang-bang* type condition for optimal controls. The following corollary easily follows from the previous one from the optimality condition (2.13) (being linear in the  $v$  variable).

**Corollary 2.12.** *Under the same assumptions of Corollary 2.11, if we further assume that  $f(x, u) = a(x)u$  and  $g(x, u) = b(x)u$  with  $a, b \in C^1(I)$ , then we have*

$$(2.14) \quad u(t) = \begin{cases} \beta & \text{if } p(t) + \mu b(x(t)) - \lambda a(x(t)) > 0 \\ \alpha & \text{if } p(t) + \mu b(x(t)) - \lambda a(x(t)) < 0 \end{cases}$$

where  $\alpha := \min\{t : t \in J\}$  and  $\beta := \max\{t : t \in J\}$ .

**Remark 2.13.** *Let us remark that if the set*

$$S := \{t, \in I : p(t) + \mu b(x(t)) - \lambda a(x(t)) = 0$$

*has zero Lebesgue measure, then  $u$  is almost everywhere determined by (2.14). This is the case for instance if  $(x, u)$  is an optimal control with a non-singular trajectory (cf. [22]).*

### 3. MAIN RESULT

The  $(\alpha, \beta)$ -regular  $N$ -gons introduced in the Example 2.7 will play a crucial role in this section.

**Theorem 3.1.** *Let  $0 < \alpha < \beta < \infty$  and  $L \in (2\pi\alpha, 2\pi\beta)$ . Then, modulo proper rigid transformations, the  $(\alpha, \beta)$ -egg is the unique minimizer of the area functional among all the  $(\alpha, \beta)$ -convex bodies in the plane with given perimeter equal to  $L$ .*

Theorem 3.1 follows easily from the following reverse isoperimetric inequality.

**Theorem 3.2.** *Let  $0 < \alpha < \beta < \infty$ . For any  $K \in \mathcal{K}^2$ , planar  $(\alpha, \beta)$ -convex body such that  $2\pi\alpha < P(K) < 2\pi\beta$ , then the following inequality holds true*

$$(3.1) \quad A(K) \geq \frac{1}{2}(\beta + \alpha)(P(K) - 2\pi\alpha) + \pi\alpha^2 - (\beta - \alpha)^2 \sin\left(\frac{P(K) - 2\pi\alpha}{2(\beta - \alpha)}\right).$$

Moreover the equality holds if and only if  $K$  is the  $(\alpha, \beta)$ -egg.

The proof of the main theorem will be a consequence of the following lemmata. From now on in this section we will implicitly assume that  $\alpha, \beta$  and  $L$  are fixed in such a way that  $0 < \alpha < \beta < \infty$  and  $L \in (2\pi\alpha, 2\pi\beta)$ .

**Lemma 3.3.** *The shape optimisation problem*

$$(3.2) \quad \min \{A(K) : K \in \mathcal{K}^2 \text{ is an } (\alpha, \beta) \text{ - convex body with } P(K) = L\}$$

*admits at least a solution.*

*Proof.* The proof follows by direct methods of Calculus of Variations. Indeed first we claim that any minimizing sequence  $K_n$  is bounded. For this is enough to observe that all the competitors are convex sets with perimeter and area equi-bounded, therefore also their diameters are equi-bounded (see for example [11, Lemma 4.1]). By Blaschke selection theorem, up to extracting a subsequence,  $K_n$  converges to a convex body  $K_\infty$  in the Hausdorff metric. Lemma 2.6 ensures that  $K_\infty$  is an admissible set and the conclusion follows by continuity of the perimeter and area functionals on  $(\mathcal{K}^2, d_H)$ .  $\square$

**Lemma 3.4.** *Let  $K \in \mathcal{K}^2$  be a minimizer for problem (3.2), then up to eventually translate  $K$ , there exists a constant  $\gamma \in \mathbb{R}$  such that*

$$(3.3) \quad \rho_K(t) = \begin{cases} \beta, & \text{for } p_K(t) < \gamma \\ \alpha, & \text{for } p_K(t) > \gamma \end{cases}, \quad \text{a.e. } t \in (0, 2\pi),$$

where with  $p_K$ , with a slight abuse of notation, we denoted the parametrised support function of the eventual translation of  $K$  and  $\rho_K(t)$  is the radius of curvature of  $\partial K$ . Moreover the set  $S := \{t \in [0, 2\pi) : p_K(t) = \gamma\}$  is finite.

*Proof.* We start observing that by Propositions 2.4 we can identify a given admissible set  $K \in \mathcal{K}^2$  with its parametric support function  $p_K$  and by Proposition 2.5 we can rephrase the minimization problem (3.2) as an optimal control problem. If  $K \in \mathcal{K}^2$  is a minimizer for problem (3.2), then the pair given by its support function and its parametric radius of curvature, i.e.  $(x, u) = (p_K, \rho_K) \in W_{per}^{2,1}(I, \mathbb{R}) \times L^\infty(I, J)$  with  $I = (0, 2\pi)$  and  $J = [\alpha, \beta]$ , form indeed an optimal control process for the functional

$$F(x, u) := \frac{1}{2} \int_0^{2\pi} u(t) x(t) dt$$

under the differential constraint

$$x(t) + \ddot{x}(t) = u(t)$$

and the isoperimetric one

$$\int_0^{2\pi} u(t) dt = L.$$

We can therefore use Corollary 2.12 to deduce that there exist  $\lambda \geq 0$ ,  $\mu \in \mathbb{R}$  and  $s \in W^{1,1}(I, \mathbb{R})$  not all of them trivial such that

$$(3.4) \quad \rho_K(t) = \begin{cases} \beta & \text{if } s(t) + \mu - \frac{\lambda}{2} p_K(t) > 0 \\ \alpha & \text{if } s(t) + \mu - \frac{\lambda}{2} p_K(t) < 0 \end{cases}$$

Moreover, from Corollary 2.11, the multiplier  $s(t)$  solves the adjoint equation

$$\ddot{s}(t) + s(t) = -\frac{\lambda}{2} \rho_K(t),$$

that, together with the differential constraint written for the pair  $(p_K, \rho_K)$ , implies that the function  $\beta := s + \frac{\lambda}{2} p_K$  is a  $2\pi$ -periodic solution of the ordinary differential equation  $y(t) + \ddot{y}(t) = 0$ . Therefore there exist constants  $c_1$  and  $c_2$ , such that

$$(3.5) \quad s(t) + \frac{\lambda}{2} p_K(t) = c_1 \cos(t) + c_2 \sin(t).$$

We can therefore rewrite (3.4), as

$$\rho_K(t) = \begin{cases} \beta & \text{if } c_1 \cos(t) + c_2 \sin(t) + \mu - \lambda p_K(t) > 0 \\ \alpha & \text{if } c_1 \cos(t) + c_2 \sin(t) + \mu - \lambda p_K(t) < 0 \end{cases}$$

We claim that  $\lambda \neq 0$ . If not, first we observe that by non-triviality condition of the Pontryagin principle,  $\mu$  and  $s(t)$  cannot be simultaneously identically zero. If  $s = 0$ , then from (3.4) we have

$$\rho_K(t) = \begin{cases} \beta & \text{if } \mu > 0 \\ \alpha & \text{if } \mu < 0 \end{cases}.$$

Being  $\mu \neq 0$ ,  $K$  is a circle of radius  $\alpha$  or  $\beta$  that is not an admissible set. If instead  $s \neq 0$ , the adjoint equation (3.5) ensures us that  $s(t) = c_1 \cos(t) + c_2 \sin(t) = A \cos(t + \phi)$ , with  $A \neq 0$  and  $\phi$  constant. The condition (3.4) becomes

$$\rho_K(t) = \begin{cases} \beta & \text{if } A \cos(t + \phi) + \mu > 0 \\ \alpha & \text{if } A \cos(t + \phi) - \mu < 0 \end{cases}.$$

Therefore, since the equation  $A \cos(t + \phi) + \mu = 0$  admits at most two solutions in the interval  $[0, 2\pi)$ , it follows that  $\partial K$  is the union of at most two arcs of circle with radii  $\alpha$  and  $\beta$ . This is impossible for an  $(\alpha, \beta)$ -convex set by the regularity Lemma 2.2 (cf. Remark 2.8). This proves the claim.

Since  $\lambda > 0$ , in a translated coordinate system centered in  $(\frac{c_1}{\lambda}, \frac{c_2}{\lambda})$ , the parametric support function of  $K$  will change in  $p_K(t) - \frac{c_1}{\lambda} \cos(t) - \frac{c_2}{\lambda} \sin(t)$ . Therefore (3.4) will read

$$\rho_K(t) = \begin{cases} \beta & \text{if } p_K(t) < \frac{\mu}{\lambda} =: \gamma \\ \alpha & \text{if } p_K(t) > \frac{\mu}{\lambda} =: \gamma \end{cases}$$

We now prove that  $S := \{t \in [0, 2\pi] : p_K(t) = \gamma\}$  is a finite set. Let  $t_0 \in [0, 2\pi] \in S^c$ , say  $p_K(t_0) > \gamma$ , and let  $(a_0, b_0)$  be the connected component of the set  $S^c$ , containing  $t_0$  (more explicitly we can define  $a_0 = \inf\{\tilde{t} : p_K(t) > \gamma \forall t \in (\tilde{t}, t_0)\}$  and  $b_0 = \sup\{\tilde{t} : p_K(t) > \gamma \forall t \in (t_0, \tilde{t})\}$ ). Observe that by continuity of  $p_K$  we deduce that

$$(3.6) \quad p_K(a_0) = p_K(b_0) = \gamma.$$

Moreover we can uniquely solve the equation  $p_K + p_K'' = \alpha$  in  $(a_0, b_0)$ , and therefore deduce the existence of two constants  $C_1$  and  $C_2$  such that  $p_K(t) = C_1 \cos t + C_2 \sin t + \alpha$  in  $[a_0, b_0]$ . We claim that  $b_0$  is an isolated point for  $S$ . The same argument could be applied for the left endpoint  $a_0$ . By contradiction, let  $\{t_m\}_{m \in \mathbb{N}}$  with  $t_m > b_0$ ,  $p_K(t_m) = \gamma$  and such that  $t_m \rightarrow b_0^+$ . The regularity of  $p_K$  ensures that the left and right derivatives of  $p_K$  in  $b_0$  agree. We can write (recalling the explicit expression of  $p_K$  in  $(a_0, b_0)$ )

$$-C_1 \sin b + C_2 \cos b = p_K'(b-) = p_K'(b+) = \lim_{m \rightarrow \infty} \frac{p_K(t_m) - p_K(b)}{t_m - b} = 0.$$

The last equality, together with (3.6), tells us that the couple  $(C_1, C_2)$  solves the following linear system

$$\begin{cases} C_1 \cos a_0 + C_2 \sin a_0 = \gamma - \alpha \\ C_1 \cos b_0 + C_2 \sin b_0 = \gamma - \alpha \\ -C_1 \sin b_0 + C_2 \cos b_0 = 0 \end{cases},$$

that is solvable only if (imposing the determinant of the full matrix to be zero)

$$(\gamma - \alpha)(\cos(a_0 - b_0) - 1) = 0$$

that in turns implies  $\gamma = \alpha$  or  $b_0 - a_0 = 2\pi$ . The last equality means that  $x$  represent a full circle of radius  $\alpha$ , that (if we choose  $L > 2\pi\alpha$ ) is not an admissible competitor for our problem. It remains to study the case  $\gamma = \alpha$ , that leads easily to a contradiction by observing that the only solution of the linear system is  $(C_1, C_2) = (0, 0)$  and therefore  $x(t) = \alpha = \gamma$  for any  $t \in (a_0, b_0)$  against the definition of  $S^c$  and this proves the claim. An analogous argument can be done when  $p_K(t_0) < \gamma$ , with  $\beta$  in place of  $\alpha$ . Since the connected components of  $S^c$  have isolated endpoints, they are finite in number. Finally we have proved that  $S^c$  is a finite union of disjoint relatively open intervals in  $[0, 2\pi]$ . Therefore its complement  $S$  is a finite union of, possibly degenerate, relatively closed intervals in  $[0, 2\pi]$ . With the same argument as above, it is easy to prove that the interior of  $S$  is empty. Indeed it is sufficient to argue by contradiction and use the regularity of  $p_K$  at the endpoints of the connected components of  $S$  with non empty interior.  $\square$

**Lemma 3.5.** *Any  $(\alpha, \beta)$ -convex body  $K \in \mathcal{K}^2$  that satisfies (3.3) is necessarily an  $(\alpha, \beta)$ -regular  $N$ -gone.*

*Proof.* From Lemma 3.4, we infer that  $\partial K$  is the union of a finite number of arcs of circles with radii  $\alpha$  and  $\beta$ , being the radius of curvature,  $\rho_K$ , a piecewise constant function with a finite number of jumps, assuming only two values. Moreover in any jump point  $t \in [0, 2\pi)$  of  $\rho_K$ , it holds  $P_K(t) = \gamma$ . By Remark 2.8 we can easily deduce that the arcs are even in number and the radii alternate between the values  $\alpha$  and  $\beta$ .

We can therefore assume that  $\partial K$  is made of  $2N$  disjoint arcs. The parametric support function of  $K$  can be written as

$$p_K(t) = \begin{cases} C_1^{2i+1} \cos t + C_2^{2i+1} \sin t + \beta, & t \in [t_{2i+1}, t_{2i+2}] \\ C_1^{2i+2} \cos t + C_2^{2i+2} \sin t + \alpha, & t \in [t_{2i+2}, t_{2i+3}] \end{cases}, \quad i \in 0, 1, \dots, N-1.$$

with  $\{t_1 < t_2 < t_3 \cdots < t_{2N+1} = t_1 + 2\pi\}$  and  $(C_1^{2i+1}, C_2^{2i+1})$  are the coordinates of centers of the disks of radius  $\beta$  and  $(C_1^{2i+2}, C_2^{2i+2})$  are the coordinates of centers of the

disks of radius  $\alpha$ . Imposing the continuity in  $t_i$  one gets

$$(3.7) \quad C_1^{2i+1} = (\gamma - \beta) \frac{\sin t_{2i+2} - \sin t_{2i+1}}{\sin(t_{2i+2} - t_{2i+1})}; \quad C_2^{2i+1} = (\gamma - \beta) \frac{\cos t_{2i+2} - \cos t_{2i+1}}{\sin(t_{2i+1} - t_{2i+2})}.$$

For  $0 \leq i \leq N - 1$ , one gets

$$(3.8) \quad C_1^{2i+2} = (\gamma - \alpha) \frac{\sin t_{2i+3} - \sin t_{2i+2}}{\sin(t_{2i+3} - t_{2i+2})}; \quad C_2^{2i+2} = (\gamma - \alpha) \frac{\cos t_{2i+3} - \cos t_{2i+2}}{\sin(t_{2i+2} - t_{2i+3})}.$$

From (3.7) and (3.8) we easily deduce that  $\gamma \neq \alpha$  and  $\gamma \neq \beta$ , otherwise the arcs of the circles of radius  $\alpha$  or  $\beta$  contained in  $\partial K$  should lie all on the same circle centered at the origin.

The continuity of the derivative of the parametric support function in  $t_j$  ensures us that, for  $0 \leq j \leq 2N - 1$ ,

$$(3.9) \quad C_1^j \sin(t_{j+1}) - C_2^j \cos(t_{j+1}) = C_1^{j+1} \sin(t_{j+1}) - C_2^{j+1} \cos(t_{j+1})$$

Combining the relations (3.7), (3.8) and (3.9), we can write, for  $0 \leq i \leq N - 1$ ,

$$\begin{cases} (\gamma - \beta) \frac{1 - \cos(t_{2i+2} - t_{2i+1})}{\sin(t_{2i+2} - t_{2i+1})} = (\gamma - \alpha) \frac{-1 + \cos(t_{2i+3} - t_{2i+2})}{\sin(t_{2i+3} - t_{2i+2})} \\ (\gamma - \alpha) \frac{1 - \cos(t_{2i+3} - t_{2i+2})}{\sin(t_{2i+3} - t_{2i+2})} = (\gamma - \beta) \frac{-1 + \cos(t_{2i+4} - t_{2i+3})}{\sin(t_{2i+4} - t_{2i+3})} \end{cases}.$$

Since the function  $t \rightarrow \frac{1 - \cos(t)}{\sin(t)}$  is strictly monotone, from the previous system we infer the existence of two positive constants  $\tau$  and  $\sigma$  such that  $t_{2i+2} - t_{2i+1} = \tau$  and  $t_{2i+3} - t_{2i+2} = \sigma$  for any  $0 \leq i \leq N - 1$ . This proves the claim.  $\square$

**Remark 3.6.** Let  $K \in \mathcal{K}^2$  be an  $(\alpha, \beta)$ -regular  $N$ -gone with perimeter  $P(K) = L$ , as in the Example 2.9. Suppose that up to a translation of  $K$ , the values of the parametric support functions in the points  $t_j$  are constants, i.e. there exists  $\lambda$ , such that  $\lambda_j = \lambda$  for any  $j \in \{0, 1, \dots, N - 1\}$ . As a byproduct of the proof of the previous lemma, we can explicitly calculate the value of  $\lambda = p_K(t_j)$ . Indeed, simply by imposing the continuity of the parametric support function in  $t_i$  and solving the linear system, we get

$$\lambda = \frac{\beta[1 - \cos(\tau)] \sin(\sigma) + \alpha[1 - \cos(\sigma)] \sin(\tau)}{[1 - \cos(\tau)] \sin(\sigma) + [1 - \cos(\sigma)] \sin(\tau)}.$$

**Lemma 3.7.** Let  $K \in \mathcal{K}^2$  be an  $(\alpha, \beta)$ -regular  $N$ -gone with  $P(K) = L$  that satisfies (3.3) then

$$(3.10) \quad A(K) = \frac{\beta + \alpha}{2} (L - 2\pi\alpha) + \pi\alpha^2 + (\beta - \alpha)^2 \frac{N \left( \cos\left(\frac{\pi}{N}\right) - \cos\left(\frac{\pi(\beta + \alpha) - L}{N(\beta - \alpha)}\right) \right)}{2 \sin\left(\frac{\pi}{N}\right)}.$$

Moreover the minimum value for the area functional is realized for  $N = 2$ , i.e. for the  $(\alpha, \beta)$ -egg.

*Proof.* Let  $K \in \mathcal{K}^2$  be an  $(\alpha, \beta)$ -regular  $N$ -gone and using the same notation as in Example 2.9 for its parametric support function and radius of curvature, we can compute

$$\begin{aligned} 2A(K) &= \sum_{i=1}^N \int_{t_{2i-1}}^{t_{2i}} \beta [C_1^{2i-1} \cos(t) + C_2^{2i-1} \sin(t) + \beta] dt \\ &\quad + \sum_{i=1}^N \int_{t_{2i}}^{t_{2i+1}} \alpha [C_1^{2i} \cos(t) + C_2^{2i} \sin(t) + \alpha] dt \\ &= \sum_{i=1}^N \left\{ \beta [C_1^{2i-1} (\sin(t_{2i}) - \sin(t_{2i-1})) + C_2^{2i-1} (\cos(t_{2i-1}) - \cos(t_{2i}))] \right. \\ &\quad \left. + \alpha [C_1^{2i} (\sin(t_{2i+1}) - \sin(t_{2i})) + C_2^{2i} (\cos(t_{2i}) - \cos(t_{2i+1}))] \right\} \\ &\quad + \beta^2 (t_{2i} - t_{2i-1}) + \alpha^2 (t_{2i+1} - t_{2i}). \end{aligned}$$

Therefore, using (3.7) and (3.8) in the previous formula and recalling that  $t_{2i} - t_{2i+1} = \tau$  and  $t_{2i+1} - t_{2i} = \sigma$ , we infer

$$A(K) = \beta(\lambda - \beta)N \frac{[1 - \cos(\tau)]}{\sin(\tau)} + \alpha(\lambda - \alpha)N \frac{[1 - \cos(\sigma)]}{\sin(\sigma)} + \frac{N}{2}(\beta^2\tau + \alpha^2\sigma).$$

Finally, using the value of  $\lambda$  given by Remark 3.6, we get

$$\lambda - \beta = (\alpha - \beta) \frac{1 - \cos(\sigma)}{\sin(\sigma) + \sin(\tau) - \sin(\sigma + \tau)} \sin(\tau)$$

and

$$\lambda - \alpha = (\beta - \alpha) \frac{1 - \cos(\tau)}{\sin(\sigma) + \sin(\tau) - \sin(\sigma + \tau)} \sin(\sigma).$$

Therefore we can write

$$(3.11) \quad A(K) = \frac{N}{2}(\beta^2\tau + \alpha^2\sigma) - \frac{N(\beta - \alpha)^2 [1 - \cos(\sigma)] [1 - \cos(\tau)]}{\sin(\sigma) + \sin(\tau) - \sin(\sigma + \tau)}.$$

Using the elementary relations

$$\sin(a) + \sin(b) + \sin(c) - \sin(a + b + c) = 4 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{b+c}{2}\right) \sin\left(\frac{a+c}{2}\right)$$

and

$$(1 - \cos(a))(1 - \cos(b)) = 4 \sin^2(a/2) \sin^2(b/2)$$

from (3.11), we get

$$A(K) = \frac{N}{2}(\beta^2\tau + \alpha^2\sigma) - N(\beta - \alpha)^2 \frac{\sin\left(\frac{\sigma}{2}\right) \sin\left(\frac{\tau}{2}\right)}{\sin\left(\frac{\sigma+\tau}{2}\right)}.$$

To make explicit the dependence on  $N$  in the expression of the area we introduce the auxiliary variable (cf. (2.5))

$$\omega := N \frac{\sigma}{2} = \frac{L - 2\pi\alpha}{2(\beta - \alpha)}.$$

Recalling that  $N(\sigma + \tau) = 2\pi$  we can finally write the area of the  $(\alpha, \beta)$ -regular  $N$ -gone  $K$  as

$$A(K) = \beta^2\pi - (\beta^2 + \alpha^2)\frac{\omega}{2} - (\beta - \alpha)^2\Phi(N, \omega),$$

where we have set

$$\Phi(N, \omega) := N \frac{\sin\left(\frac{\omega}{N}\right) \sin\left(\frac{\pi}{N} - \frac{\omega}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}$$

And this proves (3.10).

To prove that the minimum value of the area is attained when  $N = 2$ , which corresponds to the area of the  $(\alpha, \beta)$ -egg, it is sufficient to prove that

$$\Phi(N, \omega) \leq \Phi(2, \omega) = \sin(\omega).$$

To this aim, we observe that  $\omega \in [0, \pi]$  and we show that the function

$$f_N(x) := \sin(x) \sin\left(\frac{\pi}{N}\right) - N \sin\left(\frac{\pi - x}{N}\right) \sin\left(\frac{x}{N}\right)$$

is positive for  $x \in [0, \pi]$  and  $N \geq 2$ . We observe that  $f_N(0) = f_N(\pi) = 0$  and that  $f_N(x)$  is symmetric with respect to  $x_s = \frac{\pi}{2}$ . We claim that  $f_N$  is increasing in  $[0, \frac{\pi}{2}]$ . This will imply that  $f_N$  is positive on  $[0, \pi]$ . For that, we first observe that the function

$$h_N(x) := \frac{\sin\left(\frac{\pi - 2x}{N}\right)}{\sin\left(\frac{\pi - 2x}{N+1}\right)}$$

is increasing on  $(0, \frac{\pi}{2})$ . Indeed its derivative

$$h'_N(x) = -2 \frac{\sin\left(\frac{\pi - 2x}{N}\right)}{N(N+1) \sin\left(\frac{\pi - 2x}{N+1}\right)} \left[ (N+1) \cot\left(\frac{\pi - 2x}{N}\right) - N \cot\left(\frac{\pi - 2x}{N+1}\right) \right],$$

satisfies  $h'_N(x) > 0$  on  $(0, \frac{\pi}{2})$  since  $x \mapsto x \cot x$  is decreasing on  $(0, \frac{\pi}{2})$  and therefore

$$(N+1) \cot\left(\frac{\pi - 2x}{N}\right) - N \cot\left(\frac{\pi - 2x}{N+1}\right) < 0.$$

The monotonicity of  $h_N$ , implies that for  $N \geq 2$

$$\frac{\sin\left(\frac{\pi}{N+1}\right)}{\sin\left(\frac{\pi - 2x}{N+1}\right)} \geq \frac{\sin\left(\frac{\pi}{N}\right)}{\sin\left(\frac{\pi - 2x}{N}\right)},$$

that is, for any  $x \in (0, \frac{\pi}{2})$ , the sequence  $\{a_N\}_{N=2}^\infty$

$$a_N := \frac{\sin\left(\frac{\pi}{N}\right)}{\sin\left(\frac{\pi - 2x}{N}\right)}$$

is monotone in  $N$ , and therefore, for  $N \geq 2$ , it holds  $a_N \geq a_2$ , that reads as

$$\frac{\sin\left(\frac{\pi}{N}\right)}{\sin\left(\frac{\pi - 2x}{N}\right)} \geq \frac{\sin\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi - 2x}{2}\right)} = \frac{1}{\cos(x)}.$$

The last inequality is equivalent to say that

$$f'_N(x) = \cos x \sin\left(\frac{\pi}{N}\right) - \sin\left(\frac{\pi - 2x}{N}\right) > 0, \quad x \in \left(0, \frac{\pi}{2}\right),$$

proving the claim and the lemma.



□

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