



# On a Sum Involving the Sum-of-Divisors Function

Feng Zhao, Jie Wu

## ► To cite this version:

Feng Zhao, Jie Wu. On a Sum Involving the Sum-of-Divisors Function. Journal of Mathematics, 2021, 2021, pp.5574465. 10.1155/2021/5574465 . hal-03305600

**HAL Id: hal-03305600**

**<https://hal.science/hal-03305600>**

Submitted on 28 Jul 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## Research Article

# On a Sum Involving the Sum-of-Divisors Function

Feng Zhao<sup>1</sup> and Jie Wu<sup>2</sup>

<sup>1</sup>Department of Mathematics and Statistics, North China University of Water Resources and Electric Power, Jinshui E Road, Zhengzhou 450046, Henan, China

<sup>2</sup>CNRS LAMA 8050, Laboratoire D'Analyse et de Mathématiques Appliquées, Université Paris-Est Créteil, Créteil Cedex 94010, France

Correspondence should be addressed to Feng Zhao; zhaofeng@ncwu.edu.cn

Received 14 February 2021; Accepted 12 March 2021; Published 5 April 2021

Academic Editor: Tianping Zhang

Copyright © 2021 Feng Zhao and Jie Wu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let  $\sigma(n)$  be the sum of all divisors of  $n$  and let  $[t]$  be the integral part of  $t$ . In this paper, we shall prove that  $\sum_{n \leq x} \sigma([x/n]) = (\pi^2/6)x \log x + O(x(\log x)^{(2/3)}(\log_2 x)^{(4/3)})$  for  $x \rightarrow \infty$ , and that the error term of this asymptotic formula is  $\Omega(x)$ .

## 1. Introduction

As usual, denote by  $\varphi(n)$  the Euler function and by  $[t]$  the integral part of real  $t$ , respectively. Recently, Bordellès et al. [1] studied the asymptotic behaviour of the quantity

$$S_\varphi(x) := \sum_{n \leq x} \varphi\left(\left[\frac{x}{n}\right]\right), \quad (1)$$

for  $x \rightarrow \infty$ . By exponential sum technique, they proved that

$$\begin{aligned} &\left(\frac{2629}{4009} \cdot \frac{6}{\pi^2} + o(1)\right)x \log x \leq S_\varphi(x) \\ &\leq \left(\frac{2629}{4009} \cdot \frac{6}{\pi^2} + \frac{1380}{4009} + o(1)\right)x \log x, \end{aligned} \quad (2)$$

and conjectured that

$$S_\varphi(x) \sim \frac{6}{\pi^2}x \log x, \quad \text{as } x \rightarrow \infty. \quad (3)$$

Very recently, Wu [2] improved (2) and Zhai [3] resolved conjecture (3) by showing

$$S_\varphi(x) = \frac{6}{\pi^2}x \log x + O(x(\log x)^{(2/3)}(\log_2 x)^{(1/3)}), \quad (4)$$

and also proved that the error term in (4) is  $\Omega(x)$ , where  $\log_2$  denotes the iterated logarithm. Some related works can be found in [4, 5]. Since the sum-of-divisors function  $\sigma(n) := \sum_{d|n} d$  has similar properties as the Euler function  $\varphi(n)$  in many cases, it seems natural and interesting to consider its analogy of (3).

Our result is as follows.

### Theorem 1

(i) For  $x \rightarrow \infty$ , we have

$$\begin{aligned} S_\sigma(x) &:= \sum_{n \leq x} \sigma\left(\left[\frac{x}{n}\right]\right) = \frac{\pi^2}{6}x \log x \\ &+ O(x(\log x)^{(2/3)}(\log_2 x)^{(4/3)}). \end{aligned} \quad (5)$$

(ii) Let  $E(x)$  be the error term in (5). Then, for  $x \rightarrow \infty$ , we have

$$E(x) = \Omega(x), \quad \text{i.e. } \limsup_{x \rightarrow \infty} \frac{|E(x)|}{x} > 0. \quad (6)$$

Let  $\mu(n)$  be the Möbius function and define  $\text{id}(n) = n$  and  $1(n) = 1$  for all integers  $n \geq 1$ . Then,  $\varphi = \text{id} * \mu$  and  $\sigma = \text{id} * 1$ . In Zhai's approach proving (4), the inequality

$$\sum_{n \leq x} \mu(n) \ll x \exp\left\{-c\sqrt{\log x}\right\}, \quad (x \geq 1), \quad (7)$$

plays a key role, where  $c > 0$  is a positive constant. Clearly, such a bound is not true for 1. By refining Zhai's approach, we shall prove our result.

## 2. Preliminary Lemmas

As in [3], we need some bounds on exponential sums of the type  $\sum_{N \leq n < N'} e(T/n)$  where  $N < N' \leq 2N$ . For large values of  $N$ , Zhai used the theory of exponent pair, and for smaller ones the Vinogradov method. Both estimates are contained in the following general theorem of Karatsuba [6, Theorem 1], which will be a key tool for proving Theorem 1.

**Lemma 1.** *Let  $k \geq 2$  and  $M$  and  $P$  be integers,  $P$  being positive. Let  $f \in \mathcal{C}^{k+1}([M, M+P]; \mathbb{R})$ . Suppose that there exist positive absolute constants  $c_0, c_1, c_2, c_3$ , and  $c_4$  such that  $c_0 < 1$ ,  $c_1 < 1$ , and  $c_2 + c_4 < c_1$ ; an integer  $r$  such that  $c_0 k \leq r \leq k$ ; and distinct numbers  $s_j \geq 2$  ( $j = 1, \dots, r$ ) not exceeding  $k$ , such that for  $M \leq t \leq M+P$  the following inequalities are satisfied:*

- (i)  $|f^{(k+1)}(t)/(k+1)!| \leq P^{-c_1(k+1)}$ .
- (ii)  $P^{-c_2 s_j} \leq |f^{(s_j)}(t)/s_j!| \leq P^{-c_3 s_j}$ , ( $j = 1, \dots, r$ ).

Then, for each positive integer  $P_1$  not exceeding  $P$ , we have

$$\left| \sum_{M \leq m \leq M+P_1-1} e(f(m)) \right| \leq AP^{1-(c/k^2)}, \quad (8)$$

where  $e(t) := e^{2\pi i t}$  and  $A > 0$ ,  $c > 0$  are absolute constants.

The next two lemmas are essentially a special case of [7, Lemmas 2.5 and 2.6] with  $a = 1$ . The only difference is that the ranges of  $T$  and  $N$  here are slightly larger than those of [7, Lemmas 2.5 and 2.6] ( $T \geq N^2$  in place of  $T \geq N^{(3/2)}$  and  $N \leq x^{(2/3)}$  in place of  $N \leq x^{(1/2)}$ , respectively). Although the proof is completely similar, for the convenience of readers, we still reproduce a proof here.

**Lemma 2.** *Let  $e^{100} \leq N < N' \leq 2N$  and  $T \geq N^{(3/2)}$ . Then, there exists an absolute positive constant  $c_5$  such that*

$$\sum_{N \leq n < N'} e\left(\frac{T}{n}\right) \ll N \exp\left\{-\left(\frac{c_5 \log^3 N}{\log^2 T}\right)\right\}, \quad (9)$$

where the implied constant is absolute.

*Proof.* We apply Lemma 1 to  $f(t) := (T/t)$  with  $M = N$ ,  $P = N$ ,  $P_1 = N' - N$ . For this, we choose

$$\begin{aligned} c_0 &= \frac{1}{100}, \\ c_1 &= \frac{99}{100}, \\ c_2 &= \frac{87}{100}, \\ c_3 &= \frac{3}{4}, \\ c_4 &= \frac{1}{100}, \\ k &= \left\lceil 100 \frac{\log(T/n)}{\log N} \right\rceil, \end{aligned} \quad (10)$$

and take the  $s_j$  to be all integers  $s$  such that

$$4 \frac{\log(T/n)}{\log N} \leq s \leq 5 \frac{\log(T/n)}{\log N}. \quad (11)$$

Obviously the number  $r$  of  $s_j$  is between  $c_0 k$  and  $k$ . Next we shall verify that  $f(t)$  satisfies the conditions (i) and (ii) of Lemma 1 with the parameters chosen above.

For  $N \leq t \leq 2N$ , we have

$$\left| \frac{f^{(k+1)}(t)}{(k+1)!} \right| = T t^{-k-2} \leq T N^{-k-2} = N^{-\eta_1}, \quad (12)$$

where

$$\eta_1 := k + 1 - \frac{\log(T/n)}{\log N} \geq k + 1 - \frac{1}{100} k \geq \frac{99}{100} (k + 1) = c_1 (k + 1). \quad (13)$$

Similarly for  $N \leq t \leq 2N$ , we find the inequality  $|f^{(s_j)}(t)/s_j!| \leq N^{-\eta_3}$ , where

$$\eta_3 := s_j - \frac{\log(T/n)}{\log N} \geq \frac{3}{4} s_j = c_3 s_j. \quad (14)$$

For the lower bound of (ii), we have

$$\left| \frac{f^{(s_j)}(t)}{s_j!} \right| = T t^{-1-s_j} \geq T (2N)^{-1-s_j} = N^{-\eta_2}, \quad (15)$$

where

$$\begin{aligned} \eta_2 &:= s_j - \frac{\log(T/n)}{\log N} + \frac{\log 2}{\log N} (s_j + 1) \\ &\leq \frac{4}{5} s_j + \frac{\log 2}{100} (s_j + 1) \leq \frac{87}{100} s_j = c_2 s_j. \end{aligned} \quad (16)$$

From Lemma 1, there exist two positive constants  $c$  and  $A$  such that

$$\left| \sum_{N \leq n < N'} e\left(\frac{T}{n}\right) \right| \leq AN^{1-(c/k^2)} \leq AN \exp\left\{-\frac{c_5 \log^3 N}{\log^2(T/n)}\right\}, \quad (17)$$

with  $c_5 := 10^{-4}c$ . This completes the proof of Lemma 2.  $\square$

**Lemma 3.** Define  $\psi(t) := t - [t] - (1/2)$ . Let  $c_5$  be the constant defined by Lemma 2 and  $c_6 := (8/9)^2 c_5$ ,  $c^* := ((3/5)c_6)^{-(1/3)}$ . Then, we have

$$\sum_{N \leq n < N'} \frac{1}{n} \psi\left(\frac{x}{n}\right) \ll e^{-c_6 (\log N)^3 / (\log x)^2} \frac{(\log N)^3}{(\log x)^2}, \quad (18)$$

uniformly for  $x \geq 10$ ,  $\exp\{c^* (\log x)^{(2/3)}\} \leq N \leq x^{(2/3)}$  and  $N < N' \leq 2N$ .

*Proof.* By invoking a classical result on  $\psi(t)$  (see 8, page 39]) we can write, for any  $H \geq 1$ ,

$$\sum_{N \leq n < N'} \psi\left(\frac{x}{n}\right) \ll NH^{-1} + \sum_{1 \leq h \leq H} h^{-1} \left| \sum_{N \leq n < N'} e\left(\frac{hx}{n}\right) \right|. \quad (19)$$

An application of Lemma 2 with  $T = hx \geq x \geq N^{(3/2)}$  yields

$$\sum_{N \leq n < N'} \psi\left(\frac{x}{n}\right) \ll N \left( H^{-1} + e^{-c_5 (\log N)^3 / \log^2(Hx)} \log H \right). \quad (20)$$

Taking  $H = \exp\{(\log N)^3 / (\log x)^2\} \leq x^{(8/27)}$ , we easily deduce that

$$\sum_{N \leq n < N'} \psi\left(\frac{x}{n}\right) \ll N \left( e^{-(\log N)^3 / (\log x)^2} + e^{-c_6 (\log N)^3 / (\log x)^2} \frac{(\log N)^3}{(\log x)^2} \right). \quad (21)$$

The first term can be absorbed by the second, since  $c_5$  can be chosen small enough to ensure that  $c_6 < 1$  and since  $\exp\{c^* (\log x)^{(2/3)}\} \leq N$  implies  $(\log N)^3 / (\log x)^2 \geq c^*$ . Hence,

$$\sum_{N \leq n < N'} \psi\left(\frac{x}{n}\right) \ll N e^{-c_6 (\log N)^3 / (\log x)^2} \frac{(\log N)^3}{(\log x)^2}, \quad (22)$$

and an Abel summation produces the required result.  $\square$

**Lemma 4.** Let  $2 \leq z_1 < z_2 \leq x$  and  $F_x(t) := (1/t)\psi(x/t)$ . Denote by  $V_{F_x}[z_1, z_2]$  the total variation of  $F_x$  on  $[z_1, z_2]$ . Then,

$$V_{F_x}[z_1, z_2] \ll \frac{x}{z_1^2} + \frac{1}{z_1}, \quad (23)$$

where the implied constant is absolute.

*Proof.* If  $z_1 = t_0 < t_1 < \dots < t_n = z_2$  is a partition of the interval  $[z_1, z_2]$ , then

$$\begin{aligned} \sum_{k=1}^n |F_x(t_k) - F_x(t_{k-1})| &= \sum_{k=1}^n \left| \frac{1}{t_k} \psi\left(\frac{x}{t_k}\right) - \frac{1}{t_{k-1}} \psi\left(\frac{x}{t_{k-1}}\right) \right| \\ &\leq \sum_{k=1}^n \left| \left( \frac{1}{t_k} - \frac{1}{t_{k-1}} \right) \psi\left(\frac{x}{t_k}\right) \right| + \sum_{k=1}^n \frac{1}{t_{k-1}} \left| \psi\left(\frac{x}{t_k}\right) - \psi\left(\frac{x}{t_{k-1}}\right) \right|. \end{aligned} \quad (24)$$

Since  $|\psi(t)| \leq 1$  for all  $t$ , we have

$$\sum_{k=1}^n \left| \left( \frac{1}{t_k} - \frac{1}{t_{k-1}} \right) \psi\left(\frac{x}{t_k}\right) \right| \leq \frac{1}{z_1} - \frac{1}{z_2} < \frac{1}{z_1}. \quad (25)$$

On the other hand, since  $\psi(u)$  is of period 1, we have

$$\begin{aligned} \sum_{k=1}^n \frac{1}{t_{k-1}} \left| \psi\left(\frac{x}{t_k}\right) - \psi\left(\frac{x}{t_{k-1}}\right) \right| &\leq \frac{1}{z_1} \left( \frac{x}{z_1} + 1 \right) V_\psi[0, 1] \leq 2 \left( \frac{x}{z_1^2} + \frac{1}{z_1} \right). \end{aligned} \quad (26)$$

Inserting these two bounds into (24), we obtain the required result.  $\square$

### 3. Proof of Theorem 1

#### 3.1. A Formula on the Mean Value of $\sigma(n)$

##### Lemma 5

(i) For  $x \geq 2$  and  $1 \leq z \leq x^{(1/3)}$ , we have

$$\sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 - x \frac{(z - [z])^2 + [z]}{2z} + O\left(\frac{x}{z}\right) - \Delta(x, z), \quad (27)$$

where

$$\Delta(x, z) := \sum_{d \leq (x/z)} \frac{x}{d} \psi\left(\frac{x}{d}\right). \quad (28)$$

(ii) For  $x \rightarrow \infty$ , we have

$$\sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 + O(x \log x). \quad (29)$$

*Proof.* Using  $\sigma(n) = \sum_{dm=n} m$ , the hyperbole principle of Dirichlet allows us to write

$$\sum_{n \leq x} \sigma(n) = \sum_{dm \leq x} m = S_1 + S_2 - S_3, \quad (30)$$

where

$$\begin{aligned}
S_1 &:= \sum_{d \leq (x/z)} \sum_{m \leq (x/d)} m, \\
S_2 &:= \sum_{m \leq z} \sum_{d \leq (x/m)} m, \\
S_3 &:= \sum_{d \leq (x/z)} \sum_{m \leq z} m.
\end{aligned} \tag{31}$$

Firstly we have

$$S_2 = \sum_{m \leq z} m \left\lfloor \frac{x}{m} \right\rfloor = x[z] + O(z^2), \tag{32}$$

$$S_3 = \left\lfloor \frac{x}{z} \right\rfloor \frac{[z]([z] + 1)}{2} = x \frac{[z]([z] + 1)}{2z} + O(z^2). \tag{33}$$

Secondly we can write

$$\begin{aligned}
S_1 &= \frac{1}{2} \sum_{d \leq (x/z)} \left( \frac{x}{d} - \psi\left(\frac{x}{d}\right) - \frac{1}{2} \right) \left( \frac{x}{d} - \psi\left(\frac{x}{d}\right) + \frac{1}{2} \right) \\
&= \frac{1}{2} \sum_{d \leq (x/z)} \left( \frac{x^2}{d^2} - 2 \frac{x}{d} \psi\left(\frac{x}{d}\right) + \psi\left(\frac{x}{d}\right)^2 - \frac{1}{4} \right) \\
&= \frac{\pi^2}{12} x^2 - \frac{1}{2} xz - \Delta(x, z) + O(x/z),
\end{aligned} \tag{34}$$

where  $\Delta(x, z)$  is as in (28). Inserting (32), (33), and (34) into (30) and using  $z^2 \leq (x/z)$ , we get (27).

Taking  $z = 1$  in (27) and noticing that

$$\begin{aligned}
\sum_{d \leq x} \frac{1}{d^2} &= \frac{\pi^2}{6} + O\left(\frac{1}{x}\right), \\
\left| \sum_{d \leq x} \frac{x}{d} \psi\left(\frac{x}{d}\right) \right| &\ll x \log x,
\end{aligned} \tag{35}$$

we obtain the required bound. This completes the proof.  $\square$

### 3.2. Estimates of Error Terms

**Lemma 6.** Let  $N_0 := \exp\{(6/c_6)(\log x)^{(2/3)}(\log_2 x)^{(1/3)}\}$ , where  $c_6$  is given as in Lemma 3. Let  $\Delta(x, z)$  be defined by (28). Then, for  $x \geq 10$  and  $2 \leq z \leq \sqrt{N_0}$ , we have

$$\begin{aligned}
&\left| \sum_{N_0 < n \leq \sqrt{x}} \Delta\left(\frac{x}{n}, z\right) \right| + \left| \sum_{N_0 < n \leq \sqrt{x}} \Delta\left(\frac{x}{n} - 1, z\right) \right| \\
&\ll x \left( \frac{1}{(\log x)^3} + \frac{\log x}{z} \right).
\end{aligned} \tag{36}$$

*Proof.* Denote by  $\Delta_1(x, z)$  and  $\Delta_2(x, z)$  two sums on the left-hand side of (36), respectively. By (28) of Lemma 5, we can write

$$\begin{aligned}
\Delta_1(x, z) &= x \sum_{N_0 < n \leq \sqrt{x}} \sum_{d \leq (x/(nz))} \frac{1}{dn} \psi\left(\frac{x}{dn}\right) \\
&= x \sum_{d \leq (x/(N_0 z))} \frac{1}{d} \sum_{N_0 < n \leq \min(\sqrt{x}, x/(dz))} \frac{1}{n} \psi\left(\frac{x}{dn}\right) \\
&= x \Delta_1^\dagger(x, z) + x \Delta_1^\sharp(x, z),
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
\Delta_1^\dagger(x, z) &:= \sum_{d \leq (x/(N_0 z))} \frac{1}{d} \sum_{N_0 < n \leq (x/d)^{(2/3)}} \frac{1}{n} \psi\left(\frac{x}{dn}\right), \\
\Delta_1^\sharp(x, z) &:= \sum_{d \leq (x/(N_0 z))} \frac{1}{d} \sum_{(x/d)^{(2/3)} < n \leq \min\{\sqrt{x}, x/(dz)\}} \frac{1}{n} \psi\left(\frac{x}{dn}\right).
\end{aligned} \tag{38}$$

For  $0 \leq k \leq (\log((x/d)^{(2/3)}/N_0))/\log 2$ , let  $N_k := 2^k N_0$  and define

$$\mathfrak{S}_k(d) := \sum_{N_k < n \leq 2N_k} \frac{1}{n} \psi\left(\frac{x}{dn}\right). \tag{39}$$

Noticing that  $N_0 \leq N_k \leq (x/d)^{(2/3)}$ , we can apply Lemma 3 to derive that

$$\mathfrak{S}_k(d) \ll e^{-\vartheta((\log N_k)^3/(\log(x/d))^2)}, \tag{40}$$

with  $\vartheta(t) := c_6 t - \log t$ . It is clear that  $\vartheta(t)$  is increasing on  $[c_6, \infty)$ . On the other hand, for  $k \geq 0$  and  $d \geq 1$ , we have

$$(\log N_k)^3/(\log(x/d))^2 \geq (\log N_0)^3/(\log x)^2 = (6/c_6) \log_2 x. \tag{41}$$

Thus,

$$\begin{aligned}
\vartheta\left(\frac{(\log N_k)^3}{(\log(x/d))^2}\right) &\geq \vartheta\left(\left(\frac{6}{c_6}\right) \log_2 x\right) \\
&= 6 \log_2 x - \log\left(\left(\frac{6}{c_6}\right) \log_2 x\right) \geq 5 \log_2 x,
\end{aligned} \tag{42}$$

which implies that  $\mathfrak{S}_k(d) \ll (\log x)^{-5}$ . Inserting this into the expression of  $\Delta_1^\dagger(x, z)$ , we get

$$\Delta_1^\dagger(x, z) \ll \sum_{d \leq (x/(N_0 z))} \frac{1}{d} \sum_{2^k N_0 \leq (x/d)^{(2/3)}} |\mathfrak{S}_k(d)| \ll (\log x)^{-3}. \tag{43}$$

Next we bound  $\Delta_1^\sharp(x, z)$ . Let  $F(t)$  be a function of bounded variation on  $[n, n+1]$  for each integer  $n$  and let  $V_F[n, n+1]$  be the total variation of  $F$  on  $[n, n+1]$ . Integrating by parts, we have

$$\int_n^{n+1} \left(t - n - \frac{1}{2}\right) dF(t) = \frac{1}{2}(F(n+1) + F(n)) - \int_n^{n+1} F(t) dt. \quad (44)$$

From this, we can derive that

$$\frac{1}{2}(F(n+1) + F(n)) = \int_n^{n+1} F(t) dt + O(V_F[n, n+1]), \quad (45)$$

for  $n \geq 1$ . Summing over  $n$ , we find that

$$\begin{aligned} \sum_{N_1 < n \leq N_2} F(n) &= \int_{N_1}^{N_2} F(t) dt \\ &+ \frac{1}{2}(F(N_1) + F(N_2)) + O(V_F[N_1, N_2]). \end{aligned} \quad (46)$$

We apply this formula to

$$F_{(x/d)}(t) = \frac{1}{t} \psi\left(\frac{(x/d)}{t}\right), \quad (47)$$

$$N_1 = \left\lfloor (x/d)^{(2/3)} \right\rfloor,$$

$$N_2 = \left\lceil \min \left\{ \sqrt{x}, \frac{x}{(dz)} \right\} \right\rceil.$$

According to Lemma 4, we have  $V_{F_{(x/d)}}[N_1, N_2] \ll (x/d)^{-(1/3)}$ , and thus by putting  $u = (x/d)/t$ , we obtain, with the notation  $x_{d,1} = \max(\sqrt{x}/d, tz)$  and  $x_{d,2} = (x/d)^{(1/3)}$ ,

$$\begin{aligned} \sum_{(x/d)^{(2/3)} < n \leq \min\{\sqrt{x}, (x/(dz))\}} \frac{1}{n} \psi\left(\frac{x}{dn}\right) &= \int_{x_{d,1}}^{x_{d,2}} \frac{\psi(u)}{u} du + O\left(\left(\frac{x}{d}\right)^{-(1/3)}\right) \\ &\ll z^{-1} + \left(\frac{x}{d}\right)^{-(1/3)} \ll z^{-1}, \end{aligned} \quad (48)$$

where we have used the fact that  $z \leq \sqrt{N_0}$  and  $d \leq (x/(N_0 z)) \Rightarrow z \leq (x/d)^{(1/3)}$  and the bound

$$\begin{aligned} \int_{x_{d,1}}^{x_{d,2}} \frac{\psi(u)}{u} du &= \int_{x_{d,1}}^{x_{d,2}} \left( \int_{x_{d,1}}^u \psi(t) dt \right) \frac{du}{u^2} - \frac{1}{x_{d,2}^{(2/3)}} \int_{x_{d,1}}^{x_{d,2}} \psi(t) dt \\ &\ll x_{d,1}^{-1} + (x_{d,2})^{-(2/3)} \ll z^{-1} + (x/d)^{-(2/3)} \ll z^{-1}. \end{aligned} \quad (49)$$

Using (48), a simple partial integration allows us to derive that

$$\Delta_1^\#(x, z) \ll z^{-1} \sum_{d \leq x/(N_0 z)} d^{-1} \ll z^{-1} \log x. \quad (50)$$

Combining (43) and (50), it follows that

$$|\Delta_1(x, z)| \ll x(\log x)^{-3} + xz^{-1} \log x. \quad (51)$$

Similarly, we can prove the same bound for  $|\Delta_2(x, z)|$ . This completes the proof.  $\square$

**3.3. End of the Proof of Theorem 1.** Let  $c_6$  be the constant given as in Lemma 3 and  $N_0 := \exp\{(6/c_6)(\log x)^{(2/3)}(\log_2 x)^{(1/3)}\}$ . Let  $z \in [2, \sqrt{N_0}]$  be a parameter to be chosen later.

Putting  $d = \lfloor x/n \rfloor$ , we have  $(x/n) - 1 < d \leq (x/n)$  and  $x/(d+1) < n \leq (x/d)$ . We have, with the convention  $\sigma(0) = 0$ ,

$$\begin{aligned} S_\sigma(x) &= \sum_{d \leq x} \sigma(d) \sum_{(x/(d+1)) < n \leq (x/d)} 1 \\ &= \sum_{dn \leq x} \sigma(d) - \sum_{dn \leq x, d \geq 2} \sigma(d-1) \\ &= \sum_{dn \leq x} (\sigma(d) - \sigma(d-1)). \end{aligned} \quad (52)$$

By the hyperbole principle of Dirichlet, we can write

$$S_\sigma(x) = S_1(x, \sigma) + S_2(x, \sigma) - S_3(x, \sigma), \quad (53)$$

where

$$\begin{aligned} S_1(x, \sigma) &:= \sum_{d \leq \sqrt{x}, dn \leq x} (\sigma(d) - \sigma(d-1)), \\ S_2(x, \sigma) &:= \sum_{n \leq \sqrt{x}, dn \leq x} (\sigma(d) - \sigma(d-1)), \\ S_3(x, \sigma) &:= \sum_{d \leq \sqrt{x}, n \leq \sqrt{x}} (\sigma(d) - \sigma(d-1)). \end{aligned} \quad (54)$$

With the help of the bound  $\sigma(n) \ll n \log_2 n$ , we can derive that

$$S_3(x, \sigma) = \lfloor \sqrt{x} \rfloor \sigma(\lfloor \sqrt{x} \rfloor) \ll x \log_2 x. \quad (55)$$

For evaluating  $S_1(x, \sigma)$ , we write

$$\begin{aligned} S_1(x, \sigma) &= \sum_{d \leq \sqrt{x}} (\sigma(d) - \sigma(d-1)) \left\lfloor \frac{x}{d} \right\rfloor \\ &= x \sum_{d \leq \sqrt{x}} \frac{\sigma(d) - \sigma(d-1)}{d} + O\left( \sum_{d \leq \sqrt{x}} |\sigma(d) - \sigma(d-1)| \right). \end{aligned} \quad (56)$$

With the help of Lemma 5 (ii), a simple partial integration gives us

$$\begin{aligned}
\sum_{d \leq \sqrt{x}} \frac{\sigma(d) - \sigma(d-1)}{d} &= \sum_{d \leq \sqrt{x}} \frac{\sigma(d)}{d(d+1)} \\
&= \sum_{d \leq \sqrt{x}} \frac{\sigma(d)}{d^2} - \sum_{d \leq \sqrt{x}} \frac{\sigma(d)}{d^2(d+1)} \\
&= \int_{1-}^{\sqrt{x}} t^{-2} d\left(\frac{\pi^2}{12} t^2 + O(t \log t)\right) + O(1) \\
&= \frac{\pi^2}{12} \log x + O(1), \\
\sum_{d \leq \sqrt{x}} |\sigma(d) - \sigma(d-1)| \\
&\ll \sum_{d \leq \sqrt{x}} \sigma(d) \ll x.
\end{aligned} \tag{57}$$

Inserting these estimates into (56), we find that

$$S_1(x, \sigma) = \frac{\pi^2}{12} x \log x + O(x). \tag{58}$$

Finally, we evaluate  $S_2(x, \sigma)$ . For this, we write

$$S_2(x, \sigma) = S_2^\dagger(x, \sigma) + S_2^\sharp(x, \sigma), \tag{59}$$

where

$$\begin{aligned}
S_2^\dagger(x, \sigma) &:= \sum_{n \leq N_0, dn \leq x} (\sigma(d) - \sigma(d-1)), \\
S_2^\sharp(x, \sigma) &:= \sum_{N_0 < n \leq \sqrt{x}, dn \leq x} (\sigma(d) - \sigma(d-1)).
\end{aligned} \tag{60}$$

By the bound that  $\sigma(n) \ll n \log_2 n$ , we have

$$\begin{aligned}
S_2^\dagger(x, \sigma) &= \sum_{n \leq N_0} \sigma\left(\left[\frac{x}{n}\right]\right) \ll \sum_{n \leq N_0} \left(\frac{x}{n}\right) \log_2 x \\
&\ll x (\log x)^{(2/3)} (\log_2 x)^{(4/3)}.
\end{aligned} \tag{61}$$

On the other hand, (27) of Lemma 5 allows us to derive that

$$\begin{aligned}
\sum_{d \leq x} \sigma(d) - \sum_{d \leq x-1} \sigma(d) &= \frac{\pi^2}{12} (x^2 - (x-1)^2) - \Delta(x, z) \\
&\quad + \Delta(x-1, z) + O\left(\frac{x}{z}\right) \\
&= \frac{\pi^2}{6} x - \Delta(x, z) + \Delta(x-1, z) + O\left(\frac{x}{z}\right),
\end{aligned} \tag{62}$$

where  $\Delta(x, z)$  is given by (28). Thus,

$$\begin{aligned}
S_2^\sharp(x, \sigma) &= \sum_{N_0 < n \leq \sqrt{x}} \left( \frac{\pi^2}{6} \cdot \frac{x}{n} - \Delta\left(\frac{x}{n}, z\right) + \Delta\left(\frac{x}{n} - 1, z\right) + O\left(\frac{x}{nz}\right) \right) \\
&= \frac{\pi^2}{12} x \log x + O\left(x (\log x)^{(2/3)} (\log_2 x)^{(1/3)} + xz^{-1} \log x\right) - \Delta_1(x, z) + \Delta_2(x, z),
\end{aligned} \tag{63}$$

where

$$\begin{aligned}
\Delta_1(x, z) &:= \sum_{N_0 < n \leq \sqrt{x}} \Delta\left(\frac{x}{n}, z\right) \ll x (\log x)^{-3} + xz^{-1} \log x, \\
\Delta_2(x, z) &:= \sum_{N_0 < n \leq \sqrt{x}} \Delta\left(\frac{x}{n} - 1, z\right) \ll x (\log x)^{-3} + xz^{-1} \log x,
\end{aligned} \tag{64}$$

thanks to Lemma 6. Inserting these estimates into (59), we find that

$$S_2(x, \sigma) = \frac{\pi^2}{12} x \log x + O\left(x (\log x)^{(2/3)} (\log_2 x)^{(4/3)} + xz^{-1} \log x\right), \tag{65}$$

Now (5) follows from (53), (55), (58), and (66) with the choice of  $z = (\log x)^{(1/3)}$ .

**3.4. Proof of Theorem 1.** (ii) For any odd prime  $p$ , (52) allows us to write

$$\begin{aligned}
\sum_{d|p} (\sigma(d) - \sigma(d-1)) &= S_\sigma(p) - S_\sigma(p-1) \\
&= \frac{\pi^2}{6} (\log p - \log(p-1)) + E(p) - E(p-1) \\
&\geq E(p) - E(p-1) \geq -2E^*(p),
\end{aligned} \tag{66}$$

where  $E^*(p) := \max\{|E(p)|, |E(p-1)|\}$ . On the other hand, we have

$$\begin{aligned}
\sum_{d|p} (\sigma(d) - \sigma(d-1)) &= \sigma(p) - \sigma(p-1) + 1 \\
&\leq p + 1 - \left(p - 1 + \frac{1}{2}(p-1) + 2 + 1\right) + 1 \leq -\frac{1}{4}p.
\end{aligned} \tag{67}$$

Thus,  $E^*(p) \geq (1/8)p$  for all odd primes.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Acknowledgments

This study was supported in part by the National Natural Science Foundation of China (grant nos. 11871193, 11971370, and 12071375), Natural Science Foundation of Chongqing (grant no. cstc2019jcyj-msxm1651), National Natural Science Foundation of Henan Province (grant no. 202300410274), and Research Projects for Overseas Researchers of Department of Human Resources and Social Security of Henan Province 2020.

## References

- [1] O. Bordellès, L. Dai, R. Heyman, H. Pan, and I. E. Shparlinski, "On a sum involving the Euler function," *Journal of Number Theory*, vol. 202, pp. 278–297, 2019.
- [2] J. Wu, "On a sum involving the Euler totient function," *Indagationes Mathematicae*, vol. 30, no. 4, pp. 536–541, 2019.
- [3] W. Zhai, "On a sum involving the euler function," *Journal of Number Theory*, vol. 211, pp. 199–219, 2020.
- [4] J. Ma and J. Wu, "On a sum involving the Mangoldt function," *Periodica Mathematica Hungarica*, vol. 80, 2020.
- [5] J. Wu, "Note on a paper by Bordellès, Dai, heyman, Pan and Shparlinski," *Periodica Mathematica Hungarica*, vol. 80, no. 1, pp. 95–102, 2020.
- [6] A. A. Karatsuba, "Estimates for trigonometric sums by Vinogradov's method, and some applications," *Proceedings of the Steklov Institute of Mathematics*, vol. 112, pp. 251–265, 1971.
- [7] Y.-F. S. Pétermann and J. Wu, "On the sum of exponential divisors of an integer," *Acta Mathematica Hungarica*, vol. 77, no. 1/2, pp. 159–175, 1997.
- [8] S. W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge University Press, Cambridge, UK, 1991.