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# The mKdV equation and multi-parameters rational solutions

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## Abstract

$N$ -order solutions to the modified Korteweg-de Vries (mKdV) equation are given in terms of a quotient of two wronskians of order  $N$  depending on  $2N$  real parameters. When one of these parameters goes to 0, we succeed to get for each positive integer  $N$ , rational solutions as a quotient of polynomials in  $x$  and  $t$  depending on  $2N$  real parameters. We construct explicit expressions of these rational solutions for orders  $N = 1$  until  $N = 6$ .

**Key Words :** mKdV equation, wronskians, rational solutions.

## PACS numbers :

33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

## 1 Introduction

We consider the modified Korteweg-de Vries (mKdV) equation

$$u_t - 6u^2u_x + u_{xxx} = 0, \quad (1)$$

with  $u_t = \partial_t u$ ,  $u_x = \partial_x u$  and  $u_{xxx} = \partial_x^3 u$ .

The mKdV equation has many applications in various fields as in the study of waves propagating in plasma [3], the dynamics of traffic flow [4] and fluid mechanics [5]. In particular, it is used in nonlinear optics as for example to model supercontinuum generation in optical fibres [1] or to describe pulses consisting of a few optical cycles [2].

Various methods have been used to construct solutions to the mKdV equation. Hirota [6] constructed the exact soliton for the mKdV equation in 1972. In the same year, Tanaka [7] was the first to solve the mKdV equation by using the inverse scattering technique. Wadati using this same method succeeded to obtain

the exact  $N$ -soliton solution for the mKdV [8]. In 1976, rational solutions to the mKdV equation [9] were derived by Ono by using Bäcklund transformation. A limiting procedure based on bilinear results gave other type of solutions [10] in 1978. In 1983, rational solutions to the mKdV equation were determined by recurrence relations in [11] by Tanaka. Solitary wave solutions and cnoidal wave solutions were given in 1994 in [12]. Some classes of periodic solutions of mKdV have been given in [13] in 2004.

More recently, in 2012, solutions to the mKdV equation has been constructed via bilinear Bäcklund transformation in [14] and rational solutions in terms of Wronskians were obtained. In 2016, periodic solutions and rational solution of first and second order were presented in [15] by using a Darboux transformation were constructed.

Here, Darboux transformation is used to construct different type of solutions. Representations of solutions in terms of wronskians of order  $N$  depending on  $2N$  real parameters are given, using trigonometric or hyperbolic functions. Rational solutions are obtained in performing a passage to the limit when one of these parameters goes to 0. So rational solutions as a quotient of polynomials in  $x$  and  $t$ , depending on  $2N$  parameters are constructed. We give explicit solutions in the simplest cases  $N = 1, 2, 3$  and some particular rational solutions for  $N = 1$  until 6.

## 2 N-order solutions to the mKdV equation in terms of wronskians

### 2.1 N-order solutions in terms of wronskians of hyperbolic sine functions

We consider the mKdV equation

$$u_t - 6u^2u_x + u_{xxx} = 0.$$

We recall that the wronskian of order  $N$  of the functions  $f_1, \dots, f_N$  is the determinant denoted  $W(f_1, \dots, f_N)$ , defined by  $\det(\partial_x^{i-1} f_j)_{1 \leq i \leq N, 1 \leq j \leq N}$ ,  $\partial_x^i$  being the partial derivative of order  $i$  with respect to  $x$  and  $\partial_x^0 f_j$  being the function  $f_j$ .

We consider  $a_j, b_j$  arbitrary real numbers  $1 \leq j \leq N$ . We have the following result :

**Theorem 2.1** *Let  $f_j$  be the functions defined by*

$$f_j(x, t) = \sinh\left(\frac{1}{2}a_jx - \frac{1}{2}a_j^3t + b_j\right), \quad \text{for } 1 \leq i \leq N, \quad (2)$$

then the function  $u$  defined by

$$u(x, t) = \partial_x \ln \left( \frac{W(\partial_x(f_1), \dots, \partial_x(f_N))}{W(f_1, \dots, f_N)} \right) \quad (3)$$

is a solution to the mKdV equation (1) depending on  $2N$  real parameters  $a_j, b_j$ ,  $1 \leq j \leq N$ .

**Proof :** For the Korteweg de Vries (4) (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0, \quad (4)$$

the Lax pairs can be written as

$$\begin{cases} -\phi_{xx} + u\phi = \lambda\phi, \\ \phi_t = -4\phi_{xxx} + 6v\phi_x + 3v_x u\phi. \end{cases} \quad (5)$$

This system is covariant [16] by the Darboux transformation. It means that if  $\phi_1, \dots, \phi_N, \phi$  are solutions of the system (5) respectively associated to  $\lambda_1, \dots, \lambda_N, \lambda$ , then  $\phi[N]$  defined by  $\phi[N] = \frac{W(\phi_1, \dots, \phi_N, \phi)}{W(\phi_1, \dots, \phi_N)}$  is another solution of this system (5) associated to  $\lambda$  where  $u$  is replaced by  $u[N] = u - 2(\ln W(\phi_1, \dots, \phi_N))_{xx}$ . We choose  $u = 0$ , then the functions  $\phi_j = f_j$  defined in (2) verify the following system

$$\begin{cases} -\phi_{xx} = \lambda\phi, \\ \phi_t = -4\phi_{xx}. \end{cases} \quad (6)$$

We can deduce that the solution of the system (5) associated to  $\lambda$  can be written as  $\varphi(x, t) = \frac{W(f_1, \dots, f_N, f)}{W(f_1, \dots, f_N)}$ , for  $f$  any solution of (5) associated with  $\lambda$ .

From [17] we know that if  $\varphi$  is a solution of the system (5), then the function  $w$  defined by  $w(x, t) = \partial_x \ln(\varphi(x, t))$  is a solution to the equation

$$w_t - 6(\lambda + w^2)w_x + w_{xxx} = 0. \quad (7)$$

Thus the function  $u$  defined by  $u(x, t) = \partial_x \ln \left( \frac{W(f_1, \dots, f_N, f)}{W(f_1, \dots, f_N)} \right)$  is a solution of the equation (7).

If we choose for  $f$ , the constant function  $f(x, t) = 1$ , associated to  $\lambda = 0$ , then  $u(x, t) = \partial_x \ln \left( \frac{W(f_1, \dots, f_N, f)}{W(f_1, \dots, f_N)} \right)$  is a solution of the equation (1).

By expanding the determinant of the numerator of  $u(x, t)$  along his last column, the solution  $u(x, t)$  can be rewritten as  $u(x, t) = \partial_x \ln \left( \frac{W(\partial_x(f_1), \dots, \partial_x(f_N))}{W(f_1, \dots, f_N)} \right)$ , which proves the result.

## 2.2 Some examples of solutions to the mKdV equation with sine hyperbolic generating functions

In the following we only give the solutions of order 1, 2 and 3 in the case of generating hyperbolic sinus functions.

### Solution of order 1

**Proposition 2.1** *The function  $u$  defined by*

$$u(x, t) = \frac{a_1}{\sinh(-a_1 x + a_1^3 t - 2b_1)}$$

*is a solution to the mKdV equation (1) with  $a_1, b_1$  arbitrarily real parameters.*

### Solution of order 2

**Proposition 2.2** *The function  $u$  defined by*

$$u(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (8)$$

*with*

$$n(x, t) = -\frac{1}{2}(a_1^2 - a_2^2)[a_2 \sinh(-a_1 x + a_1^3 t - 2b_1) - a_1 \sinh(-a_2 x + a_2^3 t - 2b_2)]$$

*and,*

$$d(x, t) = 2(-\sinh(-1/2 a_1 x + 1/2 a_1^3 t - b_1) \cosh(-1/2 a_2 x + 1/2 a_2^3 t - b_2) a_2 + \sinh(-1/2 a_2 x + 1/2 a_2^3 t - b_2) \cosh(-1/2 a_1 x + 1/2 a_1^3 t - b_1) a_1) (\cosh(-1/2 a_1 x + 1/2 a_1^3 t - b_1) \sinh(-1/2 a_2 x + 1/2 a_2^3 t - b_2) a_2 - \cosh(-1/2 a_2 x + 1/2 a_2^3 t - b_2) \sinh(-1/2 a_1 x + 1/2 a_1^3 t - b_1) a_1)$$

*is a solution to the mKdV equation (1) with  $a_1, a_2, b_1, b_2$  arbitrarily real parameters.*

### Solution of order 3

In this case of order 3, we only present solution with  $a_1 = 1, a_2 = 2, a_3 = 3, b_1 = 0, b_2 = 0, b_3 = 0$  to shorten the paper.

**Proposition 2.3** *The function  $u$  defined by*

$$u(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (9)$$

*with*

$$n(x, t) = -150 (\cosh(-1/2 x + 1/2 t))^2 - 384 (\cosh(-x + 4 t - 2))^2 - 54 (\cosh(-3/2 x + \frac{27}{2} t - 3))^2 + 300 \sinh(-1/2 x + 1/2 t) \cosh(-3/2 x + \frac{27}{2} t - 3) \cosh(-1/2 x + 1/2 t) \sinh(-3/2 x + \frac{27}{2} t - 3) - 312 \cosh(-3/2 x + \frac{27}{2} t - 3) \sinh(-x + 4 t - 2) \sinh(-3/2 x + \frac{27}{2} t - 3) \cosh(-x + 4 t - 2) - 180 (\cosh(-1/2 x + 1/2 t))^2 (\cosh(-3/2 x + \frac{27}{2} t - 3))^2 + 480 (\cosh(-x + 4 t - 2))^2 (\cosh(-1/2 x + 1/2 t))^2 + 288 (\cosh(-x + 4 t - 2))^2 (\cosh(-3/2 x + \frac{27}{2} t - 3))^2 - 600 \sinh(-1/2 x + 1/2 t) \cosh(-x + 4 t - 2) \cosh(-1/2 x + 1/2 t) \sinh(-x + 4 t - 2)$$

*and,*

$$d(x, t) = 2(16 \sinh(-1/2 x + 1/2 t) \cosh(-x + 4 t - 2) \sinh(-3/2 x + \frac{27}{2} t - 3) - 9 \sinh(-1/2 x + 1/2 t) \cosh(-3/2 x + \frac{27}{2} t - 3) \sinh(-x + 4 t - 2) - 5 \cosh(-1/2 x + 1/2 t) \sinh(-x + 4 t - 2)$$

$2) \sinh(-3/2x + \frac{27}{2}t - 3))(-16 \cosh(-1/2x + 1/2t) \sinh(-x + 4t - 2) \cosh(-3/2x + \frac{27}{2}t - 3) + 9 \cosh(-1/2x + 1/2t) \sinh(-3/2x + \frac{27}{2}t - 3) \cosh(-x + 4t - 2) + 5 \sinh(-1/2x + 1/2t) \cosh(-x + 4t - 2) \cosh(-3/2x + \frac{27}{2}t - 3))$   
*is a solution to the mKdV equation (1).*

### 2.3 Other types of solutions

We obtain similar results with other types of generating functions whose proofs are identical.

#### 2.3.1 Solutions with hyperbolic cosine generating functions

**Theorem 2.2** *Let  $h_j, h$  be the following functions*

$$h_j(x, t) = \cosh\left(\frac{1}{2}a_j x - \frac{1}{2}a_j^3 t + b_j\right), \quad \text{for } 1 \leq i \leq N, \quad (10)$$

*then the function  $u$  defined by*

$$u(x, t) = \partial_x \ln\left(\frac{W(\partial_x(h_1), \dots, \partial_x(h_N))}{W(h_1, \dots, h_N)}\right) \quad (11)$$

*is a solution to the mKdV equation (1) with  $a_j, b_j$   $1 \leq j \leq N$  arbitrarily real parameters.*

#### 2.3.2 Solutions with trigonometric generating functions

**Theorem 2.3** *Let  $g_j$  be the following functions*

$$g_j(x, t) = \cos\left(\frac{1}{2}a_j x + \frac{1}{2}a_j^3 t + b_j\right), \quad \text{for } 1 \leq i \leq N, \quad (12)$$

*then the function  $u$  defined by*

$$u(x, t) = \partial_x \ln\left(\frac{W(\partial_x(g_1), \dots, \partial_x(g_N))}{W(g_1, \dots, g_N)}\right) \quad (13)$$

*is a solution to the mKdV equation (1) with  $a_j, b_j$   $1 \leq j \leq N$  arbitrarily real parameters.*

**Theorem 2.4** *Let  $k_j$  be the following functions*

$$k_j(x, t) = \sin\left(\frac{1}{2}a_j x + \frac{1}{2}a_j^3 t + b_j\right), \quad \text{for } 1 \leq i \leq N, \quad (14)$$

then the function  $u$  defined by

$$u(x, t) = \partial_x \ln \left( \frac{W(\partial_x(k_1), \dots, \partial_x(k_N))}{W(k_1, \dots, k_N)} \right) \quad (15)$$

is a solution to the mKdV equation (1) with  $a_j, b_j$   $1 \leq j \leq N$  arbitrarily real parameters.

### 3 Rational solutions to the mKdV equation

In the following, we replace all parameters  $a_j$  and  $b_j$ ,  $1 \leq j \leq N$  by  $\hat{a}_j = \sum_{k=1}^N a_k(je)^{2k-1}$  and  $\hat{b}_j = \sum_{k=1}^N b_k(je)^{2k-1}$  with  $e$  an arbitrary real parameter. We realize this change to obtain rational solutions to the mKdV equation (1); for this, we perform a limit when the parameter  $e$  tends to 0.

We get the following result :

**Theorem 3.1** *Let  $\psi_j$  be the functions*

$$\psi_j(x, t, e) = \sinh \left( \frac{1}{2} \sum_{k=1}^N a_k(je)^{2k-1} x - \frac{1}{2} \left( \sum_{k=1}^N a_k(je)^{2k-1} \right)^3 t + \sum_{k=1}^N b_k(je)^{2k-1} \right),$$

for  $1 \leq j \leq N$ ,

then the function  $u$  defined by

$$u(x, t) = \lim_{e \rightarrow 0} \partial_x \ln \left( \frac{W(\partial_x(\psi_1), \dots, \partial_x(\psi_N))}{W(\psi_1, \dots, \psi_N)} \right) \quad (16)$$

is a rational solution to the mKdV equation (1).

**Proof :** It is a direct consequence of the result of the previous section.

□

We have similar results with generating sine or trigonometric functions.

We can also give the expression of the rational solutions of the mKdV without the presence of a limit.

We get the following result :

**Theorem 3.2** *Let  $\psi, \varphi_j$  be the functions*

$$\psi(x, t, e) = \sinh \left( \frac{1}{2} \left( \sum_{k=1}^N a_k e^{2k-1} \right) x - \frac{1}{2} \left( \sum_{k=1}^N a_k e^{2k-1} \right)^3 t + \sum_{k=1}^N b_k e^{2k-1} \right),$$

$$\varphi_j(x, t) = \frac{\partial^{2j-1} \psi(x, t, 0)}{\partial_{2j-1} e}, \text{ for } 1 \leq j \leq N,$$

then the function  $v$  defined by

$$v(x, t) = \partial_x \ln \left( \frac{W(\partial_x(\varphi_1), \dots, \partial_x(\varphi_N))}{W(\varphi_1, \dots, \varphi_N)} \right) \quad (17)$$

is a rational solution to the mKdV equation (1) depending on  $2N$  parameters  $a_j, b_j$ ,  $1 \leq j \leq N$ .

**Proof :** For this, we combine the columns of the determinant of the previous theorem and we take a passage to the limit when  $\epsilon$  tends to 0 for each column.

□

We give some examples of rational solutions in the following.

It must be pointed out that these resulting rational solutions are singular.

These results are consequences of the previous result.

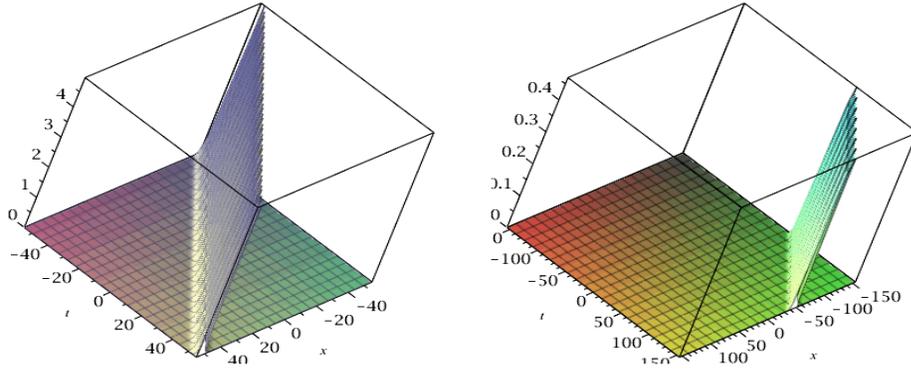
### 3.1 First order rational solutions

We have the following result at order  $N = 1$  :

**Proposition 3.1** *The function  $v$  defined by*

$$v(x, t) = \frac{-a_1}{a_1 x + 2b_1}, \quad (18)$$

*is a rational solution to the mKdV equation (1) with  $a_1, b_1$ , arbitrarily real parameters.*



**Figure 1.** Solution of order 1 to (1), on the left  $a_1 = 1, b_1 = 1$ ; on the right  $a_1 = 1, b_1 = 10^2$ .

### 3.2 Second order rational solutions

**Proposition 3.2** *The function  $v$  defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (19)$$

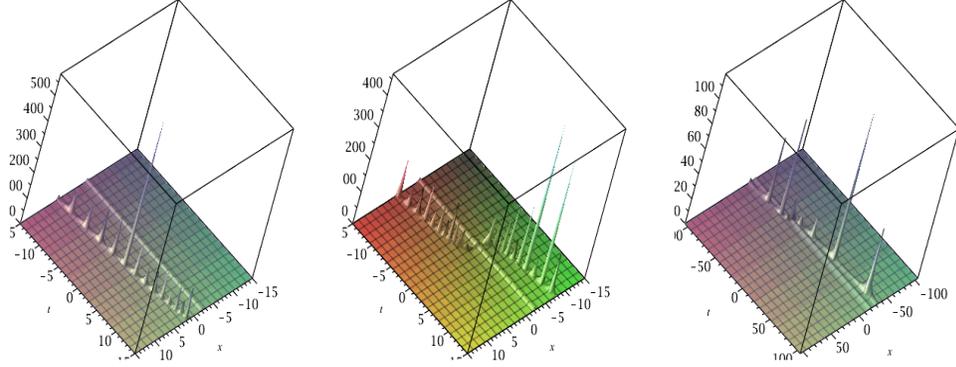
*with*

$$n(x, t) = -2a_1^5 x^3 - 12a_1^4 b_1 x^2 - 24a_1^3 b_1^2 x + 2a_1(12a_2 b_1 + 6a_1^4 t - 12b_2 a_1 - 8a_1 b_1^3),$$

*and,*

$$d(x, t) = a_1^5 x^4 + 8a_1^4 b_1 x^3 + 24a_1^3 b_1^2 x^2 + (24a_1^2 b_1^3 + a_1(24a_2 b_1 + 12a_1^4 t - 24b_2 a_1 + 8a_1 b_1^3))x + 2b_1(24a_2 b_1 + 12a_1^4 t - 24b_2 a_1 + 8a_1 b_1^3)$$

*is a rational solution to the mKdV equation (1) dependant on 4 real parameters  $a_1, a_2, b_1, b_2$ .*



**Figure 2.** Solution of order 2 to (1), on the left  $a_1 = 1, a_2 = 2, b_1 = 10, b_2 = 2$ ; in the center  $a_1 = 1, a_2 = 2, b_1 = 1, b_2 = 2$ ; on the right  $a_1 = 1, a_2 = 2, b_1 = 10, b_2 = 0$ .

### 3.3 Rational solutions of order three

In this case, we only give the rational solution with some fixed parameters in reason of the length of the general solution. We choose  $a_j = b_j = j$  for  $1 \leq j \leq 3$ . We get the following rational solution given by :

**Proposition 3.3** *The function  $v$  defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (20)$$

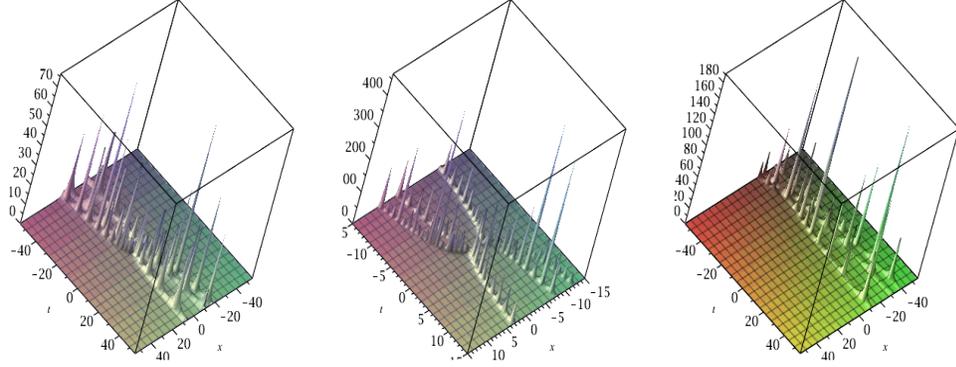
with

$$n(x, t) = 3x^8 + 48x^7 + 336x^6 + (1344 + 72t)x^5 + (720t + 3360)x^4 + (2880t + 5376)x^3 + (5760t + 5376 + 4320t^2)x^2 + (5760t + 3072 + 17280t^2)x + 768 + 2304t + 17280t^2$$

and,

$$d(x, t) = -x^9 - 18x^8 - 144x^7 + (-672 - 72t)x^6 + (-2016 - 864t)x^5 + (-4032 - 4320t)x^4 + (-4096 - 9120t + (8 + 12t)(-60t - 160) + 720t^2)x^3 + ((8 + 12t)(-240 - 360t) - 2688 - 11520t + 4320t^2)x^2 + ((8 + 12t)(-192 - 720t) - 768 - 5760t + 8640t^2)x + (8 + 12t)(-64 - 480t + 720t^2)$$

is a rational solution to the mKdV equation (1).



**Figure 3.** Solution of order 3 to (1), on the left  $a_1 = 1, a_2 = 2, a_3 = 3, b_1 = 0, b_2 = 0, b_3 = 0$ ; in the center  $a_1 = 1, a_2 = 2, a_3 = 3, b_1 = 10, b_2 = 2, b_3 = 3$ ; on the right  $a_1 = 1, a_2 = 2, a_3 = 3, b_1 = 10, b_2 = 0, b_3 = 0$ .

### 3.4 Rational solutions of order four

With the same choices of parameters  $a_j = b_j = j$  for  $1 \leq j \leq 4$ , we get the following rational solution given by :

**Proposition 3.4** *The function  $v$  defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (21)$$

with

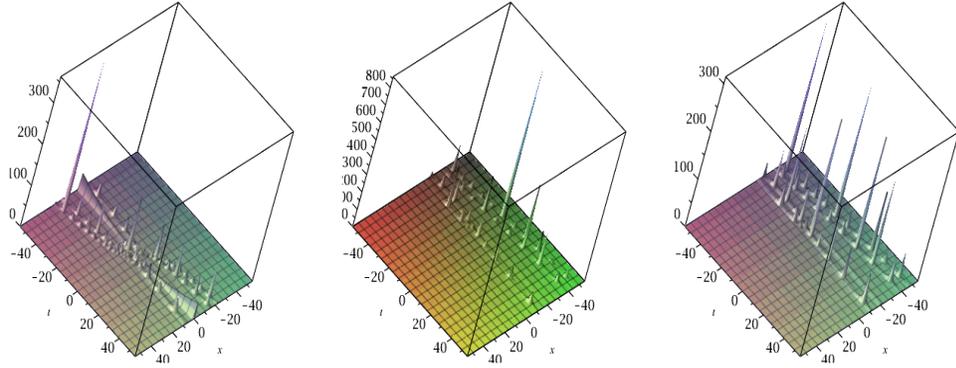
$$\begin{aligned} n(x, t) = & 4x^{15} + 120x^{14} + 1680x^{13} + (14560 + 600t)x^{12} + (14400t + 87360)x^{11} + \\ & (158400t + 384384)x^{10} + (36000t^2 + 1056000t + 1281280)x^9 + (3294720 + 648000t^2 + \\ & 4752000t)x^8 + (15206400t + 5184000t^2 + 6589440)x^7 + (-2419200t^3 + 10250240 + \\ & 24192000t^2 + 35481600t)x^6 + (12300288 + 72576000t^2 - 29030400t^3 + 60825600t)x^5 + \\ & (-145152000t^3 + 145152000t^2 + 11182080 + 76032000t)x^4 + (67584000t - 36288000t^4 + \\ & 193536000t^2 + 7454720 - 387072000t^3)x^3 + (-217728000t^4 + 40550400t + 3440640 + \\ & 165888000t^2 - 580608000t^3)x^2 + (983040 - 435456000t^4 + 82944000t^2 - 464486400t^3 + \\ & 14745600t)x + 131072 + 2457600t - 217728000t^5 + 18432000t^2 - 154828800t^3 - 290304000t^4 \end{aligned}$$

and,

$$\begin{aligned} d(x, t) = & -x^{16} - 32x^{15} - 480x^{14} + (-4480 - 240t)x^{13} + (-29120 - 6240t)x^{12} + \\ & (-139776 - 74880t)x^{11} + (-358912 + 720t^2 - 462720t + (-160 - 60t)(180t + 960))x^{10} + \\ & ((-160 - 60t)(2520t + 3360) - 1752000t - 696320 + (-360t - 240)(180t + 960) + \\ & 14400t^2)x^9 + (-1013760 + 129600t^2 - 4440960t + (-160 - 60t)(8064 + 15120t) + \\ & (-192 - 720t)(180t + 960) + (-360t - 240)(2520t + 3360))x^8 + (-302400t^3 - 7580160t - \\ & 1064960 + (-192 - 720t)(2520t + 3360) + (-360t - 240)(8064 + 15120t) + (-64 + 720t^2 - \\ & 480t)(180t + 960) + (-160 - 60t)(13440 + 50400t))x^7 + ((-64 + 720t^2 - 480t)(2520t + \\ & 3360) - 4233600t^3 - 8248320t - 753664 + (-160 - 60t)(100800t + 15360) + (-192 - \\ & 720t)(8064 + 15120t) + (-360t - 240)(13440 + 50400t))x^6 + (-25401600t^3 - 5114880t - \end{aligned}$$

$319488 + (-192 - 720t)(13440 + 50400t) + (-360t - 240)(100800t + 15360) + (-64 + 720t^2 - 480t)(8064 + 15120t) + (-160 - 60t)(120960t + 11520))x^5 + ((-64 + 720t^2 - 480t)(13440 + 50400t) + (-192 - 720t)(100800t + 15360) + (-160 - 60t)(302400t^3 + 80640t + 5120) - 61440 - 1382400t - 36288000t^3 + (-360t - 240)(120960t + 11520))x^4 + ((-360t - 240)(302400t^3 + 80640t + 5120) + (-64 + 720t^2 - 480t)(100800t + 15360) + (-192 - 720t)(120960t + 11520) + (-160 - 60t)(1024 + 23040t + 604800t^3))x^3 + ((-64 + 720t^2 - 480t)(120960t + 11520) + (-192 - 720t)(302400t^3 + 80640t + 5120) + (-360t - 240)(1024 + 23040t + 604800t^3))x^2 + ((-64 + 720t^2 - 480t)(302400t^3 + 80640t + 5120) + (-192 - 720t)(1024 + 23040t + 604800t^3))x + (-64 + 720t^2 - 480t)(1024 + 23040t + 604800t^3)$

is a rational solution to the mKdV equation (1).



**Figure 4.** Solution of order 4 to (1), on the left  $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, b_1 = 0, b_2 = 0, b_3 = 0, b_4 = 10$ ; in the center  $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, b_1 = 10, b_2 = 0, b_3 = 0, b_4 = 0$ ; on the right  $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, b_1 = 0, b_2 = 0, b_3 = 0, b_4 = 4$ .

### 3.5 Rational solutions of orders five and six

We choose parameters to get the shorter expression of the solution. For this, we consider  $a_j = j$  and  $b_j = 0$ , for  $1 \leq j \leq N$ , we get the following rational solution given by :

#### Order 5

**Proposition 3.5** The function  $v$  defined by

$$v(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (22)$$

with

$$n(x, t) = 5x^{24} + 2280tx^{21} + 352800t^2x^{18} + 4838400t^3x^{15} + 2794176000t^4x^{12} + 259096320000t^5x^9 + 5120962560000t^6x^6 - 153628876800000t^7x^3 + 460886630400000t^8$$

and,

$d(x, t) = -x^{25} - 600tx^{22} - 100800t^2x^{19} - 6955200t^3x^{16} - 254016000t^4x^{13} + 39626496000t^5x^{10} - 365783040000t^6x^7 + 76814438400000t^7x^4 + 460886630400000t^8x$   
*is a rational solution to the mKdV equation (1).*

### Order 6

**Proposition 3.6** *The function  $v$  defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (23)$$

*with*

$$\begin{aligned}
n(x, t) = & -6x^{35} - 6300tx^{32} - 2419200t^2x^{29} - 381024000t^3x^{26} - 50295168000t^4x^{23} + \\
& 192036096000t^5x^{20} + 1371137725440000t^6x^{17} + 129970029772800000t^7x^{14} - 116143430860800000t^8x^{11} + \\
& 319394434867200000000t^9x^8 + 2146330602307584000000t^{10}x^5 + 128779836138455040000000t^{11}x^2
\end{aligned}$$

*and,*

$$\begin{aligned}
d(x, t) = & x^{36} + 1260tx^{33} + 544320t^2x^{30} + 110073600t^3x^{27} + 11430720000t^4x^{24} + \\
& 329204736000t^5x^{21} + 119062379520000t^6x^{18} + 7189831434240000t^7x^{15} + 1393721170329600000t^8x^{12} - \\
& 59620294508544000000t^9x^9 - 4292661204615168000000t^{10}x^6 + 128779836138455040000000t^{12}
\end{aligned}$$

*is a rational solution to the mKdV equation (1).*

For solutions of order 5 and 6 depending respectively on 10 and 12 parameters, the structure of the polynomials of the numerators and denominators is the following.

For order 5, the numerator contains 8370 terms and the denominator 10279; the degree of the numerator in  $x$  is 24, in  $t$  is 8; the degree of the denominator in  $x$  is 25, in  $t$  is 8.

For order 6, the numerator contains 100409 terms and the denominator 119620; the degree of the numerator in  $x$  is 35, in  $t$  is 11; the degree of the denominator in  $x$  is 36, in  $t$  is 12.

Unlike other equations like the nonlinear Schrödinger equation, we do not see a specific structure of polynomials defining these solutions.

## 4 Conclusion

Two types of representations of solutions to the mKdV equation have been given. First, we have constructed solutions as a quotient of a wronskian of order  $N$  by a wronskian of order  $N$  depending on  $2N$  real parameters. Then rational solutions to the mKdV equation depending on  $2N$  real parameters, performing a passage to the limit when one parameter goes to 0.

We can mention some other recent works about this equation. In [18], first and second-order rational solutions are given as limiting cases of periodic solutions, the second one depending on one real parameter. In [19] the reduced Maxwell-Bloch (RMB) equations are considered and  $N$ th-order rational solutions containing several free parameters are presented, in particular explicit

expressions of these solutions from first to second order. Explicit periodic and rational solutions of first and second order are given in [20], and some typical nonlinear wave patterns are shown. The first four exact rational solutions of the set of rational solutions of the mKdV equation are presented in [21]. Multiple periodic solutions of the mKdV equation are given in [22] and in particular first to third-order rational solutions.

All the solutions presented in this article are different from those proposed in previous references and are new. So we get an infinite hierarchy of multiparametric families of rational solutions to the mKdV equation as a quotient of a polynomials in  $x$  and  $t$  depending on  $2N$  real parameters.

It would be relevant to better understand the structure of polynomials defining the rational solutions of this equation.

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