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# Regularity of the extremal solution for some elliptic problems with singular nonlinearity and advection

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## Abstract

In this note, we investigate the regularity of the extremal solution  $u^*$  for the semilinear elliptic equation  $-\Delta u + c(x) \cdot \nabla u = \lambda f(u)$  on a bounded smooth domain of  $\mathbb{R}^n$  with Dirichlet boundary condition. Here  $f$  is a positive nondecreasing convex function, exploding at a finite value  $a \in (0, \infty)$ . We show that the extremal solution is regular in the low dimensional case. In particular, we prove that for the radial case, all extremal solutions are regular in dimension two.

*Keywords:* singular nonlinearity, advection, extremal solution, regularity\*.

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## 1. Introduction

We consider the elliptic problem

$$\begin{cases} -\Delta u + c(x) \cdot \nabla u = \lambda f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (P_\lambda)$$

where  $\lambda > 0$ ,  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^n$  ( $n \geq 2$ ),  $c(x)$  is a smooth vector field over  $\overline{\Omega}$  and  $f : [0, a) \rightarrow \mathbb{R}_+$  with fixed  $a \in (0, \infty)$  satisfies the following condition (H):

$f$  is  $C^2$ , positive, nondecreasing and convex in  $[0, a)$  with  $\lim_{t \rightarrow a^-} f(t) = \infty$ .

In the literature,  $f$  is referred as a *singular nonlinearity*. We say that  $u$  is a regular solution if  $u \in C^2(\overline{\Omega})$ , and we also deal with solutions in the following weak sense.

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\*MSC: 35B65, 35B45, 35J60

**Definition 1.1.** We say that  $u$  is a weak solution of  $(P_\lambda)$  if  $0 \leq u \leq a$  a.e. in  $\Omega$  such that  $f(u)d(x, \partial\Omega) \in L^1(\Omega)$  and

$$-\int_{\Omega} u \Delta \phi - \int_{\Omega} u \operatorname{div}(\phi c) = \lambda \int_{\Omega} f(u) \phi, \quad \forall \phi \in C^2(\overline{\Omega}) \cap H_0^1(\Omega).$$

Moreover,  $u$  is a weak super-solution of  $(P_\lambda)$  if “=” is replaced by “ $\geq$ ” for all nonnegative functions  $\phi \in C^2(\overline{\Omega}) \cap H_0^1(\Omega)$ .

Clearly, a weak solution is regular if  $\sup_{\Omega} u < a$ . For regular solutions, we introduce a notion of stability.

**Definition 1.2.** A regular solution  $u$  of  $(P_\lambda)$  is said to be stable if the principal eigenvalue of the linearized operator  $L_{u,\lambda,c} := -\Delta + c \cdot \nabla - \lambda f'(u)$  is nonnegative in  $H_0^1(\Omega)$ .

Exploiting some ideas in [11, 10, 3], the solvability of  $(P_\lambda)$  is characterized by a parameter  $\lambda^*$ :

**Proposition 1.1.** There exists  $\lambda^* \in (0, \infty)$  such that

- For  $0 < \lambda < \lambda^*$ , the problem  $(P_\lambda)$  has a minimal solution  $u_\lambda$ ,  $u_\lambda$  is regular and the map  $\lambda \mapsto u_\lambda$  is increasing. Moreover,  $u_\lambda$  is the unique stable solution of  $(P_\lambda)$ .
- For  $\lambda = \lambda^*$ ,  $(P_{\lambda^*})$  admits a unique weak solution  $u^* := \lim_{\lambda \rightarrow \lambda^*} u_\lambda$ , called the extremal solution.
- For  $\lambda > \lambda^*$ ,  $(P_\lambda)$  admits no weak solution.

Here the minimal solution means that  $u_\lambda \leq v$  for any solution  $v$  of  $(P_\lambda)$ . We remark immediately a close similarity between  $(P_\lambda)$  and the Emden-Fowler equation with superlinear regular nonlinearity, that is

$$-\Delta u = \lambda g(u) \text{ in } \Omega \subset \mathbb{R}^n; \quad u = 0 \text{ on } \partial\Omega, \quad (1.1)$$

with  $\lambda > 0$  and  $g : [0, \infty) \rightarrow (0, \infty)$  satisfies

$$g \text{ is } C^2, \text{ nondecreasing, convex and } \lim_{t \rightarrow \infty} \frac{g(t)}{t} = \infty. \quad (1.2)$$

In fact, there exists also a critical parameter  $\bar{\lambda} \in (0, \infty)$  for (1.1) such that all conclusions in the above proposition hold true by replacing  $\lambda^*$  by  $\bar{\lambda}$  (see [2, 11]). It is well known by classical examples as  $g(u) = (1+u)^p$  with  $p > 1$  or  $g(u) = e^u$ , the extremal solution  $u^*$  can be either a regular solution or a real weak solution in the distribution sense with  $\sup_{\Omega} u = \infty$ .

For general nonlinearity  $g$  satisfying (1.2), the regularity of the extremal solution  $u^*$  to (1.1) is obtained by Nedev [13] for any bounded smooth domain  $\Omega \subset \mathbb{R}^n$  if  $n = 2, 3$ ; by Cabré [4] for convex domains in  $\mathbb{R}^4$ ; and for radial symmetry case in  $\mathbb{R}^n$  with  $n \leq 9$  by Cabré & Capella [5]. In [17], it is proved that, under mild condition on  $g$ , the extremal solution  $u^*$  is regular for any smooth bounded domain  $\Omega \subset \mathbb{R}^n$  if  $n \leq 9$ .

We can ask the same question about the problem  $(P_\lambda)$ : For  $f$  verifying  $(H)$ , is it true that the extremal solution to  $(P_\lambda)$  is regular for general vector field  $c$  and general domain  $\Omega \subset \mathbb{R}^n$  with low dimensions  $n$ ? We will partly answer this question. It is worthy to mention that for studying the explosion phenomena in a flow, Berestycki *et al.* [1] have considered the problem  $(P_\lambda)$  with a general source  $f$  verifying (1.2).

Without loss of generality, fix  $a = 1$  in the sequel. The problem  $(P_\lambda)$  can be linked to equation (1.1) up to the transformation  $v = -\ln(1 - u)$ . In fact, let  $u$  solve  $(P_\lambda)$ ,  $v$  verifies then

$$\begin{cases} -\Delta v + |\nabla v|^2 + c(x) \cdot \nabla v = \lambda e^v f(1 - e^{-v}) := \lambda g(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (Q_\lambda)$$

Therefore  $g$  verifies (1.2) and  $v^* = -\ln(1 - u^*)$  is the extremal solution for the problem  $(Q_\lambda)$ . Thus the regularity of  $u^*$  is equivalent to the boundedness of  $v^*$ , however the situation could be very different with the presence of advection terms (see [7, 16]). In last decade, a model describing the steady state of MEMS (Micro-Electro-Mechanical Systems) device given by Pelesko and Bernstein in [14], has drawn many attentions (see [9] and the references therein).

$$-\Delta u = \frac{\lambda}{(1 - u)^2} \quad \text{in } \Omega \subset \mathbb{R}^n; \quad u = 0 \quad \text{on } \partial\Omega.$$

More generally, many precise studies have been done for the singular nonlinearities with negative exponent  $f(u) = (1 - u)^{-p}$  ( $p > 0$ ) in the advection-free situation, i.e.  $c \equiv 0$ . In that case, when  $\Omega$  is moreover the unit ball in  $\mathbb{R}^n$ , it is known that  $u^*$  is regular if and only if (see [12, 10])

$$n < n_p := 2 + \frac{4p}{p+1} + 4\sqrt{\frac{p}{p+1}}. \quad (1.3)$$

Tending  $p \rightarrow 0^+$  in (1.3), we see that  $n_p \rightarrow 2$ . Therefore we cannot expect in general better than dimension two to claim the regularity of  $u^*$ .

For the radial case of  $(P_\lambda)$ , equally when  $\Omega$  is a ball and  $c(x)$  is the gradient of a smooth radial function,  $u_\lambda$  is radial by uniqueness of the minimal solution. We obtain the following optimal results which are new even for the advection-free case.

**Theorem 1.1.** *Assume that  $n = 2$ ,  $\Omega = B_1$ . Let  $\gamma$  is a smooth radial function and  $c = \nabla\gamma$ , then the extremal solution  $u^*$  is regular for any  $f$  satisfying  $(H)$ .*

**Theorem 1.2.** *For any  $f$  satisfying  $(H)$ ,  $\Omega = B_1$  and smooth radial function  $\gamma$ , there exists  $C > 0$  such that for all  $\lambda \in (0, \lambda^*)$*

$$|u'_\lambda(r)| \leq \begin{cases} Cr^{-1} & \text{if } n \geq 10; \\ Cr^{-\frac{n}{2}+1+\sqrt{n-1}} & \text{if } 3 \leq n \leq 9; \end{cases} \quad \forall r = |x| \in (0, 1]$$

where  $|\cdot|$  is the Euclidean norm in  $\mathbb{R}^n$ .

**Remark 1.1.** *The above estimates are optimal. In fact, when  $f(u) = (1-u)^{-p}$ ,  $p > 0$ ,  $\Omega = B_1$  and  $c \equiv 0$ , it is well known that  $u^*(x) = 1 - r^{\frac{2}{p+1}}$  if  $n \geq n_p$  with  $n_p$  given in (1.3), and we have*

$$n \geq n_p \quad \text{iff} \quad n \geq 10 \quad \text{or} \quad 3 \leq n \leq 9, \quad \frac{2}{p+1} \leq -\frac{n}{2} + 2 + \sqrt{n-1}.$$

But is the extremal solution  $u^*$  of  $(P_\lambda)$  regular with general singular nonlinearity  $f$  verifying  $(H)$ , vector field  $c$  and smooth bounded domains in  $\mathbb{R}^2$ ? The answer is affirmative under some additional mild condition on  $f$ .

**Theorem 1.3.** *Assume that  $f$  satisfies conditions  $(H)$  and the additional conditions,*

$$(H1) \quad \limsup_{t \rightarrow 1^-} \frac{f(t)}{f'(t)(1-t)\ln^2(1-t)} < 1$$

and

$$(H2) \quad \liminf_{t \rightarrow 1^-} \frac{f(t)f''(t)}{f'^2(t)} > 0.$$

Then  $u^*$  is regular solution to  $(P_{\lambda^*})$  if  $n = 2$ , i.e.  $\Omega \subset \mathbb{R}^2$ .

Under more precise conditions on the growth of  $f$ , the extremal solution can be showed to be regular in some higher dimensions.

**Theorem 1.4.** *Let  $f$  verify  $(H)$  and  $g(v) = e^v f(1 - e^{-v})$ . Assume that  $g$  satisfies*

$$(H3) \quad \liminf_{t \rightarrow \infty} \frac{g'(t)}{g(t)} = 1 + \delta > 1$$

and

$$(\widetilde{H2}) \quad \liminf_{t \rightarrow \infty} \frac{g''(t)g(t)}{g'^2(t)} = \mu > \frac{1}{1 + \delta}.$$

Then  $v^* = -\ln(1 - u^*)$  is bounded (so  $u^*$  is regular) when

$$n < 2 + \frac{4\delta}{1 + \delta} + \frac{4\sqrt{\delta(\mu + \mu\delta - 1)}}{1 + \delta}. \quad (1.4)$$

Consequently, if  $\mu\delta > 1$ ,  $u^*$  is regular for all  $n \leq 6$ . Furthermore, if we can tend  $\delta$  to  $\infty$ , which means  $g = o(g')$  near  $\infty$ , then  $u^*$  is regular for  $n < 6 + 4\sqrt{\mu}$  with any  $\mu > 0$ . However, we can never have  $\mu > 1$ , since otherwise  $g$  blows up at finite value and contradicts (1.2), so the best result we can expect is for  $n \leq 9$ . For example, if  $f(u) = e^{\frac{1}{1-u}}$ , then  $g(v) = e^{v+e^v}$  verifies  $\delta = \infty$  and  $\mu = 1$ .

**Theorem 1.5.** *Let  $f$  verify  $(H)$  and  $g(v) = e^v f(1 - e^{-v})$ . Assume that  $g = o(g')$  near  $\infty$ . Rewrite  $g(t) = g(0) + te^{h(t)}$  in  $(0, \infty)$ , suppose there exists  $t_0 > 0$  such that  $t^2 h'(t)$  is nondecreasing for  $t \geq t_0$ , then for any bounded smooth domain  $\Omega \subset \mathbb{R}^n$  with  $n \leq 9$ ,  $u^*$  is a regular solution.*

Furthermore, when  $g = o(g')$  near  $\infty$ , the condition  $(\widetilde{H2})$  is just equivalent to  $(H2)$ , since

$$\frac{f''(t)f(t)}{f'^2(t)} = \frac{(g'' - g')g}{(g' - g)^2}(s) = \left(\frac{g''g}{g'^2} - \frac{g}{g'}\right) \times \left(1 - \frac{g}{g'}\right)^{-2}(s), \quad \forall t = 1 - e^{-s}.$$

It is also easy to see that  $(H3)$  is equivalent to the condition

$$\liminf_{t \rightarrow 1^-} \frac{f'(t)(1-t)}{f(t)} = \delta > 0.$$

If the equality holds for the whole limit, we have the following optimal result. The case  $f(u) = (1-u)^{-2}$  was obtained in [7] with a different argument.

**Theorem 1.6.** *Assume that*

$$\lim_{u \rightarrow 1^-} \frac{f'(u)(1-u)}{f(u)} = p > 0. \quad (1.5)$$

*Then  $u^*$  is a regular solution if  $n < n_p$  where  $n_p$  is defined in (1.3).*

One of the main difficulties here is due to the vector field  $c(x)$ . When  $c \neq 0$ , the operator  $-\Delta + c \cdot \nabla$  is not self-adjoint, we use ideas from [7] to get some energy estimates. However if  $c$  is a gradient, say  $c = -\nabla \gamma$  in  $\Omega$ , then  $-\Delta + c \cdot \nabla$  can be rewritten as  $e^{-\gamma} L_\gamma$  where  $L_\gamma = -\operatorname{div}(e^\gamma \nabla)$  is a self-adjoint operator. In that case,  $(P_\lambda)$  admits a variational structure and we can expect more precise estimates of minimal solutions  $u_\lambda$ , as in the radial case.

The paper is organized as follows: In section 2, we prove quickly Proposition 1.1 and show some general consequences of the stability of  $u_\lambda$ . The section 3 is devoted to the proof of Theorems 1.3 to 1.6 for general domains. In section 4, we discuss the radial case. The norm  $\|\cdot\|_q$  denotes always the standard  $L^q$  norm for any  $q \in [1, \infty]$ . The capital letter  $C$  denotes a generic positive constant independent of  $\lambda$ , it could be changed from one line to another.

## 2. Preliminaries

As mentioned above,  $-\Delta + c \cdot \nabla$  is not a self-adjoint operator for general vector field  $c$ . However using Lemma 1 in [7], we have a kind of Hodge decomposition, which tells us that for any vector field  $c \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$ , there exist a smooth scalar function  $\gamma$  and a vector field  $b \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$  such that

$$c = -\nabla \gamma + b \quad \text{and} \quad \operatorname{div}(e^\gamma b) = 0 \quad \text{in } \overline{\Omega}. \quad (2.1)$$

Therefore the problem  $(P_\lambda)$  can be rewritten as

$$-\operatorname{div}(e^\gamma \nabla u) + e^\gamma b \cdot \nabla u = \lambda e^\gamma f(u) \quad \text{in } \Omega. \quad (P'_\lambda)$$

On the other hand, we don't have a suitable variational characterization in general to use the stability assumption. Fortunately, we can adopt an energy inequality as in [7], which is derived from a generalized Hardy inequality of [6].

**Proposition 2.2.** *Let  $u_\lambda$  be minimal solution of  $(P_\lambda)$ . For any  $1 \leq \beta < 2$ , we have*

$$\lambda \int_{\Omega} e^\gamma f'(u_\lambda) \psi^2 \leq \frac{2}{\beta} \int_{\Omega} e^\gamma |\nabla \psi|^2 + \frac{\|b\|_\infty^2}{2(2-\beta)} \int_{\Omega} e^\gamma \psi^2, \quad \forall \psi \in H_0^1(\Omega). \quad (2.2)$$

where  $b$  is the vector field in (2.1),  $\|b\|_\infty = \max_{\bar{\Omega}} |b(x)|$ .

**Proof.** We use a Hardy type inequality given by Theorem 2 in [7], which says that for a positive principal eigenfunction  $\varphi$  of  $L_{u_\lambda, \lambda, c}$ , for  $\beta \in [1, 2)$  and any  $\psi \in H_0^1(\Omega)$ ,

$$\lambda \int_{\Omega} e^\gamma f'(u_\lambda) \psi^2 \leq \frac{2}{\beta} \int_{\Omega} e^\gamma |\nabla \psi|^2 + \int_{\Omega} \left[ -\frac{2-\beta}{2} \frac{|\nabla \varphi|^2}{\varphi^2} + \frac{b \cdot \nabla \varphi}{\varphi} \right] e^\gamma \psi^2.$$

By Cauchy-Schwarz inequality, it is easy to see

$$-\frac{2-\beta}{2} \frac{|\nabla \varphi|^2}{\varphi^2} + \frac{b \cdot \nabla \varphi}{\varphi} \leq \frac{|b(x)|^2}{2(2-\beta)} \leq \frac{\|b\|_\infty^2}{2(2-\beta)},$$

so we are done.  $\square$

Another main ingredient of our approach is just the transformation  $v = -\ln(1-u)$ . Let  $\phi$  and  $\xi$  be nonnegative  $C^1$  functions satisfying  $\phi(0) = \xi(0) = 0$  and  $\xi' = \phi'^2$ . Define  $v_\lambda = -\ln(1-u_\lambda)$  and  $g(v_\lambda) = e^{v_\lambda} f(1-e^{-v_\lambda})$ . Using  $(Q_\lambda)$ , we get  $-\operatorname{div}(e^\gamma \nabla v_\lambda) + e^\gamma b \cdot \nabla v_\lambda \leq \lambda e^\gamma g(v_\lambda)$  in  $\Omega$ . Let  $\psi = \phi(v_\lambda)$  in (2.2),  $\forall \lambda \in (0, \lambda^*)$ ,

$$\begin{aligned} & \lambda \int_{\Omega} e^\gamma f'(u_\lambda) \phi^2(v_\lambda) \\ & \leq \frac{2}{\beta} \int_{\Omega} e^\gamma |\nabla \phi(v_\lambda)|^2 + \frac{\|b\|_\infty^2}{2(2-\beta)} \int_{\Omega} e^\gamma \phi^2(v_\lambda) \\ & = \frac{2}{\beta} \int_{\Omega} e^\gamma \nabla \xi(v_\lambda) \nabla v_\lambda + C_\beta \int_{\Omega} e^\gamma \phi^2(v_\lambda) \\ & = -\frac{2}{\beta} \int_{\Omega} \operatorname{div}(e^\gamma \nabla v_\lambda) \xi(v_\lambda) + C_\beta \int_{\Omega} e^\gamma \phi^2(v_\lambda) \\ & \leq \frac{2\lambda}{\beta} \int_{\Omega} e^\gamma g(v_\lambda) \xi(v_\lambda) - \frac{2}{\beta} \int_{\Omega} e^\gamma b \cdot \xi(v_\lambda) \nabla v_\lambda + C_\beta \int_{\Omega} e^\gamma \phi^2(v_\lambda) \\ & = \frac{2\lambda}{\beta} \int_{\Omega} e^\gamma g(v_\lambda) \xi(v_\lambda) + C_\beta \int_{\Omega} e^\gamma \phi^2(v_\lambda). \end{aligned}$$

The last line is due to  $\operatorname{div}(e^\gamma b) = 0$ . We claim then

**Proposition 2.3.** *Let  $1 \leq \beta < 2$ . For any  $\lambda \in (0, \lambda^*)$  and any nonnegative  $C^1$  test functions  $\phi, \xi$  verifying  $\phi(0) = \xi(0) = 0$  and  $\xi' = \phi'^2$ , there hold*

$$\lambda \int_{\Omega} e^\gamma f'(u_\lambda) \phi^2(v_\lambda) \leq \frac{2\lambda}{\beta} \int_{\Omega} e^\gamma g(v_\lambda) \xi(v_\lambda) + C_\beta \int_{\Omega} e^\gamma \phi^2(v_\lambda) \quad (2.3)$$

and

$$\lambda \int_{\Omega} e^\gamma f'(u_\lambda) \phi^2(u_\lambda) \leq \frac{2\lambda}{\beta} \int_{\Omega} e^\gamma f(u_\lambda) \xi(u_\lambda) + C_\beta \int_{\Omega} e^\gamma \phi^2(u_\lambda). \quad (2.4)$$

The proof of (2.4) is completely similar to (2.3) but using  $(P'_\lambda)$  instead of  $(Q_\lambda)$ .

We also make use the following behavior of  $f$  proved in [18].

**Lemma 2.1.** *For any  $f$  verifying (H), we have  $\lim_{t \rightarrow 1} f(t)/f'(t) = 0$ .*

Choose first  $\phi(u) = e^u - 1$  in (2.4), then  $\xi(u) = \frac{e^{2u}-1}{2}$  and

$$\lambda \int_{\Omega} e^\gamma f'(u_\lambda) (e^{u_\lambda} - 1)^2 \leq \frac{\lambda}{\beta} \int_{\Omega} e^\gamma f(u_\lambda) (e^{2u_\lambda} - 1) + C_\beta \int_{\Omega} e^\gamma (e^{u_\lambda} - 1)^2.$$

Fix  $\beta \in (1, 2)$ . By Lemma 2.1,

$$\lambda \int_{\Omega} e^\gamma f'(u_\lambda) e^{2u_\lambda} \leq C.$$

Consequently  $\|f'(u_\lambda)\|_1$  is uniformly bounded, so is  $\|f(u_\lambda)\|_1$ . Multiplying  $(P_\lambda)$  by  $u_\lambda$ ,

$$\int_{\Omega} |\nabla u_\lambda|^2 = \int_{\Omega} \frac{\operatorname{div}(c)}{2} u_\lambda^2 + \lambda \int_{\Omega} f(u_\lambda) u_\lambda \leq C,$$

which gives

**Proposition 2.4.** *The family of minimal solutions  $\{u_\lambda\}_{0 < \lambda < \lambda^*}$  is uniformly bounded in  $H_0^1(\Omega)$ .*

**Remark 2.1.** *As far as we know, it is always an open question whether the similar  $H^1$  energy estimation holds for minimal solutions of (1.1) with general regular nonlinearity satisfying (1.2) and general domain  $\Omega$  when  $n \geq 6$  (see [13] for  $n \leq 5$ ). For the advection-free case  $c = 0$ , it was proved in [18] that  $u^* \in H^2 \cap H_0^1(\Omega)$  under the condition (H), it is also true for the gradient case  $c = \nabla \gamma$  (see Lemma 4.1).*

**Sketches of proof of Proposition 1.1.** We follow the ideas coming from [1, 11, 10]. The main argument is the maximum principle for operators  $-\Delta + c \cdot \nabla$  and  $L_\gamma$  under the Dirichlet boundary condition, we use also the super-sub solution method and monotone iteration.

Let  $w \in H_0^1(\Omega)$  be the regular solution of  $-\Delta w + c \cdot \nabla w = 1$  in  $\Omega$  and fix  $\alpha > 0$  such that  $\alpha \max_{\Omega} w < 1$ . It is easy to verify that  $\alpha w$  is a supersolution of  $(P_\lambda)$  for  $\lambda > 0$  small enough. As 0 is a subsolution and  $\alpha w > 0$  in  $\Omega$ ,  $(P_\lambda)$  admits a regular solution for  $\lambda > 0$  small enough. As any regular solution  $u$  of  $(P_\lambda)$  is also a supersolution for  $(P_\mu)$  if  $\mu \in (0, \lambda)$ , the set of  $\lambda$  for which  $(P_\lambda)$  admits a regular solution is just an interval. Moreover, for these  $\lambda$ , using (H) and the monotone iteration  $v_0 = 0$ ;  $-\Delta v_{n+1} + c \cdot \nabla v_{n+1} = \lambda f(v_n)$  in  $\Omega$  with  $v_{n+1} = 0$  on  $\partial\Omega$  for  $n \in \mathbb{N}$ , we get the minimal solution  $u_\lambda = \lim_{n \rightarrow \infty} v_n$ .

If we suppose that the principal eigenvalue of  $L_{u_\lambda, \lambda, c}$  is negative, we can construct, as in [1] another solution  $v \leq u_\lambda$  using the associated first eigenfunction, this is just impossible by the definition of  $u_\lambda$ , hence  $u_\lambda$  is stable. The uniqueness of stable solution comes from Lemmas 2.16 and 2.17 in [8].



Take a positive first eigenfunction  $\varphi$  of  $L_\gamma$  with the Dirichlet boundary condition, by  $(P'_\lambda)$ ,

$$\lambda f(0) \int_{\Omega} e^\gamma \varphi \leq \int_{\Omega} \lambda e^\gamma f(u) \varphi = \int_{\Omega} \lambda_1(L_\gamma) u \varphi - \int_{\Omega} \operatorname{div}(e^\gamma b \varphi) u \leq C.$$

So  $\lambda$  is upper bounded. Define the critical threshold  $\lambda^*$  as the supremum of  $\lambda > 0$  for which  $(P_\lambda)$  admits a regular solution, as  $u^*$  is the monotone limit of  $u_\lambda$  when  $\lambda \rightarrow \lambda^*$ , we deduce that  $u^* \in H_0^1(\Omega)$  is a weak solution of  $(P_\lambda)$  by Proposition 2.4.

Suppose that  $u$  is a weak solution to  $(P_\lambda)$ . By the monotonicity of  $f$ , it is easy to verify that for any  $\delta > 1$ , the function  $v = \delta^{-1}u$  is a weak supersolution for  $(P_{\lambda/\delta})$ , then the monotone iteration will enable us a weak solution  $w$  of  $(P_{\lambda/\delta})$  satisfying  $0 \leq w \leq v \leq \delta^{-1} < 1$ . The regularity theory implies then  $w$  is a regular solution of  $(P_{\lambda/\delta})$ . This means that  $\lambda/\delta \leq \lambda^*$ . Let  $\delta$  tend to 1, we get  $\lambda \leq \lambda^*$ . Therefore, no weak solution exists for  $\lambda > \lambda^*$ .

The uniqueness of the weak solution can be proved in the very similar way as in [11] using the monotonicity and convexity of  $f$ , with the strong maximum principle for the operator  $-\Delta + c \cdot \nabla$  associated to Dirichlet boundary condition, so we omit the details.  $\square$

### 3. Regularity of $u^*$ for general $c$ and $\Omega$

For proving our results, we will choose suitable functions  $\phi$  to apply (2.3) or (2.4). We need also

**Lemma 3.1.** *For any  $q > n/2$ , there exists  $C > 0$  such that the solution  $v$  of  $(Q_\lambda)$  satisfies  $0 \leq v \leq C\|g(v)\|_q$  in  $\Omega$ .*

Indeed, let  $w$  be the solution of  $L(w) := -\Delta w + c \cdot \nabla w = \lambda g(v)$  in  $\Omega$  with  $w = 0$  on  $\partial\Omega$ . By regularity theory and Sobolev embedding,  $\|w\|_\infty \leq C\|w\|_{W^{2,q}(\Omega)} \leq C'\lambda^*\|g(v)\|_q$  because  $q > n/2 \geq 1$ . Moreover, as  $L(w - v) \geq 0$ , the maximum principle implies then  $0 \leq v \leq w \leq C\|g(v)\|_q$ .

#### 3.1. Proof of Theorem 1.3

For simplicity, we omit the index  $\lambda$  for  $u_\lambda$  or  $v_\lambda$ . Let  $\phi(u) = v = -\ln(1-u)$  in (2.4), so  $\xi(u) = (1-u)^{-1} - 1$ . Fix  $\beta \in (1, 2)$  but very close to 2. Repeating the proof of Theorem 2 in [18] with the assumption (H1), there exists  $C > 0$  such that

$$\lambda \int_{\Omega} e^\gamma \frac{f(u)}{1-u} < C + CC_\beta \int_{\Omega} e^\gamma \phi^2(u).$$

As  $\phi^2(u) = o(\xi(u)) = o(f\xi)$  when  $u \rightarrow 1^-$ ,

$$\lambda \int_{\Omega} e^\gamma \frac{f(u)}{1-u} \leq C.$$

Using the equation  $(Q_\lambda)$  and  $\partial_\nu v \leq 0$  on  $\partial\Omega$ ,

$$\begin{aligned} \int_{\Omega} |\nabla v|^2 &= \lambda \int_{\Omega} e^v f(1 - e^{-v}) + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} d\sigma - \int_{\Omega} c \cdot \nabla v \leq \lambda \int_{\Omega} \frac{f(u)}{1-u} + C\|\nabla v\|_2 \\ &\leq C + C\|\nabla v\|_2. \end{aligned}$$

Therefore  $\|\nabla v\|_2 \leq C$ , the classical Moser-Trudinger inequality enables us, as  $n = 2$

$$\int_{\Omega} e^{qv} \leq C_q, \quad \forall q \geq 1. \quad (3.1)$$

Take now  $\phi(u) = f(u) - f(0)$  in (2.4), we need to estimate

$$\begin{aligned} \zeta(u) &:= f'(u)\phi(u) - \frac{2}{\beta}\xi(u) = f'(u)\phi(u) - \frac{2}{\beta} \int_0^u f'^2(s)ds \\ &= f'(u)f(u) - \frac{2}{\beta} \int_0^u f'^2(s)ds - Cf'(u) \\ &:= I(u) - \frac{2}{\beta}J(u) - Cf'(u). \end{aligned}$$

By (H2), there exists  $\delta > 0$  such that

$$I(u) - I(0) = \int_0^u [f'^2(s) + f''(s)f(s)] ds \geq (1 + \delta)J(u) - Cf'(u), \quad \forall u \in [0, 1)$$

Let  $\frac{4}{2+\delta} < \beta < 2$ , we get  $\zeta(u) \geq CI(u) - C$ . Asserting this in (2.4),

$$\lambda \int_{\Omega} e^{\gamma} f'(u) f^2(u) \leq C \int_{\Omega} e^{\gamma} f^2(u) + C.$$

Consequently,  $\|f'(u)f^2(u)\|_1 \leq C$ . By Lemma 2.1, we deduce  $\|f(u)\|_3 \leq C$ . Combining with (3.1),  $\|g(v)\|_p \leq C$  for any  $p < 3$ . The proof is completed by Lemma 3.1 as  $n = 2$ .  $\square$

### 3.2. Proof of Theorem 1.4

Without loss of generality, we can assume that  $g(0) = 1$ . Let  $\phi(t) = g^{\alpha}(t) - 1$  where  $\alpha > 0$  is a constant to be determined later. Then

$$\begin{aligned} \xi(t) &= \int_0^t \phi'^2(s)ds \\ &= \alpha^2 \int_0^t g^{2\alpha-2}(s)g'^2(s)ds \\ &= \frac{\alpha^2}{2\alpha-1} g^{2\alpha-1}(t)g'(t) - \frac{\alpha^2}{2\alpha-1} \int_0^t g^{2\alpha-1}(s)g''(s)ds - C_{\alpha}. \end{aligned} \quad (3.2)$$

The condition  $(\widetilde{H2})$  yields: Given any  $\epsilon \in (0, \mu - \frac{1}{1+\delta})$ , there exists  $C \geq 0$  such that  $g(t)g''(t) \geq (\mu - \epsilon)g'^2(t) - C$  in  $[0, \infty)$ . Therefore

$$\begin{aligned} - \int_0^t g^{2\alpha-1}(s)g''(s)ds &\leq -(\mu - \epsilon) \int_0^t g^{2\alpha-2}(s)g'^2(s)ds + C \\ &\leq -\frac{\mu - \epsilon}{\alpha^2} \xi(t) + C. \end{aligned} \quad (3.3)$$

We divide the proof into two cases.

Case 1:  $\delta > 1$  and  $\mu > \frac{1}{1+\delta}$ ; or  $\delta \leq 1$  with  $\mu > \frac{1+\delta}{4\delta}$ .

Take  $\alpha > \frac{1}{2}$ . Combine (3.2) and (3.3),

$$\left(1 + \frac{\mu - \epsilon}{2\alpha - 1}\right) \xi(t) \leq \frac{\alpha^2}{2\alpha - 1} g^{2\alpha-1}(t) g'(t) + C,$$

consequently

$$\xi(t) \leq \frac{\alpha^2}{2\alpha - 1 + \mu - \epsilon} g^{2\alpha-1}(t) g'(t) + C, \quad \text{for any } t \geq 0. \quad (3.4)$$

According to (H3), for any  $0 < \delta' < \delta$ , there exists  $C > 0$  such that  $g'(t) \geq (1 + \delta')g(t) - C$  in  $[0, \infty)$ . Setting these estimates in (2.3), omitting the index  $\lambda$  and recalling that  $f'(u) = g'(v) - g(v)$ ,

$$\begin{aligned} & \frac{\delta' \lambda}{1 + \delta'} \int_{\Omega} e^{\gamma} g'(v) (g^{\alpha}(v) - 1)^2 - C \lambda \int_{\Omega} e^{\gamma} (g^{\alpha}(v) - 1)^2 \\ & \leq \lambda \int_{\Omega} e^{\gamma} f'(u) (g^{\alpha}(v) - 1)^2 \\ & \leq \frac{2\alpha^2 \lambda}{\beta(2\alpha - 1 + \mu - \epsilon)} \int_{\Omega} e^{\gamma} g^{2\alpha}(v) g'(v) + C \lambda \int_{\Omega} e^{\gamma} g(v) + C \int_{\Omega} e^{\gamma} (g^{\alpha}(v) - 1)^2. \end{aligned}$$

Consequently,

$$\begin{aligned} & \left[ \frac{\delta'}{1 + \delta'} - \frac{2\alpha^2}{\beta(2\alpha - 1 + \mu - \epsilon)} \right] \lambda \int_{\Omega} e^{\gamma} g'(v) g^{2\alpha}(v) \\ & \leq \frac{2\delta' C}{1 + \delta'} \int_{\Omega} e^{\gamma} g'(v) g^{\alpha}(v) + C \int_{\Omega} e^{\gamma} g(v) + C \int_{\Omega} e^{\gamma} (g^{\alpha}(v) - 1)^2. \end{aligned}$$

Choose  $\delta'$  near  $\delta$  such that

$$\text{either } \delta' > 1 \text{ and } \mu > \frac{1}{1 + \delta'} \text{ or } \delta' < \delta \leq 1 \text{ with } \mu > \frac{1 + \delta'}{4\delta'}.$$

Through direct computations, for  $\epsilon > 0$  sufficiently small and  $\beta = 2 - \epsilon$ , there exists

$$\alpha \in \left( \frac{1}{2}, \frac{\delta'}{1 + \delta'} + \frac{\sqrt{\delta'(1 + \delta')(\mu - \epsilon) - \delta'}}{1 + \delta'} \right)$$

such that

$$\left[ \frac{\delta'}{1 + \delta'} - \frac{2\alpha^2}{\beta(2\alpha - 1 + \mu - \epsilon)} \right] > 0. \quad (3.5)$$

For such  $\alpha$ , we obtain

$$\lambda \int_{\Omega} e^{\gamma} g^{2\alpha}(v) g'(v) \leq C, \quad \forall \lambda \in (0, \lambda^*). \quad (3.6)$$

Tending now  $\delta'$  to  $\delta$  and  $\epsilon$  to 0, (3.6) holds true provided that

$$\alpha < \frac{\delta}{1+\delta} + \frac{\sqrt{\delta\mu(1+\delta)-\delta}}{1+\delta}. \quad (3.7)$$

Therefore

$$\int_{\Omega} e^{\gamma} g^{2\alpha+1}(v) \leq C \int_{\Omega} e^{\gamma} g^{2\alpha}(v) g'(v) + C \leq \tilde{C},$$

which implies that  $\|g(v)\|_{2\alpha+1} \leq C$  for  $\alpha$  verifying (3.7). Applying Lemma 3.1, we conclude that for  $n < 2 + 4\alpha$  with  $\alpha$  verifying (3.7),  $v_{\lambda}$  is uniformly bounded, hence  $u^*$  is a regular solution if  $n$  satisfies (1.4).

Case 2:  $\delta \leq 1$  and  $\frac{1}{1+\delta} < \mu \leq \frac{1+\delta}{4\delta}$ .

Now we take  $\alpha \in (\frac{1}{2}(1 - \mu + \epsilon), \frac{1}{2})$ , the formulas (3.2) and (3.3) imply then

$$\left(1 + \frac{\mu - \epsilon}{2\alpha - 1}\right) \xi(t) \geq \frac{\alpha^2}{2\alpha - 1} g^{2\alpha-1}(t) g'(t) + C.$$

The inequality (3.4) still holds true. Proceeding as for Case 1, we see that for  $\delta' < \delta$  but nearby,  $\epsilon > 0$  small and  $\beta = 2 - \epsilon$ , there exists

$$\alpha \in \left(\frac{1 - \mu + \epsilon}{2}, \frac{\delta'}{1 + \delta'} + \frac{\sqrt{\delta'(1 + \delta')(\mu - \epsilon) - \delta'}}{1 + \delta'}\right) \subset \left(\frac{1 - \mu + \epsilon}{2}, \frac{1}{2}\right)$$

such that (3.5) is satisfied. Hence we conclude exactly as in Case 1.  $\square$

### 3.3. Proof of Theorem 1.5

Without loss of generality, assume again  $g(0) = 1$ . Take now  $\phi(t) = te^{\alpha h(t)}$ , where  $\alpha > 0$  is a constant to be determined, then

$$\begin{aligned} \xi(t) &= \int_0^t [1 + s\alpha h'(s)]^2 e^{2\alpha h(s)} ds \\ &= \int_0^t [1 + 2s\alpha h'(s)] e^{2\alpha h(s)} ds + \int_0^t \alpha^2 s^2 h'^2(s) e^{2\alpha h(s)} ds \\ &= te^{2\alpha h(t)} + K(t). \end{aligned}$$

Thus, for  $t \geq t_0$ ,

$$\begin{aligned} \frac{2K(t)}{\alpha} &= 2\alpha \int_0^t s^2 h'^2(s) e^{2\alpha h(s)} ds = C + \int_{t_0}^t s^2 h'(s) d\left(e^{2\alpha h(s)}\right) \\ &\leq C + t^2 h'(t) e^{2\alpha h(t)} - \int_{t_0}^t e^{2\alpha h(s)} d\left(s^2 h'(s)\right), \end{aligned}$$

where the last integration is considered in the sense of Stieltjes. The monotonicity of  $s^2 h'$  in  $[t_0, \infty)$  yields

$$K(t) \leq \frac{\alpha}{2} t^2 h'(t) e^{2\alpha h(t)} + C, \quad \forall t \geq t_0.$$

So we get

$$\xi(t) \leq C + \left[ t + \frac{\alpha}{2} t^2 h'(t) \right] e^{2\alpha h(t)}, \quad \forall t \geq 0.$$

Using (2.3) (we drop the index  $\lambda$ ),

$$\begin{aligned} & \int_{\Omega} e^{\gamma} \left[ e^{h(v)} + v h'(v) e^{h(v)} - v e^{h(v)} - 1 \right] v^2 e^{2\alpha h(v)} \\ & \leq \frac{2}{\beta} \int_{\Omega} e^{\gamma} \left( 1 + v e^{h(v)} \right) \xi(v) + C \int_{\Omega} e^{\gamma} v^2 e^{2\alpha h(v)} \\ & \leq \frac{2}{\beta} \int_{\Omega} e^{\gamma} \left( 1 + v e^{h(v)} \right) \left[ C + v e^{2\alpha h(v)} + \frac{\alpha}{2} v^2 h'(v) e^{2\alpha h(v)} \right] + C \int_{\Omega} e^{\gamma} v^2 e^{2\alpha h(v)}, \end{aligned}$$

By Young's inequality,

$$\begin{aligned} & \left( 1 - \frac{\alpha}{\beta} \right) \int_{\Omega} e^{\gamma} v^3 h'(v) e^{(2\alpha+1)h(v)} \\ & \leq C \int_{\Omega} e^{\gamma} \left[ 1 + v^2 h'(v) e^{2\alpha h(v)} + v^3 e^{(2\alpha+1)h(v)} \right]. \end{aligned} \tag{3.8}$$

Moreover,  $g = o(g')$  at infinity yields  $\lim_{t \rightarrow \infty} h'(t) = \infty$ , hence

$$\frac{t^2 h'(t) e^{2\alpha h(t)} + t^3 e^{(2\alpha+1)h(t)}}{t^3 h'(t) e^{(2\alpha+1)h(t)}} = \frac{1}{g(t) - 1} + \frac{1}{h'(t)} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Fix  $\beta \in (\alpha, 2)$ , the inequality (3.8) implies

$$\int_{\Omega} \frac{[g(v) - 1]^{2\alpha+1}}{v^{2\alpha}} = \int_{\Omega} v e^{(2\alpha+1)h(v)} \leq C + \int_{\Omega} v^3 h'(v) e^{(2\alpha+1)h(v)} \leq C.$$

Recall that  $g$  is superlinear, we obtain  $\|g(v)\|_1 \leq C$ . Consider again  $w$  satisfying  $L(w) = \lambda g(v)$  in  $\Omega$  and  $w = 0$  on  $\partial\Omega$ , as  $v \leq w$  in  $\Omega$  by maximum principle,

$$\int_{\Omega} \frac{(g(v) - 1)^{2\alpha+1}}{w^{2\alpha}} \leq C.$$

Following the proof of Lemma 2.1 in [17] (we just need a minor adjustment, say define  $\Omega_1 = \{x \in \Omega : g(v) > w^T\}$  instead, here  $T > 0$  is a suitable constant), we can obtain that if  $2\alpha + 1 > n/2$ ,  $w$  is uniformly bounded in  $L^\infty(\Omega)$ , so does  $v$ . Taking  $2 > \beta > \alpha > 7/4$ , the result holds for  $n \leq 9$ .  $\square$

#### 3.4. Proof of Theorem 1.6

Here we choose  $\phi(u) = (1 - u)^{-\alpha} - 1$  in (2.4). For  $2\lambda > \lambda^*$  and  $\epsilon > 0$ ,

$$\left( p - \frac{2\alpha^2}{\beta(2\alpha+1)} - 2\epsilon \right) \int_{\Omega} \frac{e^{\gamma}}{(1-u)^{p+2\alpha+1}} \leq C, \quad \forall \beta \in [1, 2).$$

We have used  $f'(u)(1-u) \geq (p-\epsilon)f(u) - C$  in  $[0, 1]$  by (1.5). As  $\epsilon > 0$  is arbitrary,

$$\int_{\Omega} \frac{1}{(1-u)^{p+2\alpha+1}} \leq C$$

provided that

$$p > \frac{\alpha^2}{2\alpha + 1}, \quad \text{i.e. when } \alpha < p + \sqrt{p(p+1)}.$$

Therefore  $\|(1-u)^{-1}\|_q \leq C$  if  $q < 1 + 3p + 2\sqrt{p(p+1)}$ . For any  $\epsilon > 0$ , as  $f'(u)(1-u) \leq (p+\epsilon)f(u) + C_\epsilon$  in  $[0, 1]$  by (1.5), we have  $f(u) \leq C(1-u)^{-p-\epsilon}$ , consequently

$$g(v) = e^v f(1 - e^{-v}) = \frac{f(u)}{1-u} \leq C(1-u)^{-1-p-\epsilon},$$

hence  $\|g(v)\|_r \leq C$  when

$$r < \frac{1 + 3p + 2\sqrt{p(p+1)}}{p + 1 + \epsilon}.$$

According to Lemma 3.1, the proof is done by taking  $\epsilon \rightarrow 0^+$ .  $\square$

#### 4. Radial case

As we have mentioned, when  $c = -\nabla\gamma$ , the equation  $(P_\lambda)$  is rewritten as

$$-\operatorname{div}(e^\gamma \nabla u) = \lambda e^\gamma f(u). \quad (4.1)$$

With the variational structure, the stability of minimal solutions  $u_\lambda$  is equivalent to

$$\int_{\Omega} e^\gamma |\nabla \psi|^2 \geq \lambda \int_{\Omega} e^\gamma f'(u_\lambda) \psi^2, \quad \forall \psi \in H_0^1(\Omega). \quad (4.2)$$

Moreover, for any  $C^1$  functions  $\phi$  and  $\xi$  satisfying  $\phi(0) = \xi(0) = 0$  and  $\xi' = \phi'^2$ , the estimate (2.4) is replaced by

$$\int_{\Omega} e^\gamma f'(u_\lambda) \phi^2(u_\lambda) \leq \int_{\Omega} e^\gamma f(u_\lambda) \xi(u_\lambda).$$

Taking now  $\phi(t) = f(t) - f(0)$  and working as for Theorem 1 in [18], we have

**Lemma 4.1.** *When  $c = \nabla\gamma$ , the extremal solution  $u^* \in H^2 \cap H_0^1(\Omega)$ . More precisely,*

$$\int_{\Omega} f'(u_\lambda) f(u_\lambda) \leq C, \quad \forall \lambda \in (0, \lambda^*]. \quad (4.3)$$

When  $\Omega = B_1$  is the unit ball,  $\gamma(x) = \gamma(r)$  with  $r = |x|$ ,  $u_\lambda$  is radial by uniqueness of the minimal solution and satisfies

$$-u'' - \frac{n-1}{r} u' - \gamma' u' = \lambda f(u) \quad \text{in } (0, 1], \quad (4.4)$$

with  $u'(0) = 0$  and  $u(1) = 0$ . Our main result in this section is the regularity of the extremal solution  $u^*$  for any  $f$  satisfying (H) provided  $n = 2$  and the optimal estimate for  $u'$  claimed in Theorem 1.2.

The method we use is similar to [5, 15], but the uniform boundedness of  $\|u_\lambda\|_{C^1}$  is not enough to claim the regularity of  $u^*$ , because a singular  $u^*$  could be Lipschitz in many cases (see Remark 1.1). In fact, the estimate (4.3) is crucial for our proof.

As in [5, 15], since  $u'_\lambda(r) \leq 0$  by maximum principle or equation (4.4), the boundedness of  $\|u_\lambda\|_{H_0^1}$  implies that for any  $k \in \mathbb{N}$ ,  $r > 0$ ,  $\|u_\lambda\|_{C^k(\overline{B_1} \setminus B_r)} \leq C_{k,r}$ ,  $\forall \lambda \in (0, \lambda^*]$ . So we concentrate our attention near the origin. Derivating the equation (4.4) or (4.1) with respect to  $r$ ,

$$-\operatorname{div}(e^\gamma \nabla u') = e^\gamma u' \left[ \lambda f'(u) - \frac{n-1}{r^2} + \gamma'' \right] \quad \text{in } (0, 1].$$

Using  $\psi = r\eta(r)u'_\lambda(r)$  as test function in (4.2) with  $\eta \in H_0^1(B_1) \cap C(\overline{B_1})$ , by similar calculation as for Lemma 2.1 in [5], we obtain

$$\int_{B_1} e^\gamma \left[ |\nabla(r\eta)|^2 - (n-1)\eta^2 + \gamma'' r^2 \eta^2 \right] u_\lambda'^2 \geq 0, \quad \forall \lambda \in (0, \lambda^*]. \quad (4.5)$$

#### 4.1. Proof of Theorem 1.1

For simplicity, we drop the index  $\lambda$ . All estimates below hold uniformly for  $\lambda$ . First as  $u_\lambda$  is radial, by maximum principle, we see that  $u$  is decreasing in  $r$ . Since  $f$  and  $f'$  are nondecreasing functions according to (H), the estimate (4.3) implies (as  $n = 2$ )

$$\pi r^2 f'(u(r)) f(u(r)) \leq \int_{B_r} f'(u) f(u) \leq C, \quad \forall r \in (0, 1].$$

By Lemma 2.1, we have

$$f(u(r)) \leq \frac{C}{r} \quad \text{for all } r \in (0, 1]. \quad (4.6)$$

Let  $r_0 \in (0, \frac{1}{2}]$ . Let  $\eta$  be a radial function in  $H_0^1(B_1) \cap C^0(\overline{B_1})$  such that

$$\eta(r) = \begin{cases} r_0^{-1} & \text{if } r < r_0; \\ r^{-1} & \text{if } r_0 \leq r \leq \frac{1}{2}, \end{cases}$$

and  $\eta$  be a fixed  $C^1$  function in  $\overline{B_1} \setminus B_{1/2}$ , independent of  $r_0$ . The direct calculation yields

$$|\nabla(r\eta)|^2 - \eta^2 + \gamma'' r^2 \eta^2 = \begin{cases} \gamma'' r^2 r_0^{-2} & \text{if } r < r_0; \\ \gamma'' - r^{-2} & \text{if } r_0 < r \leq \frac{1}{2}. \end{cases}$$

Using (4.5), as  $u$  is uniformly bounded in  $H^1(B_1)$  by Proposition 2.4 and  $r^2 r_0^{-2} \leq 1$  in  $[0, r_0]$ , we get

$$\int_{r_0}^{\frac{1}{2}} \frac{u'(r)^2}{r} dr \leq C.$$

Tending  $r_0$  to 0, there holds

$$\int_0^1 \frac{u'(r)^2}{r} dr \leq C. \quad (4.7)$$

Consider the following test function used in [15]: For any  $r \leq \frac{1}{2}$  and  $0 < r_0 < r$ ,

$$\eta(s) = \begin{cases} (rr_0)^{-1} & \text{if } s < r_0; \\ (rs)^{-1} & \text{if } r_0 \leq s < r; \\ s^{-2} & \text{if } r \leq s \leq \frac{1}{2}. \end{cases}$$

Applying again (4.5) and combining with (4.7), we obtain finally (with  $r_0 \rightarrow 0$ )

$$\int_0^r \frac{u'(s)^2}{s} ds \leq Cr^2, \quad \forall r \leq 1. \quad (4.8)$$

As  $(e^\gamma r u')' = -\lambda e^\gamma r f(u)$  with  $n = 2$ , so  $e^\gamma r u'$  is nonincreasing in  $r$ . Then  $u'(s) \leq C r u'(r)/s$  for  $s \in [r, 1]$ , hence  $u'(s) \leq C u'(r) \leq 0$  for any  $s \in [r, 2r]$  if  $r \leq \frac{1}{2}$ . By (4.8), for any  $0 < r \leq \frac{1}{2}$ ,

$$C_1 r^2 \geq \int_0^{2r} \frac{u'(s)^2}{s} ds \geq \int_r^{2r} \frac{u'(s)^2}{s} ds \geq \frac{C_2}{r} \int_r^{2r} u'(r)^2 ds = C_3 u'(r)^2.$$

That means

$$|u'(r)| \leq Cr \quad \text{in } [0, 1]. \quad (4.9)$$

However, we need to consider also  $u''(r)$  as explained above. Let

$$G(r) = e^\gamma r u' \quad \text{and} \quad \Psi(r) = -2G(\sqrt{r}) - M \int_0^r (r-s) f(u(\sqrt{s})) ds$$

where  $M$  is a constant to be chosen. Using  $G' = -\lambda e^\gamma r f(u)$ ,

$$\begin{aligned} \Psi''(r) &= \left[ \lambda e^{\gamma(s)} f'(u(s)) \frac{u'(s)}{2s} + \lambda e^{\gamma(s)} f(u(s)) \frac{\gamma'(s)}{2s} - M f(u(s)) \right] \Big|_{s=\sqrt{r}} \\ &\leq \left[ \lambda e^{\gamma(s)} f(u(s)) \frac{\gamma'(s)}{2s} - M f(u(s)) \right] \Big|_{s=\sqrt{r}} \\ &\leq C_0 f(u(\sqrt{r})) - M f(u(\sqrt{r})). \end{aligned}$$

For the last line, we used  $|\gamma'(s)|/s \leq C$  in  $[0, 1]$  since  $\gamma$  is a smooth function (so  $\gamma'(0) = 0$ ). Fix  $M > C_0 + 1$ ,  $\Psi$  is then concave in  $[0, 1]$ . On the other hand, by (4.6)

$$\Psi'(r) = \lambda e^{\gamma(\sqrt{r})} f(u(\sqrt{r})) - M \int_0^r f(u(\sqrt{s})) ds \geq C \lambda f(0) - CM \sqrt{r}.$$

There exists  $r_1 > 0$  small enough such that  $\Psi' \geq 0$  in  $[0, r_1]$  with  $\lambda \geq \frac{\lambda^*}{2}$ . Using (4.4), (4.6) and (4.9), for  $\lambda \geq \frac{\lambda^*}{2}$  and  $r \leq r_1$ ,

$$\begin{aligned} &-e^{\gamma(\sqrt{r})} \left[ u''(\sqrt{r}) + \frac{u'(\sqrt{r})}{\sqrt{r}} + \gamma' u'(\sqrt{r}) \right] - CM \sqrt{r} \\ &\leq \Psi'(r) \leq \frac{\Psi(r)}{r} \leq -2e^{\gamma(\sqrt{r})} \frac{u'(\sqrt{r})}{\sqrt{r}} \leq C. \end{aligned}$$



Applying one more time (4.9), we see that  $u''(\sqrt{r}) \geq -C$  for any  $\lambda \geq \frac{\lambda^*}{2}$  and  $r \leq r_1$ . Otherwise, by (4.4) and (4.9),  $u''(r) \leq -u'(r)r^{-1} - \gamma'(r)u'(r) \leq C$ , we claim then

$$\|u''\|_\infty \leq C, \quad \forall \lambda \geq \frac{\lambda^*}{2}.$$

Combining with (4.4) and (4.9), it means  $\|\lambda f(u)\|_\infty \leq C$ , no singularity will occur.  $\square$

#### 4.2. Proof of Theorem 1.2

As above, we drop the index  $\lambda$  and all estimations hold uniformly for  $\lambda$ . First, repeating the proof of Theorem 1.8, c) in [5], we obtain  $f'(u(r)) \leq Cr^{-2}$  in  $(0, 1]$ . Using Lemma 2.1 with (4.5),  $f(u(r)) \leq Cr^{-2}$  in  $(0, 1]$ . Consequently, by (4.4), for  $n \geq 3$ ,

$$0 \leq -e^{\gamma} r^{n-1} u'(r) = \int_0^r e^{\gamma(s)} s^{n-1} f(u(s)) ds \leq C \int_0^r s^{n-3} ds \leq Cr^{n-2}.$$

Hence

$$|u'(r)| \leq \frac{C}{r}. \quad (4.10)$$

Let  $\eta$  be a radial function in  $H_0^1(B_1) \cap C^0(\overline{B}_1)$  such that

$$\eta(r) = \begin{cases} r_0^{-\sqrt{n-1}} & \text{if } r < r_0; \\ r^{-\sqrt{n-1}} & \text{if } r_0 \leq r \leq r_1. \end{cases}$$

in  $\overline{B}_{r_1}$  and be a fixed  $C^1$  function in  $\overline{B}_1 \setminus B_{r_1}$ , here  $r_0$  is any constant in  $(0, r_1)$ ,  $r_1 > 0$  is a small constant to be determined. Therefore

$$|\nabla(r\eta)|^2 - (n-1)\eta^2 + \gamma''r^2\eta^2 = \begin{cases} (\gamma''r^2 + 2 - n)r_0^{-2\sqrt{n-1}} & \text{if } r < r_0; \\ (\gamma''r^2 - 2\sqrt{n-1} + 1)r^{-2\sqrt{n-1}} & \text{if } r \in [r_0, r_1]. \end{cases}$$

We fix  $r_1 > 0$  small enough such that

$$\max_{r \in [0, r_1]} \{\gamma''r^2\} < \min(n-2, 2\sqrt{n-1}-1).$$

By (4.5), as  $|\nabla(r\eta)|^2 - (n-1)\eta^2 + \gamma''r^2\eta^2 \leq 0$  for  $r \in [0, r_0]$ ,

$$\int_{r_0}^{r_1} u^2(r) r^{n-1-2\sqrt{n-1}} dr \leq C.$$

Tending  $r_0$  to 0, we have

$$\int_0^{r_1} u^2(r) r^{n-1-2\sqrt{n-1}} dr \leq C. \quad (4.11)$$

Now we take another test function used in [15],

$$\eta(r) = \begin{cases} r_0^{-\sqrt{n-1}-1} & \text{if } r < r_0; \\ r^{-\sqrt{n-1}-1} & \text{if } r_0 \leq r \leq r_1. \end{cases}$$

Combining (4.5) and (4.11), we conclude then

$$\int_0^{r_0} u'^2(r) r^{n-1} dr \leq C r_0^{2+2\sqrt{n-1}}, \quad \forall r_0 \in [0, r_1].$$

By the monotonicity of  $e^\gamma r^{n-1} u'$ , similarly as for (4.9), it holds

$$|u'(r)| \leq C r^{-\frac{n}{2}+1+\sqrt{n-1}}, \quad \forall r \in [0, 1].$$

Finally, combining with (4.10), we are done (in fact,  $-\frac{n}{2} + 1 + \sqrt{n-1} \leq -1$  for  $n \geq 10$ ).  
□

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## References

- [1] H. Berestycki, A. Kiselev, A. Novikov and L. Ryzhik, *The explosion problem in a flow*, J. Anal. Math. **110**, 31-65, (2010).
- [2] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, *Blow up for  $u_t - \Delta u = g(u)$  revisited*, Adv. Diff. Equa. **1**, 73-90, (1996).
- [3] H. Brezis and J. L. Vazquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10(2)**, 443-469, (1997).
- [4] X. Cabré, *Regularity of minimizers of semilinear elliptic problems up to dimension 4*, Comm. Pure Appl. Math. **63(10)**, 1362-1380, (2010).
- [5] X. Cabré and A. Capella, *Regularity of radial minimizers and extremal solutions of semilinear elliptic equations*, J. Funct. Anal. **238**, 709-733, (2006).
- [6] C. Cowan, *Optimal Hardy inequalities for general elliptic operators with im-provemnets*, Comm. Pure Appl. Anal. **9(1)**, 109-140, (2010).
- [7] C. Cowan and N. Ghoussoub, *Regularity of the extremal solution in a MEMS model with advection*, Methods Appl. Anal. **15(3)**, 355-360, (2008).
- [8] M. Crandall and P. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rat. Mech. **58**, 207-218, (1975).
- [9] P. Esposito, N. Ghoussoub and Y. Guo, *Mathematical analysis of partial differential equations modeling electrostatic MEMS*, Courant Lecture Notes in Mathematics **20**, ISBN: 978-0-8218-4957-6, (2010).

- [10] N. Ghoussoub and Y. Guo, *On the partial differential equations of electro MEMS devices: stationary case*, SIAM J. Math. Anal. **38**, 1423-1449, (2007).
- [11] Y. Martel, *Uniqueness of weak extremal solutions of nonlinear elliptic problems*, Houston J. Math. **23(1)**, 161-168, (1997).
- [12] F. Mignot and J. Puel, *Sur une classe de problèmes non linéaires avec non linéarité positive, croissante, convexe*, Comm. Part. Diff. Equa. **5(8)**, 791-836, (1980).
- [13] G. Nedev, *Regularity of the extremal solution of semilinear elliptic equations*, C. R. Acad. Sci. Paris **I 330**, 997-1002, (2000).
- [14] J. Pelesko and A. Berstein, *Modeling MEMS and NEMS*, Chapman Hall and CRC Press, (2002).
- [15] S. Villegas, *Shape estimates for semi-stable radial solutions of semilinear elliptic equations*, preprint arXiv:0906.1443v1, (2009).
- [16] J. Wei and D. Ye, *On MEMS equation with fringing field*, Proc. Amer. Math. Soc. **138(5)**, 1693-1699, (2010).
- [17] D. Ye and F. Zhou, *Boundedness of the extremal solution for semilinear elliptic problems*, Comm. Contemp. Math. **4(3)**, 547-558, (2002).
- [18] D. Ye and F. Zhou, *On a general family of nonautonomous elliptic and parabolic equations*, Calc. Var. and P.D.E. **37**, 259-274, (2010).