L^p-asymptotic stability analysis of a 1D wave equation with a boundary nonmonotone damping

Swann Marx, Guilherme Mazanti

L^p-asymptotic stability analysis of a 1D wave equation with a boundary nonmonotone damping

Swann Marx[1] and Guilherme Mazanti[2]

January 31, 2020

Abstract

This paper is concerned with the asymptotic stability analysis of a one dimensional wave equation with a nonlinear non-monotone damping acting at a boundary. The study is performed in an L^p-functional framework, $p \in [1, \infty]$. Some well-posedness results are provided together with exponential decay to zero of trajectories, with an estimation of the decay rate. The well-posedness results rely mainly on some results collected in [7]. Asymptotic behavior results are obtained by the use of a suitable Lyapunov functional if p is finite and on a trajectory-based analysis if $p = \infty$.

Contents

1 Problem statement

In this paper, we focus on the following boundary control wave equation

$$\begin{cases} z_{tt} = z_{xx}, & (t,x) \in \mathbb{R}_+ \times [0,1], \\ z(t,0) = 0,\ z_x(t,1) = u(t), & t \in \mathbb{R}_+, \\ z(0,x) = z_0(x),\ z_t(0,x) = z_1(x), & x \in [0,1]. \end{cases} \tag{1}$$

[1]LS2N, École Centrale de Nantes & CNRS UMR 6004, F-44000 Nantes, France.

[2]Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France & DISCO Team, Inria Saclay & Institut Polytechnique des Sciences Avancées (IPSA), 94200 Ivry-sur-Seine, France.

It is well known that choosing a feedback of the form $u(t) = -\alpha z_t(t,1)$ with $\alpha > 0$ allows one to exponentially stabilize the equilibrium point 0. We refer to [8] for the complete analysis.

Our aim in this article is to focus on the case of a non-linear feedback law, that is,

$$u(t) = -\sigma(z_t(t,1)), \qquad \forall t \geq 0, \tag{2}$$

leading to the nonlinear system

$$\begin{cases} z_{tt}(t,x) = z_{xx}(t,x), & (t,x) \in \mathbb{R}_+ \times [0,1], \\ z(t,0) = 0,\ z_x(t,1) = -\sigma(z_t(t,1)), & t \in \mathbb{R}_+, \\ z(0,x) = z_0(x),\ z_t(0,x) = z_1(x), & x \in [0,1]. \end{cases} \tag{3}$$

The nonlinearity under consideration is given by the following definition.

Definition 1 (Nonlinear damping)**.** A continuous function $\sigma : \mathbb{R} \ni s \mapsto \sigma(s) \in \mathbb{R}$ is called a *nonlinear damping* if the following properties are satisfied:

1. For all $s \in \mathbb{R}$ such that $s \neq 0$, $\sigma(s)s > 0$.
2. One has that

$$0 < \sigma_- = \liminf_{s\to 0} \frac{\sigma(s)}{s} \leq \limsup_{s\to 0} \frac{\sigma(s)}{s} = \sigma_+ < +\infty. \tag{4}$$

We will also sometimes assume in addition the following property.

3. For all distinct real numbers s_1, s_2, one has

$$\frac{\sigma(s_1) - \sigma(s_2)}{s_1 - s_2} > -1. \tag{5}$$

It is clear from Definition 1 that the nonlinearity σ is not necessarily a monotone function. Note moreover that this definition is that of [7], except for the additional second item, which is inspired by the definition given in [4,5]. As clearly explained in [7], the first item of the definition is used to ensure the well-posedness of (3) (in particular existence of solutions) while the additional item given in (5) insures uniqueness of the solution.

We aim at obtaining decay rate estimates of the trajectories in L^p-spaces, $p \in [1,\infty]$, as done in [2, 3] for a 1D wave equation subject to distributed (nonlinear) damping. L^p-spaces, $p \in [1,\infty]$, are defined as

$$H_p(0,1) := W^{1,p}_*(0,1) \times L^p(0,1), \tag{6}$$

where $W^{1,p}_*(0,1) = \{u \in L^p(0,1) \mid u' \in L^p(0,1) \text{ and } u(0) = 0\}$, and equipped with the norms

$$\begin{aligned} \|(u,v)\|_{H_p(0,1)} &:= \left(\int_0^1 (\left|u'\right|^p + |v|^p)dx\right)^{\frac{1}{p}}, \qquad p \in [1,\infty), \\ \|(u,v)\|_{H_\infty(0,1)} &:= \|u'\|_{L^\infty(0,1)} + \|v\|_{L^\infty(0,1)}. \end{aligned} \tag{7}$$

As explained in [6], the semigroup generated by the D'Alembertian $\Box z := z_{tt} - \Delta z$ associated with an open bounded subset in $\mathbb{R}^n$, $n \geq 2$, with Dirichlet boundary conditions is not defined in general for any suitable generalization for higher dimension of

$H_p(0,1)$. This is why the study of issues involving L^p spaces is only meaningful in the one-dimensional case.

In order to introduce a notion of weak solution of (3) adapted both to L^p spaces and to the one-dimensional case, let us recall the following classical result on regular solutions to the one-dimensional wave equation, which corresponds to d'Alembert decomposition into travelling waves.

Proposition 2. *Let $z : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ be such that $z \in \mathcal{C}^2(\mathbb{R}_+ \times [0,1])$. Then z satisfies $z_{xx} = z_{tt}$ in $\mathbb{R}_+ \times [0,1]$ if and only if there exist functions $f \in \mathcal{C}^1([0,+\infty))$ and $g \in \mathcal{C}^1([-1,+\infty))$ such that*

$$z(t,x) = z(0,0) + \int_0^{t+x} f(s)ds + \int_0^{t-x} g(s)ds. \tag{8}$$

Proof. Assume that z satisfies $z_{xx} = z_{tt}$ in $\mathbb{R}_+ \times [0,1]$ and let $u, v : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ be given by

$$\begin{aligned} u(t,x) &= \frac{1}{2}\left[z_t(t,x) + z_x(t,x)\right], \\ v(t,x) &= \frac{1}{2}\left[z_t(t,x) - z_x(t,x)\right]. \end{aligned} \tag{9}$$

Then $u, v \in \mathcal{C}^1(\mathbb{R}_+ \times [0,1])$ and $u_t = u_x$, $v_t = -v_x$ in $\mathbb{R}_+ \times [0,1]$. One immediately verifies that, for every $(t,x) \in \mathbb{R}_+ \times [0,1]$, the functions $h \mapsto u(t+h, x-h)$ and $h \mapsto v(t+h, x+h)$ are constant in their domains. Letting $f : [0,+\infty) \to \mathbb{R}$ and $g : [-1,+\infty) \to \mathbb{R}$ being defined by

$$f(s) = u(s,0), \qquad g(s) = \begin{cases} v(s,0) & \text{if } s \geq 0, \\ v(0,-s) & \text{if } -1 \leq s < 0, \end{cases}$$

one can easily check that $f \in \mathcal{C}^1([0,+\infty))$, $g \in \mathcal{C}^1([-1,+\infty))$, and $u(t,x) = f(t+x)$ and $v(t,x) = g(t-x)$ for every $(t,x) \in \mathbb{R}_+ \times [0,1]$. In particular, it follows from (9) that

$$\begin{aligned} z_t(t,x) &= f(t+x) + g(t-x), \\ z_x(t,x) &= f(t+x) - g(t-x). \end{aligned}$$

Hence

$$\begin{aligned} z(t,x) &= z(0,0) + \int_0^x z_x(0,s)ds + \int_0^t z_t(s,x)ds \\ &= z(0,0) + \int_0^x f(s)ds - \int_0^x g(-s)ds + \int_0^t f(s+x)ds + \int_0^t g(s-x)ds \\ &= z(0,0) + \int_0^{t+x} f(s)ds + \int_0^{t-x} g(s)ds, \end{aligned}$$

as required. Conversely, if z is given by (8), it is easy to see that $z_{tt} = z_{xx}$. □

The functions f and g from Proposition 2 are also called *Riemann invariants* in the classical literature of hyperbolic PDEs (see, for instance, [1]). Proposition 2 motivates the following definition of weak solution to (3).

Definition 3. Let $(z_0, z_1) \in H_p(0,1)$. We say that $z : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ is a *weak global solution* of (3) in $H_p(0,1)$ with initial condition (z_0, z_1) if there exist $f \in L^p_{\mathrm{loc}}(0, +\infty)$ and $g \in L^p_{\mathrm{loc}}(-1, +\infty)$ such that

$$\begin{cases} z(t,x) = \displaystyle\int_0^{t+x} f(s)ds + \int_0^{t-x} g(s)ds & \text{for all } (t,x) \in \mathbb{R}_+ \times [0,1], \\ z(t,0) = 0,\ z_x(t,1) = -\sigma(z_t(t,1)), & \text{for a.e. } t \in \mathbb{R}_+, \\ z(0,x) = z_0(x),\ z_t(0,x) = z_1(x), & \text{for a.e. } x \in [0,1]. \end{cases}$$

Note that, if z is a weak global solution of (3) in $H_p(0,1)$, then $(z(t,\cdot), z_t(t,\cdot)) \in H_p(0,1)$ for every $t \in \mathbb{R}_+$ and $z_{tt} = z_{xx}$ is satisfied in $\mathbb{R}_+^* \times (0,1)$ in the sense of distributions. In the sequel, we refer to weak global solutions of (3) simply as *solutions*.

Letting f and g be as in Definition 3, one has, for a.e. $(t,x) \in \mathbb{R}_+ \times [0,1]$,

$$\begin{aligned} z_x(t,x) &= f(t+x) - g(t-x), \\ z_t(t,x) &= f(t+x) + g(t-x). \end{aligned} \tag{10}$$

By rewriting the boundary and initial conditions of (3) in terms of the functions f and g, one obtains the following characterization of solutions to (3).

Proposition 4. *Let $(z_0, z_1) \in H_p(0,1)$, $f \in L^p_{\mathrm{loc}}(0, +\infty)$, and $g \in L^p_{\mathrm{loc}}(-1, +\infty)$. The function $z : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ defined by*

$$z(t,x) = \int_0^{t+x} f(s)ds + \int_0^{t-x} g(s)ds \tag{11}$$

is a solution of (3) in $H_p(0,1)$ with initial condition (z_0, z_1) if and only if

$$f(s) = \frac{1}{2}\left(z_1(s) + z_0'(s)\right), \qquad \text{for a.e. } s \in [0,1], \tag{12a}$$

$$g(s) = \frac{1}{2}\left(z_1(-s) - z_0'(-s)\right), \qquad \text{for a.e. } s \in [-1,0], \tag{12b}$$

$$f(s) = -g(s), \qquad \text{for a.e. } s \geq 0, \tag{12c}$$

$$f(s) - g(s-2) = -\sigma(f(s) + g(s-2)), \qquad \text{for a.e. } s \geq 1. \tag{12d}$$

2 Main results

A first result we present in this paper is the well-posedness of (3) in the functional space $H_p(0,1)$. The proof of this result was provided in [7] in the case $p = 2$ and, as we detail below, the same line of proof can be adapted to our setting. Our first theorem also states that the trace $z_t(t,1)$ is bounded almost everywhere by the initial conditions of (3).

Theorem 5 (Well-posedness and strong stability)**.** *Let p be any number in the interval $[1, \infty]$ and assume that σ satisfies 1 from Definition 1. Then, for any $(z_0, z_1) \in H_p(0,1)$, there exists a solution z of (3) in $H_p(0,1)$ with initial condition (z_0, z_1). Moreover, one has the following properties.*

1. *If σ also satisfies 3 from Definition 1, then the above solution is unique.*

2. *For every solution z of* (3) *in $H_p(0,1)$ with initial condition (z_0, z_1), one has*

$$\|(z(t,\cdot), z_t(t,\cdot))\|_{H_p(0,1)} \leq \|(z_0, z_1)\|_{H_p(0,1)}, \qquad \text{for all } t \geq 0. \tag{13}$$

3. *If $(z_0, z_1) \in H_\infty(0,1)$, then, for a.e. $t \in \mathbb{R}_+$,*

$$|z_t(t,1)| \leq \|(z_0, z_1)\|_{H_\infty(0,1)}. \tag{14}$$

4. *If $p \in [1, +\infty)$, then all solutions z of* (3) *tend to zero in $H_p(0,1)$, i.e.,*

$$\lim_{t\to+\infty} \|(z(t,\cdot), z_t(t,\cdot))\|_{H_p(0,1)} = 0. \tag{15}$$

In the result we state next, we provide some estimations of the decay rate of the solution. These estimations are however not uniform, since they depend on the H_∞-norm of the initial condition. This implies therefore to consider solutions more regular than in the case investigated in [7], but, in that paper, the authors do not provide any analysis about the decay rate of the solution.

Theorem 6 (Exponential decay rates)**.** *Consider* (3) *and assume that σ is a nonlinear damping, satisfying Items 1 and 2 of Definition 1. Let $M > 0$ and $(z_0, z_1) \in H_\infty(0,1)$ satisfying $\|(z_0, z_1)\|_{H_\infty(0,1)} \leq M$. Then, for any $p \in [1, \infty]$, there exist positive constants $C := C(M,p)$ and $\mu := \mu(M,p)$ such that any solution of* (3) *satisfies*

$$\|(z, z_t)\|_{H_p(0,1)} \leq Ce^{-\mu t}\|(z_0, z_1)\|_{H_p(0,1)}, \ \forall t \geq 0. \tag{16}$$

3 Proof of Theorem 5

Several parts of the proof of Theorem 5 already appear in [7] but, in order to make this paper self-contained, we will provide the main lines of arguments. We start with the following technical lemmas.

Lemma 7 ([7, Lemma 1])**.** *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $\sigma(s)s \geq 0$ for all $s \in \mathbb{R}$. Then,*

1. *for every real number A, the equation*

$$X + A + \sigma(X - A) = 0 \tag{17}$$

with real unknown X, has a solution $S(A)$ of smallest absolute value and the map $A \mapsto S(A)$ is lower-semi-continuous on $\mathbb{R}$.

2. *Any solution X to* (17) *satisfies*

$$|X| \leq |A|. \tag{18}$$

3. *Moreover, if the function σ satisfies* (5)*, then there exists a* unique *solution to* (17)*.*

Lemma 8. *Consider a trajectory of* (3) *associated with an initial condition* $(z_0, z_1) \in H_p(0,1)$ *for some* $p \in [1,\infty)$. *For* $t \geq 0$, *set*

$$E_p(t) = \frac{1}{2^p} \int_0^1 \Big(|z_x(t,x) + z_t(t,x)|^p + |z_x(t,x) - z_t(t,x)|^p \Big) dx. \tag{19}$$

Then, there exists positive constants C_p, C_p' *(only depending on* p*) so that, for every pair of times* $1 \leq T_1 \leq T_2$, *it holds*

$$-C_p \int_{T_1}^{T_2} |z_t(t,1)|^p M_p(\delta_t) dt \leq E_p(T_2) - E_p(T_1) \leq -C_p' \int_{T_1}^{T_2} |z_t(t,1)|^p M_p(\delta_t) dt, \tag{20}$$

where δ_t *is defined for a.e.* $t \in [T_1, T_2]$ *by* $\delta_t = \frac{\sigma(z_t(t,1))}{z_t(t,1)} > 0$ *if* $z_t(t,1) \neq 0$ *and* $\delta_t = \sigma_-$ *otherwise and* M_p *is the function defined for* $\xi \geq 0$ *by* $M_p(\xi) = \xi$ *if* $\xi \in [0,1]$ *and* $M_p(\xi) = \xi^{p-1}$ *if* $\xi > 1$.

Remark 9. The choice that $\delta_t = \sigma_-$ when $z_t(t,1) = 0$ is arbitrary and the value σ_- could be replaced by any other (fixed) real number in $[\sigma_-, \sigma_+]$ (so as to make δ continuous if σ is continuous at 0).

The functional E_p was introduced in [3], where it is shown that it is non-increasing along trajectories of 1D wave equations with Dirichlet boundary conditions and with distributed nonlinear damping. Here, we provide a more quantitative estimate of the time derivative of E_p in the situation of interest.

Proof of Lemma 8. Along any trajectory of (3) with initial condition in $H_p(0,1)$ for some $p \in [1,\infty]$, one deduces from (10) that, for $t \geq 1$,

$$\begin{aligned} E_p(t) &= \int_0^1 \Big(|f(t+x)|^p + |g(t-x)|^p \Big) dx \\ &= \int_t^{t+1} |f(\xi)|^p \, d\xi + \int_{t-1}^t |g(\xi)|^p \, d\xi. \end{aligned} \tag{21}$$

One deduces at once that, for $t \geq 1$,

$$\begin{aligned} \frac{dE_p}{dt} &= |f(t+1)|^p - |g(t-1)|^p \\ &= |z_x(t,1) + z_t(t,1)|^p - |z_x(t,1) - z_t(t,1)|^p, \end{aligned} \tag{22}$$

where we have used (12c) in the first line. It follows that, for every $1 \leq T_1 \leq T_2$,

$$E_p(T_2) - E_p(T_1) = -\int_{T_1}^{T_2} \Big(|z_x(t,1) - z_t(t,1)|^p - |z_x(t,1) + z_t(t,1)|^p \Big) dt. \tag{23}$$

We now use the boundary condition given in (3) to deduce that

$$E_p(T_2) - E_p(T_1) = -\int_{T_1}^{T_2} |z_t(t,1)|^p \Big(|1+\delta_t|^p - |1-\delta_t|^p \Big) dt.$$

Since there exists $C_p, C_p' > 0$ depending on p such that

$$C_p' M_p(\delta) \leq |1+\delta|^p - |1-\delta|^p \leq C_p M_p(\delta), \quad \text{for every } \delta \geq 0,$$

one concludes the proof of the lemma. □

We are now ready to prove Theorem 5 by extending the proof from [7] in the case $p = 2$ to other values of $p \in [1, \infty]$.

Proof of Theorem 5. Let f and g be defined in $[0, 1]$ and $[-1, 0]$, respectively, from z_0 and z_1 by (12a) and (12b). We extend f to $\mathbb{R}_+$ by setting $X = f(x)$ to be the solution of $X = S(A)$, where S is defined as in Lemma 7, $A = -g(x-2) = -\frac{1}{2}\left(z_1(2-x) - z_0'(2-x)\right)$ for $x \in [1, 2]$, and $A = f(x-2)$ for $x \geq 2$. These two equations define inductively a unique measurable function f on $\mathbb{R}_+$ and then a unique measurable function g on $[-1, \infty)$ with (12c). Letting $z(\cdot, \cdot)$ be defined by (11), in order to conclude the proof that z is indeed a solution of (3), it remains to verify that it belongs to the appropriate functional space in accordance with its initial condition. Note also that, thanks to Item 3 of Lemma 7, this solution will be unique as soon as σ satisfies Item 3 from Definition 1.

For that purpose, first note that, with Item 2 of Lemma 7, one has

$$|f(x)| \leq \frac{1}{2}\Big(|z_1(2-x)| + \big|z_0'(2-x)\big|\Big), \text{ for a.e. } x \in [1, 2], \tag{24}$$

and

$$|f(x)| \leq |f(x-2)|, \text{ for a.e. } x \geq 2. \tag{25}$$

We now split the proof into two cases: the case where $p = \infty$ and the case where $p \neq \infty$.

First case: $p = \infty$. Assume that $(z_0, z_1) \in H_\infty(0, 1)$. Combining (12a) and (24), it is easy to see that, for a.e. $x \in [0, 2]$,

$$|f(x)| \leq \frac{1}{2}\Big(\big|z_0'(\xi_x)\big| + |z_1(\xi_x)|\Big), \tag{26}$$

where $\xi_x = x$ if $x \in [0, 1]$ and $\xi_x = 2 - x$ if $x \in [1, 2]$. Then, using (25), one trivially shows that (26) actually holds for a.e. $x \geq 0$ after replacing ξ_x by ξ_{x-2k}, where k is the integer part of $x/2$. Using (12c), we deduce a similar estimate for g, namely, for a.e. $x \geq -1$

$$|g(x)| \leq \frac{1}{2}\Big(\big|z_0'(\eta_x)\big| + |z_1(\eta_x)|\Big), \tag{27}$$

where $\eta_x = -x$ if $x \in [-1, 0]$ and $\eta_x = \xi_{x-2k}$ with k the integer part of $x/2$ for $x \geq 0$. It is immediate to deduce that both $|f(x)|$ for $x \geq 0$ and $|g(x)|$ for $x \geq -1$ are smaller than $\frac{1}{2}\|(z_0, z_1)\|_{H_\infty(0,1)}$, which implies (13) for $p = \infty$ and (14) by using (10). That concludes the proof of Theorem 5 in the case where $p = \infty$.

Second case: $p \in [1, \infty)$. We suppose now that the initial conditions (z_0, z_1) belong to the space $H_p(0, 1)$ for some $p \in [1, \infty)$. Since $x \mapsto |x|^p$ is convex, one deduces from the previous equations that

$$\begin{aligned} |f(x)|^p &\leq \frac{1}{2^p}\left(\big|z_0'(\xi_{x-2k})\big|^p + |z_1(\xi_{x-2k})|^p\right), && x \geq 0, \\ |g(x)|^p &\leq \frac{1}{2^p}\left(\big|z_0'(\eta_x)\big|^p + |z_1(\eta_x)|^p\right), && x \geq -1, \end{aligned} \tag{28}$$

where ξ_{x-2k} and η_x have been defined above. Using the above equations and (10), one obtains that, for every $t \geq 0$,

$$\|h\|_{L^p(0,1)} \leq \|f(t+\cdot)\|_{L^p(0,1)} + \|g(t-\cdot)\|_{L^p(0,1)} \leq \big\|z_0'\big\|_{L^p(0,1)} + \|z_1\|_{L^p(0,1)},$$

where $h \in \{z_x(t,\cdot), z_t(t,\cdot)\}$. Then (13) follows readily, which achieves the proof of the first part of Theorem 5.

Let us now prove Item 4. The proof of (15) that we present here is inspired by that of [7] for $p = 2$. Notice first that, thanks to Lemma 8, E_p is decreasing and thus there exist $L = \lim_{t\to\infty} E_p(t)$. In particular, $\lim_{n\to\infty} E_p(2(n+1)) - E_p(2n) = 0$ and thus, by taking $T_2 = 2(n+1)$ and $T_1 = 2n$ in (20), one concludes that

$$\lim_{n\to\infty} \int_{2n}^{2(n+1)} |z_t(t,1)|^p M_p(\delta_t) dt = 0.$$

This means that

$$\lim_{n\to\infty} \int_0^2 |z_t(t+2n,1)|^p M_p(\delta_{t+2n}) dt = 0.$$

Note that

$$\begin{aligned} &|z_t(t+2n,1)|^p M_p(\delta_{t+2n}) \\ &\quad = \begin{cases} |z_t(t+2n,1)|^{p-1} |\sigma(z_t(t+2n,1))| & \text{if } |\sigma(z_t(t+2n,1))| \le |z_t(t+2n,1)|, \\ |z_t(t+2n,1)| |\sigma(z_t(t+2n,1))|^{p-1} & \text{otherwise} \end{cases} \end{aligned}$$

and thus

$$|z_t(t+2n,1)|^p M_p(\delta_{t+2n}) \ge \min\left(|\sigma(z_t(t+2n,1))|^p, |z_t(t+2n,1)|^p\right),$$

Hence

$$\lim_{n\to\infty} \int_0^2 \min\left(|\sigma(z_t(t+2n,1))|^p, |z_t(t+2n,1)|^p\right) dt = 0,$$

which can be rewritten as

$$\lim_{n\to\infty} \int_{-1}^1 \min\left(|\sigma(z_t(t+2n+1,1))|^p, |z_t(t+2n+1,1)|^p\right) dt = 0, \tag{29}$$

In order to simplify the notations, let us define, for $n \in \mathbb{N}^*$, the function $F_n : [-1,1] \to \mathbb{R}$ by $F_n(t) = f(t+2n)$. Note that, using (10), (12c), and (12d), one gets

$$\begin{aligned} z_t(t+2n+1,1) &= f(t+2n+2) + g(t+2n) = F_{n+1}(t) - F_n(t), \\ \sigma(z_t(t+2n+1,1)) &= -f(t+2n+2) + g(t+2n) = -\left(F_{n+1}(t) + F_n(t)\right), \end{aligned}$$

and in particular

$$F_{n+1}(t) + F_n(t) = -\sigma(F_{n+1}(t) - F_n(t)). \tag{30}$$

Hence (29) is equivalent to

$$\lim_{n\to\infty} \int_{-1}^1 \min\left(|F_{n+1}(t) + F_n(t)|^p, |F_{n+1}(t) - F_n(t)|^p\right) dt = 0.$$

For $n \in \mathbb{N}^*$ and $\rho > 0$, let

$$\begin{aligned} I_n &= \{t \in [-1,1] \mid |F_{n+1}(t) + F_n(t)| \le |F_{n+1}(t) - F_n(t)|\}, \\ J_n(\rho) &= \{t \in [-1,1] \mid \min\left(|F_{n+1}(t) + F_n(t)|, |F_{n+1}(t) - F_n(t)|\right) \ge \rho\}. \end{aligned}$$

Given a set $A \subset [-1,1]$, we denote $\overline{A} = [-1,1] \setminus A$. Denoting by $\mathcal{L}$ the Lebesgue measure in $\mathbb{R}$, we have $\mathcal{L}(J_n(\rho)) \to 0$ as $n \to \infty$ for every $\rho > 0$, since

$$\rho^p \mathcal{L}(J_n(\rho)) \leq \int_{J_n(\rho)} \min\left(|F_{n+1}(t) + F_n(t)|^p, |F_{n+1}(t) - F_n(t)|^p\right) dt \xrightarrow[n\to\infty]{} 0.$$

Moreover, from (28), there exists $H \in L^1(-1,1)$ depending only on (z_0, z_1) such that

$$|F_{n+1}(t) \pm F_n(t)|^p \leq H(t) \qquad \text{for every } n \in \mathbb{N}^* \text{ and a.e. } t \in [-1,1]. \tag{31}$$

We claim that

$$\lim_{n\to\infty} \int_{-1}^{1} |F_{n+1}(t) + F_n(t)|^p dt = 0. \tag{32}$$

Indeed, let $\varepsilon > 0$. Since σ is continuous, there exists $\eta > 0$ such that, if $|\xi| < \eta$, then $|\sigma(\xi)| < \varepsilon$. We write

$$\begin{aligned}\int_{-1}^{1} |F_{n+1}(t) + F_n(t)|^p dt = &\int_{I_n} |F_{n+1}(t) + F_n(t)|^p dt \\ &+ \int_{\overline{I}_n \cap J_n(\eta)} |F_{n+1}(t) + F_n(t)|^p dt + \int_{\overline{I}_n \cap \overline{J}_n(\eta)} |F_{n+1}(t) + F_n(t)|^p dt \end{aligned} \tag{33}$$

We have

$$\int_{I_n} |F_{n+1}(t) + F_n(t)|^p dt = \int_{I_n} \min\left(|F_{n+1}(t) + F_n(t)|^p, |F_{n+1}(t) - F_n(t)|^p\right) dt \xrightarrow[n\to\infty]{} 0.$$

Since $\mathcal{L}(J_n(\eta)) \to 0$ as $n \to \infty$, it follows from (31) that

$$\int_{\overline{I}_n \cap J_n(\eta)} |F_{n+1}(t) + F_n(t)|^p dt \leq \int_{\overline{I}_n \cap J_n(\eta)} H(t) dt \xrightarrow[n\to\infty]{} 0.$$

Finally, if $t \in \overline{I}_n \cap \overline{J}_n(\eta)$, then $|F_{n+1}(t) - F_n(t)| < \eta$ and thus, by (30), one has $|F_{n+1}(t) + F_n(t)| < \varepsilon$, implying that

$$\int_{\overline{I}_n \cap \overline{J}_n(\eta)} |F_{n+1}(t) + F_n(t)|^p dt < 2\varepsilon^p.$$

It now follows from (33) that

$$\limsup_{n\to\infty} \int_{-1}^{1} |F_{n+1}(t) + F_n(t)|^p dt \leq 2\varepsilon^p$$

and, since ε is arbitrary, one obtains (32).

We now claim that

$$\lim_{n\to\infty} \int_{-1}^{1} |F_{n+1}(t) - F_n(t)|^p dt = 0. \tag{34}$$

For $k > 1$, let us partition the interval $[-1,1]$ as

$$\begin{aligned} L_{1,n}(k) &= \{t \in [-1,1] \mid |F_{n+1}(t) - F_n(t)|^p \leq 1/k\}, \\ L_{2,n}(k) &= \{t \in [-1,1] \mid 1/k < |F_{n+1}(t) - F_n(t)|^p < k\}, \\ L_{3,n}(k) &= \{t \in [-1,1] \mid |F_{n+1}(t) - F_n(t)|^p \geq k\}. \end{aligned}$$

Then

$$\int_{L_{1,n}(k)} |F_{n+1}(t) - F_n(t)|^p dt \leq \frac{2}{k}.$$

Letting

$$\widetilde{L}_3(k) = \{t \in [-1,1] \mid H(t) \geq k\},$$

it follows immediately from (31) that $L_{3,n}(k) \subset \widetilde{L}_3(k)$ for every $n \in \mathbb{N}^*$ and $k > 1$. Moreover, since $H \in L^1(-1,1)$, one has $\mathcal{L}(\widetilde{L}_3(k)) \to 0$ as $k \to \infty$, and we estimate

$$\int_{L_{3,n}(k)} |F_{n+1}(t) - F_n(t)|^p dt \leq \int_{\widetilde{L}_3(k)} H(t) dt.$$

In $L_{2,n}(k)$, we decompose

$$\begin{aligned}\int_{L_{2,n}(k)} &|F_{n+1}(t) - F_n(t)|^p dt \\ &= \int_{L_{2,n}(k)\cap \overline{I}_n} |F_{n+1}(t) - F_n(t)|^p dt + \int_{L_{2,n}(k)\cap I_n} |F_{n+1}(t) - F_n(t)|^p dt\end{aligned}$$

We have

$$\begin{aligned}\int_{L_{2,n}(k)\cap \overline{I}_n} &|F_{n+1}(t) - F_n(t)|^p dt \\ &= \int_{L_{2,n}(k)\cap \overline{I}_n} \min(|F_{n+1}(t) + F_n(t)|^p, |F_{n+1}(t) - F_n(t)|^p) dt \xrightarrow[n\to\infty]{} 0\end{aligned}$$

Let

$$C_k = \min_{1/k \leq |\xi| \leq k} \left| \frac{\sigma(\xi)}{\xi} \right| > 0$$

and notice that, for every $t \in L_{2,n}(k) \cap \overline{I}_n$, one has

$$\frac{1}{C_k^p} \frac{|F_{n+1}(t) + F_n(t)|^p}{|F_{n+1}(t) - F_n(t)|^p} \geq 1,$$

which shows that

$$\begin{aligned}\int_{L_{2,n}(k)\cap I_n} |F_{n+1}(t) - F_n(t)|^p dt &\leq \frac{1}{C_k^p} \int_{L_{2,n}(k)\cap I_n} |F_{n+1}(t) + F_n(t)|^p dt \\ &= \frac{1}{C_k^p} \int_{L_{2,n}(k)\cap I_n} \min(|F_{n+1}(t) + F_n(t)|^p, |F_{n+1}(t) - F_n(t)|^p) dt \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

Hence, one deduces that

$$\limsup_{n\to\infty} \int_{-1}^{1} |F_{n+1}(t) - F_n(t)|^p dt \leq \frac{2}{k} + \int_{\widetilde{L}_3(k)} H(t) dt.$$

Since k is arbitrary and the right-hand side of the above formula tends to 0 as $k \to \infty$, one obtains (34).

One now deduces from (32) and (34) that $F_n \to 0$ in $L^p(-1,1)$ as $n \to \infty$. From the definition of F_n, (12c), and (10), one deduces that $z_t(2n,\cdot) \to 0$ and $z_x(2n,\cdot) \to 0$ in $L^p(0,1)$ as $n \to \infty$, which shows that $\lim_{n\to\infty} E_p(2n) = 0$. Since E_p is nonincreasing, we conclude that $\lim_{t\to\infty} E_p(t) = 0$, yielding (15). □

4 Proof of Theorem 6

The proof of the theorem relies on properties satisfied by the *reflection coefficient* R, defined as follows.

Definition 10. Let σ be a nonlinear damping function satisfying 1 and 2 from Definition 1 and z be a solution of (3) in $H_p(0,1)$. The *reflection coefficient* $R : \mathbb{R}_+ \to \mathbb{R}$ corresponding to z is the measurable function given by

$$R(t) = \begin{cases} \dfrac{z_t(t,1) - \sigma(z_t(t,1))}{z_t(t,1) + \sigma(z_t(t,1))}, & \text{if } z_t(t,1) \neq 0, \\ \dfrac{2 - \sigma_- - \sigma_+}{2 + \sigma_- + \sigma_+}, & \text{otherwise,} \end{cases} \tag{35}$$

where σ_- and σ_+ are the numbers defined in (4).

A first immediate property of the reflection coefficient is the following.

Lemma 11. *The reflection coefficient R takes values in $(-1, 1)$.*

Proof. The result is immediate once one writes the reflection coefficient as

$$R(t) = \frac{1 - \delta_t}{1 + \delta_t}, \tag{36}$$

where $\delta_t = \frac{\sigma(z_t(t,1))}{z_t(t,1)}$ for $z_t(t,1) \neq 0$ and $\delta_t = \frac{\sigma_- + \sigma_+}{2}$ if $z_t(t,1) = 0$. □

Remark 12. The choice of the value of $R(t)$ when $z_t(t,1) = 0$ is arbitrary and has no effect in the arguments of the proof of Theorem 6. The choice in (35) corresponds to the choice $\delta_t = \frac{\sigma_- + \sigma_+}{2}$ in (36) and is motivated by (4).

A major property satisfied by the reflection coefficient is the following.

Lemma 13. *Let z be a solution of (3), f and g be as in Definition 3, and R be as in Definition 10. Then*

$$f(t+1) = \begin{cases} R(t)g(t-1) & \text{for a.e. } t \in [0,1], \\ -R(t)f(t-1) & \text{for a.e. } t \geq 1. \end{cases} \tag{37}$$

Proof. Notice first that, if $t \geq 0$ is such that $z_t(t,1) = 0$, then, by the boundary condition in (3), one also has $z_x(t,1) = 0$, and thus, by (10), one also has $f(t+1) = g(t-1) = 0$, with also $f(t-1) = 0$ in the case $t \geq 1$ by (12c), and thus (37) is satisfied.

In the case $z_t(t,1) \neq 0$, from (35) and (10), one deduces that $R(t) = \frac{f(t+1)}{g(t-1)}$, and thus (37) follows using (12c). □

We also need to have to suitable bounds on this reflection coefficient in the case where the initial conditions are in $H_\infty(0,1)$.

Lemma 14. *Assume that, given a positive constant M, the initial conditions (z_0, z_1) are in $H_\infty(0,1)$ and satisfy*

$$\|(z_0, z_1)\|_{H_\infty(0,1)} \leq M. \tag{38}$$

We assume furthermore that the nonlinearity σ satisfies properties 1 and 2 from Definition 1. Then, there exists a positive constant $r \in (0,1)$ depending on M, such that the reflection coefficient defined in (35) satisfies, for all $t \geq 0$,

$$|R(t)| \leq r. \tag{39}$$

Proof of Lemma 14. Note that, by (14), one has, for a.e. $t \in \mathbb{R}$,

$$|z_t(t,1)| \leq M. \tag{40}$$

As a consequence of properties 1 and 2 from Definition 1, the function $\xi \mapsto \frac{\sigma(\xi)}{\xi}$ defined over $\mathbb{R}^*$ is continuous, strictly positive, and its limit points as $\xi \to 0$ belong to $[\sigma_-, \sigma_+]$. Letting

$$\delta_- = \inf_{\xi \in [-M,M]\setminus\{0\}} \frac{\sigma(\xi)}{\xi}, \quad \delta_+ = \sup_{\xi \in [-M,M]\setminus\{0\}} \frac{\sigma(\xi)}{\xi}, \tag{41}$$

one has thus $0 < \delta_- \leq \delta_+ < +\infty$. Then, using (36), one deduces that, for all $t \geq 0$,

$$\frac{1-\delta_+}{1+\delta_+} =: R_- \leq R(t) \leq R_+ := \frac{1-\delta_-}{1+\delta_-}. \tag{42}$$

Clearly, R_- and R_+ belong to $(-1,1)$, and then it follows that there exists $r \in (0,1)$ such that $|R(t)| \leq r$ for $t \geq 0$. □

As said above, this lemma is instrumental to give estimates of the decay rates of (3). To be more precise, these decay rates will depend on the bounds of the reflection coefficient R defined in (35). From the proof of Lemma 14, it is clear that these bounds are difficult to obtain without assuming the initial conditions (z_0, z_1) to be in $H_\infty(0,1)$.

We are now in position to provide the proof of Theorem 6.

Proof of Theorem 6. All along this proof, given a positive constant M, we assume that the initial conditions (z_0, z_1) are in $H_\infty(0,1)$ and satisfy

$$\|(z_0, z_1)\|_{H_\infty(0,1)} \leq M. \tag{43}$$

Our proof is divided into two steps: the first one deals with the case where $p \in [1, \infty)$ and the second one with the case where $p = \infty$. The strategies for these proofs are different: we tackle the first case by introducing a candidate Lyapunov functional inspired by [1, 3], and we treat the second case with the characteristics method.

First case: $p \in [1, \infty)$. Let us consider the following Lyapunov functional along any trajectory of (3),

$$V_p(t) = \int_0^1 e^{\mu x} F(z_x(t,x) + z_t(t,x))dx + \int_0^1 e^{-\mu x} F(z_x(t,x) - z_t(t,x))dx, \quad t \geq 0, \tag{44}$$

where μ is a positive constant and $F(s) = \left|\frac{s}{2}\right|^p$ for $s \in \mathbb{R}$. We first provide another expression for V_p before taking its time derivative along trajectories of (3). From (44), (10), and (12c), one gets, for $t \geq 1$, that

$$V_p(t) = e^{-\mu t} \int_{t-1}^{t+1} e^{\mu s} |f(s)|^p \, ds. \tag{45}$$

The time derivative of V_p along the trajectories of (3) satisfies

$$\begin{aligned} \frac{dV_p}{dt} &= -\mu V_p + e^{\mu}|f(t+1)|^p - e^{-\mu}|g(t-1)|^p \\ &= -\mu V_p + e^{\mu} F(z_t(t,1) - \sigma(z_t(t,1))) - e^{-\mu} F(z_t(t,1) + \sigma(z_t(t,1))) \\ &= -\mu V_p + e^{-\mu} F(z_t(t,1) + \sigma(z_t(t,1))) \left[|R(t)|^p e^{2\mu} - 1\right], \end{aligned} \tag{46}$$

where we have used (12c) and (35). Invoking Lemma 14, since the initial conditions $(z_0, z_1) \in H_\infty(0,1)$ are bounded as in (43), we can conclude that, for any $t \geq 0$, $|R(t)|^p \leq r^p$, Then, setting $\mu := -\frac{p}{2}\ln(r)$, one obtains that

$$\frac{dV_p}{dt} \leq -\mu V_p, \quad t \geq 1 \tag{47}$$

which implies the exponential convergence of the trajectories of (3) with a decay rate depending on the H_∞-bounds of the initial conditions (z_0, z_1).

Second case: $p = \infty$. The proof of this result is different from the other case, but relies on the properties of the reflection coefficient given by Lemmas 13 and 14. Notice first that, by (12a) and (12b), one has

$$\begin{aligned} \|f\|_{L^\infty(0,1)} &\leq \frac{1}{2}\|(z_0, z_1)\|_{H_\infty(0,1)}, \\ \|g\|_{L^\infty(-1,0)} &\leq \frac{1}{2}\|(z_0, z_1)\|_{H_\infty(0,1)}. \end{aligned} \tag{48}$$

From the above, (37), and (39), one also deduces that

$$\|f\|_{L^\infty(1,3)} \leq \frac{r}{2}\|(z_0, z_1)\|_{H_\infty(0,1)}.$$

Moreover, (37) and (39) also imply that, for every $n \in \mathbb{N}^*$,

$$\|f\|_{L^\infty(2n+1,2n+3)} \leq r\|f\|_{L^\infty(2n-1,2n+1)},$$

and thus an immediate induction shows that, for every $n \in \mathbb{N}^*$,

$$\|f\|_{L^\infty(2n-1,2n+1)} \leq \frac{r^n}{2}\|(z_0, z_1)\|_{H_\infty(0,1)}.$$

One deduces that, for $t \geq 2$,

$$\begin{aligned} \|f\|_{L^\infty(t,t+1)} &\leq \frac{1}{2}r^{\lfloor t/2 \rfloor}\|(z_0, z_1)\|_{H_\infty(0,1)}, \\ \|g\|_{L^\infty(t-1,t)} &\leq \frac{1}{2}r^{\lfloor t/2 \rfloor}\|(z_0, z_1)\|_{H_\infty(0,1)}, \end{aligned}$$

and these inequalities also hold for $t \in [0, 2)$ using (48), (37), and the fact that $|R(t)| \leq 1$ for a.e. $t \geq 0$. Hence, by (10), one deduces that, for every $t \geq 0$,

$$\|(z, z_t)\|_{H_\infty(0,1)} \leq 2r^{\lfloor t/2 \rfloor}\|(z_0, z_1)\|_{H_\infty(0,1)},$$

and hence (16) holds with $C = \frac{2}{r}$ and $\mu = -\frac{1}{2}\ln r$. □

References

[1] G. Bastin and J.-M. Coron. *Stability and boundary stabilization of 1-d hyperbolic systems*, volume 88. Springer, 2016.

[2] Y. Chitour, S. Marx, and C. Prieur. L^p-asymptotic stability analysis of a 1d wave equation with a nonlinear damping. *arXiv preprint arXiv:1907.11712*, 2019.

[3] A. Haraux. L^p estimates of solutions to some non-linear wave equations in one space dimension. *International Journal of Mathematical Modelling and Numerical Optimisation*, 1(1-2):146–152, 2009.

[4] W. Liu, Y. Chitour, and E. Sontag. On finite-gain stabilizability of linear systems subject to input saturation. *SIAM Journal on Control and Optimization*, 34(4):1190–1219, 1996.

[5] S. Marx, Y. Chitour, and C. Prieur. Stability analysis of dissipative systems subject to nonlinear damping via Lyapunov techniques. *To appear in IEEE Transactions on Automatic Control*, 2020.

[6] J.C. Peral. L^p estimates for the wave equation. *Journal of functional analysis*, 36(1):114–145, 1980.

[7] M. Pierre and J. Vancostenoble. Strong decay for one-dimensional wave equations with nonmonotone boundary damping. *Control and Cybernetics*, 29(2):473–484, 2000.

[8] C. Prieur, S. Tarbouriech, and J. M. Gomes da Silva Jr. Wave equation with cone-bounded control laws. *IEEE Trans. on Automat. Control*, 61(11):3452–3463, 2016.