Regulation of a reaction-diffusion equation with bounded observation

Hugo Lhachemi* Christophe Prieur†

Abstract

This paper solves a regulation problem for a reaction-diffusion equation. More precisely, for a given bounded observation, we design a boundary controller so that the setpoint output of the equation converges to a prescribed reference signal. The control law is finite-dimensional and is obtained by coupling a pole-placement control law with an observer. The proofs are based on a Lyapunov function and relies on the properties of Sturm-Liouville operators.

1 Introduction

Regulation control of finite-dimensional systems are very classical problems that have been widely investigated [1, 6]. This work considers the problem of regulation of distributed parameter systems [2, 3, 14, 16, 17]. This class of systems succeeds to model many dynamical systems, such as heat dynamics, chemical reactors, fluid mechanical systems, among many other potential applications (see [4] for a general reference). We focus here on a reaction-diffusion system described by parabolic partial differential equations. For this kind of systems this is very natural to not only control the internal state, but also to prescribe the output to a given reference trajectory. This is the so-called regulation problem. Depending on the control input and the to-be-regulated output, the regulation problem could be more or less complex to solve. In particular, when one of these operators is unbounded, the associated regulation problem becomes challenging and requires dedicated control techniques.

The control objective of this paper is to solve the regulation problem in the case of an unbounded control operator and a bounded to-be-regulated output. Our approach combines a finite-dimensional observer [7, 8, 10, 13] and an adequate integral component [14]. We show that such a controller can always be designed such that the resulting closed-loop system is exponentially stable, in both L^2 and H^1 norms, and achieve setpoint regulation control. These stability properties are proven using suitable Lyapunov function candidates.

This approach could likely be generalized to other control problems, as the ones considered in [5] where an infinite-dimensional dynamics is decomposed into two parts: one unstable operator having a finite-dimensional representation, and one stable operator. See also [9, 11] for control design methods exploiting this idea.

This paper is organized as follows. Section 2 recalls some results on Sturm-Liouville operators and states a technical result. Section 3 introduces the control problem under consideration and the proposed controller architecture. Section 4 gives the main results and associated stability analysis in L^2 and H^1 norms. Finally Section 5 collects some concluding remarks and presents further developments of this work.

2 Preliminaries

2.1 Sturm-Liouville operator Let the Sturm-Liouville operator defined by $\mathcal{A}f = -(pf')' + qf$ on the domain $D(\mathcal{A}) = \{f \in H^2(0,1) : f'(0) = f(1) = 0\}$ with $p \in \mathcal{C}^1([0,1])$, $p > 0$, and $q \in \mathcal{C}^0([0,1])$, $q \geqslant 0$. The eigenvalues λ_n, $n \geqslant 1$, of $\mathcal{A}$ are simple and can be sorted such that $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$. Moreover, the associated unit eigenvectors $\phi_n \in L^2(0,1)$ form a Hilbert basis. It can be easily checked that, for all $f \in D(\mathcal{A})$,

$$\sum_{n\geqslant 1} \lambda_n \langle f, \phi_n\rangle^2 = \langle \mathcal{A}f, f\rangle = \int_0^1 p(f')^2 + qf^2 \,\mathrm{d}x. \tag{2.1}$$

Let $p_*, p^*, q^* \in \mathbb{R}$ be such that $0 < p_* \leqslant p(x) \leqslant p^*$ and $0 \leqslant q(x) \leqslant q^*$ for all $x \in [0,1]$, then it holds [15]:

$$0 \leqslant \pi^2(n-1)^2 p_* \leqslant \lambda_n \leqslant \pi^2 n^2 p^* + q^* \tag{2.2}$$

for all $n \geqslant 1$.

2.2 Useful Lemma We state a lemma which was proven in a particular case in [7]. This proof also applies to the general setting described in the below lemma.

LEMMA 2.1. *Let $n, m, N \geqslant 1$, $M_{11} \in \mathbb{R}^{n\times n}$, $M_{22} \in \mathbb{R}^{m\times m}$, $M_{12} \in \mathbb{R}^{n\times m}$, $M_{14}^N \in \mathbb{R}^{n\times N}$, $M_{24}^N \in \mathbb{R}^{m\times N}$, $M_{31}^N \in \mathbb{R}^{N\times n}$, and $M_{33}^N, M_{44}^N \in \mathbb{R}^{N\times N}$. We define*

$$F^N = \begin{bmatrix} M_{11} & M_{12} & 0 & M_{14}^N \\ 0 & M_{22} & 0 & M_{24}^N \\ M_{31}^N & 0 & M_{33}^N & 0 \\ 0 & 0 & 0 & M_{44}^N \end{bmatrix}.$$

*Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France (e-mail: hugo.lhachemi@centralesupelec.fr)

†Université Grenoble Alpes, CNRS, Grenoble-INP, GIPSA-lab, F-38000, Grenoble, France (e-mail: christophe.prieur@gipsa-lab.fr). This work has been partially supported by MIAI@Grenoble Alpes (ANR- 19-P3IA-0003)

We assume that there exist constants $C_0, \kappa_0 > 0$ such that $\|e^{M_{11}t}\| \leqslant C_0 e^{-\kappa_0 t}$, $\|e^{M_{22}t}\| \leqslant C_0 e^{-\kappa_0 t}$, $\|e^{M_{33}^N t}\| \leqslant C_0 e^{-\kappa_0 t}$, and $\|e^{M_{44}^N t}\| \leqslant C_0 e^{-\kappa_0 t}$ for all $t \geqslant 0$ and all $N \geqslant 1$. Moreover, we assume that there exists a constant $C_1 > 0$ such that $\|M_{14}^N\| \leqslant C_1$, $\|M_{24}^N\| \leqslant C_1$, and $\|M_{31}^N\| \leqslant C_1$ for all $N \geqslant 1$. Then there exists a constant $C_2 > 0$ such that, for any $N \geqslant 1$, there exists $P^N \in \mathbb{R}^{n+m+2N}$ with $P^N \succ 0$ such that $P^N F^N + (F^N)^\top P^N = -I$ and $\|P^N\| \leqslant C_2$.

3 Control problem and deviation dynamics

3.1 Studied reaction-diffusion system We consider the reaction-diffusion system described by

$$z_t(t,x) = (pz_x)_x(t,x) + (q_c - q(x))z(t,x) \tag{3.3a}$$
$$z_x(t,0) = 0, \quad z(t,1) = u(t) \tag{3.3b}$$
$$z(0,x) = z_0(x) \tag{3.3c}$$
$$y(t) = \int_0^1 c(x)z(t,x)\,\mathrm{d}x \tag{3.3d}$$

where $t \geqslant 0$ and $x \in (0,1)$ stand respectively for the time and space variables, $q_c \in \mathbb{R}$ is a constant, $u(t) \in \mathbb{R}$ is the command input, $y(t) \in \mathbb{R}$ with $c \in L^2(0,1)$ is the measurement, $z(t,\cdot) \in L^2(0,1)$ is the state, and $z_0 \in L^2(0,1)$ is the initial condition. Our control objective is to design a state feedback and a finite dimensional observer to stabilize the plant and achieve the setpoint regulation of $y(t)$ to some prescribed reference signal $r(t) \in \mathbb{R}$.

3.2 Spectral reduction We introduce the change of variable

$$w(t,x) = z(t,x) - x^2 u(t). \tag{3.4}$$

Then we have

$$w_t(t,x) = (pw_x)_x(t,x) + (q_c - q(x))w(t,x)$$
$$+ a(x)u(t) + b(x)\dot{u}(t) \tag{3.5a}$$
$$w_x(t,0) = 0, \quad w(t,1) = 0 \tag{3.5b}$$
$$w(0,x) = w_0(x) \tag{3.5c}$$
$$\tilde{y}(t) = \int_0^1 c(x)w(t,x)\,\mathrm{d}x \tag{3.5d}$$

with $a, b \in L^2(0,1)$ defined by $a(x) = 2p(x) + 2xp'(x) + (q_c - q(x))x^2$ and $b(x) = -x^2$, respectively, $\tilde{y}(t) = y(t) - \gamma_0 u(t)$ with $\gamma_0 = \int_0^1 x^2 c(x)\,\mathrm{d}x$, and $w_0(x) = z_0(x) - x^2 u(0)$. Introducing the auxiliary command input $v(t) = \dot{u}(t)$, we infer that

$$\dot{u}(t) = v(t) \tag{3.6a}$$
$$\frac{\mathrm{d}w}{\mathrm{d}t}(t,\cdot) = \{-\mathcal{A} + q_c\}w(t,\cdot) + au(t) + bv(t) \tag{3.6b}$$

Introducing the coefficients of projection $w_n(t) = \langle w(t,\cdot), \phi_n\rangle$, $a_n = \langle a, \phi_n\rangle$, $b_n = \langle b, \phi_n\rangle$, and $c_n = \langle c, \phi_n\rangle$, we obtain that

$$\dot{u} = v \tag{3.7a}$$
$$\dot{w}_n = (-\lambda_n + q_c)w_n + a_n u + b_n v \tag{3.7b}$$
$$\tilde{y} = \sum_{i \geqslant 1} c_i w_i \tag{3.7c}$$

for $n \geqslant 1$.

3.3 Control design Let $N_0 \geqslant 1$ and $\delta > 0$ be given such that $-\lambda_n + q_c < -\delta < 0$ for all $n \geqslant N_0 + 1$. Let $N \geqslant N_0 + 1$ be arbitrary. We design an observer to estimate the N first modes of the plant while the state-feedback is performed on the N_0 first modes. More precisely, introducing $A_0 = \mathrm{diag}(-\lambda_1 + q_c, \ldots, -\lambda_{N_0} + q_c)$, $W^{N_0} = \begin{bmatrix} w_1, & \ldots, & w_{N_0} \end{bmatrix}^\top$, $B_{0,a} = \begin{bmatrix} a_1, & \ldots, & a_{N_0} \end{bmatrix}^\top$, and $B_{0,b} = \begin{bmatrix} b_1, & \ldots, & b_{N_0} \end{bmatrix}^\top$, we have

$$\dot{W}^{N_0} = A_0 W^{N_0} + B_{0,a} u + B_{0,b} v. \tag{3.8}$$

Let us introduce an integral component to achieve the setpoint regulation control of the system output. To do so, consider first the case of the following classical integral component:

$$\begin{aligned} \dot{z}_i &= y - r = \tilde{y} + \gamma_0 u - r \\ &= \sum_{n \geqslant 1} c_n w_n + \gamma_0 u - r \end{aligned}$$

where $r : \mathbb{R}_+ \to \mathbb{R}$ is a reference signal. Introducing $\xi_p = z_i - \sum_{n \geqslant N_0+1} \frac{c_n}{-\lambda_n + q_c} w_n$, we obtain that

$$\begin{aligned} \dot{\xi}_p &= \sum_{n \geqslant 1} c_n w_n + \gamma_0 u - r \\ &\quad - \sum_{n \geqslant N_0+1} \frac{c_n}{-\lambda_n + q_c} \dot{w}_n \\ &= \sum_{n=1}^{N_0} c_n w_n + \alpha_0 u + \beta_0 v - r \end{aligned}$$

with

$$\alpha_0 = \gamma_0 - \sum_{n \geqslant N_0+1} \frac{a_n c_n}{-\lambda_n + q_c}, \tag{3.9a}$$
$$\beta_0 = - \sum_{n \geqslant N_0+1} \frac{b_n c_n}{-\lambda_n + q_c}. \tag{3.9b}$$

Replacing w_n, which are not measured, by their estimated version $\hat{w}_n$, which will be described below, we obtain the following integral component that will be used

for control design:

$$\dot{\xi} = \sum_{n=1}^{N_0} c_n \hat{w}_n + \alpha_0 u + \beta_0 v - r. \tag{3.10}$$

We now define for $1 \leqslant n \leqslant N$ the observation dynamics:

(3.11)

$$\dot{\hat{w}}_n = (-\lambda_n + q_c)\hat{w}_n + a_n u + b_n v - l_n \left(\int_0^1 c(x) \sum_{i=1}^{N} \hat{w}_i \phi_i(x)\, \mathrm{d}x - \alpha_1 u - \tilde{y} \right)$$

with

$$\alpha_1 = \sum_{n \geqslant N+1} \frac{a_n c_n}{-\lambda_n + q_c} \tag{3.12}$$

and where $l_n \in \mathbb{R}$ are the observer gains. We set $l_n = 0$ for $N_0 + 1 \leqslant n \leqslant N$ and the initial condition of the observer as $\hat{w}_n(0) = 0$ for all $1 \leqslant n \leqslant N$. We define for $1 \leqslant n \leqslant N$ the observation error as $e_n = w_n - \hat{w}_n$. Noting that $\int_0^1 c(x) \sum_{i=1}^N \hat{w}_i \phi_i(x)\, \mathrm{d}x - \tilde{y} = -\sum_{i=1}^N c_i e_i - \zeta$ with $\zeta = \sum_{n \geqslant N+1} c_n w_n$, we infer that

$$\dot{\hat{w}}_n = (-\lambda_n + q_c)\hat{w}_n + a_n u + b_n v + l_n \sum_{i=1}^{N} c_i e_i + l_n \alpha_1 u + l_n \zeta. \tag{3.13}$$

Introducing

$$\hat{W}^{N_0} = \begin{bmatrix} \hat{w}_1 \\ \vdots \\ \hat{w}_{N_0} \end{bmatrix}, \quad E^{N_0} = \begin{bmatrix} e_1 \\ \vdots \\ e_{N_0} \end{bmatrix},$$
$$E^{N-N_0} = \begin{bmatrix} e_{N_0+1} \\ \vdots \\ e_N \end{bmatrix}, \quad L = \begin{bmatrix} l_1 \\ \vdots \\ l_{N_0} \end{bmatrix},$$
$$C_0 = \begin{bmatrix} c_1, & \ldots, & c_{N_0} \end{bmatrix}, \quad C_1 = \begin{bmatrix} c_{N_0+1}, & \ldots, & c_N \end{bmatrix},$$

we have

$$\dot{\hat{W}}^{N_0} = A_0 \hat{W}^{N_0} + B_{0,a} u + B_{0,b} v + LC_0 E^{N_0} + LC_1 E^{N-N_0} + \alpha_1 L u + L\zeta. \tag{3.14}$$

With

$$\hat{W}_a^{N_0} = \begin{bmatrix} u \\ \hat{W}^{N_0} \\ \xi \end{bmatrix}, \quad \tilde{L} = \begin{bmatrix} 0 \\ L \\ 0 \end{bmatrix}$$

and defining

(3.15)

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ B_{0,a} & A_0 & 0 \\ \alpha_0 & C_0 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 \\ B_{0,b} \\ \beta_0 \end{bmatrix}, \; B_r = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

we deduce that

$$\dot{\hat{W}}_a^{N_0} = A_1 \hat{W}_a^{N_0} + B_1 v - B_r r + \tilde{L} C_0 E^{N_0} + \tilde{L} C_1 E^{N-N_0} + \alpha_1 \tilde{L} u + \tilde{L}\zeta. \tag{3.16}$$

Setting the auxiliary command input as

$$v = K \hat{W}_a^{N_0}, \tag{3.17}$$

and defining

$$A_{\mathrm{cl}}(\alpha_1) = A_1 + B_1 K + \alpha_1 \tilde{L} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \tag{3.18}$$

we obtain that

$$\dot{\hat{W}}_a^{N_0} = A_{\mathrm{cl}}(\alpha_1) \hat{W}_a^{N_0} - B_r r + \tilde{L} C_0 E^{N_0} + \tilde{L} C_1 E^{N-N_0} + \tilde{L}\zeta \tag{3.19}$$

and

$$\dot{E}^{N_0} = (A_0 - LC_0) E^{N_0} - LC_1 E^{N-N_0} - \alpha_1 L \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \hat{W}_a^{N_0} - L\zeta. \tag{3.20}$$

Lemma 3.1. *The pair (A_1, B_1) is controllable if and only if the unique solution $z \in H^2(0,1)$ of*

$$(pz')' + (q_c - q)z = 0, \tag{3.21a}$$
$$z(0) = 1, \quad z'(0) = 0 \tag{3.21b}$$

is such that $\int_0^1 c(x) z(x)\, \mathrm{d}x \neq 0$. The pair (A_0, C_0) is observable if and only if $c_n \neq 0$ for all $1 \leqslant n \leqslant N_0$.

The proof of the lemma is reported in appendix. Hence we can compute gains K and L such that $A_1 + B_1 K$ and $A_0 - LC_0$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. Defining now

$$A_2 = \mathrm{diag}(-\lambda_{N_0+1} + q_c, \ldots, -\lambda_N + q_c)$$

$$\hat{W}^{N-N_0} = \begin{bmatrix} \hat{w}_{N_0+1} \\ \vdots \\ \hat{w}_N \end{bmatrix}, \; B_{2,a} = \begin{bmatrix} a_{N_0+1} \\ \vdots \\ a_N \end{bmatrix}, \; B_{2,b} = \begin{bmatrix} b_{N_0+1} \\ \vdots \\ b_N \end{bmatrix}$$

we obtain from (3.11) with $l_n = 0$ for $N_0 + 1 \leqslant n \leqslant N$ that

$$\dot{\hat{W}}^{N-N_0} = A_2 \hat{W}^{N-N_0} + B_{2,a} u + B_{2,b} v = A_2 \hat{W}^{N-N_0} + \left(B_{2,b} K + \begin{bmatrix} B_{2,a} & 0 & 0 \end{bmatrix} \right) \hat{W}_a^{N_0} \tag{3.22}$$

and

$$\dot{E}^{N-N_0} = A_2 E^{N-N_0}. \tag{3.23}$$

Putting together (3.19-3.20) and (3.22-3.23), we obtain with $X = \begin{bmatrix} (\hat{W}_a^{N_0})^\top & (E^{N_0})^\top & (\hat{W}^{N-N_0})^\top & (E^{N-N_0})^\top \end{bmatrix}^\top$ that

$$\dot{X} = FX + \mathcal{L}\zeta - \mathcal{L}_r r \tag{3.24}$$

where

$$F = \begin{bmatrix} A_{\rm cl}(\alpha_1) & \tilde{L}C_0 & 0 & \tilde{L}C_1 \\ -\alpha_1 L \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} & A_0 - LC_0 & 0 & -LC_1 \\ B_{2,b}K + \begin{bmatrix} B_{2,a} & 0 & 0 \end{bmatrix} & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix}$$

$$\mathcal{L} = \begin{bmatrix} \tilde{L} \\ -L \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{L}_r = \begin{bmatrix} B_r \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

3.4 Equilibrium condition and dynamics of deviations We aim at characterizing the equilibrium condition of the closed-loop system composed of the reaction-diffusion system (3.3), the auxiliary command input dynamics (3.6a), the integral action (3.10), the observer dynamics (3.11), and the state-feedback (3.17). To do so let $r(t) = r_e \in \mathbb{R}$ be arbitrary. We must solve the system of equations:

(3.25a)

$$0 = (-\lambda_n + q_c)w_{n,e} + a_n u_e + b_n v_e = 0, \quad n \geqslant 1,$$

(3.25b)

$$0 = v_e = K\hat{W}_{a,e}^{N_0},$$

(3.25c)

$$0 = \sum_{n=1}^{N_0} c_n \hat{w}_{n,e} + \alpha_0 u_e + \beta_0 v_e - r_e,$$

(3.25d)

$$0 = (-\lambda_n + q_c)\hat{w}_{n,e} + a_n u_e + b_n v_e - l_n \left\{ \sum_{i=1}^{N} c_i \hat{w}_{i,e} - \alpha_1 u_e - \tilde{y}_e \right\}, \quad 1 \leqslant n \leqslant N_0,$$

(3.25e)

$$0 = (-\lambda_n + q_c)\hat{w}_{n,e} + a_n u_e + b_n v_e, \quad N_0 + 1 \leqslant n \leqslant N,$$

(3.25f)

$$\tilde{y}_e = \sum_{n \geqslant 1} c_n w_{n,e}.$$

We first note from (3.25b) that $v_e = 0$. Then, from (3.25a) we have $w_{n,e} = -\frac{a_n}{-\lambda_n + q_c} u_e$ for all $n \geqslant N_0 + 1$. In particular, from (3.25e), we have $\hat{w}_{n,e} = w_{n,e} = -\frac{a_n}{-\lambda_n + q_c} u_e$ for all $N_0 + 1 \leqslant n \leqslant N$. Defining $e_{n,e} = w_{n,e} - \hat{w}_{n,e}$ and $\zeta_e = \sum_{n \geqslant N+1} c_n w_{n,e}$, we obtain that $e_{n,e} = 0$ for all $N_0 + 1 \leqslant n \leqslant N$. Hence, from (3.25d), we infer that $0 = (-\lambda_n + q_c)\hat{w}_{n,e} + a_n u_e + l_n \sum_{i=1}^{N_0} c_i e_{i,e} + l_n \alpha_1 u_e + l_n \zeta_e$ for all $1 \leqslant n \leqslant N_0$. Combining this latter identity with (3.25a), we obtain that $(A_0 - LC_0)E_e^{N_0} - L\alpha_1 u_e - L\zeta_e = 0$. Invoking (3.12), we note that $\alpha_1 u_e = -\sum_{n \geqslant N+1} c_n w_{n,e} = -\zeta_e$, implying that $(A_0 - LC_0)E_e^{N_0} = 0$. Since $A_0 - LC_0$ is Hurwitz, we infer that $e_{n,e} = 0$ for all $1 \leqslant n \leqslant N_0$. In particular, $\hat{w}_{n,e} = w_{n,e}$ for all $1 \leqslant n \leqslant N$. From (3.25b-3.25d) we deduce that $0 = A_{\rm cl}(\alpha_1)\hat{W}_{a,e}^{N_0} - B_r r_e + \tilde{L}\zeta_e$. Recalling that $\zeta_e = -\alpha_1 u_e$ and $A_{\rm cl}(\alpha_1)$ is defined by (3.18), we obtain that $(A_1 + B_1 K)\hat{W}_{a,e}^{N_0} = B_r r_e$. Since $A_1 + B_1 K$ is Hurwitz, we infer that $\hat{W}_{a,e}^{N_0} = \begin{bmatrix} u_e & \hat{w}_{1,e} & \dots & \hat{w}_{N_0,e} & \xi_e \end{bmatrix}^\top = (A_1 + B_1 K)^{-1} B_r r_e$. We note that $(w_{n,e})_{n \geqslant 1}, (\lambda_n w_{n,e})_{n \geqslant 1} \in l^2(\mathbb{N})$ ensuring that $w_e \triangleq \sum_{n \geqslant 1} w_{n,e}\phi_n \in D(\mathcal{A})$ and $\mathcal{A}w_e = \sum_{n \geqslant 1} \lambda_n w_{n,e}\phi_n$. Using (3.25a), we obtain that $-\mathcal{A}w_e + q_c w_e + a u_e + b v_e = 0$. Introducing the change of variable $z_e = w_e + x^2 u_e$, z_e is a static solution of (3.3a-3.3b) associated with the constant control input $u(t) = u_e$. Denoting by $y_e \triangleq \int_0^1 c(x) z_e(x)\,\mathrm{d}x = \int_0^1 c(x) w_e(x)\,\mathrm{d}x + \gamma_0 u_e$, we infer from (3.25c) while invoking (3.9) that

$$\begin{aligned} r_e &= \sum_{n=1}^{N_0} c_n \hat{w}_{n,e} + \alpha_0 u_e = \gamma_0 u_e + \sum_{n \geqslant 1} c_n w_{n,e} \\ &= \gamma_0 u_e + \tilde{y}_e = y_e. \end{aligned}$$

Hence, for an arbitrarily given constant reference signal $r(t) = r_e \in \mathbb{R}$, the equilibrium condition of the closed-loop system is unique, fully characterized by r_e, and is such that $y_e = r_e$.

We can now introduce the dynamics of deviation of the different quantities w.r.t the equilibrium condition characterized by $r_e \in \mathbb{R}$. In particular:

$$\begin{aligned} \Delta w(t,x) &= \Delta z(t,x) - x^2 \Delta u(t), \\ \Delta \dot{X} &= F\Delta X + \mathcal{L}\Delta\zeta - \mathcal{L}_r \Delta r, \\ \Delta\zeta &= \sum_{n \geqslant N+1} c_n \Delta w_n, \\ \Delta \dot{w}_n &= (-\lambda_n + q_c)\Delta w_n + a_n \Delta u + b_n \Delta v, \ n \geqslant N+1, \\ \Delta v &= K\Delta\hat{W}_a^{N_0}, \\ \Delta\tilde{y} &= \Delta y - \gamma_0 \Delta u = \sum_{n \geqslant 1} c_n \Delta w_n. \end{aligned}$$

We also have $\Delta u(t) = E\Delta X(t)$ and $\Delta v(t) = \tilde{K}\Delta X(t)$ with $E = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$ and $\tilde{K} = \begin{bmatrix} K & 0 & 0 & 0 \end{bmatrix}$. Finally, we define $G = \|a\|_{L^2}^2 E^\top E + \|b\|_{L^2}^2 \tilde{K}^\top \tilde{K}$.

4 Main results on stability analysis

4.1 Stability analysis in $L^2(0,1)$ norm

THEOREM 4.1. *Let $p \in \mathcal{C}^1([0,1])$ with $p > 0$, $q \in \mathcal{C}^0([0,1])$ with $q \geqslant 0$, $q_c \in \mathbb{R}$, and $c \in L^2(0,1)$. Consider the reaction-diffusion system described by (3.3). Let $N_0 \geqslant 1$ and $\delta > 0$ be given such that $-\lambda_n + q_c < -\delta < 0$ for all $n \geqslant N_0 + 1$. Assume that the unique solution of*

(3.21) is such that $\int_0^1 c(x)z(x)\,\mathrm{d}x \neq 0$ *and* $c_n \neq 0$ *for all* $1 \leqslant n \leqslant N_0$. *Let* $K \in \mathbb{R}^{1\times(N_0+2)}$ *and* $L \in \mathbb{R}^{N_0}$ *be such that* $A_1 + B_1K$ *and* $A_0 - LC_0$ *are Hurwitz with eigenvalues that have a real part strictly less than* $-\delta < 0$. *Assume that there exist* $N \geqslant N_0 + 1$, $P \succ 0$, *and* $\alpha, \beta, \gamma > 0$ *such that*

$$\Theta = \begin{bmatrix} F^\top P + PF + 2\delta P + \alpha\gamma G & P\mathcal{L} \\ \mathcal{L}^\top P^\top & -\beta \end{bmatrix} \prec 0, \tag{4.26}$$

$$\Gamma_n = -\lambda_n + q_c + \delta + \frac{1}{\alpha} + \frac{\beta\|c\|_{L^2}^2}{2\gamma} \leqslant 0,$$

for all $n \geqslant N+1$. *Then, for any* $\eta \in [0,1)$, *there exists* $M > 0$ *such that for all* $z_0 \in L^2(0,1)$, $u(0) \in \mathbb{R}$, *and* $r \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R})$, *the mild solution of the closed-loop system composed of the plant (3.3), the integral actions (3.6a) and (3.10), the observer dynamics (3.11) with null initial condition, and the output feedback (3.17) satisfies*

$$\begin{aligned} &\Delta u(t)^2 + \Delta\xi(t)^2 + \sum_{n=1}^{N} \Delta\hat{w}_n(t)^2 + \|\Delta z(t)\|_{L^2}^2 \\ &\quad\leqslant Me^{-2\delta t}(\Delta u(0)^2 + \Delta\xi(0)^2 + \|\Delta z_0\|_{L^2}^2) \\ &\qquad + M \sup_{\tau\in[0,t]} e^{-2\eta\delta(t-\tau)}\Delta r(\tau)^2 \end{aligned} \tag{4.27}$$

for all $t \geqslant 0$. *Moreover, the constraints* $\Theta \prec 0$ *and* $\Gamma_n \leqslant 0$ *are always feasible for* N *large enough.*

Proof. For $P \succ 0$ and $\gamma > 0$ we define

$$V(t) = \Delta X(t)^\top P \Delta X(t) + \gamma \sum_{n\geqslant N+1} \Delta w_n(t)^2.$$

Considering first classical solutions, we have

$$\begin{aligned} &\dot{V}(t) + 2\delta V(t) \\ &= \Delta X(t)^\top \left(F^\top P + PF + 2\delta P\right)\Delta X(t) \\ &\quad + 2\Delta X(t)^\top P\mathcal{L}\Delta\zeta(t) - 2\Delta X(t)^\top P\mathcal{L}_r\Delta r(t) \\ &\quad + 2\gamma \sum_{n\geqslant N+1} (-\lambda_n + q_c + \delta)\Delta w_n(t)^2 \\ &\quad + 2\gamma \sum_{n\geqslant N+1} (a_n\Delta u(t) + b_n\Delta v(t))\Delta w_n(t). \end{aligned}$$

We note that, for any $\alpha > 0$,

$$2\sum_{n\geqslant N+1} a_n\Delta w_n\Delta u \leqslant \frac{1}{\alpha}\sum_{n\geqslant N+1}\Delta w_n^2 + \alpha\|a\|_{L^2}^2\Delta u^2$$

and

$$2\sum_{n\geqslant N+1} b_n\Delta w_n\Delta v \leqslant \frac{1}{\alpha}\sum_{n\geqslant N+1}\Delta w_n^2 + \alpha\|b\|_{L^2}^2\Delta v^2.$$

Since $\Delta u(t) = E\Delta X(t)$ and $\Delta v(t) = \tilde{K}\Delta X(t)$, we have

$$\begin{aligned} &\dot{V}(t) + 2\delta V(t) \\ &\leqslant \Delta X(t)^\top \left(F^\top P + PF + 2\delta P\right)\Delta X(t) \\ &+ 2\Delta X(t)^\top P\mathcal{L}\Delta\zeta(t) - 2\Delta X(t)^\top P\mathcal{L}_r\Delta r(t) \\ &\quad + 2\gamma \sum_{n\geqslant N+1} \left(-\lambda_n + q_c + \delta + \frac{1}{\alpha}\right)\Delta w_n(t)^2 \\ &\quad + \alpha\gamma\left(\|a\|_{L^2}^2\|E\Delta X(t)\|^2 + \|b\|_{L^2}^2\|\tilde{K}\Delta X(t)\|^2\right) \\ &\leqslant \begin{bmatrix}\Delta X(t)\\ \Delta\zeta(t)\end{bmatrix}^\top \begin{bmatrix} F^\top P + PF + 2\delta P + \alpha\gamma G & P\mathcal{L} \\ \mathcal{L}^\top P^\top & 0 \end{bmatrix} \begin{bmatrix}\Delta X(t)\\ \Delta\zeta(t)\end{bmatrix} \\ &\quad - 2\Delta X(t)^\top P\mathcal{L}_r\Delta r(t) \\ &\quad + 2\gamma \sum_{n\geqslant N+1} \left(-\lambda_n + q_c + \delta + \frac{1}{\alpha}\right)\Delta w_n(t)^2. \end{aligned}$$

where $G = \|a\|_{L^2}^2 E^\top E + \|b\|_{L^2}^2 \tilde{K}^\top\tilde{K}$. Recalling that $\Delta\zeta(t) = \sum_{n\geqslant N+1} c_n\Delta w_n(t)$, we obtain that $\Delta\zeta(t)^2 \leqslant \|c\|_{L^2}^2 \sum_{n\geqslant N+1}\Delta w_n(t)^2$. Hence, for any $\beta > 0$, $\beta\|c\|_{L^2}^2\sum_{n\geqslant N+1}\Delta w_n(t)^2 - \beta\Delta\zeta(t)^2 \geqslant 0$. Combining the two latter inequalities, we obtain that

$$\begin{aligned} \dot{V}(t) + 2\delta V(t) &\leqslant \begin{bmatrix}\Delta X(t)\\ \Delta\zeta(t)\end{bmatrix}^\top \Theta \begin{bmatrix}\Delta X(t)\\ \Delta\zeta(t)\end{bmatrix} \\ &\quad - 2\Delta X(t)^\top P\mathcal{L}_r\Delta r(t) + 2\gamma\sum_{n\geqslant N+1}\Gamma_n\Delta w_n(t)^2. \end{aligned}$$

Assume that we can select $N \geqslant N_0 + 1$, $P \succ 0$, and $\alpha, \beta, \gamma > 0$ such that $\Theta \prec 0$ and $\Gamma_n \leqslant 0$ for all $n \geqslant N+1$. Then there exists $\epsilon > 0$ such that $\Theta \preceq -\epsilon I$. We deduce that

$$\begin{aligned} \dot{V}(t) + 2\delta V(t) &\leqslant -\epsilon\|\Delta X(t)\|^2 - 2\Delta X(t)^\top P\mathcal{L}_r\Delta r(t) \\ &\leqslant -\epsilon\|\Delta X(t)\|^2 + 2\|\Delta X(t)\|\|P\mathcal{L}_r\||\Delta r(t)| \\ &\leqslant \frac{\|P\mathcal{L}_r\|^2}{\epsilon}\Delta r(t)^2. \end{aligned}$$

After integration, we obtain for any $\eta \in [0,1)$ the existence of a constant $M_1 > 0$ such that

$$V(t) \leqslant e^{-2\delta t}V(0) + M_1 \sup_{\tau\in[0,t]} e^{-2\eta\delta(t-\tau)}\Delta r(\tau)^2 \tag{4.28}$$

for all $t \geqslant 0$.

On one hand we have $V(0) \leqslant \lambda_M(P)\|\Delta X(0)\|^2 + \gamma\sum_{n\geqslant N+1}\Delta w_n(0)^2$. As the initial conditions of the observer are zero, we obtain that $\|\Delta X(0)\|^2 = \Delta u(0)^2 + \Delta\xi(0)^2 + \sum_{n=1}^{N}\Delta w_n(0)^2$, hence $V(0) \leqslant \max(\gamma, \lambda_M(P))(\Delta u(0)^2 + \Delta\xi(0)^2 + \|\Delta w_0\|_{L^2}^2)$. Since, from (3.4), $\Delta w_0(x) = \Delta z_0(x) - x^2\Delta u(0)$, we obtain the existence of a constant $M_2 > 0$ such that $V(0) \leqslant M_2(\Delta u(0)^2 + \Delta\xi(0)^2 + \|\Delta z_0\|_{L^2}^2)$.

On the other hand we have $V(t) \geqslant \lambda_m(P)\|\Delta X(t)\|^2 + \gamma\sum_{n\geqslant N+1}\Delta w_n(t)^2 \geqslant \lambda_m(P)\{\Delta u(t)^2 + \Delta\xi(t)^2 + \sum_{n=1}^N(\Delta\hat{w}_n(t)^2 + \Delta e_n(t)^2)\} + \gamma\sum_{n\geqslant N+1}\Delta w_n(t)^2$. Noting that $\Delta\hat{w}_n(t)^2 + \Delta e_n(t)^2 = (\Delta w_n(t) - \Delta e_n(t))^2 + \Delta e_n(t)^2 \geqslant \frac{1}{2}\Delta w_n(t)^2$, because[1] $(x-a)^2 + a^2 \geqslant \frac{x^2}{2}$ for all $x, a \in \mathbb{R}$, we infer that $V(t) \geqslant \lambda_m(P)\{\Delta u(t)^2 + \Delta\xi(t)^2 + \frac{1}{2}\sum_{n=1}^N\Delta w_n(t)^2\} + \gamma\sum_{n\geqslant N+1}\Delta w_n(t)^2 \geqslant M_3(\Delta u(t)^2 + \Delta\xi(t)^2 + \|\Delta w(t)\|_{L^2}^2)$ for some constant $M_3 > 0$. Moreover, we deduce from (3.4) that $V(t) \geqslant M_4(\Delta u(t)^2 + \Delta\xi(t)^2 + \|\Delta z(t)\|_{L^2}^2)$ for some constant $M_4 > 0$.

Overall, we have shown for any $\eta \in [0,1)$ the existence of a constant $M > 0$, independent of the initial condition, such that (4.27) holds for classical solutions. By a classical density argument, this result also holds for mild solutions.

We finally assess that we can always select $N \geqslant N_0+1$, $P \succ 0$ and $\alpha, \beta, \gamma > 0$ such that $\Theta \prec 0$ and $\Gamma_n \leqslant 0$ for all $n \geqslant N+1$. By the Schur complement, $\Theta \prec 0$ is equivalent to $F^\top P + PF + 2\delta P + \alpha\gamma G + \frac{1}{\beta}P\mathcal{L}\mathcal{L}^\top P^\top \prec 0$. Introducing $F = F_1 + F_2$ where

(4.29a)
$$F_1 = \begin{bmatrix} A_1 + B_1K & \tilde{L}C_0 & 0 & \tilde{L}C_1 \\ 0 & A_0 - LC_0 & 0 & -LC_1 \\ B_{2,b}K + \begin{bmatrix} B_{2,a} & 0 & 0 \end{bmatrix} & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix},$$

(4.29b)
$$F_2 = \begin{bmatrix} \alpha_1\tilde{L}\begin{bmatrix}1 & 0 & 0\end{bmatrix} & 0 & 0 & 0 \\ -\alpha_1 L\begin{bmatrix}1 & 0 & 0\end{bmatrix} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $\|F_2\| \to 0$, because $\alpha_1 \to 0$, when $N \to +\infty$. We apply the Lemma of Subsection 2.2 to $F_1 + \delta I$, yielding the existence of $P \succ 0$ such that $F_1^\top P + PF_1 + 2\delta P = -I$ and $\|P\| = O(1)$ as $N \to +\infty$. Therefore, we have $F^\top P + PF + 2\delta P + \alpha\gamma G + \frac{1}{\beta}P\mathcal{L}\mathcal{L}^\top P^\top = -I + F_2^\top P + PF_2 + \alpha\gamma G + \frac{1}{\beta}P\mathcal{L}\mathcal{L}^\top P^\top$. Hence, setting $\alpha = N^{1/4}$, $\beta = N$, and $\gamma = N^{-1/2}$, and recalling that (2.2) holds, we infer the existence of $N \geqslant N_0 + 1$ such that $\Theta \prec 0$ and $\Gamma_n \leqslant 0$ for all $n \geqslant N+1$. ∎

4.2 Stability analysis in $H^1(0,1)$ norm

THEOREM 4.2. *Under the assumptions of Theorem 4.1 but with the definition of Γ_n replaced by*

$$\Gamma_n = -\lambda_n + q_c + \delta + \frac{\lambda_n}{\alpha} + \frac{\beta\|c\|_{L^2}^2}{2\gamma\lambda_n},$$

then, for any $\eta \in [0,1)$, there exists $M' > 0$ such that for all $z_0 \in H^2(0,1)$ and $u(0) \in \mathbb{R}$ such that $z_0'(0) = 0$ and $z_0(1) = u(0)$, and all $r \in \mathcal{C}^2(\mathbb{R}_+;\mathbb{R})$, the classical solution of the closed-loop system composed of the plant (3.3), the integral actions (3.6a) and (3.10), the observer dynamics (3.11) with null initial condition, and the output feedback (3.17) satisfies

$$\begin{aligned}(4.30)\quad &\Delta u(t)^2 + \Delta\xi(t)^2 + \sum_{n=1}^N\Delta\hat{w}_n(t)^2 + \|\Delta z(t)\|_{H^1}^2 \\ &\leqslant M'e^{-2\delta t}(\Delta u(0)^2 + \Delta\xi(0)^2 + \|\Delta z_0\|_{H^1}^2) \\ &\quad + M'\sup_{\tau\in[0,t]}e^{-2\eta\delta(t-\tau)}\Delta r(\tau)^2\end{aligned}$$

for all $t \geqslant 0$. Moreover, the constraints $\Theta \prec 0$ and $\Gamma_n \leqslant 0$ are always feasible for N large enough.

Proof. For $P \succ 0$ and $\gamma > 0$ we define

$$V(t) = \Delta X(t)^\top P\Delta X(t) + \gamma\sum_{n\geqslant N+1}\lambda_n\Delta w_n(t)^2.$$

Then we have

$$\begin{aligned}&\dot{V}(t) + 2\delta V(t) \\ &= \Delta X(t)^\top\left(F^\top P + PF + 2\delta P\right)\Delta X(t) \\ &\quad + 2\Delta X(t)^\top P\mathcal{L}\Delta\zeta(t) - 2\Delta X(t)^\top P\mathcal{L}_r\Delta r(t) \\ &\quad + 2\gamma\sum_{n\geqslant N+1}\lambda_n(-\lambda_n + q_c + \delta)\Delta w_n(t)^2 \\ &\quad + 2\gamma\sum_{n\geqslant N+1}\lambda_n(a_n\Delta u(t) + b_n\Delta v(t))\Delta w_n(t).\end{aligned}$$

We note that, for any $\alpha > 0$,

$$2\sum_{n\geqslant N+1}\lambda_n a_n\Delta w_n\Delta u \leqslant \frac{1}{\alpha}\sum_{n\geqslant N+1}\lambda_n^2\Delta w_n^2 + \alpha\|a\|_{L^2}^2\Delta u^2$$

and

$$2\sum_{n\geqslant N+1}\lambda_n b_n\Delta w_n\Delta v \leqslant \frac{1}{\alpha}\sum_{n\geqslant N+1}\lambda_n^2\Delta w_n^2 + \alpha\|b\|_{L^2}^2\Delta v^2.$$

Since $\Delta u(t) = E\Delta X(t)$ and $\Delta v(t) = \tilde{K}\Delta X(t)$ with $E = \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}$ and $\tilde{K} = \begin{bmatrix}K & 0 & 0 & 0\end{bmatrix}$, we have

$$\begin{aligned}&\dot{V}(t) + 2\delta V(t) \\ &\leqslant \Delta X(t)^\top\left(F^\top P + PF + 2\delta P\right)\Delta X(t) \\ &\quad + 2\Delta X(t)^\top P\mathcal{L}\Delta\zeta(t) - 2\Delta X(t)^\top P\mathcal{L}_r\Delta r(t) \\ &\quad + 2\gamma\sum_{n\geqslant N+1}\lambda_n\left(-\lambda_n + q_c + \delta + \frac{\lambda_n}{\alpha}\right)\Delta w_n(t)^2 \\ &\quad + \alpha\gamma\left(\|a\|_{L^2}^2\|E\Delta X(t)\|^2 + \|b\|_{L^2}^2\|\tilde{K}\Delta X(t)\|^2\right)\end{aligned}$$

[1] For any $x, a \in \mathbb{R}$, one has $(x-a)^2 + a^2 = x^2 - 2ax + 2a^2 = \frac{x^2}{2} + (\frac{x}{\sqrt{2}})^2 - 2\frac{x}{\sqrt{2}}\sqrt{2}a + (\sqrt{2}a)^2 = \frac{x^2}{2} + \left(\frac{x}{\sqrt{2}} - \sqrt{2}a\right)^2 \geqslant \frac{x^2}{2}$.

$$\leqslant \begin{bmatrix}\Delta X(t)\\ \Delta\zeta(t)\end{bmatrix}^\top \begin{bmatrix}F^\top P + PF + 2\delta P + \alpha\gamma G & P\mathcal{L}\\ \mathcal{L}^\top P^\top & 0\end{bmatrix}\begin{bmatrix}\Delta X(t)\\ \Delta\zeta(t)\end{bmatrix}$$
$$- 2\Delta X(t)^\top P\mathcal{L}_r\Delta r(t)$$
$$+ 2\gamma\sum_{n\geqslant N+1}\lambda_n\left(-\lambda_n + q_c + \delta + \frac{\lambda_n}{\alpha}\right)\Delta w_n(t)^2.$$

where $G = \|a\|_{L^2}^2 E^\top E + \|b\|_{L^2}^2\tilde{K}^\top\tilde{K}$. Recalling that $\Delta\zeta(t) = \sum_{n\geqslant N+1} c_n\Delta w_n(t)$, we obtain that $\Delta\zeta(t)^2 \leqslant \|c\|_{L^2}^2\sum_{n\geqslant N+1}\Delta w_n(t)^2$. Hence, for any $\beta > 0$, $\beta\|c\|_{L^2}^2\sum_{n\geqslant N+1}\Delta w_n(t)^2 - \beta\Delta\zeta(t)^2 \geqslant 0$. Combining the two latter inequalities, we obtain that

$$\dot{V}(t) + 2\delta V(t) \leqslant \begin{bmatrix}\Delta X(t)\\ \Delta\zeta(t)\end{bmatrix}^\top \Theta \begin{bmatrix}\Delta X(t)\\ \Delta\zeta(t)\end{bmatrix}$$
$$- 2\Delta X(t)^\top P\mathcal{L}_r\Delta r(t) + 2\gamma\sum_{n\geqslant N+1}\lambda_n\Gamma_n\Delta w_n(t)^2.$$

Assume that we can select $N \geqslant N_0 + 1$, $P \succ 0$ and $\alpha, \beta, \gamma > 0$ such that $\Theta \prec 0$ and $\Gamma_n \leqslant 0$ for all $n \geqslant N+1$. Then there exists $\epsilon > 0$ such that $\Theta \preceq -\epsilon I$. We deduce that

$$\begin{aligned}\dot{V}(t) + 2\delta V(t) &\leqslant -\epsilon\|\Delta X(t)\|^2 - 2\Delta X(t)^\top P\mathcal{L}_r\Delta r(t)\\ &\leqslant -\epsilon\|\Delta X(t)\|^2 + 2\|\Delta X(t)\|\|P\mathcal{L}_r\||\Delta r(t)|\\ &\leqslant \frac{\|P\mathcal{L}_r\|^2}{\epsilon}\Delta r(t)^2.\end{aligned}$$

After integration, we obtain for any $\eta \in [0,1)$ the existence of a constant $M_5 > 0$ such that

$$V(t) \leqslant e^{-2\delta t}V(0) + M_5\sup_{\tau\in[0,t]} e^{-2\eta\delta(t-\tau)}\Delta r(\tau)^2 \tag{4.31}$$

for all $t \geqslant 0$. Using (2.1), (3.4), and Poincaré inequality, we infer the existence of a constant $M' > 0$, independent of the initial condition, such that (4.30) holds.

We now show that we can always select $N \geqslant N_0+1$, $P \succ 0$ and $\alpha, \beta, \gamma > 0$ such that $\Theta \prec 0$ and $\Gamma_n \leqslant 0$ for all $n \geqslant N+1$. By the Schur complement, $\Theta \prec 0$ is equivalent to $F^\top P + PF + 2\delta P + \alpha\gamma G + \frac{1}{\beta}P\mathcal{L}\mathcal{L}^\top P^\top \prec 0$. Introducing $F = F_1 + F_2$ where F_1, F_2 are defined by (4.29) with $\|F_2\| \to 0$, because $\alpha_1 \to 0$, when $N \to +\infty$. We apply the Lemma of Subsection 2.2 to $F_1 + \delta I$, yielding the existence of $P \succ 0$ such that $F_1^\top P + PF_1 + 2\delta P = -I$ and $\|P\| = O(1)$ as $N \to +\infty$. Therefore, we have $F^\top P + PF + 2\delta P + \alpha\gamma G + \frac{1}{\beta}P\mathcal{L}\mathcal{L}^\top P^\top = -I + F_2^\top P + PF_2 + \alpha\gamma G + \frac{1}{\beta}P\mathcal{L}\mathcal{L}^\top P^\top$. Hence, setting $\alpha = N^{1/4}$, $\beta = N$, and $\gamma = N^{-1/2}$, we infer the existence of $N \geqslant N_0 + 1$ such that $\Theta \prec 0$ and $\Gamma_n \leqslant 0$ for all $n \geqslant N+1$. ∎

4.3 Setpoint regulation

THEOREM 4.3. *Under both assumptions and conclusions of Theorem 4.1, for any $\eta \in [0,1)$, there exists $M_r > 0$ such that*

$$|\Delta y(t)| \leqslant M_r e^{-\delta t}(|\Delta u(0)| + |\Delta\xi(0)| + \|\Delta z_0\|_{L^2}) + M_r\sup_{\tau\in[0,t]} e^{-\eta\delta(t-\tau)}|\Delta r(\tau)|$$

for all $t \geqslant 0$.

Proof. Noting that $\Delta y(t) = \Delta\tilde{y}(t) + \gamma_0\Delta u(t) = \sum_{n\geqslant 1} c_n\Delta w_n(t) + \gamma_0\Delta u(t)$, we have $|\Delta y(t)| \leqslant \|c\|_{L^2}\|\Delta w(t)\|_{L^2} + |\gamma_0||\Delta u(t)|$. The claimed conclusion follows from Theorem 4.1. ∎

Note that a similar result holds when invoking Theorem 4.2 but when evaluating the PDE trajectory in H^1 norm.

5 Discussion and concluding remarks

In this paper, the regulation problem has been solved for a reaction-diffusion problem with a bounded observation and a boundary control action. The proposed output feedback controller is designed so that the output achieves the setpoint tracking of a reference signal. The designed controller is computed by combining a proportional-integral finite-dimensional controller with a finite-dimensional observer.

This approach is constructive since the design method is based on explicit sufficient conditions that are numerically tractable and always feasible for a large enough dimension of the finite-dimensional observer. Let us emphasize that this approach can be extended to different regulation problem as reaction-diffusion system with unbounded measurement operators (instead of a bounded observation as in this paper) such as Dirichlet and Neumann traces. We refer the reader to [12] for full details of these results.

A Proof of Lemma 3.1

Applying [14, Lemma 2], the pair (A_1, B_1) satisfies the Kalman condition if and only if the pair $\left(\begin{bmatrix}0 & 0\\ B_{0,a} & A_0\end{bmatrix}, \begin{bmatrix}1\\ B_{0,b}\end{bmatrix}\right)$ satisfies the Kalman condition and the matrix

$$T = \begin{bmatrix}0 & 0 & 1\\ B_{0,a} & A_0 & B_{0,b}\\ \alpha_0 & C_0 & \beta_0\end{bmatrix}$$

is invertible. The former condition has been checked in [10]. Hence we only need to evaluate the invertiblity of T. Let $\begin{bmatrix}u_e & w_{1,e} & \dots & w_{N_0,e} & v_e\end{bmatrix}^\top \in \ker(T)$. We obtain that $v_e = 0$ and

$$a_n u_e + (-\lambda_n + q_c)w_{n,e} = 0, \quad 1 \leqslant n \leqslant N_0,$$

$$\alpha_0 u_e + \sum_{n=1}^{N_0} c_n w_{n,e} = 0.$$

Defining for $n \geqslant N_0+1$ the quantity $w_{n,e} = -\frac{a_n}{-\lambda_n+q_c} u_e$, we have $(-\lambda_n + q_c)w_{n,e} + a_n u_e = 0$ for all $n \geqslant 1$. Hence $(w_{n,e})_{n\geqslant 1}, (\lambda_n w_{n,e})_{n\geqslant 1} \in l^2(\mathbb{N})$ ensuring that $w_e \triangleq \sum_{n\geqslant 1} w_{n,e}\phi_n \in D(\mathcal{A})$ and $\mathcal{A}w_e = \sum_{n\geqslant 1} \lambda_n w_{n,e}\phi_n$. This shows that $-\mathcal{A}w_e + q_c w_e + a u_e = 0$. Moreover, using (3.9), we also have

$$0 = \alpha_0 u_e + \sum_{n=1}^{N_0} c_n w_{n,e} = \gamma_0 u_e + \int_0^1 c(x) w_e(x)\,\mathrm{d}x.$$

From the two last identities, we infer that

$$\begin{aligned} &(p w_e')' + (q_c - q) w_e + a u_e = 0, \\ &w_e'(0) = 0, \quad w_e(1) = 0, \\ &\gamma_0 u_e + \int_0^1 c(x) w_e(x)\,\mathrm{d}x = 0. \end{aligned}$$

With $z_e(x) = w_e(x) + x^2 u_e$, we infer that

$$\begin{aligned} &(p z_e')' + (q_c - q) z_e = 0, \\ &z_e'(0) = 0, \quad z_e(1) = u_e, \\ &\int_0^1 c(x) z_e(x)\,\mathrm{d}x = 0. \end{aligned}$$

Assume by contradiction that $z_e \neq 0$. Then $z_e(0) \neq 0$ and $z \triangleq z_e/z_e(0)$ satisfies (3.21) with $\int_0^1 c(x)z(x)\,\mathrm{d}x = 0$. This contradicts our assumption. Thus we obtain that $z_e = 0$, showing that $u_e = z_e(1) = 0$. This implies that $w_e = z_e - x^2 u_e = 0$, hence $w_{n,e} = 0$ for all $n \geqslant 1$. We have shown that $\ker(T) = \{0\}$. This allows to conclude that if $z = 0$ is the only solution of (3.21) then (A_1, B_1) is controllable. The converse holds similarly.

Since A_0 is diagonal with simple eigenvalues, the application of the Hautus test shows that (A_0, C_0) is observable is and only if $c_n \neq 0$ for all $1 \leqslant n \leqslant N_0$.

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