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ON MUIRHEAD'S INEQUALITY

ANTOINE MHANNA^{1*}

ABSTRACT. We revisit Muirhead's inequality together with some related known inequalities. We give a short proof to the Power-Mean inequality and Minkowski's determinant inequality.

1. INTRODUCTION AND PRELIMINARIES

The first section introduces some notations together with known results and inequalities starting by Muirhead's inequality. For $a = (a_1, \dots, a_n)$, $x = (x_1, \dots, x_n)$ two real n -tuples and σ a permutation in S_n , we denote by $[x_\sigma^a]$ the product $x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \cdots x_{\sigma(n)}^{a_n}$ whenever the product is well defined.

The vector \tilde{a}_j is the vector a without the a_j entry and $[x_{\tilde{a}_j}^a]$ is the product without $x_{\sigma(j)}^{a_j}$ that equals to $\frac{x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \cdots x_{\sigma(n)}^{a_n}}{x_{\sigma(j)}^{a_j}}$.

In general we only consider nonnegative entry wise vectors except mentioning otherwise.

Throughout the paper let $a = (a_1, \dots, a_n) \in \mathbb{R}^{n+}$ and $b = (b_1, \dots, b_n) \in \mathbb{R}^{n+}$ two vectors with nonnegative entries arranged in decreasing order: $a_1 \geq \dots \geq a_n$ and $b_1 \geq \dots \geq b_n$.

We say b majorizes a or $a \prec b$ if and only if

$$\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i \quad \text{for all } k, 1 \leq k \leq n \text{ and } \sum_{i=1}^n a_i = \sum_{i=1}^n b_i.$$

Muirhead's inequality: If $a \prec b$ then for any nonnegative n -tuple x

$$(1.1) \quad \sum_{\sigma \in S_n} [x_\sigma^a] \leq \sum_{\sigma \in S_n} [x_\sigma^b].$$

The inequality was first proved for integer exponents in [2], see also [1, 3]. Mainly all proofs of (1.1) use induction and rely on the following general property.

Proposition 1. *Let $a \prec b$ with precisely m different entries, then there exist at most $m - 2$ vectors $c^{(l)}$ $l \geq 1$, such that $a = c^{(0)} \prec c^{(1)} \prec c^{(2)} \prec \dots \prec c^{(j)} \prec b = c^{(j+1)}$ where $j \leq m - 1$ and for $l \geq 0$, $c^{(l)}$ differs from $c^{(l+1)}$ by only two entries.*

Proposition 1 is an important result that can be used to prove the next two classical statements.

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Corollary 1. [3, Chapter 5] *Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be two nonnegative real vectors arranged in decreasing order,*

(1) *If $a \prec b$ then $\prod_{i=k}^n a_i \geq \prod_{i=k}^n b_i$ for every k , $1 \leq k \leq n$.*

(2) *If $\prod_{i=1}^k a_i \leq \prod_{i=1}^k b_i$ for every $k = 1, \dots, n$ then $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for every $k = 1, \dots, n$.*

Proof. For (1) the case $n = 2$ is elementary, applying Proposition 1 we get $\prod_{i=k}^n c_i^{(l)} \geq$

$\prod_{i=k}^n c_i^{(l+1)}$ for every k and l , $1 \leq k \leq n$, $0 \leq l \leq j$. To prove (2) we go by induction,

the result is true for $n = 1$ assuming it holds for $n - 1$ variables and $\sum_{i=1}^n a_i > \sum_{i=1}^n b_i$

we have

$$\sum_{i=1}^{n-1} a_i \leq \sum_{i=1}^{n-1} b_i \leq \sum_{i=1}^n b_i < \sum_{i=1}^{n-1} a_i + a_n.$$

So there exist $h \leq a_n$ such that $\sum_{i=1}^n b_i = \sum_{i=1}^{n-1} a_i + a_n - h$ applying (1) for $k = 1$ to $a = (a_1, a_2, \dots, a_n - h)$ and b yields to a contradiction. \square

Next we present a proof of Muirhead's inequality similar to other known proofs see [1, 3]. Apparently the only difference is that here the sequence $c^{(l)}$ (from Proposition 1) is reversely constructed, see Example 1.

Proof of (1.1). The case $n = 2$ is proved by inspection: $(a_1, a_2) \prec (b_1, b_2)$ and $b_2 \leq a_2 \leq a_1 \leq b_1$ with $a_2 - b_2 = b_1 - a_1 = r$ we get:

$$x_1^{b_2} x_2^{b_2} (x_1^{a_1-b_2} x_2^{a_2-b_2} + x_2^{a_1-b_2} x_1^{a_2-b_2} - x_1^{b_1-b_2} - x_2^{b_1-b_2}) = x_1^{b_2} x_2^{b_2} \cdot T$$

and

$$T = (x_2^r - x_1^r)(x_1^{a_1-b_2} - x_2^{a_1-b_2}) \leq 0.$$

Since the sum is symmetric in x_i , the inequality is true if for some j , $a_j = b_j$ from the previous cases (lower dimension); this is because we can write (1.1) as

$$\sum_{i=1}^n x_i^{a_j} \left[\sum_{\sigma, \sigma(j)=i} [x_{\sigma}^{\tilde{b}_j}] - \sum_{\sigma, \sigma(j)=i} [x_{\sigma}^{\tilde{a}_j}] \right] \geq 0.$$

Thus we only need to give a vector $c = (c_1, \dots, c_n)$ such that $a \prec c \prec b$ with $a_k = c_k$ for some k and $c_h = b_h$ for some index h . We assume $n \geq 3$ and $a_i \neq b_i$ for every i .

Take p to be minimal with $a_i - b_i = r_i > 0$ for all $p \leq i \leq n$ and take q to be minimal with $a_i - b_i = d_i < 0$ for all $q \leq i \leq p - 1$. If $|r_n| \leq |d_q|$ set

$$c = \begin{cases} c_q = a_q + r_n \\ c_n = b_n \\ c_i = a_i \text{ otherwise,} \end{cases} \quad \text{if } |r_n| \geq |d_q| \text{ set } c = \begin{cases} c_n = a_n + d_q \\ c_q = b_q \\ c_i = a_i \text{ otherwise.} \end{cases} \quad \square$$

Example 1. We give an example to illustrate the construction of sequence $c^{(l)}$, say $a = (18, 16, 6, 5, 3, 3)$ and $b = (20, 15, 8, 5, 2, 1)$. We have $a \prec b$ and omitting equal entries we may apply the previous proof to get successively $c^{(1)} = (18, 16, 8, 5, 3, 1)$, $c^{(2)} = (19, 16, 8, 5, 2, 1)$.

2. SOME APPLICATIONS

We first state some applications of (1.1) that recently appeared in Amer. Math. Monthly ([6]) and College Mathematics Journal ([7]) respectively.

Problem 12066: Let n and k be two integers greater than 1. Let A be an $n \times n$ positive definite Hermitian matrix. Prove:

$$(\det(A))^{1/n} \leq \left(\frac{(\operatorname{Tr}(A))^k - \operatorname{Tr}(A^k)}{n^k - n} \right)^{1/k}.$$

We first rewrite the inequality as:

$$\left(\sum_{i=1}^n a_i^n \right)^k - \sum_{i=1}^n a_i^{kn} \geq (n^k - n) \prod_{i=1}^n a_i^k$$

where the eigenvalue of A are $\lambda_i = a_i^n$. Using the multinomial expansion:

$$\left(\sum_{i=1}^n a_i^n \right)^k - \sum_{i=1}^n a_i^{kn} = \sum_{\substack{k_1+k_2+\dots+k_n=k \\ 0 \leq k_t < k}} \binom{k}{k_1, \dots, k_n} \prod_{t=1}^n a_t^{nk_t}$$

and by symmetry we see that the sequence of powers or the vector $l = (nk_1, \dots, nk_n)$ majorizes the n -tuple $r = (k, \dots, k)$ for any partition with $k_1 + \dots + k_n = k$, in fact if $n \geq k$ we have nothing to prove if $n < k$, assume without loss of generality (upon rearranging the entries of l) that $k_1 > \dots > k_n$ and there is j less than n such that $jk > n(k_1 + \dots + k_j)$ so $n(k_{j+1} + \dots + k_n) > (n-j)k$ which means that for some i , $j+1 \leq i \leq n$, $nk_i > k$ a contradiction since we rearranged the vector l in decreasing order and supposed $jk > n(k_1 + \dots + k_j)$ where then for some $m \in [1, j]$, $nk_m < k$. The inequality holds with equality if and only if all eigenvalues are equal.

Another application of this inequality is the following:

Problem 1176: Let A be an $n \times n$ positive semidefinite Hermitian matrix having eigenvalues $(\lambda_1, \dots, \lambda_n)$, $\lambda_1 \geq \dots \geq \lambda_n$ and $\lambda_1 > 0$. For p and q nonnegative integers prove:

$$\frac{\operatorname{Tr}(A^p) + \operatorname{Tr}(A^{p+1}) + \dots + \operatorname{Tr}(A^{p+q})}{\operatorname{Tr}(A^{p+1}) + \operatorname{Tr}(A^{p+2}) + \dots + \operatorname{Tr}(A^{p+q+1})} \leq \frac{n}{\operatorname{Tr}(A)}.$$

We want to prove

$$\frac{\sum_{i=1}^n \lambda_i^p}{\sum_{i=1}^n \lambda_i^{p+q}} \leq \frac{n}{\sum_{i=1}^n \lambda_i^q}$$

for any nonnegative reals p and q , ($p \cdot q \neq 0$) which implies the result by using the next property. This is just true by Muirhead's inequality as

$$\left(\sum_{i=1}^n \lambda_i^p \right) \left(\sum_{i=1}^n \lambda_i^q \right) \leq n \sum_{i=1}^n \lambda_i^{p+q}.$$

Equality occurs only if all eigenvalues are equal.

Proposition 2 ([8]). *Let p_1, \dots, p_n respectively q_1, \dots, q_n be n real respectively positive real numbers. If $m = \min_i \frac{p_i}{q_i}$ and $M = \max_i \frac{p_i}{q_i}$ then*

$$m \leq \frac{p_1 + \dots + p_n}{q_1 + \dots + q_n} \leq M.$$

For sake of completeness we present a short proof of the classical Power-Mean inequality see [5] and the references therein;

Power-Mean Inequality: Let $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$, $\lambda \geq 1$ and for $1 \leq i \leq n$, $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$. We have

$$\sum_{i=1}^n \alpha_i a_i \leq \left(\sum_{i=1}^n \alpha_i a_i^\lambda \right)^{\frac{1}{\lambda}}.$$

Proof. Clearly the result holds for $n = 1$ suppose it is true for any n and we want to prove it for $n + 1$ variables. If for example $a_1 = a_2$ the inequality holds by assumption, otherwise $a_1 > a_2$ taking the derivative in a_1 of $f(a_1) = \left(\sum_{i=1}^n \alpha_i a_i^\lambda \right)^{\frac{1}{\lambda}} - \sum_{i=1}^n \alpha_i a_i$ we obtain

$$f'(a_1) = \alpha_1 \left(\frac{a_1^{\lambda-1}}{\left(\sum_{i=1}^n \alpha_i a_i^\lambda \right)^{\frac{\lambda-1}{\lambda}}} - 1 \right) \geq 0$$

and the inequality follows. \square

Along the same lines we can prove the Weighted Arithmetic Geometric inequality.

Weighted AM-GM Inequality: Let $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ and for $1 \leq i \leq n$, $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$. We have

$$(2.1) \quad \sum_{i=1}^n \alpha_i a_i \geq \prod_{i=1}^n a_i^{\alpha_i}.$$

Another interesting question is to ask when we can replace symmetric sums by cyclic ones. For this we define the cyclic permutation \mathfrak{c} over the set of indexes $(1, \dots, n)$ by $\mathfrak{c}(i) = i + 1$. Of course all indexes are to be assumed $(\text{mod } n)$ and in $[1; n]$. We denote by \mathfrak{c}^n the iteration of n cyclic permutation with the convention that \mathfrak{c}^0 is the identity permutation. *For example:*

$$x_{\mathfrak{c}^3(1)}^{15} x_{\mathfrak{c}^3(2)}^8 x_{\mathfrak{c}^3(3)}^7 x_{\mathfrak{c}^3(4)} = x_4^{15} x_1^8 x_2^7 x_3.$$

In the particular case when $b = (b_1, 0, \dots, 0)$ we have:

Lemma 1. *Suppose $a = (a_1, \dots, a_n) \prec b = (b_1, 0, \dots, 0)$ and $x = (x_1, \dots, x_n)$ with nonnegative entries then:*

$$\sum_{i=1}^n x_i^{b_1} \geq \sum_{i=0}^{n-1} [x_{\mathfrak{c}^i}^{a_i}].$$

Proof. By (2.1) we have:

$$\frac{a_1 x_{\mathfrak{c}^1(1)}^{b_1} + a_2 x_{\mathfrak{c}^1(2)}^{b_1} + \dots + a_n x_{\mathfrak{c}^1(n)}^{b_1}}{b_1} \geq x_{\mathfrak{c}^1(1)}^{a_1} \dots x_{\mathfrak{c}^1(n)}^{a_n},$$

summing over i we obtain the inequality. \square

Remark 1. If $a = \frac{p}{q}$ is a nonnegative rational number and q is odd, then for $x_1 \geq x_2$ it is easy to see that $x_1^a - x_2^a \geq 0$ if $x_1 \geq -x_2$.

Lemma 2. Let $a = (a_1, a_2) \prec b = (b_1, 0)$ where a_1, a_2, b_1 are positive rationals with odd denominators. If $x = (x_1, \dots, x_n)$ with $x_i + x_j \geq 0$ for every $i \neq j$, (x has at most one negative entry), then

$$\sum_{i=1}^n x_i^{b_1} \geq \sum_{i=1}^n x_i^{a_1} x_{i+1}^{a_2},$$

where $x_{n+1} \equiv x_1$.

Proof. For $n = 2$, $x_1^{b_1} + x_2^{b_1} - x_1^{a_1} x_2^{a_2} - x_2^{a_1} x_1^{a_2} = (x_1^{a_2} - x_2^{a_2})(x_1^{a_1} - x_2^{a_1}) \geq 0$, suppose the result holds for $x = (x_1, \dots, x_{n-1})$ we shall prove it for n variables. Assume without loss of generality x_1 to be the maximal variable (positive) and take the derivative of the function $f(x_1) = x_1^{b_1} - x_1^{a_1} x_2^{a_2} - x_n^{a_1} x_1^{a_2}$ to get

$$f'(x_1) = a_1 x_1^{a_1-1} (x_1^{a_2} - x_2^{a_2}) + a_2 x_1^{a_2-1} (x_1^{a_1} - x_n^{a_1}) \geq 0 \quad (\text{Remark 1}).$$

In particular we only need to consider the case $x_1 = x_2$ or $x_1 = x_n$ but in either case the inequality reduces to the case of $n - 1$ variables which is true by induction. \square

The well known Minkowski's determinant inequality -see [4, page 510]- states that for any A and B in $\mathbb{M}_n(\mathbb{C})$ positive semi-definite (Hermitian) we have

$$(2.2) \quad \text{Det}(A + B) \geq \left(\text{Det}(A)^{\frac{1}{n}} + \text{Det}(B)^{\frac{1}{n}} \right)^n.$$

Denote by $\lambda_i(M)$, $1 \leq i \leq n$ the eigenvalues of the Hermitian positive semi-definite matrix M in decreasing order. For any k , $1 \leq k \leq n$, $\sum_{i=1}^k \lambda_i(M) = \|M\|_k$ is by definition the Ky-Fan k -norm of M see [4]. We can write for $1 \leq k \leq n$:

$$\sum_{i=1}^k \lambda_i(A + B) \leq \sum_{i=1}^k \lambda_i(A) + \sum_{i=1}^k \lambda_i(B) = \sum_{i=1}^k (\lambda_i(A) + \lambda_i(B))$$

and $\sum_{i=1}^n \lambda_i(A + B) = \sum_{i=1}^n (\lambda_i(A) + \lambda_i(B)) = \text{Tr}(A + B)$. Setting $a_i = \lambda_i(A + B)$, $b_i = \lambda_i(A) + \lambda_i(B)$ and using Corollary 1 we have

$$(2.3) \quad \text{Det}(A + B) \geq \prod_{i=1}^n (\lambda_i(A) + \lambda_i(B)) \geq \left(\text{Det}(A)^{\frac{1}{n}} + \text{Det}(B)^{\frac{1}{n}} \right)^n.$$

The last bound of (2.3) can be deduced from [9, ex. 219] and is proved as follows: Suppose $\text{Det}(B) \neq 0$ (if not it is easy to conclude), dividing by $\text{Det}(B)$ and letting $\frac{\lambda_i(A)}{\lambda_i(B)} = c_i$ we need to prove

$$\prod_{i=1}^n (c_i + 1) \geq \left(\prod_{i=1}^n c_i^{\frac{1}{n}} + 1 \right)^n,$$

equivalently

$$\sum_{k=0}^n e_k(c_1, \dots, c_n) \geq \sum_{k=0}^n \binom{n}{k} \prod_{i=1}^n c_i^{\frac{k}{n}}$$

where $e_k(c_1, \dots, c_n)$ is the elementary symmetric function of order k . Clearly $e_k(c_1, \dots, c_n) \geq \binom{n}{k} \prod_{i=1}^n c_i^{\frac{k}{n}}$ with equality if and only if all c_i 's are equal.

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